1-1-1981

Core Theory with Strongly Convex Preferences

Robert M. Anderson

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 578

CORE THEORY WITH STRONGLY CONVEX PREFERENCES

Robert M. Anderson

January 23, 1981

CORE THEORY WITH STRONGLY CONVEX PREFERENCES

by

Robert M. Anderson*

Yale University and Princeton University

May 1978--Revised July 1978 and January 1981

Abstract: We consider economies with preferences drawn from a very general class of strongly convex preferences, closely related to the class of convex (but intransitive and incomplete) preferences for which Mas-Colell proved the existence of competitive equilibria [13]. We prove a strong core limit theorem for sequences of such economies with a mild assumption on endowments (the largest endowment is small compared to the total endowment) and a uniform convexity condition. The results extend corresponding results in Hildenbrand's book [8]. The proof, which is based on our earlier result for economies with more general preferences [2], is elementary.

*This paper is an outgrowth of Chapter V of the author's dissertation presented for the degree of Doctor of Philosophy in Yale University under the supervision of Professor S. Kakutani. The author gratefully acknowledges the advice and encouragement of Professors Kakutani, Donald J. Brown, M. Ali Khan, Marcel K. Richter, and Herbert Scarf. The author was supported during the writing of the dissertation by a Canada Council Doctoral Fellowship. The 1978 drafts were written while the author was a McMaster Fellow at McMaster University. The 1981 version was prepared while the author was visiting the Cowles Foundation for Research in Economics at Yale University, on leave from the Departments of Mathematics and Economics at Princeton University. Research supported in part by a grant from the National Science Foundation.

1. Introduction

In [2], the author showed that one could prove in an elementary way that core allocations can be weakly decentralized by prices, given only very weak assumptions on preferences. Roughly speaking, the theorem asserts that any allocation in the core of an exchange economy is almost competitive, in that there is a price vector so that on average the individuals' commodity bundles lie near the budget frontiers and any bundle preferred to the one allocated does not lie far below the budget frontier. This result is closely related to an earlier paper of Dierker [7].

In this paper, we show that a stronger conclusion holds if one assumes that preferences are strongly convex. Theorem 3 asserts that the average deviation between the core allocation and the demand sets tends to zero. If the preferences are "equi-spherical"--a non-differentiable class of preferences analogous to the "smooth preferences" introduced by Debreu in [5]--Theorem 5 gives a rate of convergence of $O(1/\sqrt{n})$, where n is the number of traders.

Our result extends those given in Hildenbrand's book [8] in several ways. First, the form of convergence (convergence in mean) is stronger than the convergence in measure he proves. The importance of convergence in mean is that it guarantees that supply nearly equals demand for the price vector selected; this would not necessarily be true for the convergence in measure stated by Hildenbrand. It should be noted, however, that convergence in mean can be deduced without great difficulty from Hildenbrand's main and auxiliary results.[1] Second, we are able to give a rate of convergence result without invoking generic statements, as in

[1] This fact is noted in the May 1978 draft of his chapter in The Handbook of Mathematical Economics [9].

[5]. Third, we are able to deal with sequences of economies in which the endowment of the largest individual is only assumed to be small compared to the number of traders. Hildenbrand considered uniformly integrable endowments, a condition meaning that no class with a small proportion of the population could possess a significant share of the total endowment. Heuristically, we may say that the uniform integrability assumption excludes economies with dominant classes, while our assumption only excludes economies with a dominant individual. We prove convergence in measure in this general situation, and prove convergence in mean on any sequence of coalitions for which the restrictions of endowments to these coalitions are uniformly integrable. Khan [10] and subsequently Trockel [16] gave limit theorems for sequences without a uniform integrability assumption. A difficulty in both papers is that preferences are rescaled. An equi-convex or tight set of preferences may fail to be equi-convex or tight after this rescaling. Thus, our result does not follow from theirs. Moreover, we obtain a stronger form of price decentralization. A more detailed comparison to the work of Khan and Trockel is given in the author's dissertation [1]. Fourth, our preferences need be neither complete nor transitive. Indeed, the class of preferences we consider is very close to that for which Mas-Colell [13] proved the existence of competitive equilibria without completeness or transitivity. It is rather pleasing that strong core limit theorems can be obtained in the same setting.

