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Core and Competitive Equilibria with Indivisibilities

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I. INTRODUCTION

At the end of the paper "On Cores and Indivisibility" Shapley and Scarf [9] review a series of models involving indivisible goods, which have been studied in the literature from the point of view of the core. After the exposition, they notice : "It could be interesting if a general framework could be found that would unify some or all these scattered results".

The purpose of this paper is both to present this general framework and to generalize some of the existing results.

We consider a model of an exchange economy with n agents and two goods. The first one is a perfectly divisible good which will be called money. The other is a good present in the economy in indivisible units subject to quality differenciation. The main restriction on the model is that each agent does not own initially more than one indivisible item and has no use for more than one of these items. With these restrictions we prove that, whatever the preferences of the agents, the economy is balanced and then has a non empty core.

To see how this model allows a unified study of the models reviewed in Shapley - Scarf [9] , it is convenient to classify them in two categories : the "exchange models" and the "pairing models". Enter in the first category the model of exchange of houses* without money studied by Shapley and Scarf [9] and the model of assignment between buyers and sellers presented by Shapley and Shubik [10], and generalized by Kaneko [4] [5]. In the second category, we find the "college admission" model of Gale-Shapley [2], with the particular case of "marriage", interpreted after introduction of money as a "job matching model" by Crawford and Knoer. The "problem of roommates", for which Gale and Shapley [2] show that there may exist no stable solution, enters also this category.

The exchange models are special cases of the general model presented above. Their characteristic features can be captured by making restrictive assumptions on the distribution of initial resources and on the preferences of the agents. These restrictions do not alter the result of existence of the core. What we prove is that a market for one kind of indivisible goods has in general a non empty core. No assumption of complete symmetry as in [9], or of complete dissymetry as in [10] or of transferable utility is needed for this result.

On the opposite, to adapt our model to the pairing models, we have to impose restrictions on the allocations which are feasible in the exchange. To give an image, we have to

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* All the models quoted in the introduction will be described in the main body of the paper.

translate the fact that if A is married to B, then B of course is married to A. We will show how this restriction intervenes in the proof of balancedness, implying that this proof no longer works. We will study that on the counter-example given by Gale and Shapley [2] for the problem of roommates. We also prove that the core still exists in models involving two types of agents - men an women for a marriage model, firms and workers for a job market, college and students for the "college admission model".

The study of the relation between the core, when it exists, and the competitive equilibria is the subject of the last section of the paper. The main result of this section is that the core coincides with the competitive equilibrium allocations in an exchange economy of the type presented above, as soon as money enters really in the model. In fact we need for this result two assumptions. The first one ensures that money enters really in the preferences of the agents. The other implies that the initial resources in money are in some sense "sufficient".

The other conclusion of this section is that the "pairing models", even with two types of agents, have a complete different behaviour with respect to decentralization. To prove this, we give an example of a pairing model with money which has no competitive equilibrium.

II. THE MODEL OF EXCHANGE

Let us consider an exchange economy with n agents and two goods. The first good is a perfectly divisible good which will be called money. The second good exists only in indivisible units. These units can be different in quality but have all the same function for the consumers. In consequence, we assume that each agent has no use for more than one unit of the second good. A typical example of such a good is the good "house". Following the terminology of Shapley- Scarf, an indivisible unit of the second good will be called an item. We make the restrictive assumption that initially an agent does not own more than one item. The number of items in the economy is therefore inferior or equal to the number of agents.

Let ω_i be the initial endowment of agent i in money. If agent i owns one item before the exchange, this item is denoted e^i. Let us rank the agents in such a way that the first q ones own initially an item, and the others have none. The initial resources of agent i are (ω_i , e^i) if i belongs to $[1 , \dots , q]$, and $(\omega_i , 0)$ if i belongs to $[q + 1 , \dots , n]$. In the following, it will be convenient to use the notation e^i for the initial resources of all the agents. Then, it will be understood that if $q + 1 \leq i \leq n$, then $e^i = 0$.

The preferences of agent i are represented by a utility function u_i defined on $\mathbb{R}_+ \times \{e^1, \dots, e^q, 0\}$. For every $e^j \in \{e^1, \dots, e^q, 0\}$, it is assumed that $u_i(\cdot, e^j)$ is continuous and non decreasing with respect to the quantity of money.

An allocation for this economy is a vector $(m_i, e_i^j)_{i=1,\dots,n}$ in $\mathbb{R}_+^n \times \{e^1, \dots, e^q, 0\}^n$. e_i^j means that agent i has been allocated the item (or possibly the absence of item) which was owned initially by agent j. For example if $j \in [q+1, \dots, n]$, then $e_i^j = 0$.

An allocation is feasible if there exists a permutation σ of $N = \{1, \dots, n\}$ such that the allocation is of the form $(m_i, e_i^{\sigma(i)})_{i=1,\dots,n}$ with $\sum_{i=1}^{n} m_i < \sum_{i=1}^{n} \omega_i$. The existence of the permutation σ implies that each item has been attributed to one and only one agent.

Since we want to study the core of the economy we must describe the allocations feasible for a coalition S. An allocation $(m_i, e_i^j)_{i=1,\dots,n}$ is feasible for a coalition S if $\sum_{i\in S} m_i < \sum_{i\in S} \omega_i$ and if there exists a permutation σ_S of S such that : $\forall i \in S \;\; e_i^j = e_i^{\sigma_S(i)}$.

Let us denote A(S) the feasible allocations of the coalition S and Σ_S the set of permutations of S.

$$A(S) = \left\{ (m_i, e_i^j)_{1 \leq i \leq n} \;\middle|\; \begin{array}{l} \exists\, \sigma_S \in \Sigma_S \quad j = \sigma_S(i) \quad \forall\, i \in S \\ \text{and} \quad \sum_{i \in S} m_i \leq \sum_{i \in S} \omega_i \end{array} \right\}$$

The core of the economy consists of those allocations which are feasible (for N) and such that no coalition S can find in $A(S)$ an allocation which is strictly preferred by all its members.

We associate to the economy the game without side-payments whose characteristic function V is defined by

$$V(S) = \left\{ v = (v_i)_{1 \leq i \leq n} \in \mathbb{R}^n \;\middle|\; (m_i, e_i^j) \in A(S),\ v_i \leq u_i\ (m_i, e_i^j)\ \forall i \in S \right\}$$

The game is well defined in the sense that it has the following properties

a) $V(S)$ is closed in $\mathbb{R}^n$

b) If $v \in V(S)$ and $v'_i \leq v_i \quad \forall\, i \in S$, then $v' \in V(S)$

c) $\mathrm{Proj}_{\mathbb{R}^S} \left[V(S) - \bigcup_{i \in S} \mathrm{Int}\, V(\{i\}) \right]$ is non empty and bounded.

