Fiscal and Monetary Policy in a General Equilibrium Model

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by

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Introduction

This is a theoretical study of monetary and fiscal policy in a general equilibrium model with rational expectations, with perfect markets for current goods, but with restrictions on borrowing and insurance and with a Clower constraint on payments. Fiscal actions are understood to be manipulations of taxes and subsidies. Monetary policy is understood to be the purchase and sale of government debt or control of the banking system's ability to lend.

The model contains the following elements. There are overlapping generations, each of which lives many periods. Consumers face individual risk. There is a continuum of individuals and the individual risks are mutually independent. For this reason, individual risks do not cause fluctuations in the aggregate. There is also uncertainty at the aggregate level. This uncertainty appears in production functions and preference orderings and is generated by a random variable which everyone observes. The individual uncertainty faced by one person is not verifiable by other agents and so is uninsurable. The aggregate uncertainty is insurable in one version of the model and is not insurable

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in a second version. When agents can insure, they can sell insurance only against the collateral of assets held. Such sales of insurance can also be used to borrow. Individuals are never permitted to borrow against future wage income. Individuals hold assets for self-insurance and perhaps as a reserve to be spent during retirement. The assets which can be held are money, government debt and equity in firms. There is a Clower constraint on payments, which gives money a role besides that of a store of value. All income is received in money at the end of a period. Even though all markets always clear, there is a form of unemployment. There are several industries, and when a worker transfers from one industry to another, he has to pass through a fixed period of unemployment. This period may be interpreted as search time, but is not modelled as such. It is simply part of the technology and is included only to give a likely mechanism by which aggregate uncertainty could give rise to trade fluctuations.

By "trade fluctuations," I mean what is commonly called business cycles.[1] It is easy to imagine how something resembling trade fluctuations could occur in the model, even though all markets clear. Suppose there is a change of consumer tastes and that one industry must reduce output. Wages in that industry would fall and some people working there would find it advantageous to change jobs and so would suffer a period of unemployment. The budgets of the temporarily unemployed would slowly deteriorate and these people would be obliged to decrease steadily their spending. The same might be true of the people who remained at their jobs and earn reduced wages. Those working in unaffected industries would have extra savings, for they would no longer spend as much on the newly unpopular commodity. If they do not spend these savings, they will accumulate assets. This accumulation could increase their spending on other goods, offsetting the decline in spending by the unemployed. But the savings might be kept, perhaps to buy a new product that would satisfy the new tastes when it became available. If the net result of the changes in expenditure were a steady decline in aggregate spending, deflation would result. The deflation could in turn increase real interest rates and so discourage investment. So, temporary deflation (or inflation) could result from simple changes of taste, and such episodes could affect investment.

One may well ask "so what?", for in an overlapping generations model, Pareto optimality does not require that the real interest rate be constant. Fluctuations in investment resulting from fluctuations in the real interest rate do not indicate lack of Pareto optimality. This is so even though fluctuations in investment may cause fluctuations in employment as labor shifts in and out of investment good industries. In the model of this paper, unemployment is simply part of the production process. One can argue that fluctuations in real interest rates decrease welfare only if one has in mind a social welfare function which assigns relative weight to the welfare of different generations.

With some such welfare function in mind, I prove, roughly speaking, that there exists a monetary and fiscal policy which fixes the price level and the nominal interest rate on government bonds.[2] This result is similar in spirit to the second welfare theorem of general equilibrium theory. There is an ambiguity as to what is meant by "price level." If the object is to fix the real rate of interest, then the usual Euler equation of optimal growth theory seems to indicate that one should fix consumers' average marginal utility of expenditure.[3]

This is what I do, but I could equally well have fixed some conventional price index. It would have the advantage of being observable.

I do not maximize any explicit social welfare function. I believe that doing so would lead to enormous complications. It seems that in order to have an interesting model with money and asset holding, one must include features which preclude also Pareto optimality of equilibrium. Optimization of a social welfare function in such a model would involve all the problems of second best optimization. A number of features of the model I use prevent Pareto optimality. They include constraints on insurance and borrowing, the Clower constraint on payments, the lack of compensation for time spent unemployed, and the fact that taxes are not lump-sum.

The proof that a stabilizing policy exists is also a proof that equilibrium exists. One may wonder whether equilibrium exists only because of the policy. For this reason, I also prove existence of equilibrium when government policy is neutral. A neutral policy is one which leaves constant the money supply and supply of government debt.

This result may be viewed as a generalization of the many theorems which exist proving the existence of equilibrium with money and uninsured risk. It generalizes them by including stocks and a special form of insurance sold against collateral. Papers containing existence theorems with money and uninsured risk include Lucas (1980), Hellwig (1982), Townsend (1983), and Bewley (1980a).

As stated earlier, I consider two versions of the model, one with insurance of aggregate uncertainty and one without. My primary reason for introducing insurance is to show what insurance looks like when modelled in a way which seems to me realistic. The realistic aspect is that insurance can be sold only against transferable collateral. Future wages cannot be used as collateral. In my inexpert opinion, these restrictions reflect roughly what occurs in practice. There are obvious problems of moral hazard which make it dangerous to accept future wages as collateral. The insurance I define is not equivalent to unlimited trading on Arrow-Debreu markets in contingent claims. People may not be able to buy full coverage, for in order to buy a great deal of insurance against one state of nature, they might have to sell insurance against other states. A seller of insurance would already have to have accumulated assets, and he may not have accumulated many assets or he might not have sufficient incentive to do so. If his rate of pure time preference were high relative to interest rates, he would accumulate few assets. The insurance I define would certainly decrease the magnitude of the sort of deflationary episodes described earlier, but would not necessarily prevent them.

Another reason for including insurance is that it makes possible a kind of asset pricing formula. Without such insurance, no such formula seems possible.

When one permits insurance one permits borrowing as well, for borrowing is the same as the sale of equal amounts of insurance against all states of nature. Borrowing in turn makes it possible to distinguish inside from outside money. The distinction may be important, for an increase in the aggregate supply of inside money need not have the same effects as an increase in outside money. However, the possibilities opened by this distinction are not pursued very far here.

Four types of equilibria are possible, corresponding to whether insurance is allowed and to whether the government has a neutral or stabilizing policy. I prove equilibrium existence theorems for two of these cases, that with no insurance and a stabilizing policy and that with insurance and neutral policy. The other two cases could be handled in a similar fashion.

Models with incomplete markets have their own special set of difficulties. One is that different shareholders may not agree on what a firm should do if they face uninsured risk which is correlated with the firm's profits. Each shareholder would want his firm to emphasize periods or states of nature where he had the greatest need. I avoid this problem by making special assumptions. Each firm is finite lived and corresponds to an investment project. Firms do not choose investments. Consumers do so when they invest in new projects. The technology of each firm is such that it cannot affect the temporal or stochastic distribution of its profits.

Another problem is that it is hard to prove that prices of capital goods are finite. This problem does not occur when markets are complete, for in that case prices are the dual variables of a social maximization problem. Here they are not. I avoid this problem by assuming that the only capital inputs are labor.

A number of other special assumptions are made in order to avoid technical problems. I cannot pretend to be sure that the conclusions drawn would survive removal of the special assumptions.

The plan of the paper is as follows. The next section contains the formal model. The assumptions and theorems follow. In Sections 5-13, I discuss the model and the results. In particular, in Section 6, I discuss the fact that Pareto optimality puts almost no restrictions on interest rates. In Section 7, I show that trade fluctuations may occur in the model. Here, I draw on a recent paper by Scheinkman and Weiss (1983). In Section 8, I show that one may include inside money in the model, and discuss briefly the idea of Sargent and Wallace (1980) that the money supply should be market determined. In Section 10, I discuss briefly an asset pricing formula. In Section 11, I show that transaction costs in the trading of assets would cause a discontinuity in individual consumer demand. The proofs are in Sections 14 and 15. They are long and complicated, but they use arguments typical of equilibrium theory.

2. The Model

Time

Time is discrete. Time periods should be interpreted as short, such as a day or a week.

Consumers

A continuum of consumers is born in each period t . Each consumer lives N periods, where N is a positive integer. The consumer born in period t is indexed by (i,t) , where $i \in [0,1]$.

Random Disturbances

Each consumer (i,t) observes a random variable $s_n^{(i,t)}$ in the $(n+1)^{st}$ period, where $n = 0, 1, \ldots, N-1$. Nobody but consumer (i,t) observes $s_n^{(i,t)}$, so that it represents individual uninsurable risk. The sequences $(s_0^{(i,t)}, \ldots, s_{N-1}^{(i,t)})$ are mutually independent and identically distributed as i and t vary. Each $s_n^{(i,t)}$ belongs to a finite set S , which is the same for all (i,t) . Very often, the superscript (i,t) is suppressed from the symbol " $s_n^{(i,t)}$."

All consumers alive at time t observe a random variable θ_t. Each of the θ_t belongs to a finite set Θ. The stochastic process $(\theta_t)_{t=1}^{\infty}$ is independent of each of the processes $(s_n^{(i,t)})_{t=0}^{N-1}$.

The finite sequence $(\theta_1, \theta_2, \ldots, \theta_t)$ is denoted $\underset{\sim}{\theta}_t$. Similarly, $\underset{\sim}{s}_n^{(i,t)}$ denotes $(s_0^{(i,t)}, \ldots, s_n^{(i,t)})$.

Consumers born at time t should really be distinguished by the event $(\underset{\sim}{\theta}_t, s_0^{(i,t)})$ in which they are born. This distinction is made implicitly when it is required that a consumer's allocation be measurable with respect to these events. Hence, I will speak of the consumer (i,t) as if he were one person.

Commodities

The same finite set of commodities may be traded at any time. The set of these commodities is C. The set of produced goods is denoted C_p, and $C \sim C_p$ is denoted by C_w. C_w is the set of types of labor. The set of consumption goods is denoted C_c.

R^C denotes the Euclidean space of real-valued functions on C. The symbols R_+^C, R^{C_c} and so on have the obvious meanings.

Utility

Each consumer has a utility function which is additively separable with respect to time. The one period utility of consumer (i,t) is $u_i : R_+^{C_c} \times \Theta \times S \to R$. He discounts future utility by a factor δ, where $0 < \delta < 1$. The lifetime utility of consumer (i,t), born in period t, is $E \sum_{n=n_t}^{N} \delta^n u_i(x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n^{(i,t)}), \theta_{t+n}, s_n) \equiv U_i(x)$, where $x = (x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n))_{n=n_t}^{N-1}$ and where $n_t = \max(0, -t+1)$.

Production

There are J industries, where J is a positive integer. For each j, there is a technology, which is described by i) $K_j \in R_-^C$, ii) a positive integer N_j, and iii) an input-output possibility set $Y_j(\theta)$, for each $\theta \in \Theta$. K_j represents the initial capital of one unit of investment. N_j is the lifetime of the investment. $Y_j(\theta)$ is the input-output possibility set made possible by one unit of investment.

A firm in industry j is represented by the date-event pair of its birth $(t, \underset{\sim}{\theta}_t)$ and by an initial capital γK_j, for some $\gamma > 0$. The firm then lives N_j periods after period t and has the use of the production possibility sets $\gamma Y_j(\theta_{t+n})$ in each of those periods.

Endowment

The only endowment consumers have is labor. Endowments are defined by a function $k : [0,1] \to C_w$. A consumer (i,t) can offer labor of type $k(i)$.

Whether a consumer can work in any period is also a random event. Consumer (i,t) can work in period $t+n$ if $\omega_{in}(s_n^{(i,t)}) = 1$, where $\omega_{in} : S \to \{0,1\}$. If $\omega_{in}(s_n^{(i,t)}) = 0$, he cannot work. (He may be sick or retired.)

When a consumer works, he offers one unit of labor per period.

Unemployment

Workers may be employed either directly by other consumers or by one of the J industries. This makes $J+1$ employment sectors. If a worker changes employment from one sector to another, he must pass through a period of unemployment. A worker offering labor of type

$k \in C_w$ must wait T_k periods before obtaining a new job, where $T_k \geqq 0$. After T_k periods he may obtain a job in any sector which uses his labor. Labor markets always clear and workers always change jobs voluntarily.

<u>The Extended Set of Commodities</u>

It is necessary to treat the same kind of labor emyloyed in different sectors as distinct commodities earning different wages. For this reason, I define an extended set of commodities, $CE = (C \setminus C_w) \cup \{(k,j) | k \in C_w, j = c, 1, \ldots, J\}$. The symbol (k,j) stands for labor of type k used in sector j . The sector c is the consumer sector. From now on, all commodity vectors referring to one period will be thought of as belonging to R^{CE} . The input-output possibility sets $Y_j(\theta)$ will be thought of as subsets of R^{CE} by the obvious embedding.

<u>Initial Conditions in Production</u>

For $t \leqq 0$, $w_0^{(i,t)}$ denotes the number of consecutive periods individual (i,t) has been unemployed in the periods immediately preceding period 1.

For $j = 1, \ldots, J$ and $m = 0, \ldots, N_j - 1$. $\bar{h}_{jm0}$ will denote the size of the investment in industry j made in period $-m$. That is, $\bar{h}_{jm0}K_j$ was invested in industry j in period $-m$.

<u>Resource Allocations</u>

A resource allocation for industry j is represented by a vector $y_j = (y_{jmt}(\underset{\sim}{\theta}_t))$, where t varies over the positive integers, m varies over $0, 1, \ldots, N_j$ and $\underset{\sim}{\theta}_t$ varies over all histories of the pr[illegible] θ_t . $y_{jmt}(\underset{\sim}{\theta}_t)$ represents the input-output possibility vector in period t and event $\underset{\sim}{\theta}_t$ of a production process initiated at time $t-m$. The allocation y_j is feasible for industry j if the following are true.

For all m, either $\bar{h}_{jm0} = 0$ and $y_{j,m+1,1}(\theta_1) = 0$ or $\bar{h}_{jm0} > 0$ and $\bar{h}_{jm0}^{-1} y_{j,m+1,1}(\theta_1) \in Y_j(\theta_1)$.

For all t and $\underset{\sim}{\theta}_t$, there exists $\bar{h}_{j0t}$ such that $y_{j0t}(\underset{\sim}{\theta}_t) = -\bar{h}_{j0t}K_j$ and for $m > 0$, either $\bar{h}_{j0t} = 0$ and $y_{jm,t+m}(\underset{\sim}{\theta}_{t+m}) = 0$ or $\bar{h}_{j0t} > 0$ and $\bar{h}_{j0t}^{-1} y_{jm,t+m} \in Y_j(\theta_{t+m})$.

A resource allocation to consumer (i,t) is of the form $(x,L) = (x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n^{(i,t)}), L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n^{(i,t)}))$, where n varies over $n_t, n_t + 1, \ldots, N$, for $n_t = \max(0, -t+1)$. $\underset{\sim}{\theta}_{t+n}$ varies over histories of the θ process and $\underset{\sim}{s}_n^{(i,t)}$ varies over histories of the $s^{(i,t)}$ process. x_{t+n} is consumption and L_{t+n} is employment status, both in period $t+n$. The amount of labor offered in period $t+n$ is $\omega_{in}(s_n^{(i,t)})L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n^{(i,t)})$. The allocation (x,L) is feasible for the consumer if (2.1) and (2.2) below are true.

(2.1) For all n and $(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$, $x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) \in R_+^{C_c}$ and $L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$ equals zero or the $(k(i), j)^{th}$ standard basis vector of R^{CE}, for some $j = c, 1, \ldots, J$.

I now define variables $w_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1})$ which indicate the number of consecutive periods the consumer has been unemployed at the end of period $t+n$. The definition is by induction on n. If $t < 1$ and $t+n = 0$, $w_0^{(i,t)}$ is defined by the initial conditions. If $t \geq 1$, $w_0^{(i,t)} = 0$. Suppose by induction that $w_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1})$ has been defined. Then, $w_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = w_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1}) + 1$ if $L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = 0$ and $w_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = 0$, otherwise.

(2.2) $L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) \neq 0$ only if $n = 0$ or $w_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1})$
$\geq T_{k(i)}$ or if $L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = L_{t+n-1}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1})$.

This condition says that periods of unemployment must last at least $T_{k(i)}$ periods, where $k(i)$ is the type of labor offered by the consumers.

A resource allocation is of the form $((y_j)_{j=1}^J, (x^{(i,t)}, L^{(i,t)})_{i\in[0,1],t=-N+2,-N+3,\ldots})$, where each y_j is a feasible allocation for industry j and each $(x^{(i,t)}, L^{(i,t)})$ is a feasible allocation for consumer (i,t) . The functions $x_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{n+n}, \underset{\sim}{s}_n)$ and $L_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$ are required to be measurable with respect to i .

Aggregation over Consumers

The aggregate consumption associated with consumer allocation $(x^{(i,t)}, L^{(i,t)})$ in period t and event θ_t is

$$(2.3) \quad \overline{X}_t(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1} \int_0^1 E[x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n^{(i,t-n)})|\underset{\sim}{\theta}_t]di .$$

Similarly, the aggregate supply of labor in period t and event $\underset{\sim}{\theta}_t$ is

$$(2.4) \quad \overline{L}_t(\underset{\sim}{\theta}_t) = \sum_{n=1}^{N-1} \int_0^1 E[\omega_n(s_n^{(i,t-n)})L_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n^{(i,t-n)})|\underset{\sim}{\theta}_t]di .$$

Notice that the variation over $\underset{\sim}{s}_n^{(i,t-n)}$ is averaged out. This definition may be justified by appealing to the strong law of large numbers. The aggregates are what would be obtained in the limit if one took averages over ever larger random samples of consumers. For details, see Bewley (1980b).

Feasibility

A resource allocation $((y_j), (x^{(i,t)}, L^{(i,t)}))$ is feasible if

$$(2.5) \quad \bar{x}_t(\underset{\sim}{\theta}_t) - \bar{L}_t(\underset{\sim}{\theta}_t) = \sum_{j=0}^{J} \sum_{m=0}^{N_j} y_{jmt}(\underset{\sim}{\theta}_t) \ ,$$

where $\bar{x}_t$ and $\bar{L}_t$ are as defined above.

The Timing of Financial Flows

In describing financial flows, a distinction is made between the beginning and end of each time period. Taxes are paid and transactions are made at the beginning of each period. All goods and assets bought are paid for immediately at the beginning of the period. The seller of an asset receives payment immediately. However, income in the form of wages, interest or dividends is received only at the end of the period. Stocks and bonds bought in period t pay dividends or interest in that period to the buyer.

Securities

The securities are government bonds and shares in firms. Consumer (i,t)'s holdings of government bonds at the end of period $t+n$ is denoted by $h^{(i,t)}_{g,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$. The symbol $h^{(i,t)}_{jm,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$ denotes his holdings of shares of firms in industry j which first invested at time $t+n-m$. One such share represents ownership of a process resulting from an initial investment of K_j. One government bond is a promise to pay one unit of money at the end of the period, so that bonds live only one period.

The symbol $h^{(i,t)}$ stands for consumer (i,t)'s holdings of all assets in all periods of life. An investment allocation is of the form

$(h^{(i,t)})$, where (i,t) varies over all consumers. It is required that every component of $(h^{(i,t)})$ be measurable with respect to i .

Given $(h^{(i,t)})$, $\bar{h}_{jmt}(\underset{\sim}{\theta}_t)$ is defined by

$$(2.6)\quad \bar{h}_{jmt}(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1} \int_0^1 E[h_{jmt}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)|\underset{\sim}{\theta}_t]di .$$

This is the aggregate holdings in period t of stock in firms of age m in industry j . The aggregate holdings of government bonds, $h_{gt}(\underset{\sim}{\theta}_t)$, are defined in a similar way.

Part of the initial conditions of the model are the securities held at the end of period zero, $h_{g0}^{(i,t)} \geqq 0$ and $h_{jm0}^{(i,t)} \geqq 0$, for $t = -N+2, \ldots, 0$. The initial conditions $\bar{h}_{jm0}$ defined earlier are simply the aggregates corresponding to the $h_{jm0}^{(i,t)}$.

Insurance

I now introduce insurance on the aggregate or universal uncertainty. I follow up on Arrow's original paper (1963-4) on contingent claims by placing in an intertemporal context his markets for contingent claims on units of account.

An insurance program for consumer (i,t) is of the form $a = (a_{t+n}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n))_{n=n_t}^{N}$, where $n_t = \max(0, -t+1)$ and where each $a_{t+n}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n)$ is a number and represents the amount of money received (>0) or paid (<0) in event $\underset{\sim}{\theta}_{t+n+1}$ at the beginning of period $t+n$. The commitments $a_{t+n+1}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n)$ are arranged in period $t+n$ and in event $(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$.

Part of the initial conditions of the model are the insurance contracts falling due in the initial period. Let $a_0^{(i,t)}(\theta_1)$ be the amount of money due to consumer (i,t) at the beginning of period 0, for $t \geq 0$. It is required that $\sum_{t=-N+2}^{0} \int_0^1 a_0^{(i,t)}(\theta_1) = 0$, for all θ .

An <u>insurance allocation</u> is of the form $(a^{(i,t)})$, where each $a^{(i,t)}$ is an insurance program for consumer (i,t) . It is required that the components of $a^{(i,t)}$ be measurable with respect to i . Given $(a^{(i,t)})$, the aggregate insurance commitment is defined to be

$$(2.7) \quad \bar{a}_t(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1} E[a_t^{(i,t-n)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) | \underset{\sim}{\theta}_{t+n}] di \ .$$

<u>Allocation</u>

An allocation is of the form $((y_j), (x^{(i,t)}, L^{(i,t)}, h^{(i,t)}, a^{(i,t)}))$, where $((y_j), (x^{(i,t)}, L^{(i,t)}))$ is a resource allocation, $(h^{(i,t)})$ is an investment allocation and $(a^{(i,t)})$ is an insurance allocation. The allocation is feasible if

$((y_j), (x^{(i,t)}, L^{(i,t)}))$ is feasible as a resource allocation,

$y_{j0t}(\underset{\sim}{\theta}_t) = -\bar{h}_{j0t}(\underset{\sim}{\theta}_t)K_t$, for all j , t and $\underset{\sim}{\theta}_t$,

$\bar{h}_{jmt}(\underset{\sim}{\theta}_t) = h_{j,m+1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1})$, for all j, $m = 0, \ldots, N_j - 1$, and for $t \geq 0$ and all $\underset{\sim}{\theta}_t$, θ_{t+1} ,

$\bar{a}_t(\underset{\sim}{\theta}_{t+1}) = 0$, for all t and $\underset{\sim}{\theta}_{t+1}$.

The last condition implies that there is no aggregate borrowing or lending.

Prices

All prices are in terms of money.

A commodity price system is of the form $p = (p_1(\underset{\sim}{\theta}_1), p_2(\underset{\sim}{\theta}_2), \ldots)$ where $p_t(\underset{\sim}{\theta}_t) \in R_+^{CE}$, for all t and $\underset{\sim}{\theta}_t$. A security price system is of the form $q = (q_1(\underset{\sim}{\theta}_1), q_2(\underset{\sim}{\theta}_2), \ldots)$, where $q_t(\underset{\sim}{\theta}_t) = (q_{gt}(\underset{\sim}{\theta}_t), (q_{jmt}(\underset{\sim}{\theta}_t)))$. $q_{gt}(\underset{\sim}{\theta}_t)$ is the price of a government bond in period t. $q_{jmt}(\underset{\sim}{\theta}_t)$ is the price in period t of a share of a firm in industry j which invested initially in period $t-m$. A system of insurance premiums is of the form $v = (v_1(\underset{\sim}{\theta}_2), v_2(\underset{\sim}{\theta}_3), \ldots)$ where $v_t(\underset{\sim}{\theta}_t, \theta_{t+1})$ is the amount of money paid in period t and event $\underset{\sim}{\theta}_t$ from one unit of money in period $t+1$ and event $(\underset{\sim}{\theta}_t, \theta_{t+1})$.

