5-1-1985

Mass Economies with Vital Small Coalitions; The f-Core Approach

Peter J. Hammond

Mamoru Kaneko

Myrna Holtz Wooders

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 752

MASS-ECONOMIES WITH VITAL SMALL COALITIONS; THE f-CORE APPROACH

by

Peter J. Hammond

Mamoru Kaneko

and

Myrna Holtz Wooders

May, 1985

Mass-Economies with Vital Small Coalitions; The f-Core Approach*

Peter J. Hammond (Stanford University)
Mamoru Kaneko (University of Tsukuba)
Myrna Holtz Wooders (University of Toronto)

Abstract

A mass-economy is one with many, many agents where each agent is negligible and each trading group is also negligible with respect to the mass-economy. Feasible allocations are those which are virtually attainable by trades only among members of coalitions contained in feasible ("measure-consistent") partitions of the agent set. A feasible allocation is in the core, called the f-core, if it cannot be improved upon by any finite coalition. We show that in a private goods economy with indivisibilities and without externalities, the f-core, the A-core (Aumann's core concept) and the Walrasian allocations coincide. In the presence of widespread externalities, the f-core and the Walrasian allocations coincide but the definition of the A-core is problematic. The conceptual significance of these results will be discussed.

* Research support from the National Science Foundation of the USA, the Office of Naval Research Contract Number N00014-77-0518, and from the Social Sciences and Humanities Research Council and the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. Hammond also wishes to express his gratitude to Ed Green and Yair Tauman for discussions which prompted his interest in this work. Kaneko and Wooders express their gratitude to Martin Shubik for helpful discussions and his continued encouragement and assistance in facilitating their joint research.

1. Introduction

A mass-economy is one with many, many agents, where each agent is negligible and every trading group is also negligible with respect to the mass-economy.* This picture is formulated by Kaneko-Wooders [1984] in a rigorous manner. The set of agents is an atomless measure space but vital coalitions are finite (contain only a finite number of agents) and thus are negligible relative to the mass-economy. A similar sort of formulation in economics was developed by Aumann[1964]. His insightful original motivation (p.39) was to capture the notion of perfect competition, a motivation we share. However, as we will discuss later, our mass-economy framework is formally distinct from Aumann's continuum framework and conceptually better captures the original motivation. In this paper we will apply our structure to exchange economies, investigate the core of a mass-economy, and compare our framework to Aumann's.

Specifically, we test the core concept, called the f-core, of a mass economy with vital small coalitions on economies with and without externalities. Our objective is to examine the properties of the f-core relative to the A-core, the core concept due to Aumann [1964] where coalitions are sets of positive measure. There are two main results of this test: (1) Not surprisingly, the f-core, the A-core, and the Walrasian equilibrium allocations coincide in pure exchange economies; (2) In the presence of widespread externalities, the f-core and the Walrasian equilibrium have natural definitions and coincide, but the definition of the A-core is problematic. We discuss the interpretation of these results in the concluding section of the paper. As a by-product of our work, we also provide a new existence theorem for Walrasian equilibrium in a continuum economy with indivisibilities.

In the remainder of this introduction we will discuss our framework,

---

* We use the term mass - economy as an intuitive expression and also to distinguish our framework from the continuum framework due to Aumann and subsequently used by many others.

concepts and results in more detail. Since we are dealing with finite coalitions in an atomless measure space of agents, two aspects of the model require some discussion; the definition and interpretation of the individual agent, and the appropriate treatment of feasible allocations when only finite coalition formation is allowed.

We consider allocations which are almost achievable by trades (i.e., reallocations) within finite coalitions and which cannot be improved upon by finite coalitions; this set of allocations is the f-core. More precisely, we define "measure-consistent" partitions of the set of agents into finite coalitions. These partitions are defined in a manner consistent with the underlying measure on the space of agents. Relative to a measure-consistent partition p a feasible allocation f (a measurable function from the set of agents to the commodity space) must satisfy $\sum_{a \in S} f(a) \leq \sum_{a \in S} e(a)$ for every coalition S in p where e is the initial endowment function. The set of feasible allocations is the closure (in terms of convergence in measure) of the set of all feasible allocations relative to measure-consistent partitions. The concept "can improve upon" is now standard; a finite coalition S can improve upon an allocation f if there is a reallocation of endowments within S which is preferred by each agent a in S to $f(a)$. The f-core is the set of feasible allocations which cannot be improved upon by any finite coalition.

Following Aumann [1964] it has become standard to define the core of a continuum economy as the set of allocations which cannot be improved upon by any coalition <u>of positive measure</u>. To maintain the idea of the individual agent as effective in coalition formation, it is then necessary to interpret the agent as an arbitrarily "small" set of positive measure. This, however, creates two problems. First, it leaves the individual agent imprecise. In Aumann [1964, p. 41] and Aumann and Shapley [1974, p. 176] appeal is made to some physical theories of mechanics, e.g. fluid mechanics, to justify this approach. However, even in the physical world, if it is necessary to look at problems from the level of particles, then the treatment to which Aumann and Shapley refer is inappropriate. From the viewpoint of microeconomics, it is necessary to look at problems from the level of individual agents. Second, interpreting the agent as an (albeit arbitrarily small) set of positive measure is not completely consistent with Aumann's own original motivation to model perfect competition by making the individual agent negligible.

A second interpretation of an agent in a continuum economy is due to Hildenbrand [1974,1982]. He, as we, regards each agent as a single point in the continuum of agents. We note that this is consistent with the definition of Walrasian equilibrium where an agent is again viewed as a point. Yet the only coalitions allowed to form are those with positive measure so the idea of the agent as effective in coalition formation is not captured; individuals are negligible in coalitions.

In our mass-economy, the individual agent is a single point and thus precise but nevertheless only one of the mass. Since coalitions are finite, individuals can be viewed as effective within coalitions. Our concept of measure-consistent partitions enables us to describe feasible allocations in finite terms (by finite sums of quantities) both at the level of an individual agent and a finite coalition and at the level of the aggregate economy. This approach is discussed further in Kaneko and Wooders [1984], where the game-theoretic framework used here was initially introduced and motivated.

The economic environment we consider allows indivisible goods. It is closely related to that of Mas-Colell [1977] and Khan and Yamazaki [1981], except that we do not restrict the number of divisible goods. A result of Kaneko and Wooders [1984, Lemma 3.1] is applied to show that our set of feasible allocations coincides with the set of feasible allocations in Aumann's sense. We then show equivalence of the f-core, the A-core, and the Walrasian allocations and demonstrate existence under almost the same conditions as Khan and Yamazaki [1981]. We remark that a very similar proof applies to economies with only divisible goods, as considered by Aumann [1964], Hildenbrand [1982] and others.

