4-1-1982

Stochastic Games I: Foundations

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COWLES FOUNDATION DISCUSSION PAPER NO. 623

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1.

This paper strives to provide a sound underpinning for the theory of stochastic games. Section 2 is a reworking of the Bewley-Kohlberg result integrated with Shapley's; the "black magic" of Tarski's principle is replaced by the "gray magic" of the Hilbert Nullstellensatz. Section 3 explicates the underlying topology and measure theory; I believe it is as necessary for Mertens and Neyman's proof of the minmax theorem [8] as it is for mine [9, 10]. Finally, Section 4 establishes a result on this sort of structure which may be of some independent interest; in any case, it is critical for the argument in [10].

This work was strongly influenced by Andrew Gleason. Other acknowledgments may be found in [10].

2.

Fix a stochastic game. In this section we study the r-discount game, following [11] and [3].

Let S, the state set, A (resp. B) the choice set each turn for A (resp. B), and C, with fixed measurable function $q : [0,1] \to C$ all be finite sets; let $\delta : S \times A \times B \times C \to S$ and $d : S \times A \times B \times C \to [-\tilde{M}, \tilde{M}]$ give the state and the outcome resulting from choices (starting in state $s \in S$) of $a \in A$, $b \in B$ and random $c \in C$. Without loss of generality, these sets and functions describe the game completely. Consider now the stochastic game, with fixed starting state s, and discount rate $r \in (0,1)$; that is, let the payoff function be $\sum_{i=1}^{\infty} d_i(1-r)^{i-1}$, where d_i the payoff on the i^{th} step.

As $\sum_{i=1}^{\infty} d_i(1-r)^{i-1}$ is a continuous payoff function on a pair of compact strategy spaces (see Section 3) the min-max theorem follows immediately (this well-known result follows rapidly from [11]); let the value of the game, for starting state s, be $V_s(r)$. Following [3] and [11] we characterize the values and optimal strategies, by considering the system of equations:

$$U_s(r) = \operatorname{val}(\operatorname*{Exp}_C(d(s,a,b,c)) + (1-r)\sum_{\bar{s}\in S} P(\bar{s})\cdot U_{\bar{s}}(r)) \ . \qquad (2.1s)$$

Here $P(\bar{s}) = \text{measure}(\{t \in [0,1] : \delta(s,a,b,q(t)) = \bar{s}\})$ depends upon a, b, and s; val is just the ordinary min-max value of the matrix. For fixed r, think of the system as one equation in a variable in $c^{|S|}$. Applying the Contraction Mapping Theorem ([6], p. 229), it must have a unique solution $(\tilde{U}_s(r))$. Indeed, as the contraction constant $\|1-r\|$ is bounded away from 1 on any interval $[r^*, 1]$, and indeed on a set $\{x+yi \in C : r^* \leq x \leq 1, 0 \leq y \leq \delta(r^*)\}$, $(\tilde{U}_s(r))$ is a continuous function of r in (a neighborhood of) $(0,1)$.

Proposition 2.2: For $r \in (0,1)$, this solution is precisely $(V_s(r))$.

Proof: It is clearly enough to show the following, for fixed $r \in (0,1)$:

> By playing as his i^{th} move an optimal strategy in the game given by the matrix of $(2.1s_{i-1})$, Mr. A guarantees himself a payoff of at least $\tilde{U}_{s_0}(r)$. (2.3)

(2.3) is clearly equivalent to:

$$\tilde{U}_{s_0}(r) \geq V_{s_0}(r) \quad \text{for all} \quad s_0 \in S . \tag{2.4}$$

So suppose (2.4) is false. Let $\alpha = \max_S (V_{s_0}(r) - \tilde{U}_{s_0}(r))$; choose $\tilde{s}$ so that the max is achieved.

Then

$$\tilde{U}_{\tilde{s}}(r) = \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(\tilde{s},a,b,c)) + \sum_{\bar{s}\in S}(1-r)P(\bar{s})\cdot\tilde{U}_{\bar{s}}(r))$$

$$= -(1-r)\alpha + \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(\tilde{s},a,b,c)) + \sum_{\bar{s}\in S}(1-r)P(\bar{s})\cdot(\tilde{U}_{\bar{s}}(r) + \alpha))$$

$$\geq -(1-r)\alpha + \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(\tilde{s},a,b,c)) + \sum_{\bar{s}\in S}(1-r)P(s)\cdot V_{\bar{s}}(r)) .$$

So

$$\tilde{U}_{\tilde{s}}(r) + \alpha > \underset{C}{\mathrm{val}}(\mathrm{Exp}(d(\tilde{s},a,b,c)) + \sum_{\bar{s}\in S}(1-r)P(s:a,b,\tilde{s})\cdot V_{\bar{s}}(r)) . \tag{2.5}$$

But since both sides of (2.5) equal $V_{\tilde{s}}(r)$, we have a contradiction.