A price system is (p,q,v), where p, q and v are as above.

Profits

For $p \in R_+^{CE}$, let $\pi_j(p,\theta) = \sup\{p\cdot y | y \in Y_j(\theta)\}$. If $p = (p_t(\underset{\sim}{\theta}_t))$ is a price system, then $\pi_j(p_t(\underset{\sim}{\theta}_t), \theta_t)$ is the dividend paid at the end of period t on one share of stock in a firm in industry j which produces during period t.

Taxes

The government chooses a sequence of tax functions $\tau = (\tau_1(\cdot, \theta_1), \tau_2(\cdot, \theta_2), \ldots)$, where $\tau_t(W, \theta_t)$ is a function of a consumer's wealth, W. Taxes will always be either proportional to wealth or lump-sum subsidies.

Initial Money Balances

Additional initial conditions are the end of period 0 money balances $M_0^{(i,t)} \geqq 0$ of consumers alive in period zero.

Wealth and Money Balances

Let a price and tax system (p,q,v,τ) be given and fix an allocation (x,L,h,a) for a consumer (i,t). I now define corresponding wealths and money balances. $M_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$ denotes money balances held at the end of the $(n+1)^{st}$ period of life. $I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$ denotes the quantity of money held during the $(n+1)^{st}$ period of life and as an investment. $W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1})$ is wealth at the beginning of the $(n+1)^{st}$ period of life. I now define these quantities by induction on n.

If consumer (i,t) is alive at time zero, his initial money balances are $M_0^{(i,t)}$. Otherwise, they are $M_{t-1}^{(i,t)} = 0$. Suppose by induction that $M_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1})$ has been defined, for $n \geq 0$. Then,

$$(2.8)\quad W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}) = M_{t+n-1}^{(i,t)}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1}) + a_{t+n-1}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1})$$
$$+ \sum_{j=1}^{J} \sum_{m=0}^{N_{j-1}} q_{j,m+1,t+n}(\underset{\sim}{\theta}_{t+n}) h_{jm,t+n-1}(\underset{\sim}{\theta}_{t+n-1}, \underset{\sim}{s}_{n-1}) .$$

Also,

$$(2.9)\quad I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, s_n) - p_{t+n}(\underset{\sim}{\theta}_{t+n}) \cdot x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$$
$$- q_{g,t+n}(\underset{\sim}{\theta}_{t+n}) \cdot h_{g,t+n}(\underset{\sim}{\theta}_{t+n}, s_n) - \sum_{j=1}^{J} \sum_{m=0}^{N_j} q_{j,m,t+n}(\underset{\sim}{\theta}_{t+n}) \cdot h_{jm,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$$
$$= \sum_{\theta_{t+n+1}} v_{t+n}(\underset{\sim}{\theta}_{t+n}, \theta_{t+n+1}) a_{t+n}(\underset{\sim}{\theta}_{t+n}, \theta_{t+n+1}, \underset{\sim}{s}_n) - \tau_t(W^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}), \underset{\sim}{\theta}_{t+n}) .$$

Finally,

$$(2.10)\quad W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) + p_{t+n}(\underset{\sim}{\theta}_{t+n})\omega_{in}(\underset{\sim}{s}_n)L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$$

$$+ h_{g,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) + \sum_{j=1}^{J}\sum_{m=1}^{N_j}\pi_j(p_{t+n}(\underset{\sim}{\theta}_{t+n}), \theta_{t+n})h_{j,m,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)$$

The Budget Constraints

The budget set of consumer (i,t) is $\beta^{(i,t)}(p,q,v,\tau)$ $= \{(x,L,h,a)\,|\,(x,L,h,a)$ is an allocation for consumer (i,t), which is feasible for him and $W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}) \geqq 0$ and $I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) \geqq 0$, for all n, $\underset{\sim}{\theta}_{t+n}$ and $\underset{\sim}{s}_n\}$, where $W_{t+n}^{(i,t)}$ and $I_{t+n}^{(i,t)}$ are as just defined. The constraint $I_{t+n}^{(i,t)} \geqq 0$ means that the consumer never holds negative money balances. The constraint $W_{t+n}^{(i,t)} \geqq 0$ means that all payments on insurance contracts (or loans) are fully secured by negotiable collateral. One cannot pay off insurance contracts by selling new ones. Nor can one use future wages as collateral. This last restriction may be justified by referring to the random variation $\omega_{in}(\underset{\sim}{s}_n^{(i,t)})$ in labor endowments.

Consumer Demand

The consumer demand correspondence, $\xi^{(i,t)}(p,q,v,\tau)$, is the set of solutions of the problem $\max\{U_i(x)\,|\,(x,L,h,a) \in \beta^{(i,t)}(p,q,v,\tau)\}$.

Supply

The supply correspondence for industry j is η_j, where for $p \in R_+^{CE}$ and $\theta \in \Theta$. $\eta_j(p,\theta)$ is the set of solutions of the problem $\max\{p\cdot y\,|\,y \in Y_j(\theta)\}$.

Equilibrium

An equilibrium consists of $((y_j),\ (x^{(i,t)},\ L^{(i,t)},\ h^{(i,t)},\ a^{(i,t)}),\ (p,q,v,\tau))$ such that

1) $((y_j),\ (x^{(i,t)},\ L^{(i,t)},\ h^{(i,t)},\ a^{(i,t)}))$ is a feasible allocation,

2) (p,q,v) is a price system,

3) τ is a tax system,

4) if any good or security is in excess supply, its price is zero,

5) $y_{j0t}(\underset{\sim}{\theta}_t) = -\overline{h}_{j0t}(\underset{\sim}{\theta}_t)K_j$, for all j , t and $\underset{\sim}{\theta}_t$,

6) $y_{jmt}(\underset{\sim}{\theta}_t) = h_{jmt}(\underset{\sim}{\theta}_t)\tilde{y}_{jmt}(\underset{\sim}{\theta}_t)$, where $\tilde{y}_{jmt}(\underset{\sim}{\theta}_t) \in \eta_j(p_t(\underset{\sim}{\theta}_t),\ \underset{\sim}{\theta}_t)$, for all $j,\ m > 0$, t and $\underset{\sim}{\theta}_t$,

7) $q_{j0t}(\underset{\sim}{\theta}_t) \leqq p_t(\underset{\sim}{\theta}_t)K_t$, for all j , t , and $\underset{\sim}{\theta}_t$, with equality if $\overline{h}_{j0t}(\underset{\sim}{\theta}_t) > 0$,

8) $(x^{(i,t)},\ L^{(i,t)},\ h^{(i,t)},\ a^{(i,t)}) \in \xi^{(i,t)}(p,q,v,\tau)$, for almost every i and all t .

Nothing yet has been said about equilibrium in the market for government bonds or about the money supply.

The Money Supply

The appropriate definition of money supply seems to be after tax money supply. The after tax money supply of consumer (i,t) at the beginning of period t is $M^{A(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n},\ \underset{\sim}{s}_n) = M^{(i,t)}_{t+n-1}(\underset{\sim}{\theta}_{t+n-1},\ \underset{\sim}{s}_{n-1}) - \tau_t(W^{(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n},\ \underset{\sim}{s}_n),\ \underset{\sim}{\theta}_{t+n})$. The aggregate money supply in period t is

$$(2.11)\quad \overline{M}^A_t(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1}\int_0^1 E[M^{A(i,t-n)}_t(\underset{\sim}{\theta}_t,\ \underset{\sim}{s}_n)\,|\,\underset{\sim}{\theta}_t]di\ .$$

Because government debt is repaid every period, $\overline{M}^A_t$ represents the government's entire liability, including debt.

The Marginal Utility of Expenditure

Policy intervention uses the aggregate marginal utility of expenditure as an index of the price level. Given $p \in R_+^{CE}$ and $x \in R_+^{C_c}$, the marginal utility of expenditure of an individual of type i is

$$\alpha_i(p,x,\theta,s) = \max_{k \in C_c} p_k^{-1} \frac{\partial u_i(x,\theta,s)}{\partial x_k},$$

assuming that the derivative exists. In an equilibrium $((y_j), (x^{(i,t)}; L^{(i,t)}, h^{(i,t)}, a^{(i,t)}), (p,q,v,\tau))$, the marginal utility of expenditure of consumer (i,t) in period $t+n$ is $\alpha_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = \alpha_i(p_{t+n}(\underset{\sim}{\theta}_{t+n}), x_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n), \theta_{t+n}, \underset{\sim}{s}_n)$. The aggregate marginal utility of expenditure is

$$(2.12) \quad \bar{\alpha}_t(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1} \int_0^1 E[\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) | \underset{\sim}{\theta}_t] di .$$

Types of Equilibrium

Let $\alpha > 0$ and Q be such that $0 < Q < 1$. An (α,Q) controlled equilibrium is an equilibrium such that $\bar{\alpha}_t(\underset{\sim}{\theta}_t) = \alpha$ and $q_{gt}(\underset{\sim}{\theta}_t) = Q$, for all t and $\underset{\sim}{\theta}_t$. That is, the aggregate marginal utility of expenditure is always α and the price of government bonds is always Q. The government is assumed to provide in each period t and state $\underset{\sim}{\theta}_t$ the quantity $\bar{h}_{gt}(\underset{\sim}{\theta}_t)$ of government bonds demanded. One can imagine that the tax system is chosen so as to achieve $\alpha_t(\underset{\sim}{\theta}_t) = \alpha$. In this equilibrium, the money supply fluctuates freely in response to changes in tax collections and the demand for government debt.

A (G,M) balanced budget equilibrium is an equilibrium such that $\bar{M}_t^A(\underset{\sim}{\theta}_t) = M$ and $\bar{h}_{gt}(\underset{\sim}{\theta}_t) = G$, for all t and $\underset{\sim}{\theta}_t$ where $M > G \geq 0$.

That is, the government debt and money supply are fixed at G and M, respectively. It follows that the government's budget always balances.

Other kinds of equilibria exist. One can fix the tax system and money supply and allow government debt to vary, or one could fix the tax system and allow the money supply to vary. One cannot fix all three independently, of course. One can define equilibria without insurance simply by excluding the symbols $a^{(i,t)}$ and v_t everywhere above.

3. Assumptions

Assumptions about Utility

(3.1) For all i, θ and s, the function $u_i(\cdot,\theta,s) : R_+^{C_c} \to R$ is continuously differentiable, strictly increasing and concave.

(3.2) There is $\overline{u} > 0$ such that $\frac{\partial u_i(x,\theta,s)}{\partial x_k} \leqq \overline{u}$, for all i, k, x, θ and s.

(3.3) There is $\underline{u} > 0$ such that $\frac{\partial u_i(x,\theta,s)}{\partial x_k} \geqq \underline{u}$, for all i, k, θ and s and for x such that $x_n \leqq 1$, for all n.

(3.4) There is a number $\overline{MS} > 0$ such that $\frac{\partial u_i(x,\theta,s)}{\partial x_k} \leqq \overline{MS}\frac{\partial u_i(x,\theta,s)}{\partial x_n}$, for all i, k, n, θ, s and x.

This assumption says that marginal rates of substitution are uniformly bounded.

Assumptions about Production

Associated with each technology j there are sets $C_j^- \subset C$ and $C_j^+ \subset C_p$. Industry j uses the goods in C_j^- as inputs and produces goods in C_j^+. Let $X_j = \{y \in R^C | y_k \leqq 0 \text{ if } k \in C_j^- \text{ and } y_k \geqq 0 \text{ if } k \in C_j^+$ and $y_k = 0$ otherwise$\}$.

(3.5) There is $g_j(\cdot,\theta) : X_j \to R$ such that $Y_j(\theta) = \{y \in X_j | g_j(y,\theta) \leqq 0\}$.

I remind the reader that in the discussions of equilibrium, $Y_j(\theta)$ should be thought of as a subset of R^{CE}.

(3.6) For all j and θ, $g_j(\cdot,\theta) : X_j \to R$ is continuously differentiable and convex. Also, for all k and y, $\frac{\partial g_j(y,\theta)}{\partial y_k} \geqq 0$; for k with strict inequality for some k. Finally, $g_j(0,\theta) = 0$, for all j and θ (so that $0 \in Y_j(\theta)$).

(3.7) There is $B > 0$ such that for all j and θ and all $y \in Y_j(\theta)$, $y_k < B$, for all $k \in C_j^+$. Also, if $k \in C_j^-$, then $\frac{\partial g_j(y,\theta)}{\partial y_k} = 0$ whenever $y_k \leqq -B$.

(3.8) There is a number $\overline{MP} > 1$ such that $\frac{\partial g_j(y,\theta)}{\partial y_k} \leqq \overline{MP}\frac{\partial g_j(y,\theta)}{\partial y_n}$, for all j and θ, for all $y \in Y_j(\theta)$ and for all $n \in C_j^+$ and $k \in C_j^-$.

That is, marginal products are uniformly bounded.

(3.9) $K_j \in R_+^{C_W}$, for all j.

That is, the only investment goods are kinds of labor.

(3.10) $K_j > 0$, for all j .

Measurability Assumptions

(3.11) The function $\frac{\partial u_i(x,\theta,s)}{\partial x_k}$ is jointly measurable in i and x for all θ , s and k .

(3.12) The functions $k(i)$, $\omega_{in}(s)$, $w_0^{(i,t)}$, $h_{g0}^{(i,t)}$, $h_{jm0}^{(i,t)}$, $a_0^{(i,t)}(\theta_1)$, and $M_0^{(i,t)}$ are all measurable with respect to i .

Other Assumptions

(3.13) $N > 2$.

That is, people live more than two periods.

(3.14) $\omega_{iN}(s) = 0$, for all i and s .

That is, people don't work in the last period of life.

(3.15) The stochastic processes $(s_n^{(i,t)})_{n=0}^{N-1}$ and $(\theta_t)_{t=1}^{\infty}$ are all mutually independent. The processes $(s_n^{(i,t)})_{n=0}^{N-1}$ are all identically distributed as $(s_n)_{n=0}^{N-1}$.

(3.16) There is $\eta > 0$ such that for all t and $\underset{\sim}{\theta}_t$, $\text{Prob}[\theta_{t+1}|\underset{\sim}{\theta}_t] > \eta$ whenever $\text{Prob}[\theta_{t+1}|\underset{\sim}{\theta}_t] > 0$.

Without loss of generality, it may be assumed that for all n and $\underset{\sim}{s}_n$, $\text{Prob}[s_{n+1}|\underset{\sim}{s}_n] > \eta$ whenever $\text{Prob}[s_{n+1}|\underset{\sim}{s}_n] > 0$.

(3.17) Good 1 belongs to $C_w \cap C_c$ and is not used as an input by any industry (that is, $1 \notin C_j^-$ and $K_{j1} = 0$, for all j). Also, $\text{mes}\{i \in [0,1] | k(i) = 1\} > 0$, where mes denotes Lebesgue measure. Finally, there is $\underline{\omega} > 0$ such that $E[\omega_{in}(s_n)] > \underline{\omega}$, for all i such that $k(i) = 1$ and for $n < N-1$.

Good 1 may be thought of as domestic help. This assumption guarantees a minimum level of economic activity.

(3.18) The functions $h_{g0}^{(i,t)}$, $h_{jm0}^{(i,t)}$, $a_0^{(i,t)}(\theta_1)$ and $M_0^{(i,t)}$ are uniformly bounded and $\overline{a}_0(\theta_1) = 0$, for all θ_1 . Also $a_0^{(i,t)}(\theta_1) + M_0^{(i,t)} \geqq 0$, for all i , t and θ_1 .

4. Theorems

I prove the existence of two of the several types of equilibria possible. Assume that assumptions (3.1)-(3.18) apply.

(4.1) Theorem. For every $\alpha > 0$ and every Q such that $0 < Q \leqq 1$, there exists an (α,Q) controlled equilibrium with no insurance.

Assume now that $\int_0^1 (M_0^{(i,N-2)} + a_0^{(i,N-2)}(\theta_1))di > 0$ and $\int_0^1 (M_0^{(i,N-3)} + a_0^{(i,N-3)}(\theta_1))di > 0$, for every θ_1 .

(4.2) Theorem. For every G and M such that $M > G \geq 0$, there is a (G,M) balanced budget equilibrium.

After having read the proofs of these theorems, one will be able to see how to prove the existence of an (α,Q) controlled equilibrium with insurance and a (G,M) balanced budget equilibrium without insurance. One will also see how to prove the existence of a controlled equilibrium where instead of keeping $\alpha_t(\underset{\sim}{\theta}_t)$ constant, one fixes a consumer price index of the form $[\sum_{k\in C_c} \bar{x}_{tk}(\underset{\sim}{\theta}_t)]^{-1} p_t(\underset{\sim}{\theta}_t)\cdot\bar{x}_t(\underset{\sim}{\theta}_t)$.

In an (α,Q)-controlled equilibrium, the control of interest rates becomes ineffective if no one buys government bonds. I include no condition which guarantees that bonds are held. One can, however, manipulate the tax system so as to create as large a demand as one likes for assets. One simply gives a large subsidy to young consumers which is taxed away later in life--a kind of reverse social security system.

5. Indeterminacy

The use of the word "controlled" in the definition of an (α,q) controlled equilibrium is somewhat deceptive, for the equilibrium may not be determinate. In that equilibrium, the government pegs the interest rate on bonds and announces and institutes a certain tax system. There may be many equilibria corresponding to these choices. The work of Geanakoplos and Polemarchakis (1983) and Kehoe and Levine (1982) makes clear why this is so. In the simplest overlapping generations model, there may exist a continuum of equilibria, even if we exclude inflationary equilibria. Thus, the government cannot be sure that the outcome of its policy will be the desired equilibrium.

In the model of this paper, it is possible to exclude inflationary equilibria by placing an upper bound on the money supply. The argument is simple. Assumption (3.17) puts a lower bound on the level of economic activity, and the Clower constraint says that a certain level of real balances is needed to support this activity. In the equilibrium of Theorem 4.1, the money supply is in fact bounded.

6. Why Stabilize the Real Interest Rate?

I now indicate why it is that one needs a social welfare function to justify interest rate stabilization in the model of this paper. Consider a Diamond (1965) overlapping generations model in which people live two periods. One person is born each period. There are two commodities, labor and a produced good. Each consumer has an endowment of $\overline{L}_i$ units of labor in the i^{th} period of life. Each has a utility function $u(x_1, x_2)$, where x_i is consumption of produced good in the i^{th} period of life. The production function is $y_t = f(K_{t-1}, L_t)$, where y_t is output in period t, K_{t-1} is output set aside as capital in period $t-1$ and L_t is labor used in period t. Assume that f is homogeneous of degree one. Consumers can hold capital goods or government debt as assets. The debt held from period t to $t+1$ earns interest at real interest rate r_t. Government debt and r_t may be negative. Let the numeraire in each period be the produced good in that period. Suppose that there is a lump-sum tax of τ_t paid by everyone in period t. The budget constraint of a consumer born in period t is $x_{t1} + x_{t2}(1+r_t)^{-1} \leqq w_t\overline{L}_1 - \tau_t + (1+r_t)^{-1}(w_{t+1}\overline{L}_2 - \tau_{t+1})$, where x_{ti} is consumption of produced good in the i^{th} period of life and w_t is the wage in period t. The technology is operated so as to maximize $f(K_{t-1}, L_t) - (1+r_t)K_{t-1} - w_tL_t$, for every t.

If appropriate regularity restrictions are made, then every equilibrium of this economy is Pareto optimal provided

$$(6.1) \quad 0 < \sum_{t=1}^{\infty} \prod_{n=1}^{t} (1+r_t)^{-1} < \infty .$$

The equilibrium allocation simply maximizes the social welfare function.

$$(6.2)\quad \sum_{t=1}^{\infty} \prod_{n=1}^{t-1} (1+r_n)^{-1} u(x_{t1}, x_{t2}) + u(x_{01}, x_{02}) ,$$

where x_{01} is given as an initial condition. In fact, condition (6.1) is stronger than necessary. (See Balasko and Shell (1980).) The above is a version of the first welfare theorem of equilibrium theory. A version of the second also applies. Let the interest rates r_t be prescribed so that (6.1) is valid. There is a feasible allocation maximizing (6.2), and one may choose the taxes τ_t so that the maximizing allocation is the allocation of an equilibrium with these taxes and the interest rates r_t and with the price of the consumption good always equal to one. One may renormalize prices so that interest rates are always non-negative. The taxes pay the interest on the government debt. They may be negative just as may be the government debt. Negative debt corresponds to lending by the government to consumers. The tax τ_t is chosen so as to balance the budget of the consumer born at time $t-1$. A version of the second welfare theorem is in Diamond (1973). Another version is proved in Balasko and Shell (1980). The theorem holds in great generality, with many commodities, people living many periods, and so on.

Thus, Pareto optimality puts no practical restriction on the real interest rates r_t. Periods of inflation $(r_t < 0)$ may alternate with periods of deflation $(r_t > 0)$, or the capital stock may be driven to zero $(\lim_{t\to\infty} r_t = \infty)$. In either of these cases, equilibrium would be Pareto optimal.

These two welfare theorems may be interpreted just as are the usual welfare theorems. Thinking of the first theorem, we can assert that there is no such thing as a just interest rate. We have only equilibrium interest rates. Thinking of the second theorem, we can assert that if other considerations tell us what real interest rates we want, then these rates can be engineered by appropriate choices of taxes and subsidies. The distribution of welfare among generations is just as subject to manipulation as is that among contemporaries. Since the real interest rate determines the rate of economic growth and the welfare of future generations, it seems quite appropriate to think of it as an object of policy.

Theorem 4.1 is of interest only if one adopts the view expressed by the second welfare theorem. The point of view expressed by the first theorem is, of course, equally valid. It has found expression in the literature, as in Sargent and Wallace (1982).

There is no reason that a social welfare function should specify constant interest rates, as I have done in Theorem 4.1. Interest rates were made constant there simply for economy of exposition.

If consumers were immortal, the situation would be different. For instance, if all consumers discounted future utility at rate ρ , then ρ would be the matural target real interest rate. With restrictions on borrowing, the equilibrium real interest rate could be less than ρ , but can be greater than ρ only temporarily. An appropriate policy prescription in this case would be to drive real interest rates as high as possible.

7. The Possibility of Trade Fluctuations

Examples of trade fluctuations in models like that of this paper have been provided by Scheinkman and Weiss (1983) and Grandmont (1983). Grandmont demonstrates that cyclic fluctuations can occur in the simplest Samuelson consumption loan model, provided the utility function has special properties. The fluctuations in the model of Scheinkman and Weiss are similar to the trade fluctuations mentioned in the introduction, which would be generated by changes in taste and unemployment. Scheinkman and Weiss consider an economy with two immortal consumers who use labor to produce a single consumption good. Labor is the only input in production. Labor itself causes disutility. At any time only one worker can work, and the ability to work passes from one to the other randomly. The person who works accumulates money to be spent when he is unemployed. Since he discounts future utility, there is a limit to how much money he is willing to accumulate. As he reaches this limit, he consumes more and works less. Thus, the output of the economy depends on the distribution of wealth among the two consumers.

I have not included a disutility of work and prefer to think in terms of fluctuations in output as being caused by fluctuations in investment due to changes in real interest rates. By including investment in the model of Scheinkman and Weiss, one can obtain such fluctuations. Imagine that time is discrete and that every n^{th} period the ability to work passes from one consumer to the other. Imagine for the moment that there is no investment, but that goods are produced directly from labor, as in Scheinkman and Weiss. Finally, imagine that the model is otherwise as in Section 2 above, with a Clower payments lag. Then, it is easy to see that there is a cyclic equilibrium with period $2n$.

