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ARBITRAGE PRICING THEORY IN A FINITE ECONOMY

John J. Beggs

June 9, 1981

ARBITRAGE PRICING THEORY IN A FINITE ECONOMY*

by

John J. Beggs

1. Introduction

In a recent paper Huberman (1980) has re-examined the central result of the arbitrage pricing theory (Ross, 1976), namely that there exists a linear-like relationship among mean returns of assets and the covariances of those returns. Huberman establishes that such a linear relationship is "approximately" true for "most" assets in the limit economy. The procedure involves establishing a bound on the squared deviations from an exact linear relationship and showing that such a bound is independent of the numbers of assets being traded in the economy. Hence as the number of assets becomes large the given amount of deviation from the linear relationship between mean returns and covariances must be "spread" over a large number of equations. This spreading involves, for example, making the deviations from linearity on each asset very small or, perhaps, fixing the deviations from linearity to be all associated with one or several assets and have all the remaining assets fit the linear relationship exactly. Huberman avoids making assumptions about preferences and defines arbitrage as a situation where there exists a sequence of arbitrage portfolios whose expected returns go to infinity as the variance on returns goes to zero. One important consequence of this definition of arbitrage is that it does not allow one to constructively establish the "size" of the bound on the squared deviations from linearity, hence it is not possible to constructively

*This paper has benefited from discussions with Stephen Ross and Gregory Connor.

establish how many assets would be required before the bound becomes relatively insignificant. Connor (1980) has examined this linear-like relationship, again in the context of sequences of economies, and has, with the aid of assumptions on preferences shown that in the equilibrium state for the limit economy the linear relationship among mean returns on assets and the covariances of those returns will hold exactly. Connor also requires further assumptions upon the asset structure of the economy which admit the possibility of all asset holding portfolios being well-diversified in the limit economy.

The notions of sequences of economies, and limit economies, while useful theoretical constructs, do not relate well to real-world market phenomena. For these reasons it is most useful to establish the arbitrage pricing theory result for a fixed finite economy. In a finite economy the linear relationship of mean returns to covariances does not obtain. Rather one can establish an exact linear relationship among mean returns, covariances and equilibrium portfolios.

The chief result for quadratic preferences can be summarized as follows. Let α be an N-dimensional vector containing the proportions of wealth held in each of N assets. If E is an N-dimensional vector of expected returns per dollar invested in each asset, and B is an (N×k) matrix of factor weights for the variance-covariances of asset returns, and D is a diagonal matrix containing the variance of the idiosyncratic risk terms (defined more formally in Section 2), then an equilibrium portfolio which is efficient in the sense that it exploits all desirable arbitrage possibilities may be expressed as

$$(1) \quad E = \gamma_0 E + \gamma_1 D\alpha + B\gamma_2$$

where γ_0 and γ_1 are scalars and γ_2 is a k -dimensional vector. The vector e is an N -dimensional vector of units. This result is powerful in that it suggests a testable hypothesis for the theory. Generalizations to other utility functions are considered in the text of the paper.

It is useful to link this result to the "limit" economy case. Connor establishes sufficient conditions for a well diversified economy, so that in the equilibrium of the limit economy all asset holding portfolios will be well-diversified. By well diversified one means that the proportion of wealth held in any one asset becomes small. Connor's well diversified economy has the further property that the proportion of total idiosyncratic risk due to any given asset is small relative to the sum of the idiosyncratic risk on all assets. This, in effect, ensures that none of the elements of D , defined above become large. It is immediate from (1) that these conditions ensure the linear relationship among expected returns and covariances of returns. If in equation (1) $\alpha = 0$, the familiar arbitrage pricing result obtains exactly.

The presentation proceeds first with the result for a quadratic utility function. This allows an attractive diagrammatic exposition of the relevant propositions. Extensions to more general utility functions are then considered. Section 4 of the paper indicates the close relationship between the arbitrage pricing theory and the CAPM, while Section 5 discusses the significance of the results in terms of the testing of asset pricing models.

