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Inefficiency of Nash Equilibria in a Private Goods Economy

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1. Introduction

In this paper we consider the following question: to what extent are the Nash equilibria (N.E.) of a private goods economy (given in the form of a smooth strategic market game) efficient? Suppose there are n players (traders) and that the j^{th} player has a strategy set S_j and an outcome space Y_j, where Y_j is a smooth manifold. A market mechanism is a set of n maps

$$\phi_j : S \to Y_j \quad (j = 1, \ldots, n)$$

where $S = S_1 \times \ldots \times S_n$. $\phi_j(\bar{s})$ is the j^{th} player's outcome when the strategies $\bar{s} = (s_1, \ldots, s_n)$ are chosen. Utility functions are defined on the outcome spaces Y_j. When the Y_j and ϕ_j are the same for all j, this may be viewed as a public goods economy (and has been discussed in [3]). The focus of the present paper is on the private goods case, when the ϕ_j and Y_j are distinct.

In Sections 2 and 3, we assume that the Y_j are the spaces of allocations of privately owned commodities. Thus each $Y_j = \mathbb{R}_+^{\ell}$ where ℓ is the number of commodities and the market mechanism $\phi = (\phi_1, \ldots, \phi_n)$

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defines reallocations of the initial endowments. A basic assumption on the ϕ_j (Assumption 1 of Section 1) is that each player j can, by changing his own strategy, span a submanifold of Y_j of codimension one when all other players remain fixed. This is analogous to the "budget plane" of Walrasian analysis and was also made in [2]. We show (2.2) that there is a disjoint decomposition $S = S_{UO} \cup S_{UI}$, which depends only on the market mechanism ϕ, such that: for any choice of utilities (satisfying some standard assumptions) if a Nash Equilibrium lies in S_{UO} it is efficient, and it is inefficient if it lies in S_{UI}. We therefore nickname S_{UO} and S_{UI} the set of ultra-optimal and ultra-inoptimal points. The analysis of the efficiency of Nash Equilibria is reduced to that of S_{UO}. Typically, S_{UO} is contained in a finite union of submanifolds of codimension at least $t > 0$ in S and then, as is shown in Section 2, the set of efficient N.E. is contained in a finite union of submanifolds of codimension at least t in the set of all N.E.

In fact, two notions of efficiency ("economic" and "game-theoretic," see Section 2) are considered. The conclusions above also carry over to game-theoretic efficiency and here it is no longer necessary to assume that the range of ϕ is the space of reallocations. Depending again only on ϕ, there is a decomposition $S = SG_{UO} \cup SG_{UI}$ (5.1). For any choice of utilities, all efficient N.E. lie in SG_{UO} and all N.E. which lie in SG_{UI} are inefficient (but now there may exist N.E. in SG_{UO} which are also inefficient). SG_{UO} typically has positive codimension in S and when this is the case, the set of (game-theoretically) efficient N.E. has the same codimension in the set of all N.E.

In Sections 4 and 5, we apply the above results to the Shapley-Shubik models (bid-offer and sell-all). In both cases we find that a Nash equilibrium can be economically efficient only if it leads to the outcome which leaves each player with his initial endowment and can be game-theoretically efficient only if at least one player is left with his initial endowment. Such Nash equilibria can exist robustly (in utilities) only for the bid-offer model. This clarifies the inefficiency result for the sell-all model in [1], which is a precursor to the present paper, and throws light on the nature of efficient Nash outcomes.

Throughout we deal only with "interior" Nash Equilibria. This is largely a matter of technical convenience and the method of this paper can be applied when the S_j have "nice" boundaries, e.g. are simplices, and when ϕ is defined smoothly in a neighborhood of S. Here we restrict outselves, when the S_j are simplices, to Nash Equilibria occurring in the interior. See [1] and [3] for treatment of similar questions without this restriction.

2. Ultra-Optimal and Ultra-Inoptimal Points

We begin with the following set-up: a private goods economy in strategic game form. Assume that there are n players and that associated to each player are:

(i) a strategy set S_j

(ii) an outcome space Y_j

(iii) a map $\phi_j : S_1 \times \dots \times S_n \to Y_j$.

Let $S = S_1 \times \dots \times S_n$. A choice of strategies $\bar{s} = (s_1, \dots, s_n) \in S$ determines the outcome $\phi_j(\bar{s}) \in Y_j$ for the j^{th} player.

We assume throughout that Y_j is a smooth manifold and that the utility function u_j of the j^{th} player is a C^2-function on Y_j . A choice of utilities $\bar{u} = (u_1, \dots, u_n)$ together with the maps ϕ_j , defines the strategic market game. Let $Y = Y_1 \times \ldots \times Y_n$ and $\phi : S \to Y$ be the map $\phi_1 \times \ldots \times \phi_n$.

