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Stable Disequilibrium Prices: Macroeconomics and Increasing Returns I

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COWLES FOUNDATION DISCUSSION PAPER NO. 650

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by

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Abstract

The paper establishes conditions for the existence of a set of prices which are stable and at which markets do not clear, providing a rigorous foundation for the existence of fixed-price equilibria via the macroeconomics of increasing returns. It analyzes a Walrasian price adjustment process for a general equilibrium economy with economies of scale in production.

*University of Essex and the Cowles Foundation, Yale. In writing this paper, I have benefitted from discussions with Don Brown, Graciela Chichilnisky, Jeff Frank, Frank Hahn, Birgit Grodal, Alan Kirman, Barry Murphy, Heraklis Polemarchakis, and participants in the Essex conference on "Recent Developments in Mathematical Economics."

1. Introduction

The last decade has seen considerable interest in general equilibrium models in which prices are assumed to be fixed, either exactly or within certain bounds, by forces exogenous to the model. In particular, such models have been very extensively used as a vehicle for the development of the microeconomic foundations of macroeconomic theory. Perhaps the most influential such developments have been those of Barro and Grossman [2] and Malinvaud [13]: for their theoretical foundations, these works have drawn on the more general and abstract formulations of Drèze [7] and Benassy [3]. This literature has been extensively surveyed by Grandmont (e.g., [8]), and Hahn [9] has evaluated it and attempted to retain some of its features while developing it further.

The appeal of such models clearly lies partly in their descriptive realism, and partly in their ability to yield an interesting synthesis of neoclassical and Keynesian results, showing each as a special case of a more general model and thus affording a prospect of order and unity in an apparently fragmented field.

Their principal weakness has of course been widely noted--namely, that the crucial feature of rigid prices is a property of the model that should in principle be *derived* from underlying theory rather than *assumed*. After all, in a market economy, prices are supposed to change in response to market disequilibrium, so that to assume fixed prices is to assume a major malfunctioning of the market system. If you can assume that, say the critics of this approach, then you can assume anything--and, in particular, you could just as well assume the possibility of unemployment equilibrium. So the assumption of exogenously fixed prices prevents the models from going very deeply into the very phenomena at which they are directed. Perhaps partly as a reaction to this obviously *ad hoc* feature of these fixed-price models, macroeconomic theorizing has more recently been conducted in the context of models where both prices and expectations are assumed to have a quite remarkable facility to adjust so as to equilibrate the system.

I shall argue in this paper that in fact the fixed-price assumption can be given theoretical foundations, and that *one can provide a fully rigorous proof of the existence of fixed non-market-clearing prices, even in an economy where prices adjust in the usual Walrasian fashion in response to differences between supply and demand.* It will, however, be shown that not any prices can be fixed prices, but that there are *some* prices which may be an equilibrium of the Walrasian adjustment process even though they do not clear all markets. So by assuming arbitrary fixed prices, the original fixed-price models were assuming too much: only certain prices can be fixed non-market-clearing prices, and this places considerable structure on the kinds of quantity-constrained equilibria which may emerge at these prices. A characterization of the associated equilibria is the subject of a separate paper (Heal [11]).

It may at first sound paradoxical that there are prices which are stable yet which do not clear all markets, in an economy where prices are assumed to respond in the usual fashion to excess supply or demand. However, the basic analytical point is in fact extremely simple, and hinges upon the possibility of increasing returns to scale in production. In this case, firms' supply responses will be discontinuous, and with discontinuous supply responses it is straightforward to establish the existence of prices which are Walrasian stable yet not market clearing. The assumption of economies of scale in production is obviously a very mild and reasonable one,[1] particularly for the production of manufactured goods, which are precisely those goods whose prices appear to be very stable. It is widely accepted that the production of many such goods is characterized by fixed costs and a period of efficiency increasing as scale increases, until eventually diminishing returns set it. For a single-product firm, this would give rise to--and would be implied by--the traditional textbook U-shaped average cost curve.