The core of the economy is non empty if and only if the game V has a non empty core. We will prove this property by proving that the game V is balanced (SCARF [7]).

<u>Theorem 1</u> : <u>The game V is balanced ; hence the exchange economy has a non empty core</u>.

To prove Theorem 1, we have to prove that if a vector v belongs to $\bigcap_{S \in B} V(S)$ for a balanced family B of coalitions, then v belongs to $V(N)$. Since $\bigcap_{S\in B} V(S) \subset \bigcap_{S\in B'} V(S)$ if B' is a subfamily of B, it is enough to prove this property for a minimal balanced family of coalitions. This proof holds on the following property of minimal balanced families of coalitions :

<u>Lemma 1</u> : <u>Let $M = (m_{ij})$ be a $n \times n$ real matrix. Let B be a minimal balanced family of coalitions of $N = [1,\dots,n]$, associated to the balancing weights $(\delta_S)_{S\in B}$. Then, for any family $(\sigma_S)_{S\in B}$ of permutations of the sets S, we have the following inequality :</u>

$$\min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} < \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)} .$$

Before proving Lemma 1, let us show how it implies Theorem 1.

<u>Proof of Theorem 1</u> : Let B be a minimal balanced family of coalitions of N associated to the weights $(\delta_S)_{S\in B}$ and let v be a vector in $\bigcap_{S\in B} V(S)$. To the vector v, let us associate the matrix $M(v) = (m_{ij}(v))$ defined as follows :

$$m_{ij}(v) = \inf \{ m_i \in \mathbb{R}_+ \mid u_i(m_i, e^j) > v_i \}$$

with the convention that if the set $\{m_i \in \mathbb{R}_+ \mid u_i(m_i, e^j) > v_i\}$ is empty then $m_{ij}(v) = +\infty$.

The interpretation of the matrix $M(v)$ is clear :

$m_{ij}(v)$ is the minimum amount of money that must be given to agent i in conjunction with the item e^j, initially owned by agent j, to garantee the utility level v_i to agent i.

Given a coalition S, it follows from the definitions of M(v) and V(S) that v belongs to V(S) if and only if there exists a permutation σ_S of S such that

$$\sum_{i\in S} m_{i,\sigma_S(i)} \leq \sum_{i\in S} \omega_i .$$

Hence we know from the fact that $v \in \bigcap_{S\in B} V(S)$ that there exists a family $(\sigma_S)_{S\in B}$ of permutations of the sets S such that

$$\forall\, S \in B \qquad \sum_{S\in B} m_{i,\sigma_S(i)} \leq \sum_{i\in S} \omega_i .$$

From Lemma 1 this implies that :

$$\min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} \leq \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)} \leq \sum_{S\in B} \delta_S \sum_{i\in S} \omega_i$$

$$= \sum_{i=1}^{n} \omega_i \Big(\sum_{S|i\in S} \delta_S \Big)$$

$$= \sum_{i=1}^{n} \omega_i$$

Therefore, there exists at least one permutation σ of N such that

$$\sum_{i=1}^{n} m_{i,\sigma(i)} \leq \sum_{i=1}^{n} \omega_i$$

and thus v belongs to V(N).

∎

Before proving Lemma I we need the following definitions : let S be a coalition of $N = [1,\dots,n]$. A S-permutation matrix is a n by n zero-one matrix containing one 1 in each row and each column indexed by a member of S and only zeros in lines and columns indexed by members of $N\backslash S$. If σ_S is a permutation of the set S, the S-permutation matrix $A_{\sigma_S} = (a_{ij})$ associated with σ_S is such that $a_{ij} = 1$ if and only if $i \in S$ $j \in S$ and $j = \sigma_S(i)$.

A N-permutation matrix will be simply called a permutation matrix.

A matrix is said to be doubly stochastic if all its components are positive real numbers, and if all its columns and all its lines sum up to 1.

Proof of Lemma 1

Let $M = (m_{ij})$ be a $n \times n$ real matrix and B a minimal balanced family of coalitions of $N = [1,\dots,n]$.

It is easy to deduce from the proof of uniqueness of the associated balancing weights $(\delta_S)_{S\in B}$ (SHAPLEY [8] that the weights (δ_S) are rational numbers. After reduction to the same denominator d, the numbers (δ_S) can be written as

$$\forall\, S \in B \quad \delta_S = \frac{c_S}{d} \qquad c_S \in N^*,\ d \in N^*.$$

For each $S \in B$, let σ_S be a permutation of S.

We have to prove that :

$$\min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} < \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)} .$$

Let us remark first that if this property holds for the matrix $\tilde{M} = (\tilde{m}_{ij})$ defined by

$\tilde{m}_{ij} = m_{ij}$ if there exists $\sigma_S \in \{\sigma_S/S\in B\}$ such that $j=\sigma_S(i)$

$\tilde{m}_{ij} = +\infty$ otherwise,

then the property holds for M since

$$\min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} < \min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} \tilde{m}_{i,\sigma(i)} .$$

Since $\tilde{M}$ has some coefficients which are $+\infty$, not all the permutations σ of Σ_N will give a finite sum. In order to find the relevant permutations for $\tilde{M}$ let us define the matrix $A = (a_{ij})$ in the following way.

For each $S \in B$, let $B_S = (b^S_{ij})$ be the S-permutation matrix associated with σ_S. Then A is defined by :

$A = \sum_{S\in B} c_S \; B_S$.

This matrix A has positive integer coefficients and has the following properties :

1°) The significant coefficients of M – i.e. those which are not $+\infty$ – correspond exactly to the indices a_{ij} such that $a_{ij} \neq 0$.

Formally : $[\tilde{m}_{ij} < +\infty] \Leftrightarrow [a_{ij} > 0]$.

<u>Proof</u> :

$$a_{ij} > 0 \Leftrightarrow \exists S \quad b^S_{ij} \neq 0 \Leftrightarrow \exists \sigma_S \quad j = \sigma_S(i)$$
$$\Leftrightarrow \tilde{m}_{ij} = m_{ij} \text{ and } m_{ij} < +\infty$$
$$\Leftrightarrow \tilde{m}_{ij} < +\infty$$

This property implies trivially that :

2°) The permutations $\sigma \in \Sigma_N$ which give a finite result for $\sum_{i=1}^{n} \tilde{m}_{i,\sigma(i)}$ are such that $A_\sigma \leqslant A$ (where A_σ is the permutation matrix associated with σ).

Formally : $[\sum_{i=1}^{n} \tilde{m}_{i,\sigma(i)} < +\infty] \Leftrightarrow [A_\sigma \leqslant A]$.