Generally, if M is a manifold of dimension d , we will use the following notation:

a) $T_m(M)$ is the tangent space to M at m (it is a vector space of dimension d)

b) $T_m^*(M)$ is the cotangent space to M at m .

Recall that the cotangent space $T_m^*(M)$ is, by definition, the dual vector space to $T_m(M)$. For $v \in T_m(M)$ and $w \in T_m^*(M)$, we denote the value of w at v by $w \cdot v \in \mathbb{R}$.

For this section, there is no need to impose any differentiable structure on the S_j or ϕ_j . <u>We consider market mechanisms</u> ϕ <u>which satisfy the following assumption</u>:

<u>Assumption 1</u>: If all but the j^{th} player fix their strategies, then the j^{th} player can, by changing his own strategy, span a submanifold of Y_j of codimension one. (A submanifold of codimension one is called a hypersurface.)

Let $\bar{s} = (s_1, \ldots, s_n) \in S$. Let $Y_j(\bar{s})$ denote the hypersurface in Y_j that the j^{th} player can span when the k^{th} player remains fixed at s_k for all $k \neq j$. We will call $Y_j(\bar{s})$ the j^{th} player's holding hypersurface at $\bar{s}$. For $y \in Y_j(\bar{s})$, the tangent space $T_y(Y_j(\bar{s}))$ is a codimension one linear subspace of $T_y(Y_j)$. Set

$$V_j(\bar{s}) = \{w \in T^*_{\phi_j(\bar{s})} : w \cdot v = 0 \text{ for all } v \in T_{\phi_j(\bar{s})}(Y_j(\bar{s}))\} .$$

Thus $V_j(\bar{s})$ is the one-dimensional space of linear functions on $T_{\phi_j(\bar{s})}(Y_j)$ which vanish on $T_{\phi_j(\bar{s})}(Y_j(\bar{s}))$.

Let ∇u_j be the gradient of u_j, that is, the gradient of derivatives of u_j with respect to outcome variables on Y_j. For $y \in Y_j$, $\nabla u_j(y) \in T_y^*(Y_j)$ and $\nabla u_j(Y)\cdot v$ is the directional derivative of u_j in the direction v.

A choice of strategies $\bar{s} = (s_1, \ldots, s_n) \in S$ is called a <u>Nash Equilibrium</u> for the utility functions $\bar{u} = (u_1, \ldots, u_n)$ if for all j,

$$u_j(\phi_j(s_1, \ldots, s_n)) \geq u_j(\phi_j(s_1, \ldots, s_{j-1}, t, s_{j+1}, \ldots, s_n))$$
$$\text{for all } t \in S_j .$$

Let $N(\bar{u})$ denote the subset of S of Nash equilibria for the utilities $\bar{u}$.

<u>Lemma 2.1</u>: Let $\bar{s} \in N(\bar{u})$. Then

$$\nabla u_j(\phi_j(\bar{s})) \in V_j(\bar{s})$$

for all $j = 1, \ldots, n$.

<u>Proof</u>: Suppose that for some j and $\bar{s} \in S$, $\nabla u_j(\phi_j(\bar{s})) \notin V_j(\bar{s})$. Then there is a vector $v \in T_{\phi_j(\bar{s})}(Y_j(\bar{s}))$ such that $\nabla u_j(\phi_j(\bar{s}))\cdot v > 0$, and v defines a direction in $Y_j(\bar{s})$ along which u_j is increasing. Player j can move in this direction by changing only his own strategy, hence $\bar{s} \notin N(\bar{u})$.

<u>Remark</u>: Suppose that the $Y_j(\bar{s})$ are contained in Euclidean space $\mathbb{R}^m$ and the sets $\{x \in R^m : x \leq y \text{ for some } y \text{ in } Y_j(\bar{s})\}$ are convex. Then if utilities are concave and non-decreasing in each variable, any local maximum on $Y_j(\bar{s})$ will be a global maximum and the condition of Lemma 2.1 in this case is also sufficient for $\bar{s}$ *to be in* $N(\bar{u})$.

Definition: Given utilities $\bar{u} = (u_1, \ldots, u_n)$, a strategy choice $\bar{s} \in S$ is called:

a) economically efficient if there does not exist a reallocation $y = (y_1, \ldots, y_n) \in Y$ such that

$$u_j(y_j) \geq u_j(\phi_j(\bar{s})) \quad \text{for all} \quad j = 1, \ldots, n$$

with strict inequality for some j .

b) game-theoretically efficient if there does not exist a strategy choice $\bar{s}' \in S$ such that

$$u_j(\phi_j(\bar{s}')) \geq u_j(\phi_j(\bar{s})) \quad \text{for all} \quad j = 1, \ldots, n$$

with strict inequality for some j .