Figures 1 and 2 show the intuitive basis for the arguments which will be developed in detail in the remaining sections of the paper. Figure 1 shows the production possibility set of a firm using a single input to produce a single output, and operating under a mixture of increasing and decreasing returns to scale. At a given output price, its demand curve for the input is shown in Figure 2. It is clearly possible that the supply curve for this input may pass through the discontinuity, as shown: a completely detailed numerical example of a general equilibrium model with this property is given in Heal [10]. Now consider the price p^* indicated in Figure 2: clearly for $p > p^*$, supply exceeds demand, and vice versa. Hence under the normal assumptions about price response to excess demand, p^* is a stable price in the sense that any price not equal to p^*, will be changed in the direction of p^*, and this change will continue until p^* is reached. Because of the discontinuity at p^*, p^* is not in fact an equilibrium, but it is stable in the sense of being approached from any initial conditions.[2] Obviously, it is not a market-

[1] Although it is one which has traditionally been avoided by theorists on the grounds that it is unmanageable analytically, and hard to obtain simple conclusions from. This paper shows that this is not necessarily the case.

[2] The use of discontinuity to establish price stability in a rather different sense, can be traced back to Sweezy [16], who argued that discontinuities in the marginal revenue curve would cause price rigidity.

clearing price. To summarize: I model the determination of a rigid, non-market-clearing price vector which is determined endogenously by the normal forces of supply and demand. I do this in a standard Arrow-Debreu general equilibrium model, with a full set of markets and with no imperfections of any sort. The only departure from the conventional framework, is the dropping of the rather restrictive assumption that production always occurs under conditions of non-increasing returns to scale.

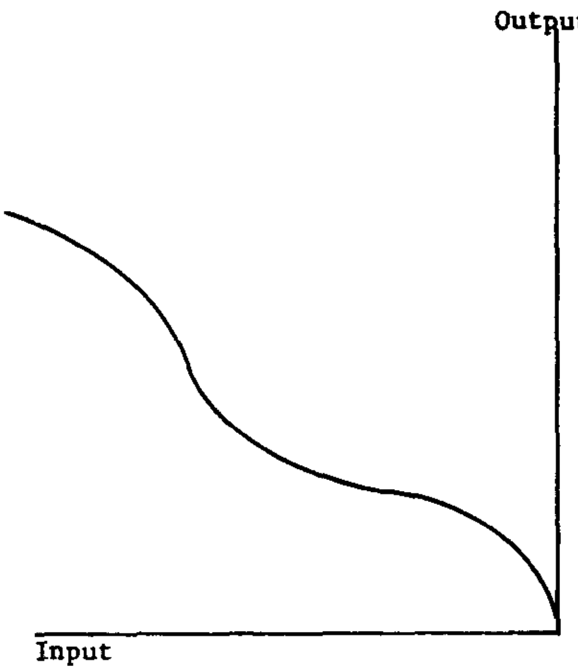

FIGURE 1: A non-convex production possibility set.

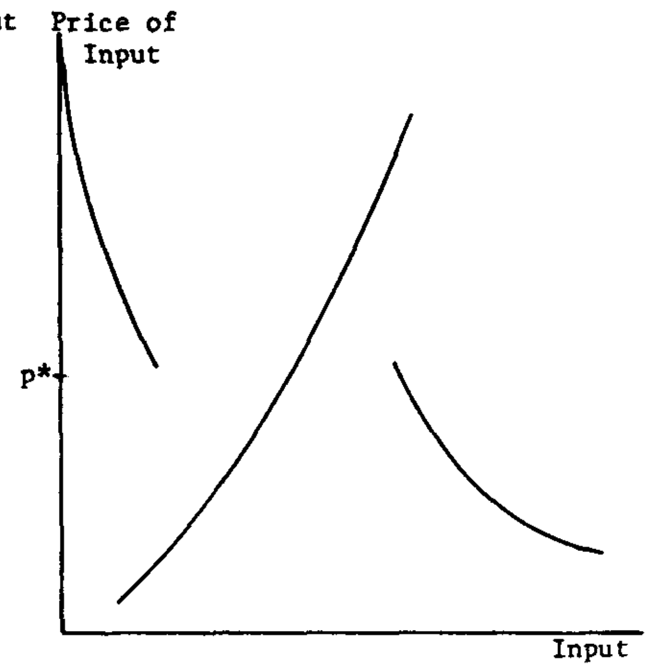

FIGURE 2: The demand curve derived from Figure 1.