The proof relies critically on the simple core equivalence theorem which we gave in [2]. The additional argument is completely elementary, though possibly somewhat unmotivated. It arose as the translation of a nonstandard argument. Since the original nonstandard proof is, to those familiar with the methodology, more natural, we give it in an appendix.

This paper is an outgrowth of Chapter V of the author's dissertation [1]. The dissertation contains a slightly weaker form of Theorem 3. The proof given there relies in an important way on nonstandard analysis, and was inspired by the work of Brown-Robinson [4], Khan-Rashid [11], and Rashid [14]. A more detailed account of the extensive debt to those papers is given in the dissertation.

2. Results

We begin with some notation. Suppose $x, y \in R^k$, $A, B \subset R^k$.

x^i denotes the i^{th} component of x

$\|x\| = \max_{1 \le i \le k} |x^i|$, $\|x\|_1 = \sum_{i=1}^{k} |x^i|$, $\|A\| = \inf\{\|x\| : x \in A\}$

$x \le y$ if $x^i \le y^i$ $(1 \le i \le k)$

$x << y$ if $x^i < y^i$ $(1 \le i \le k)$

$A + B = \{x + y : x \in A, y \in B\}$

con A is the convex hull of A

$R_+^k = \{x \in R^k : x \ge 0\}$, $B(x,\delta) = \{y \in R_+^k : \|y-x\| < \delta\}$.

Let P denote the set of preferences (i.e. binary relations on R_+^k) satisfying the following conditions:

(i) irreflexivity: $x \not\succ x$

(ii) weak monotonicity: $x >> y \Longrightarrow x \succ y$

(iii) convexity: $\{y : y \succ x\}$ is convex for all x

(iv) upper semi-continuity: $\{y : x \succ y\}$ is relatively open in R_+^k for each x.

Let P' denote the set of preferences in P satisfying the following additional condition

(v) strong convexity: if $x \neq y$, then either $\frac{x+y}{2} \succ x$ or $\frac{x+y}{2} \succ y$.

Note that our preferences need be neither complete nor transitive.

Definition: An exchange economy is a map $\varepsilon : A \to \mathcal{P} \times R^k_+$, where A is a finite set. For $a \in A$, let $\succ_a$ be the projection of $\varepsilon(a)$ onto $\mathcal{P}$, and $e(a)$ the projection of $\varepsilon(a)$ onto R^k_+ . $\succ_a$ is interpreted as the preference of trader a , and $e(a)$ his initial endowment. An allocation is a map $f : A \to R^k_+$ such that $\sum_{a\in A} f(a) = \sum_{a\in A} e(a)$. A coalition is a non-empty subset of A . An allocation, f , is blocked by a coalition S if there exists $g : S \to R^k_+$ with $\sum_{a\in S} g(a) = \sum_{a\in S} e(a)$ such that $g(a) \succ_a f(a)$ for all $a \in S$. The core of ε , $\mathcal{C}(\varepsilon)$, is the set of all allocations which are not blocked by any coalition. A price p is an element of R^k_+ such that $\|p\|_1 = 1$. Let $\mathcal{S}$ be the set of all prices, $\mathcal{S}^\circ = \{p \in \mathcal{S} : p >> 0\}$. The demand of an individual at a price p , $D_a(p)$, is the set of all elements in$\{x : p\cdot x \leq p\cdot e(a)\}$ maximal with respect to $\succ_a$. Let $M = \max\{\|e(a_1) + \ldots + e(a_k)\| : a_1, \ldots, a_k$ are distinct elements of $A\}$ (recall k is the number of commodities).

The following theorem asserts that if $\succ_a \in \mathcal{P}$, $D_a(p)$ is non-empty, and if $\succ_a \in \mathcal{P}'$, $D_a(p)$ contains a unique vector. Part (i) is due to Sonnenschein [15], but the proof we give is taken from the author's dissertation [1].

Theorem 1: Suppose $p \in \mathcal{S}^\circ$.

(i) If $\succ_a \in \mathcal{P}$, $D_a(p) \neq \phi$.

(ii) If $\succ_a \in \mathcal{P}'$, $D_a(p)$ contains exactly one vector.