Let p be the price of the consumption good in the period in which one of the workers first gets paid after a period of unemployment. The price of the consumption good will be $\delta^k p$ in the k^{th} period thereafter, for $k = 1, \ldots, n-1$, provided δ is sufficiently close to one, where δ is the discount factor applied to the future utility. If there is no disutility of labor, output remains constant. Imagine now that one introduces a little bit of investment. Then, the prices of the cyclic equilibrium will be nearly the same as before, and investment would jump upward during the periods of inflation (when the price jumps from $\delta^{n-1}p$ to p). A specific example of this kind is given in the appendix (example A.1).

One might well ask why on earth labor endowments should ever fluctuate as in the example. However, the example can be modified so as to reflect a story closer to that involving changes in consumer tastes which was told in the introduction. Let there be four produced goods and suppose that consumer 1 is in the industry producing goods 1 and 2 and that consumer 2 is in the industry producing goods 3 and 4. Suppose that consumer demand is such that the following are true: good 1 is demanded in periods $4kn+1, 4kn+2, \ldots, (4k+2)n$; good 2 is demanded in periods $(4k+2)n+1, \ldots, 4(k+1)n$; good 3 is demanded in periods $(4k+1)n+1, \ldots, (4k+3)n$; and good 4 is demanded in periods $(4k+3)n+1, \ldots, (4k+5)n$; for k any integer. Suppose also that it takes n periods for a worker to switch from the production of one good to that of another. Then, a pattern of employment will result much like that previously described.

The secondary effects of unemployment would be reduced by the insurance defined in Section 2. However, even perfect insurance could not eliminate fluctuations in real interest rates. In order to see that this is so, it is sufficient to realize that in the model of this paper the real interest rate depends on all the elements of the economy. If any of these change, the interest rate may also change. An example of this sort is given in the appendix (example A.2). There a change in a production function causes a change in the interest rate.

8. Inside Money

It is possible to include in the model a distinction between inside and outside money. Inside money is created by borrowing by individuals. Outside money is an obligation of the government.

The number $\bar{a}_t(\underset{\sim}{\theta}_t, \theta_{t+1})$, defined by equation (2.7), is the aggregate indebtedness of individuals in period t and event $\underset{\sim}{\theta}_t$ for period $t+1$ and state $(\underset{\sim}{\theta}_t, \theta_{t+1})$. In Theorem 4.2, the $\bar{a}_t(\underset{\sim}{\theta}_t, \theta_{t+1})$ were all required to be zero. Now let there be numbers $A_t(\underset{\sim}{\theta}_t) \leqq 0$, for each t and $\underset{\sim}{\theta}_t$ and require that $\bar{a}_t(\underset{\sim}{\theta}_t, \theta_{t+1}) = A_t(\underset{\sim}{\theta}_t)$, for all t, $\underset{\sim}{\theta}_t$ and θ_{t+1}. Then, the aggregate borrowing of individuals in period t and event $\underset{\sim}{\theta}_t$ is $-A_t(\underset{\sim}{\theta}_t) \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})$. Define this quantity to be the quantity of <u>inside money</u> in period t and event $\underset{\sim}{\theta}_t$, and call it $M_t^I(\underset{\sim}{\theta}_t)$. Outside money is defined to be $M_t^O(\underset{\sim}{\theta}_t) = \bar{M}_t^A(\underset{\sim}{\theta}_t) - M_t^I(\underset{\sim}{\theta}_t)$, where $\bar{M}^A(\underset{\sim}{\theta}_t)$ is defined by equation (2.11).

In interpreting inside money, one may think of the government as owning the banking system. Think of banks as making non-contingent loans each period to be repaid at the end of the beginning of the following period. The aggregate value of the loans is the quantity of inside money. The rate of interest paid on loans in period t and event $\underset{\sim}{\theta}_t$

is $[\sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})]^{-1} - 1$.

Since the government owns the banking system, the interest on loans is its revenue. The government's budget is balanced when tax collections at the beginning of period t equal interest paid on government debt minus that earned on bank loans during the previous period. If the government's budget is always in balance, then the quantity of outside money remains constant. (Recall that because government debt is repaid every period, $\bar{M}_t^A$ includes government debt.) However, the quantity of inside money can fluctuate, so that a balanced budget may call for fluctuating taxes. The taxes may, in fact, be negative. That is, subsidies may be called for.

The quantity of outside money may also be negative. Negative outside money may be interpreted as demand deposits with the banking system held by the government.

An increase in outside money at the expense of inside money increases the taxes of a balanced budget and, in simple examples, it raises interest rates. Interest rates increase because the supply of loans decreases. Thus, inside and outside money seem to be separate instruments of monetary policy. However, it is not always possible for the government to control the quantities of both inside and outside money, for the simple reason that if enough deflation were to occur people might not be willing to borrow at all, even if the interest were zero, as is shown by example A.3 of the appendix. In order to control inside money, it would be necessary to use fiscal policy to prevent deflation.

Sargent and Wallace (1982) have argued that the level of inside money should not be controlled at all. They give an example of an overlapping generations model in which the real rate of interest and borrowing fluctuate widely and nevertheless equilibrium is Pareto optimal. They argue that the aggregate borrowing should be determined by market forces alone.[4]

If banks may not pay interest on loans, then competition among banks would reduce the nominal interest rate to zero, or to a level which just covered the cost of doing business. This is what occurs in the paper of Sargent and Wallace. If banks are allowed to pay interest on money, then there seems to be a problem of indeterminacy of equilibrium. Example A.4 in the appendix shows why. I do not know how general this indeterminacy may be.

9. Consumer Durables

It is easy to include consumer durables in the model. These would be goods which could be bought by consumers only in discrete quantities. Also, they would have a lifetime of more than one period and holders of a durable good would derive utility from its services. It would be necessary to include a market for each consumer good of each age. Because there is a continuum of consumers, the discreteness of purchases of durables does not cause a discontinuity in aggregate demand. I have reluctantly excluded durables from the model in order to save space and reduce notation. The advantage of including durables is that one can make quite convincing examples in which changes in taste for durables lead to temporary deflation. It is natural to suppose that if such a change of taste occurred, then people would save money in order to buy new durables corresponding to their new tastes when these new durables became available.

10. An Asset Pricing Formula

Suppose that there is insurance of the aggregate events $\underset{\sim}{\theta}_t$. Let $q_t(\underset{\sim}{\theta}_t)$ be the equilibrium price of any security in period t and let $Q_{t+1}(\underset{\sim}{\theta}_{t+1})$ be the total return of the same security in period $t+1$ (that is, principal plus interest on price plus dividend). Then,

(10.1) $q_t(\underset{\sim}{\theta}_t) \geqq \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})Q_{t+1}(\underset{\sim}{\theta}_t, \theta_{t+1})$, with equality if a positive quantity of the asset is held,

where $v_t(\underset{\sim}{\theta}_{t+1})$ is the equilibrium insurance premium. A simple arbitrage argument demonstrates this inequality. It follows that in equilibrium all consumers are indifferent between holding securities or buying insurance.

If there is no insurance of aggregate insurance, then there is no equation relating an asset's price to its total returns.

11. Transactions Costs in Financial Assets

It would be desirable from the point of view of realism to include transactions costs in security trades. Doing so would give a Baumol-Tobin motive for holding money. However, such transaction costs introduce a discontinuity into consumer demand functions, as the example below shows. Thus, equilibrium may not exist. In order that the discontinuity in individual demand not appear as a discontinuity in aggregate demand, it would be necessary that only a set of consumers of measure zero have a discontinuity at any given vector of values for prices and taxes. In order to prove a result of this kind, it would be necessary that the individual random variables s_{it} vary over continuum.

Example. A single consumer lives two periods. He buys one consumption good. His utility function is $\log(x_1 + 1) + \log(x_2 + 1)$; where x_t is consumption in period t . He can hold money or a single security. He holds one unit of each at the end of period zero and has no other source of income or wealth. One unit of the security held at the beginning of period one yields a dividend of one at the end of period one. The consumer must pay a tax of one half of his wealth at the beginning of period one. The price of the consumption good is one in each period. That of the security is q in each period. Each transaction in the security costs one unit of money. The tax base is wealth net of transaction costs.

If $q \leq 1$, the consumer prefers to sell his security in period 2 and his consumption is $x_1 = \frac{1-q}{2}$, $x_2 = q$. If $q > 1$, he is forced to sell in period 1 and his consumption is $x_1 = x_2 = \frac{1}{4}q$. Hence, demand is discontinuous at $q = 1$. More precisely, the demand correspondence is not upper semi-continuous.

12. The Clower Constraint

The model of Section 2 is asymmetric in that the Clower payment lag applies to payments of income but not to the sale of securities. The proceeds from sales of assets may be spent immediately.

The Clower constraint has been restricted in order to reduce notation and the complexity of the arguments. Theorems 4.1 and 4.2 remain valid if the Clower constraint is extended to transactions in securities. However, the asset pricing formula (10.1) must be replaced by two inequalities.

Extending the Clower constraint to transactions in securities would increase the demand for money and make it possible to have a more interesting demand for money in examples. For instance, the demand for money would depend on the volume of transactions in securities.

The Clower lag would also be extended to borrowing and sales of insurance. However, doing so would require major changes in the model. Loans or insurance contracts would have to be for at least two periods.

13. Critique of Assumptions

The Nature of Firms

As was pointed out in the introduction, firms and technologies are described in such a way that firms cannot affect the temporal or stochastic distribution of their profits. This was done so that consumers facing uninsurable risk could agree on the objective of their firms.

Bounds on Relative Prices

Assumptions 3.4, 3.8 and 3.9 put bounds on relative prices. Assumptions 3.4 and 3.9 are especially restrictive. A similar restriction is that there is no durable productive asset, such as land, in the model.

These restrictions could be dropped if one knew that there was a positive lower bound on the amount of any good that could be produced in any period. This is the so-called interiority property. In turnpike theory, it is common practice to assume that optimal paths have this property. The property seems plausible, but conditions guaranteeing it are so complicated that one usually just assumes it. However, one can hardly assume interiority when proving th existence of equilibrium.

The Lower Bound on Economic Activity

Assumption 3.17 guarantees a minimum level of economic activity. Some such assumption is always needed in equilibrium existence theorems. In the case of the model of this paper, it is easy to make up examples in which no economic activity is possible in equilibrium. Think of an overlapping generations model in which there is no production without capital and in which consumers earn all their wage income in the last period of life. Then, unless one allows borrowing, no one would be able to buy any capital and so there would be no production.

Individual Uncertainty

One might ask why individual uncertainty was included in the model. It is not necessary for the existence of equilibrium. I included this extra bit of generality both because I believe that individual uncertainty is important in reality and because individual uncertainty is useful in making up illustrative examples.

Continuum of Individuals

The continuum of individuals was included so that individual uncertainty would not cause random fluctuations in prices. It also enables one to include discrete individual actions involving unemployment or durable goods.

Minor Restrictions

A number of minor restrictions have been made simply to economize notation. For instance, utility functions could be more general and there could be disutility for labor.

Fixed Lifetimes

It should be possible to generalize Theorems 4.1 and 4.2 to a model in which lifetimes are random. However, it might be difficult to do so. In proving the theorems, I frequently use the fact that people consume all their wealth in the last period of life.

Major Implicit Restrictions

Two very important assumptions underlie the whole model. These are that expectations are rational and that information about aggregate uncertainty is fully symmetric. Clearly, these assumptions do not apply strictly in reality. But they still may give a useful approximation of reality.

14. Proof of Theorem 4.1

The rough plan of the proof is as follows. I truncate the horizon so that the model has only finitely many periods. I fix the price of government bonds at Q . A fixed point argument proves the existence of an equilibrium for the truncated model. I calculate upper and lower bounds for the prices and then allow the horizon to go to infinity. A compactness or Cantor diagonal argument yields an equilibrium in the limit. The same plan has been used before in proving existence of equilibrium in infinite horizon models. See, for instance Bewley (1980), Balasko and Shell (1980), Balasko, Cass and Shell (1980), or Wilson (1981).

Without loss of generality, I may assume that $\alpha = 1$, where α is as in the definition of an (α,Q) controlled equilibrium.

The argument is broken up into a number of steps.

Step 1: Truncation

I do not go through this in detail. Let $T > 3$ and imagine that all economic activity stops after period T. A T-period equilibrium is defined in the obvious way.

Step 2: List of Bounds

In applying a fixed point argument, a number of bounds are used. These are listed here in order to show that the proof is not circular.

(4.1) $\overline{H}$ is any number greater than 1 such that for each $j = 1, \ldots, J$,

$$\overline{H}K_{jk} > N-1 \text{ , for some } k \text{ .}$$

By assumption 3.10, $\overline{H}$ exists.

Let B be as in assumption 3.7 and let $\hat{Y}_j(\theta) = \{y \in Y_j(\theta) \,|\, |y_k| \leq B,$ for all $k\}$. Let $\hat{\eta}_j(p,\theta)$ be the set of solutions of $\max\{p\cdot y \,|\, y \in \hat{Y}_j(\theta)\}$. By assumption 3.7, $\hat{\eta}_j(p,\theta) \subset \eta_j(p,\theta)$, for all j and θ.

(4.2) $$b \equiv \max(\overline{H}B(\sum_{j=1}^{J} N_j) + 1, N+1) \text{ .}$$

(4.3) $$\hat{p} = N(\overline{MS})\overline{u} + 1 \text{ ,}$$

where $\overline{u}$ is as in assumption 3.2 and $\overline{MS}$ is as in assumption 3.4.

(4.4) $$\overline{p}_p = 1 + (1 + \overline{MP})^{|C_p|}\hat{p} \text{ ,}$$

where $\overline{MP} > 1$ is as in assumption 3.8, C_p is the cardinality of C_p.

(4.5) $$\overline{p}_w = (1 + \max\{N_j K_{jk}^{-1} \,|\, j = 1, \ldots, J \text{ and } K_{jk} > 0\})|C_p|B\overline{p}_p \text{ .}$$

Let $\varepsilon_i = \inf\left\{\frac{\partial u_i(x,\hat{e},s)}{\partial x_k} \middle| k \in C_c,\ \hat{e} \in \Theta,\ s \in S, \text{ and } 0 \leqq x_n \leqq 4|C_c|b\right.$ for all $\left. n \in C_c\right\}$. By assumptions 3.1 and 3.11, ε_i is positive for all i and measurable. Let $\underline{\varepsilon} > 0$ be such that $\text{mes}\{i|\varepsilon_i \geqq 2\underline{\varepsilon}\} > \frac{3}{4}$, where mes denotes Lebesgue measure.

(4.6) $\underline{p} = \frac{1}{3}\min(\underline{u},\underline{\varepsilon})$,

where $\underline{u}$ is as in assumption 3.3.

(4.7) $\overline{q} = 1 + B\overline{p}_p|C_p|\max_j N_j$,

where B is as in assumption 3.7 and $\overline{p}_p$ is as in (14.4)

(14.8) $L_1 = \underline{\omega}\ \text{mes}\{i|k(i) = 1\} > 0$,

where $\underline{\omega}$ is as in assumption 3.17. That assumption implies that $L_1 > 0$.

(14.9) $\underline{R}$ is any positive number less than $\underline{p}\ \min(L_1(N-1),\ 1)$,

where $\underline{p}$ is as in (14.6) and N is the lifetime of consumers.

Step 3: Tax Functions

I define tax functions depending on a parameter c. If $c \geqq 0$, let $\tau(W,c) = c$. If $c \geqq 0$, let $\tau(W,c) = cW$. The parameter c will vary over $[\underline{c},1]$, where

(14.10) $\underline{c} = -\hat{p}|C_c|b + 1$,

where b and $\hat{p}$ are defined in (14.2) and (14.3), respectively.

Step 4: The Fixed Point Argument

Let $\underline{q}$ be any small number less than $\overline{q}$, where $\overline{q}$ is as in (14.7). Let $\Delta_{\underline{q}} = \{(p,q,c) | p \in R_+^{CE},\ p_k \geq \underline{p}$, if $k \in C_c$, $p_k \leq \overline{p}_p$, for all $k \in C_p$, $p_k \leq \overline{p}_w$, for $k \in C_w$, $q = (q_g, (q_{jm})_{j=1,\ldots,J;m=0,1,\ldots,N_j})$, $q_g = Q$, $q_{j0} = \max(\underline{q}, p\cdot K_j)$, all j, $\underline{q} \leq q_{jm} \leq \overline{q}$, for all j and m, and $\underline{c} \leq c \leq 1\}$, where $\overline{p}$ is as in (14.4), $\underline{p}$ is as in (14.6), and $\underline{c}$ as in (14.10). Let $\Delta_{\underline{q}}^T = \{(p_t(\underset{\sim}{\theta}_t), q_t(\underset{\sim}{\theta}_t), c_t(\underset{\sim}{\theta}_t))_{t=1}^T | (p_t(\underset{\sim}{\theta}_t), q_t(\underset{\sim}{\theta}_t), c_t(\underset{\sim}{\theta}_t)) \in \Delta_{\underline{q}}$, for all t and $\underset{\sim}{\theta}_t\}$. To each $(p,q,\tau) \in \Delta_{\underline{q}}^T$, there corresponds a T-period price and tax system $(p, q, \tau(c))$, where $\tau_t(W, \underset{\sim}{\theta}_t) = \tau(W, c_t(\underset{\sim}{\theta}_t))$.

If $(p,q,c) \in \Delta_{\underline{q}}^T$, let $\xi^{T(i,t)}(p, q, \tau(c))$ be the demand of consumer (i,t) in the T-period economy. Since the prices of assets and consumption goods are bounded away from zero, $\xi^{T(i,t)}$ has closed graph and $\xi^{T(i,t)}(p, q, \tau(c))$ is always non-empty. Since the asset prices and subsidies are bounded, the graph of $\xi^{T(i,t)}$ is bounded uniformly in i and t. Here, use is made of assumption 3.18. The proofs of these key facts are routine.

Let

$$\hat{\xi}^{(i,t)}(p, q, \tau(c)) = \left\{(x,L,h,\alpha) \middle| (x,L,h) \in \xi^{T(i,t)}(p, q, \tau(c))\right.$$

and $\alpha = (\alpha_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n))$, where $\alpha_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) =$

$$\left. = \max_{k \in C_c} p^{-1}_{t+n,k}(\underset{\sim}{\theta}_{t+n}) \frac{\partial u_i(x_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n), \theta_{t+n}, s_n)}{\partial x_k} \right\}.$$

$\hat{\xi}^{(i,t)}$ will also be referred to as a demand correspondence, even though one of the components is the marginal utility of expenditure. $\hat{\xi}^{(i,t)}$ is again non-empty valued and has a closed graph which is bounded uniformly in i . Assumption 3.12 implies that $\hat{\xi}^{(i,t)}(p, q, \tau(c))$ has measurable graph as a correspondence of i , p , q and c . Hence, $\hat{\xi}^{(i,t)}(p, q, \tau(c))$ has a measurable selection $(x^{(i,t)}, L^{(i,t)}, h^{(i,t)}, \alpha^{(i,t)})$. (See Hildenbrand (1974), p. 54, Theorem 1.) Choose such a selection for each t . Let $\overline{x}_t(\underset{\sim}{\theta}_t)$, $\overline{L}_t(\underset{\sim}{\theta}_t)$, $\overline{h}_{jmt}(\underset{\sim}{\theta}_t)$, and $\overline{\alpha}_t(\underset{\sim}{\theta}_t)$ be the corresponding aggregates defined by equations (2.3), (2.4), (2.6), and (2.12). Next, let $h_{0jt}(\underset{\sim}{\theta}_t) = \min(\overline{H}, \overline{h}_{0jt}(\underset{\sim}{\theta}_t))$, where $\overline{H}$ is defined by equation (14.1), and denote $\hat{h}_{j0t}(\underset{\sim}{\theta}_t)$ by $\overline{h}_{j0t}(\underset{\sim}{\theta}_t)$ again. Finally, let $\overline{h}_t(\underset{\sim}{\theta}_t) = (\overline{h}_{jmt}(\underset{\sim}{\theta}_t))_{j=1,\ldots,J,m=0,1,\ldots,N_j}$ and let $(\overline{x},\overline{L},\overline{h},\overline{\alpha}) = (\overline{x}_t(\underset{\sim}{\theta}_t), \overline{L}_t(\underset{\sim}{\theta}_t), \overline{h}_t(\underset{\sim}{\theta}_t), \overline{\alpha}_t(\underset{\sim}{\theta}_t))_{t=1}^{T}$. Aggregate demand is $\overline{\xi}(p, q, \tau(c)) = \{(\overline{x},\overline{L},\overline{h},\overline{\alpha}) | (\overline{x},\overline{L},\overline{h},\overline{\alpha})$ is obtained as above from measurable selections from the correspondences $\xi^{(i,t)}(p, q, \tau(c))\}$. Since the measure space of all consumers is atomless, $\overline{\xi}$ is convex valued. (See Hildenbrand (1974), p. 62, Theorem 3.) Since the $\xi^{(i,t)}$ are uniformly bounded, the graph of the correspondence $\overline{\xi}$ is bounded. Since the correspondences $\xi^{(i,t)}$ have closed graph and are uniformly bounded, $\overline{\xi}$ is compact valued and has closed graph. (See Hildenbrand (1974), p. 73, propositions 7 and 8.)

Next, let $Z(p,q,c) = \Big\{(z, \Delta h, \alpha) = (z_t(\underset{\sim}{\theta}_t), \Delta h_t(\underset{\sim}{\theta}_t), \alpha_t(\underset{\sim}{\theta}_t))_{t=1}^{T} |$ there is $(\overline{x},\overline{L},\overline{h},\overline{\alpha}) \in \overline{\xi}(p, q, \tau(c))$ and for each j, t and $\underset{\sim}{\theta}_t$, there is $\tilde{y}_t(\underset{\sim}{\theta}_t) \in \hat{\eta}_j(p_t(\underset{\sim}{\theta}_t), \underset{\sim}{\theta}_t)$ such that the following are true:

1) $z_t(\underset{\sim}{\theta}_t) = \overline{x}_t(\underset{\sim}{\theta}_t) + \sum_{j=1}^{J} \overline{h}_{j0t}(\underset{\sim}{\theta}_t)K_j - \overline{L}_t(\underset{\sim}{\theta}_t) - \sum_{j=1}^{J} \sum_{m=1}^{N_j} \overline{h}_{jmt}(\underset{\sim}{\theta}_t)\tilde{y}_j(\underset{\sim}{\theta}_t)$,

2) $\Delta h_t(\underset{\sim}{\theta}_t) = (\Delta h_{jmt}(\underset{\sim}{\theta}_t)) = (\overline{h}_{j,m+1,t}(\underset{\sim}{\theta}_t) - \overline{h}_{j,m,t-1}(\underset{\sim}{\theta}_{t-1}))_{j=1,\ldots,J,m=0,\ldots}$

and 3) $\alpha = \overline{\alpha}\Big\}$. (The symbol $\hat{\eta}_j(p,\theta)$ was defined just before equation (14.2).) Z again is non-empty and convex values and has bounded and closed graph on $\Delta_{\underline{q}}^T$. (Ξ is convex-valued in spite of the product hy, which appears above. In order to see why, notice that if $0 < \gamma < 1$, $h_1 > 0$, $h_2 > 0$ and y_1 and y_2 are vectors, then

$\gamma h_1 y_1 + (1-\gamma)h_2 y_2 = (\gamma h_1 + (1-\gamma)h_2)[\gamma h_1(\gamma h_1 + (1-\gamma)h_2)^{-1} y_1$
$+ (1-\gamma)h_2(\gamma h_1 + (1-\gamma)h_2)^{-1} y_2]$.)