We then consider an environment with "widespread externalities"; i.e. the utility of an individual depends on his own component of an allocation and the entire allocation (up to null sets). The f-core and Walrasian allocations are shown to coincide. We argue that is difficult to give a natural definition of the A-core for this model.

In the next section we introduce the model and theorems. The third section contains the proofs. In the final section of the paper we evaluate our test results and make some additional remarks.

2. The Model and The Theorems

2.1 Agents and Measure-Consistent Partitions of Agents

Let $(A,\mathcal{A},\mu)$ be a measure space, where A is a Borel subset of a complete separable metric space, $\mathcal{A}$, the σ-algebra of all Borel subsets of A ; and μ , a nonatomic measure with $0 < \mu(A) < +\infty$ * . Each element in A is called an agent . The measure μ represents the distribution of agents. The σ-algebra $\mathcal{A}$ is necessary for measurability arguments but does not play any important game-theoretic role.

Let F be the set of all finite subsets of A . Each element S in F is called a finite coalition or simply a coalition. As mentioned in Section 1, only finite subsets of players can form coalitions in our model.

Remark 2.1. Since a singleton set is closed in A , every coalition is measurable.

Since only finite subsets of players are allowed to form coalitions, allocations are attained by trades within finite coalitions in partitions of the agent set A into finite coalitions. Several conditions must be imposed to ensure that these partitions are consistent with the distribution of agents described by the measure μ . We first define measure-consistent partitions and then motivate these conditions via an example.

Recall that a function ψ from a set B in $\mathcal{A}$ to a set C in $\mathcal{A}$ is a measure-preserving isomorphism from B to C iff (i) ψ is a measure-theoretic isomorphism, i.e., ψ is 1 to 1, onto, and measurable in both directions, and (ii) $\mu(T) = \mu(\psi(T))$ for all $T \subset A$ with $T \in \mathcal{A}$. A partition p of A is measure-consistent iff for any positive

---

* Under our assumptions, $(A,\mathcal{A})$ is measure-theoretically isomorphic to $([0,1],\mathcal{C})$, where $\mathcal{C}$ is the σ-algebra of all Borel subsets of the interval [0,1] (see Parthasarathy (1967), pp. 12-14).

integer k

$$A_k^p \equiv \bigcup_{\substack{S \in p \\ |S|=k}} S \text{ is a measurable subset of } A \text{ ; and}$$

$$\begin{aligned} &\text{each } A_k^p \ (k=1,2,\ldots) \text{ has a partition } \{A_{kt}^p\}_{t=1}^k \text{ , where} \\ &\text{each } A_{kt}^p \text{ is measurable, with the following property:} \\ &\text{there are measure-preserving isomorphisms } \psi_{k1}^p, \psi_{k2}^p, \ldots, \psi_{kk}^p \\ &\text{from } A_{k1}^p \text{ to } A_{k1}^p, \ldots, A_{kk}^p \text{ , respectively, such that} \\ &\{\psi_{k1}^p(a), \ldots, \psi_{kk}^p(a)\} \in p \text{ for all } a \in A_{k1}^p \text{ .} \end{aligned} \tag{2.1}$$

Note that (2.1) implies that for any $S \in p$ with $|S| = k$, we have $S = \{\psi_{k1}^p(a), \ldots, \psi_{kk}^p(a)\}$ for some $a \in A_{k1}^p$. Therefore, for each integer k , A_k^p consists of all the members of k-agent coalitions and A_{kt}^p consists of the t^{th} members of these coalitions. The requirement that all the sets A_{kt}^p have equal measure then captures the idea that coalitions of size k should have as "many" (i.e. the same measure) first members as second members, as many second members as third members, etc.

Let Π denote <u>the set of measure-consistent partitions</u>.

To see the necessity of measure-consistent partitions, consider an economy where the set of agents is $[0,3)$ with Lebesgue measure. The agents indexed by real numbers in $[0,1)$ each own a right-hand glove (RHG) and those in $[1,3)$, a left-hand glove (LHG). Clearly $\{\{a,1+2a\} : a \in [0,1)\}$ is a partition of $[0,3)$ which pairs every agent owning a LHG with an agent owning a RHG. However, this partition violates the meaning of the measure on the space of agents since it makes a set of agents with measure one "equivalent" to (i.e. the same size as) a set of agents with measure two. Were such partitions allowed we would violate the idea of relative scarcities of goods reflected by the measure.

2.2 The Economy and Allocations

Let Z_+ be the set of nonnegative integers and $\Omega = Z_+^I \times R_+^D$ be the consumption set where I is a finite index set for indivisible commodities and D, a finite index set for divisible commodities with $|D|$, the cardinality of D, greater than zero. Let $M = I \cup D$. We use the following conventional symbols for the orderings of elements of Ω: $>>$, $>$, $\geq$. We define the vector $1_M := (\overbrace{1,1,\ldots,1}^{|M|})$, and define 1_I and 1_D analogously.

A preference relation $\succ$ is a subset of $\Omega \times \Omega$ which is an irreflexive, transitive, and continuous (i.e., $\succ$ is open) binary relation. We denote the set of all preference relations by P. The topology on P is the one induced by closed convergence on the class $\{\Omega\times\Omega \backslash \succ : \succ \in P\}$ of closed subsets of $\Omega \times \Omega$ as in Hildenbrand [1974, pp. 96-98].*

For technical convenience and ease, we define $\bar{\Omega} = Z_+^I \times R_+^D \cup \{\infty 1_M\}$, where $\infty 1_M = (\overbrace{\infty,\infty,\ldots,\infty}^{|M|})$, and view this as the consumption set.** Since an allocation will take the value $\infty 1_M$ only on a set of measure zero, this is not a substantive modification. Also, we topologize $\bar{\Omega}$ so that the point $\infty 1_M$ is a finite distance from the set $Z_+^I \times R_+^D$.*** Every preference $\succ$ is extended to $\bar{\Omega}$ so that $\infty 1_M \succ x$ but not $x \succ \infty 1_M$ for all x in $Z_+^I \times R_+^D$ and $\infty 1_M \not\succ \infty 1_M$. Let $\bar{P}$ denote the set of extended preference relations.