This then is the basic insight which the following sections formalize and establish in a more general framework. Although the underlying idea is simple, some of the analysis is technically quite complex: this is because the excess demand correspondence of a non-convex economy can be very badly behaved. The analysis in fact proceeds by considering a convexified version of the original economy in which every production set is replaced by its convex hull.[1] This has a convex-valued, upper semi-continuous excess demand correspondence, for which one can establish existence and stability results about solutions. I then use the facts that the excess demand correspondence of the original non-convex economy is everywhere contained in that of the convexified economy, and that the two correspondences can be shown to be identical for almost all prices. The behavior of the Walrasian price adjustment process for the non-convex economy, can then be inferred from its behavior in the convexified case. In particular, it can be shown that, under certain assumptions, the convexified economy will have a stable competitive equilibrium and that this implies the existence of stable non-market-clearing prices (as in Figure 2) for the original economy.

2. The Model

We shall consider an economy with f firms, indexed by i, $i \in \{1, \ldots, f\}$ and c consumers, indexed by j, $j \in \{1, \ldots, c\}$. The notation will be standard. R^n is the commodity space, $Y_i \subset R^n$ firm i's production set, and individual j is described by a consumption set $X_j \subset R^n$, a utility function $U_j : X_j \to R$, an endowment vector $w_j \in R^n$, and a set of shareholdings in firms, θ_{ji} being his or her share in the i^{th} firm. R^n_+ and R^n_- will denote the non-negative and non-positive cones of R^n respectively. For any subset S of R^n, $\overline{S}$ will denote the closed[2] convex hull of S. The following assumptions

[1] This is analogous to the approach used by Starr [15] in his study of existence of competitive equilibria for economies with non-convex preferences.

[2] We need the closed convex hull because the convex hull of a closed set need not be closed--see Lay [12], p. 21. I am grateful to Birgit Grodal for this point.

will be used.

(A1) For all firms i :

(i) Y_i is closed.

(ii) Y_i admits free disposal, i.e., $R^n_- \subset Y_i$.

(iii) $\overline{Y}_i \cap R^n_+ = \{0\}$.

(iv) $\overline{Y}_i \cap (-\overline{Y}_i) = \{0\}$.

(A2) For all consumers j ,

(i) X_j is closed.

(ii) X_j is convex.

(iii) X_j is bounded below.

(A3) For all consumers j ,

(i) U_j is continuous.

(ii) U_j is strictly quasi-concave.

(iii) U_j is non-satiated.

(iv) $w_j >> 0$.

An economy satisfying (A1) to (A3) will be denoted by $E = \{Y_i, X_j, U_i, \ _j, \theta_{ij}\}$. $\overline{E}$ will denote the economy obtained from E by replacing each production set Y_i by its convex hull $\overline{Y}_i$, i.e. $\overline{E} = \{\overline{Y}_i, X_j, U_j, w_j, \theta_{ij}\}$. A price vector will be denoted p , an element of S , the simplex of R^n . Individual j's demand vector, $D_j(p)$, is defined as:

$$D_j(p) = x_j(p) - w_j$$

where x_j solves

Maximize $U_j(x)$ subject to

$$p \cdot x \leq p \cdot w_j + \sum_i \theta_{ji} p \cdot y_i(p)$$

where $y_i(p)$ is firm i's profit-maximizing production plan at prices p . Firm i's supply correspondence $y_i(p)$ is simply the solution of:

Maximize $p \cdot y$, $y \in Y_i$.

The excess demand of the economy is a mapping $Z : S \to R^n$ defined by

$$Z(p) = \sum_j D_j(p) - \sum_i y_i(p) .$$

We shall be concerned with the adjustment process

$$\frac{dp}{dt} = \dot{p} \in Z(p) \tag{1}$$

and will say that a price vector p is *globally stable* if every solution to the process (1) has p as its limit. Definitions of solution and stability concepts for differential correspondences are given in more detail in the appendix. p is a *market clearing price vector* (mcp) if $0 \in Z(p)$.

We denote by $Z(p)$ the excess demand of the economy E , and by $\overline{Z}(p)$ that of the convexified economy $\overline{E}$. $y_i(p)$ and $\overline{y}_i(p)$ will be their supply functions.

