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Approximate Equilibria with Bounds Independent of Preferences

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INDEPENDENT OF PREFERENCES

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ABSTRACT

We prove the existence of approximate equilibria in exchange economies, giving bounds on the excess demand in terms of the number of traders and and norms of the endowments, but independent of the preferences.

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1. Introduction

There have been many papers proving existence of approximate equilibria in exchange economies (e.g., Starr [1969], Hildenbrand-Schmeidler-Zamir [1973], Khan [1975], Anderson [1981]). All of these papers have expressed bounds on the norm of the excess demand, directly or indirectly, in terms of the preferences. For example, Starr expresses the bound on excess demand in terms of a measure of how non-convex the preferences are; the other papers named above assume the preferences come from a compact family of monotone preferences, with the bound on excess demand determined (in a rather complicated way) by the compact family.

Recently, Khan and Rashid [1981] obtained an approximate existence theorem for exchange economies with indivisibilites, without making compactness assumptions on preferences. The purpose of this article is to show that their nonstandard proof can be translated to give an elementary standard theorem. We show that, independent of the preferences, there exists a price vector p so that the per capita excess demand is bounded above by $C/\sqrt{n}$, where n is the number of traders and C is a constant determined solely by the endowments.

2. The Result

The commodity space will be $\mathbb{R}_+^k$. Let $\mathcal{P}$ denote the set of preferences, i.e., binary relations on $\mathbb{R}_+^k$ satisfying

(i) continuity: $\{(x, y): x \succ y\}$ is relatively open in $\mathbb{R}_+^k$

(ii) transitivity: $x \succ y$, $y \succ z \Rightarrow x \succ z$

(iii) irreflexivity: $x \not\succ x$

An exchange economy is a map $\varepsilon: A \to \mathcal{P} \times \mathbb{R}^k_+$, where A is a finite set. Let $n = |A|$. The open price simplex is $\Delta = \{p \in \mathbb{R}^k_+ : \max p^i = 1 , p^i > 0$ for all $i\}$; this somewhat unusual normalization will be convenient. Given $a \in A$, define $(\succ_a, e_a) = \varepsilon(a)$. Given $a \in A$ and $p \in \Delta$, the excess demand set is $d(p, a) = \{x - e_a : x \in \mathbb{R}^k_+ , px \leq p \cdot e_a , y \succ_a x \Rightarrow p \cdot y > p \cdot e_a\}$. $||x||_1$ denotes $\sum_{i=1}^{k} |x^i|$.

Theorem: Given the exchange economy just described, there exists $p \in \Delta$ and a selection $f(a) \in d(p, a)$ such that

$$\frac{1}{n} \sum_{i=1}^{k} \max \left\{ \sum_{a \in A} f(a)^i , 0 \right\} \leq \frac{k+1}{\sqrt{n}} \max_{a \in A} ||e_a||_1 .$$

Proof: Let $\Delta' = \{p \in \Delta : p^i \geq 1/\sqrt{n}$ for all $i\}$, and $X = \left\{ x \in \mathbb{R}^k : ||x||_1 \leq (n^{3/2} + n) \max_{a \in A} ||e_a||_1 \right.$. Define $D(p) = \sum_{a \in A} d(p, a)$. Define the correspondence $\phi : \Delta' \times X \to \Delta' \times X$ by $\phi(p, x) = (q, \text{con } D(p))$, where q maximizes $q \cdot x$ over Δ' . It is easy to check $\text{con } D(p) \subset X$; by standard arguments, ϕ is convex-valued and upper semi-continuous. By Kakutani's Fixed Point Theorem (Kakutani [1941]) , ϕ has a fixed point, i.e., $(p, x) \in \phi(p, x)$. Thus, $x \in \text{con } (D(p))$ and $q \cdot x \leq p \cdot x \leq 0$ for all $q \in \Delta'$.

By the Shapley-Folkman Theorem (Starr [1969]), we can write

$x = \sum_{a \in A'} x_a + \sum_{j=1}^{k} y_{a_j}$, where $A = A' \cup \{a_1, \ldots, a_k\}$, $x_a \in d(p, a)$, $y_{a_j} \in$ con $(d(p, a_j))$. Choose arbitrarily $x_{a_j} \in d(p, a_j)$. Then $x_{a_j} - y_{a_j} =$

$$\left(x_{a_j} + e_{a_j}\right) - \left(y_{a_j} + e_{a_j}\right) \leq x_{a_j} + e_{a_j} \text{ , so}$$

$$\sum_{i=1}^{k} \max\left(x_{a_j}^{i} - y_{a_j}^{i}, 0\right) \leq \sum_{i=1}^{k} x_{a_j}^{i} + e_{a_j}^{i} \leq \frac{p \cdot e_{a_j}}{\min\{p^1, \ldots, p^k\}}$$

$\leq \sqrt{n}\, ||e_a||_1$. Hence $q \cdot \sum_{a \in A} x_a = q \cdot x + q \cdot \left(\sum_{j=1}^{k} x_{a_j} - y_{a_j}\right) \leq k\sqrt{n} \max_{a \in A} ||e_a||_1$

for all $q \in \Delta'$. Letting $q^i = 1$ if $\left(\sum_{a \in A} x_a\right)^i > 0$, and $q^i = 1/\sqrt{n}$

otherwise, we get

$$\sum_{i=1}^{k} \max\left\{\sum_{a \in A} x_a^i , 0\right\} \leq k\sqrt{n} \max_{a \in A} ||e_a||_1 + \frac{1}{\sqrt{n}} \left|\left|\sum_{a \in A} e_a\right|\right|_1 \leq (k+1)\sqrt{n} \max_{a \in A} ||e_a||_1 .$$

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