3°) For every couple (i, j), a_{ij} is the coefficient of m_{ij} in the sum $\sum_{S\in B} c_S \sum_{i\in S} m_{i,\sigma_S(i)}$.

Formally :

$$\sum_{S\in B} c_S \sum_{i\in S} m_{i,\sigma_S(i)} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} m_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \tilde{m}_{ij}$$

Proof :

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, m_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{S\in B} c_S\, b_{ij}^{S} \right) m_{ij}$$

$$= \sum_{S\in B} c_S \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^{S}\, m_{ij}$$

$$= \sum_{S\in B} c_S \sum_{i\in S} m_{i,\sigma_S(i)}$$

The equality $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \tilde{m}_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, m_{ij}$ follows from Property 1°).

4°) The sum of each row and each column of the matrix A is equal to q.

Proof :

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \left(\sum_{S\in B} c_S\, b_{ij}^{S} \right) = \sum_{S\in B} c_S \sum_{j=1}^{n} b_{ij}^{S}$$

$$= \sum_{S\in B \mid i\in S} c_S = d$$

The same reasoning gives the result for the columns.

To finish the proof we need the following lemma :

<u>Lemma 2</u> : <u>If a n × n matrix $A = (a_{ij})$ is such that the coefficients a_{ij} are integers and all rows and all columns sum to a positive integer d, then there exist d permutations $\sigma_1 , \dots , \sigma_d$ such that</u> :

$$A = A_{\sigma_1} + \dots + A_{\sigma_d}$$

<u>Proof of Lemma 2</u> :

The proof is by induction on d. If d = 1, the matrix A is itself a permutation matrix. Let us suppose that the property holds for d and let us consider a matrix A whose lines and columns sum to d + 1. We certainly have :

$$A \geq \frac{1}{d+1} A.$$

$\frac{1}{d+1} A$ is a doubly stochastic matrix. It is proved in Shapley-Scarf [9] that a matrix with integer coefficients superior to a doubly stochastic matrix is superior to at least one permutation matrix. Let us call A_{σ_1} such a matrix. The induction property applied to $A - A_{\sigma_1}$ gives the result. ∎

Lemma 2 proves first that the set of permutations which give a finite result for $\sum_{i=1}^{n} \tilde{m}_{i,\sigma(i)}$ is non empty since there exists a permutation matrix inferior to A. Let $\bar{\sigma}$ be the permutation for which the minimum of this sum is reached. Then $A - A_{\bar{\sigma}}$ have positive integers coefficients and its lines and columns sum to $d-1$. Then, from Lemma 2, there exists permutations $\sigma_2, \dots, \sigma_d$ such that $A = A_{\bar{\sigma}} + A_{\sigma_2} + \dots + A_{\sigma_d}$. Then, from property 3°), we have the following :

$$\sum_{S \in B} c_S \sum_{i \in S} m_{i,\sigma_S(i)} = \sum_{i=1}^{n} \sum_{j=n}^{n} a_{ij} \tilde{m}_{ij}$$

$$= \sum_{i=1}^{n} \tilde{m}_{i,\bar{\sigma}(i)} + \sum_{i=1}^{n} \tilde{m}_{i,\sigma_2(i)} + \dots + \sum_{i=1}^{n} \tilde{m}_{i,\sigma_d(i)}$$

$$\geq d \sum_{i=1}^{n} \tilde{m}_{i,\bar{\sigma}(i)}.$$

Since $\delta_S = \frac{c_S}{d}$ and $M \leq \tilde{M}$, this implies :

$$\min_{\sigma \in \Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} \leq \sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}$$

∎

Let us turn now to the exchange models studied in the literature. Shapley and Scarf [9] consider a model with n traders, each with initially one item (for example a house). Each trader has a preference ordering on the n items available in the economy and has no use for more than one. The problem is to find a redistribution of the items in accordance with the preferences of the traders. It is proved that the problem has a solution since the model has a non empty core.

This model is a special case of our general model with $m = n$ and $\omega_i = 0$ for all i. Theorem 1 proves that money can be introduced in the model without altering the result of existence of the core.

At the opposite of the symmetric model of Shapley - Scarf, we find the completely dissymmetric model of Shapley and Shubik [10] . Here there are two kinds of traders : m sellers, each one with with initially one item, and p buyers which own initially the money. Sellers and buyers have dissymmetric preferences. A seller values only his own item. A buyer has no use for more than one item but his preferences hold on all available items. The problem is to find a redistribution of the items with compensations in money which cannot be improved by any coalition of buyers and sellers. Shapley and Shubik make the assumption of transferable utility and prove the non-emptiness of the core using linear programming. Kaneko [4] generalized the model and the result to the case of non-transferable utility.

The model just described corresponds to the following specification of our model :

- $\forall i \; 1 \leq i \leq n$, $\forall j \; 1 \leq j \leq n$, $u_i(m_i , e^j)$ is increasing in m_i
- $\forall i \; 1 \leq i \leq q$, $\forall j \; 1 \leq j \leq q$, $j \neq i$, $\forall m_i > 0$,

$$u_i(m_i , e^j) < u_i(m_i , 0) < u_i(m_i , e^i)$$

The dissymmetry in the preferences of buyers and sellers implies that a seller will never buy the item of another seller than himself. An allocation in the core of the economy such that buyer i ($q + 1 \leq i \leq n$) gets the item of seller j must be of the form $(\omega_j + c , 0)$ $(\omega_i - c , e_i^j)$ with $c > 0$. If not, it would be blocked either by the coalition $\{i , j\}$ or by the coalition $N - \{i , j\}$. Then the only coalitions relevant for the problem are singletons or pairs of agents of different types.

The problem can thus be solved as a "pairing problem" of assignment between buyers and sellers.

Theorem 1 proves that the dissymmetry in the preferences of buyers and sellers is not necessary for the existence of a core. We may allow a seller to sell its own item and buy another one he prefers without altering the result of existence of the core. We will generalize in Section IV, the result of equivalence between core allocations and competitive allocations proved, for the model presented above, by Shapley - Shubik [10] in the case of transferable utility and by Kaneko [4] in the case of non-transferable utility.

III. THE PAIRING MODEL

The "pairing" models quoted in the introduction can enter in the framework of the model presented in the preceding section only if we make restrictions on the feasible allocations. Let us consider for example the simple model of marriage of Gale and Shapley [2] . "A certain community consists of n women and n men. Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner. The problem is to find a satisfactory way of marrying off all members of the community."