An economy E is a measurable mapping of the measure space $(A,\mathcal{A},\mu)$

---

* \ denotes the set theoretic relative complement.

** We could avoid introducing $\infty 1_M$ by taking the completion of the measure space $(A,\mathcal{A},\mu)$ and using additional assumptions, e.g. strict monotonicity on the indivisible goods. (The basic lemmas on the existence of a measure-preserving isomorphism of Kaneko and Wooders [1984, Section 2.3] hold for the completion of $(A,\mathcal{A},\mu)$.)

*** If we take the metric on $Z_+^I \times R_+^D$ as bounded, then this is possible.

into the space of agents' attributes $\bar{P} \times \Omega$, i.e. E assigns to each agent a a preference relation $\succ_a$ and an initial endowment $e(a)$. We assume that $\int_A e$ is finite and strictly positive.

Let $L(A,\bar{\Omega})$ be the set of measurable functions from A to $\bar{\Omega}$. A function f in $L(A,\bar{\Omega})$ is called an allocation with respect to a partition p ($p \in \Pi$) iff

$$\sum_{a' \in p(a)} f(a') \leq \sum_{a' \in p(a)} e(a') \quad \text{for all } a \in A , \tag{2.2}$$

where $p(a)$ denotes the member of the partition containing $a \in A$. An allocation f with respect to p can be attained by trading commodities only within coalitions in p . Define the sets $F(p)$ ($p \in \Pi$) , F and F* by

$$F(p) = \{f \in L(A,\bar{\Omega}) : f \text{ is an allocation with respect to } p\}; \tag{2.3}$$

$$F = \bigcup_{p \in \Pi} F(p) ; \tag{2.4}$$

$$F^* = \{f \in L(A,\bar{\Omega}) : \text{ for some sequence } \{f^\nu\} \text{ in } F , \{f^\nu\} \text{ converges in measure to } f \}. \tag{2.5}$$

Note that $F(p) \subset F \subset F^*$ for all $p \in \Pi$ and F* is the closure of F with respect to convergence in measure. Also, since the endowment e satisfies (2.2) , we have $F(p) \neq \emptyset$. Recall that "$\{f^\nu\}$ converges in measure to f " means that for any $\varepsilon > 0$, $\mu(\{a \in A: d(f^\nu(a),f(a)) > \varepsilon\}) \to 0$ as $\nu \to \infty$.* In interpretation we consider an element of F* as approximately

* If $\{f^\nu\}$ converges in measure to f , then $\{f^\nu\}$ has a subsequence which converges pointwise to f a.e. and, conversely, if $\{f^\nu\}$ converges pointwise to f a.e., then $\{f^\nu\}$ itself converges in measure to f .

feasible in the sense that it is in the closure of F and not necessarily in F itself.

The following example illustrates that F may not be closed with respect to convergence in measure so it may be the case that $F \neq F^*$.

<u>Example 2.1: F is not closed</u>

Suppose that $A = [0,1)$, that $\Omega = \mathbb{R}^2_+$, μ is Lebesgue measure, and

$$e(a) = \begin{cases} (1,0) & (0 \leq a < \alpha) \\ (0,1) & (\alpha \leq a < 1) \end{cases}$$

for some irrational number α. Consider the allocation $f(a) = (\alpha, 1-\alpha)$ (all $a \in A$) which satisfies $\int_A (f-e) = 0$.

Each finite coalition in any measure-consistent partition has an aggregate endowment (m,n) for some pair of integers m,n. To achieve the allocation $(\alpha,1-\alpha)$ for each of the $m+n$ agents, one needs $(m+n)(\alpha,1-\alpha) = (m,n)$ which is impossible because α is irrational. Therefore $f \notin F$. It can be verified that $f \in F^*$. (It can also be verified by appropriate choice of preferences that there may be no Pareto-optimal allocations in F.)

We say that a coalition S in $\mathcal{F}$ <u>can improve upon</u> a function f in $L(A,\bar{\Omega})$ iff there is a vector $(x^a)_{a\in S}$ such that

$$x^a \in \Omega \text{ for all } a \in S ; \qquad (2.6)$$

$$\sum_{a \in S} x^a \leq \sum_{a \in S} e(a) \qquad (2.7)$$

$$x^a \succ_a f(a) \text{ for all } a \in S . \qquad (2.8)$$

The <u>f-core of the economy E</u> is defined to be

$$C_f = \{f \in F^*: \text{no coalition in } F \text{ can improve upon } f\} .$$

An allocation f in the f-core C_f is stable in the sense that no coalition can improve upon f and is approximately feasible in the sense that $f \in F^*$. More precistly there is a sequence $\{f^\nu\}$ in F converging to f with the properties: for any $\varepsilon > 0$ there is a ν_o such that for all $\nu \geq \nu_o$ $\mu(\{a \in A: d(f^\nu(a), f(a)) > \varepsilon\}) < \varepsilon$ and no finite coalition in $A \setminus \{a \in A: d(f^\nu(a), f(a)) > \varepsilon\}$ can "approximately-improve" upon f.

The f-core of an economy is in the same spirit as the f-core of a game, both introduced by Kaneko and Wooders [1984]. There, they demonstrate that with a mild restriction, a type assumption, the f-core of an economy is nonempty. This result could be applied to the model herein if we imposed sufficient conditions to ensure the existence of utility functions.

2.3 The Relationship of the f-core, the A-Core and the Walrasian Allocations

In this subsection, we demonstrate conditions on E under which the f-core, the A-core and the set of Walrasian allocations E_W coincide. As we will discuss further later, in other economic environments this coincidence may not hold. First we give definitions of the A-core and the Walrasian allocations.

The A-core of the economy is the set of allocations

$$C_A = \{f \in L(A,\bar{\Omega}) : \int_A f \leq \int_A e \quad \text{and no subset of } A \text{ of positive measure can improve upon } f\}, \tag{2.9}$$

where a subset S of positive measure is said to be able to improve upon f iff for some $g \in L(S,\Omega)$, we have $\int_S g \leq \int_S e$ and $g(a) \succ_a f(a)$ for all a in S.

A function f in $L(A,\bar{\Omega})$ is called a Walrasian allocation iff for some $p \in R_+^M$,

$$p \cdot f(a) \leq p \cdot e(a) \quad \text{a.e. in } A; \tag{2.10}$$

$$\text{a.e. in } A, \; x \succ_a f(a) \quad \text{for all } x \in \Omega \text{ with } p \cdot x \leq p \cdot e(a); \tag{2.11}$$

$$\int_A f \leq \int_A e. \tag{2.12}$$

Let E_W denote the set of Walrasian allocations.

