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Stochastic Games II: The Minmax Theorem

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1. INTRODUCTION

A two-person, zero-sum stochastic game consists of a (finite) set S of states; each state S is a (finite) matrix game. The entries of these matrices consist of

1) a payoff (from the column-chooser, B , to the row-chooser, A) and
2) a lottery on S , determining which state will be played next.

Shapley [1953] introduced this concept, studying stochastic games which terminate with probability 1 after finitely many steps; equivalently, these games could be thought of as infinite in duration, but with a non-zero discount rate. In this case the min-max theorem is straightforward (Shapley [1953], Monash [1979, 1981]). Gillette [1957] studied stochastic games with zero stop probabilities, establishing the min-max theorem in a couple of special cases. In these cases, the optimal strategies are stationary (i.e., dependent only upon the current state, rather than the history); thus the game "should" go into a Markov chain. The payoff can be defined either as the Cesaró limit $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} d_N$ or the Abel limit $\lim_{r\to 0} r \sum_{i=1}^{\infty} d_i (1-r)^{i-1}$, where d_i = the payoff on the i^{th} play, since, with best play, these limits exist and are equal (compare Royden [1963]).

In The Big Match, Blackwell and Ferguson [1968] considered a more difficult example. Although this game still has a value, it cannot be guaranteed by stationary strategies; furthermore, no strategy is better than ε-optimal. Extending these methods, Bewley and Kohlberg [1976] showed that the Cesaró limit of the values of the N-stage games exists, and equals the Abel limit of the values of the r-discounted games;

furthermore, no strategy for either player can guarantee an average payoff (in any sense) better than this number v_∞. Thus v_∞ is the only candidate for min-max value. Finally, the min-max theorem for stochastic games was proved by Monash [1979] and independently by Mertens and Neyman [1980]. This paper is a revision of Monash [1979].

2. DEFINITIONS

Without loss of generality, a stochastic game can be described by finite sets S, A, B, C and measurable functions

$$d : S \times B \times B \times C \to [-\tilde{M}, \tilde{M}] ,$$

$$s : S \times A \times B \times C \to S , \text{ and}$$

$$q : [0,1] \to C$$

such that:

1) S is the state space;
2) Player A (resp. B) chooses a move from his choice set A (resp. B);
3) s, composed with q, reproduces the lottery in each entry of each state matrix; and
4) d is the payoff function.

A state $s^* \in S$ is absorbing if $\delta(s^*, a, b, c) = s^*$ for all a, b, c and $d(s^*, a, b, c) = v(s^*)$, a constant. $S^* \subset S$ is the set of absorbing states $S_\infty = S - S^*$.

A play of the game is just a sequence $s_0, a_1, b_1, c_1, s_1, a_2, b_2, c_2, s_2, \ldots,$ where $s_i = \delta(s_{i-1}, a_i, b_i, c_i)$, for all i; let $d_i = d(s_{i-1}, a_i, b_i, c_i)$, the payoff on the i^{th} turn. Writing $t_i = (s_{i-1}, a_i, b_i, c_i) \in T = S \times A \times B \times C$, we denote a play by

$\mathcal{t} = (t_1, t_2, t_3, \ldots)$; thus

$$T^{\infty} = S \times A \times B \times C \times S \times A \times B \times C \times S \times \ldots$$
$$= \{\text{all possible plays}\} .$$

The subsequence $(t_1, \ldots, t_n)$ is denoted by $\mathcal{t}(n)$; we use this notation even if we are thinking of this subsequence as belonging to many different possible plays.

Strategies for A will always be denoted by σ , and strategies for B by τ . These strategies will always be of the form

$$\text{Prob}(a \in A \text{ (resp. } b \in B) \text{ on turn } k) = \text{function}(t_1, \ldots, t_{k-1}) .$$

Thus, by the Kolmogorov Extension Theorem (see Kolmogorov [1950] or Monash [1981]), a pair (σ,τ) determines a probability measure $\mu(\sigma,\tau)$ on T^{∞} . Unless otherwise noted, all expectations below are with respect to this measure. Let

$$T^* = \{\mathcal{t} \in T : s_i \in S^* \text{ for some } i\} ,$$

and T_{∞} the complement. In the next section we write $P(*) = \mu(T^*)$.

Following Bewley and Kohlberg [1976] or Monash [1981], recall that for all $s \in S$, for all $r \in (0,1)$, $V_s(r)$ = the value of the r-discount game, starting in s , satisfies

$$V_s(r) = \operatorname{val}_c(\exp(d(s,a,b,c) + (1-r)\sum_{\bar{s}\in S} P(\bar{s})V_{\bar{s}}(r)) , \qquad (2.1s)$$

where $P(\bar{s})$ = the probability that $\delta(s,a,b,c) = \bar{s}$, and val is the ordinary min-max value. For some $\tilde{r} > 0$, all the $V_s(r)$ are algebraic, as are the optimal strategies in the games (2.1s). Thus, on $(0,\tilde{r})$,

$$V_s(r) = V_\infty(s)\cdot r^{-1} + (\ )r^{-1+\frac{1}{n}} + \dots$$

$$= V_\infty(s)\cdot u^{-n} + (\ )u^{-n+1} + \dots ,$$

where $u = r^{1/n}$. Let $0 \leq \tilde{u} \leq \tilde{r}^{1/n}$; on $(0,\tilde{u})$, we write $W_s(u) = V_s(u^n)$, so that $\lim_{u\to 0^+} u^n W_s(u) = \lim_{r\to 0^+} rV_s(r) = v_\infty(s)$.

In Sections 4 through 6, we assume $v_\infty(s) = 0$ for all $s \in S_\infty$. In that case we have $\lim_{u\to 0^+} u^{n-1}W_s(u) < \infty$ for all s ; thus, writing $\overline{W}(u) = \max_{s\in S_\infty} |W_s(u)|$, we have $\lim_{u\to 0^+} u^{n-1}\overline{W}(u) < \infty$, also.

3. STATEMENT OF THEOREM

Our main result is

Theorem I: For any starting state $s_0 \in S$, for any $\varepsilon > 0$, there exists a strategy σ for A such that, for any strategy τ for B ,

$$\liminf_{N\to\infty} \exp\left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) > v_\infty(s) - \varepsilon .$$

Theorem I clearly follows from the following two propositions:

Proposition 3.1: Suppose, for all $s \notin S^*$, $v_\infty(s) = 0$. Then the conclusion of Theorem I holds.

Proposition 3.2: Proposition 3.1 $\Longrightarrow$ Theorem I.

In this section, we prove Proposition 3.2; the remainder of the paper is devoted to Proposition 3.1.

The proof of Proposition 3.2 depends upon

Lemma 3.3: Let G be a stochastic game, with state set S. Let H be another stochastic game, identical to G except for the following modification: Replace a single state $x \in S$ by an absorbing state y such that $v(y) = v_\infty(x)$. Then, for all $s \in S$,

$$v_{\infty,H}(s) = v_{\infty,G}(s) ,$$

where $v_{\infty,G}(s)$ (resp. $v_{\infty,H}(s)$) is simply $v_\infty(s)$ in the game G (resp. H).