□

It is well-known that the value of a matrix is one of a finite list of rational functions of the coefficients (namely the value of some square submatrix; see [8], p. 76). Thus the system may be viewed as a set of polynomial equations--identifying S as $\{1, \ldots, k\}$, where $k = |S|$ -- $g_1(x_1, \ldots, x_k) = 0, \ldots, g_k(x_1, \ldots, x_k) = 0$, where the coefficients lie in the algebraically closed field F . Now, if the ideal $(g_1, \ldots, g_k)$ --that is, $\{\sum_{i=1}^{k} h_i g_i : h_i \in F, i = 1, \ldots, k\}$ --equals the unit ideal (1), then some linear combination $\sum_{i=1}^{k} h_i g_i = 1$, $h_i \in F$ for $i = 1, \ldots, k$; this is impossible because $\sum_{i=1}^{k} h_i g_i((V_s)) = \sum_{i=1}^{k} h_i \cdot 0 \equiv 0$ <u>when viewed as functions of</u> r .

By the Hilbert Nullstellenstaz (weak form; see [2], p. 69), then, the only other possibility is that $\{(x_1, \ldots, x_k) \in F^k : g_1(x_1, \ldots, x_k) = 0, \ldots, g_k(x_1, \ldots, x_k) = 0\}$ is non-empty. On the other hand, we saw above that the solution is unique in the much larger field of <u>all</u> functions of r ; hence it is certainly unique in F , and precisely equal to $(V_s(r))$. But an algebraic function which maps reals to reals must indeed lie in F' = the field of <u>real</u> algebraic functions $\subset F$. Finally, observe that each $V_s(r)$ equals, at each point, one of a finite list of algebraic functions; each pair of these can cross each other only finitely often (for their difference is an algebraic function, which can only have finitely many zeroes). Thus by the continuity of the $V_s(r)$, there exists an interval $(0,\tilde{r})$ on which the $V_s(r)$ are algebraic, for all $s \in S$.

F' has a natural ordering: $f > g$ if and only if there exists an $\tilde{\tilde{r}}$ such that $f > g$ on the interval $(0,\tilde{\tilde{r}})$. Consider the games (2.1s) with the values $V_s(r)$ substituted in place of the $U_s(r)$; then these are ordinary two-person zero-sum games, with coefficients in F' , and hence have optimal strategies $\{f(a,s;r)\}$; by construction, for each a and s , $f(a,s;r)$ belongs to F' . For each r , then, we have optimal stationary strategies, algebraic in r, for the r-discount game. Finally, it is well known that algebraic functions have Puiseaux (fractional power) series expansions—see [1]--and so we have checked the

Theorem of Bewley and Kohlberg: The values of the r-discount game, and the stationary optimal strategies, have Puiseaux expansions.

3.

More generally, consider an infinite decision tree such that, at each node, the choice set is finite. By possibly duplicating certain branches, we may assume that, without loss of generality, there exist finite sets Z_i , $i = 1, 2, \ldots$, such that Z_1 consists simply of the first node, Z_2 is the choice set at the first node, Z_3 is the choice set at the second node, <u>no matter which node was chosen as the second node</u>, and so on. Let Q be the space of paths on the tree; $Q = Z_1 \times Z_2 \times \ldots$ is compact by Tychonoff's theorem ([9], p. 144).

Simultaneously fix probability distributions on the choice set at each node. Then these distributions induce a sequence of functions $\mu_i : Z_1 \times \ldots \times Z_i \to \mathbb{R}$ such that

$$\sum_{z \in Z_i} \mu_i(z_1, z_2, \ldots, z_{i-1}, z) = \mu_{i-1}(z_1, \ldots, z_{i-1})$$

for any fixed $z_1, \ldots, z_{i-1}$, for any $i = 1, 2, \ldots$.

The following is a special case of the Kolmogorov Extension Theorem ([5]).

<u>Proposition 3.1</u>: There exists a measure μ on Q such that $\mu(\{z_1\} \times \{z_2\} \times \ldots \times \{z_i\} \times Z_{i+1} \times Z_{i+2} \times \ldots) = \mu_i(z_1, \ldots, z_i)$, for all $z_1, \ldots, z_i$, for all i .