An important preliminary is a comparison of the excess demands $Z(p)$ and $\overline{Z}(p)$. Clearly by (A3(ii)), $D_j(p)$ is a continuous function and for the convex economy $\overline{E}$, $y_i(p)$ is a convex-valued upper-semi-continuous correspondence, by standard results. Hence $\overline{Z}(p)$ is a convex-valued upper-semi-continuous correspondence. We shall need $\overline{Z}(p)$ to be compact-valued: clearly it is closed-valued, but the rest has to be assumed. Hence:

(A4) $Z(p)$ and $\overline{Z}(p)$ are compact-valued.

It is clear that $Z(p)$ and $\overline{Z}(p)$ differ only where $y_i(p)$ and $\overline{y}_i(p)$ differ, for some i. The set of $p \in S$ where Z and $\overline{Z}$ differ, will be denoted $\overline{S} \subset S$, and may be expressed as:

$$\overline{S} = \{p \in S : \overline{y}_i(p) \cap (\partial\overline{Y}_i/\partial Y_i) \neq \phi, \text{ some } i\} .$$

Here $\partial\overline{Y}_i/\partial Y_i$ is the set-theoretic difference of the boundaries of $\overline{Y}_i$ and Y_i --i.e., those points in the boundary of $\overline{Y}_i$ but not in the boundary of Y_i. $\overline{S}$ is therefore the set of prices such that for some firm i, there are points in the supply correspondence generated by the convex hull, which are in the boundary of the convex hull but not in that of the original set. Figure 3 illustrates.

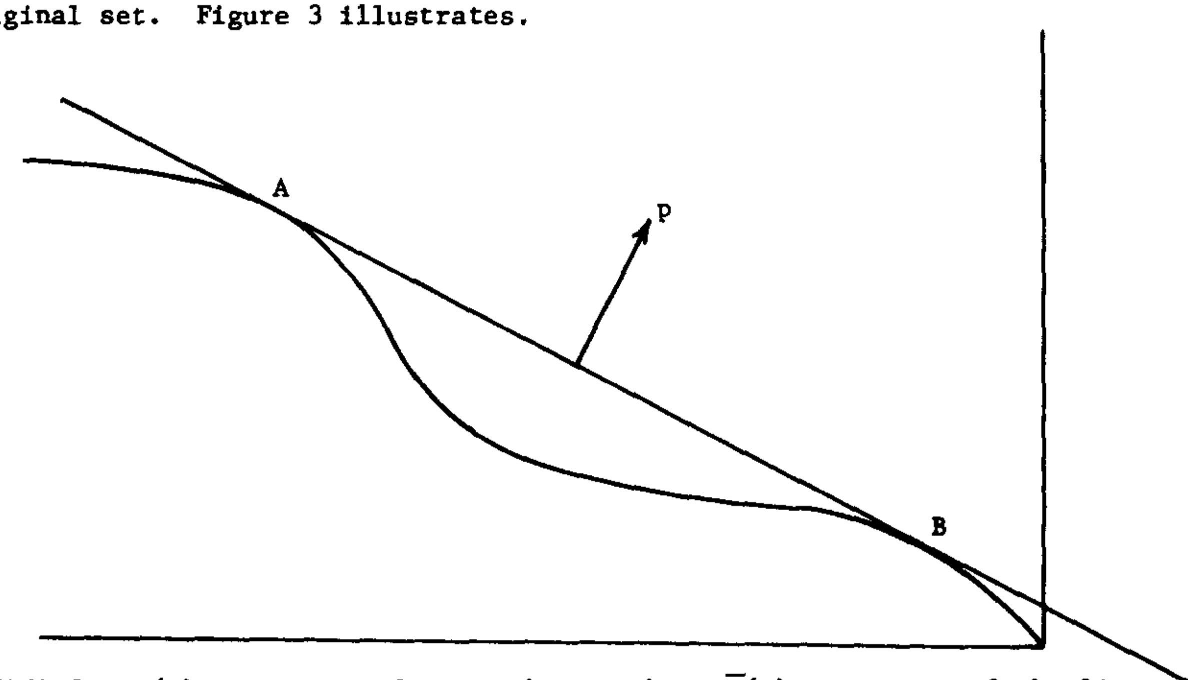

FIGURE 3: $y(p)$ consists of A and B, but $\overline{y}(p)$ consists of the line segment AB, which is in the boundary of $\overline{Y}$.