Ξ maps into a Euclidean space. Let K be a compact convex set in this space containing the range of Ξ. Let $F : \Delta_{\underline{q}}^T \times K \to \Delta_{\underline{q}}^T \times K$ be defined to be $F(p, q, c; z, \Delta h, \alpha) = \{\mu(p, q, c; z, \Delta h, \alpha)\} \times \Xi(p,q,c)$, where μ is defined as follows. $\mu(p, q, c; z, \Delta h, \alpha)$

$= (\hat{p}_t(\underset{\sim}{\theta}_t), \hat{q}_t(\underset{\sim}{\theta}_t), \hat{c}_t(\underset{\sim}{\theta}_t))_{t=1}^T$, where

1) $\hat{p}_{tk}(\underset{\sim}{\theta}_t) = \min(\underline{p}, \max(\overline{p}, p_{tk}(\underset{\sim}{\theta}_t) + z_{tk}(\underset{\sim}{\theta}_t)))$,

2) $q_{gt}(\underset{\sim}{\theta}_t) = Q$, $q_{j0t}(\underset{\sim}{\theta}_t) = \max(\underline{q}, p_t(\underset{\sim}{\theta}_t)\cdot K_j)$, for all j ,

3) $\hat{q}_{jmt}(\underset{\sim}{\theta}_t) = \min(\underline{q}, \max(\overline{q}, q_{jmt}(\underset{\sim}{\theta}_t) + \Delta h_{jmt}(\underset{\sim}{\theta}_t)))$, for all j and for $m \geq 1$, and

4) $c_t(\underset{\sim}{\theta}_t) = \min(\underline{c}, \max(1, c_t(\underset{\sim}{\theta}_t) - \alpha_t(\underset{\sim}{\theta}_t) + 1))$.

F is a convex and non-empty valued correspondence with closed graph from a compact convex set to itself. Hence, by the Kakutani fixed point theorem, it has a fixed point. That is, there is $(p, q, c, z, \Delta h, \alpha) \in \Delta_{\underline{q}}^T \times K$ such that $(p, q, c, z, \Delta h, \alpha) \in F(p, q, c; z, \Delta h, \alpha)$. Next, let $\tilde{y}_{jt}(\underset{\sim}{\theta}_t) \in \hat{\eta}_j(p_t(\underset{\sim}{\theta}_t, \theta_t))$ and let the measurable selections $(x^{(i,t)}, L^{(i,t)}, h^{(i,t)}, \alpha^{(i,t)})$ from $\xi^{(i,t)}(p, q, \tau(c))$ be such that $(z, \Delta h, \alpha)$ is obtained from them in the manner described above. Next let $y_{j0t}(\underset{\sim}{\theta}_t) = \min(\overline{H}, \overline{h}_{j0t}(\underset{\sim}{\theta}_t))K_j$ and $y_{jmt}(\underset{\sim}{\theta}_t) = \overline{h}_{jmt}(\underset{\sim}{\theta}_t)\tilde{y}_{jmt}(\underset{\sim}{\theta}_t)$, for $m > 0$. Let (y_j) be the corresponding production allocation.

I will show that $((y_j), (x^{(i,t)}, L^{(i,t)}, h^{(i,t)}), (p, q, \tau(c)))$ satisfies most of the conditions of a (1,Q) controlled equilibrium for the T-period economy. Specifically, it satisfies all the conditions in periods $t \leqq T-1$, except that there may be excess supplies of stock.

Step 5: $\bar{h}_{j,m+1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1}) \leqq \bar{h}_{jmt}(\underset{\sim}{\theta}_t)$, for all j, m, t and $\underset{\sim}{\theta}_t$, θ_{t+1}.

That is, there is no excess demand for shares of stock.

The bounds on output and prices imply that the dividend from a share of stock may never exceed $\bar{d} \equiv B|C_p|\bar{p}$. It now follows by backward induction on the lifetime of a stock that no one would buy a stock whose price exceeded $\bar{d}$ times the stock's lifetime. If the price exceeded this number, money would be a better investment. But $\bar{q}$ exceeded $\bar{d}$ times the lifetime of any stock (see (14.7), and by the definition of the fixed point map F, any stock in excess demand has price $\bar{q}$. Hence, one obtains a contradiction if a stock is in excess demand.

Step 6: Supply is bounded by b, where b is defined by equation (14.2).

By the construction of $\bar{\xi}$, $h_{j0t}(\underset{\sim}{\theta}_t) \leqq \bar{H}$. Since stocks are never in excess demand, $\bar{h}_{jmt}(\underset{\sim}{\theta}_t) \leqq \bar{H}$, for $m > 0$. If k is any produced good, aggregate supply is $\sum_{j=1}^{J} \sum_{m=1}^{N_j} \bar{h}_{jmtk}(\underset{\sim}{\theta}_t) y_{jmtk}(\underset{\sim}{\theta}_t) \leqq B\bar{H} \sum_{j=1}^{J} N_j < b$.
If k is not a produced good, it is a kind of labor, and its supply is bounded by $N-1 < b$. This proves Step 6.

Let $\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)$ and $\bar{\alpha}_t(\underset{\sim}{\theta}_t)$ be the individual and aggregate marginal utility of expenditure corresponding to the candidate equilibrium obtained as a fixe[illegible] oint.

Step 7: If $\bar{\alpha}_t(\theta_t) \geqq 1$, then for $k \in C_c$, $x_{tk}(\theta_t) = 0$ whenever $p_{tk}(\theta_t) \geqq \hat{p}$, where $\hat{p}$ is defined by equation (14.3)

Suppose that for some t and θ_t , $\bar{\alpha}_t(\theta_t) \geqq 1$ and that for some $k \in C_c$, $p_{tk}(\theta_t) \geqq \hat{p}$ and $\bar{x}_{tk}(\theta_t) > 0$, say for $k = 2$. I now show that $p_{tk}(\theta_t) \geqq (\overline{MS})^{-1}\hat{p}$, for all $k \in C_c$, where $\overline{MS}$ is as in assumption 3.4. Since $\bar{x}_{tk}(\theta_t) > 0$, there are $(i, t-n)$ and s_n such that $x_{t2}^{(i,t-n)}(\theta_t, s_n) > 0$. Then, for all $k \in C_c$,

$$p_{tk}^{-1}(\theta_t)\frac{\partial u_i(x_t^{(i,t-n)}(\theta_t, s_n), \theta_t, s_n)}{\partial x_k} \leqq \alpha_t^{(i,t-n)}(\theta_t, s_n)$$

$$= p_{t2}^{-1}(\theta_t)\frac{\partial u_i(x_t^{(i,t-n)}(\theta_t, s_n), \theta_t, s_n)}{\partial x_2} \leqq p_{2t}^{-1}(\theta_t)\overline{MS}\frac{\partial u_i(x_t^{(i,t-n)}(\theta_t, s_n), \theta_t, s_n)}{\partial x_k} .$$

where the second inequality holds by assumption 3.4. Therefore, $p_{tk}(\theta_t) \geqq (\overline{MS})^{-1}p_{t2}(\theta_t) \geqq (\overline{MS})^{-1}\hat{p}$.

Because of the inequality just demonstrated, the following is true for all $(i,t-n)$ and s_n , $\alpha_t^{(i,t-n)}(\theta_t, s_n)$

$$= \max_{k \in C_c} p_{tk}^{-1}(\theta_t)\frac{\partial u_i(x_t^{(i,t-n)}(\theta_t, s_n), \theta_t, s_n)}{\partial x_k} \leqq \overline{MS}\hat{p}^{-1}\bar{u} < N^{-1}$$, where $\bar{u}$

is as in assumption 3.2. The last inequality follows from the choice of $\hat{p}$. Hence, $\bar{\alpha}_t(\theta_t) < 1$, contrary to hypothesis. This proves Step 7.

Step 8: $\tau_t(\theta_t) > \underline{c}$, for all $t < T$ and for all θ_t .

Suppose that $c_t(\theta_t) = \underline{c}$, for some t and θ_t . By the choice of the fixed point function F , $\bar{\alpha}_t(\theta_t) \geqq 1$. Each person of age N alive in period T and event θ_t has after tax wealth of at least $-\underline{c}$ at the beginning of the period. A person of age N spends all his after tax wealth on consumption. Since there is a mass one of persons of age N and each spends at least $-\underline{c}$ units of money, it follows that at least $-|C_c|^{-1}\underline{c}$ is spent in the aggregate on some good k. Then, $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0$, so that by Step 7, $p_{tk}(\underset{\sim}{\theta}_t) < \hat{p}$. Hence, $\overline{x}_{tk}(\underset{\sim}{\theta}_t) \geqq -\hat{p}^{-1}|C_c|\underline{c} > b$, by the definition of $\underline{c}$ (equation (4.10)). Hence, by Step 6, there is excess demand for good k. Therefore, by the construction of the fixed point map F, $p_{tk}(\underset{\sim}{\theta}_t)$ equals $\overline{p}_p$ or $\overline{p}_w$, both of which exceed $\hat{p}$. This contradiction proves Step 8.

The after tax wealth of consumer $(i,t-n)$ in period t is $R_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \equiv W_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) - \tau(W_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n), c_t(\underset{\sim}{\theta}_t))$. The corresponding aggregate after tax wealth is

$$\overline{R}_t(\underset{\sim}{\theta}_t) = \sum_{n=0}^{N-1} \int_0^1 E[R_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)|\underset{\sim}{\theta}_t]di .$$

<u>Step 9</u>: For $t < T$, if $\overline{R}_t(\underset{\sim}{\theta}_t) \leqq \underline{R}$, then $\overline{\alpha}_t(\underset{\sim}{\theta}_t) > 1$, where $\underline{R}$ is defined by equation (14.9).

Suppose that $\overline{R}_t(\underset{\sim}{\theta}_t) \leqq \underline{R}$. Let $Z = \{(i,t-n,s_n)|R_t^{(i,t-n)}(\underset{\sim}{\theta}_t, s_n) \leqq 2N^{-1}\underline{R}\}$ Then, $\text{mes } Z \geqq \frac{1}{2}N$, where mes is the product of Lebesgue measure over i, counting measure over n and the probability on $\underset{\sim}{s}_n$.

Since prices of consumption goods are bounded, people always have an incentive to earn wages (except in period T). Since $p_{t1}(\underset{\sim}{\theta}_t) \geq \underline{p} > 0$, the supply of good 1 in period t is at least $(N-1)L_1$, where L_1 is defined by equation (14.8). The demand for good 1 in period t and event $\underset{\sim}{\theta}_t$ is at most $p_{t1}^{-1}(\underset{\sim}{\theta}_t)\overline{R}_t(\underset{\sim}{\theta}_t) \leqq \underline{p}^{-1}\underline{R} < L_1(N-1)$, where the last inequality follows from equation (14.9). Therefore, there is an excess supply of good 1 and $p_t(\underset{\sim}{\theta}_t) = \underline{p}$.

If $(i, t-n, \underset{\sim}{s}_n) \in Z$, then $x_{tk}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leqq 2\underline{p}^{-1}N^{-1}\underline{R} < 1$, for all k, where the last inequality is by equation (14.9). Hence, by assumption 3.3, $\frac{\partial u_i(x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)}{\partial x_1} \geq \underline{u}$ and so $\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)$ $\geq \underline{p}^{-1}\underline{u} > 2$. Therefore, $\overline{\alpha}_t(\underset{\sim}{\theta}_t) \geq 2 \text{ mes } \Xi > N > 1$. This proves Step 9.

<u>Step 10</u>: $\tau_t(\underset{\sim}{\theta}_t) < 1$, for all $t < T$ and $\underset{\sim}{\theta}_t$.

If $\tau_t(\underset{\sim}{\theta}_t) = 1$, then by the definition of the fixed point map F , $\overline{\alpha}_t(\underset{\sim}{\theta}_t) \leq 1$. But if $\tau_t(\underset{\sim}{\theta}_t) = 1$, then $\overline{R}_t(\underset{\sim}{\theta}_t) = 0$, so that by the previous step, $\overline{\alpha}_t(\underset{\sim}{\theta}_t) > 1$. This contradiction proves Step 10.

<u>Step 11</u>: $\overline{\alpha}_t(\underset{\sim}{\theta}_t) = 1$, for all $t < T$ and $\underset{\sim}{\theta}_t$.

This follows immediately from Steps 8 and 10 and the nature of the fixed point map F .

<u>Step 12</u>: $p_{tk}(\underset{\sim}{\theta}_t) > \underline{p}$, for all $k \in C_c$ and for all $t < T$ and $\underset{\sim}{\theta}_t$.

I will prove that if $p_{tk}(\underset{\sim}{\theta}_t) = \underline{p}$, for some $k \in C_c$, then $\overline{x}_{tn}(\underset{\sim}{\theta}_t) > b$, for some $n \in C_c$. This assertion proves Step 12, for if $\overline{x}_{tn}(\underset{\sim}{\theta}_t) > b$, then by Step 6, there is excess demand for good n , so that $p_{tn}(\underset{\sim}{\theta}_t)$ equals $\overline{p}_p$ or $\overline{p}_w$, both of which exceed $\hat{p}$. Since $\overline{\alpha}_t(\underset{\sim}{\theta}_t) = 1$, it follows from Step 7 that $\overline{x}_{tn}(\underset{\sim}{\theta}_t) = 0$, which is a contradiction.

Suppose that $p_{tk}(\underset{\sim}{\theta}_t) = \underline{p}$, for some k , say for $k = 2$. Let $m_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) = \frac{\partial u_i(x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n), \theta_t, s_n)}{\partial x_2}$. For all $(i, t-n, \underset{\sim}{s}_n)$, $p_{t2}(\underset{\sim}{\theta}_t)\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \geq m_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)$, so that $\underline{p} = p_{t2}(\underset{\sim}{\theta}_t)$ $= p_{t2}(\underset{\sim}{\theta}_t)\sum_{n=0}^{N-1}\int_0^1 E[\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)|\theta_t]di \geq \sum_{n=0}^{N-1}\int_0^1 E[m_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)|\theta_t]di$. Hence $\text{mes}\{(i, t-n, \underset{\sim}{s}_n)|m_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leq 2\underline{p}\} \geq \frac{1}{2}$, where mes here refers to the product of Lebesgue measure over i , counting measure on n and the probability on $\underset{\sim}{s}_n$. Let ε_i and $\underline{\varepsilon}$ be as just before equation (14.6). Finally, let $Z = \{(i, t\text{-}n, \underset{\sim}{s}_n) | m_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leqq 2\underline{p}$ and $\varepsilon_i \geqq 2\underline{\varepsilon}\}$. Then, mes $Z > 1/4$, by the definition of $\underline{\varepsilon}$. Since $\underline{p} < \underline{\varepsilon}$, it follows that if $(i, t\text{-}n, s_n) \in Z$, then $x_{tk}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) > 4|C_c|b$, for some $k \in C_c$. Hence, for some k, $\text{mes}\{(i, t\text{-}n, \underset{\sim}{s}_n) | x_{tk}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) > 4|C_c|b\} > (4|C_c|)^{-1}$. Therefore $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > b$, as was to be proved. This completes the proof of Step 12.

Let $z_t(\underset{\sim}{\theta}_t) \in R^{CE}$ be the excess demand vector for period t and event $\underset{\sim}{\theta}_t$ corresponding to the candidate equilibrium obtained from the fixed point. The definition of $z_t(\underset{\sim}{\theta}_t)$ is contained in the definition of the map $Z(p,q,c)$.

<u>Step 13</u>: For all $t < T$ and all $\underset{\sim}{\theta}_t$, $z_t(\underset{\sim}{\theta}_t) \leqq 0$ and $p_t(\underset{\sim}{\theta}_t) \cdot z_t(\underset{\sim}{\theta}_t) = 0$

Fix t and $\underset{\sim}{\theta}_t$. From the previous step and the nature of the fixed point map F, it follows that $z_{tk}(\underset{\sim}{\theta}_t) \geqq 0$, for all $k \in C_c$.

Let $C_0 = \{k \in C_c | \overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0\}$. By Step 7, $p_{tk}(\underset{\sim}{\theta}_t) < \hat{p}$, for $k \in C_0$.

I now define sets of commodities C_n, for $n = 1, 2, \ldots$. Suppost that C_n has been defined, for $n \geqq 0$. Let $C_{n+1} = C_n \cup \{k \in CE |$ for some j, $\tilde{y}_{tjk}(\underset{\sim}{\theta}_t) < 0$ and $\tilde{y}_{jtm}(\underset{\sim}{\theta}_t) > 0$, for some $m \in C_n\}$. Since $C_n \subset C_{n+1}$, for all n, it follows that $C_n = C_{n-1}$, for $n \geqq |C_p|$.

It follows from assumption 3.8 and by induction on n that if $k \in C_n$, then $p_{tk}(\underset{\sim}{\theta}_t) \leqq (\overline{MP})^n \hat{p}$. Hence, for $k \in C_{|C_p|}$, $p_{tk}(\underset{\sim}{\theta}_t) \leqq (\overline{MP})^{|C_p|} \hat{p} < \min(\overline{p}_p, \overline{p}_w)$. The second inequality follows from equations (14.4) and (14.5). Hence, by the nature of the fixed point map F, $z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$ and $p_{tk}(\underset{\sim}{\theta}_t) z_{tk}(\underset{\sim}{\theta}_t) = 0$, for all $k \in C_{|C_p|}$.

I now show that $z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$, for $k \in C' \equiv C_p \setminus C_{|C_p|}$. Let

$J' = \{j | \tilde{y}_{jtk}(\underset{\sim}{\theta}_t) < 0$, for some $k \in C'\}$. If $j \in J'$, then $\tilde{y}_{jtm}(\underset{\sim}{\theta}_t) \leqq 0$, for all $m \in C_{|C_p|} \cup C_w$. Therefore, for $j \in J'$, $p_t(\underset{\sim}{\theta}_t) \cdot \tilde{y}_{jt}(\underset{\sim}{\theta}_t)$ $\leqq \sum_{k \in C'} p_{tk}(\underset{\sim}{\theta}_t) \tilde{y}_{jtk}(\underset{\sim}{\theta}_t)$. If $k \in C'$ and $z_{tk}(\underset{\sim}{\theta}_t) < 0$, then $k \notin C_c$, since $z_{tk}(\underset{\sim}{\theta}_t) \geqq 0$, for all $k \in C_c$. Hence $k \in C'$ and $z_{tk}(\underset{\sim}{\theta}_t) < 0$ imply $p_{tk}(\underset{\sim}{\theta}_t) = 0$, since zero is the lower bound for prices not in C_c . (I have here once again used the nature of the fixed point map F .) Hence $\sum_{k \in C'} p_{tk}(\underset{\sim}{\theta}_t) z_{tk}(\underset{\sim}{\theta}_t) \geqq 0$. If $j \notin J'$, then $\tilde{y}_{jtk}(\underset{\sim}{\theta}_t) \geqq 0$, for all $k \in C'$, by the definition of J' . Therefore, for $k \in C'$, $z_{tk}(\underset{\sim}{\theta}_t) = - \sum_{j=1}^{J} y_{jtk}(\underset{\sim}{\theta}_t) \leqq - \sum_{j \in J'} y_{jtk}(\underset{\sim}{\theta}_t)$. Finally, since $0 \in Y_j(\theta_t)$ and since $\tilde{y}_{jt}(\underset{\sim}{\theta}_t) \in \eta_j(p_t(\underset{\sim}{\theta}_t), \theta_t)$, it follows that $p_t(\underset{\sim}{\theta}_t) \cdot \tilde{y}_{jt}(\underset{\sim}{\theta}_t) \geqq 0$, for all j . In summary, $0 \leqq \sum_{j \in J'} p_t(\underset{\sim}{\theta}_t) \cdot y_{jt}(\underset{\sim}{\theta}_t) \leqq \sum_{j \in J'} \sum_{k \in C'} p_{tk}(\underset{\sim}{\theta}_t) y_{jtk}(\underset{\sim}{\theta}_t)$ $\leqq - \sum_{k \in C'} p_{tk}(\underset{\sim}{\theta}_t) z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$, so that $\sum_{k \in C'} p_{tk}(\underset{\sim}{\theta}_t) z_{tk}(\underset{\sim}{\theta}_t) = 0$. Hence, for $k \in C'$, $z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$ and $p_{tk}(\underset{\sim}{\theta}_t) z_{tk}(\underset{\sim}{\theta}_t) = 0$, and so the same statements are true for all produced goods k .

Now suppose that $k \in C_w \setminus (C_c \cup C_{|C_p|})$ and that $t \leqq T-2$. Then, either $z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$ and $p_{tk}(\underset{\sim}{\theta}_t) = 0$ or $p_{tk}(\underset{\sim}{\theta}_t) > 0$ and for some j , $y_{jtk}(\underset{\sim}{\theta}_t) < 0$ or $h_{j0t}(\underset{\sim}{\theta}_t) K_{jk} > 0$. Suppose that $p_{tk}(\underset{\sim}{\theta}_t) > 0$. If $y_{jtk}(\underset{\sim}{\theta}_t) < 0$, then since profits are non-negative it follows that $y_{jtm}(\theta_t) > 0$, for some $m \in C_p$. Then, by assumption 3.8, $p_{tk}(\underset{\sim}{\theta}_t) \leqq (\overline{MP}) p_{tm}(\underset{\sim}{\theta}_t) \leqq (\overline{MP})^{|C_p|+1} \hat{p} < \overline{p}_w$. Thus, $z_{tk}(\underset{\sim}{\theta}_t) = 0$. Dividends are bounded by $\overline{\pi} \equiv |C_p| B \overline{p}_p$. Therefore, if $h_{j0t}(\underset{\sim}{\theta}_t) > 0$, it follows that $p_t(\underset{\sim}{\theta}_t) \cdot K_j < N_j \pi$, so that $p_{tk}(\underset{\sim}{\theta}_t) \leqq K_{jk}^{-1} N_j \overline{\pi} < \overline{p}_w$. (The strict inequality follows its definition of $\overline{p}_w$ in (14.5).) Therefore, $z_{tk}(\underset{\sim}{\theta}_t) = 0$ in this case as well.

The case $k \in (C_w \cap C_c) \setminus C_{|C_p|}$ remains to be considered. Since

$k \in C_c$, $z_{tk}(\underset{\sim}{\theta}_t) \geqq 0$. Also $\overline{x}_{tk}(\underset{\sim}{\theta}_t) = 0$, since $k \notin C_0$. Hence, if $z_{tk}(\underset{\sim}{\theta}_t) > 0$, it follows that good k is used as an input into production or as an investment good, and one may repeat the argument just made. This completes the proof of Step 13.

Step 14: $\overline{h}_{j0t}(\underset{\sim}{\theta}_t) < \overline{H}$, for all j and all $t \leq T-2$, and all $\underset{\sim}{\theta}_t$.

This says that the demand for new investment equals the supply. Recall that in defining the candidate equilibrium, the supply of new capital was restricted to be no more than $\overline{H}$.

I now prove Step 14. By assumption 3.9, $K_{jk} > 0$ only if $k \in C_W$, where C_W is the set of kinds of labor. The supply of any type of labor is at most $N-1$. Since markets for goods clear in periods before T , $\overline{h}_{j0t}(\underset{\sim}{\theta}_t)K_{jk} < N-1$, for $t < T$. By the definition of $\overline{H}$ (equation (14.1)), $\overline{h}_{j0t}(\underset{\sim}{\theta}_t) < \overline{H}$. This completes the proof of Step 14.

I have now shown that in periods $t < T$, all the conditions of a (1,Q)-controlled equilibrium are satisfied except that there may be excess demand for shares of stock. I now proceed to obtain some bounds that are independent of T and $\underline{q}$.

Step 15: For each t , there is $\overline{c}_t < 1$ such that $c_t(\underset{\sim}{\theta}_t) \leqq \overline{c}_t$ for all $\underset{\sim}{\theta}_t$ and $\overline{c}_t$ is independent of T and q .