We may try to describe this problem with our model. Let us consider an exchange economy with 2 n agents, each agent endowed initially with no money and one -namely himself- indivisible item. We can translate the fact that each person has preferences only for persons of the opposite sex by the following assumption :

$$\forall i \; 1 \leq i \leq n \quad \forall j \; \underset{i \neq j}{1 \leq j \leq n} \quad \forall k \; n+1 \leq k \leq 2n \quad \forall k' \; \underset{k \neq k'}{n+1 \leq k' \leq 2n}$$

$$u_i\,(0\,,e^j) < u_i\,(0\,,e^i) < u_i\,(0\,,e^k)$$

$$u_k\,(0\,,e^{k'}) < u_k\,(0\,,e^k) < u_k\,(0\,,e^i)$$

If we study the core of such an economy, we will find allocations of the form $(0\,,e_i^{\sigma(i)})_{1 \leq i \leq n}$ where σ is a permutation of $N = \{1\,,\ldots,\,2\,n\}$. The assumption made on preferences will imply that if $i \in [1\,,\ldots,\,n]$ then $\sigma(i) \in [n+1\,,\ldots,\,2\,n]$.

But nothing in the model can ensure that the agents are paired. We have to impose the condition : $\sigma(i) = j \Rightarrow \sigma(j) = i$ *.

Hence the following definition :

Definition

A model as described in Section II is called a "pairing" model if the set of feasible allocations for the coalition $S \subseteq N$ is restricted by the condition $\sigma_S \circ \sigma_S = Id_S$ where Id_S is the identity mapping of the set S.

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* In spite of the apparent similarity between the model of marriage and the model of assignment between buyers and sellers, this difficulty does not appear when we translate the model of Shapley and Shubik in our exchange model. In this case, if j is a "seller" ($j \in [1, \ldots, q]$) and if i and i' are two "buyers" ($i \in [q+1, \ldots, n]$ and $i' \in [q+1, \ldots, n]$), the allocations $((m_j, e_j^{i'}), (m_i, e_i^j))$ and $((m_j, e_j^i), (m_i, e_i^j))$ are the same since $e^i = e^{i'} = 0$. In other words, a "seller" is indifferent between the "no item" of i or the "no item" of i'. The assumptions of the preferences and initial resources then imply that an allocation in the core can always be represented by a permutation σ such that $j = \sigma(i) \Rightarrow i = \sigma(j)$. At the contrary, in the marriage model, the "man" j typically will not be indifferent between the "woman" i and the "woman" i', and then the allocation $((m_j, e_j^{i'}), (m_i, e_i^j))$ is really different from $((m_j, e_j^i), (m_i, e_i^j))$.

The set of feasible allocations for a coalition S is then :

$$\bar{A}(S) = \left\{ (m_i , e_i^j)_{i=1,\dots,n} \;\middle|\; \begin{array}{l} \exists\, \sigma_S \in \Sigma_S \;\; \sigma_S \circ \sigma_S = Id_S \;\; j = \sigma_S(i) \;\; \forall i \in S \\ \text{and} \;\; \sum_{i\in S} m_i < \sum_{i \in S} \omega_i \end{array} \right\}$$

The associated game will be denoted $\bar{V}(S)$.

In the literature we find some pairing models which have an empty core. The mathematical reason for that is the following. If we try to prove that the game $\bar{V}$ is balanced, we will make the same construction of the matrix $M(v)$* as in the proof of Theorem I and we will have to prove :

$$\min_{\substack{\sigma\in\Sigma_N \\ \sigma\circ\sigma=Id_N}} \sum_{i=1}^{n} m_{i,\sigma(i)}(v) < \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)}(v)$$

This restriction on the admissible permutations σ can change the result. In fact to prove the above inequality we should prove that the matrix A (defined in the same way and with the same properties than in Theorem I) can be decomposed into the sum of d symmetric permutation matrices. And this is not always possible.

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* The interpretation of $m_{ij}(v)$ is now : $m_{ij}(v)$ is the minimum amount of money needed by agent i to reach the utility level v_i when paired with agent j.

As an example, let us consider the "problem of roommates" presented by GALE and SHAPLEY [2]. "An even number of boys wish to divide up into pairs of roommates. A set of pairing is called stable if under it there are no two boys who are not roommates and who prefer each other to their actual roommates. An easy example shows that there can be situations in which there exists no stable pairing. Namely consider boys α β γ and δ, where α ranks β first, β ranks γ first, γ ranks α first and α β and γ all rank δ last. Then regardless of δ's preferences there can be no stable pairing, for whoever has to room with δ will want to move out, and one of the other two will be willing to take him in."

The stable pairings of this problem are the core allocations of our pairing model with the following specifications :

$$u_1(0,e^1) < u_1(0,e^4) < u_1(0,e^3) < u_1(0,e^2)$$

$$u_2(0,e^2) < u_2(0,e^4) < u_2(0,e^1) < u_2(0,e^3)$$

$$u_3(0,e^3) < u_3(0,e^4) < u_3(0,e^2) < u_3(0,e^1)$$

and for example :

$$u_4(0,e^4) < u_4(0,e^3) < u_4(0,e^2) < u_4(0,e^1).$$

Let us consider the balanced family $B = (\{1,2\} \{1,3\} \{2,3\} \{ 4 \})$ with balancing coefficients $(1/2 , 1/2 , 1/2 , 1)$.

Let us consider a utility vector $v \in \bigcap_{S \in B} V(S)$. A two-person coalition has two possibilities : either each agent stays alone or the two agents form a pair. The utility functions are such that the second solution always raises a larger utility for both the agents than the first one. So let us suppose that v is such that {1,2}, {1,3}, {2,3} can achieve v only by forming pairs. Then the corresponding permutations $(\sigma_S)_{S \in B}$ have matrices :

$$\underset{\sigma_{\{1,2\}}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\sigma_{\{1,3\}}}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\sigma_{\{2,3\}}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\sigma_{\{4\}}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

The matrix A is then : $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

and the matrix M(v) can only be : $\begin{bmatrix} +\infty & 0 & 0 & +\infty \\ 0 & +\infty & 0 & +\infty \\ 0 & 0 & +\infty & +\infty \\ 0 & 0 & 0 & 0 \end{bmatrix}$

One can observe that M(v) is such that for any choice of a symmetric permutation σ, $\sum_{i=1}^{n} m_{i,\sigma(i)} = +\infty$. This corresponds to the fact that we cannot find any symmetric permutation matrix inferior to A although A is itself symmetric.

However , in the literature, the results concerning the existence of core of the pairing models involving two different types of agents are positive (marriage problem, college admission problem, job matching). We can prove that this result is general.

Definition

We will say that the pairing model involves two types of agents if there exist integer numbers a and b such that n = a + b and if the utility functions have the following property :

$\forall i \quad 1 < i < a \; , \; \forall j \quad 1 < j < a \quad i \neq j$

$\forall h \quad a+1 < h < n \; , \; \forall k \quad a+1 < k < n \quad k \neq h$

$\forall m \in \mathbb{R}_+$

$$u_i \, (m , e^j) < u_i \, (m , e^h)$$

$$u_h \, (m , e^k) < u_h \, (m , e^i)$$

This assumption means that agents of one type always prefer strictly to be paired with any agent of the other type than to be paired with an agent of his own type.

