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Abstract

Sufficient conditions are given for large replica games without side payments to have non-empty approximate cores for all sufficiently large replications. No "balancedness" assumptions are required. The conditions are superadditivity, a very weak boundedness condition, and convexity of the payoff sets. An example is provided to show that under these conditions, the (exact) core well may be empty.

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1. Introduction

Cooperative behavior lies at the very heart of economics and the fundamental concept of a cooperative social equilibrium is the core. However, the power of the core concept is limited by the fact that the non-emptiness of the core can be assured only in certain ideal environments. In this paper, it is shown that all members of a class of games with many players and relatively few types of players have non-empty approximate cores and the approximation can be made better as the number of players increases.

Since Shapley and Shubik [22] first introduced concepts of approximate cores, a number of authors have demonstrated sufficient conditions for non-emptiness of approximate cores of large economies, cf. Kannai [12, 13], Hildenbrand, Schmeidler and Zamir [9], and Khan and Rashid [14]. With the exception of Khan's and Rashid's work, all these results deal only with the exchange of private goods; Khan and Rashid consider production with the firms exogenously given. Recently, this author in [29] obtained sufficient conditions for non-emptiness of approximate cores of large replica economies with a local public good and endogenous jurisdiction formation. All these results are, of course, dependent on the particular formulations of the economies considered.

In this paper we use the framework of n-person game theory. This framework is sufficiently general to accommodate a variety of departures from the classical model of a private goods exchange economy, including increasing returns, coalition production, and the presence of local and pure public goods. The following example illustrates a simple economic model to which the results of this paper can be applied.

Suppose $A = \{1, \ldots, n\}$ is the set agents in the economy and there are two goods, say x and y. Each agent is initially endowed with

1 unit of good x. A unit of good y is produced from a unit good x but the production of y requires input of x and also the joint effort of two agents. One agent, by himself, cannot produce any positive quantity of good y; he can only dispose of the initially endowed good. If three agents work together, some one of the agents only impedes the work of the other two. To formally define production technologies which are consistent with this description, let S be a non-empty subset of agents and let $|S|$ denote the number of agents in S. Then define $Y[S] = \{(x,y) : x \leq 0,\ y \leq -x\}$ if $|S|$ is even and $Y[S] = \{(x,y) : x \leq 0,\ y \leq -x-1\}$ if $|S|$ is odd. In Figure 1, the production technology sets for both cases are depicted. The utility function of agent i is $u^i(x,y) = y$ for each i; agents do not derive any utility from the initially endowed good. From this economic data, for each coalition of agents S, we can define a set $V(S) \subset \mathbb{R}^n$ where $V(S)$ represents the utility levels achievable by the members of S using their own initial endowments. Define $V(S) = \{\bar{u} \in \mathbb{R}^A_+ : \sum_{i \in S} \bar{u}^i \leq |S|$ if $|S|$ is even and $\sum_{i \in S} \bar{u}^i \leq |S| - 1$ if $|S|$ is odd$\}$ (we follow the convention that coordinates of $V(S)$ not associated with members of S are unrestricted). The pair (A,V) is a game (a formal definition of a game is provided later). It is easy to see that for this simple model, if $\bar{u}$ is in $V(S)$ then the members of S, using only their own initial endowments, can produce $(y^i : i \in S)$ such that $u^i(y^i) = \bar{u}^i$ for each $i \in S$. In Figure 2, the maximal sum of the utilities achievable by the members of a coalition S, using their own resources is sketched (ignoring indivisibilities of agents). Some features of this model to note are: (1) $V(S)$ is convex for each coalition S; and (2) $V(S) \cap V(S') \subset V(S \cup S')$ for any disjoint coalitions S and S', i.e. V is superadditive. Observe that even though preferences are convex, production sets for each coalition S are convex, and $V(S)$ is convex

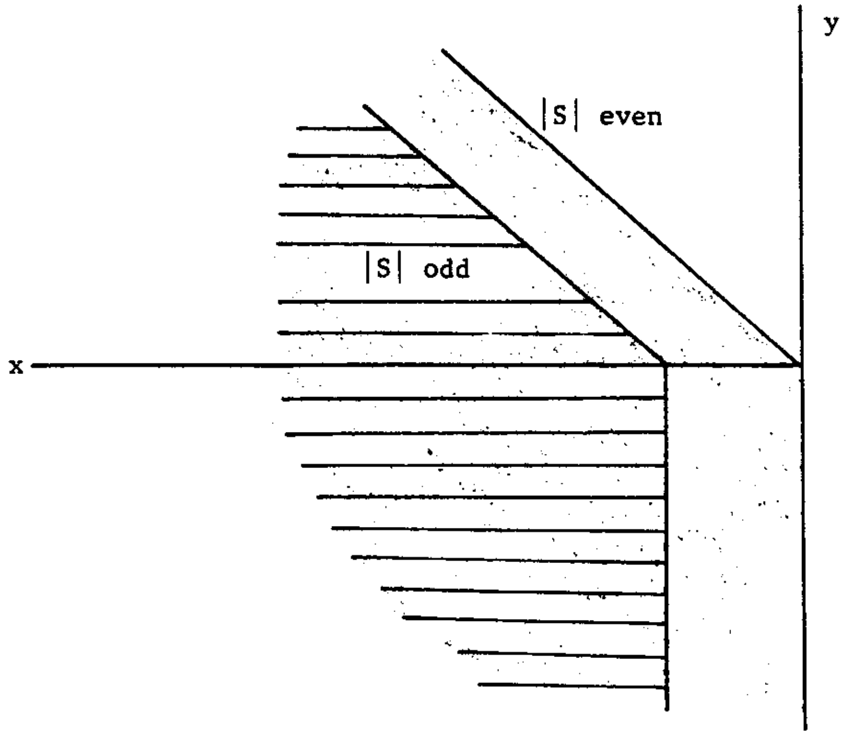

FIGURE 1

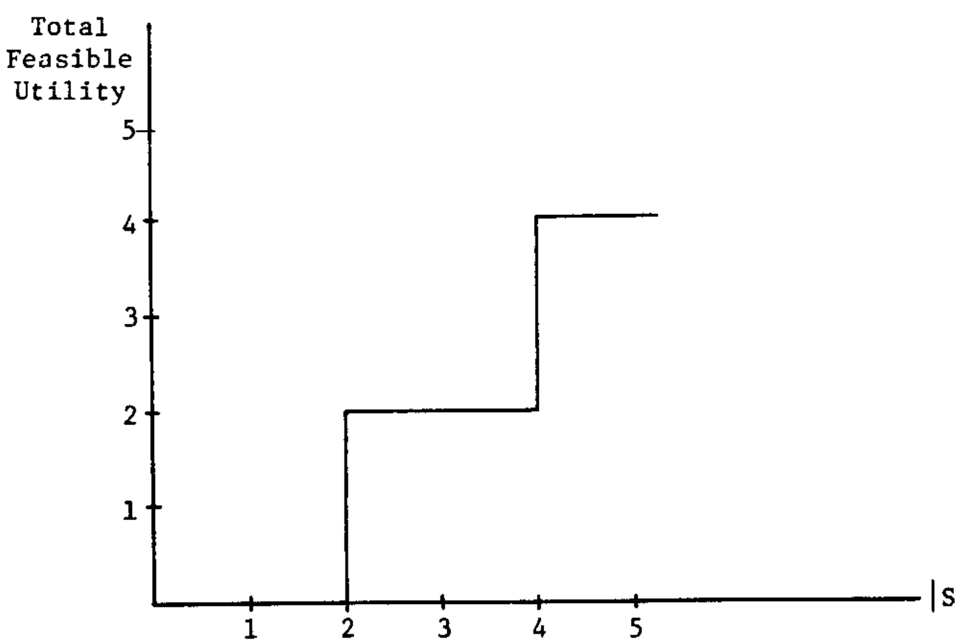

FIGURE 2

for all coalitions S of agents, for $n > 2$ the core of the game is non-empty if and only if n is an even number. As our main theorem states, however, given any $\varepsilon > 0$ for all sufficiently large n an approximate ε-core is non-empty.