The first step in showing equivalence of the core concepts and the Walrasian allocations is to demonstrate that $f \in F^*$ iff f satisfies the mean excess demand condition (2.12). This is proved, under slightly less restrictive assumptions on Ω *, in Kaneko and Wooders [1984, Lemma 3.1]. Therefore we have:

$$F^* = \{f \in L(A,\bar{\Omega}) : \int_A f \leq \int_A e\} . \tag{2.13}$$

Additional conditions are required on the economy E : For all $a \in A$, we have

(A.1) $\succ_a$ is strictly monotone on R^D_+ (the divisible commodities);

(A.2) for all $(x_I, x_D) \in \Omega = Z^I_+ \times R^D_+$, there is a $y_D \in R^D_+$ such that $(0_I, y_D) \succ_a (x_I, x_D)$; and

(A.3) $e(a) \succ_a (x_I, 0_D)$ for all $x_I \in Z^I_+$.

Assumption (A.1) is standard. (A.2) is simply that for any commodity bundle (x_I, x_D) , there is a commodity bundle with divisible goods only $(0_I, y_D)$ such that the second bundle is preferred -- "enough" of the divisible goods is better than the given bundle. Our third assumption

---

* The number of divisible goods may be zero.

(A.3) is that the initial endowment is preferred to any commodity bundle with only indivisible goods.

Before stating our theorems, we have a final notation: given two subsets B and C of $L(A,\bar{\Omega})$, we say $B \underset{ae}{\subset} C$ if for any $b \in B$, there is a $c \in C$ such that $b(a) = c(a)$ a.e. in A.

Theorem 1. Under assumptions (A.1), (A.2), and (A.3), we have

(i) $C_f \subset E_W$ and $E_W \underset{ae}{\subset} C_f$; and

(ii) $C_A = E_W$

For our model, the sets of allocations associated with all three concepts coincide (up to null sets). Later, at the end of Section 3, we will remark on a more standard model of an economy with only divisible goods.

It is evident that our assumptions are similar to those of Khan and Yamazaki [1981]. (In fact, if the cardinality $|D|$ of D is equal to one, our assumptions imply theirs.) These authors showed the equivalence of the A-core and the set of "weakly-competitive allocations", and the non-emptiness of the A-core; they employed an example due to Mas-Colell [1977] to illustrate that the set of Walrasian allocations may be empty. Therefore, with their assumptions the A-core and the set of Walrasian equilibrium allocations might not coincide.Our assumption (A.3) rules out Mas-Colell's example.

Our first theorem shows the equivalence of all three solution concepts; our next theorem shows existence.

Theorem 2. Under assumptions (A.1), (A.2), and (A.3), we have

(i) $C_f \neq \emptyset$ and $C_A = E_W \neq \emptyset$.

2.4 An Economy with Widespread Externalities

In this subsection we consider an economy with "widespread" externalities -- externalities depending upon the actions of subsets of agents of positive measure. For this economy we have the equivalence of the f-core and the Walrasian allocations, and we will argue that it is difficult to give a natural definition of the A-core.

We will extend the consumption set $\bar{\Omega}$ so that preferences of agents are defined over both their own consumptions of commodities (in $\bar{\Omega}$) and the allocations in $L(A,\bar{\Omega})$. Therefore we assume

$$\succ_a \subset (\bar{\Omega} \times L(A,\bar{\Omega})) \times (\bar{\Omega} \times L(A,\bar{\Omega}))$$

and typically write

$$[x,f] \succ_a [x',f']$$

to denote the preference between two members of the new consumption set. Given $f \in L(A,\bar{\Omega})$ and $a \in A$ we define the <u>conditional preference</u> $\succ_a(f)$ <u>of</u> $\succ_a$ by

$$x \succ_a (f)\, y \Leftrightarrow [x,f] \succ_a [y,f] .$$

We assume the conditional preference $\succ_a(f)$ is in $\bar{\mathcal{P}}$ and make assumptions (A.1), (A.2), and (A.3) on $\succ_a(f)$ for each $f \in L(A,\bar{\Omega})$ and $a \in A$. We require the additional assumption

(A.4) if $f=g=h$ a.e. in A, then $[x,f] \succ_a [y,g] \Leftrightarrow [x,h] \succ_a [y,h]$.

Let $\mathcal{E}'$ denote an economy satisfying these assumptions.

An allocation $f \in F^*$ is in the <u>f-core</u> of $\mathcal{E}'$, denoted by C_f', if there is no $S \in \mathcal{F}$ such that S has an allocation $(x^a)_{a\in S}$ with the properties:

$$\sum_{a\in S} x^a \leq \sum_{a\in S} e(a) \tag{2.7'}$$

$$x^a \succ_a(f)\ f(a) \quad \text{for all } a \in S\ . \tag{2.8'}$$

Of course if the members of S change their part of the allocation f to $(x^a)_{a\in S}$, then the resulting allocation agrees with f only on $A\setminus S$. However, since S is a finite coalition (2.8') is appropriate by (A.4).

A function f in $L(A,\bar{\Omega})$ is called a <u>Walrasian allocation</u> iff for some $p \in R_+^M$, conditions (2.10), (2.12) and the following hold a.e. in A,

$$x \not\succ_a(f)\ f(a) \quad \text{for all } x \in \Omega \text{ with } p\cdot x \leq p\cdot e(a)\ . \tag{2.11'}$$

Let E'_W denote the set of Walrasian allocations for E'.

<u>Theorem 1'</u>. Under assumptions (A.1) to (A.4), we have

(i') $C'_f \subset E'_W$ and $E'_W \underset{ae}{\subset} C'_f$.

The proof of Theorem 1' can be obtained by obvious modifications of the proof of Theorem 1.

We now discuss the feature that, in the context of this economy, there is no obvious natural definition of the A-core. If the members of a coalition S of positive measure change that part of an allocation under their control, the allocation received by the complementary coalition changes almost necessarily to maintain feasibility of the resulting allocation. Moreover, in the presence of widespread externalities the utilities realized by the members of S depend on the actions of the members of the complementary coalition. Therefore the definition of "can improve upon" depends upon the hypotheses made about the actions of the complementary coalition. This is a general problem in the definition of the characteristic function except in market games (without externalities).