Definitions a) and b) differ only when the map $\phi : S \to Y$ is not onto. Let $E(\bar{u})$ and $E_G(\bar{u})$ be the sets of economically efficient and game-theoretically efficient strategies respectively. Then $E(\bar{u}) \subseteq E_G(\bar{u})$. Set

$$EN(\bar{u}) = E(\bar{u}) \cap N(\bar{u})$$

$$EN_G(\bar{u}) = E_G(\bar{u}) \cap N(\bar{u}) \ .$$

For the rest of this section, we consider a market game where the Y_j represent the spaces of final holdings of the players. Let $\mathbb{R}^{\ell}_{++} = \{y = (y^1, \ldots, y^{\ell}) \in \mathbb{R}^{\ell} : y^i > 0 \text{ for all } i\}$ and assume that $Y_j = \mathbb{R}^{\ell}_{++}$ for all i . For each player j , let $a_j = (a_j^1, \ldots, a_j^{\ell}) \in Y_j$ be the "initial endowment" of player j and set

$$Y_0 = \{(y_1, \ldots, y_n) \in Y_1 \times \ldots \times Y_n : \sum_{j=1}^{n} y_j = \sum_{j=1}^{n} a_j\} \ .$$

Y_0 represents the space of reallocations of the ℓ commodities such that each player holds a positive amount in each commodity. Assume that ϕ maps S to $Y_0 \subset Y_1 \times \ldots \times Y_n$, i.e.,

$$\sum_{j=1}^{n} \phi_j(\bar{s}) = \sum_{j=1}^{n} a_j$$

for all $\bar{s} \in S$.

Definition: A strategy $\bar{s} \in S$ is called

a) Ultra-optimal if the one-dimemsional subspaces $V_j(\bar{s})$ coincide for all j.

b) Ultra-inoptimal if for some pair j and k, $V_j(\bar{s}) \neq V_k(\bar{s})$.

Let S_{UO} and S_{UI} denote the subsets of ultra-optimal and ultra-inoptimal points of S. It is clear that $S = S_{UO} \cup S_{UI}$ and $S_{UO} \cap S_{UI} = \phi$.

Let U be the space of C^2-functions u on $\mathbb{R}^\ell$ such that:

a) u is strictly concave: $u(tP + (1-t)Q) > tu(P) + (1-t)u(Q)$ for all $0 < t < 1$ and $P, Q \in \mathbb{R}^\ell$.

b) $\nabla u(y) = (\partial u(y)/\partial x_1, \ldots, \partial u(y)/\partial x_\ell)$ is a vector with strictly positive components for all $y \in \mathbb{R}^\ell$.

Proposition 2.2: Let $\phi : S \to Y_0$ be a market mechanism satisfying Assumption 1. Then the decomposition $S = S_{UO} \cup S_{UI}$ has the following property: for all $\bar{u} = (u_1, \ldots, u_n) \in U^n$

$$EN(\bar{u}) = S_{UO} \cap N(\bar{u}) .$$

In other words, if $\bar{s} \in N(\bar{u}) \cap S_{UO}$, then $\bar{s}$ is economically efficient and if $\bar{s} \in N(\bar{u}) \cap S_{UI}$, then $\bar{s}$ is economically inefficient.

Proof: Let $\bar{s} \in N(\bar{u}) \cap S_{UO}$. By Lemma 1.1, $\nabla u_j(\phi_j(\bar{s})) \in V_j(\bar{s})$ for all j and since the $V_j(\bar{s})$ all coincide, there is a single vector v with positive components such that

$$\nabla u_j(\phi_j(\bar{s})) = \lambda_j v$$

for some positive number λ_j , for all j . If $\bar{s}$ were not economically efficient there would exist vectors $x_1, \ldots, x_n \in \mathbb{R}^\ell$ such that $\sum_{j=1}^{n} x_j = 0$, $\phi_j(\bar{s}) + x_j \in Y_j$, and such that

$$u_j(\phi_j(\bar{s}) + x_j) \geq u_j(\phi_j(\bar{s})) \quad \text{for all } j$$

with strict inequality for some j . Since the u_j are strictly concave

$$u_j(\phi_j(\bar{s}) + tx_j) > u_j(\phi_j(\bar{s}))$$

for all $0 < t < 1$ with strict inequality for some j . Hence $\nabla u_j(\phi_j(\bar{s}))\cdot x_j \geq 0$ for all j with strict inequality for some j . Since $\nabla u_j(\phi_j(\bar{s})) = \lambda_j v$, we have $v\cdot x_j \geq 0$ for all j with strict inequality for some j and this contradicts the assumption $\sum_{j=1}^{n} x_j = 0$.