The subset $\overline{S}$ of S plays a crucial role in the formal argument of the paper. It is shown in Lemma 1 that the excess demand correspondences differ only for $p \in \overline{S}$, and in Lemma 2 that $\overline{S}$ is a set of measure zero. Lemma 2 in fact establishes that for all p not in $\overline{S}$, $z(p)$ and $\overline{z}(p)$ are identical and single valued. Once we know that the vector fields for the price adjustment processes are identical and single valued almost everywhere (indeed, in many cases everywhere except on a discrete set of points), it is possible to establish everything we need about the price adjustment process for the non-convex economy, by studying that for the convexified economy. Studying the stability of the price adjustment process for the convexified economy is not entirely straightforward, as this is not strictly convex, so that it generates an excess demand correspondence rather than function. However, standard techniques for the analysis of stability with excess demand functions (see Arrow and Hahn [1], Chapters 11 and 12) can be extended to this more general case by the use of a stability theorem for differential correspondences due to Champsaur, Drèze and Henry [6]. In the remainder of this section we establish the preliminary Lemmas 1 and 2: the application to stability results comes in the next section.

<u>LEMMA 1</u>. $Z(p) = \overline{Z}(p)$ *if and only if* $p \notin \overline{S}$.

<u>PROOF</u>. Z and $\overline{Z}$ differ if and only if $y_i(p)$ and $\overline{y}_i(p)$ differ for at least one i. Now $y_i(p) \in \partial Y_i$ and $\overline{y}_i(p) \in \partial \overline{Y}_i$. If $\overline{y}_i(p)$ contains points in $\partial \overline{Y}_i$ not in ∂Y_i, as is the case when $p \in \overline{S}$, then $\overline{y}_i(p) \neq y_i(p)$. Hence $p \in \overline{S}$ implies $\overline{Z}(p) \neq Z(p)$. Conversely, suppose p . $\overline{S}$. Then for all i, $\overline{y}_i(p) \cap (\partial \overline{Y}_i / \partial Y_i) = \phi$ so that $\overline{y}_i(p) \subset (\partial \overline{Y}_i \cap \partial Y_i)$ and in particular $\overline{y}_i(p) \subset \partial Y_i$. It is then immediate that $y_i(p) = \overline{y}_i(p)$, as required.

LEMMA 2. $\overline{S}$ *is of Lebesgue measure zero in* S *, and its complement is dense in* S *. Furthermore, for* $p \notin S$ *, the excess demand correspondence* Z(p) *is single-valued.*

PROOF.[1] By Lemma 1, $\overline{S} = \{p \in S : y(p) \neq \overline{y}(p)\}$. Define $g : S \to R$ and $\overline{g} : S \to R$ by $g(p) = p \cdot y(p)$ and $\overline{g}(p) = p \cdot \overline{y}(p)$. Clearly $g(p) = \overline{g}(p)$, is a convex function. We restrict our attention to the behavior of g on the interior of S , int S , as a set of Lebesgue measure zero in int S (or nowhere dense in int S) is of Lebesgue measure zero (or nowhere dense) in S . Now, the set of points I in int S where g is differentiable is dense in int S , and its complement is of Lebesgue measure zero in int S (Rockafellar [14], p. 246, Theorem 25.5). Now for $p \in I$, $Dg(p) = D\overline{g}(p)$ where Dis the gradient, as in I g(p) and $\overline{g}(p)$ are equal and differentiable. But $Dg(p) = y(p) = D\overline{g}(p) = \overline{y}(p)$. It follows that $\overline{S}$ is contained in the complement of I , which is of measure zero. Hence $\overline{S}$ is of measure zero, and its complement is dense, as required. Note finally that in I , $\overline{y}(p)$ is single valued, as it equals Dg(p) . This completes the proof.

Remark. Under mild regularity conditions on the efficiency frontiers $eff(Y_i)$, $\overline{S}$ can be shown to be a discrete set. If the commodity space is of dimension 2, $\overline{S}$ is always a discrete set.

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[1] I am grateful to Barry Murphy for the following proof, which is simpler than my earlier one using polar duality theory.

3. Results

We have now established two economies, E and $\overline{E}$, the latter the convexification of the former. The excess demands of these economies agree everywhere except on a set of measure zero, denoted by $\overline{S}$. The next stage is to establish conditions under which $\overline{E}$ has a Walrasian stable mcp, say p^* : it will then be possible to argue that as the Walrasian adjustment process (1) is identical almost everywhere for the two economies, then p^* must also be Walrasian stable for E .