Proof: Let $B = \{z : p\cdot z \leq p\cdot e(a)\}$. B is compact and convex. Suppose that, for all $z \in B$, there exists $x_z \in B$, such that $x_z \succ_a z$. Since $\succ_a$ is upper semi-continuous, there exists $\delta_z > 0$ such that $x_z \succ_a B(z, \delta_z)$. Since B is compact, there exist $z_1, \ldots, z_n$ such that $B \subset B(z_1, \delta_1) \cup \ldots \cup B(z_n, \delta_n)$, where $\delta_n = \delta_{z_n}$. Let $x_i = x_{z_i}$.

Let $g_i(z) = \max\{0, \delta_i - \|z - z_i\|\}$, and $f_i(z) = g_i(z)/(\sum_{j=1}^{n} g_j(z))$. Then each f_i is continuous and $\sum_i f_i(z) = 1$ for all $z \in B$; moreover, $f_i(z) > 0 \Longrightarrow z \in B(z_i, \delta_i)$.

Let $f(z) = \sum_i f_i(z)x_i$. Since $f_i(z) > 0 \Longrightarrow x_i \succ z$ and $\succ_a$ is convex, $f(z) \succ z$. Moreover, $f : B \to B$ is continuous. By Brouwer's Fixed Point Theorem, there exists z such that $f(z) = z$, so $z \succ z$, contradicting the irreflexivity assumption. Hence there exists $z \in B$ so that $x \succ z \Longrightarrow x \notin B$, i.e. $x \in D_a(p)$.

Now suppose $\succ_a \in P'$, and $z_1 \neq z_2 \in B$. Since $\succ_a$ is strongly convex, $\frac{z_1 + z_2}{2} \succ z_1$ or $\frac{z_1 + z_2}{2} \succ z_2$, so at most one of z_1 and z_2 can be maximal. Hence $D_a(p)$ contains a unique vector. This completes the proof of Theorem 1.

Our central tool is the following special case of Anderson [2, Theorem 1].

Theorem 2: Let $\varepsilon : A \to P \times R^k_+$ be a finite exchange economy. If $f \in C(\varepsilon)$, there exists $p \in S$ such that

(i) $\sum_{a\in A} |p\cdot(f(a) - e(a))| \leq 2M_\varepsilon$.

(ii) $\sum_{a\in A} |\inf\{p\cdot(x - e(a)) : x \succ_a f(a)\}| \leq 2M_\varepsilon$.

Proof: This result will be seen to be a special case of [2, Theorem 1] if we can show that any $\succ \in \mathcal{P}$ satisfies the following condition, called free disposal in [2]:

$$x >> y\ ,\quad y \succ z \Longrightarrow x \succ z\ .$$

Let $\alpha > 1 + [\|z\|/\min_i |x^i - y^i|]$, $w = y + \alpha(x-y)$. Since $x >> y$, $w >> z$. Hence $w \succ z$. Since $x = w/\alpha + y(1 - 1/\alpha)$ and $\succ$ is convex, $x \succ z$. This proves Theorem 2.

In order to get control of how far individual demands are from core allocations, we need to introduce a measure of how convex a preference is. This is done in the following definition.

Definition: For $x \neq y \in R^k_+$, let $\sigma_{xy}(\succ) = \sup\left\{\delta : B\left(\frac{x+y}{2}, \delta\right) \succ B(x,\delta) \text{ or } B\left(\frac{x+y}{2}, \delta\right) \succ B(y,\delta)\right\}$. If $P \subset \mathcal{P}$, let $\sigma_{xy}(P) = \inf\{\sigma_{xy}(\succ) : \succ \in P\}$. We say that P is equi-convex if $\sigma_{xy}(P) > 0$ for all $x \neq y \in R^k_+$.

Remark: Note that since $B(x,\delta) = \bigcup_{\delta'<\delta} B(x, \delta')$, we will have either $B\left(\frac{x+y}{2}, \sigma_{xy}(\succ)\right) \succ B(x, \sigma_{xy}(\succ))$ or $B\left(\frac{x+y}{2}, \sigma_{xy}(\succ)\right) \succ B(y, \sigma_{xy}(\succ))$. Note also that $\sigma_{xy}(P)$ is a continuous function of x and y , for all P . Suppose $\succ \in \mathcal{P}'$, and in addition $\succ$ is continuous (i.e. $\{(u,v) : u \succ v\}$ is relatively open in R^k_+). Then given $x \neq y$, either $\frac{x+y}{2} \succ x$ or $\frac{x+y}{2} \succ y$; in the first case, $B\left(\frac{x+y}{2}, \delta\right) \succ B(x,\delta)$, while in the second $B\left(\frac{x+y}{2}, \delta\right) \succ B(y,\delta)$ for some δ .