Theorem 2 : The pairing model with two types of agents has a non empty core.

Remark :

This theorem is an other form of the result of Kaneko [4] that the "Central Assignment Game" has a non empty core. It seems however interesting to give a proof of this result which is coherent with the logic of this paper.

Proof :

The steps of the proof are the same than in the proof of Theorem 1. The only difference is that the permutations $(\sigma_S)_{S \in B}$ verify $\sigma_S \circ \sigma_S = Id_S$ and that we have to prove that :

$$\min_{\substack{\sigma \in \Sigma_N \\ \sigma o \sigma = Id_N}} \sum m_{i,\sigma(i)}(v) \leq \sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}(v)$$

As explained in the first part of this section we can prove that this inequality holds if we can prove that the matrix A can be decomposed in a sum of symmetric permutation matrices.

The assumption made on preferences implies that the permutation σ_S can always be chosen such that :

$1 \leq i \leq p \quad \sigma_S(i) = i \quad$ or $\quad \sigma_S(i) \in [p+1, \dots, n]$

$p+1 \leq i \leq n \quad \sigma_S(i) = i \quad$ or $\quad \sigma_S(i) \in [1, \dots, p]$

The associated matrices B_S are thus of the form

$$\left[\begin{array}{c|c} D_S & {}^tM_S \\ \hline M_S & D'_S \end{array}\right]$$

where D_S is an $a \times a$ diagonal matrix, D'_S is a $b \times b$ diagonal matrix and M_S a $a \times b$ matrix.

$A = \sum_{S \in B} c_S B_S$ is also of this form.

Then the proof of Theorem 2 is given by the following lemma.

Lemma 3 : If a $n \times n$ matrix $A = (a_{ij})$ is such that the coefficients a_{ij} are integers and all rows and columns sum to a positive integer d, if A can be written as :

$$\begin{bmatrix} D & {}^tM \\ M & D' \end{bmatrix}$$

where D is a $a \times a$ diagonal matrix, D' a $b \times b$ diagonal matrix and M a $b \times a$ matrix, then there exists symmetric permutations $\sigma_1, \dots, \sigma_d$ of N such that :

$$A = A_{\sigma_1} + \dots + A_{\sigma_d}$$

Proof

The proof is by induction on d. If $d = 1$, A is itself a symmetric permutation matrix. Let us suppose that the property holds for d. Let A be a matrix with the properties of the Lemma whose lines and columns sum to $d + 1$. We know from Lemma 2 that there exists a permutation σ such that $A \geq A_\sigma$. If σ is not symmetric let us consider σ' defined by the following :

$$1 \leq i \leq p \qquad \sigma'(i) = \sigma(i)$$

$$p + 1 \leq i \leq n \quad \sigma'(i) = \sigma^{-1}(i)$$

We have to prove first that $\sigma' \circ \sigma' = Id_N$.

- If $i \in [1, \ldots, p]$ $\sigma' \circ \sigma'(i) = \sigma'(\sigma(i))$

As A_σ is inferior to A which has the form indicated in the Lemma, $i \in [1, \ldots, p] \Rightarrow \sigma(i) = i$ or $\sigma(i) \in [p + 1, \ldots, n]$

If $\sigma(i) = i$ $\sigma'(\sigma(i)) = \sigma'(i) = \sigma(i) = i$

If $\sigma(i) \in [p + 1, \ldots, n]$ $\sigma'(\sigma(i)) = \sigma^{-1}(\sigma(i)) = i$

- If $i \in [p + 1, \ldots, n]$ $\sigma' \circ \sigma'(i) = \sigma'(\sigma^{-1}(i))$

For the same reason, $i \in [p + 1, \ldots, n] \Rightarrow [\sigma^{-1}(i) = i$ or $\sigma^{-1}(i) \in [1, \ldots, p]]$

If $\sigma^{-1}(i) = i$ $\sigma'(\sigma^{-1}(i)) = \sigma'(i) = \sigma^{-1}(i) = i$

If $\sigma^{-1}(i) \in [1, \ldots, p]$ $\sigma'(\sigma^{-1}(i)) = \sigma(\sigma^{-1}(i)) = i$

To prove that $A_{\sigma'} < A$, let us remark that, from the construction of σ', the non zero coefficients of $A_{\sigma'}$ correspond to positive coefficients of A_σ or $A_{\sigma^{-1}}$. $A_{\sigma^{-1}}$ is the transposed matrix of A_σ. As A is symmetric, $A_\sigma < A$ implies $A_{\sigma^{-1}} < A$ and therefore $A_{\sigma'} < A$.

The induction property applied to $A - A_{\sigma'}$ gives the definitive result.

Theorem 2 gives an alternative proof of the result of Gale and Shapley [2] that the marriage problem has a stable solution.

Apparently, the college admission problem does not enter the framework of the pairing model since more than one student go to the same college. The problem is the following : "A set of n applicants is to be assigned among m colleges where q_i is the quota of the i-th college. Each applicant ranks the colleges in order of his preferences (...). Each college similarly ranks the students who have applied to it in order of preference (...). An assignment of applicants to colleges will be called unstable if there are two applicants α and β who are assigned to colleges A and B, respectively, although β prefers A to B and A prefers β to α."

The existence of a stable assignment can nevertheless be deduced from Theorem 2, by considering a pairing model with no endowments in money, where the m students are the agents of one type, and where there are $q_1 + q_2 + \dots + q_n$ agents of the other type, college i being replicated q_i times. Of course, the rank of two replica of the same college is the same in the preferences of the students, and two replicated colleges have the same ranking of the students. A core allocation of this model gives a stable assignment of students to colleges.

However, the constructive proof given by Gale and Shapley [2] of the existence of stable assignments for the college admission problem and the marriage problem is more interesting than the proof by balancedness since it gives a procedure to reach these stable assignments. The main interest here is more the comparison of the structures of the exchange models and the pairing models than the result of Theorem 2 itself.

A version of the college admission model with money is studied by Crawford and Knoer [1 [and Kelso and Crawford [5] with an interpretation in terms of job matching. One type of agents is composed by firms and the other by workers. The money enter in the model under the form of salaries given by firms to the workers. When it is assumed that each firm hire only a worker, the existence of a stable assignment can be deduced from Theorem 2, without any assumption on the utility functions.But when firms can hire more than one worker, it becomes difficult to use the pairing model to get results. The replication of the firms is not possible if the preferences of firms on the workers are not separable and if the firms have budget constraints (problem which is not taken into account in the two papers quoted above). Here again, the constructive proof based on a generalization of the Gale - Shapley algorithm is interesting.