A simplifying feature of the above example is that the game (A,V) is one with side-payments, i.e. for each non-empty subset of agents S there is a real number, say $v(S)$, such that $\bar{u} \in V(S)$ if and only if $\sum_{i \in S} \bar{u}^i \leq v(S)$. It is easy to generate examples of games without side payments to which our results apply. For example, for each non-empty subset S of A when $|S|$ is even define $V(S) = \{\bar{u} \in \mathbb{R}^A : \sum_{i \in S} \bar{u}_i^{-2} \leq K|S|\}$ and, when $|S|$ is odd define $V(S) = \{\bar{u} \in \mathbb{R}^A : \sum_{i \in S} \bar{u}_i^{-2} \leq K(|S| - 1)\}$ where K is some positive constant. In Shubik and Wooders [25], we provide additional examples of games to which our results can be applied. For the present purposes, the point we wish to make via the example is that even though $V(S)$ is convex for all coalitions S and the game is superadditive, the game may have an empty core and such games can arise as games derived from economic models. We remark that Scarf [16] has shown that "convex economies" have non-empty cores but this result depends on the economy being a private goods exchange economy.

In this paper, we develop the concept of a sequence of replica games without side payments (i.e., with not-necessarily-transferable utilities). More specifically, we consider a sequence of games $(A_r, V_r)_{r=1}^{\infty}$ where A_r is the set of players of the r^{th} game, consisting of r players of each of T "types," and V_r is a correspondence from subsets of A_r to $\mathbb{R}^{rT}$. Given any coalition S contained in A_r, the set $V_r(S)$ describes the utility vectors achievable by the members of S. We assume that $A_r \subset A_{r+1}$ for all r. The sequence is then said to be a sequence of replica games if (a) all players of the same type are substitutes for each other, and (b) $V_r(S)$ does not "decrease" as r increases; i.e., if $S \subset A_r$ and $S \subset A_{r'}$, where $r \leq r'$, then the projection of $V_r(S)$ on the subspace associated with the members of S is contained in that of $V_{r'}(S)$.[1] Simple and quite general conditions are demonstrated under which given any $\varepsilon > 0$ there is an r^* such that for all $r \geq r^*$ the game (A_r, V_r) has a non-empty ε-core. The conditions are that the games are superadditive, that $V_r(A_r)$ is convex for all r, and a "per-capita boundedness" condition. For the present, we remark that for games with side payments, the per-capita boundedness assumption puts an upper limit on the average utility obtainable by the players of the game.[2] We also remark that our conditions are sufficiently general to include, as a special case, sequences of games derived from sequences of replica economies as in Debreu and Scarf [6], and Shapley and Shubik [22].

Before concluding this introduction, we briefly relate our results to other results concerning non-emptiness of cores and approximate cores of games.

Bondareva [4 , 5] and independently, Shapley [18] introduced the concept of "balancedness" for games with side payments and showed that a game with side payments is balanced if and only if it has a non-empty

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[1]This definition of a sequence of replica games is sufficiently general to include games derived from sequences of private goods economies as in Shubik [23] and Debreu and Scarf [6], of coalition production economies as in Boehm [3], of economies with local public goods as in Wooders [29], and also economies with pure public goods as in Wooders [30]. It does not include the games derived from sequences of economies with a pure public good as developed in, for example, Milleron [15] because the method used there of "replicating" the economy is different than that used in the other papers referenced (this is discussed further in Wooders [30]).

[2]In the side payments case, this is simply the assumption that for all r, $\frac{v_r}{r} \leq K$ for some constant K where v_r is a real number such that $V_r = \{\bar{u} \in \mathbb{R}^{A_r} : \sum_{(t,q)\in A_r} \bar{u}^{tq} \leq v_r\}$.

core. In [16], Scarf extended the concept of balancedness to games without side payments and showed that if such a game is balanced, it has a non-empty core. Other authors have demonstrated other conditions sufficient to ensure non-emptiness of the core of a game. In particular, some variations of the concept of balancedness have been studied and shown to ensure non-emptiness of the core; cf. Billera [1, 2]. Shapley [19] introduced the concept of a convex game and showed that convex games have non-empty cores; these results have been extended to games without side payments by Vilkov [26].[3] Shapley and Scarf in [21] showed that games derived from a certain class of economies with indivisibilities are balanced. Numerous other results have been obtained showing that games derived from particular classes of economies have non-empty cores; however, numerous results have shown that games derived from economic models may well have empty cores, cf. Shubik [23, 24], Shapley and Shubik [22], Shapley and Scarf [21], and Greenberg [7].

In [28], this author introduced a concept of "approximate" balancedness, called ε-balancedness, and extended Shapley's and Shubik's result (for the weak ε-core) by showing that under extraordinarily simple conditions, large replica games with side payments are ε-balanced and have non-empty ε-cores. The conditions are simply that the sequences of games are superadditive and per-capita bounded. This present paper begins the study of approximate cores of replica games without side payments. In Shubik and Wooders [25], alternative and less restrictive concepts of approximate cores than the one used herein are investigated and results analogous to the main result of this paper are obtained without any convexity assumptions.

[3] Convexity of $V(A)$ does not imply that the game (A,V) is a convex game.

We remark that other authors, in particular Weber [27] and Ichiishi and Schäffer [10], have shown conditions under which games without side payments and with measure spaces of agents have non-empty approximate cores. These authors have, however, initially assumed the games were balanced, using extensions of the concept of balancedness introduced by Kannai [11] and Schmeidler [17]. In contrast, we require no assumptions of balancedness.

The paper is divided into several sections. In the next section, we introduce some notation. The third section consists of a statement of the model and results. All the results are proven in Section 4. Section 5 concludes the paper. In the appendix, a technical result used in the paper is developed.

2. Notation

The following notation will be used: $\mathbb{R}^n$: Euclidean n-dimensional space; $\mathbb{R}^n_+$: the non-negative orthant of $\mathbb{R}^n$; given $K \subset \mathbb{R}^n$, int K denotes the interior of the set K ; given a finite set S , $|S|$ denotes the cardinal number of S and $\mathbb{R}^S$ is the Euclidean $|S|$-dimensional space. Define $\underline{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$, we denote the (sup) norm of x by $\|x\|$ where $\|x\| = \max_i \mathrm{ab}(x_i)$ and $\mathrm{ab}(x_i)$ denotes the absolute value of $x_i \in \mathbb{R}^1$.

Given x and y in $\mathbb{R}^n$, we write $x \geq y$ if $x_i \geq y_i$ for all i ; $x > y$ if $x \geq y$ and $x \neq y$; and $x >> y$ if $x_i > y_i$ for all i .

3. The Model and the Results

A game without side payments, or simply a game, is an ordered pair (A,V) where A, called the set of players, is a finite set and V is a correspondence from the set of non-empty subsets of A into subsets of $\mathbb{R}^A$ such that:

(i) for every non-empty $S \subset A$, $V(S)$ is a non-empty, proper, closed subset of $\mathbb{R}^A$ containing some member, say x, where $x >> 0$;

(ii) if $x \in V(S)$ and $y \in \mathbb{R}^A$ with $x^i = y^i$ for all $i \in S$, then $y \in V(S)$;

(iii) $V(S)$ is bounded relative to $\mathbb{R}_+^S$ i.e., for each S, there is a vector $k(S) \in \mathbb{R}^A$, where, for all $x \in V(S)$, $x^i \leq k^i(S)$ for all $i \in S$.

(iv) if $x \in V(S)$ then there is a $y \in V(S) \cap \mathbb{R}_+^A$ such that $y \geq x$.

The above definition differs from the usual definitions of a game in that we've required each payoff set $V(S)$ to contain a strictly positive member and in that we've imposed property (iv). Both these requirements are simply for technical convenience.

Let (A,V) be a game. A vector $x \in \mathbb{R}^A$, where the coordinates of x are superscripted by the members of A, is called a payoff for the game. A payoff x is feasible if $x \in V(A)$. Given a payoff x and players i and j, let $\sigma[x;i,j]$ denote the payoff formed from x by permuting the values of the coordinates associated with i and j. Players i and j are substitutes if: for all $S \subset A$ where $i \notin S$ and $j \notin S$, given any $x \in V(S \cup \{i\})$, we have $\sigma[x;i,j] \in V(S \cup \{j\})$; and, for all $S \subset A$ where $i \in S$ and $j \in S$, given any $x \in V(S)$, we have $\sigma[x;i,j] \in V(S)$. The game is superadditive if whenever S and S' are disjoint, non-empty subsets of A, we have $V(S) \cap V(S') \subset V(S \cup S')$. It is comprehensive if for any non-empty subset S of A, if $x \in V(S)$ and $y \leq x$ then $y \in V(S)$.