2. Model

In this section the basic model of the asset returns generating process is introduced along with some propositions on the opportunity set created by arbitrage possibilities. Individuals are assumed to believe that the random returns on the set of assets being considered are governed by a k -factor generating model of the form:

$$(2) \quad R = E + B\delta + \epsilon .$$

The first term in (2), R , is an N -dimensional vector of returns on an initial one dollar purchase of each of the N assets traded in the market. E is an N -dimensional vector of expected returns. The term δ represents a k -dimensional vector of random variables which are the random factors in the model. These factors capture the systematic components of risk in the model. It is assumed that

$$(3a) \quad E(\delta) = 0$$

$$(3b) \quad E(\delta\delta') = \Lambda_{kk} .$$

B is an $(N \times k)$ matrix of weighting coefficients which reflect the sensitivity of each asset's returns to the movements in the factors. The final term, ϵ , is an N -dimensional vector of random variables which capture the unsystematic risk component which is idiosyncratic to each asset. It is assumed to reflect the random influence of information that is unrelated to other assets. It is assumed that

$$(4a) \quad E(\epsilon) = 0$$

$$(4b) \quad E(\epsilon\epsilon') = \text{diag}[\sigma_{\epsilon_1}^2, \dots, \sigma_{\epsilon_N}^2] = D$$

$N \times N$

and further that ϵ is distributed independently of δ . Finally it is assumed that N , the number of assets being traded, is much greater than k , the number of factors.

An individual's wealth holdings are denoted by the portfolio vector α , an N -dimensional vector, where each element shows the proportion of total wealth held in an asset. When considering an alteration of that portfolio any new portfolio will differ from the old portfolio by amounts contained in an N -dimensional vector, x . The sum of the elements of x will be zero since the old and the new portfolios put the same wealth into N -assets. In other words, additional purchases of assets must be financed by sales of others. Portfolios, such as x , which use no wealth are called arbitrage portfolios.

The vector of expected returns can always be written

$$(5) \quad E = e\gamma_0 + D\alpha\gamma_1 + B\gamma_2 + u$$

where γ_0 and γ_1 are arbitrary scalars, γ_2 is an arbitrary k -dimensional vector and e is an N -dimensional vector of units. The elements of u , an N -dimensional vector, can then be chosen to ensure the equality. The proposition introduced in Section 1 is established by showing that in equilibrium there exist values of γ for which the elements of u will be zero.

Consider now an arbitrage portfolio, x , constructed so as to have zero systematic risk (orthogonal to B) and having returns which are uncorrelated with the returns on the weighted initial portfolio, $D\alpha$ (orthogonal to $D\alpha$). Formally, then x should satisfy the conditions

$$(6) \quad x'[e:D\alpha:B] = 0$$

where $[e:D\alpha:B]$ is a partitioned $(N \times (k+2))$ matrix. It is also useful

to scale x so that it satisfies the "size" requirement that

$$(7) \quad x'x = 1 .$$

The exercise of an arbitrage opportunity, or the realignment of a wealth holding portfolio, alters the expected return and the risk associated with the wealth holding portfolio. The net expected return from any alteration of the portfolio from α to $(\alpha+x)$ is given by $x'E$.* In view of the restrictions placed on x it can be seen from equations (5) and (6) that

$$(8) \quad x'E = x'u .$$

The risk associated with the initial wealth holding portfolio α , and with the realigned portfolio $(\alpha+x)$ are given by the formulae in equations (9) and (10) below.

$$(9) \quad \text{Risk}(\alpha) = \text{Var}[\alpha'R] = \alpha'BAB'\alpha + \sigma_{\epsilon}^2 \alpha'D\alpha$$

$$(10) \quad \text{Risk}(\alpha+x) = \text{Var}[(\alpha+x)'R] = \alpha'BAB'\alpha + \alpha'D\alpha + x'Dx .$$

The simplification of the expression for the variance in (10) is due to the orthogonality conditions imposed on the arbitrage portfolio, x , in equation (6).