Proof: Let $V_{G,s}(r)$ (resp. $V_{H,s}(r)$) be $V_s(r)$ in the game (resp. H). Define

$$\hat{V}(r) = r^{-1} \cdot v_\infty(x) - V_{G,x}(r)$$

$$\overline{V}(r) = \min_{s \in S-\{x\}} (V_{H,s}(r) - V_{G,s}(r)) .$$

Then, for any $s \in S - \{x\}$, (2.1s) gives

$$V_{H,s}(r) = \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(s,a,b,c))) + (1-r)\sum_{\overline{s} \in S-\{y\}} P(\overline{s}) \cdot V_{H,\overline{s}}(r)$$

$$+ r^{-1} \cdot (1-r) P(y) \cdot v(y))$$

$$\geq \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(s,a,b,c)) + (1-r)\sum_{\overline{s} \in S} P(\overline{s}) \cdot V_{G,\overline{s}}(r))$$

$$+ (1-r) \min_{P \in [0,1]} ((1-P) \cdot \overline{V}(r) + P \cdot \hat{V}(r) ,$$

where P corresponds to $P(x : a,b,s)$,

$$= V_{G,s}(r) + (1-r)((1-P^*) \cdot \overline{V}(r) + P^* \cdot \hat{V}(r)) ,$$

for some $P^* \in [0,1]$.

Picking $\tilde{s}$ now so that

$$V_{H,\tilde{s}}(r) - V_{G,\tilde{s}}(r) = \overline{V}(r) \quad \text{in some interval} \quad [0,\tilde{r}) ,$$

$$\overline{V}(r) = V_{H,\tilde{s}}(r) - V_{G,\tilde{s}}(r)$$

$$\geq (1-r)((1-P^*)\cdot\overline{V}(r) + P^*\cdot\tilde{V}(r))$$

$$r\overline{V}(r) \geq (1-r)\cdot P^*\cdot(\tilde{V}(r) - \overline{V}(r)) .$$

So either $\overline{V}(r) \geq 0$, or $\tilde{V}(r) - \overline{V}(r) < 0$. In either case,

$$\min_{s\in S-\{x\}} (v_{\infty,H}(s) - v_{\infty,G}(s))$$

$$= \min_{s\in S-\{x\}} \lim_{r\to 0^+} (rV_{H,s}(r) - rV_{G,s}(r))$$

$$= \lim_{r\to 0^+} r \min_{s\in S-\{x\}} (V_{H,s}(r) - V_{G,s}(r))$$

$$= \lim_{r\to 0^+} r\overline{V}(r)$$

$$\geq 0 \quad \text{or} \quad \geq \lim_{r\to 0^+} r\tilde{V}(r) = 0 .$$

So, for all $s \in S - \{x\}$, $v_{\infty,H}(s) - v_{\infty,G}(s) \geq 0$; that is,

$$v_{\infty,H}(s) \geq v_{\infty,G}(s) .$$

But, by symmetry (i.e., interchanging the names A and B),

$$-v_{\infty,H}(s) \geq -v_{\infty,G}(s) .$$

Hence $v_{\infty,H}(s) = v_{\infty,G}(s)$, for all $s \in S - \{x\}$.

Since Lemma 3.3 is clearly true for state x , we are done.

□

Proof of Proposition 3.2:

We now proceed by induction on $|S_\infty|$, the number of non-absorbing states.

$|S_\infty| = 0$. Trivially true.

So assume for $|S_\infty| - 1$, and prove for $|S_\infty|$.

To every state s, associate a number $\alpha(s)$ such that

$$v_\infty(s) - \alpha(s) = \sup_\sigma \inf_\tau \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \exp(d_i) .$$

By the Bewley-Kohlberg result, $\alpha(s) \geq 0$ for all $s \in S$.

Want to show: $\alpha(s) = 0$ for all s. If so, done.

So suppose otherwise.

Definition: We will call a strategy σ, starting in state s, ε-optimal (for s) if

$$\inf_\tau \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \exp(d_i) \geq v_\infty(s) - \varepsilon .$$

Case 1: There exist states s_1, s_2 such that $\alpha(s_1) > \alpha(s_2) \geq 0$.

Let $\varepsilon = \frac{1}{3}(\alpha(s_1) - \alpha(s_2))$.

Consider the modified game H, where s_2 is replaced by an absorbing state y such that $v(y) = v_\infty(s_2)$. Then H, by induction, has an ε-optimal strategy, for any initial state.

Consider, then, the following strategy, for the game G starting in state s_1: Play the ε-optimal strategy for H, until "absorbed" in "y"; this is meaningful because G and H are identical outside of state s_2 . Once in s_2 , play an $(\alpha(s_2)+\varepsilon)$-optimal strategy, which exists by the definition of $\alpha(s_2)$. Then this strategy is clearly

$$(\alpha(s_2)+2\varepsilon)\text{-optimal for } s_1 .$$

But

$$\alpha(s_2) + 2\varepsilon = \tfrac{2}{3}\alpha(s_1) + \tfrac{1}{3}\alpha(s_2) < \alpha(s_1) ,$$

contradicting the definition of $\alpha(s_1)$.

Hence the only possibility is:

<u>Case 2</u>: There exists $\bar{\alpha} > 0$ such that, for all $s \in S_\infty$, $\alpha(s) = \bar{\alpha}$.

Now, let $v_0 = \min_{s\in S_\infty} v_\infty(s)$.

Let $S_0 \subseteq S_\infty$ be $\{s \in S_\infty : v_\infty(s) = v_0\}$; let $\tilde{S}$ be the complement.

<u>Case 2a</u>: $\tilde{S}$ is non-empty. Then let $v_1 = \min_{s\in\tilde{S}} (v_\infty(s))$; $v_1 > v_0$. Let

$$\beta = \frac{v_1 - v_0}{2(v_1 + \tilde{M})} \le \frac{1}{2} .$$

By repeated applications of Lemma 3.3, replace the states in $\tilde{S}$ by absorbing states with the same v_∞ . Then the states in S_0 still have value v_0 . Assuming Proposition 3.1, this new game has an ε-optimal strategy, where

$$\varepsilon = \min\left(\frac{v_1 - v_0}{4}, \frac{\beta\bar{\alpha}}{4}\right) .$$

Play this strategy until "absorbed," and an $(\bar{\alpha}+\varepsilon)$-optimal strategy thereafter (unless the "absorption" is genuine). Fixing (any) τ , we have two cases:

Case 1: Expected value if "absorbed" $\geq \frac{v_0+v_1}{2}$ or $P(*) = 0$. Then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \exp(d_i) \geq P(*)\left(\frac{v_0+v_1}{2} - (\bar{\alpha}+\varepsilon)\right) + (1-P(*))(v_0-\varepsilon)$$

$$= v_0 - \bar{\alpha} + P(*)\left(\frac{v_1-v_0}{2} - \varepsilon\right) + (1-P(*))(\bar{\alpha}-\varepsilon)$$

$$\geq v_0 - \bar{\alpha} + \min\left(\frac{v_1-v_0}{4}, \frac{7\bar{\alpha}}{8}\right) ;$$

since τ was arbitrary, this contradicts the definition of $\bar{\alpha}$.