Thus, if $\succ$ is a continuous preference in $\mathcal{P}'$, $\sigma_{xy}(\succ) > 0$ for all x , y . Finally, the reader familiar with the topology of closed convergence on the space of preferences (see Hildenbrand [8]) will have no difficulty verifying that any compact set of preferences which are strongly convex in Hildenbrand's sense is equi-convex.

We can now state the first of our main results. The introduction of the sets E_n makes the result more general. It says that, provided any part of the individuals have well-behaved preferences and endowments, any core allocation will be well-behaved on that part. In order to understand the statement of the theorem, however, the reader may wish to take $E_n = A_n$ at the first reading.

The most important part of the theorem is conclusion 2), which asserts that core and demand are close in mean provided the endowments are uniformly integrable. Since f_n is an allocation this means that supply nearly equals demand for p_n. Indeed, if only a part of the sequence is uniformly integrable, we see that core and demand are close in mean on that part. In any case, we see that core and demand are close for most traders, by conclusion 1).

Theorem 3: Let $\varepsilon_n : A_n \to P \times R^k_+$ be a sequence of exchange economies satisfying

(i) $M_{\varepsilon_n} / |A_n| \to 0$

(ii) $\sup_n \| \sum_{a \in A_n} e_n(a) \| / |A_n| < \infty$.

If $f_n \in C(\varepsilon_n)$, there exist prices $\{p_n\}$, contained in a compact subset of $S°$, so that for any collection $\{E_n\}$ of subsets of $\{A_n\}$ satisfying

(iii) $\inf_n |E_n|/|A_n| > 0$

(iv) for all $\delta > 0$ there exists P equi-convex such that $|\{a \in E_n : \succ_a \in P\}|/|E_n| > 1-\delta$ for all n

(v) there exists $\delta > 0$ such that $\{|a \in E_n : e_n(a)^i > \delta\}/|E_n| > \delta$ for all n and all i

the following holds:

1) for all $\gamma > 0$, $|\{a \in E_n : \|f_n(a) - D_a(p_n)\| > \delta\}|/|A_n| \to 0$.

If in addition we have

$$\text{(vi)}\quad E_n' \subset E_n ,\quad |E_n'|/|A_n| \to 0 \Longrightarrow \|\sum_{a \in E_n'} e_n(a)\|/|A_n| \to 0$$

then

$$\text{2)}\quad \|\sum_{a \in E_n} f_n(a) - D_a(p_n)\|/|A_n| \leq \sum_{a \in E_n} \|f_n(a) - D_a(p_n)\|/|A_n| \to 0 .$$

The first step in the proof of Theorem 3 is the following lemma, which says that for sequences of economies described in Theorem 3, the sequence of prices determined by Theorem 2 stays within a compact subset of S° .

<u>Lemma 4</u>: Let $\varepsilon_n : A_n \to P \times R_+^k$ be as described in Theorem 3, and assume there exists a collection $\{E_n\}$ with the properties described. If p_n is a price for ε_n satisfying the conclusion of Theorem 2, then $\{p_n\}$ is contained in a compact subset of S° .

<u>Proof of Lemma 4</u>: If the lemma is false, we can (by passing to a subsequence) assume that $p_n \to p \in S - S^\circ$. Assume without loss of generality that $p^1 = 0$, $p^2 > 0$. There exists $\delta > 0$ such that $p^2 \geq 2\delta$ and, if $S_n = \{a \in E_n : e_n(a)^2 > \delta\}$, $|S_n|/|E_n| \geq \delta$ for all n . Choose P equi-convex so that $|\{a \in E_n : \succ_a \in P\}|/|E_n| > 1 - \delta/4$ for all n . Since $\sup_n \|\sum_{a \in A_n} e_n(a)\|/|A_n| < \infty$, $\sup_n \|\sum_{a \in A_n} f_n(a)\|/|A_n| < \infty$. Thus, there exists $\alpha \in R$ so that

$$|\{a \in E_n : \|e_n(a)\| < \alpha,\ \|f_n(a)\| < \alpha\}|/|E_n| > 1 - \delta/4$$

for all n .