For notational purposes the arbitrage portfolio has been normalized (equation (7)) to be of unit length. In making investment decisions the individual investor will select an arbitrage portfolio ϕx , where ϕ is

*The net expected return $x'E$ will be taken to be non-negative. Suppose an arbitrage portfolio satisfied (6) in such a way that $x'E$ were negative. Then an equivalent solution would be $\hat{x} = -x$, for which $\hat{x}'E$ would be positive and \hat{x} would continue to satisfy (6).

a positive scalar. The ϕ term has the effect of scaling the portfolio up or down in the desired size. The net expected return from any arbitrage or realignment of the portfolio, is linear in ϕ , while the net increment to the variance of asset returns is quadratic in ϕ .

Arbitrage portfolios are the device by which an individual can alter a wealth holding portfolio so as to take on alternative combinations of expected returns and risk. Making use of equations (8), (9) and (10) the opportunity set created by the presence of arbitrage portfolios, for which there are positive net expected returns, can be graphed as in the diagram in Figure 1. The diagram shows such an opportunity set for a given arbitrage portfolio x satisfying the conditions in equation (6).

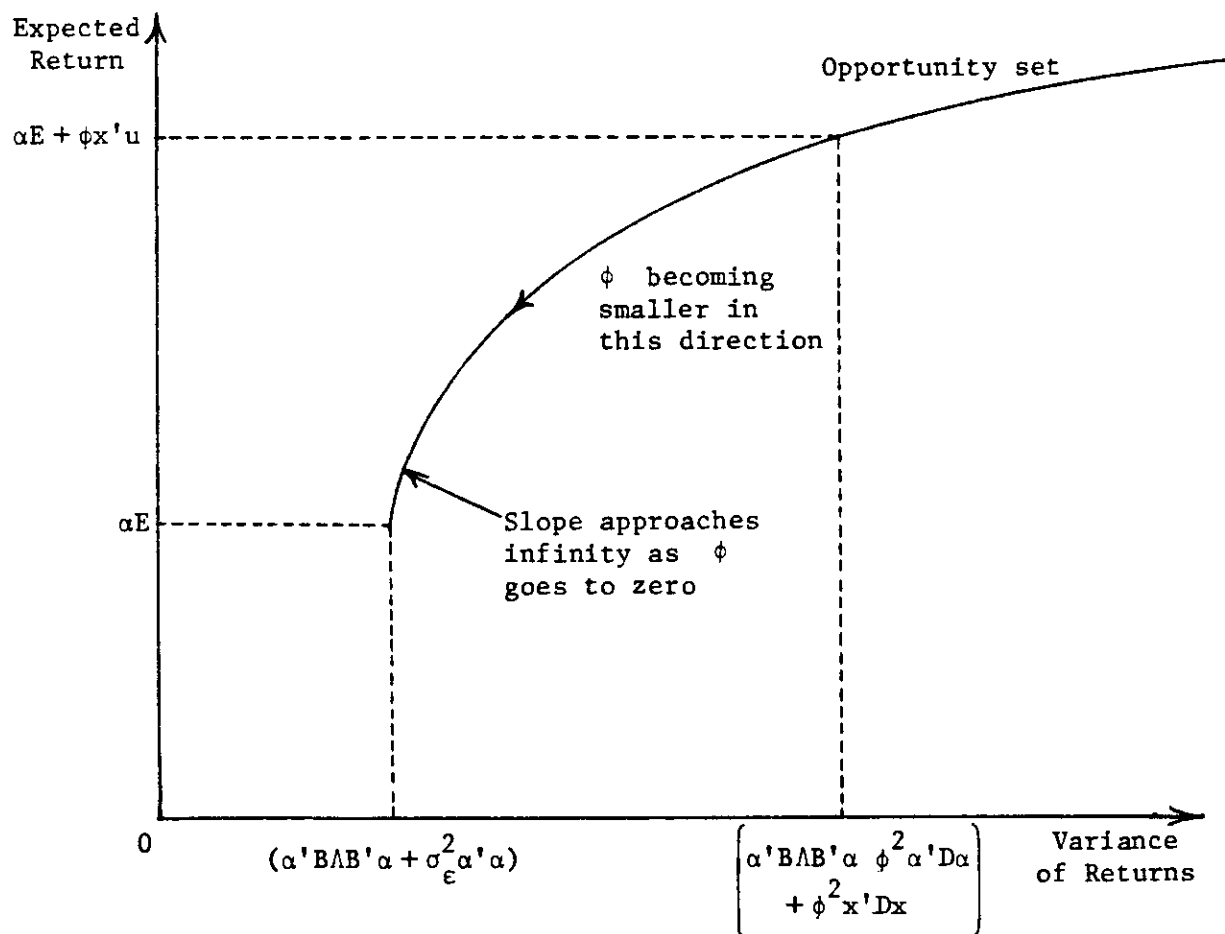


FIGURE 1: Opportunity Set Created by Arbitrage Portfolio x , ($x'u > 0$).