Case 2: Expected value if "absorbed" $< \frac{v_0+v_1}{2}$ and $P(*) > 0$.

Let $\gamma = \frac{\text{prob(genuine absorption)}}{P(*)}$. Then

$$\frac{v_0+v_1}{2} > \text{Expected value if "absorbed"}$$

$$\geq \gamma(-\tilde{M}) + (1-\gamma)\cdot v_1 \;; \quad \text{i.e.,}$$

$$v_1 - \frac{v_1-v_0}{2} > v_1 - \gamma(v_1+\tilde{M})$$

$$\gamma > \frac{(v_1-v_0)}{2(v_1+\tilde{M})} = \beta .$$

Hence

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \exp(d_i) \geq v_0 - \varepsilon - P(*)(1-\gamma)(\bar{\alpha}+\varepsilon)$$

$$\geq v_0 - \varepsilon - (1-\beta)(\bar{\alpha}+\varepsilon)$$

$$> v_0 - \bar{\alpha} + (\beta\bar{\alpha} - 2\varepsilon) ;$$

since $\varepsilon \leq \frac{\beta\bar{\alpha}}{4}$, this is a contradiction.

Case 2b: $\tilde{S} = \phi$.

Deducting v_0 from all payoffs, this is exactly the case of Proposition 3.1. Hence there exists an $\frac{\bar{\alpha}}{2}$-optimal strategy, for our final contradiction.

So $\alpha(s) = 0$ for all $s \in S^*$.

But this is exactly what we wanted to prove.

□

4. PRELIMINARY COMPUTATIONS

For the rest of this paper, we will assume $v_\infty(s) = 0$ for all $s \in S_\infty$. We will always choose A's strategy σ to be Prob(a) = f(a,s,u) = the optimal (stationary) strategy in the u^n-discount game, for some u , in the current state s . Without loss of generality (see Monash [1979] or (1981]), B's strategy τ is pure: $b_k = \text{function}(\ell(k-1))$.

Let us now focus on one move of the game. Fix $s \in S_\infty$, $u \in (0,\tilde{u})$, and $b \in B$, with A playing strategy $\{f(a,s,u)\}$. Let $P_*(u) = \text{Prob}(\delta(s,a,b,c) \in S^*)$, given the probability distributions f(a,s,u) on A and q on C . In Sections 5 and 6, if a play ℓ is understood along with a sequence of u's , we will let

$$P_*(i) = \begin{cases} P_*(u) \quad \text{on the } i^{th} \text{ turn, if } s_{i-1} \in S_\infty \\ 0 \qquad\qquad\qquad\qquad\quad \text{, if } s_{i-1} \in S^* . \end{cases}$$

Meanwhile, let $\bar{s} = \delta(s,a,b,c)$.

We distinguish three cases:

1) $P_*(u) \equiv 0$ on $(0,\tilde{u})$;

2) Not 1, and

$$\lim_{u \to 0^+} \exp(v_\infty(\bar{s}) : \bar{s} \in S^*) = 0 ;$$

3) Not 1 or 2.

We further distinguish between:

A. Either Case 1, or $\text{order}(P_*(u)) \geq n$;

B. Not Case 1, and $\text{order}(P_*(u)) \leq n-1$.

Observe that $P_*(u)$ is a rational function of u , and thus has finitely many zeroes; without loss of generality, none of them occur on $(0,\tilde{u})$. Define $\delta(u)$ as follows (where we suppress the dependence upon s and b):

If Case A, then

$$\delta(u) = -\exp(v_\infty(\bar{s}) : \bar{s} \in S^*) \cdot P_*(u) \cdot u^{-n} ;$$

if Case B, then

$$\delta(u) = -\exp(v_\infty(\bar{s}) : \bar{s} \in S^*) \cdot P_*(u) + (1 - (1-u^n)(1-P_*(u))W_s(u) \cdot u^{-n} .$$

The point of this definition may be found in the following propositions (where we write $\exp(d : S_\infty)$ for $\exp(d : \bar{s} \in S_\infty)$, and so forth):

<u>Proposition 4.1</u>: $\exp(d : S_\infty)(1 - P_*(u)) \geq \delta(u) - \exp(W_{\bar{s}} - W_s(u) : S_\infty)(1 - P_*(u)) + \eta(u)$, for $u \in (0,\tilde{u})$, where $\lim_{u \to 0^+} \eta(u) = 0$.

and

Proposition 4.2:

1. If Case 1 (above), then $\delta(u) = 0$ and $P_*(u)\cdot\exp(v_\infty(\bar{s}) : S^*) = 0$;

2. If Case 2, then

$$|\exp(v_\infty(\bar{s}) : S^*)| < o(u^0)$$

and

$$\left|\frac{\delta(u)\cdot u^n}{P_*(u)}\right| < o(u^0) .$$

3. If Case 3,

$$\frac{-P_*(u)\cdot\exp(v_\infty(\bar{s}) : S^*)\cdot u^{-n}}{\delta(u)} = 1 + o(u^0) .$$

From Equation (2.1s), we have

$$W_s(u) \le (1 - P_*(u))\cdot(\exp(d : S_\infty) + (1 - u^n)\cdot\exp(W_{\bar{s}}(u) : S_\infty)) + P_*(u)\cdot\exp(v_\infty(\bar{s}) : S^*)u^{-n} . \quad (4.3)$$

Proof of Proposition 4.1:

Rearranging (4.3), we have

$$(1 - P_*(u))\exp(d : S_\infty) \ge (1 - (1 - P_*(u))(1 - u^n))W_s(u) - (1 - P_*(u))(1 - u^n)\exp(W_{\bar{s}}(u) - W_s(u) : S_\infty) - P_*(u)\exp(v_\infty(\bar{s}) : S^*) \quad (4.4)$$

If Case A holds, then

$$(4.4) = \delta(u) - (1 - P_*(u))\exp(W_{\bar{s}}(u) - W_s(u) : S_\infty) + (P_*(u) + u^n - u^nP_*(u))W_s(u) + u^n(1 - P_*(u)) \cdot\exp(W_{\bar{s}}(u) - W_s(u) : S_\infty) . \quad (4.5)$$

If Case B holds, then (4.4) equals

$$\delta(u) - (1 - P_*(u))\cdot\exp(W_{\overline{s}}(u) - W_s(u) : S_\infty)$$
$$+ u^n(1 - P_*(u))\cdot\exp(W_{\overline{s}}(u) - W_s(u) : S_\infty) \ . \qquad (4.6)$$

Let $\overline{P} > 0$ be such that $|P_*(u)| \leq \overline{P}u^n$ whenever Case A holds. Writing

$$\eta(u) = -(\overline{P}+4)u^n\overline{W}(u) \ ,$$

and observing that

$$(4.5) \geq \delta(u) - (1 - P_*(u))\cdot\exp(W_{\overline{s}} - W_s(u) : S_\infty) + \eta(u) \ ,$$

$$(4.6) \geq \delta(u) - (1 - P_*(u))\cdot\exp(W_{\overline{s}} - W_s(u) : S_\infty) + \eta(u)$$

and $$\lim_{u\to 0^+} \eta(u) = 0 \ ,$$

we are done.