Since $\sum_{a \in A_n} |p_n \cdot (f_n(a) - e_n(a))|/|A_n| \leq 2M_{\varepsilon_n}/|A_n| \to 0$, there exist

$\beta_n \to 0$ such that $|\{a \in E_n : |p_n \cdot (f_n(a) - e_n(a))| < \beta_n\}|/|E_n| > 1 - \delta/4$, $|\{a \in E_n : |\inf\{p_n \cdot (x - e_n(a)) : x \succ_a f_n(a)\}| < \beta_n\}|/|E_n| > 1 - \delta/4$, and $p_n^1 < \beta_n$ for n sufficiently large.

Thus, for n sufficiently large, there exists $a_n \in E_n$ simultaneously satisfying

(i) $e_n(a_n)^2 > \delta$

(ii) $\succ_{a_n} \in P$

(iii) $\|e_n(a_n)\| \leq \alpha$, $\|f_n(a_n)\| \leq \alpha$

(iv) $|p_n \cdot f_n(a_n) - p_n \cdot e_n(a_n)| < \beta_n$

(v) $|\inf\{p_n \cdot x : x \succ_a f_n(a)\} - p_n \cdot e_n(a_n)| < \beta_n$

Moreover, for n sufficiently large, $p_n^2 > p^2/2 \geq \delta$, so $p_n \cdot e_n(a_n) \geq p_n^2 e(a_n)^2 > \delta^2$.

Hence $p_n \cdot f_n(a) > \delta^2 - \beta_n$, so there exists j such that $p_n^j \cdot f_n(a_n)^j > (\delta^2 - \beta_n)/k$. Hence $p_n^j > (\delta^2 - \beta_n)/k\alpha$, and $f_n(a_n)^j > (\delta^2 - \beta_n)/k$.

Since P is equi-convex, there exists $\lambda \in (0, \delta^2/2k)$ such that $B\left(\frac{x+y}{2}, \lambda\right) \succ_{a_n} B(x,\lambda)$ or $B\left(\frac{x+y}{2}, \lambda\right) \succ_a B(y,\lambda)$ whenever $\|x-y\| \geq 1/2$ and $\|x\|, \|y\| \leq \alpha+1$. $f_n(a_n)^j > (\delta^2 - \beta_n)/k \to \delta^2/k$ as $n \to \infty$, hence $f_n(a_n)^j > \lambda$ for n sufficiently large. Let $w = (1, 0, \ldots, 0)$, $x = f_n(a_n) + w$, $y = f_n(a_n) + w/2$. Then $B(y,\lambda) \succ_{a_n} B(x,\lambda)$ or $B(y,\lambda) \succ_{a_n} B(f_n(a_n), \lambda)$; we claim it can not be the first alternative. If $B(y,\lambda) \succ_{a_n} B(x,\lambda)$, then $y \succ_{a_n} x + \frac{\lambda}{2}(1, \ldots, 1)$. Then $x + \frac{\lambda}{2}(1, \ldots, 1) \gg y \succ_{a_n} x + \frac{\lambda}{2}(1, 1, \ldots, 1)$, so $x + \frac{\lambda}{2}(1, 1, \ldots, 1) \succ_{a_n} x + \frac{\lambda}{2}(1, 1, \ldots, 1)$ by free disposal (as defined in the proof of Theorem 2.) We have thus contradicted irreflexivity.

Thus, $B(y,\lambda) \succ_{a_n} B(f_n(a_n), \lambda)$. Let $z = y - (0, \ldots, \lambda/2, 0, \ldots, 0)$, where the $\lambda/2$ occurs in the j^{th} place. $f_n(a_n)^j > \lambda$, so $z \geq 0$, hence $z \in B(y,\lambda)$, and thus $z \succ_{a_n} f_n(a_n)$. $p_n \cdot z = p_n \cdot y - \frac{\lambda}{2} p_n^j$ $= p_n \cdot f_n(a_n) + p_n^1/2 - \frac{\lambda}{2} p_n^j \leq p_n \cdot e_n(a_n) + \beta_n + \beta_n/2 - \frac{\lambda}{2}(\delta^2 - \beta_n)/k\alpha$ $= p_n \cdot e_n(a_n) + \beta_n\left[\frac{3}{2} + \frac{\lambda}{2k\alpha}\right] - \frac{\lambda}{2k\alpha}\delta^2 \to p_n \cdot e_n(a_n) - \frac{\lambda}{2k\alpha}\delta^2$. Hence, $p_n \cdot z < p_n \cdot e_n(a_n) - \beta_n$ for n sufficiently large. But $\inf\{p_n \cdot (x - e_n(a_n)) : x \succ_{a_n} f_n(a_n)\} \geq -\beta_n$, contradiction. This proves the lemma.