3. Linearity Relationships in "Arbitrage-Efficient" Portfolios

As indicated in Section 1, let us now assume that individuals have mean-variance utility functions with the usual concavity properties. Indifference curves can then be drawn on a diagram similar to that shown in Figure 1. Figure 2, below, shows the individual's indifference curve which passes through the point $(\alpha' BAB\alpha' + \sigma_{\epsilon}^2 \alpha' D\alpha, \alpha'E)$ the variance and expected return associated with the initial wealth holding portfolio α . Superimposed on Figure 2 is the opportunity set (for realignment of the portfolio) associated with the presence of an arbitrage portfolio x which satisfies the conditions of equation (6) and for which $x'u > 0$. The shaded area on the opportunity set represents all those possible realignments of the

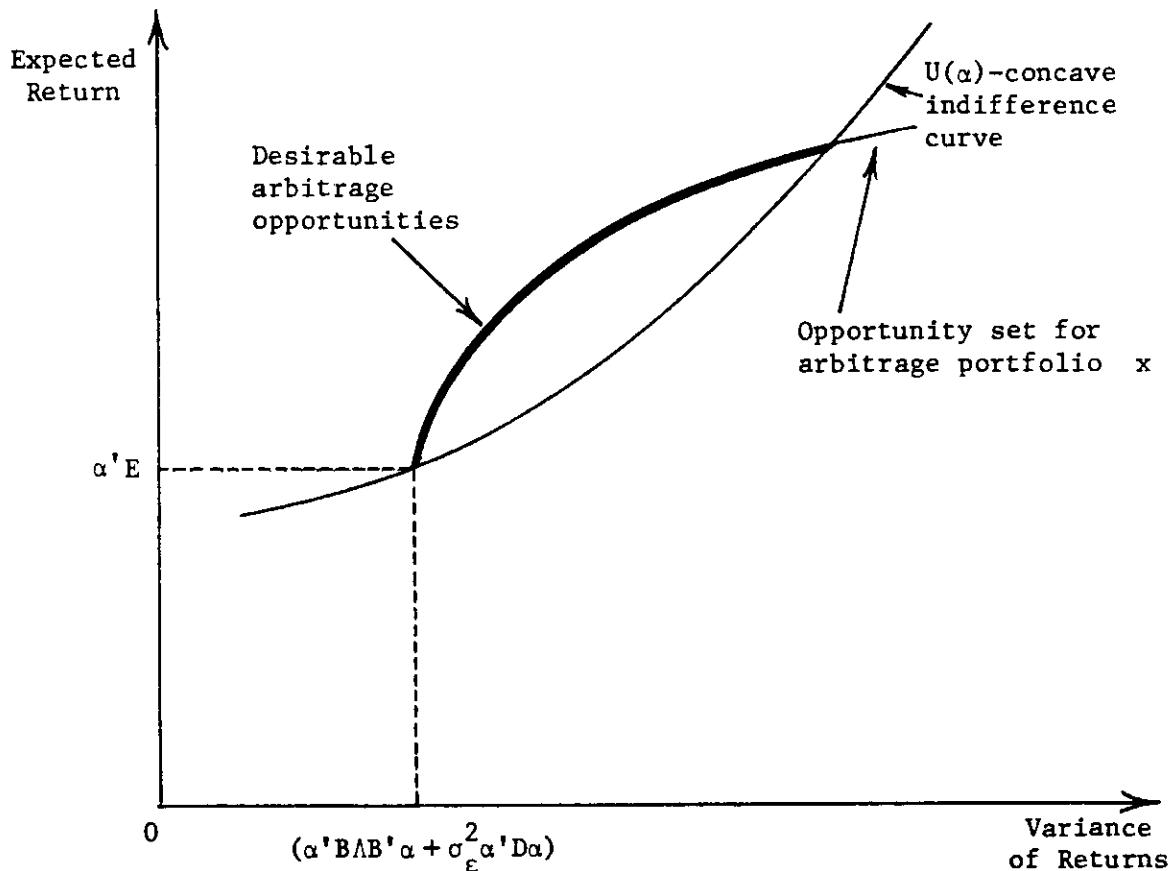


FIGURE 2. Desirable Arbitrage Opportunities Created by Arbitrage Portfolio x , $(x'u > 0)$

wealth holding portfolio (associated with different values of ϕ in the arbitrage portfolio ϕx) which would allow the individual to obtain a higher level of utility. Since the slope of the opportunity set approaches infinity as ϕ approaches zero such a realignment of the portfolio will be desirable as long as the indifference curve between expected return and risk is concave. This is in accord with the notion that the individual will accept any small orthogonal risk which yields a positive expected return.

It is apparent from the above discussion that equilibrium will not obtain (opportunities remain to be exploited) while there exist arbitrage portfolios such as x for which $x'u$ is positive (recollecting that x is selected to ensure that $x'u$ is non-negative). Referring back to equation (6) it is known that there exist $(N-k-2)$ linearly independent arbitrage portfolios which can satisfy the orthogonality conditions. These arbitrage portfolios can be collected in a matrix

$$(11) \quad X = [x_1, x_2, \dots, x_{N-k-2}]$$

which is of dimension $(N \times N-k-2)$. For an equilibrium to obtain a necessary condition then is that

$$(12) \quad X'u = 0 .$$