□

Proof of Proposition 4.2:

1. Suppose Case 1 holds: $P_*(u) \equiv 0$. Then so does Case A, and

$$\delta(u) = -\exp(v_\infty(\overline{s}) : \overline{s} \in S^*)\cdot P_*(u)\cdot u^{-n}$$
$$= 0 \quad \text{for all} \quad u \ ,$$

and so done.

2. Suppose, then, Case 2 holds. Since $\exp(v_\infty(\overline{s}) : S^*)$ is a power series in u , with limit 0 as $u \to 0$, it is indeed $o(u^0)$. Now, if Case A, then

$$\left|\frac{\delta(u)\cdot u^n}{P_*(u)}\right| = \left|-\exp(v_\infty(\bar{s}) : S^*)\right|$$
$$< \mathcal{O}(u^0) \;;$$

while, if Case B, then

$$\left|\frac{\delta(u)\cdot u^n}{P_*(u)}\right| = \left|-\exp(v_\infty(s) : S^*) + \frac{P_*(u)\cdot W_s(u)\cdot u^n}{P_*(u)} + \text{higher order terms}\right|$$
$$< \mathcal{O}(u^0) + |u^n W_s(u)| + \text{higher order terms}$$
$$< \mathcal{O}(u^0) \;.$$

3. Suppose Case 3 holds. If Case A, then

$$\frac{-P_*(u)\cdot\exp(v_\infty(\bar{s}) : S^*)\; u^{-n}}{\delta(u)} \equiv 1 \;, \quad \text{by definition.}$$

So suppose Case B: order $P_*(u) \leq n-1$. It is clearly enough to check

$$\left|\frac{(P_*(u) + u^n - u^n P_*(u))W_s(u)}{-P_*(u)\cdot\exp(v_\infty(\bar{s}) : S^*)u^{-n}}\right| < \mathcal{O}(u^0) \;.$$

Then order $(P_*(u) + u^n - u^n P_*(u))$ = order $(P_*(u))$

order $(\exp(v_\infty(\bar{s}) : S^*)) = 0$,

and so the order of the left-hand-side is

$\geq$ order $(P_*(u))$ + order$(W_s(u))$ - order $(P_*(u))$ - 0 - order (u^{-n})

= order $(u^n W_s(u))$

≥ 1 .

Hence done.

□

5. THE ABSORBING CASE

Recall that a fixed strategy pair (σ,τ) induces a probability measure $\mu(\sigma,\tau)$ on T^{∞}, the space of all possible plays. If $s_0 \in S^*$, Proposition 3.1 is trivial; thus it follows immediately from

Proposition 5.1: For any starting state $s \in S_{\infty}$, for any $\varepsilon > 0$, there exists a strategy σ for A such that

$$\inf_{\tau} \liminf_{N\to\infty} \int_{T^{\infty}} \frac{1}{N}\sum_{i=1}^{N} d_i \, d\mu(\sigma,\tau) > -(6\tilde{M}+3)\varepsilon .$$

Proof of Proposition 5.1:

As remarked earlier, the strategy σ will be the form $\text{Prob}(a) = f(a,s,u)$, the optimal strategy in the u^n-discount game, for u cleverly chosen. Specifically, writing u_N for the u prevailing on the $N+1^{st}$ move, we set $u_N = u_0(1-\frac{1}{2}\varepsilon)^{\nu(N)}$, for u_0 sufficiently small and $\nu(N)$ a non-negative integer depending upon the history of the first $N-1$ moves.

Write $q = 1 - \frac{1}{2}\varepsilon$. Recalling Proposition 4.2, choose $R > 0$ and $\tilde{u}$ sufficiently small so that each $O(u^0)$ is $< Ru$. Assume $\varepsilon < 1$. Then $u_0 \in (0,\tilde{u}) \subseteq (0,1)$ must satisfy the following four conditions:

1. For every $u \in (0,u_0], \eta(0) > -\varepsilon$
2. For every $u \in (0,u], \overline{W}(u) < \frac{\varepsilon}{4}\cdot u^{-n+\frac{1}{2}}$
3. $Ru_0 < \varepsilon$
4. $(1+\varepsilon)^3 \cdot \dfrac{u_0^{1/2}}{1-q^{1/2}} < \varepsilon$.

To define $\nu(N)$, we first define a set of benchmarks $\tilde{m}$ on 1, 0, 1, 2, ... by:

$$\tilde{m}(-1) = -\infty$$

$$\tilde{m}(0) = 0$$

$$\tilde{m}(i) = \tilde{m}(i-1) + (u_0 q^{i-1})^{-n+\frac{1}{2}} , \quad \text{for} \quad i = 1, 2, 3, \ldots .$$

Next, define sequences $\overline{m}_0, \overline{m}_1, \overline{m}_2, \ldots$ and $\mathcal{L} = (\ell_0 = 0, \ell_1, \ell_2, \ldots)$, $\mathcal{L}$ increasing, in conjunction with the sequences $u_0, u_1, u_1, \ldots$ and $v(0), v(1), v(2), \ldots$ by:

1) $\overline{m}_0 = 0$

2) $\nu(0) = 0$

3) $u_N = u_0 q^{\nu(N)}$ for $N = 1, 2, 3, \ldots$

4) If $\overline{m}_{N-1} + \delta_N(u_{N-1}) > \tilde{m}(\nu(N-1)+1)$, then $\nu(N) = \nu(N-1)+1$ and $N \in \mathcal{L}$; if $\overline{m}_{N-1} + \delta_N(u_{N-1}) < \tilde{m}(\nu(N-1)-1)$, then $\nu(N) = \nu(N-1) - 1$ and $N \in \mathcal{L}$; otherwise $\nu(N) = \nu(N-1)$ and $N \notin \mathcal{L}$.

5) If $N \notin \mathcal{L}$, then $\overline{m}_N = \overline{m}_{N-1} + \delta_N(u_{N-1})$.

6) If $N = \ell_i \in \mathcal{L}$, then $\overline{m}_N = \overline{m}_{N-1} + \delta_N(u_{N-1}) + W_{s_{\ell_{i-1}}}(u_{N-1})$ $-W_{s_{\ell_i}}(u_{N-1})$.