Given a game (A,V) and $\varepsilon \geq 0$, a payoff x is in the ε-core of (A,V) if (a) x if feasible and if, (b) for all non-empty subsets S of A, there does not exist an $x' \in V(S)$ such that $x' \gg x + \varepsilon \underline{1}$. When $\varepsilon = 0$, we call the ε-core simply the core. When (A,V) is comprehensive, condition (b) is equivalent to the condition that $x + \varepsilon \underline{1} \notin \text{int } V(S)$ and our definition of the ε-core corresponds to that used by other authors, such as Weber [27]. When (A,V) is comprehensive and, in addition, $\varepsilon = 0$, the ε-core is equivalent to the (exact) core in Scarf [16].

Given a game (A,V), define $V^P(S)$ by $V^P(S) = \{x \in \mathbb{R}^S$: for some $x' \in V(S)$, x is the projection of x' on $\mathbb{R}^S\}$ where $\mathbb{R}^S$ is the subspace of $\mathbb{R}^A$ associated with the members of S. (Note that if (A,V) and (A,V') are two games where $V^P(S) = V'^P(S)$ for all $S \subset A$, then the correspondences V and V' are identical by condition (ii) above.)

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of games where, for each r, $A_r \subset A_{r+1}$ and $A_r = \{(t,q) : t \in \{1, \ldots, T\}, q \in \{1, \ldots, r\}\}$. Write $x = (x_1, \ldots, x_q, \ldots, x_r)$ for a payoff for the r^{th} game where $x_q = (x^{1q}, \ldots, x^{tq}, \ldots, x^{Tq})$ and x^{tq} is the component of the payoff associated with the $(t,q)^{th}$ player. Given r and t, define $[t]_r$ by $[t]_r = \{(t,q) \in A_r : q \in \{1, \ldots, r\}\}$; the set $[t]_r$ consists of the players of type t of the r^{th} game. The sequence $(A_r, V_r)_{r=1}^{\infty}$ is a sequence of replica games if:

(a) for each r and each $t = 1, \ldots, T$, all players of type t of the r^{th} game are substitutes for each other;[4]

(b) for any r' and r'' where $r' < r''$ and any $S \subset A_{r'}$, we have $V^P_{r'}(S) \subset V^P_{r''}(S)$ (i.e., the set of utility vectors achievable by

[4]Players of different types might also be substitutes for each other; thus the requirement that (A_1, V_1) has one player of each type is not as restrictive as it might at first appear.

the coalition S does not decrease as r increases).[5]

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. A payoff x for the game (A_r, V_r) is said to have the <u>equal-treatment property</u> if, for each t, we have $x^{tq'} = x^{tq''}$ for all q' and q"; players of the same type are allocated the same amount. The sequence of games is <u>superadditive</u> if (A_r, V_r) is a superadditive game for all r. The sequence is <u>per-capita bounded</u> if there is a constant K such that for all r and for all equal-treatment payoffs x in $V_r(A_r)$ we have $x^{tq} \leq K$.[6]

The sequence has a <u>non-empty asymptotic core</u> if given any $\varepsilon > 0$ there is an r^* such that for all $r \geq r^*$, (A_r, V_r) has a non-empty ε-core.

<u>Theorem 1</u>. Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive, per-capita bounded replica games where $V_r(A_r)$ is convex for all r. Then the asymptotic core is non-empty.

When a sequence of replica games satisfies the conditions of Theorem 1, and, in addition, the games are comprehensive, we have equal-treatment payoffs in the ε-core (where $\varepsilon > 0$) for all sufficiently large replications. Our next theorem provides a stronger result concerning equal-

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[5] In an initial draft of this paper, we required that $V^P_{r'}(S) = V^P_{r''}(S)$ -- what a coalition S can ensure for its members is independent of the size of the game containing that coalition. This property is common for games derived from sequences of replica economies; cf. Debreu and Scarf [6] and Wooders [29]. The weaker restriction, that when $r' < r''$ we have $V^P_{r'}(S) \subset V^P_{r''}(S)$, permits some "positive enternalities" to benefit the coalition as the set of players is replicated. This is sufficient to permit our results.

[6] It can be easily verified that games derived from sequences of replica economies, such as in Debreu and Scarf [6], Boehm [3], and Wooders [29] all satisfy the per-capita boundedness property. We note that the per-capita boundedness assumption does not rule out the possibility that the sequence $(V_r(A_r))$ is unbounded from above.

treatment payoffs in the ε-cores.

Theorem 2. Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games satisfying the conditions of Theorem 1 and, in addition, assume the games are comprehensive. Then given $\varepsilon > 0$ there is a $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_T) \in \mathbb{R}_+^T$ such that x_r is in the ε-core of (A_r, V_r) for all sufficiently large r where x_r is defined by its coordinates $x_r^{tq} = \bar{x}_t$ for each $(t,q) \in A_r$.

Part of the strategy of the proof of Theorems 1 and 2 is to construct other sequences of games with additional properties and to approximate the games in the original sequence by the constructed games. Since games having these additional properties are of some interest themselves, we introduce these properties here and state an additional result.

We first review the concepts of balancedness and the balanced cover of a game. Let (A,V) be a game. Consider a family β subsets of A and let $\beta_i = \{S \in \beta : i \in S\}$. A family β of subsets of A is balanced if there exists positive "balanced weights" w_S for S in β with $\sum_{S\in\beta_i} w_S = 1$ for all $i \in A$. Let $\mathbb{B}(A)$ denote the collection of all balanced families of subsets of A . Define $\hat{V}(A) = \bigcup_{\beta\in\mathbb{B}(A)} \bigcap_{S\in\beta} V(S)$. Define $\hat{V}(S) = V(S)$ for all $S \subset A$ with $S \neq A$. Then $\hat{V}$ maps subsets of A into $\mathbb{R}^A$ and is called the balanced cover of V . The game $(A,\hat{V})$ is called the balanced cover of (A,V) . If the game (A,V) has the property that $\hat{V}(A) = V(A)$, the game (A,V) is balanced, and from Scarf's theorem [16], the core of the game is non-empty.

A sequence of games $(A_r, V_r)_{r=1}^{\infty}$ satisfies the assumption of quasi-transferable utility, QTU, if, given any r and any S in A_r , if $x > 0$ and x is in the boundary of $V_r^P(S)$, then $V_r^P(S) \cap \{x' \in \mathbb{R}^S : x' \geq x\} = x$. This is called the assumption of quasi-transferable utility because it has the implication that for any $x > 0$, if $x \in V_r^P(S)$ for some subset S and $x' \in V_r^P(S)$ where $x' > x$, then there is an $x'' \in V_r^P(S)$ where $x'' \gg x$ —a property of games with transferable utility. The QTU assumption rules out the possibility that segments of the boundary of $V_r^P(S)$ in $\mathbb{R}_+^S$ are parallel to the coordinate planes.

Given a game (A,V) and any positive number δ, let (A, V^δ) be a game with the QTU property where, for all non-empty subsets S of A, we have $V_r(S) \subset V_r^\delta(S)$ and the Hausdorff distance (with respect to the sup norm) between $V_r(S)$ and $V_r^\delta(S)$ is less than δ.[7] Then (A, V^δ) is called a δ-QTU cover of (A,V). In the appendix, we show that if (A,V) is a comprehensive game, then there is a comprehensive δ-QTU cover of (A,V).

Let $(A_r, V_r)_{r=1}^\infty$ be a sequence of replica games and let S be a non-empty subset of A_r for some r. Define the vector $s \in \mathbb{R}^T$ by the coordinates $s_t = |S \cap [t]_r|$ for each $t \in \{1, \ldots, T\}$; the vector s is called the profile of S. Define $\rho(S) = s$ so $\rho(\cdot)$ maps subsets into their profiles.

A sequence of games $(A_r, V_r)_{r=1}^\infty$ is said to satisfy the assumption of minimum efficient scale (for coalitions), MES, if there is an r^* such that for all $r \geq r^*$, given $x \in \tilde{V}_r(A_r)$ there is a balanced collection $\mathcal{B}$ of subsets of A_r with the properties that (1) $\rho(S) \leq \rho(A_{r^*})$ for all $S \in \mathcal{B}$ and (2) $x \in \bigcap_{S \in \mathcal{B}} V_r(S)$. We call r^* an MES bound. Informally, a sequence of games satisfies the MES property if all "increasing returns to coalition size" are eventually exhausted.