Now suppose that $\bar{s} \in S_{UI} \cap N(\bar{u})$. Then there is a pair j and k such that $v_j(\bar{s}) \neq v_k(\bar{s})$, and since $\nabla u_j(\phi_j(\bar{s})) \in V_j(\bar{s})$ and $\nabla u_k(\phi_k(\bar{s})) \in V_k(\bar{s})$, there is a vector $x \in \mathbb{R}^\ell$ such that

$$\nabla u_j(\phi_j(\bar{s}))\cdot x > 0$$

$$\nabla u_k(\phi_k(\bar{s}))\cdot(-x) > 0 .$$

Hence for t sufficiently small and positive, $\phi_j(\bar{s}) + tx \in Y_j$, $\phi_k(\bar{s}) - tx \in Y_k$, and:

$$u_j(\phi_j(\bar{s}) + tx) > u_j(\phi_j(\bar{s}))$$

$$u_k(\phi_k(\bar{s}) - tx) > u_k(\phi_k(\bar{s}))$$

and the reallocation assigning $u_i(\phi_i(\bar{s}))$ to player i for $i \neq j, k$ and $\phi_j(\bar{s}) + tx$ (resp. $\phi_k(\bar{s}) - tx$) to player j (resp. k) shows that $\bar{s}$ is not economically efficient.

3. Inefficiency of Nash Equilibria

In this section we retain the set-up of Proposition 1.2 and assume in addition that the S_j are smooth manifolds and that the ϕ_j are smooth maps. When the initial endowment vectors a_j are fixed, the set of possible reallocations to the j^{th} player is bounded in $\mathbb{R}^{\ell}_{++}$. In the next proposition, we take the space of utility functions U to be as in Section 1 except that now we must also require that all $u \in U$ be C^{r+2}-functions where $r = \dim S - n(\ell-1)$. The topology on U will be as follows. Choose a compact set $C \in \mathbb{R}^{\ell}$ which contains the set of all possible reallocations and let

$$||u|| = \sup_{D} \sup_{y \in C} |Du(y)|$$

where Du ranges over all derivatives of u of order $0, 1, \ldots, r+2$. The compact set C exists but will depend on the initial endowment vectors. The norm $||u||$ makes U into a Banach space.

Proposition 3.1: Let $\phi : S \to Y_0$ be a market mechanism as in Proposition 2.2. Then there exists a dense set (open dense if the S_j are compact) $U_0 \subset U^n$ such that for all $\bar{u} = (u_1, \ldots, u_n) \in U_0$:

1) $N(\bar{u})$ is contained in a submanifold of codimension $n(\ell-1)$ in S .

2) If S_{UO} is contained in a union of submanifolds of S of codimension at least t , then $EN(\bar{u})$ is contained in a union of submanifolds of codimension at least $n(\ell-1) + t$ (the submanifolds containing S_{UO} must be compact if we want to take U_0 open dense).

Proof: Let $U_1 = C^{r+2}(C)$ and let ψ be the map

$$U_1^n \times S \xrightarrow{\psi} S \times \mathrm{Mat}(n,\ell)$$

$$(\bar{u},\bar{s}) \longmapsto \left(\bar{s}, \begin{bmatrix} \nabla u_1 \\ \vdots \\ \nabla u_n \end{bmatrix}\right)$$

where $\mathrm{Mat}(n,\ell)$ denotes the set of $n \times \ell$ matrices. For $A \in \mathrm{Mat}(n,\ell)$, $A_1, \ldots, A_n$ will denote the rows of A . Let

$$N = \{(\bar{s},A) \in S \;\; \mathrm{Mat}(n,\ell) : A_j \in V_j(\bar{s}) \;\; \text{for} \;\; j = 1, \ldots, n\} .$$

Then N is a submanifold of $S \times \mathrm{Mat}(n,\ell)$ of codimension $n(\ell-1)$. It is clear that ψ is transverse to all submanifolds of $S \times \mathrm{Mat}(n,\ell)$ and that for fixed $\bar{u} \in U$, $N(\bar{u}) \subset \{\bar{s} \in S : \psi(\bar{u},\bar{s}) \in N\}$. Conclusion 1) follows from the transversal density theorem since U is open in U_1 . If $S_{UO} \subset M_1 \cup \ldots \cup M_r$, where the M_j are submanifolds of S of codimension at least t , set

$$EN = \{(\bar{s},A) \in (\bigcup_{j=1}^{r} M_j) \times Mat(n,\ell) : A_j \in V_j(\bar{s}) \quad \text{for} \quad j = 1, \ldots, n\} .$$

Then $EN(\bar{u}) \subset \{\bar{s} \in S_{UO} : \psi(\bar{s},\bar{u}) \in EN\}$ and conclusion 2) also follows from the transversal density theorem.