Fix σ as above, and any (pure) τ . Proposition 5.1 follows instantly (by redefining ε) from:

<u>Proposition 5.2</u>: $\lim_{N\to\infty} \int_{T^*} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu \geq -\varepsilon$

and

<u>Proposition 5.3</u>: $\liminf_{N\to\infty} \int_{T_\infty} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu > -\varepsilon$.

We now prove Proposition 5.2, deferring Proposition 5.3 to the next section.

<u>Proof of Proposition 5.2</u>:

Let $T_k = \{\mathbf{t} = (t_1, t_2, \ldots) \in T^* : \mathfrak{s}(t_k) \quad S^* \text{ but } \mathfrak{s}(t_{k-1}) \notin S^*)\}$; thus $T^* = T_1 \cup T_2 \cup T_3 \cup \ldots$.

So
$$\lim_{N\to\infty} \int_{T^*} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu$$

$$= \lim_{N\to\infty} \sum_{k=1}^{\infty} \int_{T_k} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu$$

$$= \sum_{k=1}^{\infty} \lim_{N\to\infty} \int_{T_k} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu \ ,$$

by the Lebesgue Dominated Convergence Theorem (Royden [1963]),

$$= \sum_{k=1}^{\infty} \int_{T_k} \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} d_i d\mu$$

$$= \sum_{k=1}^{\infty} \int_{T^\infty} P_*(k)\cdot\exp(v_\infty(s_t) : \mathbf{t} \in T_k) d\mu \ . \tag{5.4}$$

The following is a special case of Proposition 4.1 of Monash [1981] (identifying $Z_i^* = \mathfrak{s}^{-1}(S^*)$ for all i).

<u>Proposition 5.5</u>: There exists a probability measure $\tilde{\mu}$ on T such that, for all N , for all $f_N : T^{N-1} \to \mathbb{R}$ such that $\mathfrak{s}(t_{N-1}) \in S^*$ implies $f_N(t_1, \ldots, t_{N-1}) = 0$,

$$\int_{T^\infty} f_N(\mathbf{t}(N-1))\, d\tilde{\mu}(H) = \int_{T^\infty} f_N(\mathbf{t}(N-1)) \cdot \prod_{i=1}^{N-1} (1 - P_*(i))\, d\tilde{\mu}(\mathbf{t})$$

Now, assume temporarily,

<u>Proposition 5.6</u>: For all N, for all t,

$$\sum_{k=1}^{N} \left(P_*(k)\cdot \exp(v_\infty(s_k) : t \in T_k)\cdot \prod_{i=1}^{k-1} (1 - P_*(i))\right) > -\varepsilon$$

Let

$$f_k(t(k-1)) = \begin{cases} P_*(k)\cdot \exp(v_\infty(s) : t \in T_k) & \text{if } s(t_{k-1}) \notin S^* \\ 0 & \text{if } s(t_{k-1}) \in S^* . \end{cases}$$

Then f_k satisfies the hypothesis of Proposition 5.5. Thus, for all N,

$$\sum_{k=1}^{N} P_*(k)\cdot \exp(v_\infty(s_k) : t \in T_K)$$

$$= \sum_{k=1}^{N} \int_{T^\infty} f_k(t(k-1))d\mu$$

$$= \sum_{k=1}^{N} \int_{T^\infty} f_k(t(k-1))\cdot \prod_{i=1}^{k-1} (1 - P^*(i))d\tilde{\mu}$$

$$= \int_{T^\infty} \sum_{k=1}^{N} P_*(k)\cdot \exp(v_\infty(s_k) : t \in T_k)\cdot \prod_{i=1}^{k-1} (1 - P^*(i))d\tilde{\mu}$$

$$> \int_{T^\infty} (-\varepsilon)d\tilde{\mu} \text{ , by Proposition 5.6,}$$

$$= -\varepsilon \text{ ;}$$

as these are the partial sums of equation (5.4), this establishes Proposition 5.1.

So we pass to the

Proof of Proposition 5.6:

Fix $\mathcal{t}$ and N . Recalling Proposition 4.2, we make the simplifying assumption that Cases 1 or 3 hold everywhere (for fullest detail see Monash [1979]); thus, for $k = 1, \ldots, N$,

$$\text{either} \quad \delta(u_{k-1}) = P_*(k)\cdot\exp(v_\infty(s_k) : T_k) = 0$$

$$\text{or} \quad \left|\frac{-P_*(k)\cdot\exp(v_\infty(s_k) : T_k)\cdot u_{k-1}^{-n}}{\delta(u_{k-1})}\right| \in (1 - Ru_{k-1},\ 1 + Ru_{k-1})$$

$$\subseteq (1-\varepsilon,\ 1+\varepsilon) \ . \qquad (5.8)$$

Writing $F_k = P_*(k)\ \exp(v_\infty(s_k) : T_k)\cdot \prod_{i=1}^{k-1} (1 - P_*(i))$, our task is to bound $\sum_{k=1}^{N} F(k)$ below. We spread out this sum as the integral of a step function by defining $A(z)$ on $[0,N)$: $A(z) = F([z]+1)$, where $[z]$ is the usual greatest integer function. Thus $\int_0^N A(z)dz = \sum_{k=1}^{N} F_k$.

First, observe that, for $\ell_j \in \mathcal{L}$,

$$\frac{\bar{m}_{\ell_{j+1}} - \bar{m}_{\ell_j}}{\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})} = 1 + \frac{W_{s_{\ell_j}}(u_{\ell_j}) - W_{s_{\ell_{j+1}}}(u_{\ell_j})}{\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})} \in (1-\varepsilon,\ 1+\varepsilon) \ ,$$

since
$$|W_{s_{\ell_j}}(u_{\ell_j}) - W_{s_{\ell_{j+1}}}(u_{\ell_j})| \le 2\bar{W}(u_{\ell_j}) < \frac{\varepsilon}{2}\, u_{\ell_j}^{-n+\frac{1}{2}} \ ,$$

while
$$\left|\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})\right| \approx u_{\ell_j}^{-n+\frac{1}{2}} \ , \text{ by definition.}$$