[7] The definition of the Hausdorff distance can be found in Hildenbrand [8], p. 16.

Theorem 3. Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive replica games satisfying the assumptions of QTU and MES with MES bound r^*. For any $r > r^*$, the core of the game $(A_r, \hat{V}_r)$ is non-empty and if x is a payoff in the core, then x has the equal treatment property.

The non-emptiness of the core of the game $(A_r, \hat{V}_r)$ is Scarf's result [16]. It is well-known that for games with side payments, the core is non-empty if and only if it contains a payoff with the equal-treatment property. The result that the MES and QTU properties ensure that all payoffs in the core of a balanced game have the equal-treatment property is new.

5. Proofs of the Theorems

Throughout this section, we let $(A_r, V_r)_{r=1}^{\infty}$ denote a sequence of superadditive replica games with T types of players and let $(A_r, \hat{V}_r)_{r=1}^{\infty}$ be the associated sequence of balanced cover games. We continue to let $\underline{1}$ denote the vector of ones and the reader is to infer from the context the dimension of the space in which $\underline{1}$ is contained. Given r and a positive integer n, we write (A_{nr}, V_{nr}) for the game $(A_{r'}, V_{r'})$ where $r' = nr$.

Given a payoff x for the r^{th} game, (A_r, V_r), when we write $y = \prod_{i=1}^{n} x$ it is to be understood that the coordinates of y are superscripted so that y is a payoff for the nr^{th} game.

Throughout the following, given any $S \subset A_r$ it is to be understood that S is non-empty.

The proofs proceed through several lemmas. The main result of these lemmas is that in the case where $V_r(A_r)$ is convex, the equal-treatment payoffs in $V_r(A_r)$ converge to those in $\hat{V}_r(A_r)$. This result and other lemmas used along the way towards this result are then used to prove the theorems.

Lemma 1. Given r, let $y \in V_r(S)$ for some $S \subset A_r$ where y has the equal-treatment property. Let $S' \subset A_r$ where S' has the same profile as S. Then $y \in V_r(S')$.

Proof. Since S and S' have the same profile, we can define a one to one mapping, say ψ, of S onto S' such that if $\psi((t,q)) = (t', q')$, then $t = t'$. Since for each player (t,q) in S, the player $\psi((t,q))$ is a substitute for (t,q), the payoff y' is in $V_r(S')$ where y' is constructed from y by permuting the values of the coordinates of y associated with (t,q) and $\psi((t,q))$ for each (t,q) in S. Since y has the equal treatment property, $y' = y$ and, since y' is in $V_r(S')$, we have y in $V_r(S')$.

Q.E.D.

Lemma 2. Given r and a positive integer n, let $S \subset A_r$ and let $S' \subset A_{nr}$ where, for some $j \in \{1, \ldots, n\}$, we have $S' = \{(t,q)$: for some $(t', q') \in S$, $t = t'$ and $q = (j-1)r + q'\}$. Then $\prod_{i=1}^{n} V_r(S) \subset V_{nr}(S')$.

Proof. Let $B_j = \{(t,q) \in A_{nr} : t = 1, \ldots, T$ and $(j-1)r < q \leq jr\}$. Let $x' \in \prod_{i=1}^{n} V_r(S)$ and let x denote the projection of x' on the coordinates associated with members of B_j. Observe that $x \in V_r(S)$ and $\prod_{i=1}^{n} x \in V_{nr}(S)$ since $V_r^P(S) \subset V_{nr}^P(S)$ and from (ii) of the definition of a game. Let $y = \prod_{i=1}^{n} x$. From the construction of y, given any (t, q') in A_r, we have $y^{tq} = y^{tq'}$ where $q = (j-1)r + q'$; therefore $\sigma[y; (t, q'), (t,q)] = y$. Since each player (t, q') in A_r is a substitute for the player (t,q) in A_{nr} it follows that $y \in V_{nr}(S')$. Since the projection of y on the coordinates associated with members of S' equals the projection of x' on the same coordinates, $x' \in V_{nr}(S')$. Therefore $\prod_{i=1}^{n} V_r(S) \subset V_{nr}(S')$.

Q.E.D.

Lemma 3. Given any r and any positive integer n, we have

$$\prod_{i=1}^{n} V_r(A_r) \subset V_{nr}(A_{nr}) .$$

Proof. Given $j \in \{1, \ldots, n\}$, let $B_j = \{(t,q) \in A_{nr} : t = 1, \ldots, T$ and $(j-1)r < q \leq jr\}$. From Lemma 2, $\prod_{i=1}^{n} V_r(A_r) \subset V_{nr}(B_j)$ for each $j = 1, \ldots, n$. Since $\{B_j : j = 1, \ldots, n\}$ is a partition of A_{nr}, from superadditivity $\prod_{i=1}^{n} V_r(A_r) \subset \bigcap_{j=1}^{n} V_{nr}(B_j) \subset V_{nr}(A_{nr})$.

Q.E.D.

Given a finite set A, a balanced family $\mathcal{B}$ of subsets of A is a minimal balanced family of subsets if no proper subset of $\mathcal{B}$ is balanced. Our next lemma is a restatement and an easy extension of a result due to Shapley [18, Corollary to Lemma 2], and is stated without proof.

Lemma 4. Let A denote a finite set and let $\mathbf{B}$ denote the collection of all balanced families of subsets of A. Let $\{\mathcal{B}^1, \ldots, \mathcal{B}^\ell, \ldots, \mathcal{B}^L\}$ denote the minimal balanced families of subsets of A. Then (1) $\mathcal{B} \in \mathbf{B}$ if and only if for some subset $L' \subset \{1, \ldots, L\}$, we have $\mathcal{B} = \bigcup_{\ell \in L'} \mathcal{B}^\ell$ and (2) for each ℓ, there is a unique set of balancing weights, w_S^ℓ for $S \in \mathcal{B}^\ell$, and each w_S^ℓ is a rational number.

The next lemma is a key lemma since it relates the sequence of balanced cover games to a subsequence of the sequence of games.

Lemma 5. Given any r, there is a positive integer n such that if $x \in \tilde{V}_r(A_r)$, then $\prod_{i=1}^{n} x \in V_{nr}(A_{nr})$.

Proof. Given r, let $\{\beta^1, \ldots, \beta^\ell, \ldots, \beta^L\}$ be the set of all minimal balanced families of subsets of A_r. From Lemma 4, for each ℓ, there is a unique set of balancing weights, w_S^ℓ for $S \in \beta^\ell$, and each w_S^ℓ is a rational number. Since all the weights w_S^ℓ are rational, we can choose a positive integer n such that nw_S^ℓ is an integer for each $S \in \beta^\ell$ and for all ℓ. We claim that this n satisfies the requirements of the lemma. More specifically, we claim that given any ℓ, there is a partition of A_{nr}, say P, such that given any $x \in \bigcap_{S \in \beta^\ell} V_r(S)$, we have $\prod_{i=1}^{n} x \in \bigcap_{S \in P} V_{nr}(S)$. (The result then follows from superadditivity and properties of minimal balanced families.) We next prove this claim.

Given ℓ, let $\beta^\ell = \{S_1, \ldots, S_k, \ldots, S_K\}$ and, for ease in notation, for each k let w_k denote the associated (rational) balancing weight for $S_k \in \beta^\ell$. We now construct a partition P of A_{nr} such that P contains nw_k members with the same profile as S_k for each $S_k \in \beta^\ell$. For each $(t,q) \in A_r$, let $[(t,q); n] = \{(t', q') : t' = t$ and, for some $j \in \{1, \ldots, n\}$, $q' = (j-1)r + q\}$. Observe that $[(t,q); n]$ contains n players, all of whom are substitutes for each other. For each k, choose nw_k subsets, say $D_k^1, \ldots, D_k^m, \ldots, D_k^{nw_k}$, such that

(1) for each m, if $(t,q) \in S_k$, then $|D_k^m \cap [(t,q); n]| = 1$ and if $(t,q) \notin S_k$, then $|D_k^m \cap [(t,q); n]| = 0$;

(2) for each k', each $m' \leq nw_{k'}$, and each $m \leq nw_k$ we have $D_k^m \cap D_{k'}^{m'} = \phi$ whenever $k \neq k'$ or $m \neq m'$ (or both).