Remark: For $\bar{s} \in S$, the condition that all $V_j(\bar{s})$ coincide is defined "in general" by $(n-1)(\ell-1)$ equations. It seems likely that if the S_j are bounded open subsets of Euclidean space, then a rigorous argument could be given to show that the codimension of S_{UO} in S is $(n-1)(\ell-1)$ for a generic class of market mechanisms. We indicate briefly how this might be done. Let Ω be the set of smooth maps ϕ from S to Y_0 which satisfy Assumption 1. Then $\phi = \phi_1 \times \ldots \times \phi_n$ where $\phi_j : S \to Y_j$ and for each $\bar{s} \in S$, $V_j(\bar{s})$ defines a line in $\mathbb{R}^{\ell}$. Let $\mathbb{P}^{\ell-1}$ denote the projective space of all lines in $\mathbb{R}^{\ell}$. It is a compact manifold of dimension $(\ell-1)$ whose points correspond to lines in $\mathbb{R}^{\ell}$. Let ψ be the map

$$\psi : \Omega \times S \longrightarrow S \times (\mathbb{P}^{\ell-1})^n$$

$$(\phi,\bar{s}) \longrightarrow (s, V_1(\bar{s}), \ldots, V_n(\bar{s})) .$$

and let $UO = \{(\bar{s}, V_1, \ldots, V_n) \in S \times (\mathbb{P}^{\ell-1})^n : V_1 = V_2 = \ldots = V_n\}$. Then for all $\phi \in \Omega$,

$$S_{UO}(\phi) = \{\bar{s} \in S : \psi(\phi,\bar{s}) \in UO\}$$

where $S_{UO}(\phi)$ denotes the set of ultra-optimal points in S with respect to the market mechanism ϕ . It can probably be shown that, with respect to suitable topologies, there is a dense open set $\Omega_0 \subset \Omega$ such that Ω_0

is a Banach manifold. One would then show that ψ restricted to $\Omega_0 \times S$ is transverse to the submanifold UO of $S \times (\mathbb{P}^{\ell-1})^n$. Since the codimension of UO in $S \times (\mathbb{P}^{\ell-1})^n$ is $(n-1)(\ell-1)$, the desired conclusion would follow from the transversal density theorem.

4. The Shapley-Shubik Model

We now examine the results of Sections 1 and 2 in a special case: the Shapley-Shubik bid-offer model. There are n players and ℓ commodities, where the ℓ^{th} commodity is treated as money. The initial endowment vectors $a_j = (a_j^1, \ldots, a_j^\ell) \in Y_j = \mathbb{R}_{++}^\ell$ are given and the j^{th} player's strategies consist of a bid b_j^i of money to purchase commodity i and an offer to sell a quantity q_j^i of commodity i $(1 \leq i \leq \ell-1)$. Player j's strategy is represented by two vectors

$$b_j = (b_j^1, \ldots, b_j^{\ell-1}) \ , \quad q_j = (q_j^1, \ldots, q_j^{\ell-1})$$

and

$$S_j = \{(b_1, q_j) \in \mathbb{R}_{++}^{\ell-1} \times \mathbb{R}_{++}^{\ell-1} : q_j^i < a_j^i \text{ and } \sum_{i=1}^{\ell-1} b_j^i < a_j\} \ .$$

S_j is an open set in $\mathbb{R}^{2(\ell-1)}$ and $\dim S = 2n(\ell-1)$. Let

$$B^i = \sum_{j=1}^n b_j^i \ , \quad Q^i = \sum_{j=1}^n q_j^i$$

be the total amounts bid and offered on commodity i . Then

$$p^i = \frac{B^i}{Q^i}$$

is the price formed on commodity i .

The outcome is given by distributing the total amount of each commodity offered among the players in proportion to their bids and the total amount of momey bid on each commodity in proportion to the offers. The outcome of player j will be denoted by $X_j = (x_j^1, \ldots, x_j^n) \in Y_j$ and is given by:

$$x_j^i = a_j^i - q_j^i + \frac{b_j^i}{p^i} \quad (1 \leq i \leq \ell-1)$$

$$x_j^\ell = a_j^\ell - \sum_{i=1}^{\ell-1} b_j^i + \sum_{i=1}^{\ell-1} p^i q_j^i .$$

We compute the hypersurfaces $Y_j(\bar{s})$ and the lines $V_j(\bar{s})$ (or equivalently, the normals to the holding hypersurfaces $Y_j(\bar{s})$) in the next lemma.

Set

$$B_j^i = \sum_{k \neq j} b_k^i , \quad Q_j^i = \sum_{k \neq j} q_k^i , \quad p_j^i = \frac{B_j^i}{Q_j^i} .$$

Then p_j^i is the price on commodity i formed by the players other than j .