Since the γ terms introduced in equation (5) are entirely arbitrary we can now proceed to make a judicious choice of values. For example, one can define γ in the following fashion. Let

$$(13) \quad \tilde{\gamma} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} e' \\ \alpha'D \\ B' \end{bmatrix} [e: :B]^{-1} \begin{bmatrix} e' \\ \alpha'D \\ B' \end{bmatrix} E .$$

It is a simple result in linear algebra to then show that the residual terms, u , defined

$$(14) \quad u = E - [e:D\alpha:D]\tilde{\gamma}$$

will be orthogonal to $[e:D\alpha:B]$. That is

$$(15) \quad [e:D\alpha:D]'u = 0.$$

As the arbitrage portfolios were selected to be orthogonal to $[e:D\alpha:B]$, equations (12) and (15) together form N linearly independent homogeneous equations. For γ defined as in (13) the solution of the equations ensures that the vector of residual terms, u , is zero valued. That is, there is an exact linear relationship among expected returns, covariances of returns and equilibrium portfolios (appropriately normalized for idiosyncratic risk)

$$(16) \quad E = [e:D\alpha:B]\tilde{\gamma}.$$

Alternatively this exact linear relationship can be restated as

$$(17) \quad D\alpha = [e:E:B]\lambda$$

where the λ are the obvious transformations of $\tilde{\gamma}$.

Equations (16) and (17) are then the central results of this paper. They extend the exact linearity results of the arbitrage pricing theory to the finite economy. The result applies to the equilibrium portfolios of each trader so that the vectors α and $\tilde{\gamma}$ could well be subscripted by i to read α^i , $\tilde{\gamma}^i$ where $i = 1, 2, \dots, M$ and M is the number of investors in the market. Since linearity holds for each individual it must be true for the portfolios of any subset of traders and, of course,

for the market portfolio.

The result in (16) is quite similar to the arbitrage pricing theory results for limit economies. These results require that the number of investment opportunities become large and that portfolios be well-diversified so that the proportion of wealth held in any one asset (the value of any one element of the vector α) becomes very small. Exact linearity among expected returns and covariances is achieved as the elements of α go individually to zero.