So far, the two approaches most frequently used to define a core (or variations of a core) are the von Neumann-Morgenstern [1953] minimax criterion (more precisely, the α-core, β core, due to Aumann [1967]) and the strong equilibrium, also due to Aumann [1959]. However, in our context, for the reasons of feasibility of a new allocation made by a coalition and the complementary coalition, the strong equilibrium concept cannot be naturally defined. As is well-known, (except for two-person zero sum games) the minimax criterion is not particularly persuasive. In the remainder of this subsection, we will illustrate the minimax criterion for our model.

If preferences are monotonic increasing on the externalities, then the definition according to the minimax criterion of "can improve upon" would be to assume the complementary coalition disposes of all its private goods. More formally, an allocation $f \in F^*$ is in the <u>A-core</u> of E', denoted by C'_A, if there is no subset S of A with positive measure and no allocation $g \in L(A,\tilde{\Omega})$ where

$$\int_S g \leq \int_S e$$

$$g(a) = 0 \quad \text{a.e. in } A\backslash S \text{ and}$$

$$[g(a),g] \succ_a [f(a),f] \quad \text{for all } a \in S .$$

We observe that an allocation in the A-core is Pareto-optimal whereas those in the f-core are not necessarily. Also, the A-core and the Walrasian allocations do not coincide except in very special situations. Even under our simple monotonicity assumptions on the externalities, the minimax criterion does not lead to a natural definition of the A-core since we cannot justify the resulting assumption that a complementary coalition disposes of all its private goods.

For a simple example of an economy with widespread externalities, a two-city housing market, see Kaneko and Wooders [1984].

Remark 2.2 The existence of the Walrasian equilibrium and the f-core can be obtained by imposing only a slight restriction on the externalities. Specifically, we assume that there is a given finite family of subsets of positive measure of A, say $T_1,\ldots,T_n$, such that the externalities depend on an allocation only via the value of its integral over each of the subsets, i.e.

$$\text{(A.4')}\qquad [x,f] \succ_a [y,g] \Leftrightarrow [x,\textstyle\int_{T_1} f,\ldots,\int_{T_n} f] \succ_a [y,\int_{T_1} g,\ldots,\int_{T_n} g] \quad \text{for all } [x,f],\ [y,g] \in \bar{\Omega} \times L(A,\bar{\Omega})$$

and assume preferences are open in $(\bar{\Omega}\times\mathbb{R}_+^{|M|n})\times(\bar{\Omega}\times\mathbb{R}_+^{|M|n})$. Then essentially the same proof of the existence of equilibrium can be applied by using Liapunov's theorem to convexify over the externalities.

3. Proofs

We will prove that $E_W \underset{ae}{\subset} C_f$ and $C_f \subset E_W$. The proof that $E_W \subset C_A$ is standard so it is omitted, as is the proof that $C_A \subset E_W$ since it is essentially the same as the proof that $C_f \subset E_W$. The proof of Theorem 2 can be obtained by modification of Khan and Yamazaki's proof of the existence of the "weakly competitive allocations" and using part of our proof of equivalence. We will remark further on this later.

Proof of $E_W \underset{ae}{\subset} C_f$

Let (p,f) be a Walrasian equilibrium. Let N be a null subset of A such that for all $a \in A \setminus N$,

$$x \succ_a f(a) \Rightarrow p \cdot x > p \cdot f(a) \tag{3.1}$$

Replacing $f(a)$ by $(\infty,\ldots,\infty)$ for all $a \in N$, we obtain a new feasible allocation f' from f,

$$f'(a) = \begin{cases} f(a) & \text{if } a \in A \setminus N \\ (\infty,\ldots,\infty) & \text{otherwise.} \end{cases}$$

Since f' is different from f only on the null set N, (p,f') is also a Walrasian equilibrium. From (2.13) f' is in F^*. It is now standard to prove that the allocation f' is in the f-core. Indeed, if there is a finite coalition S and a vector $(y^a)_{a \in S} \in \Omega^S$ such that $\sum_{a \in S} y^a \leq \sum_{a \in S} e(a)$ and $y^a \succ_a f'(a)$ for all $a \in S$ then $S \subset A \setminus N$. It follows from (3.1) that $p \cdot y^a > p \cdot e(a)$ for all $a \in S$, so $p \cdot \sum_{a \in S} y^a > p \cdot \sum_{a \in S} e(a)$. This contradicts $\sum_{a \in S} y^a \leq \sum_{a \in S} e(a)$. □

Proof of $C_f \subset E_W$

Let f be an allocation in the f-core. Define the set $\psi(a)$ for each agent $a \in A$ by

$$\psi(a) = \{x \in Z^I \times R^D : x + e(a) \succ_a f(a)\} \cup \{0\} .$$

(which is equivalent to Hildenbrand's definition of $\psi(a)$ in [1974, Theorem 1, p. 133]). Then the following Lemma holds.

Lemma 1. There is no $x \in \int\psi$ such that $x << 0$.

From this lemma it follows that the set $\int\psi$ can be separated from the strictly negative orthant of R^M by a hyperplane, because $\int\psi$ is convex by Liapunov's theorem. That is, there is a price vector p in R^M , $p > 0$, such that

$$0 \leq p \cdot z \quad \text{for all} \quad z \in \int\psi . \tag{3.2}$$

Proof of Lemma 1: Suppose there exists a vector $x \in \int\psi$ such that $x << 0$. Then there exists a measurable function $t: A \to Z^I \times R^D$ such that $t(a) \in \psi(a)$ a.e. in A and $\int_A t = x << 0$.

Define $S = \{a \in A: t(a)+e(a) \succ_a f(a)\}$. Then $\mu(S) > 0$ and $x = \int_S t << 0$ and the following claim holds.