The first stage, then, is to establish stability results for $\overline{E}$. This is not immediate, as its adjustment process is defined by a differential correspondence (1) and not by a differential equation: existing results on Walrasian stability (as for example in Arrow and Hahn [1]) assume excess demand mappings to be single-valued. We therefore need to invoke a theorem of Castaing-Valadier [4] (C-V) on the existence of solutions to differential correspondences, and a theorem of Champsaur-Drèze-Henry [6] (C-D-H) on the stability of adjustment processes defined by such correspondences. These theorems are stated, and the appropriate solution concepts defined, in the appendix.

It is routine to establish that, as S is compact, the adjustment process

$$\frac{dp}{dt} \in \overline{Z}(p) \qquad (2)$$

satisfies the conditions of the C-V theorem. The set of solution paths is therefore non-empty and compact, and the solution correspondence upper-semi-continuous. In view of this, the existence of a Lyapunov function for $\overline{E}$ would enable us to use the C-D-H theorem to establish that the process

(2) is quasi-stable. In fact non-negativity of the price vector requires that one studies a more complex adjustment process than (2), namely

$$\dot{p}_i = 0 \quad \text{if} \quad p_i = 0 \quad \text{and} \quad \overline{Z}_i(p) < 0$$
$$\dot{p}_i \in \overline{Z}_i(p) \quad \text{otherwise.} \qquad (2')$$

The right-hand side of (2') may fail to be upper-semi-continuous on the boundary of S . Champsaur-Drèze-Henry [6] show how the existence and stability theorems involved here, can also be applied to cases of this type. We shall therefore neglect complications posed by the non-negativity of the price vector, and focus only on those features unique to the present problem, namely the fact that the right-hand side of (2) is a set because the economy $\overline{E}$, being a convexified non-convex economy, is not strictly convex.

The existence of a Lyapunov function for convex economies is discussed at length in Arrow and Hahn [1], and we need do no more here than user their results. They show that in essentially two types of cases, there is a competitive equilibrium which is globally stable under a single-valued Walrasian adjustment process. These two cases are when the economy is *Hicksian*, and when it satisfies *gross substitutability*. The economy is Hicksian if it behaves as if there were a single consumer, i.e., if preferences can be aggregated.[1] It satisfied gross substitutability if $Z(p)$ is single-valued and

$$\frac{\partial Z_i(p)}{\partial p_j} > 0 \qquad \text{(GS)}$$

[1] For conditions for preference aggregation, see for example Chichilnisky and Heal [6] and references therein to earlier work by Chipman and Gorman.

for all i, j, with $i \neq j$. As the excess demand correspondences in the present model are single-valued only almost everywhere, the economy E will be said to satisfy gross substitutability if (GS) holds for almost all $p \in S$. An equivalent definition applies to E. The concept of being Hicksian requires no modification for the present context, as it is a function purely of the properties of consumer preferences, about which we make standard convexity assumption.

We are now in a position to establish the main result, which is the following.

THEOREM 1. *If the convexified economy* $\bar{E}$ *has a Walrasian-stable competitive equilibrium, then the non-convex economy* E *has either a Walrasian-stable non-market-clearing price vector or a stable competitive equilibrium.*

An important complement to this, which establishes that it is not vacuous is:

THEOREM 2. *If the convexified economy* $\bar{E}$ *is Hicksian (which is equivalent to* E *being Hicksian) or if it satisfies gross substitutability, then it has a Walrasian-stable competitive equilibrium.*

Proof of Theorem 1

By assumption there exists $p^* \in S$ such that $\lim_{t\to\infty} p(t) = p^*$ for any trajectory $p(t)$ satisfying $\dot{p}(t) \in \bar{Z}(p(t))$ for almost all t. We need to establish two points:

(i) that if $p(t)$ satisfies $\dot{p}(t) \in \bar{Z}(p(t))$ for almost all t, then it also satisfies $\dot{p}(t) \in Z(p(t))$ for almost all t, and so is a solution to the adjustment equations of the economy E.

(ii) that p^* is a non-market-clearing price vector for E or a competitive equilibrium.

Point (ii) is immediate so that only (i) needs proving.