Exact linearity among expected returns and covariances may be achieved in a finite economy in the special situation in which, at equilibrium, there are traders with no wealth and no position in the market, $\alpha = 0$, who will swoop in to take advantage of any arbitrage opportunity should it arise. It is certainly a traditional dichotomy in economics to think of capitalists and workers. In the present context the workers might be imagined at the fringe of the market, having no capital wealth, but willing to exercise any arbitrage possibility which arises (picking up the crumbs if you will). In many cases this may be a plausible empirical approximation to real world markets. An exact linear relationship will also obtain if any individual trader were in equilibrium, holding a portfolio, α , such that $D\alpha$ were spanned by the columns of $[e:B]$. Unfortunately there is little theory to guide us as to when this happy event would occur.

The first order conditions for optimum portfolio selection under quadratic utility functions bear some resemblance to the results in equations (16) and (17). The problem posed, in its simplest form is:

$$(18) \quad \underset{\alpha}{\text{Max:}} \quad \text{Expected Utility} = \psi_1 \alpha' E + \psi_2 \alpha' (BAB' + D)$$

$$\text{Subject to: } \alpha' e = 1 .$$

The solution to (18) yields the following familiar first order condition:

$$(19) \quad (BAB' + D)\alpha = \theta_1 e + \theta_2 E$$

where the parameters θ_1 and θ_2 contain the parameters of the utility function ψ_1 and ψ_2 and a Lagrangian multiplier. For the case where the processes generating factor return does not contain factors ($B = 0$), but only idiosyncratic risk, then (19) simplifies to a comparable result for (17) where $B = 0$. For the general case where $B \neq 0$ (19) gives a highly non-linear relationship between portfolio shares and factor weights and bears no immediate relationship to the equilibrium statement in (17).

4. Extensions to More General Utility Functions

The constructive application* of the arbitrage pricing theory requires some bounds being placed on the curvature of the utility function. Ross (1975), for example, assumes bounded relative risk aversion while Connor (1980) requires bounded absolute risk aversion and bounded third derivatives of the utility function. The extensions of the results in Section 3 go through in cases where the utility function is a polynomial function of bounded order. The exact linearity result requires that some higher order derivative of the utility function becomes zero. In so far as the utility function is well approximated by a polynomial of bounded order the results stated in this paper can be reinterpreted as approximate linearity.

*To reiterate the point made in the Introduction: Huberman provides an existence theorem without recourse to conditions upon the utility function. Though this is indeed a parsimonious proof of existence it does not leave any indication as to how "large" an economy needs to be before the arbitrary pricing theory "approximation" can in fact be regarded as an "approximation."

Consider the individual, having a wealth holding portfolio α , who is considering an alteration of that portfolio, via an arbitrage portfolio x , to $(\alpha + \phi x)$. If α is to be an equilibrium portfolio it must be that the expected marginal utility of any alteration to the portfolio will be zero. If x is defined to satisfy the normalization in equation (7) and if ϕ is a scalar of the form defined earlier, equilibrium then requires that

$$(20) \quad \left. \frac{\partial E\{U[(\alpha + \phi x)'R]\}}{\partial \phi} \right|_{\phi=0} = 0 ,$$

that is,

$$(21) \quad E\{U'[\alpha'R](x'R)\} = 0 .$$

Consider now a Taylor Series expansion of $U'(\cdot)$ around the terms $\alpha'(E + B\delta)$. If the arbitrage portfolio, x , is chosen as before to be orthogonal to the factor weights and the unit vector we can write $x'R$ as $(x'E + x'\epsilon)$. If the marginal utility function $U'(\cdot)$ can be written as a J^{th} order Taylor's Series expansion equation (21) can be stated

$$(22) \quad E\{U'[\alpha'R]\}(x'E) + E\left\{ \sum_{j=0}^J (x'\epsilon)(\alpha'\epsilon)^j \frac{U^{(1+j)}[\alpha'(E + B\delta)]}{j!} \right\} = 0$$

where $U^{(1+j)}(\cdot)$ is the $(1+j)^{\text{th}}$ partial derivative of the utility function $U(\cdot)$. The similarity to the result for quadratic utility functions is now apparent. In Section 2 the arbitrage portfolio was selected to be orthogonal only to the vector of the wealth holding portfolio α . Now consider placing additional restrictions on the arbitrage portfolio.