Less formally, the sets D_k^m are selected so that each set D_k^m contains one and only one member of $[(t,q); n]$ for each $(t,q) \in S_k$ and no player appears in any two of the sets $\{D_k^m : 1 \leq k \leq K; 1 \leq m \leq nw_k\}$. We observe that from (1) each set D_k^m has the same profile as S_k. We are now going to show this selection is possible. For each $(t,q) \in A_r$, let $K(t,q) \subset \{1, \ldots, K\}$ be such that $k \in K(t,q)$ if and only if $(t,q) \in S_k$.

Observe that $\left|[(t,q); n] \cap (A_{nr} - \bigcup_{k'=1}^{k-1} \bigcup_{m=1}^{nw_{k'}} D_{k'}^m)\right| = n - \sum_{\substack{k'<k \\ k' \in K(t,q)}} nw_{k'}$.

Since β^ℓ is balanced, $\sum_{k' \in K(t,q)} w_{k'} = 1$ and we have

$n - \sum_{\substack{k'<k \\ k' \in K(t,q)}} nw_{k'} = n \sum_{\substack{k' \geq k \\ k' \in K(t,q)}} w_{k'}$. It follows that it is possible to select subsets $D_k^1, \ldots, D_k^{nw_k}$ satisfying the requirements (1) and (2), i.e, for each k and each $(t,q) \in S_k$, there are enough players in $[(t,q); n]$ and not in $\bigcup_{k'=1}^{k-1} \bigcup_{m=1}^{nw_{k'}} D_{k'}^m$ so that nw_k players can be selected from those remaining players who have not previously been selected. Moreover, since $n - \sum_{k \in K(t,q)} nw_k = 0$ for each $(t,q) \in A_r$, all agents in each set $[(t,q); n]$ are eventually taken in the construction of the sets D_k^m. Therefore the collection $P = \{D_k^m : 1 \leq k \leq K, 1 \leq m \leq nw_k\}$ is a partition of A_{nr}.

Given $\beta^\ell = \{S_1, \ldots, S_k, \ldots, S_K\}$ and P as above, let $x \in \bigcap_{k=1}^{K} V_r(S_k)$. Define $y = \prod_{i=1}^{n} x$. Observe that $y^{t'q'} = x^{tq}$ for all $(t', q') \in [(t,q); n]$ for each $(t,q) \in A_r$. For each D_k^m, the fact that D_k^m consists of one member of $[(t,q); n]$ for each $(t,q) \in S_k$ defines a one to one mapping, say ψ, of the set of agents in S_k onto the set of agents in D_k^m such that for $\psi((t,q)) = (t, q')$ for some (t, q') in D_k^m and

$y^{tq} = y^{tq'}$. Since for each player (t,q) in S_k the player $\psi((t,q))$ is a substitute for (t,q) , the payoff y' is in $V_{nr}(D_k^m)$, where y' is constructed from y by permuting the values for the coordinates of y associated with (t,q) and $\psi((t,q))$ for each $(t,q) \in S_k$. However, since $y^{tq} = y^{tq'}$ when $\psi((t,q)) = (t, q')$, $y' = y$. Therefore $y \in V_{nr}(D_k^m)$. From superadditivity, since $P = \{D_k^m\}$ is a partition, $y \in V_{nr}(A_{nr})$.

We have shown in the above that given any minimal balanced family of subsets of A_r , say β^ℓ , if $x \in \bigcap_{S\in\beta^\ell} V_r(S)$, then $\prod_{i=1}^n x \in \bigcap_{S\in P} V_{nr}(S)$ for some partition P of A_{nr} . From superadditivity, $\prod_{i=1}^n x \in V_{nr}(A_{nr})$. Now let β be any balanced family of subsets of A_r and let $x \in \bigcap_{S\in\beta} V_r(S)$. From Lemma 4, for some subset $L' \subset \{1, \ldots, L\}$, we have $\beta = \bigcup_{\ell\in L'} \beta^\ell$. Therefore $x \in \bigcap_{S\in\beta} V_r(S) = \bigcap_{\ell\in L'} \bigcap_{S\in\beta^\ell} V_r(S)$ so, for $\ell \in L'$, $x \in \bigcap_{S\in\beta^\ell} V_r(S)$. From the above result, $\prod_{i=1}^n x \in V_{nr}(A_{nr})$.

Q.E.D.

We remark that none of the preceding lemmas required comprehensiveness and therefore can be applied to not-necessarily-comprehensive games.

Define $E(r)$ and $\tilde{E}(r)$ to be subsets of $\mathbb{R}^T$ representing the equal-treatment payoffs in $V_r(A_r)$ and $\tilde{V}_r(A_r)$ respectively, i.e., $E(r) = \{x \in \mathbb{R}^T : \prod_{i=1}^r x \in V_r(A_r)\}$ and $\tilde{E}(r) = \{x \in \mathbb{R}^T : \prod_{i=1}^r x \in \tilde{V}_r(A_r)\}$. Define $E_+(r) = E(r) \cap \mathbb{R}_+^T$ and $\tilde{E}_+(r) = \tilde{E}(r) \cap \mathbb{R}_+^T$.

From the preceding lemma, we immediately have the result that given any r , there is a positive integer n such that if $x \in \tilde{E}(r)$, then $x \in E(nr)$.

We remark that when (A_r, V_r) is comprehensive and, consequently, $E(r)$ is comprehensive, the set $E(r) + \varepsilon\{\underline{1}\}$ used in the following lemma is the ε-neighborhood of $E(r)$; i.e. $E(r) + \varepsilon\{\underline{1}\} = \{x \in \mathbb{R}^T :$ for some $y \in E(r)$, $\|x-y\| \leq \varepsilon\}$.

Lemma 6. Suppose $V_r(A_r)$ is a convex and comprehensive set for all r . Then, given any $\varepsilon > 0$ and any r' , there is an r^* such that for all $r \geq r^*$, we have $\tilde{E}(r') \subset E(r) + \varepsilon\{\underline{1}\}$.

Proof. Let r' and $\varepsilon > 0$ be given.

The following observations will be relevant. From Lemma 5, we can select a positive integer n' such that $\tilde{E}(r') \subset E(n'r')$. Let $r'' = n'r'$. From Lemma 3, for any positive integer n , we have $E(r'') \subset E(nr'')$; therefore $\tilde{E}(r') \subset E(nr'')$ for all positive integers n . Given any $r > r''$, let n and j be non-negative integers such that $r = nr'' + j$ where $j \in \{1, \ldots, r''\}$. Observe that, from superadditivity, given any $x \in \tilde{E}(r')$ and any $z \in V_1(A_1)$, we have

$$\left(\prod_{i=1}^{nr''} x, \prod_{i=1}^{j} z\right) \in V_r(A_r) .$$

Since $V_r(S)$ is closed and bounded relative to $\mathbb{R}^S_+$ for all S , we have $\tilde{V}_r(A_r) \cap \mathbb{R}^{rT}_+$ compact. It follows that $\tilde{E}_+(r')$ is compact. Therefore there are a finite number of points, say $x^1, \ldots, x^\ell, \ldots, x^L$, in $\tilde{E}_+(r')$ such that $\tilde{E}_+(r') \subset \bigcup_{\ell=1}^{L} \{x \in \mathbb{R}^T : \|x - x^\ell\| < \varepsilon/2\}$. Arbitrarily select $z \in V_1(A_1)$. Now given r'' and $r = nr'' + j$ as above, given any $\ell \in \{1, \ldots, L\}$ we have $\left(\prod_{i=1}^{nr''} x^\ell, \prod_{i=1}^{j} z\right) \in V_r(A_r)$. Since players of the same type are substitutes, it follows that any vector with nr'' components (in $\mathbb{R}^T$) equal to x^ℓ and any j components (also in $\mathbb{R}^T$) equal to z is also in $V_r(A_r)$ and there are C such vectors where