Thus we can define a function $m(z)$ on $[0,N]$ such that

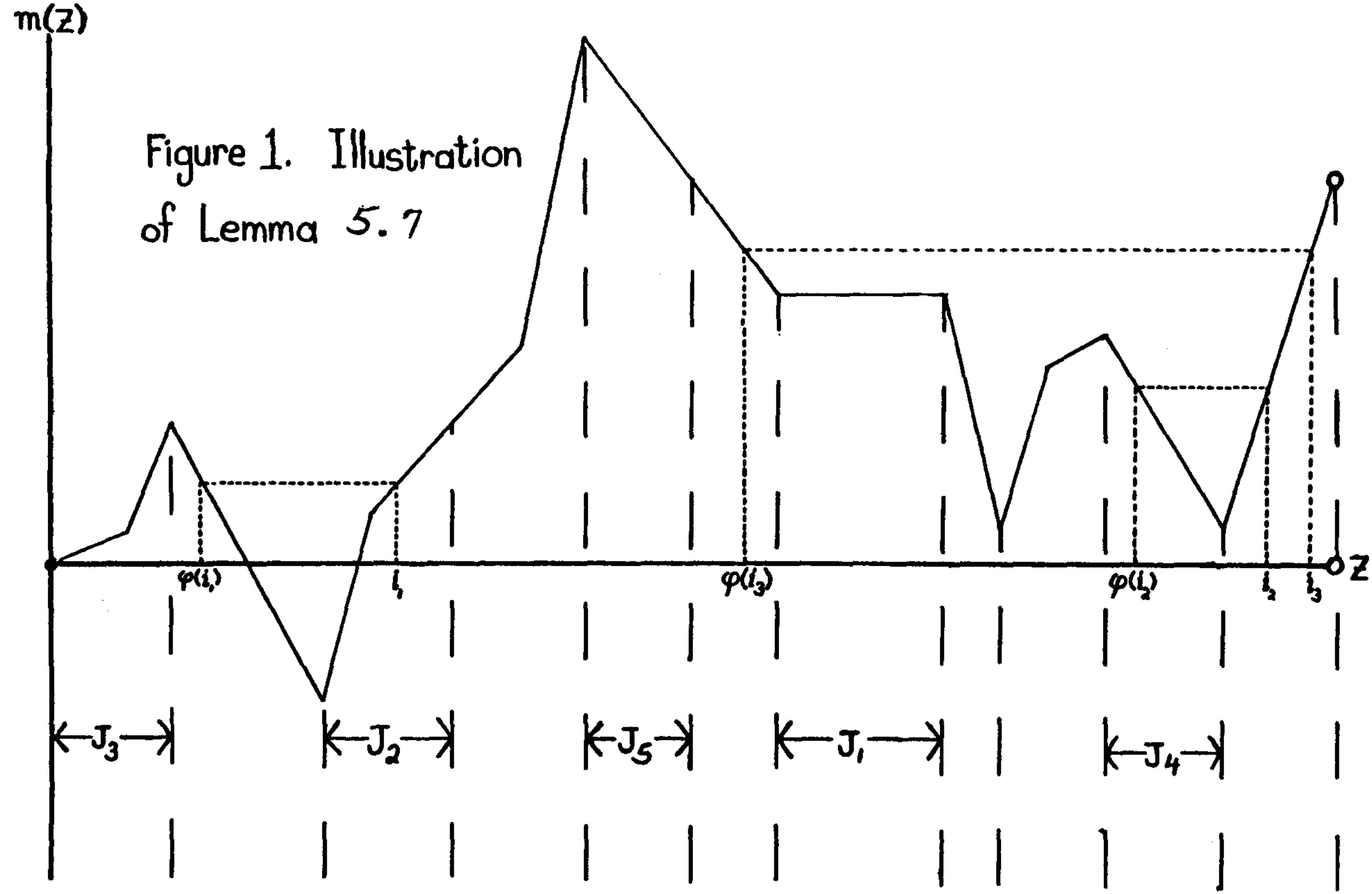

Figure 1. Illustration of Lemma 5.7

1) m is linear on $[k, k+1]$, for $k = 0, 1, \ldots, N-1$.

2) $m(\ell_j) = \overline{m}_{\ell_j}$, for $\ell_j \in \mathcal{L}$.

3) $\dfrac{m(k+1) - m(k)}{\delta_{k+1}(u_k)} \in (1-\varepsilon, 1+\varepsilon)$, for $k = 0, 1, \ldots, N-1$. (5.9)

We now want a finite, increasing sequence $\mathcal{J} = \{0 = j_0, j_1, j_2, \ldots, N\}$, containing every integer $0, 1, \ldots, N$; $\mathcal{J}$ should be partitioned into five sets, with the following properties:

1) For $j_i \in \mathcal{J}_1$, m is constant on $[j_i, j_{i+1}]$.

2) For $j_i \in \mathcal{J}_2$ or $\mathcal{J}_3$, m is increasing on $[j_i, j_{i+1}]$.

3) For $j_i \in \mathcal{J}_4$ or $\mathcal{J}_5$, m is decreasing on $[j_i, j_{i+1}]$.

4) There exists a bijection $\phi : \mathcal{J}_2 \to \mathcal{J}_4$ such that if $\phi(j_h) = j_i$,

   1) $i < h$

   2) $m(j_h) = m(j_{i+1})$

   3) $m(j_{h+1}) = m(j_i)$.

5) For any $j_h, j_i \in \mathcal{J}_3$ with $h < i$, $m(j_h) < m(j_i)$.

Let $J_i = \bigcup_{j_h \in \mathcal{J}_i} [j_h, j_{h+1})$, $i = 1, 2, 3, 4, 5$.

6) The sets J_i cover $[0,N)$.

<u>Lemma 5.7</u>. There exists such a sequence $\mathcal{J}$.

<u>Proof</u> (See Figure 1):

We will prove this lemma by induction on the number H of maxima attained by m on the interval $[0,N]$ (this number is $\leq N+1$, and hence finite).

Clearly it will be enough to construct the sets $J_1, \ldots, J_5$.

<u>$H = 1$</u> : Then m is either monotonic non-increasing or monotonic non-decreasing. In the first case, let J_5 be the set on which m is

strictly decreasing, and J_1 the balance; in the second let J_3 be the set where m is strictly increasing, and J_1 the balance.

<u>Inductive step</u>: Assume true for H.

So suppose the maxima occur at $y_1, \ldots, y_{H+1}$, and the minima at $(x_0), x_1, x_2, \ldots, x_H, (x_{H+1})$, so that $x_{i-1} < y_i < x_i$, for $i = 1, \ldots, H+1$ (x_0 or x_{H+1} may not exist). Apply induction to $[0, x_H]$ (recall that m changes direction only at integral arguments), and construct a tentative $J_1, \ldots, J_5$. Then, for every point j in $[x_H, y_{H+1}]$, either there exists $i \in J_5 \subset [0, x_H)$ such that $m(i) \geq m(j)$, or else not. In the first case, put j into J_2 and move i into J_4; in the second, put j into J_3. Finally, put $[y_{H+1}, N)$ into J_5.

It is clear that this is the desired partition.

□

Our result is now clear (again redefining ε) from the following four lemmas:

<u>Lemma 5.10a</u>: $\int_{J_1} A(z)dz = 0$.

<u>Lemma 5.10b</u>: $\int_{J_5} A(z)dz \geq 0$.

<u>Lemma 5.10c</u>: $\int_{J_3} A(z)dz > -\varepsilon$.

<u>Lemma 5.10d</u>: $\int_{J_2 \cup J_4} \geq -6\tilde{M}\varepsilon$.