Let $\overline{W}_t(\underset{\sim}{\theta}_t)$ be aggregate before tax wealth at the beginning of period t in the T-period equilibrium. Wealth can be money or stocks. Since government bonds are one period bonds, their value would show up as money at the beginning of a period. Stocks are bounded in value by $\overline{q}\,\overline{H}\sum_{j=1}^{J}(N_j+1)$. Let $\overline{M}_{t-1}(\underset{\sim}{\theta}_{t-1})$ be the aggregate value of the money held before taxes at the beginning of period t . $\overline{M}_{t-1}(\underset{\sim}{\theta}_{t-1})$ can increase only from interest in the government debt, from subsidies and by the value of the excess supply of stock. The interest rate is $r = Q^{-1} - 1$. The value of the subsidy is bounded by $-\underline{c}$. The value of the excess supply of stock is bounded by $\overline{q}\,\overline{H}\,\sum_j (N_j + 1)$. Let m_t solve the difference equation $m_{t+1} = (1+r)m_t - \underline{c} + \overline{q}\,\overline{H}\,\sum_j (N_j + 1)$, with $m_1 = \overline{h}_{g0} + \overline{M}_0$ being given by the initial conditions of the model. Then, $\overline{W}_t(\underset{\sim}{\theta}_t) \leqq w_t \equiv \overline{H}\,\overline{q}\,\sum_j (N_j + 1) + m_t$. Let $\overline{c}_t$ solve $(1 - \overline{c}_t)w_t = \underline{R}$, where $\underline{R}$ is defined by equation (14.9). By Step 9, if $c_t(\underset{\sim}{\theta}_t) \geq \overline{c}_t$, then $\overline{a}_t(\underset{\sim}{\theta}_t)$, which is impossible. This proves Step 15.

<u>Step 16</u>: For every $\beta > 0$, there is an $\varepsilon > 0$, independent of T and $\underline{q}$, such that for all $t \leqq T-1$, and for all $\underset{\sim}{\theta}_t$,

$$\text{mes}\left\{(i, t-n, \underset{\sim}{s}_n) \,\middle|\, \min_{k \in C_c} \frac{\partial u_i(x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n), \theta_t, s_n)}{\partial x_k} \leqq \varepsilon\right\} \leqq \beta ,$$ where

mes is the obvious measure on $[0,1] \times \{0, \ldots, N-1\} \times \{s_n\}$.

Let $\overline{B} > 2|C_c| b\beta^{-1}$, where b is defined by equation (14.2). Let $\varepsilon_i = \inf\left\{\frac{\partial u_i(x,\theta,s)}{\partial x_k} \,\middle|\, k \in C_c,\ x \in R_+^{C_c},\ x_n \leqq \overline{B}, \text{ for all } n \in C_c, \theta \in \Theta, \text{ and } s \in S\right\}$. ε_i is measurable as a function of i and is positive, by assumptions 3.1 and 3.11. Let $\varepsilon > 0$ be so small that $\text{mes}\{i | \varepsilon_i \leqq \varepsilon\} < \beta/2N$. Suppose that for some $t \leq T-1$ and some $\underset{\sim}{\theta}_t$, the inequality of Step 16 were false. Then, $\text{mes}\{(i, t-n, \underset{\sim}{s}_n) | x_{tk}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) > B, \text{ some } k\} > \beta/2$. Hence, for some k , $\text{mes}\{(i, t-n, \underset{\sim}{s}_n) | x_{tk}^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) > \overline{B}\} > \beta(2|C_c|)^{-1}$, so that $\overline{x}_{tk}(\underset{\sim}{\theta}_t) \geqq \overline{B}\beta(2|C_c|)^{-1} > b$. This contradicts the fact that markets for goods clear and that no more than b of any good may be supplied. This proves Step 16.

Step 17: For every $\hat{\varepsilon} > 0$, there is an $\varepsilon > 0$, independent of T and $\underline{q}$, such that for all $t \leq T-1$ and all $\underset{\sim}{\theta}_t$,
$\text{mes}\{(i, t-n, \underset{\sim}{s}_n) | \alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leqq \varepsilon\} \leqq \hat{\varepsilon}$.

If $\alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leqq \varepsilon$, then $\max_k \dfrac{\partial u_i(x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n), \theta_t, s_n)}{\partial x_k}$ $\leqq \max(\overline{p}_c, \overline{p}_w)\varepsilon$, where $\overline{p}_c$ and $\overline{p}_w$ are defined in equations (14.4) and (14.5). Hence, Step 17 follows from Step 16.

Step 18: For $t \leq T-2$, there is a positive number $\rho_t < \infty$, independent of T and $\underline{q}$, such that for all j,
$\pi_{jt}(p_t(\underset{\sim}{\theta}_t), \theta_t) + q_{j,m+1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1}) \leqq \rho_t q_{jmt}(\underset{\sim}{\theta}_t)$, for all j, $m \geqq 1$, $\underset{\sim}{\theta}_t$ and θ_{t+1}. Similarly, $q_{j1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1}) \leqq \rho_t q_{j0t}(\underset{\sim}{\theta}_t)$, for all j, $\underset{\sim}{\theta}_t$ and θ_{t+1}.

That is, there is an upper bound on rates of return on investments.

At the beginning of each period $t \leq T-1$, there is a population of at least L_1 of consumers of age $N-1$ who have provided good 1 in the previous period and so have before tax wealth at least $\underline{p}$. (It is here that I use the assumption $N > 2$.)

The marginal utility of money of consumer $(i, t-n)$ in period t is denoted by $\lambda_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)$. The marginal utility of money may exceed the marginal utility of expectations if the person does not consume but only saves or invests his money.

By the previous step, there is $\varepsilon > 0$ such that for all t and $\underset{\sim}{\theta}_t$, $\text{mes}\{(i, t-n, \underset{\sim}{s}_n) | \alpha_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) \leqq \varepsilon\} < L_1\eta^2(2|\Theta||S|)^{-1}$, where η is as in assumption 3.16.

Fix $t \leq T-2$ and $\underset{\sim}{\theta}_t$. Yet $Z = \{(i, t-N+2, \underset{\sim}{s}_{N-2}) | W_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{N-2})$ $\geqq \underline{p}$ and $\alpha_{t+1}^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_{N-2}, s_{N-1}) > \varepsilon$, . all θ_{t+1} and $s_{N-1}\}$.

where $W_t^{(i,t-N+2)}$ is defined by equation (2.8). Then, mes $Z > \frac{1}{2}L_1$, where mes here refers to the obvious measure over $[0,1] \times \{s_{N-1}\}$.

Since a person spends all his money in the last period of life, we have that $\lambda_{t+1}^{(i,t-N+2)}(\tilde{\theta}_t, \theta_{t+1}, \tilde{s}_{N-2}, s_{N-1})$
$= \alpha_{t+1}^{(i,t-N+2)}(\tilde{\theta}_t, \theta_{t+1}, \tilde{s}_{N-2}, s_{N-1}) \leq \underline{p}^{-1}\overline{u}$, where $\overline{u}$ is as in assumption 3.2 and $\underline{p}$ is defined by equation (14.6).

Let $\rho_t > (\underline{p}\eta\varepsilon)^{-1}(1-\overline{c}_{t+1})^{-1}\overline{u}\max(Q^{-1}, \delta^{-1}, 2(1-\overline{c}_t)^{-1}(L_1\underline{p})^{-1}\overline{q}\,\overline{H}\sum_{j=1}^{J}(N_j+1))$. Here $\overline{c}_t$ and $\overline{c}_{t+1}$ are as in Step 13.

Suppose that the inequality of Step 18 is violated at date t and in event $\tilde{\theta}_t$ by some stock (j,m) and event θ_{t+1}. Denote $q_{jmt}(\tilde{\theta}_t)$ by $q_t(\tilde{\theta}_t)$ and $\pi_t(p_t(\tilde{\theta}_t), \theta_t) + q_{j,m+1,t+1}(\tilde{\theta}_t, \theta_{t+1})$ by $Q_{t+1}(\tilde{\theta}_t, \theta_{t+1})$. If $m = 0$, let $Q_{t+1}(\tilde{\theta}_t, \theta_{t+1}) = q_{j1,t+1}(\tilde{\theta}_t, \theta_{t+1})$. Then, $Q_{t+1}(\tilde{\theta}_t, \theta_{t+1}) > \rho_t q_t(\tilde{\theta}_t)$.

If $(i, t-N+2, \tilde{s}_{N-2}) \in Z$, then

$$(14.11) \quad \lambda_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-2}) > (1-\overline{c}_{t+1})\rho_t\delta\eta\varepsilon .$$

I now show that consumers in Z do not buy government bonds in period t and event $\tilde{\theta}_t$. Suppose that $h_{gt}^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-2}) > 0$, for some $(i, t-N+2, \tilde{s}_{N-2}) \in Z$. Let $c_t^+(\tilde{\theta}_t) = \max(0, c_t(\tilde{\theta}_t))$. Then,
$(1-c_{t+1}^+(\tilde{\theta}_t, \theta_{t+1}))Q^{-1}\underline{p}^{-1}\overline{u} \geq \lambda_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-2}) \geq (1-c_{t+1}^+(\tilde{\theta}_t, \theta_{t+1})\rho_t\delta\eta\varepsilon$, so that $\rho_t \leq (Q\underline{p}\eta\varepsilon)^{-1}\overline{u}$, contrary to the definition of ρ_t. Thus, no bonds are bought. _A fortiori_, no money is held as an investment from period t to $t+1$.

Next, I show that $x_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-1}) = 0$, if $(i, t-N+2, s_{N-2}) \in Z$. Suppose that $x_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-1}) > 0$, for some $(i, t-N+2, \tilde{s}_{N-2}) \in Z$. Then, by (14.11), $(1-\overline{c}_{t+1})\rho_t\delta\eta\varepsilon$
$\leq \lambda_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-2}) = \alpha_t^{(i,t-N+2)}(\tilde{\theta}_t, \tilde{s}_{N-2}) \leq \underline{p}^{-1}\overline{u}$, or

$\rho_t \leqq (1 - \bar{c}_{t+1})^{-1}(\delta \underline{p}\eta\varepsilon)^{-1}\underline{u}$, contrary to the definition of ρ_t .

I now show that if a consumer in Z buys a share of stock at time t and in event $\underset{\sim}{\theta}_t$, then the price of a share is bounded above by $\hat{q} = (1 - \bar{c}_{t+1})^{-1}(\rho_t \underline{p}\eta\varepsilon)^{-1}\bar{q}\,\bar{u}$. Let v be the price of the stock in question. We know that the sum of the dividend in period t and price in period $t+1$ is bounded by $\bar{q}$. Hence, if a consumer $(i, t-N+2, s_{N-2}) \in Z$ buys the stock, we have that $v^{-1}\bar{q}\,\underline{p}^{-1}\bar{u}\delta \geqq \lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{N-2}) \geqq (1 - \bar{c}_{t+1})\rho_t \delta\eta\varepsilon$. This implies that $v \leq \hat{q}$.

In period t and event $\underset{\sim}{\theta}_t$, the consumers in Z spend all their money on stocks. The price of any stock they buy is at most $\hat{q}$. The total supply of shares is at most $\bar{H}\sum_{j=1}^{J}(N_j + 1)$. Hence, they spend at most $\hat{q}\bar{H}\sum_j (N_j + 1)$ on stocks. But their wealth is at least $\frac{1}{2}(1 - \bar{c}_t)L_1\underline{p}$. By the choice of ρ_t , $\hat{q}\bar{H}\sum_j (N_j + 1) < \frac{1}{2}(1 - \bar{c}_t)L_1\underline{p}$, so that there is an excess demand for stocks. This contradicts Step 5 and so proves Step 18.

Step 19: Allow the horizon to go to infinity.

Fix $\underline{q} > 0$. For each T , let $((y_j^T), (x^{T(i,t)}, L^{T(i,t)}, h^{T(i,t)}), (p^T, q^T, \tau(c^T)))$ be a T-period equilibrium as obtained by the fixed point argument of Step 4. The vectors $\tilde{y}_{jt}^T(\underset{\sim}{\theta}_t)$, $\bar{x}_t^T(\underset{\sim}{\theta}_t)$, $\bar{L}_t^T(\underset{\sim}{\theta}_t)$, $\bar{h}_t^T(\underset{\sim}{\theta}_t)$, $p_t^T(\underset{\sim}{\theta}_t)$, $q_t^T(\underset{\sim}{\theta}_t)$ and the number $c_t^T(\underset{\sim}{\theta}_t)$ corresponding to these equilibria are bounded for each t . Hence, by a Cantor diagonal argument, one may obtain limit vectors and numbers corresponding to $T = \infty$. Denote them as above but without the superscript T . By the generalized Fatou's lemma, there exist $(x^{(i,t)}, L^{(i,t)}, h^{(i,t)}) \in \xi^{(i,t)}(p, q, \tau(c))$, which are measurable with respect to i and such that the corresponding aggregat , are $(\bar{x}_t, \bar{L}_t, \bar{h}_t)$. (See

Hildenbrand (19), p. 73, proposition 8.) Then, $((y_j), (x^{(i,t)}, L^{(i,t)}, h^{(i,t)}), (p, q, \tau(c))$ satisfies all the conditions of a (1,Q)-controlled equilibrium except that there may be excess supply for shares of stock. Also, all share prices are at least $\underline{q}$ and the rate of return from buying a stock in period t and holding it one period does not exceed $c_t - 1$.

Since the arguments proving these facts are standard, I do not give details.

Step 20: Allow $\underline{q}$ to go to zero.

Once again, I apply a Cantor diagonal argument. Since the rates of return on stocks and wages are bounded, the wealths of individuals at any fixed time are bounded. Hence, purchases of goods are bounded If the price of a stock converges to zero as $\underline{q}$ goes to zero, an individual's purchases of it may diverge to infinity. However, the aggregate purchases are bounded by $\overline{H}$. Since the Cantor diagonal argument is applied only to aggregates and prices and input-output vectors, this argument may be carried out. Let the limit objects be denoted $\tilde{y}_{jmt}(\underset{\sim}{\theta}_t)$, $\overline{x}_t(\underset{\sim}{\theta}_t)$, $\overline{L}_t(\underset{\sim}{\theta}_t)$, $\overline{h}_t(\underset{\sim}{\theta}_t)$, $p_t(\underset{\sim}{\theta}_t)$, $q_t(\underset{\sim}{\theta}_t)$, and $c_t(\underset{\sim}{\theta}_t)$. Call this the limit array.

I now cannot apply at once the generalized Fatou's lemma, for the individual demand correspondences are discontinuous at stock prices equal to zero. However, it is possible to take advantage of the bound on aggregate demands for stocks in order to pass from the aggregate limit demands to individual demands.

First of all, I modify the limit aggregate holdings of stock, $\overline{h}_{jmt}(\underset{\sim}{\theta}_t)$, by setting $\overline{h}_{jmt}(\underset{\sim}{\theta}_t) = 0$ if $q_{jmt}(\underset{\sim}{\theta}_t) = 0$. Call these modified holdings $\bar{h}_{jmt}(\underset{\sim}{\theta}_t)$ again.

Observe that because of the bounds $\rho_t - 1$ on rates of return, if a stock ever has price zero in the limit array, then it has price zero and earns no dividend in all the succeeding periods and events.

Now fix attention on one generation, say on that born in period one. For simplicity, suppose that only one stock has price zero during their lifetime and that there is a single initial date and event, $\bar{t}$ and $\underset{\sim}{\bar{\theta}}_{\bar{t}}$, at which the price is zero. Once the argument is understood, in this case, it will be clear how to proceed in the general case. Let the name of the stock be $(\bar{j},\bar{m})$ in period $\bar{t}$. Let $q_t(\underset{\sim}{\theta}_t)$ be the price of the stock in each period and let $\pi_t(\underset{\sim}{\theta}_t)$ be its dividend. Then, $q_t(\underset{\sim}{\theta}_t) = \pi_t(\underset{\sim}{\theta}_t) = 0$, for $t \geq \bar{t}$ and $\underset{\sim}{\theta}_t$ equal to or following $\underset{\sim}{\bar{\theta}}_t$. Let $q^k_t(\underset{\sim}{\theta}_t)$ be the corresponding stock prices in the k^{th} equilibrium of the Cantor subsequence and let $h^{k(i,1)}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$ be the individual demands in this equilibrium. Let $\bar{h}^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$ $= \int_0^1 E[h^{k(i,1)}_t(\underset{\sim}{\theta}_t, s_{t-1})|\underset{\sim}{\theta}_t]di$. Let t' be the last date the stocks in question exists. Finally, let $b_k = \bar{H}\max\{q^k_t(\underset{\sim}{\theta}_t)|\bar{t} \leq t \leq t', \underset{\sim}{\theta}_t$ follows $\underset{\sim}{\bar{\theta}}_{\bar{t}}\}$.

We may assume that $\bar{h}^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$ converges, say to $\bar{h}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$.

Observe that $\lim_{k\to\infty} b_k = 0$. By passing to a subsequence if necessary, one may assume that $\lim_{K\to\infty} \sum_{k=K}^{\infty} \sqrt{b_k} = 0$.

Let $Z_k = \{(i, 1, s_0)$ for some $t \geq \bar{t}$ and for some $\underset{\sim}{\theta}_t$ equal to or following $\underset{\sim}{\bar{\theta}}_{\bar{t}}$ and for some $\underset{\sim}{s}_t$ following s_0, $q^k_t(\underset{\sim}{\theta}_t)h^{k(i,1)}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) > \sqrt{b_k}\}$. In order to obtain a bound on mes Z_k , notice that
$b_k \geq q^k_t(\underset{\sim}{\theta}_t)\bar{h}^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) \geq \sqrt{b_k}$ mes$\{(i, 1, \underset{\sim}{s}_t)|q^k_t(\underset{\sim}{\theta}_t)h^{k(i,1)}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$ $> \sqrt{b_k}\}$. It now follows that mes $Z_k \leq A\sqrt{b_k}$, where

$A = \sum_{t=\bar{t}}^{N} (n^{-2}|S||\Theta|)^t$.

Let $\hat{Z}_k = \bigcup_{n=k}^{\infty} Z_n$. Then, $\text{mes}\ \hat{Z}_k \leq A \sum_{n=k}^{\infty} \sqrt{b_n}$, so that $\lim_{k\to\infty} \text{mes}\ \hat{Z}_k = 0$. Define the sets K_n by induction on n as follows.

$K_1 = \{(i, 1, s_0) | (i, 1, s_0) \notin \hat{Z}_1\}$. Given K_n , $K_{n+1} = \hat{Z}_n \setminus \hat{Z}_{n+1}$.

The sets K_n are mutually disjoint and $\text{mes} \bigcup_{n=1}^{\infty} K_n = \text{mes}\{(i,1,s_0) | i \in [0,1], s_0 \in S\} = 1$.

I now apply the generalized Fatou's lemma on each of the sets K_n . If $(i, 1, s_0) \in K_n$, then for $k \geq n$, $q_t^k(\underset{\sim}{\theta}_t) \cdot h_t^{k(i,1)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) \leq \sqrt{b_k}$, for all $t \geq \bar{t}$ and for $\underset{\sim}{\theta}_{\bar{t}}$ equal to or following $\underset{\sim}{\bar{\theta}}_{\bar{t}}$ and for all $\underset{\sim}{s}_{t-1}$.

Let $\xi^{k(i,1)}(p,q,\tau)$ be the demand correspondence for consumer $(i,1)$ with the added restriction that $q_{j,\bar{m}+m,\bar{t}+m}(\underset{\sim}{\theta}_{\bar{t}+m}) \cdot h_{j,\bar{m}+m,\bar{t}+m}(\underset{\sim}{\theta}_{\bar{t}+m}, \underset{\sim}{s}_{\bar{t}+m-1}) \leq \sqrt{b_k}$, for $m = 0, \ldots, N_j - \bar{m}$ and for $\underset{\sim}{\theta}_{\bar{t}+m}$ following $\underset{\sim}{\bar{\theta}}_{\bar{t}}$. Let $\xi^{kP(i,1)}$ be the correspondence obtained from $\xi^{k(i,1)}$ by setting $h_{j,\bar{m}+m,\bar{t}+m}(\underset{\sim}{\theta}_{\bar{t}+m}, \underset{\sim}{s}_{\bar{t}+m-1}) = 0$, for $m = 0, \ldots, N_j - \bar{m}$ and for $\underset{\sim}{\theta}_{\bar{t}+m}$ following $\underset{\sim}{\bar{\theta}}_{\bar{t}}$. Let $\xi^{P(i,1)}$ be obtained from $\xi^{(i,1)}$ in the same way. (The " P " stands for "projection.")

If $(i, 1, \bar{s}_0) \in K_n$, then the demand, $x^{k(i,t)}$, of $(i, 1, \bar{s}_0)$ in the k^{th} equilibrium of the Cantor sequence belongs to $\xi^{k(i,1)}(p^k, q^k, \tau(c^k))$, for $k \geq n$. (More precisely, the part of consumer $(i,1)$'s demand conditional on $s_0 = \bar{s}_0$ would belong to a demand in $\xi^{k(i,1)}(p^k, q^k, \tau(c^k))$.) It is easy to see that the limit points of $\xi^{kP(i,1)}(p^k, q^k, \tau(c^k))$ belong to $\xi^{P(i,1)}(p, q, \tau(c))$, which is in turn contained in $\xi^{(i,1)}(p, q, \tau(c))$, where p , q and c belong to the limit equilibrium.

Now let ${}_n x^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) = \int_{K_n} x^{k(i,1)}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})di$ and make similar definitions for the other components of consumer demand, except for the demand for the stock in question on and after date $\bar{t}$ and event $\underset{\sim}{\bar{\theta}}_{\bar{t}}$. By passing to a subsequence if necessary, it may be assumed that $\lim_{k\to\infty} {}_n x^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) = {}_n x_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$ exists for all n , t , $\underset{\sim}{\theta}_t$ and $\underset{\sim}{s}_{t-1}$, and that similar limits exist for the other components of consumer demand. Individual demands for all commodities are uniformly bounded, except for the demands for the stock in question. It follows that there is some number $D > 0$ such that

$$(14.12) \quad {}_n x^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) \leqq D \text{ mes } K_n , \quad \text{for all } k , \ t , \ \underset{\sim}{\theta}_t , \text{ and } \underset{\sim}{s}_{t-1} .$$

Also, $\sum_{n=1}^{\infty} {}_n x^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) = x^k_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$. Because of (14.12), this infinite series converges uniformly in k , and so $\sum_{n=1}^{\infty} {}_n x_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1}) = x_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$. Similar arguments apply to the other components of consumer demand.

By the generalized Fatou's lemma (Hildenbrand (1974), p. 73, proposition 8) applied to $\xi^{kP(i,1)}(p^k, q^k, \tau(c^k))$ and to $\xi^{P(i,1)}(p, q, \tau(c))$ for each n , there is a measurable solution $(x^{(i,1)}, L^{(i,1)}, h^{(i,1)})$ from $\xi^{P(i,1)}(p, q, \tau(c)) \subset \xi^{(i,1)}(p, q, \tau(c))$ such that $\int_{K_n} x^{(i,1)}_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})di = {}_n x_t(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{t-1})$, for all t , $\underset{\sim}{\theta}_t$ and $\underset{\sim}{s}_{t-1}$. This defines consumer demand in the limit equilibrium for all $(i, 1, s_0) \in \bigcup_{n=1}^{\infty} K_n$. Since this set is of full measure and the individual demands aggregate to the limit aggregate demand, we are done.

After this operation has been applied to consumers born in every period, there is no demand for any stock of price zero. There might be an excess demand for these stocks, but this does not contradict the definition of equilibrium.

Q.E.D.

15. Proof of Theorem 4.2

The outline of the proof is the same as that of the previous one. I truncate the horizon, obtain a finite horizon equilibrium, calculate bounds, and pass to the limit.