Lemma 4.1: a) The hypersurface $Y_j(\bar{s})$ is defined by the equation

$$x_j^\ell = a_j^\ell + \sum_{i=1}^{\ell-1} \frac{B_j^i(a_j^i - x_j^i)}{Q_j^i + a_j^i - x_j^i} .$$

b) A vector normal to $Y_j(\bar{s})$ at $\phi_j(\bar{s})$ is given by:

$$((p^1)^2 p_j^1, (p^2)^2 p_j^2, \ldots, (p^{\ell-1})^2 p_j^{\ell-1}, 1) .$$

<u>Proof</u>: To prove a), we have to show that for $i = 1, \ldots, \ell-1$,

$$p^i q_j^i - b_j^i = \frac{B_j^i(a_j^i - x_j^i)}{Q_j^i + a_j^i - x_j^i}$$

and then summing over i gives the result. Since $p^i = B^i/Q^i$, $B_j^i = B^i - b_j^i$, and $Q_j^i = Q^i - q_j^i$, we have to check that

$$\frac{B^i}{Q^i} q_j^i - b_j^i = \frac{(B^i - b_j^i)(a_j^i - x_j^i)}{(Q^i - q_j^i + a_j^i - x_j^i)}$$

Since

$$a_j^i - x_j^i = q_j^i - \frac{b_j^i}{p^i} = q_j^i - \frac{b_j^i Q^i}{B^i} \,, \quad \text{this is easily verified.}$$

To prove b), note that B_j^i and Q_j^i do not depend on player j and a normal to a hypersurface in parametric form, as in a), is given by $(\partial x_j^\ell/\partial x_j^1, \ldots, \partial x_j^\ell/\partial x_j^{\ell-1}, -1)$ where x_j^ℓ is a function of $x_j^1, \ldots, x_j^{\ell-1}$ as in a). From a), we have

$$\frac{\partial x_j}{\partial x_j^i} = \frac{-B_j^i Q_j^i}{(Q_j^i + a_j^i - x_j^i)^2}$$

and it is easy to check that $Q_j^i + a_j^i - x_j^i = B_j^i/p^i$. Hence a normal is given by $((p^1)^2 p_j^1, \ldots, (p^{\ell-1})^2 p_j^{\ell-1}, 1)$.

A strategy is ultra-optimal when the lines spanned by the normals to the hypersurfaces $Y_j(\overline{s})$ coincide. Since the prices p^i coincide for all players, $\overline{s} \in S_{UO}$ if and only if the quantities p_j^i are independent of j for $i = 1, \ldots, \ell-1$. This gives the $(n-1)(\ell-1)$ independent equations defining S_{UO}.

Suppose that p_j^i is independent of j for all i, say $p_j^i = \lambda_i$. Then we have

$$\frac{B^i - b_j^i}{Q^i - q_j^i} = \lambda_i \text{ , or}$$

$$(*) \qquad B^i = b_j^i = \lambda_i(Q^i - q_j^i)$$

for all j. Summing (*) over j shows that $\lambda_i = p^i$. Substituting this back in (*) gives:

$$p_j^i = \frac{b_j^i}{q_j^i} = p^i \text{ .}$$

When this holds, the outcome to each player is simply his initial endowment. We have:

Proposition 4.2. In the bid-offer model, the set S_{UO} of ultra-optimal strategies consists of those strategies such that each player's outcome is his initial endowment, that is, such that:

$$b_j^i - p^i q_j^i \quad \text{for all } i \text{ and } j \text{ .}$$

Corollary 4.3. The codimension of S_{UO} in S in the bid-offer model is $(n-1)(\ell-1)$.

A variant of the bid-offer model is the sell-all model, in which all players are required to offer their entire endowment for sale, i.e., $q_j^i = a_j^i$ for all i and j.

Corollary 4.4. In the sell-all model, S_{U0} consists of the strategies such that $b_j^i = a_j^i p^i$ for all i and j and each such strategy gives each player his initial endowment as the outcome. The codimension of S_{U0} in S is $(n-1)(\ell-1)$.

It turns out that the market mechanism in both of these models blows up at strategies at which the total bid or offer (or both) is zero for any commodity. However if we confine ourselves to the subset V of U given by $V = \{u \in U : a < \nabla u < b\}$, for some positive vectors a and b , then there exist positive numbers c and d such that (in either model)

$$N(\bar{u}) \subset T = \{\bar{s} \in S : c < p^i(\bar{s}) < d \quad \text{for} \quad i = 1, \ldots, \ell-1\}$$

for $\bar{u} \in V^n$. This is shown in Lemma 1 of [1] for the sell-all model and can be shown for the bid-offer model in the same way. By applying Proposition 3.1 to the set T for all $d > c \geq 0$, we obtain the next proposition. Furthermore, the holding hypersurfaces $Y_j(\bar{s})$ are concave in the Shapley-Shubik models and hence, according to the remark following Lemma 2.1, the condition of Lemma 1.1 defines the set $N(\bar{u})$.