$C = \frac{(nr''+j)!}{(nr'')!j!}$. In this collection of vectors, given $p \in \{1, \ldots, nr''+j\}$, we have x^ℓ in the p^{th} position in $\frac{nr''}{nr''+j}C$ of the vectors and z in the p^{th} position in $\frac{j}{nr''+j}C$ of the vectors. From the convexity assumption, a convex combination of these vectors is in $V_r(A_r)$. In particular, the convex combination formed by taking the sum of these vectors times $1/C$ is in $V_r(A_r)$. The vector thus formed has the equal-treatment property and each component (in $\mathbb{R}^T$) of the vector is $\frac{nr''x^\ell}{nr''+j} + \frac{jz}{nr''+j}$. Let $z_{\ell j}(n) = \frac{nr''x^\ell}{nr''+j} + \frac{jz}{nr''+j}$. It follows that $z_{\ell j}(n)$ is in $E(r)$. Also, it is obvious that given any $j \in \{1, \ldots, r''\}$, $z_{\ell j}(n)$ converges to x^ℓ as n becomes large. Since $j \leq r''$ for all n , we can select r , and therefore n , sufficiently large so that $\|z_{\ell j}(n) - x^\ell\| < \varepsilon/2$. Let n^* be sufficiently large so that for all ℓ , all j , and all $n \geq n^*$, we have $\|z_{\ell j}(n) - x^\ell\| < \varepsilon/2$. Suppose $r > n^*r''$ so $n \geq n^*$ where $r = nr'' + j$ for some $j \in \{1, \ldots, r''\}$. Let x be an arbitrary element of $\tilde{E}_+(r')$. Then there is an x^ℓ such that $\|x^\ell - x\| < \varepsilon/2$. Since $n \geq n^*$, $\|z_{\ell j}(n) - x^\ell\| < \varepsilon/2$ so $\|z_{\ell j}(n) - x\| < \varepsilon$. Therefore $\tilde{E}_+(r') \subset \{x : \text{for some } x' \in E(r) , \|x - x'\| \leq \varepsilon\} = E(r) + \varepsilon\{\underline{1}\}$ and from property (iv) of the definition of a game and comprehensiveness, $\tilde{E}(r') \subset E(r) + \varepsilon\{\underline{1}\}$.

Q.E.D.

<u>Lemma 7</u>. For all r , we have $\tilde{E}(r) \subset \tilde{E}(r+1)$.

<u>Proof</u>. Given any r , let $x \in \tilde{E}(r)$. For some balanced family β of subsets of A_r , we have $\prod_{i=1}^{r} x \in \bigcap_{S \in \beta} V_r(S)$. We construct a balanced family of subsets of A_{r+1} , say β^* , such that if $S \in \beta^*$ then S has the same profile as some $S' \in \beta$. We then use Lemma 1 to show

that $\prod_{i=1}^{r+1} x \in V_{r+1}(S)$ for all $S \in \beta^*$; therefore $x \in \tilde{E}(r+1)$.

Let $\beta = \{S_1, \ldots, S_k, \ldots, S_K\}$ and let w_k denote the weight for $S_k \in \beta$. Given $q \in \{1, \ldots, r+1\}$, let $B_q = \{(t,q) : t = 1, \ldots, T\}$. Given $\ell \in \{1, \ldots, r+1\}$, we construct a balanced family of K subsets of $\bigcup_{\substack{q=1 \\ q \neq \ell}}^{r+1} B_q$, say $\beta^\ell = \{S_1^\ell, \ldots, S_k^\ell, \ldots, S_K^\ell\}$, where $(t,q) \in S_k^\ell$ if and only if (a) $(t,q) \in S_k$ with $q \neq \ell$, or (b) $q = r+1$ and $(t,\ell) \in S_k$. Informally, β and β^ℓ are the same except that for each type t , each player (t,ℓ) is replaced by $(t, r+1)$. Note that the weights w_k for $S_k^\ell \in \beta^\ell$ balance β^ℓ . In this manner we construct $r+1$ balanced families, $\beta^1, \ldots, \beta^{r+1}$, of $\bigcup_{q=2}^{r+1} B_q, \ldots, \bigcup_{q=1}^{r} B_q$ respectively (where, of course, $\beta^{r+1} = \beta$). Let $\beta^* = \bigcup_{\ell=1}^{r+1} \beta^\ell$. We claim that this is a balanced family of subsets of A_{r+1} . Clearly $A_{r+1} = \bigcup_{\ell=1}^{r+1} \bigcup_{k=1}^{K} S_k^\ell = \bigcup_{S \in \beta^*} S$. Note that given $(t,q) \in A_{r+1}$ there are r members of $\{ \bigcup_{\substack{q'=1 \\ q' \neq \ell}}^{r+1} B_{q'} : \ell = 1, \ldots, r+1\}$ containing (t,q) . Since, if $(t,q) \in \bigcup_{\substack{q'=1 \\ q' \neq \ell}}^{r+1} B_{q'}$, the sum of the weights $\sum_{\{k:(t,q) \in S_k^\ell\}} w_k$ is one, we have $\sum_{\{k,\ell:(t,q) \in S_k^\ell\}} w_k = r$ and the weights $\frac{w_k}{r}$ for $S_k^\ell \in \beta^*$ balance β^* . From the definition of a sequence of replica games, $\prod_{i=1}^{r+1} x \in V_{r+1}(S_k)$ for each $k = 1, \ldots, K$. From Lemma 1, $\prod_{i=1}^{r+1} x \in V_{r+1}(S_k^\ell)$ for all k and ℓ so $\prod_{i=1}^{r+1} x \in \bigcap_{\ell=1}^{L} \bigcap_{k=1}^{K} V_{r+1}(S_k^\ell) = \bigcap_{S \in \beta^*} V_{r+1}(S)$. Since β^* is balanced, we have $\prod_{i=1}^{r+1} x \in \tilde{V}_{r+1}(A_{r+1})$ and therefore $x \in \tilde{E}(r+1)$.

Q.E.D.

In the following, we use the concept of the closed limit of a sequence of sets. A definition of this concept and some properties can be found in Hildenbrand [8], pp. 15-18. We also employ the theorem that a sequence of subsets (F_n) of a compact metric space converges to a subset F with respect to the Hausdorff distance if and only if the closed limit of the sequence exists and equals F (see Hildenbrand [8, p. 17]). The closed limits, which we will show exist, of $(\tilde{E}(r))$ and $(E(r))$ are denoted by $L(\tilde{E})$ and $L(E)$ respectively. We denote the Hausdorff distance between two sets, say F and G (with respect to the sup norm metric) by $\|F,G\|$.[8]

Lemma 8. Assume that $(A_r, V_r)_{r=1}^{\infty}$ is per-capita bounded and that $V_r(A_r)$ is convex and comprehensive for all r . Then the closed limits $L(\tilde{E})$ and $L(E)$ exist and are equal and $\|\tilde{E}(r), E(r)\|$ converges to zero as r goes to infinity.

Proof. Since $\tilde{E}(r) \subset \tilde{E}(r+1)$ for all r and from the per-capita boundedness assumption, it can easily be shown that the closed limit, $L(\tilde{E})$, exists. Also, from the per-capita boundedness assumption there is a compact set, say K , such that $\tilde{E}_+(r) \subset K$ for all r . It follows that given $\varepsilon > 0$, there is an r' sufficiently large so that for all $r \geq r'$, $\|\tilde{E}_+(r), L(\tilde{E}_+)\| < \varepsilon/2$. From property (iv) of a game and comprehensiveness, we have $\|\tilde{E}(r), L(\tilde{E})\| < \varepsilon/2$ for all $r \geq r'$.

[8] We've used $\|\cdot\|$ to denote the sup norm and $\|\cdot,\cdot\|$ to denote the Hausdorff distance. This should create no confusion, however, since in the first case the variable is a vector and, in the second, two sets.

Given $\varepsilon > 0$, let r' be sufficiently large so that for all $r \geq r'$, $\|\tilde{E}(r), L(\tilde{E})\| < \varepsilon/2$. From Lemma 6, there is an $r^* \geq r'$ so that for all $r \geq r^*$, $\tilde{E}(r') \subset E(r) + \frac{\varepsilon}{2}\{\underline{1}\}$. Since $E(r) \subset \tilde{E}(r)$, for all $r \geq r^*$ we have

$$\tilde{E}(r') \subset E(r) + \frac{\varepsilon}{2}\{\underline{1}\} \subset \tilde{E}(r) + \frac{\varepsilon}{2}\{\underline{1}\} \subset L(\tilde{E}) + \frac{\varepsilon}{2}\{\underline{1}\} .$$

Since $\|\tilde{E}(r'), L(\tilde{E})\| < \varepsilon/2$, it follows that $\|\tilde{E}(r), E(r)\| < \varepsilon$ for all $r \geq r^*$. (We've squeezed $E(r)$ and $\tilde{E}(r)$ between $\tilde{E}(r')$ and $L(\tilde{E}) + \frac{\varepsilon}{2}\{\underline{1}\}$.) It follows that $L(\tilde{E}) = L(E)$.