These constructive proofs will appear even more appealing at the end of Section IV where it will be shown that, at the difference of the exchange model, the core allocations of a pairing model cannot in general be decentralized by means of competitive prices. Then other procedures have to be used.

IV. COMPETITIVE PRICES

In both models of exchange with indivisibilities studies by Shapley-Scarf [9] and Shapley-Shubik [10], the relation between the core allocations and the competitive equilibrium allocations was studied. The results were different from one paper to another.

For the model without money of Shapley-Scarf, it was proved tnat at least one core allocation can be decentralized as a competitive equilibrium allocation (and this implies the existence of a competitive equilibrium for the model). But Shapley and Scarf give an example where some core allocation cannot be decentralized by mean of prices. As noticed by the authors, the main reason for that comes from the definition of core allocations. Core allocations are defined as allocations which cannot be improved strictly by all members of a coalition. In a model without money but with only indivisible items, this implies that some core allocations are weak Pareto Optima but not Pareto Optima. This fact introduces complexities in the question of decentralization by prices of core allocations, problem which has been studied more completely by Roth and Postlewaite [6].

On the contrary, in the model of "assignment" between buyers and sellers with compensation in money presented by Shapley and Shubik, it was proved that all core allocations could be decentralized as competitive equilibrium allocations. This was proved in the case of linear utility of money in the original paper [10] , and in the case of non transferable utility by Kaneko [4] under assumptions which ensure that money enter really in the model.

We will prove in Theorem 3 that this result can be generalized with the same assumptions (assumption A.1. and A.2. below) to the general model of exchange. Before stating this Theorem, we define precisely what is a competitive equilibrium in our model.

<u>Definition</u>

Let $E = \{(\omega_i , e^i)_{1 \leq i \leq n} , (u_i)_{1 \leq i \leq n} , q\}$ be an exchange economy, where q is an integer belonging to $[1 , \ldots , n]$ such that : $i > q \Rightarrow e^i = 0$. A <u>price vector</u> for this economy is a vector $p = (p_1 , \ldots , p_n) \in \mathbb{R}^n$ such that :
$i > q \Rightarrow p_i = 0$.

A competitive equilibrium is a couple of a price vector $\bar{p}$ and a feasible allocation $(\bar{m}_i , e_i^{\sigma(i)})_{1 \leq i \leq n}$ such that

- $\bar{m}_i + \bar{p}_{\sigma(i)} \leq \omega_i + \bar{p}_i \qquad \forall\, i \in [1 , \ldots , n]$
- $[u_i (m_i , e_i^j) > u_i (\bar{m}_i , e_i^{\sigma(i)})] \Rightarrow [m_i + \bar{p}_j > \omega_i + \bar{p}_i]$
$\forall\, i \in [1 , \ldots , n]$

A competitive equilibrium allocation is an allocation $(\bar{m}_i , e_i^{\sigma(i)})_{i=1,\dots,n}$ for which one can find a price vector $\bar{p}$ such that $(\bar{p} , (\bar{m}_i , e_i^{\sigma) i)})_{1 \leq i \leq n})$ is a competitive equilibrium.

Theorem 3 :

Let $E = \{(\omega_i , e^i)_{1 \leq i \leq n} , (u_i)_{1 \leq i \leq n} , q\}$ be an exchange economy. Under the following assumptions :

A.1. the functions u_i are increasing with respect to the money

A.2. $u_i \ (\omega_i , e^i) > u^i \ (0 , e^j)$ for every i and j in $[1 , \dots , n]$

the set of core allocations and the set of competitive equilibrium allocations of E coincide.

Assumption A.2. can be justified as in Kaneko [4] . It is argued in this paper that a model with two goods must be considered as a partial analysis model where the money is a composite good of all other commodities which are not considered explicitly in the model. Then it is not "normal" for an agent to enjoy an indivisible item (a house for example) but to consume nothing else.

Proof of Theorem 3 *

- The usual reasoning can be applied to prove that a competitive equilibrium allocation is in the core. We are interested in the proof of the inverse implication.

- Let $(\bar{m}_i , e_i^{\sigma(i)})_{1 \leq i \leq n}$ be a core allocation.

Let us define for all i and j in [1 , ... , n] the quantity m_{ij} by

$$m_{ij} = \min \{m \in \mathbb{R}_+ \mid u_i (m , e_i^j) \geq u_i (\bar{m}_i , e_i^{\sigma(i)})\}$$

In the notation os Section II, $m_{ij} = m_{ij}(\bar{v})$, where $\bar{v}$ is defined by :

$$\bar{v}_i = u_i (\bar{m}_i , e_i^{\sigma(i)}).$$

If $j = \sigma(i)$, $m_{i,\sigma(i)} = \bar{m}_i$.

Assumption A.2. implies that, if $m_{ij} < + \infty$, then

$$u_i (m_{ij} , e_i^j) = \bar{v}_i.$$

To prove that $p = (p_1 , \ldots , p_n)$ is an equilibrium price associated with the allocation $(\bar{m}_i , e_i^{\sigma(i)})$, it is enough, from Assumption A.1., to prove that :

---

* This proof is due in large part to David Gale who introduced me to the notion of shortest path and allowed me, by his suggestions, to simplify considerably my original proof.

1) $\bar{m}_i + p_{\sigma(i)} = \omega_i + p_i \quad \forall\, i \in [1, \dots, n]$

2) $m_{ij} + p_j \geq \omega_i + p_i \quad \forall\, i \in [1, \dots, n] \quad \forall\, j \in [1, \dots, n]$

3) $p_i = 0$ if $i > q$

To find a price vector which satisfies these conditions, let us consider the directed graph with nodes $1, \dots, n$ and such that the "transportation cost" or "length" from i to j is $m_{ji} - \omega_j$ (see Figure 1). Let us denote ℓ_{ij} this length.

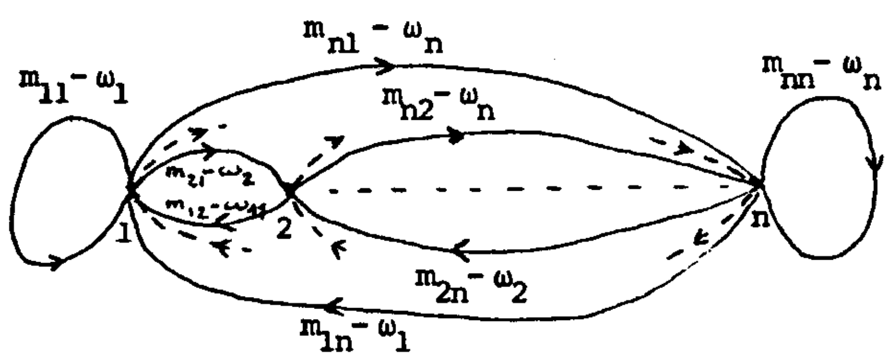

Figure 1

A path from i to j is a sequence $(i, i_1, i_2, \dots, i_m, j)$

The length of the path is then $\ell_{ii_1} + \ell_{i_1 i_2} + \dots + \ell_{i_m j}$.