Claim. There is a finite partition $(S_0, S_1, \ldots, S_m)$ of S and a simple function $\bar{t}: S \to Z^I \times R^D$ such that

$$\mu(S_1) = \mu(S_2) = \ldots = \mu(S_m) > 0 ; \tag{3.3}$$

$$\bar{t}(a) = \bar{t}(a') \quad \text{if} \quad a, a' \in S_j \quad (j=1,\ldots,m) ; \tag{3.4}$$

$$\bar{t}(a) = 0 \quad \text{for all} \quad a \in S_0 \tag{3.5}$$

$$\bar{t}_D(a) >> t_D(a) \quad \text{for all} \quad a \in \bigcup_{j=1}^{m} S_j$$

$$\bar{t}_I(a) = t_I(a) \quad \text{for all} \quad a \in \bigcup_{j=1}^{m} S_j \ ;$$

$$\int_S \bar{t}(a) << 0 \ . \tag{3.6}$$

Proof of Claim. We begin by partitioning a set consisting of "most" agents into subsets so that if a and a' are in the same subset, then $t(a)$ is approximately equal to $t(a')$. For any positive integer n , define

$$K_n = \{-n,-n+\frac{1}{2^n},\ldots,-\frac{1}{2^n},0,\frac{1}{2^n},\frac{2}{2^n},\frac{3}{2^n},\ldots,n-\frac{2}{2^n},n-\frac{1}{2^n}\} \ ;$$

$$K_n^D = \overbrace{K_n \times \ldots \times K_n}^{|D|}$$

$$Z_n = \{-n,-n+1,\ldots,-1,0,1,\ldots,n-1\} \ ; \text{ and}$$

$$Z_n^I = \overbrace{Z_n \times \ldots \times Z_n}^{|I|} \ .$$

Define a simple function $t^n = (t_I^n, t_D^n): S \to Z^I \times R^D$ by

$$t_I^n(a) = \begin{cases} t_I(a) & \text{if} \quad -n1_M \leq t(a) << n1_M \\ 0_I & \text{otherwise} \ , \end{cases} \tag{3.7}$$

$$t_D^n(a) = \begin{cases} k_D + \frac{1}{2^n} 1_D & \text{if } -n1_I \leq t_I(a) \ll n1_I \text{ and} \\ & t_D(a) \in \prod_{d\in D} [k_d, k_d + \frac{1}{2^n}) \text{ for} \\ & \text{some } k_D \in K_n^D \\ 0_D & \text{otherwise .} \end{cases} \tag{3.8}$$

Then, for each $a \in A$, there is an $\bar{n}$ such $-\bar{n}1_M \leq t(a) \ll \bar{n}1_M$. Therefore the sequence $\{t^n(a)\}$ is nonincreasing for all $n \geq \bar{n}$ and converges to $t(a)$. This implies that the sequence $\{\int_S t^n\}$ converges to $\int_S t \ll 0$. Therefore there is an integer n_o such that

$$\int_S t^{n_o} \ll 0 . \tag{3.9}$$

For $(z_I, k_D) \in Z_{n_o}^I \times K_{n_o}^D$, denote the set $\{a \in A: t_I(a) = z_I$ and $t_D(a) \in \prod_{d\in D} [k_d, k_d + \frac{1}{2^{n_o}})\}$ by $T(z_I, k_D)$. Then it follows from (3.7), (3.8) and (3.9) that

$$\sum_{(z_I,k_D)\in Z_{n_o}^I \times K_{n_o}^D} (z_I, k_D + \frac{1}{2^{n_o}} 1_D)\mu(T(z_I,k_D)) \tag{3.10}$$

$$= \int_S t^{n_o} \ll 0 .$$

From each $T(z_I,k_D)$ $((z_I,k_D) \in Z_{n_o}^I \times K_{n_o}^D)$, we can choose a subset $S(z_I,k_D)$ so that

$$\mu(S(z_I,k_D)) \text{ is a rational number ;} \tag{3.11}$$

$$\sum_{(z_I,k_D)\in Z_{n_o}^I \times K_{n_o}^D} (z_I, k_D + \frac{1}{2^{n_o}} 1_D)\mu(S(z_I,k_D)) \ll 0 . \tag{3.12}$$

Since each $\mu(S(z_I, k_D))$ is a rational number (possibly zero), we can find a subpartition $(S_1, \ldots, S_m)$ of $\{S(z_I, k_D) : (z_I, k_D) \in Z^I_{n_o} \times K^D_{n_o}$ and $\mu(S(z_I, k_D)) > 0\}$ such that $\mu(S_1) = \cdots = \mu(S_m) > 0$. Define S_0 and $\bar{t}$ by

$$S_0 = S - \bigcup_{j-1}^{m} S_j ;$$

$$\bar{t}(a) = \begin{cases} t^{n_o}(a) & \text{if } a \in \bigcup_{j=1}^{m} S_j \\ 0 & \text{otherwise} . \end{cases}$$

It immediately follows that $(S_0, S_1, \ldots, S_m)$ and $\bar{t}$ satisfy conditions (3.3), (3.4) and (3.5). From (3.12) we have

$$\int_S \bar{t} = \sum_{(z_I, k_D) \in Z^I_{n_o} \times K^D_{n_o}} (z_I, k_D + \frac{1}{2^{n_o}} 1_D) \mu(S(z_I, k_D)) << 0 .$$

This completes the proof of the Claim. □

Now select one agent $a_j \in S_j$ for each $j=1,\ldots,m$, and write $C = \{a_1, \ldots, a_m\}$.* Since $\bar{t}_D(a_j) >> t_D(a_j)$ and $\bar{t}_I(a_j) = t_I(a_j)$ for all $j=1,\ldots,m$ by (3.5), we have, by Assumption (A.1),

$$\bar{t}(a_j) + e(a_j) \succ_{a_j} t(a_j) + e(a_j) \succ_{a_j} f(a_j) \quad \text{for all } j=1,\ldots,m .$$

---

* To modify the proof of this theorem to obtain $C_A \subset E_W$, replace C by $\bigcup_{j=1}^{m} S_j$.

The feasibility of the allocation $(\bar{t}(a_j)+e(a_j))_{a_j \in C}$ follows from conditions (3.3) and (3.6); indeed

$$\sum_{a_j \in C} (\bar{t}(a_j)+e(a_j)) = \frac{1}{\mu(S_1)} \sum_{j=1}^{m} \bar{t}(a_j)\mu(S_j) + \sum_{a_j \in C} e(a_j)$$

$$= \frac{1}{\mu(S_1)} \int_S \bar{t} \quad + \sum_{a_j \in C} e(a_j) << \sum_{a_j \in C} e(a_j) \ .$$

Therefore the coalition C can improve upon f, which is a contradiction to the supposition that f is in C_f. This completes the proof of the Lemma. □

Recall that from Lemma 1 it follows that there is a price vector $p \in R^M$ satisfying (3.2). We will now show that p is an equilibrium price vector.