Point (i) follows immediately from the fact that the vector fields $\dot{p} \in Z(p)$ and $\dot{p} \in \overline{Z}(p)$ agree and are single valued for almost all $p \in S$, and for a dense subset of S, and so give rise to identical trajectories.

Proof of Theorem 2

This is an immediate application of standard existence results, and of the results of Chapter 12 of Arrow and Hahn [1] and the stability theorem of Champsaur-Drèze-Henry [5], reproduced in the appendix.

We have established conditions under which the non-convex economy E has either a stable non-market-clearing price vector, or a stable competitive equilibrium. E, being non-convex, does not meet the known sufficient conditions for the existence of a competitive equilibrium, and it is routine to construct examples of non-convex economies with no competitive equilibrium. This is done in Heal [11] and in Arrow and Hahn [1], Chapter 11. Theorem 1 establishes that if we construct such an economy so that it is Hicksian--something we can clearly do--then it will have a stable non-market-clearing price vector.

5. Concluding Remarks

I have shown above the increasing returns to scale in production can provide a rigorous explanation of the existence of stable non-market-clearing prices, even in an economy where prices are free to respond in the normal manner to excess supplies or demands. Of course, many other explanations have been offered for price stickiness--implicit contracts arguments, references to adjustment costs or to kinked demand curves, etc. But the present argument seems to be at least as simple and general as any of the alternatives, and so to merit serious consideration.

In a companion piece to this paper, I have also argued that increasing returns can provide clear and simple reasons for agents to introduce quantity rationing in their dealings with other agents (Heal [10]). These two points together imply that a model with increasing returns may be a very natural one for analyzing the economic consequences of rationing and price rigidity, and for examining the macroeconomic phenomena often linked with these. A preliminary step in this direction is made in Heal [11]. Weitzman [17] has recently stated a related intuition about the role of increasing returns in providing a basis for the micro foundations of macro theory.

There is one final point that I would like to make about the possibility of sharpening Theorem 1. This established the existence of either a stable disequilibrium price vector, or a stable competitive equilibrium. In fact, the two are not mutually exclusive if we think of local rather than global stability, as Figure 4 shows. Here the demand curve for the

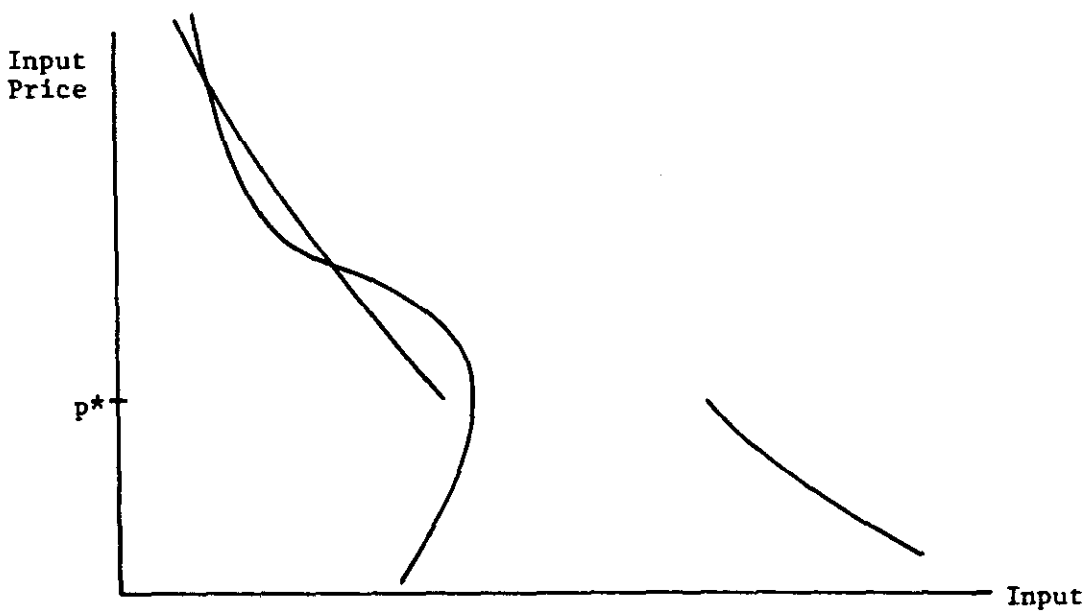

FIGURE 4: Demand and supply curves corresponding to a case where there is at least one competitive equilibrium, and where p* is a locally stable non-market-clearing price.

input is discontinuous, as before. The supply curve for the input (which one might think of as labor) is backward rising: it passes through the discontinuity, but also intersects the continuous segments of the demand curve. In this case p^* is a *locally stable* non-market-clearing price.