Step 1: The Reduced Economy

In this proof, I take advantage of the fact that asset pricing formula (10.1) must hold in equilibrium. This implies that consumers are indifferent to their portfolio. They would be just as glad to buy only insurance. So, I eliminate all securities from the consumers' domain of choice until the end of the proof. In order to do so, I define a reduced economy in which consumers buy only goods and insurance and in which the production sector is entirely autonomous from consumers.

Let p and v be a commodity price system and system of insurance premiums, respectively. These imply asset prices q by formula (10.1). Call these the derived prices. They are defined as follows.

$q_{gt}(\underset{\sim}{\theta}_t) = \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})$. Stock prices are defined by backward induction on m, the age of the stock. $q_{jN_jt}(\underset{\sim}{\theta}_t) = \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})\pi_j(p_t(\underset{\sim}{\theta}_t), \theta_t)$.

Given $q_{j,m+1,t+1}$, $q_{jmt}(\underset{\sim}{\theta}_t) = \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})(q_{j,m+1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1})$

$+ \pi_j(p_t(\underset{\sim}{\theta}_t), \theta_t))$, if $m > 0$, and

$q_{j0t}(\underset{\sim}{\theta}_t) = \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})q_{j1,t+1}(\underset{\sim}{\theta}_t, \theta_{t+1})$.

Consumers are restricted to reduced allocations. A reduced allocation for a consumer is of the form (x,L,a) , where (x,L) is a resource allocation for the consumer and a is a non-negative insurance allocation (that is, $a_t(\underset{\sim}{\theta}_{t+1}) \geqq 0$).

I now define wealth $W_{t+n}^{(i,t)}$ and investment money balances $I_{t+n}^{(i,t)}$. Let (p,r,τ) be given, where p is a commodity price system, v is a system of insurance premiums, and τ is a tax system. Let q be the derived system of security prices. Finally, let (x,L,a) be a reduced allocation for consumer (i,t) . The definitions of wealths and balances are by induction on n . If $t \geqq 1$, $W_t^{(i,t)} = 0$. If $t < 1$, $W_1^{(i,t)}(\theta_1) = M_0^{(i,t)} + a_0^{(i,t)}(\theta_1) + \sum_{j=1}^{J} \sum_{m=0}^{M} q_{j,m+1,1}(\theta_1) h_{jm0}$. Suppose by induction on n that $W_{t+n}^{(i,t)}$ has been defined. Then, investment money balances are

$$(15.1) \quad I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1})$$

$$+ \sum_{\theta_{t+n+1}} v_{t+n}(\underset{\sim}{\theta}_{t+n}, \theta_{t+n+1}) p_{t+n}(\underset{\sim}{\theta}_{t+n}) L_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) \omega_{i,t+n}(\underset{\sim}{\theta}_n)$$

$$- \sum_{\theta_{t+n+1}} v_{t+n}(\underset{\sim}{\theta}_{t+n}, \theta_{t+n+1}) a_{t+n}(\underset{\sim}{\theta}_{t+n}, \theta_{t+n+1})$$

$$- p_{t+n}(\underset{\sim}{\theta}_{t+n}) \cdot x_{t+n}(\underset{\sim}{\theta}_{t+n}) - \tau_{t+n}(W_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{t+n}), \underset{\sim}{\theta}_{t+n}) .$$

Notice that it is assumed that the consumer sells insurance on his end of period wages. Thus, his insurance allocation includes these wages. Given, $I_{t+n}^{(i,t)}$, beginning period, before tax wealth is

$$W_{t+n+1}^{(i,t)}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n) = a_{t+n}(\underset{\sim}{\theta}_{t+n+1}) + I_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) .$$

Because wages have been sold, they are not included in the formula for wealth.

The budget set of consumer (i,t) is

$$\beta^{(i,t)}(p,v,\tau) = \{(x,L,a) \mid (x,L,a) \text{ is a reduced consumer allocation, feasible for consumer } (i,t) \text{ and } I^{(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) \geq 0, \text{ for all } n, \underset{\sim}{\theta}_{t+n} \text{ and } \underset{\sim}{s}_n\} .$$

By maximizing utility on this budget set, one obtains the reduced demand correspondence $\xi^{(i,t)}(p,v,\tau)$.

A vector of initial investments is a vector $(h_{j0}) = (h_{j0t}(\underset{\sim}{\theta}_t))$, where j varies over $1, \ldots, J$ and t varies over integers at least as large as $-N_j$, and where $h_{j0t}(\underset{\sim}{\theta}_t) \geq 0$, for all j , t , and $\underset{\sim}{\theta}_t$. The vector is said to be feasible if for $t < 0$, $h_{j0t}(\underset{\sim}{\theta}_t) = \bar{h}_{j,-t,0}$, where $\bar{h}_{j,-t,0}$ is given by the initial conditions of the model.

A reduced equilibrium consists of $((y_j), (h_{j0}), (x^{(i,t)}, L^{(i,t)}, a^{(i,t)}), (p,v,\tau))$ such that

1) $((y_j), (x^{(i,t)}, L^{(i,t)}))$ is a feasible resource allocation,
2) p , v and τ are a commodity price system, an insurance premium system and a tax system, respectively,
3) (h_{j0}) is a feasible vector of initial investments,
4) $y_{j0t}(\underset{\sim}{\theta}_t) = -h_{j0t}(\underset{\sim}{\theta}_t)K_j$, all t and $\underset{\sim}{\theta}_t$,
5) $y_{jmt}(\underset{\sim}{\theta}_t) = h_{j0,t-m}(\underset{\sim}{\theta}_{t-m})\tilde{y}_{jmt}(\underset{\sim}{\theta}_t)$, where $\tilde{y}_{jmt}(\underset{\sim}{\theta}_t) \in \eta_j(p_t(\underset{\sim}{\theta}_t), \theta_t)$, for all j , $m > 0$, t and $\underset{\sim}{\theta}_t$,
6) $q_{j0t}(\underset{\sim}{\theta}_t) \leqq p_t(\underset{\sim}{\theta}_t) \cdot K_j$, with equality if $h_{j0t}(\underset{\sim}{\theta}_t) > 0$, for all j , t and $\underset{\sim}{\theta}_t$, where q_{j0t} is the derived stock price, and
7) $(x^{(i,t)}, L^{(i,t)}, a^{(i,t)}) \in \xi^{(i,t)}(p,v,\tau)$, for almost every i and all t .

Step 2: Passage from Reduced to Ordinary Equilibrium

I now define what I will call (G,M)-balanced budget, reduced equilibrium and show how it may be transformed into an ordinary balanced budget equilibrium.

Let $((y_j), (h_{j0}), (x^{(i,t)}, L^{(i,t)}, a^{(i,t)}), (p,\nu,\tau))$ be a reduced equilibrium with derived security prices q. The after tax wealth of consumer (i,t) in period $t+n$ is

$$(15.2)\quad W^{A(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}) = W^{(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}) - \tau_t(W^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_{n-1}), \underset{\sim}{\theta}_{t+n}) .$$

The corresponding aggregate wealth at time t is $\overline{W}^A_t(\underset{\sim}{\theta}_t)$. The after tax aggregate money supply in period t is

$$(15.3)\quad \overline{M}^A_t(\underset{\sim}{\theta}_t) = \overline{W}^A_t(\underset{\sim}{\theta}_t) - \sum_{j=1}^{J} \sum_{m=1}^{N_j} q_{jmt}(\underset{\sim}{\theta}_t) h_{j0,t-m}(\underset{\sim}{\theta}_{t-m}) .$$

I define the aggregate excess demand for insurance in period t to be

$$(15.4)\quad \Delta a_t(\underset{\sim}{\theta}_{t+1}) = \overline{a}_t(\underset{\sim}{\theta}_{t+1}) - [p_t(\underset{\sim}{\theta}_t)\overline{L}_t(\underset{\sim}{\theta}_t) + \sum_{j=1}^{J} \pi_j(p_t(\underset{\sim}{\theta}_t), \theta_t) \sum_{m=1}^{N_j} h_{j0,t-m}(\underset{\sim}{\theta}_{t-m}) + \sum_{j=1}^{J} \sum_{m=1}^{N_j} q_{jm,t+1}(\underset{\sim}{\theta}_{t+1}) h_{j0,t-m+1}(\underset{\sim}{\theta}_{t-m+1}) + G] ,$$

where $\overline{a}_t$ is the aggregate corresponding to the insurance allocation $(a^{(i,t)})$. A (G,M) balanced budget, reduced equilibrium is defined to be a reduced equilibrium such that

(15.5) $\overline{M}_t^{\lambda}(\underset{\sim}{\theta}_t) = M$, for all t and $\underset{\sim}{\theta}_t$ and

(15.6) $\Delta a_t(\underset{\sim}{\theta}_{t+1}) = 0$, for all t and $\underset{\sim}{\theta}_{t+1}$.

Given a (G,M) balanced budget, reduced equilibrium as above, one obtains an ordinary (G,M) balanced budget equilibrium as follows. Let $h^{(i,t)}_{g,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = N^{-1}G$ and let $h^{(i,t)}_{jm,t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) = N^{-1}h_{j0,t+n-m}(\underset{\sim}{\theta}_{t+n-m})$, for all j , m , (i,t) , n , $\underset{\sim}{\theta}_{t+n}$ and $\underset{\sim}{s}_n$. That is, everyone has the same porfolio. Let

$$\hat{a}^{(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n) = a^{(i,t)}_t(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n) - p_{t+n}(\underset{\sim}{\theta}_{t+n})L^{(i,t)}_{t+n}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n) + \omega_{in}(s_n)$$

$$- N^{-1}[\sum_{j=1}^{J} \pi_j(p_{t+n}(\underset{\sim}{\theta}_{t+n}), \theta_{t+n}) \sum_{m=1}^{N_j} h_{j0,t+n-m}(\underset{\sim}{\theta}_{t+n-m})$$

$$+ \sum_{j=1}^{J} \sum_{m=1}^{N_j} q_{jm,t+n+1}(\underset{\sim}{\theta}_{t+n+1}) h_{j0,t+n+1-m}(\underset{\sim}{\theta}_{t+n+1-m}) + G] .$$

Then, $((y_j), (x^{(i,t)}, L^{(i,t)}, h^{(i,t)}, \hat{a}^{(i,t)}), (p,y,\tau))$ forms a (G,M) balanced budget equilibrium.

The rest of the proof demonstrates that a reduced (G,M)-balanced equilibrium exists.

<u>Step 3: Truncation</u>

The truncation at time T is carried out just as in the proof of Theorem 4.1.

<u>Step 4: The Tax Function</u>

The tax function $\tau(W,c)$ is defined just as in Step 3 of the proof of Theorem 4.1.

Step 5: List of Bounds

Let H be as in (14.1). Let

(15.7) $\hat{p} = 1 + 2L_1^{-1}(M + |\Theta|G)$,

where L_1 is defined by equation (14.8) . Let

(15.8) $\overline{p} = ((1 + (\overline{MP})^{|C_p|})\overline{MS} + 1)\hat{p}$

where $\overline{MP}$ is as in assumption 3.8 and $\overline{MS}$ is as in assumption 3.4. Let

(15.9) $\overline{\pi} = |C_p|B\overline{p}$,

where B is as in assumption 3.7. Let

(15.10) $\overline{q} = \overline{\pi}(1 + \max_j N_j)$.

Let $B_1 = 1 + \max\{K_{jk}^{-1}|K_{jk} > 0,\ j = 1, \ldots, J,\ k \in C_w\}$. Let

(15.11) $\overline{p}_w = \max(\overline{p}, B_1\overline{q})$.

Let

(15.12) $B_2 = [N - 1 + (2 + \max_j N_j)(NB_1 + \overline{H}\sum_j N_j)|C_p|B][(1 + (\overline{MP})^{|C_p|})\overline{MS} + 1]$

Let

(15.13) $\overline{c} = 1 - \frac{1}{2}M[M + G + \overline{p}_w N + (\overline{\pi} + 2\overline{q})\overline{H}\sum_j N_j]^{-1}$.

Let

$$(15.14)\quad \underline{p}_1 = (\overline{MS}|C_c|b)^{-1}(1-\overline{c})\min_{\theta_1}\int_0^1 (M_0^{(i,N-1)} + a_0^{(i,N-1)}(\theta_1))di\ ,$$

where $M_0^{(i,N-1)}$ and $a_0^{(i,N-1)}(\theta_1)$ are given by the initial conditions and b is defined by equation (14.2).

It will be shown (Step 14) that the ε_1 and ε_2 defined in (15.15) and (15.16) below exist.

(15.15) The number $\varepsilon_1 > 0$ is such that for any $t-1$ and $\underset{\sim}{\theta}_{t-1}$, $\text{mes}\ Z^1_{t-1}(\underset{\sim}{\theta}_{t-1}) > \frac{1}{2}L_1$, where $Z^1_{t-1}(\underset{\sim}{\theta}_{t-1}) = \Big\{(i,\ t-N+2,\ \underset{\sim}{s}_{N-3})\ |$ $k(i) = 1$, $\omega_{i,N-3}(\underset{\sim}{s}_{N-3}) = 1$ and

$$\frac{\partial u_i(x^{i,t-N+2}_{t+1}(\underset{\sim}{\theta}_{t-1},\ \theta_t,\ \theta_{t+1};\ \underset{\sim}{s}_{N-3},\ s_{N-2},\ s_{N-1}),\ \theta_{t+1},\ s_{N-1})}{\partial x_k} > \varepsilon_1,$$

for all θ_t, θ_{t+1}, s_{N-2}, s_{N-1} and for all $k \in C_c\Big\}$,

where L_1 is defined by equation (14.8) and mes is the obvious measure on $[0,1] \times \{\underset{\sim}{s}_{N-3}\}$.

(15.16) The number ε_2 is such that for any t and $\underset{\sim}{\theta}_t$, $\text{mes}\ Z^2_t(\underset{\sim}{\theta}_t) > \frac{1}{2}L_1$, where $Z^2_t(\underset{\sim}{\theta}_t)$ $= \Big\{(i,\ t-N+2,\ \underset{\sim}{s}_{N-2})\,|\,k(i) = 1,\ \omega_{i,N-2}(\underset{\sim}{s}_{N-2}) = 1$ and

$$\frac{\partial u_i(x^{(i,t-N+2)}_{t+1}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2},\ s_{N-1}),\ \theta_{t+1},\ s_{N-1})}{\partial x_k} > \varepsilon_2,$$

for all θ_{t+1} and s_{N-1} and all $k \in C_c\Big\}$.

I now define lower bounds $\underline{v}_2$ and $\underline{p}_t$ by induction on t. Equation (15.14) defines $\underline{p}_1$. Suppose that $\underline{p}_1, \ldots, \underline{p}_t$ and $\underline{v}_1, \ldots, \underline{v}_{t-1}$ have been defined.

$$(15.17)\quad \underline{v}_t = \tfrac{1}{2}\min[(2|\Theta|)^{-1},\ B_2^{-1}L_1,\ \underline{p}_{t-1}\varepsilon_1(\eta\delta(1-\overline{c}))^2(\overline{p}\,\overline{u})^{-1},\ \delta\eta^2\varepsilon_1(1-\overline{c})^2$$

$$\cdot(\overline{u}\,\overline{p})^{-1}\underline{v}_{t-1}\ \min(\underline{p}_t\delta\eta\varepsilon_1(1-\overline{c})(\overline{u})^{-1},\ \underline{p}_{t-1}\delta\eta\varepsilon_1$$

$$\cdot(2|\Theta|^2\overline{u}\,\overline{MS}\,|C_c|b)^{-1}(1-\overline{c})^3L_1)\ ,$$

where ε_1 is as in definition (15.15), B_2 is as defined by equation (15.12), η is as in assumption (3.16), δ is the discount factor on utility, $\overline{c}$ is defined by equation (15.13), $\overline{u}$ is as in assumption 3.2, $\overline{MS}$ is as in assumption 3.4, and L_1 is defined by equation (14.8).

Now suppose that $\underline{p}_1, \ldots, \underline{p}_t$ and $\underline{v}_1, \ldots, \underline{v}_t$ have been defined.

$$(15.18)\quad \underline{p}_{t+1} = \tfrac{1}{2}\underline{p}_t\ \min(\delta\eta\varepsilon_2(1-\overline{c})\overline{u}^{-1},\ [\underline{v}_t(1-\overline{c})\varepsilon_2\eta\overline{u}^{-1}]^{|\Theta|}$$

$$\cdot(2|\Theta|\overline{MS}|C_c|b)^{-1}(1-\overline{c})L_1)\ .$$

$$(15.19)\quad \underline{M} = -[M + 1 + \overline{q}\,\overline{H}\sum_{j=1}^{J}N_j]\ .$$

Step 6: The Fixed Point Argument

I first define a domain for prices, then I define demand and the fixed point map.

The domain of prices and the tax parameter in period t is $\Delta_t = \{(p,v,c)\,|\,p \in R_+^{CE},\ p_k \geq \underline{p}_t,$ for all $k \in C_c$, $p_k \leq \overline{p}$, for all $k \in C_p \cup C_c$, and $p_k \leq \overline{p}_w$ otherwise, $v = (v(\theta))_{\theta\in\Theta}$, $v(\theta) \geq \underline{v}_t$, for all θ and $\sum_{\theta\in\Theta} v(\theta) \leq 1$, and $-\underline{M} \leq c \leq 1\}$, where $\underline{M}$ is defined by equation (15.19). The domain of prices in the T-period economy $\Delta^T = \{(p_t(\underset{\sim}{\theta}_t),\ v_t(\underset{\sim}{\theta}_t),\ c_t(\underset{\sim}{\theta}_t))_{t=1}^T\,|\,(p_t(\underset{\sim}{\theta}_t),\ v_t(\underset{\sim}{\theta}_t),\ c_t(\underset{\sim}{\theta}_t)) \in \Delta_t$, for all t and $\underset{\sim}{\theta}_t\}$.

Given $(p,v,c) \in \Delta^T$, let $\xi^{T(i,t)}(p,\ v,\ \tau(c))$ be the reduced demand for the T-horizon economy. It is defined just as is reduced demand in the infinite horizon economy, except that one must take account of the T-period horizon. Let $q_{jmt}(\underset{\sim}{\theta}_t)$ be the security prices derived from p and v . For each j , t and $\underset{\sim}{\theta}_t$, let

$$\varphi_{j0t}(p, v, \underset{\sim}{\theta}_t) = \begin{cases} 0 & \text{, if } q_{j0t}(\underset{\sim}{\theta}_t) < p_t(\underset{\sim}{\theta}_t)\cdot K_j \text{ ,} \\ [0,\overline{H}] & \text{, if } q_{j0t}(\underset{\sim}{\theta}_t) = p_t(\underset{\sim}{\theta}_t)\cdot K_j \text{ , and} \\ \overline{H} & \text{, if } q_{j0t}(\underset{\sim}{\theta}_t) > p_t(\underset{\sim}{\theta}_t)\cdot K_j \text{ .} \end{cases}$$

Let $\hat{\eta}_j(p,\theta)$ be as defined just before equation (14.2).

I may now define the appropriate "excess demand" function for the fixed point argument. Given $(p,q,c) \in \Delta^T$, let $Z(p,q,c)$ $= \{(z, \Delta a, M^A) = (z_t(\underset{\sim}{\theta}_t), \Delta a_t(\underset{\sim}{\theta}_{t+1}), M_t^A(\underset{\sim}{\theta}_t)) \mid$ there is a measurable selection $(x^{(i,t)}, L^{(i,t)}, a^{(i,t)})$ from $\xi^{(i,t)}(p, q, \tau(c))$, for all t, and there are $h_{j0t}(\theta_t) \in \varphi_{j0t}(p, v, \underset{\sim}{\theta}_t)$ and $\tilde{y}_{jt}(\underset{\sim}{\theta}_t) \in \hat{\eta}_j(p_t(\underset{\sim}{\theta}_t), \theta_t)$, for all j, t and $\underset{\sim}{\theta}_t$ such that

1) $z_t(\underset{\sim}{\theta}_t) = \overline{x}_t(\underset{\sim}{\theta}_t) + \sum_{j=1}^{J} h_{j0t}(\underset{\sim}{\theta}_t)K_j - \overline{L}_t(\underset{\sim}{\theta}_t) - \sum_{j=1}^{J}\sum_{m=1}^{N_j} h_{j0,t-m}(\underset{\sim}{\theta}_{t-m})\tilde{y}_{jt}(\underset{\sim}{\theta}_t)$,

2) $\Delta a_t(\underset{\sim}{\theta}_{t+1})$ is defined by equation (15.4), and

3) $M_t^A(\underset{\sim}{\theta}_t)$ is defined by equations (15.2) and (15.3) ,

when $\overline{x}_t$, $\overline{L}_t$ and $\overline{a}_t$ are the aggregates corresponding to the selections $(x^{(i,t)}, L^{(i,t)}, a^{(i,t)})$ and where $h_{j0t}(\underset{\sim}{\theta}_t) = \overline{h}_{j,-t,0}$, for $t < 0$. (The $\overline{h}_{j,-t,0}$ are part of the initial conditions of the model.) Also, in applying equation (15.4) to define Δa_T , it is understood that $q_{jm,T+1} = 0$.

Because the prices of insurance and consumption goods are bounded away from zero, the individual demand correspondences $\xi^{(i,t)}(p, v, \tau(c))$ are non-emp-- valued and have closed graph. Because prices are bounded as well, these demand correspondences are uniformly bounded. Because the set of consumers is atomless, Z is convex-valued. It is uniformly bounded and by the generalized Fatou's lemma it has closed graph (Hildenbrand (1974), p. 73). Let K be a compact convex set containing the range of Z and define $\mu : \Delta^T \times K \to \Delta^T$ as follows $\mu(p, v, c; z, \Delta a, \overrightarrow{M}^A) = (\hat{p}_t(\underset{\sim}{\theta}_t), \hat{v}_t(\underset{\sim}{\theta}_{t+1}), \hat{c}_t(\underset{\sim}{\theta}_t))$, where the components are defined as follows.

1) $\hat{p}_{tk}(\underset{\sim}{\theta}_t) = \min(\underline{p}_t, \max(\overline{p}, p_{tk}(\underset{\sim}{\theta}_t) + z_{tk}(\underset{\sim}{\theta}_t)))$.

2) $\hat{c}_t(\underset{\sim}{\theta}_t) = \min(-M, \max(1, c_t(\underset{\sim}{\theta}_t) + M_t^A(\underset{\sim}{\theta}_t) - M))$.

3) $\hat{v}_t$ is more complicated to define. Fix $\underset{\sim}{\theta}_t$ and let $v_\theta = v_t(\underset{\sim}{\theta}_t, \theta)$, $\hat{v}_\theta = \hat{v}_t(\underset{\sim}{\theta}_t, \theta)$, and $\Delta a_\theta = \Delta a_t(\underset{\sim}{\theta}_t, \theta)$. If $\sum_\theta (v_\theta + \Delta a_\theta) \leq 1$, let $b_\theta = \Delta a_\theta + v_\theta$. If $\sum_\theta (v_\theta + \Delta a_\theta) > 1$, let $b_\theta = \Delta a_\theta + v_\theta - |\Theta|^{-1}[\sum_\theta(\Delta a_\theta + v_\theta) - 1]$. Let $c_\theta = \min(\underline{v}_t, b_\theta)$. Then, $\hat{v}_\theta = \underline{v}_t + D^{-1}(c_\theta - \underline{v}_t)$, where $D = \max(1, (1 - |\Theta|\underline{v}_t)^{-1}(\sum_\theta c_\theta - |\Theta|\underline{v}_t)$.

Since $\underline{v}_t < \frac{1}{2}|\Theta|^{-1}$, $\hat{v}_t$ is well defined.