Let us denote Π_{ij} the set of all paths from i to j.

Let us choose p_n arbitrary, and let us take $p_i - p_n$ as the length of the shortest path from n to i .

It is well known in graph theory that the minimization problem $\min_{\Pi_{ij}} \ell_{ii_1} + \ell_{i_1 i_2} + \dots + \ell_{i_m j}$, referred to, in the literature, as the Bellman's equations has a solution if and only if there is no cycle of negative length. Let us prove that this is case here.

Suppose that there exists a cycle $(i , i_1 , \dots , i_m , i)$ such that $\ell_{ii_1} + \dots + \ell_{i_m i} = m_{i_1 i} - \omega_{i_1} + \dots + m_{ii_m} - \omega_i < 0.$

This is equivalent to $\omega_{i_1} + \dots + \omega_i > m_{i_1 i} + \dots + m_{ii_m}$.

This means that the coalition $\{i_1 , \dots , i_m , i\}$ can ensure the utility level $\bar{v}$ to its members by giving to i_1 the item of i and the amount of money $m_{i_1 i}$, to i_2 the item of i_1 and the amount $m_{i_2 i_1}$ of money, ..., to i the item of i_m and the amount m_{ii_m} of money. This coalition will moreover have a positive surplus of money, which can be used to increase strictly the utility of each of its members. Then this coalition block the allocation $(\bar{m}_i , \ell_i^{\sigma(i)})_{1 \leq i \leq n}$, which is impossible since the allocation is in the core.

Therefore $p_i - p_n = \min_{\Pi_{ni}} \ell_{ni_1} + \dots + \ell_{i_m i}$ defines p_i without ambiguity.

Let us prove that p_i so defined satisfies (1) and (2).

Suppose that $\exists\, i \in [1, \ldots, n]$ $\exists\, j \in [1, \ldots, n]$ such that :

$$m_{ij} + p_j < \omega_i + p_i$$

This is equivalent to : $p_i > p_j + m_{ij} - \omega_i$, or to : $p_i - p_n > p_j - p_n + \ell_{ji}$, which contradicts the definition of p_i. Then (2) holds. To prove (1) let us consider the following inequalities :

$$\bar{m}_i + p_{\sigma(i)} \geqslant \omega_i + p_i$$

$$\bar{m}_{\sigma(i)} + p_{\sigma^2(i)} \geqslant \omega_{\sigma(i)} + p_{\sigma(i)}$$

$$\bar{m}_{\sigma^2(i)} + p_{\sigma^3(i)} \geqslant \omega_{\sigma^2(i)} + p_{\sigma^2(i)}$$

$$\vdots$$

As σ is a permutation of a finite set $[1, \ldots, n]$ there exists, for each i, a minimum number $\lambda_i \geqslant 0$ such that $\sigma^{\lambda_i}(i) = i$.

The coalition $S^i = \{i, \sigma(i), \ldots, \sigma^{\lambda_i - 1}(i)\}$ is a "trading cycle" for the core allocation that we consider, in the sense that the exchange of indivisible items takes place inside this coalition S^i. The item of $\sigma^{\lambda_{i_1} - 1}(i)$ goes to $\sigma^{\lambda_i - 2}(i)$, ... , the item of $\sigma(i)$ goes to i , and the item of i to $\sigma^{\lambda_i - 1}(i)$. Then the core allocation must be such that $\sum_{j \in S^i} \bar{m}_j = \sum_{j \in S^i} \omega_j$ since if $\sum_{j \in S^i} \bar{m}_j < \sum_{j \in S^i} \omega_j$, S^i would block the allocation, and if $\sum_{j \in S^i} \bar{m}_j > \sum_{j \in S^i} \omega_j$, $N \setminus S$ would block the allocation.

Therefore, adding the inequalities :

$$\bar{m}_i + p_{\sigma(i)} \geq \omega_i + p_i$$

$$\bar{m}_{\sigma(i)} + p_{\sigma^2(i)} \geq \omega_{\sigma(i)} + p_{\sigma(i)}$$

$$\vdots$$

$$\bar{m}_{\sigma^{\lambda_i - 1}(i)} + p_i \geq \omega_{\sigma^{\lambda_i - 1}(i)} + p_{\sigma^{\lambda_i - 1}(i)}$$

we must obtain an equality, which is possible only if all inequalities are equalities. This proves (1).

If $q = n$, whatever the choice of p_n, the price p defined above is an equilibrium prive.

If $q < n$, we have to deal with condition (3). This condition imposes to choose $p_n = 0$ and we must prove that $p_{q+1}, \ldots, p_{n-1}$ are then equal to 0.

Let i be an index such that $i \in [q + 1, \ldots, n - 1]$. There exists $k \in [1, \ldots, n]$ such that $\sigma(k) = i$. We must have, from (1) :

$$\bar{m}_k + p_i = \omega_k + p_k \qquad (\text{since } \bar{m}_k = m_{ki})$$

Since both i and n have no house as initial resources, $m_{kn} = m_{ki} = \bar{m}_k$ and since $p_n = 0$, we have, from (2) :

$$\bar{m}_k = m_{kn} \geq \omega_k + p_k$$

which implies, with the above equality, that $p_i = 0$.

Remark 1 :

The proof of Theorem 3 shows -and this can be seen directly- that, if $q = n$, if p is an equilibrium price associated with an allocation $(m_i , e_i^{\sigma(i)})_{1<i<n}$, then for every $a \in \mathbb{R}$, $p + a$ is also an equilibrium price. (Only the differences $p_i - p_n$ are significant). Then, in this case, the equilibrium prices can be chosen to be positive.

On the contrary, if $q < n$, the condition $p_i = 0$ if $i \quad q$ imposes a normalization. Then, to ensure that equilibrium prices are non negative, we should have a "desirability" condition on the indivisible items, for example the assumption

$$\text{(A.3.)} \quad \forall m > 0 \quad \forall i \in [1 , \dots , n] \quad \forall j \in [1 , \dots , q]$$
$$u_i \ (m , e^j) > u_i \ (m , 0)$$

Remark 2 :

The reason for which Assumptions A.1. and A.2. allows to get rid of the problems encountered by Shapley - Scarf in the model of exchange without money is that they ensure that the core and the strong core of E coincide. Let us define precisely the terms.