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APPENDIX

(from Champsaur-Drèze-Henry [5])

We first give a result about the existence of solutions to differential correspondences. Let G be an upper hemi-continuous correspondence from R^n to the set of non-empty, convex and compact subsets of R^n (i.e., $G : R^n \to PR^n : z \to G(z)$, with $G(z)$ compact and convex), such that there exists a positive number α with, $\forall z \in R^n$,

(B.1) $$\sup_{w \in G(z)} \|w\| \leq \alpha(1 + \|z\|) .$$

Consider the system of differential equations

(B.2) $$\frac{dz}{dt} \in G(z) ,$$

and denote by $S_T(z^0)$, where $z^0 \in R^n$, the set of all the solutions of system (B.2), solutions on $[0,T]$, $T > 0$, and starting from z^0 . The word "solution" must be understood here in the following sense: A function $z : [0,T] \to R^n : t \to z(t)$ is a solution of (B.2) on $[0,T]$ if it is absolutely continuous on $[0,T]$ and if, for almost every t in $[0,T]$ $\frac{d}{dt}z(t) \in G(z(t))$. We then have

Theorem A.1. (Castaing-Valadier): There exists $T > 0$ such that

A1.1. $\forall z^0 \in R^n$, $S_T(z^0)$ is non-empty and compact in $C([0,T]; R^n)$, the space of continuous functions from $[0,T]$ to R^n endowed with the uniform convergence topology.

A1.2. $\forall A \subset R^n$, A compact, the correspondence $S_T : A \to PC([0,T]; R^n)$: $z^0 \to S_T(z^0)$ is upper hemi-continuous.

Next we state the theorem about stability of solutions used in the text.

G is a correspondence between the points of a subset E of R^n and the subsets of R^n ; z being any point in E , $G(z)$ is a non-empty subset of R^n ;

(B.3) $$\frac{dz}{dt} \in G(z)$$

is a multivalued system of differential equations defining a dynamic process P .

We shall call *equilibrium* of P any point $\bar{z}$ in E such that $0 \in L(\bar{z})$ and *trajectory* of P any solution of (B.3) on $[0,+\infty[$. From the definition of L such a trajectory, if it exists, is included in E . Let us consider a point z^0 in E and a trajectory $z(z^0;\cdot)$ starting from z^0 : This notation means that, at time t , this trajectory passes through point $z(z^0;t)$; of course $z(z^0;0) = z^0$. We shall call *limit point* of trajectory $z(z^0;\cdot)$ a point $\bar{z}$ for which there exists a sequence of times t^n such that $t^n \xrightarrow[n\to+\infty]{} +\infty$ and $z(z^0;t^n) \xrightarrow[n\to+\infty]{} \bar{z}$. Since E is closed, $\bar{z}$ is a point in E . P is said to be *quasi-stable* if any limit point of any trajectory is an *equilibrium*. A *Lyapunov function* for P is a continuous function V from E to R such that

--for any point z^0 in E and any trajectory $z(z^0;\cdot)$, the function $v(z^0;\cdot) : [0,+\infty[\to R : t \to v(z^0;t) = V(z(z^0;t))$ converges to a unique limit when $t \to +\infty$.

--if there exist $T \in]0,+\infty[$ and a trajectory $z(z^0;\cdot)$ such that $v(z^0;\cdot)$ is constant on $[0,T]$, then z^0 is an equilibrium of P .

Let $S_T(z^0)$ denote the set of all trajectories starting from z^0 but restricted to $[0,T]$. Let also $S_T(\cdot)$ denote the correspondence $E \to PC_u([0,T]; E) : z^0 \to S_T(z^0)$; we then have:

Theorem A.2. (Champsaur-Drèze-Henry). If there exists a Lyapunov function for P , if $S_T(z^0)$ is non-empty and compact in $C([0,T]; E)$ for every z^0 in E and if $S_T(\cdot)$ is upper hemi-continuous, then P is quasi-stable.