Lemma 5.10a is obvious, since $A(z) \equiv 0$ on J_1 by construction. Similarly, $A(z)$ is increasing on J_5, and so Lemma 5.10b holds.

Proof of Lemma 5.10c:

$$\int_{J_3} A(z)dz = \sum_{j_i \in J_3} A([j_i])\cdot(j_{i+1} - j_i)$$

$$\geq -(1+\varepsilon)^2 \sum_{j_i \in J_3} u^n_{[j_i]}(m(j_{i+1}) - m(j_i))\cdot \prod_{i=1}^{[j_i]} (1 - P_*(i)) ,$$

by (5.8) and (5.9),

$$\geq -(1+\varepsilon)^2\cdot(1+\varepsilon) \sum_{\ell=0}^{\infty} (u_0 q^{\ell})^n (\tilde{m}(\ell+1) - \tilde{m}(\ell))$$

$$= -(1+\varepsilon)^3 \sum_{\ell=0}^{\infty} (u_0 q^{\ell})^{1/2}$$

$$= -(1+\varepsilon)^3 \frac{u_0^{1/2}}{1 - q^{1/2}}$$

$$> -\varepsilon .$$

Proof of 5.10d: Let $\gamma : J_4 \to J_2$ be ϕ^{-1} .

$$\int_{J_2 \cup J_4} A(z)dz = \sum_{j_i \in J_2} A([j_i])\cdot(j_{i+1} - j_i) + \sum_{j_i \in J_4} A([j_i]\cdot(j_{i+1} - j_i)$$

$$\geq - \sum_{j_i \in J_2} (1+\varepsilon)^2(m(j_{i+1}) - m(j_i))\cdot u^n_{[j_i]} \prod_{i=1}^{[j_i]} (1 - P_*(i))$$

$$- \sum_{j_i \in J_4} (1-\varepsilon)^2(m(j_{i+1}) - m(j_i))\cdot u^n_{[j_i]} \prod_{i=1}^{[j_i]} (1 - P_*(i))$$

$$\geq \sum_{j_i \in J_4} ((1-\varepsilon)^2 u^n_{[j_i]} - (1+\varepsilon)^2 u^n_{[\gamma(j_i)]})(m(j_{i+1}) - m(j_i)) \prod_{i=1}^{[j_i]} (1 - P_*(i)) \tag{5.11}$$

by the defining properties of ϕ..

Now, $u_{[\gamma(j_i)]} = q^{\lambda} u_{[j_i]}$, where $\lambda = -1, 0, 1$ (to see this, observe that if $\tilde{m}(i) \leq \bar{m}_{N-1} \leq \tilde{m}(i+1)$, $\nu(N)$ must $= i$ or $i+1$). Thus (5.11)

$$\geq ((1-\varepsilon)^2 - q^{-n}(1+\varepsilon)^2) \cdot \sum_{j_i \in J_4} u^n_{[j_i]}(m(j_{i+1}) - m(j_i) \cdot \prod_{i=1}^{[j_i]} (1 - P_*(i))$$

$$\geq -5\varepsilon \sum_{k=1}^{\infty} \hat{M}(1+\varepsilon)^2 P_*(k) \prod_{i=1}^{k-1} (1 - P_*(i))$$

by the properties of m and the fact that $|v_\infty(s^*)| \leq \hat{M}$ for all $s^* \in S^*$,

$$\geq -6\hat{M}\varepsilon \ .$$

This completes the proof of Lemma 5.10d, hence of Proposition 5.6, and hence of Proposition 5.2. □

6. THE NON-ABSORBING CASE

We now prove

Proposition 5.3:

$$\liminf_{T\to\infty} \int_{T_\infty} \frac{1}{N}\sum_{i=1}^{N} d_i \, d\mu > -\varepsilon \ .$$

Proof:

Lemma 6.1:

Let $\{e_{ik}\}_{k\to\infty}$ converge uniformly.

Let $E = \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} (\lim_{k\to\infty} e_{ik})$

Then $E = \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} e_{jj}$.

Proof: Easy.

□

Identifying e_{ik} as $\int_{(T_1 \cup \ldots \cup T_k)^c} d_i d\mu$, we observe that

$$\left|\int_{T_\infty} d_i d\mu - \int_{(T_1 \cup \ldots \cup T_k)^c} d_i d\mu\right| \leq \hat{M} \sum_{k+1}^{\infty} \mu(T_k) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus Lemma 6.1 gives

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \int_{T_\infty} d_i d\mu = \liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \int_{(T_1 \cup \ldots \cup T_i)^c} d_i d\mu \ .$$

Let $\omega_i(u) = W_{s_i}(u) - W_{s_{i-1}}(u)$. Then, by Proposition 4.1,

$$\int_{(T_1 \cup \ldots \cup T_i)^c} d_i d\mu \geq \int_{(T_1 \cup \ldots \cup T_i)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1}) + \eta(u_{i-1})) d\mu$$

$$\geq \int_{(T_1 \cup \ldots \cup T_i)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1})) d\mu - \varepsilon \cdot \mu((T_1 \cup \ldots \cup T_c)^c) \ .$$

Of course, also,

$$\int_{(T_1 \cup \ldots \cup T_i)^c} d_i d\mu \geq -\tilde{M}\mu((T_1 \cup \ldots \cup T_c)^c) \ .$$

Thus, setting

$$f_k(t(k-1)) = \begin{cases} \max(-\tilde{M},\ \delta_k(u_{k-1}) - \omega_k(u_{k-1})) & \text{if } s(t_{k-1}) \notin S^* \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\int_{(T_1 \cup \ldots \cup T_i)^c} d_i \, d\mu \geq \int_{(T_1 \cup \ldots \cup T_i)^c} f_i(t(i-1))\, d\mu - \varepsilon$$

$$= \int_{T^\infty} f_i(t(i-1))\, d\mu - \varepsilon \ .$$

Hence, to establish Proposition 5.3, it is

Enough to show:

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_{T^\infty} f_k(t(k-1))\, d\mu \geq 0 \ . \tag{6.2}$$

Applying Proposition 5.5, for each N ,

$$\int_{T^\infty} \sum_{k=1}^{N} f_k(t(k-1))\, d\mu = \int_{T^\infty} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k-1} (1 - P_*(i))\, d\mu \ .$$

Thus

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1} \int_{T^\infty} f_k \, d\mu = \liminf_{N\to\infty} \int_{T^\infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k-1} (1 - P_*(i))\, d\mu$$

$$\geq \int_{T^\infty} \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f_k \prod_{i=1}^{k-1} (1 - P_*(i))\, d\mu \ ,$$

by Fatou's Lemma (Royden [1963]).

So, if we establish

Lemma 6.3: For all $t \in T^\infty$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f_k(t(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) \geq 0 \ ,$$

we are done.