Proposition 4.5. There is an open dense set $U_0 \subset U^n$ such that for all $\bar{u} = (u_1, \ldots, u_n) \in U_0$:

1) In the bid-offer model, $N(\bar{u})$ is either a submanifold of S of dimension $n(\ell-1)$ or is empty, and $EN(\bar{u})$ is either a union of submanifolds of S of dimension $(\ell-1)$ or is empty.

2) In the sell-all model; $N(\bar{u})$ is a finite set and $EN(\bar{u})$ is empty.

(Note: The fact that $N(\bar{u})$ and $EN(\bar{u})$ are either empty or of the dimension stated follows from the definition of transversality and the proof of Proposition 3.1.)

5. Game-Theoretic Efficiency

Assume now that strategy sets S_j, outcome spaces Y_j, and maps $\phi_j : S \to Y_j$ satisfying Assumption 1 are given, and assume that the S_j are smooth manifolds and that the ϕ_j are smooth maps. To examine game-theoretic efficiency, we will define a decomposition $S = SG_{UI} \cup SG_{UO}$ similar to the decomposition $S = S_{UI} \cup S_{UO}$ defined in Section 1 when Y_0 is a space of reallocations.

Let $d\phi_j$ be the Jacobian of the map ϕ_j. We may write

$$d\phi_j = [d\phi_{j1} \cdots d\phi_{jn}]$$

where $d\phi_{ji}$ is the matrix of partial derivatives of the j^{th} player's outcomes with respect to the i^{th} player's strategies. Assumption 1 of Section 2 will now be changed to:

<u>Assumption 1'</u>: For all j and all $\bar{s} \in S$, $d\phi_{jj}(\bar{s})$ has rank equal to $(\dim Y_j - 1)$.

This is the infinitesimal version of Assumption 1.

Let $\nabla_s u_j$ be the gradient of partial derivatives of u_j with respect to the strategic variables. By the chain rule:

$$\nabla_s u_j = (\nabla u_j) \cdot d\phi_j \ .$$

According to a simple lemma of Smale, if $\bar{s} \in E_G(\bar{u})$, then the vectors $\nabla_s u_j(\bar{s})$ are linearly dependent. Let $v_j(\bar{s})$ denote a non-zero vector in $V_j(\overline{s})$; it is determined up to scalar multiples. Smale's lemma and Lemma 2.1 yield the following. (Concavity of utilities is not needed here, only that the gradients be nowhere-vanishing.)

Lemma 5.1: Let SG_{UO} be the set of $\overline{s}$ S with the property: the vectors $(v_j(\overline{s}))\cdot d\phi_j$ are linearly dependent. Then for all utilities $\overline{u} = (u_1, \ldots, u_n)$:

$$N(\overline{u}) \cap E_G(\overline{u}) \subset SG_{UO} .$$

If we let SG_{UI} be the complement of SG_{UO} in S , then $S = SG_{UI} \cap SG_{UO}$. For all choices of utilities (with nowhere-vanishing gradients), the efficient Nash equilibria, if there are any, all lie in SG_{UO} . An analogue of Proposition 3.1 is also true in the context of game-theoretic efficiency, where S_{UO} is replaced by SG_{UO} . Generically, the set of efficient Nash equilibria will have codimension t in $N(\overline{u})$ if SG_{UO} has codimension t in S . Let $m = \dim S$. Since $m \geq n$, the condition that the n vectors $v_j(\overline{s})\cdot d\phi_j$ be linearly dependent is defined by $\binom{m}{n}$ equations--those obtained by setting the determinants of all $n \times n$ minors of the matrix with rows $v_j(\overline{s})\cdot d\phi_j$ $(j = 1, \ldots, n)$ equal to zero. These equations may not define independent conditions on $\overline{s}$, but one may expect that for a generic class of ϕ , SG_{UO} has positive codimension in S . This is true for the Shapley-Shubik models, as we shall see below, and this explains the inefficiency result of [1], according to which, generically in utilities, the Nash equilibria are inefficient in the sell-all model.