<u>Proof</u>: Consider the ring generated by $\mathcal{Z}$ = the collection of sets of the form

$$\{z_1\} \times \ldots \times \{z_i\} \times Z_i \times Z_{i+1} \times \ldots \ .$$

We clearly have a finitely additive set function μ with the desired properties; to check that μ is a measure, we need only confirm that for any descending tower $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \dots$, $Y_j \in \mathcal{Z}$ for all j , $\bigcap_{j=1}^{\infty} Y_j = \phi$ implies $\lim_{j\to\infty} \mu(Y_j) = 0$ ([4], p. 39). But every element of $\mathcal{Z}$ is both open and closed in the direct product topology; hence the same is true of every member of the ring. Thus, by the compactness of Q , any tower descending to 0 has $Y_j = \phi$ for all j sufficiently large; hence $\lim_{j\to\infty} \mu(Y_j) = 0$ trivially. Finally, since $\mathcal{Z}$ is a base for the direct product topology, the Caratheodory Extension Theorem extends to the Borel field ([4], p. 54).

□

Denote by τ a function which assigns to every node in N , the set of nodes, a probability distribution on its choice set; denote the set of all possible τ by M , the space of <u>mixed strategies</u>. Then every $\tau \in M$ induces a probability measure μ_τ on Q . Let $P \in M$, the space of <u>pure strategies</u>, be the set of all $\pi \in M$ such that, at every node, π selects some alternative with probability one. Then there exists a canonical mapping $f : P \to Q$, sending $\pi \in P$ into the path determined by π . τ also induces a probability measure θ_τ on P , and we have $\theta_\tau = \mu_\tau \circ f$.

Partition N into two disjoint subsets $N_B \cup N_C$ where N_B will be thought of as the set of <u>decision nodes</u>, and N_C as the set of <u>chance nodes</u>. Then we can write $M = M_B \times M_C$, where an element $\tau_B(\tau_C)$ of $M_B(M_C)$ is an assignemnt to each node in $N_B(N_C)$ a probability distribution on its choice set. Similarly $P = P_B \times P_C$.

$\theta_{(\tau_B,\tau_C)}$ then decomposes as a product measure:

$$\theta_{(\tau_B,\tau_C)} = \theta_{\tau_B} \times \theta_{\tau_C} .$$

Assume now that τ_C is fixed. Letting d_i , $i = 1, 2, \ldots$ be a sequence of measurable functions on Q , bounded below, we have

Proposition 3.2:

$$\inf_{\tau_B \in \mathcal{M}_B} \liminf_{N\to\infty} \int_Q \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\mu_{(\tau_B,\tau_C)}$$

$$= \inf_{\pi_B \in \mathcal{P}_B} \liminf_{N\to\infty} \int_Q \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\mu_{(\pi_B,\pi_C)} .$$

Thus thinking of $\mathcal{N}_B$ as the nodes controlled by the decision-maker and $\mathcal{N}_C$ as those controlled by nature (with nature's "strategy" known in advance, the decision-maker "might as well" play a pure strategy as a mixed one. This is true even in a slightly stronger sense:

Corollary 3.3: If there exists a mixed strategy such that the infinum is achieved, then it is also achieved by a pure strategy.

Proof of Proposition 3.2: For any i , for any τ_B ,

$$\int_Q d_i d\mu_{(\tau_B,\tau_C)} = \int_{\mathcal{P}} d_i(f(\pi))d\theta_{(\tau_B,\tau_C)}(\pi)$$

$$= \int_{\mathcal{P}_B} \left(\int_{\mathcal{P}_C} d_i(f(\pi_B, \pi_C))d\theta_{\tau_B}(\pi_C)\right)d\theta_{\tau_B}(\pi_B)$$

by Fubini's Theorem ([9], p. 233). Thus of course

$$\int_Q \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\mu_{(\tau_B,\tau_C)} = \int_{\mathcal{P}_B}\int_{\mathcal{P}_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\,\theta_{\tau_C} d\theta_{\tau_B} .$$

Observing that the support of θ_{π_B} is a subset of P_B on which $f(\pi, \pi_C) \equiv f(\pi_B, \pi_C)$, for any $\pi_C \in P_C$, we see:

$$\inf_{\pi_B \in P_B} \liminf_{N\to\infty} \int_{P_B} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C}\, d\theta_{\tau_B}$$

$$= \inf_{\pi_B \in P_B} \liminf_{N\to\infty} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C} \; .$$