Consider the vector defined

$$(23) \quad \alpha^{(j)} = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \\ \vdots \\ \alpha_N^j \end{bmatrix}$$

which is a vector containing the proportion of wealth held in each of the N -assets raised to the j^{th} power. Let $D^{(j)}$ be a diagonal matrix where the diagonal elements contain the j^{th} moments of the idiosyncratic disturbance terms. Now the arbitrage portfolios will be required to satisfy the orthogonality condition*

$$(24a) \quad x' [e : B : D^{(2)} \alpha^{(1)} : \dots : D^{(J)} \alpha^{(J-1)}] = 0$$

$$(24b) \quad x' \Omega = 0 .$$

There will be $(N-k-J)$ linearly independent arbitrage portfolios which will satisfy condition (24). For arbitrage portfolios satisfying equation (24) the independence of the idiosyncratic disturbances ε and the factors δ ensures that the second term in the expression in equation (22) will be zero. Taking the expected marginal utility to at all times be positive it follows that the $x'E$ term must be zero. Collecting the $(N-k-J)$ linearly independent arbitrage portfolios satisfying (22) in an $(N \times (N-k-J))$ matrix, X , we can write that

$$(25) \quad x'E = 0$$

Writing E as the regression of E on the matrix Ω we have

$$(26) \quad E = \Omega \tilde{\gamma} + u$$

* $D^{(2)} \alpha^{(1)}$ corresponds to D_α defined in Section 2 of the paper.

where u is a vector of residual term which will be orthogonal to Ω .

That is,

$$(27) \quad u' \Omega = 0 .$$

From (25) and the orthogonality conditions imposed on X ,

$$(28) \quad \begin{aligned} X'E &= X'\hat{\Omega}\gamma + X'u \\ &= X'u = 0 . \end{aligned}$$

Since X and Ω are linearly independent by definition of X , equations (27) and (28) give N linearly independent homogeneous of degree zero equations in u which ensures that the solution for u in equilibrium will be zero. Restating the exact linearity result we have that in equilibrium, where α is an equilibrium wealth holding portfolio

$$(29) \quad E = [e : B : D_{\alpha}^{(2)} : \dots : D_{\alpha}^{(J)}] \gamma$$

where γ is a $(K+J)$ dimensional vector of parameters.

As discussed for the quadratic utility function, diversification of the type envisaged by Connor (1980) which made individual wealth holdings in any one asset small would cause there to be near linearity among expected returns and the covariances.*

*The above result requires finite moments of the idiosyncratic risk term up to the J^{th} order. Connor (1980) requires only finite third moments. Intuitively, this parsimony with moments is achieved by assuming bounded third derivatives of the utility function. In the case described in this paper finite moments to the third order would only be adequate if the utility function were a fourth order polynomial.

5. The Not-So "Grey Eminence" Behind CAPM

In a recent paper Roll and Ross (1981) assert that

There are two major differences between the APT and the original Sharpe (1963) diagonal model, a single factor model which we believe is the intuitive grey eminence behind the CAPM. First, and most simply, the APT allows more than just one generating factor. Second, the APT demonstrates that since any market equilibrium must be consistent with no arbitrage profits, every equilibrium will be characterized by a linear relationship between each asset's expected returns and its return's response amplitudes, or loadings on common factors.

Indeed, the relationship between the APT and CAPM models can be made most transparent in the context of presentation of the arbitrage pricing theory offered in Sections 3 and 4 of this paper. The linearity result of CAPM is of the form (see, for example, Roll, 1977).

$$(30) \quad E = \theta_0 e + (r_m - \theta_0)\beta$$

where E is the vector of expected returns, θ_0 is the rate of return on the riskless asset, r_m is the expected rate of return on the market portfolio so that

$$(31) \quad r_m = \alpha'_m E$$

where α_m is the market portfolio, and α is a vector containing elements defined

$$(32) \quad \beta_i = \frac{\left\{ \begin{array}{l} \text{Covariance between return on } i^{\text{th}} \text{ asset} \\ \text{and return on market portfolio} \end{array} \right\}}{\left(\text{Variance of return on market portfolio} \right)}, \quad i = 1, \dots, N.$$

Consider the case where the basic asset return generating process is of the type employed in the APT and which was described in equation (2).