It follows from (3.2) and Hildenbrand [1974, p. 63, Proposition 6] that

$$\inf_{z \in \int \psi} p \cdot z = \int \inf_{x \in \psi(\cdot)} p \cdot x \quad .$$

It also follows from (3.2) that $0 \leq \int_A \inf p \cdot \psi$. Since $\psi(a)$ contains 0, we have $\inf p \cdot \psi(a) \leq 0$. Hence we have $\int_A \inf p \cdot \psi = 0$, which implies that $\inf p \cdot \psi(a) = 0$ a.e. in A . Then this implies that a.e. in A ,

$$x \succ_a f(a) \Rightarrow p \cdot e(a) \leq p \cdot x \text{ , or equivalently,} \tag{3.13}$$

$$p \cdot e(a) > p \cdot x \Rightarrow x \not\succ_a f(a) \ .$$

Lemma 2. (i) $p \cdot e(a) = p \cdot f(a)$ a.e. in A (budget constraints are satisfied).

Proof. For all $a \in A$, by Assumption (A.1) we can choose a sequence $\{y^n\}$ such that $\{y^n\}$ converges to $f(a)$ and $y^n \succ_a f(a)$ for all n. Then we have, from (3.13), $p \cdot e(a) \leq p \cdot y^n$ for all n. This implies $p \cdot e(a) \leq p \cdot f(a)$ a.e. in A. If $p \cdot e(a) < p \cdot f(a)$ for a set of agents with positive measure, then we obtain $p \cdot \int e < p \cdot \int f$. Since f is in the f-core $\int_A e \geq \int_A f$ by (2.13) and we have a contradiction. □

Remark 3.1. We have now shown f is a "weak Walrasian equilibrium allocation". i.e., it satisfies (3.13) and budget constraints. In the remainder of our arguments, we show that f is a Walrasian equilibrium allocation without using directly the fact that f is in the f-core. Therefore our arguments also show that a weak equilibrium allocation is a Walrasian allocation. Consequently, Khan and Yamazaki's proof [1981, pp. 223-224, Proof of Proposition 2] of the existence of the weak equilibrium provides a proof of the existence of the Walrasian equilibrium in our framework.

Lemma 3. $p_D >> 0_D$.

Proof. First we prove that $p_D > 0_D$. Suppose $p_D = 0_D$. Since $\int_A e >> 0$ and $p > 0$ there is a subset S of A such that $p \cdot e(a) > 0$ for all $a \in S$ and $\mu(S) > 0$. Choose an agent a from S for whom (3.13) holds and

$p\cdot f(a) = p\cdot e(a) > 0$. By Assumption (A.2), there is a vector $y = (0_I, y_D) \in \Omega$ such that $y \succ_a f(a)$. Since $p_D = 0_D$, we have $0 = p\cdot y < p\cdot f(a) = p\cdot e(a)$, which contradicts (3.13). This shows that $p_D > 0_D$.

Now let us show that $p_D >> 0_D$. Since $\int_A e \geq \int_A f$ by (2.13) and $p \cdot \int_A e = p \cdot \int_A f$ by Lemma 2, we have, for all $k \in I \cup D$,

$$p_k > 0 \Rightarrow \int_A e_k = \int_A f_k \tag{3.14}$$

Since $p_D > 0_D$ and $\int_A e_D >> 0_D$, it follows from (3.14) that $p_D\cdot\int_A f_D = p_D\cdot\int_A e_D > 0$. Therefore there is a subset S of A with postive measure such that $p_D\cdot e_D(a) = p_D\cdot f_D(a) > 0$ and (3.13) holds for all $a \in S$.

Consider an arbitrary agent a in S . We shall prove that

$$x \succ_a f(a) \text{ and } x_I = f_I(a) \Rightarrow p\cdot x > p\cdot f(a). \tag{3.15}$$

This implies $p_D >> 0_D$; indeed, if $p_d = 0$ for some $d \in D$, then x , given by $x_d = f_d(a)+1$ and $x_k = f_k(a)$ for all $k \neq d$, violates (3.15) because of Assumption (A.1). Therefore we prove (3.15). Suppose on the contrary that $x \succ_a f(a)$, $x_I = f_I(a)$ and $p\cdot x \leq p\cdot f(a)$ for some x . Because $\succ_a$ is continuous, there exists $\lambda \in (0,1)$ near 1 so that $\lambda(x_I,0) + (1-\lambda)x \succ_a f(a)$ and $p\cdot[\lambda(x_I,0)+(1-\lambda)x] = \lambda p_I\cdot x_I + (1-\lambda)p\cdot x \leq \lambda p_I\cdot f_I(a) + (1-\lambda)p\cdot f(a) = p\cdot f(a) - \lambda p_D\cdot f_D(a) < p\cdot f(a)$. This is a contradiction. So $p_D >> 0_D$. □

Lemma 4. a.e. in A , $p\cdot x \leq p\cdot e(a) \Rightarrow x \not\succ_a f(a)$ (utility maximization on the budget sets).

Proof. It follows from Assumption (A.3) that $e_D(a) > 0_D$ for all $a \in A$. This together with Lemma 3 implies that $p \cdot e(a) > 0$ for all $a \in A$.

Consider the agents a for whom (3.12) ($p \cdot x < p \cdot f(a) \Rightarrow x \not\succ_a f(a)$) holds. It suffices to show that, for these agents, $p \cdot x = p \cdot e(a) \Rightarrow x \not\succ_a f(a)$. Suppose that $p \cdot x = p \cdot e(a)$, $x_D > 0_D$ and $x \succ_a f(a)$ for some x. By continuity of $\succ_a$ and Lemma 3 we can find an x' such that $p \cdot x' < p \cdot e(a)$ and $x' \succ_a f(a)$, which is a contradiction. Finally suppose that $p \cdot x = p \cdot e(a)$, $x_D = 0_D$ and $x \succ_a f(a)$ for some x. Then Assumption (A.3) implies that $e(a) \succ_a x \succ_a f(a)$, so $e(a) \succ_a f(a)$ by transitivity of $\succ_a$. (This contradicts our assumption that f is in the f-core, so we have shown Lemma 4. So that we can apply our proofs to the existence of a Walrasian equilibrium we do not use this argument.) Since $e_D(a) > 0_D$, by continuity of $\succ_a$ we can find an x' such that

$$x'_D < e_D(a) \text{ , } x'_I = e_I(a) \text{ and } x' \succ_a f(a) \text{ .}$$

Since $p_D \gg 0_D$ by Lemma 3, we have $p \cdot x' < p \cdot e(a)$, which contradicts (3.12). □

Remark 3.1. In more standard environments with only divisible goods, c.f. Hildenbrand [1974], Theorem 1 can be demonstrated without transitivity and by replacing our assumptions (A.1), (A.2), (A.3) on preferences by either one of the following: (i) $x > y \Rightarrow x \succ y$ or (ii) $x \succ y$ if there exists a $z \in R^M_+$ such that $x \gg z$ and $z \succ y$. The main step of the proof herein, the Claim of Lemma 1, can be modified as indicated in the footnote, and then one can apply standard arguments.