But for any fixed τ_B ,

$$\liminf_{N\to\infty} \int_{P_B} \int_{P_C} \left(\frac{1}{N}{}_{i=1}^{N} d_i\right) d\theta_{\tau_C}\, d\theta_{\tau_B}$$

$$\geq \inf_{\pi_B \in P_B} \liminf_{N\to\infty} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C} \; ,$$

by Fatou's Lemma ([9], p. 199); of course, the same inequality holds if we replace the left-hand side by the $\inf_{\tau_B \in M_B}$. But, since $P_B \subset M_B$, the reverse inequality also holds and so

$$\inf_{\tau_B \in M_B} \liminf_{N\to\infty} \int_{P_B} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C}\, d\theta_{\tau_B}$$

$$= \inf_{\pi_B \in P_B} \liminf_{N\to\infty} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C} \; , \text{ and done.}$$

□

<u>Proof of Corollary 3.3</u>: For τ_B such that the infinum is achieved, the set of π in the support of θ_{τ_B} such that $\liminf_{N\to\infty} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C} \geq I + \frac{1}{k}$ is clearly of measure zero, where I is the infinum, and k an arbitrary

positive integer. So the set where

$$\liminf_{N\to\infty} \int_{P_C} \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right) d\theta_{\tau_C} \neq I$$

is indeed of measure zero. Since the support of θ_{τ_B} is of course of measure one, we are done.

□

Note 3.4: The results of this section remain valid if "finite" is replaced by "compact."

Note 3.5: Our "mixed strategies" may not be the most general the decision-maker could adopt. However, strategies with more complicated forms of mixing can be disposed of in the same fashion using the compact form of Proposition 3.2 (see Note 3.4).

4.

Let $\{Z_i\}$, Q , $\{\mu_i\}$, and μ be as in the previous section. Suppose each Z_i has a distinguished subset Z_i^* such that $(Z_1 \times \ldots \times Z_{i-1} \times Z_i^* \times (Z_{i+1}^*)^c \times Z_{i+2}^* \times \ldots) = 0$; in other words, once a node in <u>some</u> Z_i^* is reached, the path remains in the $\{Z_i^*\}$ with probability 1. For a given path $z = (z_1, z_2, \ldots)$, let

$$P_*(i;z) = \begin{cases} \dfrac{\mu(\{z_1\} \times \ldots \times \{z_{i-1}\} \times Z_i^* \times Z_{i+1} \times \ldots)}{\mu(\{z_i\} \times \ldots \times \{z_{i-1}\} \times Z_i \times Z_{i+1} \times \ldots)} & \text{if } z_{i-1} \notin Z_{i-1}^* \\ 0 & \text{if } z_i \in Z_{i-1}^* \end{cases}$$

the probability (conditional on $z_1, \ldots, z_{i-1}$) of entering Z_i^* . Suppose we wish to integrate a function over Q which vanishes outside $(Z_1^*)^c \times (Z_2^*)^c \times \ldots$. Then, intuitively, we can integrate a smaller function, on a new measure which is bigger on $(Z_1^*)^c \times (Z_2^*)^c \times \ldots$, to achieve the same result. This can be helpful if we wish to estimate the integral by estimating the (smaller) function directly.

More precisely, we prove this result in terms of truncated sequences; it is applied in Section 5 of [10]. For any $z = (z_1, z_2, \ldots) \in Q$, define the truncations $z(k) = (z_1, \ldots, z_k) \in Z_1 \times \ldots \times Z_k$. Write also $Q(z_1, \ldots, z_k) = \{z_1\} \times \{z_2\} \times \ldots \times \{z_k\} \times Z_{k+1} \times Z_{k+2} \times \ldots \subset Q$. Then

<u>Proposition 4.1</u>: There exists a probability measure $\tilde{\mu}$ on Q such that, for all k , for all $f_k : Z_1 \times \ldots \times Z_k \to \mathbb{R}$ such that $f_k : Z_1 \times \ldots \times Z_{k-1} \times Z_k^* \to \{0\}$,

$$\int_Q f_k(z(k))d\mu = \int_Q f_k(z(k)) \cdot \prod_{i=1}^{k} (1 - P_*(i;z))d\tilde{\mu} .$$

<u>Proof</u>: By Proposition 2.1, to specify $\tilde{\mu}$ we need only specify $\tilde{\mu}(Q(z_1, \ldots, z_i))$, for all $z_1, \ldots, z_i$, for all i, while checking that

$$\sum_{\bar{z}_i \in Z_i} \tilde{\mu}(Q(z_1, \ldots, z_{i-1}, \bar{z})) = \tilde{\mu}(Q(z_1, \ldots, z_{i-1})) \qquad (4.2)$$

So we define $\tilde{\mu}(Q(z_1, \ldots, z_k))$ inductively on k.