Proof of Theorem 3. If there is no collection $\{E_n\}$ of subsets satisfying conditions (iii)-(v), then the theorem is vacuously true. Hence we may assume that such a collection exists. Choose p_n for ε_n according to Theorem 2. By Lemma 4, $\{p_n\}$ is contained in a compact subset of S°. Hence $C = \sup_n\{\max\{1/p_n^1, \ldots, 1/p_n^k\}\}$ is finite.

We shall suppose at first that there is some equi-convex P such that $\succ_a \in P$ for all $a \in E_n$ and all n, and that $\sup_n \max_{a \in E_n} \|e_n(a_n)\| = \alpha < \infty$.

Fix $\gamma > 0$. Since $\sigma_{xy}(P)$ is continuous in x and y, there exists $\delta \in (0, \gamma/2)$, $\delta \leq C$, such that $\sigma_{xy}(P) > \delta$ whenever $\|x-y\| > \gamma$ and $\|x\|, \|y\| \leq C(\alpha+1)$. Suppose $p_n \cdot f_n(a) < p_n \cdot e_n(a) + \delta/C$. Then $\|f_n(a)\| \leq C(\alpha+1)$; moreover, $\|D_a(p_n)\| \leq C\alpha$. If $\|D_a(p_n) - f_n(a)\| > \gamma$, then $B(y,\delta) \succ_a B(D_a(p_n), \delta)$ or $B(y,\delta) \succ_a B(f_n(a), \delta)$, where $y = (D_a(p_n) + f_n(a))/2$. Since $p_n \cdot f_n(a) < p_n \cdot e_n(a) + \delta/C$, there exists $z \in B(y,\delta)$ such that $p_n \cdot z \leq p_n \cdot e_n(a)$. Hence, we can't have $z \succ_a D_a(p_n)$. Hence, we must have $B(y,\delta) \succ_a f_n(a)$. But $\inf p_n \cdot B(y,\delta) \leq p_n \cdot y - \delta/C$, unless $0 \in B(y,\delta)$, in which case the inf is 0.
$p_n \cdot y - \delta/C = (p_n \cdot f_n(a) + p_n \cdot e_n(a))/2 - \delta/C < p_n \cdot e_n(a) - \delta/2C$. If

$0 \in B(y,\delta)$, then $\|y\| < \delta$, so $\|D_a(p_n)\| < 2\delta$ and $\|f_n(a)\| < 2\delta$, and hence $\|D_a(p_n) - f_n(a)\| < 2\delta < \gamma$.

We have shown the following: if $a \in E_n$ and $\|D_a(p_n) - f_n(a)\| > \gamma$, then either $p_n \cdot f_n(a) \geq p_n \cdot e_n(a) + \delta/C$ or $\inf\{p_n \cdot x : x \succ_a f_n(a)\} < p_n \cdot e_n(a) - \delta/2C$. By Theorem 2, $|\{a \in A_n : p_n \cdot f_n(a) \geq p_n \cdot e_n(a) + \delta/C\}|/|A_n| \to 0$, and $|\{a \in A_n : \inf\{p_n \cdot x : x \succ_a f_n(a)\} < p_n \cdot e_n(a) - \delta/2C\}|/|A_n| \to 0$. Hence $|\{a \in E_n : \|D_a(p_n) - f_n(a)\| > \gamma\}|/|A_n| \to 0$.

Now consider the general case for $\{E_n\}$. Fix $\gamma, \delta > 0$. By the assumptions, there exists an equi-convex P and $\alpha > 0$ such that $|\{a \in E_n : \succ_a \notin P \text{ or } \|e_n(a)\| > \alpha\}|/|A_n| < \delta/2$. Applying the special case, we see that $|\{a \in E_n : \|D_a(p_n) - f_n(a)\| > \gamma\}|/|A_n| < \delta$ for sufficiently large n . Since δ is arbitrary, this establishes conclusion 1).