Now let $F : \Delta^T \times K \to \Delta^T \times K$ be defined by $F(p, v, c; z, \Delta a, \overrightarrow{M}^A) = \mu(p, v, c; z, \Delta a, \overrightarrow{M}^A) \times Z(p,v,c)$. By the Kakutani fixed point theorem, F has a fixed point. To this fixed point, there correspond input-output vectors $\tilde{y}_{jt}(\underset{\sim}{\theta}_t)$, excess demand vectors $z_t(\underset{\sim}{\theta}_t)$, and a candidate T-period equilibrium $((y_j), (x^{(i,t)}, L^{(i,t)}, a^{(i,t)}), (p, v, r($ The corresponding aggregates will be denoted $\overline{x}_t(\underset{\sim}{\theta}_t)$, $\overline{L}_t(\underset{\sim}{\theta}_t)$, $\overline{a}_t(\underset{\sim}{\theta}_t)$, $\Delta a_t(\underset{\sim}{\theta}_{t+1})$ and so on. Throughout what follows, $c_t^+(\underset{\sim}{\theta}_t)$ denotes $\max(0, c_t(\underset{\sim}{\theta}_t))$.

Step 7: $\overrightarrow{M}_t^A(\underset{\sim}{\theta}_t) = M$, for all t and $\underset{\sim}{\theta}_t$.

If $\overrightarrow{M}_t^A(\underset{\sim}{\theta}_t) > M$, then $c_t(\underset{\sim}{\theta}_t) = 1$ and so $\overrightarrow{M}_t^A(\underset{\sim}{\theta}_t) \leq \overrightarrow{W}_t^A(\underset{\sim}{\theta}_t) = 0 < M$, which is a contradiction.

If $\overrightarrow{M}_t^A(\underset{\sim}{\theta}_t) < M$, then $c_t(\underset{\sim}{\theta}_t) = -\underline{M}$, so that by equations (15.3) and (15.19), $\overrightarrow{M}_t^A(\underset{\sim}{\theta}_t) > M$, which is a contradiction. This proves Step 7.

Step 8. If $\Delta a_t(\underset{\sim}{\theta}_{t+1}) < 0$, then $\underline{v}_t(\underset{\sim}{\theta}_{t+1}) = \underline{v}_t$.

Proof. Let $\hat{v}_\theta$, v_θ, Δa_θ, b_θ, and D be as in part 3 of the definition of the function μ. If $\Delta a_\theta < 0$, then $b_\theta < v_\theta$. If $b_\theta \leqq \underline{v}_t$, then $v_\theta = \hat{v}_\theta = \underline{v}_t$, as desired. If $b_\theta > \underline{v}_t$, then since $D \geqq 1$, $\hat{v}_\theta = \underline{v}_t + D^{-1}(b_\theta - \underline{v}_t) \leqq b_\theta < v_\theta$, which is impossible by the fixed point property. This proves Step 8.

Step 9: If $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) > 0$, for some θ_{t+1}, then $\sum_\theta v_t(\underset{\sim}{\theta}_t, \theta) = 1$.

I continue to use the notation used in the previous step.

Suppose that $\sum_\theta v_\theta < 1$. Then, it must be that $\sum_\theta (v_\theta + \Delta a_\theta) \leq 1$, for otherwise $\sum_\theta v_\theta = \sum_\theta \hat{v}_\theta = 1$. Also $D > 1$, for if $D = 1$ and $\Delta a_\theta > 0$, then $\hat{v}_\theta = \min(v_t, v_\theta + \Delta a_\theta) > v_\theta$, which is impossible. But, if $D > 1$, then $\sum_\theta v_\theta = \sum_\theta \hat{v}_\theta = 1$. This proves Step 9.

Step 10: If $v_t(\underset{\sim}{\theta}_t, \theta_{t+1}) > \underline{v}_t$, for all θ_{t+1}, then $\Delta a_t(\underset{\sim}{\theta}_t, \theta) = \Delta a_t(\underset{\sim}{\theta}_t, \theta')$, for all θ and θ'.

This is obvious from part 3 of the definition of the map μ.

Step 11. For all t and $\underset{\sim}{\theta}_t$,

$$M = \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) + p_t(\underset{\sim}{\theta}_t)\cdot\overline{x}_t(\underset{\sim}{\theta}_t)$$

$$+ \sum_{j=1}^{J} q_{j0t}(\underset{\sim}{\theta}_t)h_{j0t}(\underset{\sim}{\theta}_t) + G\sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1}) + \overline{I}_t(\underset{\sim}{\theta}_t) .$$

This is proved by aggregating the individual budget constraints and by using (15.3) and (15.5).

Step 12. $z_{tk}(\underset{\sim}{\theta}_t) \leqq 0$, whenever $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0$, for $k \in C_c$, $t \leq T-1$ and all $\underset{\sim}{\theta}_t$.

I will prove that $p_{t1}(\underset{\sim}{\theta}_t) \leq \hat{p}$, where $\hat{p}$ is defined by equation (15.7). From this, Step 15 follows. For suppose that $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0$, for $k \in C_c$ and $k \neq 1$. By assumption 3.4, $p_{tk}(\underset{\sim}{\theta}_t) \leqq \overline{MS}\hat{p} < \overline{p}$, where $\overline{p}$ is defined by equation (15.8). But, if $z_{tk}(\underset{\sim}{\theta}_t) > 0$, then $p_{tk}(\underset{\sim}{\theta}_t) = \overline{p}$.

In order to prove that $p_{t1}(\underset{\sim}{\theta}_t) \leq \hat{p}$, I need an upper bound on the purchasing power of consumers. By Step 11, $p_t(\underset{\sim}{\theta}_t)\overline{x}_t(\underset{\sim}{\theta}_t)$ $\leqq M - \sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1})$. If $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) < 0$, then $v_t(\underset{\sim}{\theta}_t, \theta_{t+1}) = \underline{v}_t$. Also, by the definition of Δa_t (equation (15.4), $-\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) \leqq \overline{p}_w(N-1) + \overline{\pi}\overline{H}\sum_j N_j + \overline{q}\overline{H}\sum_j N_j + G$, where $\overline{p}_w$, $\overline{\pi}$, and $\overline{q}$ are defined by equations (15.11), (15.9) and (15.10), respectively. It follows that, $p_t(\underset{\sim}{\theta}_t)\cdot\overline{x}_t(\underset{\sim}{\theta}_t) \leq M + |\Theta|\underline{v}_t G + \underline{v}_t B_2\hat{p}$, where B_2 is as in (15.12). Use has been made of the fact that $\overline{p}_w$, $\overline{\pi}$ and $\overline{q}$ are all constants times $\hat{p}$.

Clearly, if $p_{t1}(\underset{\sim}{\theta}_t) = \underline{p}_t$, we are done, since $\underline{p}_t < \hat{p}$. So suppose that $p_{t1}(\underset{\sim}{\theta}_t) > \underline{p}_t$. Then ther is no excess supply of good 1.

At least $L_1(N-1)$ of good 1 is offered (since $t < T$), where L_1 is defined by equation (14.7). Therefore, $p_{t1}(\underset{\sim}{\theta}_t)L_1(N-1) \leq M + \underline{v}_t(|\theta|G + B_2\hat{p})$. If $p_{t1}(\underset{\sim}{\theta}_t) > \hat{p}$, then $\hat{p}L_1(N-1) \leq M + \underline{v}_t(|\theta|G + B_2\hat{p})$, or $\hat{p}(L_1(N-1) - \underline{v}_t B_2) < M + \underline{v}_t|\theta|G$. By (15.17), $\underline{v}_t \leq 1$ and $\underline{v}_t B_2 \leq \frac{1}{2}L_1(N-1)$, so that $\frac{1}{2}\hat{p}L_1 \leq \hat{p}(L_1(N-1) - \underline{v}_t B_2) < M + |\theta|G$. This contradicts the definition of $\hat{p}$ (equation (15.7)). This proves Step 12.

Let b be as defined in equation (14.2).

Step 13: Supply is bounded by b.

This follows from assumption 3.7 and the fact that no more than $\overline{H}$ units of production are installed in any industry at any time.

Step 14: The statement of Step 16 of the proof of Theorem 4.1 applies here, for $t < T$.

The proof of this statement relied on the fact that consumers could not demand more than b units of any good. By Step 12, this fact is true here.

Step 15: $c_t(\underset{\sim}{\theta}_t) < \overline{c}$, for all t and $\underset{\sim}{\theta}_t$, where $\overline{c}$ is as defined by equation (15.13).

I must calculate an upper bound on $\overline{W}_t(\underset{\sim}{\theta}_t) = \overline{I}_{t-1}(\underset{\sim}{\theta}_{t-1}) + \overline{a}_t(\underset{\sim}{\theta}_t, \theta_{t+1})$. First of all, since $\overline{M}^A_{t-1}(\underset{\sim}{\theta}_{t-1}) = M$, it follows from equation (15.3) and aggregation of the consumers' period t-1 budget constraints that $\overline{I}_{t-1}(\underset{\sim}{\theta}_{t-1}) \leqq M + \overline{q}\,\overline{H}\sum_{j=1}^{J} N_j + \overline{p}_w$. Secondly, observe that I may assume that $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) \leqq 0$. If $\sum_\theta v_t(\underset{\sim}{\theta}_t, \theta) < 1$, then that is true by Step 9. If $\sum_\theta v_t(\underset{\sim}{\theta}_t, \theta) = 1$ and if $\Delta a_t(\underset{\sim}{\theta}_t, \quad_{+1}) = \beta > 0$, then I can reduce $a_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta, \underset{\sim}{s}_n)$ by $N^{-1}\beta$ and increase all $I_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n)$ by $N^{-1}\beta$, for all $(i,t-n)$, θ , and $\underset{\sim}{s}_n$. In so doing, I need change no other action of any consumer at any time, all consumers are equally well off and I do not change $\overline{W}_t(\underset{\sim}{\theta}_t)$. After having made this change, I have $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) \leq 0$. If $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) \leq 0$, then $\overline{a}_t(\underset{\sim}{\theta}_t, \theta_{t+1}) < \overline{p}_W(N-1) + (\overline{\pi}+\overline{q})\overline{H}\sum_j N_j + G$. Therefore, $\overline{W}_t(\underset{\sim}{\theta}_t) \leq M + G + \overline{p}_W N + (\overline{\pi} + 2\overline{q})\overline{H}\sum_j N_j$. Call this upper bound $\overline{W}$.

Clearly, if $(1-c)\overline{W}_t^A(\underset{\sim}{\theta}_t) < M$, then $c_t(\underset{\sim}{\theta}_t) < c$. Since $\overline{c}$ satisfies $(1-\overline{c})\overline{W} < M$, it follows that $c_t(\underset{\sim}{\theta}_t) \leq \overline{c}$, for all t and $\underset{\sim}{\theta}_t$. This proves Step 15.

I now introduce the numbers

$$A_{t+n+1}^{(i,t)}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n) = W_{t+n+1}^{(i,t+n)}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n)$$

$$- p_{t+n}(\underset{\sim}{\theta}_{t+n}) L_{t+n}^{(i,t)}(\underset{\sim}{\theta}_{t+n}, \underset{\sim}{s}_n)\omega_{in}(\underset{\sim}{s}_n) ,$$

$A_{t+n+1}^{(i,t)}(\underset{\sim}{\theta}_{t+n+1}, \underset{\sim}{s}_n)$ is the wealth transferred from period $t+n$ and one event $\underset{\sim}{\theta}_{t+n}$ to period $t+n+1$ and event $\underset{\sim}{\theta}_{t+n+1}$.

<u>Step 16</u>: If $A_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_n) \leq 0$, for all θ_{t+1} , and if $A_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_n) < 0$, for some θ_{t+1} , then $x_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) > 0$.

This step follows immediately from the equation

$$\sum_{\theta_{t+1}} v_t(\underset{\sim}{\theta}_t, \theta_{t+1})A_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_{t+1}, \underset{\sim}{s}_n) = W_t^{A(i,t-n)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_n) - p_t(\underset{\sim}{\theta}_t)\cdot x_t^{(i,t-n)}(\underset{\sim}{\theta}_t,$$

which follows from the definition of the consumers' wealth. This proves Step 16.

The marginal utility in period t and event $\underset{\sim}{\theta}_t$ of transferring wealth to period $t+1$ and event $\underset{\sim}{\theta}_{t+1}$ is

$$(15.20)\quad \Lambda_t^{(i,t-n)}(\underset{\sim}{\theta}_{t+1}, \underset{\sim}{s}_n) = \delta v_t^{-1}(\underset{\sim}{\theta}_{t+1})(1 - c_{t+1}^{+}(\underset{\sim}{\theta}_{t+1}))$$

$$\cdot E[\lambda_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_{t+1}, \underset{\sim}{s}_n, s_{n+1}) | \underset{\sim}{\theta}_{t+1}, \underset{\sim}{s}_n] \mathrm{Prob}[\theta_{t+1} | \underset{\sim}{\theta}_t] ,$$

where $\lambda_t^{(i,t-n)}$ denotes the marginal utility of money. The following assertion should be clear.

Step 17: If $A_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_n) \leqq 0$ and $A_{t+1}^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta'_{t+1}, \underset{\sim}{s}_n) > 0$, then $\Lambda_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_n) \leq \Lambda_t^{(i,t-n)}(\underset{\sim}{\theta}_t, \theta'_{t+1}, \underset{\sim}{s}_n)$.

Step 18: If the people of age N in period $t \leq T-1$ and event $\underset{\sim}{\theta}_t$ have before tax wealth $W > 0$, then $p_{tk}(\underset{\sim}{\theta}_t) > (\overline{MS}|\,|C_c|b)^{-1}(1-\overline{c})W$, for all $k \in C_c$.

Since these people of age N spend all their money on consumption, $p_t(\underset{\sim}{\theta}_t)\ \overline{x}_t(\underset{\sim}{\theta}_t) \geqq W - \tau(W, c_t(\underset{\sim}{\theta}_t)) > W(1-\overline{c})$. Since $\overline{x}_{tk}(\underset{\sim}{\theta}_t) \leqq b$, for all k (Steps 12 and 13), we have that $p_t(\underset{\sim}{\theta}_t)\cdot\overline{x}_t(\underset{\sim}{\theta}_t) \leqq b\sum p_{tk}(\underset{\sim}{\theta}_t)$, where the sum is over the set of k such that $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0$. Hence for some such k, say $k = 2$, $p_{t2}(\underset{\sim}{\theta}_t) > (|C_c|b)^{-1}(1-\overline{c})W$. If $k \in C_c$ and $p_{tk}(\underset{\sim}{\theta}_t) < (\overline{MS})^{-1}p_{t2}(\underset{\sim}{\theta}_t)$, then $\overline{x}_{t2}(\underset{\sim}{\theta}_t) = 0$, by assumption 3.4. This contradiction proves Step 18.

Step 19: $p_{1k}(\theta_1) > \underline{p}_1 > 0$, for all θ_1 and all $k \in C_c$.

This follows from the previous step, the definition of $\underline{p}_1$ (equation (15.14)) and the special assumption made before the statement of Theorem 4.2.

The next two steps are the hard steps of the proof. Let $\underline{v}_n$ and $\underline{p}_n$ be as defined in (15.17) and (15.18), respectively.

Step 20: Suppose by induction on t that for $n = 1, \ldots, t$, $p_{nk}(\underset{\sim}{\theta}_n) > \underline{p}_n$, for all $\underset{\sim}{\theta}_n$ and all $k \in C_c$ and that for $n = 1, \ldots, t-1$ $v_n(\underset{\sim}{\theta}_{n+1}) > \underline{v}_n$, for all $\underset{\sim}{\theta}_{n+1}$. Then, $v_t(\underset{\sim}{\theta}_{t+1}) > \underline{v}_t$, for all $\underset{\sim}{\theta}_{t+1}$, provided $t \leq T-2$.

Clearly, if $\text{Prob}[\theta_{t+1} | \underset{\sim}{\theta}_t] = 0$, then $v_t(\underset{\sim}{\theta}_t, \theta_{t+1})$ and $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1})$ may be set equal to anything we like. So, I will always assume that $\text{Prob}[\theta_{t+1} | \underset{\sim}{\theta}_t] > 0$.

Fix $\underset{\sim}{\theta}_{t-1}$ and let $\varepsilon_1 > 0$ and $Z = Z^1_{t-1}(\underset{\sim}{\theta}_{t-1})$ be as in (15.15).

Throughout what follows, λ stands for the marginal utility of money.

Case 1: There is $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z$ such that $x^{(i,t-N+2)}_{t-1}(\underset{\sim}{\theta}_{t-1}, \underset{\sim}{s}_{N-3}) > 0$.

I will show that for all $\underset{\sim}{\theta}_t$ following $\underset{\sim}{\theta}_{t-1}$,

$$(15.21)\quad E[\lambda^{(i,t-N+2)}_t(\underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}, s_{N-2} | \underset{\sim}{\theta}_{t-1}, \theta_t, s_{N-3}] \leq \overline{u}(\underline{p}_{t-1}\delta(1-\overline{c})\eta)^{-1}$$

Since consumption is positive, the marginal utility of money equals that of expenditure. As a result, $\overline{u}\underline{p}^{-1}_{t-1} \geqq \alpha^{(i,t-N+2)}_{t-1}(\underset{\sim}{\theta}_{t-1}, \underset{\sim}{s}_{N-3})$ $= \lambda^{(i,t-N+2)}_{t-1}(\underset{\sim}{\theta}_{t-1}, \underset{\sim}{s}_{N-3}) \geqq \Lambda^{(i,t-N+2)}_{t-1}(\underset{\sim}{\theta}_{t-1}, \underset{\sim}{s}_{N-3})$ $\geqq \delta(1-\overline{c})\eta E[\lambda^{(i,t-N+2)}_t(\underset{\sim}{\theta}_{t-1}, \theta_t; \underset{\sim}{s}_{N-3}, s_{N-2}) | \underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}]$, for all $\underset{\sim}{\theta}_t$, where $\Lambda^{(i,t-N+2)}_{t-1}$ is defined by equation (15.20). This proves (15.21).

I now prove that

$$(15.22)\quad v_t(\underset{\sim}{\theta}_{t-1}, \theta_t, \theta_{t+1}) \geqq \underline{p}_{t-1}\varepsilon_1(\eta\delta(1-\overline{c}))^2(\overline{p}\,\overline{u})^{-1}.$$

Let $(i, t-N+2, s_{N-3})$ be as in Case 1. In period $t+1$, this persor [illegible] of age N and so spends all his wealth on consumption. Hence,

his marginal utilities of money and expenditure are the same in that period. It follows that for all θ_t and θ_{t+1},
$\lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}, s_{N-2}) \geqq \delta(1-\bar{c})\varepsilon_1\bar{p}^{-1}\eta v_t^{-1}(\underset{\sim}{\theta}_{t-1}, \theta_t, \theta_{t+1})$, where ε_1 is as in (15.15). Hence, if inequality (15.22) is violated for some θ_t and θ_{t+1}, we know that $\lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}, s_{N-2}) | \underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}] > \bar{u}(\underline{p}_{t-1}\delta(1-\bar{c})\eta)^{-1}$, for all s_{N-2}. This contradicts inequality (15.21) and so proves inequality (15.22). Since the right hand side of inequality (15.22) exceeds $\underline{v}_t$, Step 20 is proved in Case 1.

<u>Case 2</u>: $x_{t-1}^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \underset{\sim}{s}_{N-3}) = 0$, for every $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in \Xi$.

Then, by Step 16, for each $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in \Xi$,
$A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}) \geqq 0$, at some θ_t. Hence, there is $\bar{\theta}_t$ such that mes $Z(\bar{\theta}_t) \geqq |\Theta|^{-1}$mes $\Xi \geqq L_1(2|\Theta|)^{-1}$, where
$Z(\bar{\theta}_t) = \{(i, t-N+2, \underset{\sim}{s}_{N-3}) \in \Xi | A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}) \geqq 0\}$. The last inequality above follows from the definition of Ξ in (15.15).

I now prove the following

$$(15.23)\quad v_t(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta_{t+1})p_{t+1,k}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta_{t+1})$$

$$\geqq d \equiv \min(\underline{p}_t\delta\eta\varepsilon_1(1-\bar{c})(\bar{u})^{-1}, \underline{p}_{t-1}\delta\eta\varepsilon_1(2|\Theta|^2 u\,\mathrm{MS}\,|C_c|b)^{-1}(1-\bar{c})^3 L_1),$$

for all θ_{t+1} and for all $k \in C_c$.

Let $V(\theta_{t+1}) = v_t(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta_{t+1})$ and let $P(\theta_{t+1})$
$= \min_{k \in C_c} p_{t+1,k}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta_{t+1})$.

<u>Subcase 1</u>: $x_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2}) > 0$, for some $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z(\bar{\theta}_t)$ and for some s_{N-2}.

I now show that in this subcase,

(15.24) $V(\theta_{t+1})P(\theta_{t+1}) > \underline{p}_t\delta\eta\varepsilon_1(1-\overline{c})(\overline{u})^{-1}$

Let $(i, t-N+2, \underset{\sim}{s}_{N-3})$ and s_{N-2} be as in Subcase 1. Then, $\overline{u}\underline{p}_t^{-1} = \alpha_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \overline{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2}) = \lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \overline{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2})$ $> \delta(1-\overline{c})\eta\varepsilon_1 V^{-1}(\theta_{t+1})P^{-1}(\theta_{t+1})$, for some θ_{t+1} . This prove inequality (15.24) and so proves inequality (15.23) in Subcase 1.

<u>Subcase 2</u>: $x_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \overline{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2}) = 0$, for all $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in \mathcal{I}(\overline{\theta}_t)$ and for all s_{N-2} .

Let $A(\theta_{t+1}) = \int_{\mathcal{I}(\overline{\theta}_t)} E[A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \overline{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_{N-3}, s_{N-2}) | \underset{\sim}{\theta}_{t-1}, \overline{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_{N-3}] di\, d\underset{\sim}{s}_{N-3}$. Notice that each person in $\mathcal{I}(\overline{\theta}_t)$ has at least $(1-\overline{c})\underline{p}_{t-1}$ units of wealth at the beginning of period t in event $(\theta_{t-1}, \overline{\theta}_t)$. Hence, collectively the people in $\mathcal{I}(\overline{\theta}_t)$ have at least $(1-\overline{c})\underline{p}_{t-1}L_1(2|\Theta|)^{-1}$ at this date-event pair. Let $\Theta' = \{\theta | A(\theta) > 0\}$. From the defining conditions of the subcase it follows that $\sum_{\theta\in\Theta'} V(\theta)A(\theta) \geqq \sum_{\theta\in\Theta} V(\theta)A(\theta) \geqq (1-\overline{c})\underline{p}_{t-1}L_1(2|\Theta|)^{-1}$. From Step 18, it follows that if $A(\theta) > 0$, then $P(\theta) > (\overline{MS}|C_c|b)^{-1}(1-\overline{c})A(\theta)$, so that

(15.25) $$\sum_{\theta\in\Theta'} V(\theta)P(\theta) \geqq \underline{p}_{t-1}(2|\Theta|\overline{MS}|C_c|b)^{-1}(1-\overline{c})L_1 .$$

I will show that if $A(\theta') > 0$, then

(15.26) $$V(\theta)P(\theta) \geqq (\overline{u})^{-1}\delta\eta\varepsilon_1(1-\overline{c})V(\theta')P(\theta') , \text{ for all } \theta .$$

The previous two inequalities prove inequality (15.23) in Subcase 2. For let $\theta' \in \Theta'$ be such that $V(\theta')P(\theta') \geqq V(\theta)P(\theta)$, for all $\theta \in \Theta'$. Then, for any θ , $V(\theta)P(\theta) \geqq (\overline{u})^{-1}\eta\varepsilon_1\delta(1-\overline{c})V(\theta')P(\theta')$

$\geq (|\Theta|\bar{u})^{-1}\eta\varepsilon_1\delta(1-\bar{c}) \sum_{\theta\in\Theta} V(\theta)P(\theta) \geq \underline{p}_{t-1}\delta\eta\varepsilon_1(2|\Theta|^2\bar{u}\,\overline{NS}\,|C_c|b)^{-1}(1-\bar{c})^3L_1$,

which is no smaller than the right hand side of inequality (15.25).