$\tilde{\mu}(Q) = 1$. Fix $z = (z_1, \ldots) \in Q$

$$\tilde{\mu}(Q(z_1)) = \begin{cases} \frac{1}{1 - P_*(1;z)} \cdot \mu(Q(z_1)) & \text{if } z_1 \notin Z_1^* \\ 0 & \text{if } z_1 \in Z_1^* \end{cases} \qquad (4.3)$$

unless $P_*(1;z) = 1$. In that case,

$$\tilde{\mu}(Q(z_1)) = \mu(Q(z_1)) .$$

More generally, suppose we have specified $\tilde{\mu}(Q(z_1, \ldots, z_{i-1}))$. If this equals 0, we let $\tilde{\mu}(Q(z_1, \ldots, z_{i-1}, z_i)) = 0$. If not, then also $\mu(Q(z_1, \ldots, z_{i-1})) \neq 0$, and we let

$$\tilde{\mu}(Q(z_1, \ldots, z_i)) = \begin{cases} \frac{1}{1 - P_*(i;z)} \cdot \frac{\tilde{\mu}(Q(z_1, \ldots, z_{i-1}))}{\mu(Q(z_1, \ldots, z_{i-1}))} \cdot \mu(Q(z_1, \ldots, z_i)) & \\ & \text{if } z_i \notin Z_i^* \\ 0 & \text{if } z_i \in Z_i^* \end{cases}$$

unless, again, $P_*(i;z) = 1$, in which case we let $\tilde{\mu}(Q(z_1, \ldots, z_i)) = \mu(Q(z_1, \ldots, z_i))$.

Now

$$\sum_{\bar{z}_i \in Z_i} \tilde{\mu}(z_1, \ldots, z_{i-1}, \bar{z}_i)$$

$$= \frac{1}{1 - P_*(i;z)} \cdot \frac{\tilde{\mu}(Q(z_1, \ldots, z_{i-1}))}{\mu(Q(z_1, \ldots, z_{i-1}))} \cdot \sum_{\bar{z}_i \notin Z_i^*} \mu(Q(z_1, \ldots, z_{i-1}, \bar{z}_i)) .$$

But $$\sum_{\bar{z}_i \notin Z_i^*} \mu(Q(z_1, \ldots, z_{i-1}, \bar{z}_i)) = (1 - P_*(i;z))\mu(Q(z_1, \ldots, z_{i-1}))$$

by the definition of $P_*(i;z)$, and so (4.2) is checked, and we have defined a measure $\tilde{\mu}$.

It remains to show that

$$\int_Q f_k(z(k))d\mu = \int_Q f_k(z(k)) \cdot \prod_{i=1}^{k} (1 - P_*(i;z))d\tilde{\mu} .$$

We start by observing that

$$\int_Q = \sum_{z_1 \times \ldots \times z_k} \int_{Q(z_1 \times \ldots \times z_k)} .$$

But on $Q(z_1 \times \ldots \times z_k)$, f_k is just a constant, as is $\prod_{i=1}^{k} (1 - P_*(i;z))$, and hence we are asking whether $\sum f(z_1, \ldots, z_k) \cdot \tilde{\mu}(Q(z_1, \ldots, z_k))$ $= \sum f(z_1, \ldots, z_k) \cdot \prod_{i=1}^{k} (1 - P_*(i;z))\tilde{\mu}(Q(z_1, \ldots, z_k))$. This of course follows if

$$\mu(Q(z_1, \ldots, z_k)) = \prod_{i=1}^{k} (1 - P_*(i;z)) \cdot \tilde{\mu}(z_1, \ldots, z_k)) \qquad (4.4)$$

whenever $f_k(z_1, \ldots, z_k) \neq 0$ --i.e., whenever $z_k \notin Z_k^*$.

When $k = 1$, (4.4) is immediate by (4.3). So assume proved for $k-1$, and check for k. Then, for any $z_1 \notin Z_1^*$, we define $\bar{\mu}$ on

$Q(z_1)$ by $\bar{\mu} \equiv \frac{1}{1 - P_*(1;z)} \cdot \mu$. Defining $\tilde{\mu}$ exactly as before, we apply induction to get

$$\prod_{i=1}^{k} (1 - P_*(1;z)) \cdot \tilde{\mu}(Q(z_1, \ldots, z_k))$$

$$= (1 - P_*(1;z)) \cdot \bar{\mu}(Q(z_1, \ldots, z_k)) = \mu(Q(z_1, \ldots, z_k)) \ ,$$

and we are done.

□

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ACKNOWLEDGMENTS

Partially supported by the Office of Naval Research Contract Number N00014-77-0518 and a National Science Foundation Graduate Fellowship.