It is then a routine exercise to demonstrate that equation (30) may be rewritten in the form of the result for the arbitrage pricing theory given in equation (16). That is,

$$(33) \quad E = \theta_0 e + \theta_1 D\alpha_m + B\theta_2$$

where, θ_0 is as defined above, and

$$(34a) \quad \theta_1 = \frac{(\alpha_m' E - \theta_0)}{(\alpha_m' BAB' \alpha_m' + \alpha_m' D\alpha_m)}$$

$$(34b) \quad \theta_2 = \frac{(\alpha_m' E - \theta_0)}{(\alpha_m' BAB' \alpha_m' + \alpha_m' D\alpha_m)} \cdot AB' \alpha_m .$$

θ_1 is a scalar and θ_2 is a k -dimensional vector.

It seems that some disservice is done to the CAPM by describing it as a single factor returns generating model. An inspection of equations (30) and (33) makes quite clear that the CAPM β simply serves to aggregate, ex post, all the information pertaining to factor specific effects. The linearity between expected returns and factor effects is again achieved with the aid of the terms in the vector $(D\alpha_m)$. The simplicity of the β decomposition has apparently not been fully realized in earlier literature on this subject.

6. Empirical Testing of the Arbitrage Pricing Theory

In view of the above results for a finite economy it is useful to reflect upon the empirical testing of APT and CAPM as reported in the literature. The central concern of Roll and Ross (1980) and, for example, Miller and Scholes (1972) has been the apparent ability of the idiosyncratic

risk term to explain expected returns. In the APT model, as formulated by Ross (1976), expected returns were thought to be fully explained by "factor" weights, while in CAPM expected returns were thought to be fully explained by the covariance with the market portfolio as measured by beta. As Miller-Scholes and Roll-Ross indicate part of their apparently contradictory results may be due to sample bias in the estimation of variance in non-symmetric distributions.

A reading of the results in Sections 3 and 4 indicates that in a finite economy there may be good theoretical reasons to believe that idiosyncratic risk plays a role in explaining expected returns in the arbitrage pricing framework. Empirical studies which regress expected returns on factor weights in a world with quadratic preferences clearly omit the variable $(D\alpha)$, a vector containing idiosyncratic variances in each element. When such idiosyncratic variance terms are included in the regression one would expect the improvement in explanatory power of the regression of the type reported by Roll and Ross (1981).

As the utility function is generalized to higher order polynomials additional terms must be included in the regression (of the form $[D^{(2)}_{\alpha}{}^{(1)}; D^{(3)}_{\alpha}{}^{(2)}; \dots; D^{(J)}_{\alpha}{}^{(J-1)}]$, see equations (23) and (24)) to ensure the linearity of the relationship. The CAPM automatically includes the terms $(D^{(2)}_{\alpha}{}^{(1)})$ but if the utility function is more general, additional terms need to be included. Indeed the omission of these terms may explain the Miller-Scholes result if variance is acting as a proxy for other omitted higher order moments.

Future testing of asset pricing models will certainly require that attention be paid to the curvature of the utility function, as indicated by the order of the polynomial, and to the resultant higher moment terms which appear in the relationship between expected returns and factor weights.

7. Conclusions

Since it is never clear what real world phenomenon corresponds to the notion of a sequence of economies, it is valuable to have insights into relevant theoretical mechanisms for the case of a finite economy. The chief result of the arbitrage pricing theory, namely a linear relationship among expected returns and covariances of returns does not hold exactly in a finite economy. However, for a large class of utility functions it is possible to construct comparable linear relationships, where the gap between expected returns and covariance is "figuratively" filled by terms involving the equilibrium wealth holding portfolio and the variances and higher moments of the idiosyncratic disturbance term. Since it is possible to empirically examine wealth holding portfolios and to obtain estimates of the moments of the idiosyncratic risk term for each asset this information can be incorporated into any empirical investigation of the arbitrage pricing theory in real world finite economies.

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