Consider the bid-offer model with n players and ℓ commodities (the ℓ^{th} commodity is money). With notation as before, let:

$$C^i_{jk} = \begin{bmatrix} \frac{\partial x^1_j}{\partial b^i_k} \\ \vdots \\ \frac{\partial x^\ell_j}{\partial b^i_k} \end{bmatrix}, \quad B^i_{jk} = \begin{bmatrix} \frac{\partial x^1_j}{\partial q^i_k} \\ \vdots \\ \frac{\partial x^\ell_j}{\partial q^i_k} \end{bmatrix}$$

so that the $\ell \times 2(\ell-1)$ matrix $d\phi_{jk}$ is given by

$$d\phi_{jk} = [C^1_{jk} \ldots C^{\ell-1}_{jk} B^1_{jk} \ldots B^{\ell-1}_{jk}] .$$

It is easy to compute that

$$C^i_{jj} = \Big[\underbrace{0, \ldots, 0,}_{i^{th} \text{ place}} \frac{B^i - b^i_j}{B^i p^i}, \ldots, 0, \frac{q^i_j - Q^i}{Q^i}\Big]$$

$$C^i_{jk} = \Big[\underbrace{0, \ldots, 0,}_{i^{th} \text{ place}} \frac{-b^i_j}{B^i p^i}, \ldots, 0, \frac{q^i_j}{Q^i}\Big] \quad \text{for} \quad j \neq k$$

$$B^i_{jj} = \Big[\underbrace{0, \ldots, 0,}_{i^{th} \text{ place}} \frac{b^i_j - B^i}{B^i}, \ldots, 0, p^i\left(\frac{Q^i - q^i_j}{Q^i}\right)\Big]$$

$$B^i_{jk} = \Big[0, \ldots, 0, \frac{b^i_j}{B^i}, \ldots, 0, -p^i\left(\frac{q^i_j}{Q^i}\right)\Big] \quad \text{for} \quad j \neq k .$$

As we saw in Section 4,

$$v_j(\bar{s}) = \left[(p^1)^2 \left(\frac{q_j^1 - Q^1}{B^1 - b_j^1} \right), \ldots, (p^{\ell-1})^2 \left(\frac{q_j^{\ell-1} - Q^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}} \right), -1 \right]$$

defines a vector in $V_j(\bar{s})$. For $j \neq k$, set

$$w_j = \left[\frac{b_j^1 - p^1 q_j^1}{B^1 - b_j^1}, \ldots, \frac{b_j^{\ell-1} - p^{\ell-1} q_j^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}}, p^1 \left(\frac{p^1 q_j^1 - b_j^1}{B^1 - b_j^1} \right), \ldots, p^{\ell-1} \left(\frac{p^{\ell-1} q_j^{\ell-1} - b_j^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}} \right) \right] .$$

A short calculation shows that in the n 2n(-1) matrix $\begin{bmatrix} V_1(\bar{s}) \\ \vdots \\ V_n(\bar{s}) \end{bmatrix} \cdot d\phi(\bar{s})$ is equal to

$$\begin{bmatrix} 0 \ldots 0 & w_1 & w_1 & & w_1 \\ w_2 & 0 \ldots 0 & w_2 & & w_2 \\ w_3 & w_3 & 0 \ldots 0 & \cdots & w_3 \\ \vdots & \vdots & \vdots & & \vdots \\ w_n & w_n & w_n & & 0 \ldots 0 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{2(\ell-1)}$$

In order for $\bar{s}$ to lie in SG_{UO} , the above matrix must have linearly dependent rows. It is easily checked that this is possible only if $w_j = (0, 0, \ldots, 0)$ for some j , in other words, only if $b_j^i = p^i q_j^i$ for $i = 1, \ldots, \ell-1$, for at least one player j .

Proposition 5.2. In the Shapley-Shubik bid-offer and sell-all models, SG_{UO} consists of those strategies such that the outcome for at least one player is his initial endowment, that is, for at least one j ,

$$b_j^i = p^i q_j^i \quad \text{for} \quad i = 1, \ldots, \quad -1 \; .$$

The codimension of SG_{UO} is $(\ell-1)$.

This result and Proposition 4.2 show clearly the difference between economic and game-theoretic efficiency of Nash equilibria. Just as in Proposition 4.5, $EN_G(\overline{u})$ is either empty or a union of manifolds of codimension $(\ell-1)$ in $N(\overline{u})$, hence of dimension $(n-1)(\ell-1)$, for a generic class of $\overline{u}$, in the bid-offer model.

As a final remark, we note that the game-theoretic efficiency of Nash equilibria for market mechanisms where $Y_j(\overline{s})$ has codimension greater than one in Y_j can also be analyzed by the methods of this paper. To do so, it would be necessary to examine the vector spaces $\{v \in T^*_{\phi_j(\overline{s})}(Y_j) :$ $v \cdot w = 0$ for all $w \in T_{\phi_j(\overline{s})}(Y_j(\overline{s}))\}$ which would then have dimension greater than one.

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