<u>Proof</u>: Suppose we know

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_k(\mathcal{t}(k-1))\cdot \prod_{i=1}^{k} (1-P_*(i)) \geq 0 .$$

Then either there exists N such that $k > N$ implies that $1 - P_*(k) > \frac{1}{2}$, in which case we are done immediately, or else

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_k(\mathcal{t}(k-1))\cdot \prod_{i=1}^{k-1} (1-P_*(i))$$

$$\geq \liminf_{N\to\infty} \left(-\frac{\tilde{M}}{N}\sum_{k=1}^{N} \prod_{i=1}^{k-1} (1-P_*(i)) \right) = 0 .$$

But we can in fact show the stronger

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f'_k(\mathcal{t}(k-1))\cdot \prod_{i=1}^{k} (1-P(\text{abs} : \mathcal{t},\ i-1)) \geq 0 , \qquad (6.3')$$

where $f'_k = \delta_k - \omega_k \leq f_k$.

Letting $P_i = P_*(i)$, for all i, we have

$$\sum_{k=1}^{N} f'_k \cdot \prod_{i=1}^{k} (1-P_i) = \sum_{k=1}^{N} f'_k \cdot \prod_{i=1}^{N} (1-P_i)$$

$$+ \sum_{j=1}^{N-1} \Big(\sum_{k=1}^{j} f'_k \prod_{i=1}^{j} (1-P_i)\cdot P_j \Big) . \qquad (6.4)$$

<u>Lemma 6.5</u>: There exists a number Ω_0 such that for all $\mathcal{t}$, for all N, for all N_0, $\sum_{k=1}^{N_0} f'_k \leq 0$ implies that

$$\sum_{k=N_0+1}^{N} f'_k \cdot \prod_{i=N_0+1}^{N} (1-P_*(i)) > -\Omega_0 .$$

(In particular, this conclusion holds when $N_0 = 0$.)

Proof: $\sum_{k=1}^{N_0} f'_k \leq 0$ implies that $u_{N_0} = u_0$ or u_1 . Write $\sum_{k=N_0+1}^{N} f'_k = -M - 2\overline{W}(u_1)$; assume

$$M \geq 0 (\text{else} \sum_{k=N_0+1}^{N} f'_k \cdot \prod_{i=N_0+1}^{N} (1 - P_i) \geq -2\overline{W}(u_1)) .$$

Then

$$\sum_{k=N_0+1}^{N} \delta_k \leq \sum_{k=N_0+1}^{N} (\delta_k - \omega_k) + 2\overline{W}(u_1)$$

$$= -M - 2\overline{W}(u_1) + 2\overline{W}(u_1)$$

$$= -M . \qquad (6.6)$$

Recalling Proposition 4.2, and noting that

$$|v_\infty(s)| \leq M \quad \text{for all} \quad s \in S ,$$

(6.6) implies that

$$\sum_{k=N_0+1}^{N} P_*(k) \geq \frac{(1-\varepsilon)}{\tilde{M}} \cdot Mu_1^n$$

$$> \frac{Mu_1^n}{2\tilde{M}} ;$$

hence

$$\sum_{k=N_0+1}^{N-1} (1 - P_*(k) \leq \prod_{k=N_0+1}^{N-1} (1 - P_*(k))$$

$$\leq e^{-\frac{Mu_1^n}{2\tilde{M}}} ,$$

by a well-known inequality (which can be derived immediately from the observation $\ln(1-P) < -P$ for $0 < P < 1$). Thus

$$\sum_{k=N_0+1}^{N} f'_k \prod_{i=N_0+1}^{N} (1-P_i) > (-M - 2\overline{W}(u_1))e^{-\frac{Mu_1^n}{2\tilde{M}}}$$

$$- \frac{2\tilde{M}}{u_1^n}\cdot\frac{1}{e} - 2\overline{W}(u_1) \ .$$

So if we set $\Omega_0 = 3\tilde{M}u_1^{-n}$, we are done.

Returning now to the proof of (6.3'), we distinguish two cases:

<u>Case 1</u>: $\sum_{k=1}^{\infty} P_k < \infty$.

Then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1-P_i)$$

$$\geq \liminf_{N\to\infty} \frac{1}{N}(-\Omega_0 + \sum_{j=1}^{N-1} (-\Omega_0)\cdot P_{j+1}) \ ,$$

by (6.4) and Lemma 6.5,

$$\geq \liminf_{N\to\infty} \left(-\frac{\Omega_0}{N}\right)\cdot\left(1 + \sum_{k=2}^{\infty} P_k\right)$$

$$= 0 \ .$$

<u>Case 2</u>: $\sum_{k=1}^{\infty} P_k = \infty$.

Case 2a: There exists N_1 such that $N > N_1$ implies $\sum_{k=1}^{N} f_k > 0$.

Then

$$\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i)$$

$$= \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{N} (1 - P_i) + \sum_{j=1}^{N_1} \left(\sum_{k=1}^{j} f_k \cdot \prod_{i=1}^{j} (1 - P_i) \cdot P_{j+1} \right)$$

$$+ \sum_{j=N_1+1}^{N-1} \left(\sum_{k=1}^{j} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \cdot P_{j+1} \right)$$

$$\geq -\Omega_0 + \sum_{j=1}^{N_1} (-\Omega_0) \cdot P_{j+1} + \text{positive terms;}$$

hence

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i)$$

$$\geq \liminf_{N \to \infty} \frac{\text{constant}}{N} = 0 \ .$$

Case 2b: There exists no such N_1 . Then for arbitrary $\varepsilon^* > 0$, there exists N^* such that

$$\sum_{i=1}^{N^*} (1 - P_i) < \varepsilon^* \quad \text{and}$$

$$\sum_{k=1}^{N^*} f_k \leq 0 \ .$$

Let $N > N^*$.

Then

$$\sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i)$$

$$= \sum_{k=1}^{N^*} f_k \cdot \prod_{i=1}^{k} (1 - P_i) + \sum_{k=N^*+1}^{N} f_k \cdot \prod_{k=N^*+1}^{k} (1 - P_i) \cdot \prod_{i=1}^{N} (1 - P_i)$$

$$\geq -\Omega_0 - \sum_{j=1}^{N^*-1} \Omega_0 \cdot P_{j+1} - \varepsilon^* - \Omega_0 - \sum_{j=N^*+1}^{N-1} \Omega_0 \cdot P_{j+1}$$

by Lemma 6.5 and a slight extension of (6.4),

$$\geq -\text{constant} - N\varepsilon^*\Omega_0 \ ;$$

hence $$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \geq -\Omega_0 \varepsilon^* \ .$$

But ε^* was arbitrary:

hence $$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f_k \cdot \prod_{i=1}^{k} (1 - P_i) \geq 0 \ .$$

This completes the proof of equation (6.3'), hence of Lemma 6.3, hence of Proposition 5.3, hence of Proposition 5.1, hence of Proposition 3.1, and hence of Theorem I.

Q.E.D. □

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