Q.E.D.

We remark that nothing we have done so far depends on the assumptions of QTU and MES. Therefore, it is possible to use these lemmas in situations where some or all of these assumptions are not made.

The theorems are now proven--in reverse order to their statements since each proof uses the preceeding one.

<u>Proof of Theorem 3</u>

Select $r > r^*$ and let x be a payoff in the core of $(A_r, \tilde{V}_r)$; from Scarf's theorem there is such a payoff. Note that $x > 0$ from the assumption that x is in the core and (i) of the definition of a game. Let β be a balanced family of subsets of A_r such that $x \in \bigcap_{S\in\beta} V_r(S)$ and such that $\rho(S) \leq \rho(A_{r^*})$ for all $S \in \beta$; from the MES assumption such a balanced family β exists. Suppose x does not treat all players of type t equally, i.e., for some q' and q'' , we have $x^{tq'} \neq x^{tq''}$. We consider two cases which cover all possibilities and in both cases obtain a contradiction.

Case 1: Suppose for some S' in β we have $(t, q') \in S'$ and $(t, q'') \notin S'$. Since β is balanced, it follows that there must be an S'' in β such that $(t, q'') \in S''$ and $(t, q') \notin S''$; otherwise β could not be a balanced family since no set of "balancing" weights could sum to one over both these members of the family containing (t, q') and those containing (t, q''). Suppose $x^{tq'} > x^{tq''}$. Let $S^* = (S' - \{(t, q')\}) \cup \{(t, q'')\}$. Then, since (t, q') and (t, q'') are substitutes, the payoff $x' = \sigma[x; (t, q'), (t, q'')]$ is in $V_r(S^*)$. Since $x'^{t'q} \geq x^{t'q}$ for all (t', q) in S^* and $x'^{tq''} > x^{tq''}$, from the QTU assumption there is a payoff x'' in $V_r(S^*)$ where $x'' >> x$. This contradicts the assumption that x is in the core. Therefore $x^{tq'} < x^{tq''}$. However, since $(t, q'') \in S''$ and $(t, q') \notin S''$, by reasoning as above, we can again obtain a contradiction to the assumption that x is in the core. Therefore, for Case 1, we have the result that $x^{tq'} = x^{tq''}$.

Case 2: Suppose for some S in β we have both (t, q') and (t, q'') in S. Since $|S \cap [t]_r| \leq r^*$ there is a player of type t, say (t,q), not in S. From Case 1, we have $x^{tq} = x^{tq'}$ and $x^{tq} = x^{tq''}$ so $x^{tq'} = x^{tq''}$.

Q.E.D.

For the proof of Theorems 1 and 2 we require the following definition.

Let $\mathcal{P}(A_r; r')$ denote the collection of partitions of A_r into non-empty subsets where, given P in $\mathcal{P}(A_r; r')$, for each $S' \in P$, we have $\rho(S') \leq \rho(A_{r'})$. Given a sequence of replica games $(A_r, V_r)_{r=1}^{\infty}$ and given r', we define the r'^{th} <u>truncation of</u> V_r by the correspondence $V_r(\cdot; r')$, where, for each non-empty subset S of A_r, we have $V_r(S; r') = \bigcup_{P \in \mathcal{P}(A_r; r')} \bigcap_{S' \in P} V_r(S')$. It is easily verified that the sequence

$(A_r, V_r(\cdot; r'))_{r=1}^{\infty}$ is a sequence of superadditive replica games and that the sequence of balanced cover games $(A_r, \tilde{V}_r(\cdot; r'))_{r=1}^{\infty}$ satisfies the assumption of MES with MES bound r'.[9] Define $E(r; r') = \{x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in V_r(A_r; r')\}$ and $\tilde{E}(r; r') = \{x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in \tilde{V}_r(A_r; r')\}$.

Observe that $\tilde{E}(r) \subset \tilde{E}(r+1; r)$; this is an application of Lemma 7. It is immediate that $E(r+1; r) \subset \tilde{E}(r+1; r)$ and $\tilde{E}(r+1; r) \subset \tilde{E}(r+1)$.

We note that from Theorem 4 in the appendix, given $\delta > 0$ and a sequence of comprehensive replica games $(A_r, V_r)_{r=1}^{\infty}$, we can select a sequence of comprehensive games $(A_r, V_r^{\delta})_{r=1}^{\infty}$ where each game (A_r, V_r^{δ}) is a δ-QTU cover of (A_r, V_r) .

For ease in exposition, since comprehensiveness is assumed for Theorem 2, we use the fact that a feasible payoff x is in the ε-core of a comprehensive game (A,V) if and only if $x + \varepsilon \underline{1} \notin \text{int } V(S)$ for all $S \subset A$.

Proof of Theorem 2

Given $\varepsilon > 0$, select a positive number δ such that $\delta < \frac{\varepsilon}{4}$. Let $(A_{r+1}, \tilde{V}_{r+1}^{\delta}(\cdot;r))$ be the balanced cover of a comprehensive, δ-QTU cover of $(A_{r+1}, V_{r+1}(\cdot;r))$ for each r . Let $\tilde{E}^{\delta}(r+1; r) = \{x \in \mathbb{R}^T : \prod_{i=1}^{r+1} x \in \tilde{V}_{r+1}^{\delta}(A_{r+1}; r)\}$ and let $L(\tilde{E}^{\delta})$ denote the closed limit of the sequence $(\tilde{E}^{\delta}(r+1; r))$. It is easily verified that $\|\tilde{V}_{r+1}(A_{r+1}: r), \tilde{V}_{r+1}^{\delta}(A_{r+1}; r)\| < \varepsilon/4$ for each r , so $\|\tilde{E}(r+1; r), \tilde{E}^{\delta}(r+1; r)\| < \varepsilon/4$ for each r , and since $L(\tilde{E}) = L(E)$ we have $\|L(E), L(\tilde{E}^{\delta})\| < \varepsilon/4$.

9 $(A_r, \tilde{V}_r(\cdot; r'))$ is the balanced cover of $(A_r, V_r(\cdot; r'))$, i.e., we "truncate" then "balance."

From Theorem 3, for each r we can select $y^r \in \tilde{E}^{\delta}(r+1;\ r)$ such that $\prod_{i=1}^{r+1} y^r$ is in the core of $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$. Since for each r, $y^r \geq 0$ and since $L(\tilde{E}^{\delta}) \cap \mathbb{R}^T_+$ is compact, (y^r) has a convergent subsequence; let $\bar{y}$ denote the limit of some such subsequence. Define $\bar{x} = (\bar{y} - \frac{\varepsilon}{2}\underline{1})$.

We now show that for all r sufficiently large $\prod_{i=1}^{r+1} \bar{x}$ is in the ε-core of $(A_{r+1}, \tilde{V}^{\delta}_{r+1}(\cdot;r))$. Since $\tilde{E}^{\delta}(r+1;\ r)$ converges to $L(\tilde{E}^{\delta})$, for all r sufficiently large, say $r \geq r^*$, $\bar{x} \in \tilde{E}^{\delta}(r+1;\ r)$. Therefore, if $\prod_{i=1}^{r'+1} \bar{x}$ is not in the ε-core of $(A_{r'+1}, \tilde{V}^{\delta}_{r'+1}(\cdot;\ r'))$ for some $r' \geq r^*$, for some $S \subset A_{r'+1}$ we have $\prod_{i=1}^{r'+1} \bar{x} + \varepsilon\underline{1} \in \text{int}\ \tilde{V}^{\delta}_{r'+1}(S;\ r')$. Consequently $\prod_{i=1}^{r'+1} (\bar{y} + \frac{\varepsilon}{2}\underline{1}) \in \text{int}\ \tilde{V}^{\delta}_{r'+1}(S;\ r')$. Select a member of the sequence (y^r), say $y^{r''}$, such that $r'' \geq r'$ and such that $\|\bar{y} - y^{r''}\| < \varepsilon/2$. From (b) of the definition of a sequence of replica games, and since $\|\bar{y} - y^{r''}\| < \varepsilon/2$ and $\prod_{i=1}^{r'+1} (\bar{y} + \frac{\varepsilon}{2}\underline{1}) \in \text{int}\ \tilde{V}^{\delta}_{r'+1}(S\text{:}\ r')$, we have $\prod_{i=1}^{r''+1} y^{r''} \in \text{int}\ \tilde{V}^{\delta}_{r''+1}(S;\ r'')$. This is a contradiction.