4. Conclusions

In this section, we will first discuss our test results and then make some additional remarks.

In the context of private goods exchange economies without externalities we obtain equivalence of the f-core, the A-core, and the Walrasian allocations in reasonably general environments -- in fact, our theorem for the A-core and the Walrasian allocations by itself extends the existing literature. (As a by-product, we obtain a new existence result for the Walrasian equilibrium.)

Our equivalence result for the f-core and the Walrasian allocations can be viewed as the direct limit version of convergence results for large but finite economies, cf. Shubik [1959], Debreu and Scarf [1963] and Anderson [1978]. For large but finite economies, no matter how large the economy "improve upon" is defined for finite coalitions and agents are nonnegligible within coalitions. Our mass-economy model preserves these properties; all improvements must be done by finite coalitions in which individual agents are nonnegligible. There is no discontinuity in the nature of a coalition as one goes from the large finite to the mass-economy. In Aumann's continuum economy, although the core is, in effect, the same, in contrast coalitions are uncountably infinite and there is a discontinuity in the nature of a coalition when one goes from the large finite to the continuum economy. Mas-Colell's [1979] results on core-equilibrium equivalence in large (but finite) economies where the relative sizes of blocking coalitions are constrained to be arbitrarily small further reinforces our position.* The conclusion from the first test,

---

* One may be tempted to regard Schmeidler's [1972] results on core-equilibrium equivalence where coalitions are constrained to be of "small" positive measure as a continuum form of Mas-Colell's. However, the discontinuity in the nature of a coalition as one goes from the finite to the continuum case still holds.

the application of the mass-economy framework and the f-core concept to an economy without externalities, is that we capture the behavior of the core in large, finite economies while maintaining the precision and effectiveness of the individual agent.

In the context of a situation with widespread externalities, equivalence of the f-core and the Walrasian allocations and the lack of a natural definition of the A-core is demonstrated. These test results indicate that the mass-economy and the f-core concepts capture the notion of "perfect" competition and are fundamentally distinct from the continuum economy and the A-core concepts.

Remark 4.1. Another type of consideration was in fact the original source of one author's interest in finite coalitions. This concerns the power of finite coalitions to manipulate resource allocation mechanisms in continuum economies. When considering questions of group incentive compatibility, the size of permissible coalitions -- finite, or of positive measure -- is critical; see Hammond [1983].

Remark 4.2. In "overlapping generations" economies, following the original model of Samuelson [1958], it seems that the f-core and the A-core may not coincide. We expect the f-core to coincide with the set of "Walrasian equilibria" just as Chae's [1983] "bounded core" does. The A-core, however, consists entirely of Pareto efficient allocations, while the Walrasian equilibria in such economies may be Pareto inefficient.

Remark 4.3. Other authors have investigated finite coalitions in infinite economies without a measure-theoretic structure, for example Keiding [1976]. Keiding's approach however, can give counterintuitive results in situations with a measure space of agents. To illustrate, in Keiding's framework for our glove example of Section 2.1, the partition which pairs every agent owning a LHG with an agent owning a RHG is feasible even though the measure of LHGs is twice that of RHGs.

References

Anderson, R.M. [1978], "An Elementary Core Equivalence Theorem", Econometrica, 46, 1483-1487.

Aumann, R.J. [1959], "Acceptable Points in General Cooperative n-person Games", in A.W. Tucker and R.D. Luce, eds. Contributions to the Theory of Games Vol. IV, Ann. Math. Stud. 40, Princeton University Press, Princeton, N.J., 287-324.

__________ [1964], "Markets with a continuum of traders", Econometrica, 32, 39-50.

__________, [1967], "Cooperative Games without Side Payments", Essays in Mathematical Economics in Honor of Oskar Morgenstern, ed. M. Shubik, Princeton University Press, Princeton, New Jersey.

__________ and L.S. Shapley, [1974], Values of Nonatomic Games, Princeton University Press, Princeton, New Jersey.

Chae, S. [1983], "Bounded core equivalence and core characterizations", University of Pennsylvania, CARESS Working paper #83-02.

Debreu, G. and H. Scarf [1963], A limit theorem on the core of an economy, International Economic Review 4, 235-246.

Hammond, P.J. [1983], "Multilateral Incentive Compatability in Continuum Economies", Stanford University Institute for Mathematical Studies in the Social Sciences, Economics Technical Report No. 435.

Hildenbrand, W. [1974], Core and Equilibria of a Large Economy, Princeton University Press.

Hildenbrand, W. [1982], "The Core of an Economy", ch. 18, pp. 831-877 of K.J. Arrow and M.D. Intriligator (eds), Handbook of Mathematical Economics, Vol. II (Amsterdam: North-Holland).

Kahn, M.A. and A. Yamazaki [1981], On the cores of economies with indivisible commodities and a continuum of traders, Journal of Economic Theory 24, 218-225.

Kaneko, M. and M.H. Wooders [1984], "The core of a game with a continuum of players and finite coalitions; the model and some results", IMA Preprint Series No. 111, Institute for Mathematics and its Applications, University of Minnesota.

Keiding, H. [1976], "Cores and equilibria in an infinite economy", in J. Los and M.W. Los (eds), Computing Equilibria: How and Why (Amsterdam: North-Holland), 65-73.

Mas-Colell, A. [1977], "Indivisible commodities and general equilibrium theory", Journal of Econometric Theory 16, 443-456.

Mas-Colell, A. [1979], "A Refinement of the Core Equivalence Theorem", Economics Letters, 3, 307-310.

Parthasarathy, K.R. [1967], _Probability Measures on Metric Spaces_, Academic Press.

Samuelson, P.A. [1958], "An Exact consumption-loan model of interest with or without the social contrivance of money", _Journal of Political Economy_ 66, 467-482.

Schmeidler, D. [1972], "A Remark on the Core of an Atomless Economy", _Econometrica_ 40, 579-580.

von Neumann, J. and O. Morgenstern [1953], _Theory of Games and Economic Behavior_, 3rd ed., Princeton University Press, Princeton.