Now suppose hypothesis (vi) also holds. We know $\sum_{a \in A_n} |p_n \cdot (f_n(a) - e_n(a))| \leq 2M_{\varepsilon_n}$. But $\|f_n(a)\| \leq \|e_n(a)\| + \|f_n(a) - e_n(a)\| \leq \|e_n(a)\| + C|p_n \cdot (f_n(a) - e_n(a))|$. Also $\|D_a(p_n)\| \leq C\|e_n(a)\|$. Hence $\|f_n(a) - D_a(p_n)\| \leq \|e_n(a)\|(1+C) + C|p_n \cdot (f_n(a) - e_n(a))|$.

Let $E_n' = \{a \in E_n : \|D_a(p_n) - f_n(a)\| > \gamma\}$. Then $|E_n'|/|E_n| \to 0$, and so $\sum_{a \in E_n'} \|e_n(a)\|/|A_n| \to 0$. Thus $\sum_{a \in E_n'} \|D_a(p_n) - f_n(a)\|/|A_n| \to 0$. Consequently, $\sum_{a \in E_n} \|D_a(p_n) - f_n(a)\|/|A_n|$ is eventually less than 2γ ; since γ is arbitrary, the sum in fact tends to 0, establishing conclusion 2) and concluding the proof of Theorem 3.

We can now state a theorem giving a rate of convergence. The principal concept is the notion of an equi-spherical set of preferences.

Definition: $P \subset \mathcal{P}$ is said to be equi-spherical if it is equi-convex and, for any compact $K \in R_{+}^{k}$, there exists $\beta > 0$ such that $\forall x, y \in K \quad \forall \succ \in P \quad B\left(\frac{x+y}{2}, \beta\|x-y\|^{2}\right) \succ x$ or $B\left(\frac{x+y}{2}, \beta\|x-y\|^{2}\right) \succ y$.

Remark: Equi-spherical preferences are essentially non-differentiable analogues of Debreu's smooth preferences used in proving the rate of core convergence [6]. The square exponent $\|x-y\|^{2}$ is essentially a curvature condition analogous to Debreu's non-vanishing Gaussian curvature condition. The following is a sketch of the relationship.

Debreu's preferences are defined only on the interior of R_{+}^{k}, and don't extend continuously to R_{+}^{k}; his indifference curves have closure in the interior of R_{+}^{k}, whereas the analogues of indifference curves in our setting (or at least some of them) are forced by the equi-convexity condition to cut the boundary of R_{+}^{k}. However, Debreu's preferences satisfy the equi-spherical definition, relativized to the interior of R_{+}^{k}. In other words, if $\succ$ is one of Debreu's preferences, $\sigma_{xy}(\succ) > 0$ for all $x, y \gg 0$ and for all compact K in the interior of R_{+}^{k}, there is $\beta > 0$ such that $\forall x, y \in K \quad B\left(\frac{x+y}{2}, \beta\|x-y\|^{2}\right) \succ x$ or $B\left(\frac{x+y}{2}, \beta\|x-y\|^{2}\right) \succ y$. To see this, one need only note (as Debreu does) that, because of the Gaussian curvature condition, there is $\gamma > 0$ such that, $\forall x \in K$, $\{y \in K : y \succ x\}$ is contained in a ball of radius γ, with x on the boundary of the ball; the rest of the argument is analytic geometry, which we leave to the reader.

The equi-spherical notion is closely related to the notion of sphero-convexity, developed independently by Vial [7].

<u>Theorem 5</u>: Suppose $\varepsilon_n : A_n \to P \times R^k_+$ satisfies

(i) $\sup_n \max_{a\in A_n} \|e_n(a)\| = \alpha < \infty$, $|A_n| \to \infty$

(ii) there exists P equi-spherical such that $\succ_a \in P$ for all $a \in A_n$ and all n

(iii) $\inf_n \sum_{a\in A_n} e_n(a)^i/|A_n| > 0$ for each i .