I now prove inequality (15.26). Let $\theta' = \theta'_{t+1}$. Since $A(\theta'_{t+1}) > 0$, $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta'_{t+1}, \underset{\sim}{s}_{N-3}, s_{N-2}) > 0$, for some $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z(\bar{\theta}_t)$ and some s_{N-2} . But then $\lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2}) = \delta(1 - c^+_{t+1}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta'_{t+1}))v_t^{-1}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta'_{t+1})$ $\text{Prob}[\theta'_{t+1}|\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t]E[\alpha_{t+1}^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta'_{t+1}, \underset{\sim}{s}_{N-3}, s_{N-2}, s_{N-1})|\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \theta'_{t+1}, \underset{\sim}{s}_{N-3}, s_{N-2}] \leq \bar{u}V^{-1}(\theta'_{t+1})P^{-1}(\theta'_{t+1})$. But $\lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}, s_{N-2}) \geq \delta\eta\varepsilon_1(1-\bar{c})V^{-1}(\theta_{t+1})P^{-1}(\theta_{t+1})$, for all θ_{t+1} . These two inequalities prove inequality (15.26).

This completes the proof of inequality (15.23). I now prove Step 20 in Case 2.

Suppose that for some θ_t and θ_{t+1} , $v_t \equiv v_t(\underset{\sim}{\theta}_{t-1}, \theta_t, \theta_{t+1}) \leq \underline{v}_t$. For every $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z$,

$$\text{(15.27)} \quad \lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \theta_t, \underset{\sim}{s}_{N-3}, s_{N-2}) \geq \delta\eta\varepsilon_1(1-\bar{c})(\bar{p})^{-1}v_t^{-1} .$$

Let $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z(\bar{\theta}_t)$ (that is, $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}) \geq 0$). We know that this consumer has positive wage income at the end of period t-1 in the event $\underset{\sim}{\theta}_{t-1}$. Therefore

$$\text{(15.28)} \quad \Lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}) \geq \Lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_{t-1}, \bar{\theta}_t, \underset{\sim}{s}_{N-3}) .$$

The left hand side of inequality (15.28) is less than or equal to $\underline{v}_{t-1}^{-1}\bar{u}\max(\underline{p}_t^{-1}, d^{-1}) = \underline{v}_{t-1}^{-1}\bar{u}d^{-1}$, where d is as in inequality (15.23). The right hand side of (15.28) is at least as big as $\delta\eta^2\varepsilon_1(1-\bar{c})(\bar{p})^{-1}v_t^{-1}$. Putting these two inequalities together one has that

$v_t \geqq \delta\eta^2\varepsilon_1(1-\bar{c})^2(\bar{u}\,\bar{p})^2(\bar{u}\,\bar{p})^{-1}d\underline{v}_{t-1} > \underline{v}_t$. This contradicts the hypothesis that $v_t \leqq \underline{v}_t$.

This completes the proof of Step 20.

Step 21: Suppose by induction of t that for $n = 1, \ldots, t$, $p_{nk}(\underset{\sim}{\theta}_n) > \underline{p}_n$, for all $k \in C_c$ and for all $\underset{\sim}{\theta}_n$ and that $v_n(\underset{\sim}{\theta}_{n+1}) > \underline{v}_n$, for all $\underset{\sim}{\theta}_{n+1}$. Then $p_{t+1,k}(\underset{\sim}{\theta}_{t+1}) > \underline{p}_{t+1}$, for all $k \in C_c$ and all $\underset{\sim}{\theta}_{t+1}$, if $t \leq T-3$.

Fix $\underset{\sim}{\theta}_t$ and let $\varepsilon_2 > 0$ and $Z = Z_t^2(\theta_t)$ be as in definition (15.16).

Case 1: There is $(i, t-N+2, \underset{\sim}{s}_{N-3}) \in Z$ such that $x_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{N-2}) > 0$.

Then, for all θ_{t+1} , $\bar{u}\underline{p}_t^{-1} \geqq \alpha_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{N-2}) = \lambda_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \underset{\sim}{s}_{N-2})$ $\geqq \delta\eta\varepsilon_2(1-\bar{c})\min_k(p^{-1}_{t+1,k}(\underset{\sim}{\theta}_t, \theta_{t+1}))$, so that $p_{t+1,k}(\underset{\sim}{\theta}_t, \theta_{t+1})$ $\geqq \underline{p}_t\delta\eta\varepsilon_2(1-\bar{c})(\bar{u})^{-1} > \underline{v}_t$, for all $k \in C_c$. This proves Step 21 in Case 1.

Case 2: $x_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \theta_{t+1}) = 0$, for all $(i, t-N+2, \underset{\sim}{s}_{N-2}) \in Z$.

Arguing as in Case 2 of the proof of the previous step, there is θ_{t+1} such that mes $Z(\theta_{t+1}) \geqq |\Theta|^{-1}$mes $Z \geqq L_1(2|\Theta|)^{-1}$, where $Z(\theta_{t+1}) = \{(i, t-N+2, \underset{\sim}{s}_{N-2}) \in Z | A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t, \theta_{t+1}, \underset{\sim}{s}_{N-2}) \geqq 0\}$. Again applying Step 18 as in the proof of Step 20, we have that

$$(15.29)\quad p_{t+1,k}(\underset{\sim}{\theta}_t, \theta_{t+1}) > \underline{p}_t(1-\bar{c})(2|\Theta|\overline{MS}|C_c|b)^{-1}L_1 ,$$

for every θ_{t+1} such that mes $Z(\theta_{t+1}) \geqq L_1(2|\Theta|)^{-1}$.

I now define sets Θ_n by induction on n . Let $\Theta_1 = \{\theta_{t+1} | \text{mes } Z(\theta_{t+1}) \geqq L_1(2|\Theta|)^{-1}\}$. Given Θ_n , let $\Theta_{n+1} = \Theta_n \cup \{\theta_{t+1} |$ there is $(i, t-N+2, \underset{\sim}{s}_{N-2}) \in Z$ such that

$A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) \leqq 0$, and for some $\theta'_{t+1} \in \Theta_n$,
$A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta'_{t+1},\ \underset{\sim}{s}_{N-2}) > 0\}$.

I now show that

(15.30) for every $\theta_{t+1} \in \Theta_n$, $p_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta_{t+1})$

$$> [\underline{v}_t(1-\overline{c})\varepsilon_2 n(\overline{u})^{-1}]^{n-1}\underline{p}_t(2|\Theta|\overline{MS}|C_c|b)^{-1}(1-\overline{c})L_1 .$$

The proof is by induction on n . The case $n = 1$ is inequality (15.29). Suppose that (15.30) is true for n . Let $\theta_{t+1} \in \Theta_{n+1} \setminus \Theta_n$, $\theta'_{t+1} \in \Theta_n$ and $(i,\ t-N+2,\ \underset{\sim}{s}_{N-2}) \in Z$ be as in the definition of Θ_{n+1} . By Step 17 and because people in Z are in the last period of life in period $t+1$, we have that $(1-\overline{c})\varepsilon_2 n \max_{k\in C_c}(p^{-1}_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta_{t+1}))$
$\leqq A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) \leqq A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2})$
$\leqq \underline{v}_t^{-1}\overline{u} \max_{k\in C_c} p^{-1}_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta'_{t+1})$. Hence, $\min_{k\in C_c} p_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta_{t+1})$
$\geqq \underline{v}_t\varepsilon_2 n(1-\overline{c})(\overline{u})^{-1} \min_{k\in C_c} p_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta'_{t+1}) \geqq [\underline{v}_t\varepsilon_2 n(1-\overline{c})(\overline{u})^{-1}]^n$
$\underline{p}_t(2|\Theta|\overline{MS}|C_c|b)^{-1}(1-\overline{c})L_1$, where the last inequality follows from the induction hypothesis. This proves inequality (15.30).

For some $n \leq |\Theta|$, the sequence Θ_n stops growing. Therefore,

(15.31) for every $\theta_{t+1} \in \bigcup_n \Theta_n$, $p_{t+1,k}(\underset{\sim}{\theta}_t,\ \theta_{t+1})$

$$\geqq [\underline{v}_t(1-\overline{c})\varepsilon_2 n(\overline{u})^{-1}]^{|\Theta|}\underline{p}_t(2|\Theta|\overline{MS}|C_c|b)^{-1}(1-\overline{c})L_1 > \underline{p}_{t+1} .$$

I now show that $\Theta = \bigcup_n \Theta_n$. Let Θ' be the complement of $\bigcup_n \Theta_n$ and suppose that Θ' is not empty. Then, there exists $(i,\ t-N+2,\ \underset{\sim}{s}_{N-2}) \in Z$ such that $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N+2}) < 0$, for all $\theta_{t+1} \in \Theta'$. Otherwise, there is some $\theta_{t+1} \in \Theta'$ such that

$\text{mes}\{(i,\ t\text{-}N+2,\ \underset{\sim}{s}_{N-2}) \in Z \mid A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) \geqq 0\} \geqq |\Theta|^{-1}\text{mes } Z_1$, and $\theta_{t+1} \in \Theta$, which is impossible. If $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) < 0$ for all $\theta_{t+1} \in \Theta'$, then $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) \leqq 0$, for all $\theta_{t+1} \in \bigcup_n \Theta_n$, for otherwise $\Theta \in \bigcup_n \Theta_n$, which is impossible. So, $A_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \theta_{t+1},\ \underset{\sim}{s}_{N-2}) \leqq 0$, for all θ_{t+1} , which means by Step 16 that $x_t^{(i,t-N+2)}(\underset{\sim}{\theta}_t,\ \underset{\sim}{s}_{N-2}) > 0$, which contradicts the defining condition of Case 2. This proves Step 21.

<u>Step 22</u>: For $t \leq T\text{-}2$, $z_t(\underset{\sim}{\theta}_t) \leq 0$ and $p_t(\underset{\sim}{\theta}_t)\cdot z_t(\underset{\sim}{\theta}_t) = 0$.

By Steps 20 and 21, $p_{tk}(\underset{\sim}{\theta}_t) > \underline{p}_t$, for all $t \leq T\text{-}2$. Hence, $z_{tk}(\underset{\sim}{\theta}_t) \geq 0$, if $k \in C_c$. By Step 12, $z_{tk}(\underset{\sim}{\theta}_t) \leq 0$ whenever $\overline{x}_{tk}(\underset{\sim}{\theta}_t) > 0$, one may now proceed exactly as in the proof of Step 13 of the proof of Theorem 4.1.

<u>Step 23</u>: For $t \leq T\text{-}3$, $\Delta a_t(\underset{\sim}{\theta}_{t+1}) \geqq 0$, for all $\underset{\sim}{\theta}_{t+1}$.

This is so because by Steps 20 and 21, $v_t(\underset{\sim}{\theta}_t,\ \theta_{t+1}) > \underline{v}_t$, for $t \leqq T\text{-}3$.

<u>Step 24</u>: One may assume that $\Delta a_t(\underset{\sim}{\theta}_{t+1}) = 0$, for $t \leq T\text{-}3$ and for all $\underset{\sim}{\theta}_{t+1}$.

Since $v_t(\underset{\sim}{\theta}_{t+1}) > \underline{v}_t$, for all $\underset{\sim}{\theta}_{t+1}$, it follows from Step 10 that $\Delta a_t(\underset{\sim}{\theta}_t,\ \theta) = \Delta a_t(\underset{\sim}{\theta}_t,\ \theta')$, for all θ and θ' . Suppose that $\Delta a_t(\underset{\sim}{\theta}_t, \theta_{t+1}) = \beta > 0$, for all θ_{t+1} . Then, as in the proof of Step 15, I can increase every consumer's money balance by $N^{-1}\beta$ and reduce his purchases of insurance by $N^{-1}\beta$ for every state θ_{t+1} . After I do so, the consumers still are making optimal choices and the excess demand for insurance is eliminated.

I have now satisfied all the conditions for a reduced, (G,M) balanced equilibrium for periods $t \leq T-3$.

Step 25: Allow the horizon to go to infinity.

One applies a Cantor diagonal argument and obtains in the limit an infinite horizon reduced (G,M) balanced equilibrium. By Step 2, this completes the proof of the theorem.

Q.E.D.

16. Conclusion

The main result of this paper, Theorem 4.1, may be misinterpreted as saying macroeconomics is easy. Here is a model in which phenomena resembling trade cycles may occur, and it is proved that policy may prevent them. Furthermore, the result is not at all surprising once one realizes that it is an analogue of the second welfare theorem. However, one should be aware that there are strong hypotheses underlying the model. These are that expectations are rational and that random changes of aggregate importance are revealed to everyone simultaneously. The latter assumption, of course, precludes the asymmetric information which underlies the so-called island models of macroeconomic theory.

APPENDIX--EXAMPLES

A.1) Example. There are two immortal consumers and two goods, labor and a produced good. There are two production functions. One is $y_t = L_t$, where y_t and L_t are period t output and labor output, respectively. The second is $y_{t+1} = (1.09)^{-1}L_t$. Use of this process involves investment since output arrives with a lag. Each consumer has a utility function of the form $\sum_{t=0}^{\infty} (0.9)^t \log x_t$, where x_t is consumption of produced goods in period t. Only one consumer can work at any time. Consumer 1 can work in periods $t = 4n$ or $4n+1$, for n any integer. Consumer 2 can work in periods $t = 4n+2$ or $4n+3$. There is no borrowing or insurance and there is a one period Clower lag in the payment of wages or dividends. Wages and dividends are paid in money. There are no government bonds.

The following is a cyclic equilibrium. The prices of labor and the consumption good are always the same. The price is 1.09 in odd periods and 1 in even periods. The second production process is used only in even periods. In those periods, 0.01 units of labor are invested in the second process, yielding $(0.01)(1.09)^{-1}$ units of output in the following period. There are two kinds of securities, shares of newly invested labor and shares of old investment. Shares of old investment are shares in the proceeds from sale of the output of the investment. Let us say that one share corresponds to an original investment of one unit of labor. The price of one share of new investment is one in even periods and 1.09 in odd periods. The price of old investment is 1.09 in even periods and 1 in odd periods. The real rate of interest is $(1.09)^{-1} - 1 < 0$ in even periods and 0.09 in even periods.

The money supply is 1.925304.

Let x_1 be the consumption of a consumer in the first period in which he has a wage to spend after two periods with no wage. Let x_2, x_3 and x_4 be his consumption in each of the succeeding periods. Then, $x_1 = 0.5575548$, $x_2 = 0.546913$, $x_3 = 0.4516193$, $x_4 = 0.4430386$.

Output of the consumption good is 0.99 in even periods and 1.0091673 in odd periods.

A2) Example. There are two immortal consumers and two goods, labor and a produced good. Consumer 1 has one unit of labor in even periods and none in odd periods. Consumer 2 has one unit of labor in odd periods and none in even periods. The utility function of each consumer is $\sum_{t=0}^{\infty} \frac{1}{2^t} \log x_t$, where x_t is consumption of produced good in period t. The production function is of the form $y_t = K_{t-1}^{\alpha} L_t^{1-\alpha}$, where $0 < \alpha < 1$, y_t is output in period t, L_t is labor input in period t, and K_{t-1} is produced good set aside as capital in period $t-1$. There is one unit of money and no government debt. Wages and dividends are paid at the end of the period. There are two kinds of shares, shares in newly invested capital and shares in capital invested in the previous period. One share corresponds to an original investment of one unit of produced good.

The following is an equilibrium if $\alpha = 1/7$; $r = 0$, $p = 1.3830874$, $w = 6/7$, $\pi = q_1 = q_2 = p$, $y = 0.7230201$, $K = 0.1032886$, $x_1 = 0.4131543$, and $x_2 = 0.2065772$, where r is the real interest rate, p is the price of the produced good, w is the wage, q_1 is the price of one share of new capital, q_2 is the price of one share of old capital, π is the dividend on one share of old capital, y is output, K is capital, x_i is consumption in the i^{th} period after having received a wage. New capital pays no dividend.

Using the same notation, the following is an equilibrium when $\alpha = 1/6$: $r_2 = 0.13066239$, $p = 1.5030153$, $w = 5/6$, $q_1 = p$, $q_2 = (1+r)q_1$, $\pi = (1+r)^2 q_1$, $y = 0.6653292$, $K = 0.08674$, $x_1 = 0.3696273$, and $x_2 = 0.2089618$.

If one checks these equilibria, one must realize that the marginal product of capital is $(1+r)^2$, not $1+r$, since an investment does not yield dividends until the end of the period after the period of investment. For this reason, the second equilibrium is not Pareto optimal.

In the above equilibria, all shares are held by the consumer who just earned a wage. For this reason, $w = x_1 + q_1 K + q_2 K$.

One sees that in the above example an increase in the share of capital in output, α , increases the real interest rate. Imagine that the economy were in a stationary equilibrium with $\alpha = 1/6$ and $r = 0$ and that α increased to 1/7. Presumably, the economy would adjust toward the new equilibrium with $r = 0.13066239$.

A3) Example. This is an overlapping generations model in which people live two periods and one person is born each period. There are two commodities, labor and a produced good. Individuals are endowed with one unit of labor when they are young and none when they are old. The utility function of each individual is $\log x_1 + \log x_2$, where x_i is the consumption of produced good in the i^{th} period of life. The production function is $y_t = 4\sqrt{K_{t-1}L_t}$, where y_t is output in period t , K_{t-1} is capital created in period $t-1$ and L_t is labor used in period t . The capital is produced good.

The following is an equilibrium. The quantity of capital set aside during each period is one. Individuals consume one unit of produced good in youth and two in old age. The price of the produced good in period t is 2^{-t} . The wage in period t is 2^{-t+1} . The nominal rate of interest is zero.

The equilibrium financial flows are as follows. A youth in period t earns a wage of 2^{-t+1} at the end of the period. Against this he borrows 2^{-t+1} at the beginning of the period. He spends this on two units of produced good, one of which he consumes and the other he sets aside to be used as capital in period $t+1$. At the end of the period he uses his wages to pay off his debt and he is left with one unit of capital. At the beginning of period $t+1$, he borrows 2^{-t} against the 2^{-t} of profit that his capital will pay him at the end of the period. He spends this on two units of consumption and pays off his debt at the end of the period. (Alternatively, the old could sell their capital to the young, who would finance the purchase by borrowing.)

The money supply at the end of period t is 2^{-t+1} , being the sum of wages and profits paid at that time. The aggregate quantity of inside money in period t is the sum of the debts of the young and the old, which is 2^{-t+2} again. Thus, there is no outside money. Inside money declines exponentially to zero. Since the nominal interest rate is zero, it is impossible to prevent the decline by reducing interest rates.

A4) Example. This example is exactly the same as the previous example, except that the production function is $y_t = 3K_{t-1}^{1/3}L_t^{2/3}$.

The initial conditions are as follows. At the beginning of period one, the person who was young in period zero possesses one unit of money and K_0 units of capital set aside in period zero. He does not receive the profits earned from this capital until the end of period one.

For each $r > 0$, the following allocation is that of an equilibrium with money earning interest at rate $r > 0$ and with balanced government budget. The capital set aside during period t is $K_t = (1+r)^{-1}K_{t-1}^{1/3}$, for $t > 0$. (This difference equation converges.) The person who is young in period t consumes $x_{t1} = (2+r)^{-1}(1+r)^{-1}(2+3r)K_{t-1}^{1/3}$. The person who is old in period t consumes $x_{t2} = (2+r)^{-1}(2+3r)K_{t-1}^{1/3}$. Output in period t is $y_t = 3K_{t-1}^{1/3}$. (This allocation is obtained by maximizing the social welfare function $\log x_{02} + \sum_{t=0}^{\infty}(1+r)^{-t}(\log x_{t1} + \log x_{t2})$.

The equilibrium prices are as follows. The price of produced good in period t is $p_t = (2r)^{-1}(2+r)K_{t-1}^{-1/3}$. The wage paid at the end of period t is $w_t = r^{-1}(2+r)(1+r)$. The price of a unit of old capital at the beginning of period t is $q_t = (1+r)p_t = (2r)^{-1}(2+r)K_{t-1}^{-1}$. The price of one unit of new capital in period t is p_t.

There are also government debt and capital. One government bond is the right to receive one unit of money at the end of the period. The price of one bond at the beginning of each period is $(1+r)^{-1}$. Also, consumers pay a tax of r at the beginning of their old age.

The equilibrium financial flows are as follows. At the beginning of period t, the firm who is producing that period sells its output receiving $p_t y_t = 3(2r)^{-1}(2+r)$. This revenue is deposited in a bank and held until the end of the period, when it is paid to consumers as wages and dividends. This money earn terest at rate r, so that

$(1+r)p_t y_t = 3(2r)^{-1}(2+r)(1+r)$ is paid out, $r^{-1}(2+r)(1+r)$ as wages and $(2r)^{-1}(2+r)(1+r)$ as dividends.

Consider a young consumer. His only income is wages earned at the end of his youth. At the beginning of his youth, he borrows $3(2r)^{-1}(2+r)$ from banks. He spends this as follows: $p_t x_{t1} = 3(2r)^{-1}(1+r)^{-1}(2+3r)$ on consumption, $p_t K_t = (2r)^{-1}(1+r)^{-1}(2+r)$ on new capital, $q_t K_{t-1} = (2r)^{-1}(2+r)$ on the capital which is currently productive, and 1 on $1+r$ units of government bonds. His wages and purchased assets serve as collateral for the loan. He pays interest at rate r to the bank, so that at the end of his youth he owes $3(2r)^{-1}(2+r)(1+r)$ to the banks. He also receives at this moment wages of $r^{-1}(2+r)(1+r)$ and dividends of $q_t K_{t-1}(1+r) = (2r)^{-1}(2+r)(1+r)$. These add up precisely to his debt, which he pays off. He also receives $1+r$ on his bonds and pays a tax of r . Finally, his investment in K_t units of new capital now gives him the rights to dividends at the end of period $t+1$. Those rights he sells for $q_{t+1}K_t = (2r)^{-1}(2+r)$. In all, he has $(2r)^{-1}(2+3r)$ units of money, which is exactly the cost of his consumption in old age, $p_{t+1}x_{t2}$.

Consider now the government. In each period, it collects in taxes exactly what it incurred in interest obligation during the previous period. In each period, it pays to the old consumer the principal of one on its old debt and borrows one from the young consumer.

The total money supply at the beginning of period t is $p_t y_t = 3(2r)^{-1}(2+r) + 1$. The quantity of inside money is $3(2r)^{-1}(2+r)$. The quantity of outside money is one.

Any of these equilibria is possible as long as the government always balances the budget by setting its taxes in period

$t+1$ so as to equal the interest obligation incurred in the previous period. Thus, as long as the government is passive in the above sense, interest rates are indeterminate.

Remark: One could define a balanced budget in a different manner, so that the face value of government debt would be constant. Above I made the amount of government borrowing constant. The interest rate remains indeterminate if the alternative definition is adopted.

FOOTNOTES

[1]The term "business cycle" is misleading, for it seems to imply periodicity.

[2]This result is related to the literature on whether a monetary policy fixing interest rates leads to some sort of indeterminacy or instability. (See Sargent and Wallace (1975) and McCallum (1983).) That literature is not expressed in terms of the kind of full general equilibrium model which is used here, but I think the main difference is that I allow variation of both monetary and fiscal policy, whereas in that literature only one instrument is used, the money supply.

[3]It would also be natural to fix the average marginal utility of money, but I have been unable to prove that one can do so.

[4]I may be mistaken in my interpretation of the work of Sargent and Wallace, for in their model there is no formal distinction between inside and outside money. The money in their model is outside money according to the definition given here.

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