Now select r sufficiently large, say r^{**}, such that $r^{**} \geq r^*$ and for all $r \geq r^{**}$, $\bar{x} \in E(r)$. From comprehensiveness, the fact that $E(r) \subset \tilde{E}^{\delta}(r+1;\ r)$ and $\|L(E), L(\tilde{E}^{\delta})\| < \varepsilon/4$ it is routine to verify that there is such an r^{**}. Then for all $r \geq r^{**}$, $\prod_{i=1}^{r} \bar{x} \in E(r)$. Suppose for some $r \geq r^{**}$ and some $S \subset A_r$, we have $\prod_{i=1}^{r} \bar{x} \in \text{int}\ V_r(S) + \varepsilon\{\underline{1}\}$. But then $\prod_{i=1}^{r+1} \bar{x} \in \text{int}\ \hat{V}_{r+1}(S;\ r) + \varepsilon\{\underline{1}\}$ $\subset \text{int}\ \tilde{V}^{\delta}_{r+1}(S;\ r) + \varepsilon\{\underline{1}\}$ which is a contradiction to the result of the preceeding paragraph. Therefore $\prod_{i=1}^{r} \bar{x}$ is in the ε-core of (A_r, V_r) for all sufficiently large r.

Q.E.D.

Our final result is now easy.

Proof of Theorem 1

Let (A_r, V_r^c) denote the "comprehensive cover" of each game (A_r, V_r) in the sequence, i.e., $V_r^c(S) = \{x \in \mathbb{R}^{rT} : \text{for some } y \in V_r(S)\ ,\ x \leq y\}$. Note that since $V_r(A_r)$ is convex, $V_r^c(A_r)$ is convex. From Theorem 2, given $\varepsilon > 0$ we can select r^* such that for all $r \geq r^*$, the ε-core of (A_r, V_r^c) is non-empty. Given $r > r^*$ and a payoff x in the ε-core of (A_r, V_r^c) , let $x' \in V_r(A_r)$ where $x' \geq x$. From the definition of the comprehensive cover, there is such an x' . Since x is in the ε-core of (A_r, V_r^c) , it is immediate that for all $S \subset A_r$, there does not exist an $x'' \in V_r(S)$ where $x'' \gg x + \varepsilon \underline{1}$. Therefore x' is in the ε-core of (A_r, V_r) . Q.E.D.

Conclusions

In this paper, we have shown that quite simple conditions--convexity of the payoff set for the entire set of players, superadditivity, and per-capita boundedness--ensure that sequences of replica games have non-empty asymptotic cores. Of these conditions, we view the superadditivity and per-capita boundedness ones as particularly non-restrictive. We view the convexity assumption as more troublesome. As Shapley [20] has pointed out, although convexity of the sets $V(S)$ often arises in application, convexity is not an ordinal concept since it depends on the topological structure of $\mathbb{R}^n$. Moreover, it is easy to generate examples of economic models whose derived games do not satisfy the convexity requirement, cf. Shapley and Scarf [21]. On the other hand, the notion of a non-empty asymptotic core used in this paper is quite restrictive. To illustrate this, observe that from the proof of Theorem 2 it follows that given any $\varepsilon > 0$ there is an r sufficiently large so that for some equal-treatment imputation x^* in the ε-core of the balanced cover game there is a feasible payoff x for the game such that

$x^{tq} = x^{*tq}$ for <u>all</u> players (t,q). An obvious choice for an approximate core concept is one where some feasible payoff x is in the approximate core if for "most" agents, for some x^* in the ε-core of the balanced cover game $x^{tq} = x^{*tq}$. In Shubik and Wooders [25], such an approximate core concept is introduced and it is shown that non-emptiness of the approximate core obtains for all sufficiently large replications. It would be of interest, however, to determine less restrictive conditions than convexity under which the asymptotic core introduced in this paper is non-empty; at this point we have no results of this nature for sequences of games.

APPENDIX

From the following theorem it is immediate that given any $\delta > 0$ and any comprehensive game (A,V), there is a comprehensive δ-QTU cover of the game.

Theorem 4. Let $K \subset \mathbb{R}^n_+$ be compact and comprehensive in $\mathbb{R}^n_+$, i.e., if $x \in K$, then $y \in K$ for all $y \in \mathbb{R}^n_+$ with $y \leq x$. Then there is a compact, comprehensive subset $K' \subset \mathbb{R}^n_+$ such that $K \subset K'$, $\|K, K'\| < \delta$, and K' satisfies the property that if x is in the boundary of the set $\{y \in \mathbb{R}^n : \text{ for some } z \in K, y \leq z\}$ and $x > 0$, then

$$K \cap \{x' : x' \geq x\} = \{x\} .$$

In other words, the upper boundary of K' contains no line segments parallel to the coordinate axis.

Proof. There is no loss of generality in assuming that K is a subset of the unit ball in $\mathbb{R}^n$. Let f be any continuous, real-valued function on K such that

$$1 < f(x) < 1+\delta \quad \text{for all } x \in K \text{ where } x > 0 \text{ and}$$
$$x < y \Longrightarrow f(x) > f(y) .$$

(For example, we could use the function $f(x) = 1 + \dfrac{\varepsilon/2}{1 + \sum_{i=1}^{n} x_i}$.) Define $T : K \to \mathbb{R}^n_+$ by $T(x) = f(x)x$; this is continuous, so $T(K)$ is compact. Let $K' = T(K)$. Since $\|x - T(x)\| < \delta$ for $x \in K$ (because we have assumed K is a subset of the unit ball) we have $\|K, K'\| < \delta$.

We note that for $x \in K$, $T(x)$ lies on the ray from 0 through x and (unless $x = 0$) is further out than x. Thus T maps each ray to itself, continuously.

To see that K' is comprehensive, it suffices (because K is comprehensive) to prove

(*) if $x \in K$ and $a \in \mathbb{R}^n_+$ where $a < T(x)$, then there is a y in K, $y < x$, with $T(y) = a$. To see this, write $T(x) = f(x)x$ and set $z = \frac{1}{f(x)}a$. Then $z \in \mathbb{R}^n_+$ and $z < x$ so $z \in K$ and $f(z) > f(x)$. Hence $T(z) = f(z)z > f(x)z = a$; since the points 0, a, and $T(z)$ all lie on the same ray, and in that order, we can apply the Intermediate Value theorem to T on this ray to conclude that there is a y on the ray with $T(y) = a$ and $y < z < x$, as desired.

It remains to prove the assertion about the boundary of K'. If there were a line segment in the boundary of K' parallel to a coordinate axis and not lying in a coordinate plane, its endpoints, say a and b, would give two points in K', with, say $b < a$ and no coordinate of a equal to zero. Since we can replace b by the midpoint of this segment, we assume also that no coordinate of b is equal to zero. Let $x \in K$ such that $a = T(x)$; use (*) to find $y \in K$ such that $T(y) = b$ and $y < x$.

Case 1. $y < x$ but $y \not\ll x$; that is, some coordinate of y is equal to the corresponding coordinate of x; say $y_i = x_i$. Then $b_i = f(y)y_i > f(x)x_i = a_i$, since $f(y) > f(x)$ and no coordinate of a is zero (so no coordinate of x is zero). This is a contradiction.

Case 2. $y \ll x$. On the ray from zero through y there is a unique point w with $w \leq x$ but $w \not\ll x$; this is the first point on the ray with some coordinate equal to the corresponding coordinate of x; say

$w_i = x_i$. If in fact $w = x$ then y and x lie on the same ray, so b and a lie on this ray with $b < a$ which guarantees $b \ll a$. But then b belongs to the open set $\{x' \in \mathbb{R}^n_+ : x' \ll a\}$ which is in K' (since $a \in K'$ and K' is comprehensive); hence b would not be in the boundary of K' . Therefore we may assume $w \neq x$. Thus $f(w) > f(x)$; set $c = T(w)$ so, as in Case 1, $c_i = f(w)w_i > f(x)x_i = a_i \geq b_i$. Since c and b lie on the same ray, we conclude that $b \ll c$ so b lies in the open set $\{x' \in \mathbb{R}^n_+ : x' \ll c\}$, which, again, lies in K' , contradicting our assumption that b is in the boundary of K' .

Q.E.D.

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