Let V be a non-sidepayment game with n players.

An imputation $v = (v_1 , \dots , v_n)$ is strongly dominated by a coalition S if there exists an imputation $v' \in V(S)$ such that $v'_i > v_i \quad \forall i \in S$.

An imputation v is in the core of V if

- $v \in V(N)$ (feasibility)
- v is not strongly dominated by any coalition S.

An imputation v is <u>weakly dominated</u> by a coalition S if there exists an imputation $v' \in V(S)$ such that $v'_i \geq v_i$ $\forall\ i \in S$, with at least a strict inequality.

An imputation v is in the <u>strong core</u> if :

- $v \in V(N)$
- v is not weakly dominated by any coalition S.

If the game comes from an economy, it is straightforward to adapt these definitions to the core and the strong core of the economy.

<u>Lemma 4</u> : <u>If an economy</u> <u>$E = \{(\omega_i, e^i)_{1 \leq i \leq n}, (u_i)_{1 \leq i \leq n}, q\}$</u> <u>satisfies assumptions A.1. and A.2., the core and the strong core of the economy E coincide.</u>

<u>Proof</u>

The strong core is always included in the core. We have only to prove the inverse inclusion.

Let $(\bar{m}_i , e_i^{\sigma(i)})_{1 \leq i \leq n}$ be a core allocation. Suppose it is weakly blocked by a coalition S. Then there exists an allocation $(m'_i , e_i^{\sigma_S(i)})_{1 \leq i \leq n}$ such that :

- $\sigma_S \in \Sigma_S$

- $\sum_{i \in S} m'_i \leq \sum_{i \in S} \omega_i$

- $u_i \ (m'_i , e_i^{\sigma_S(i)}) \geq u_i(\bar{m}_i , e_i^{\sigma(i)}) \quad \forall \ i \in S$, with at least one strict inequality for an agent $i_o \in S$.

A core allocation being individually rational we have :

$$u_i \ (m'_{i_o} , e_{i_o}^{\sigma_S(i_o)}) > u_i \ (\bar{m}_{i_o} , e_{i_o}^{\sigma(i_o)}) \geq u_i \ (\omega_{i_o} , e^{i_o})$$

From assumption A.2., this implies $m'_{i_o} > 0$.

From continuity of u_i with respect to money, there exists $\varepsilon > 0$ such that :

$$u_i \ (m'_{i_o} - \varepsilon , e_{i_o}^{\sigma_S(i_o)}) > u_i \ (\bar{m}_{i_o} , e_{i_o}^{\sigma(i_o)}).$$

Distributing ε between the other members of S, we will replace, from assumption A.1., all inequalities by strict inequalities.

Therefore a core allocation cannot be weakly blocked. This proves Lemma 3.

The property of the exchange model that all core allocations can be decentralized by mean of prices if we introduce money, is not true for the pairing model with two types, for which the core is non empty. To illustrate this, let us consider the following example of a pairing model with money, which has no competitive equilibrium at all.

There are two "men" α_1 and α_2 and two "women" A_1 and A_2. Each person owns initially one unit of money. To build the preference relation, let us consider the following ranking matrix :

$$\begin{array}{cc} & \begin{array}{cc} A_1 & A_2 \end{array} \\ \begin{array}{c} \alpha_1 \\ \\ \alpha_2 \end{array} & \begin{bmatrix} 1\,,\,2 & 2\,,\,1 \\ & \\ 2\,,\,1 & 1\,,\,2 \end{bmatrix} \end{array}$$

The first number in each couple in the matrix gives the ranking of women by the men, the second number the ranking of the men by the women.

The utility functions are defined by :

$$u_{A_i}\ (m\ ,\ \alpha_j) = \frac{1}{c_{A_i \alpha_j}}\ m$$

where $c_{A_i \alpha_j}$ is the rank of α_j in the ordering of A_i given by the matrix .

$$u_{A_i}(m, A_j) = \frac{m}{4}$$

In the same way :

$$u_{\alpha_i}(m, A_j) = \frac{1}{c_{\alpha_i A_j}} m$$

$$u_{\alpha_i}(m, \alpha_j) = \frac{m}{4}$$

These utility functions are such that :

- for a given amount of money, the ranking of one person on the possible partners of the other sex is the one given in the matrix ;

- for a given amount of money, a person always prefer to be paired with a person of the other sex than to stay alone or to be paired with a person of the same sex ;

- assumptions A.1. and A.2. are fulfilled.

It is clear from this properties, that core allocations must be associated with one of the pairings (α_1, A_1) (α_2, A_2) or (α_1, A_2) (α_2, A_1).

Let $(p_{A_1}, p_{A_2}, p_{\alpha_1}, p_{\alpha_2})$ be prices attached to each person. As a competitive equilibrium allocation is in the core, these prices, if they are competitive prices, must decentralize an allocation of the form :

$$(m_{\alpha_1}, A_1) \ (m_{A_1}, \alpha_1) \quad \text{for the pair } (\alpha_1, A_1)$$

$$(m_{\alpha_2}, A_2) \ (m_{A_2}, \alpha_2) \quad \text{for the pair } (\alpha_2, A_2)$$

or

$$(m_{\alpha_1}, A_2) \ (m_{A_2}, \alpha_1) \quad \text{for the pair } (\alpha_1, A_2)$$

$$(m_{\alpha_2}, A_1) \ (m_{A_1}, \alpha_2) \quad \text{for the pair } (\alpha_2, A_1)$$

In the first case we must have :

$$m_{\alpha_1} + p_{A_1} = 1 + p_{\alpha_1}$$

$$m_{A_1} + p_{\alpha_1} = 1 + p_{A_1}$$

$$m_{\alpha_2} + p_{A_2} = 1 + p_{\alpha_2}$$

$$m_{A_2} + p_{\alpha_2} = 1 + p_{A_2}$$

If $p_{\alpha_1} < p_{\alpha_2}$, then (m_{A_2}, α_1) would satisfy the budget constraint of A_1 and is preferred by A_2 to (m_{A_2}, α_2).

If $p_{\alpha_2} < p_{\alpha_1}$, (m_{A_1}, α_2) would satisfy the budget constraint of A_2 and is preferred by A_1 to (m_{A_1}, α_1)

Then $(p_{\alpha_1}, p_{\alpha_2}, p_{A_1}, p_{A_2})$ cannot decentralize any allocation associated with the pairs (α_1, A_1) (α_2, A_2). Changing the role of men and women the same reasoning proves that these prices cannot decentralize neither an allocation associated with the pairs (α_1, A_2) (α_2, A_1). Then there is no competitive equilibrium.

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