Then for any $f_n \in C(\varepsilon_n)$, there exist prices p_n , contained in a compact subset of S° , such that

$$\sum_{a\in A_n} \|f_n(a) - D_a(p_n)\|/|A_n| = O(1/\sqrt{|A_n|}) .$$

<u>Proof</u>: Since an equi-spherical set is equi-convex, it is easy to see that the hypotheses of Theorem 3 are satisfied. Thus, the constant C (as defined in the first paragraph of the proof of Theorem 3) is again finite. Accordingly, for any $a \in A_n$, $\|f_n(a)\| \le \|f_n(a) - e_n(a)\| + \|e_n(a)\| \le C|p_n\cdot(f_n(a) - e_n(a))| + \alpha \le 2Ck\alpha + \alpha = (2Ck+1)\alpha$.

It is not hard to see that there exists $\beta > 0$ such that

$$\sum_{a\in A_n} \|f_n(a) - D_a(p_n)\| \le \beta \sum_{a\in A_n} (|p_n\cdot(f_n(a) - e_n(a))|^{1/2} + |\inf\{p_n\cdot(x - e_n(a)) : x \succ_a f_n(a)\}|^{1/2}) ;$$

the proof of this fact is similar to the third paragraph of the proof of Theorem 3, but easier.

By the concavity of the square root function,

$$\sum_{a\in A_n} |p_n\cdot(f_n(a)-e_n(a))|^{1/2} + |\inf\{p_n\cdot(x-e_n(a)) : x \succ_a f_n(a)\}|^{1/2}$$

$$\leq \sqrt{|A_n|}[(\sum_{a\in A_n} |p_n\cdot(f_n(a)-e_n(a))|)^{1/2}$$

$$+ (\sum_{a\in A_n} |\inf\{p_n\cdot(x-e_n(a)) : x \succ_a f_n(a)\}|)^{1/2}]$$

$$\leq \sqrt{|A_n|}(\sqrt{2k\alpha}+\sqrt{2k\alpha}) < 4\sqrt{k\alpha}\ \sqrt{|A_n|} .$$

Hence $\frac{1}{|A_n|}\sum_{a\in A_n} \|f_n(a)-D_n(p_a)\| = O(1/\sqrt{|A_n|})$.

Appendix: The Nonstandard Proof

In this appendix, we give a single combined proof of Lemma 4 and Theorem 3, using Nonstandard Analysis. The proof given in the body of the paper was obtained by translating this proof.

Proof of Lemma 4 and Theorem 3: Let $\varepsilon_n : A_n \to P \times R^k_+$ satisfy hypotheses (i)-(ii) in the statement of Theorem 3, $f_n \in C(\varepsilon_n)$ and let $\{E_n\}$ be a collection of subsets of $\{A_n\}$ satisfying (iii)-(v). Choose $n \in {}^*N - N$; for simplicity of notation, let $A = A_n$, $\varepsilon = \varepsilon_n$, $f = f_n$, $E = E_n$, $\omega = |A|$. Let ν be the normalized counting measure on E , $\nu(S) = |S|/|E|$, and μ the associated Loeb measure ([12]) .

By Theorem 2 and the Transfer Principle, there exists $p \in {}^*S$ such that

$$\frac{1}{\omega}\sum_{a\in A} |p\cdot(f(a)-e(a))| \leq \frac{2M_\varepsilon}{\omega} \simeq 0$$

$$\frac{1}{\omega}\sum_{a\in A} |\inf\{p\cdot(x-e(a)) : x \succ_a f(a)\}| \leq \frac{2M_\varepsilon}{\omega} \simeq 0 .$$

Since $|E|/\omega \not\simeq 0$, $p\cdot f(a) \simeq p\cdot e(a) \simeq \inf\{p\cdot x : x \succ_a f(a)\}$ for μ-almost all $a \in E$.

We claim ${}^\circ p >> 0$. If not, we can assume without loss of generality that $p^1 \simeq 0$, $p^2 \not\simeq 0$. There exists $\delta \in R$, $\delta > 0$ such that, letting $S = \{a \in E : e(a)^2 > \delta\}$, $\nu(S) > \delta$. Choose P equi-convex so that $\nu(\{a \in E : \succ_a \in {}^*P\}) \gtrsim 1-\delta$. Since $\|\sum_{a\in A} e(a)\|/\omega$ is finite, so is $\|\sum_{a\in A} f(a)\|/\omega$, and ${}^\circ\|e(a)\| < \infty$, ${}^\circ\|f(a)\| < \infty$ for μ-almost all $a \in E$. Hence, there exists $a \in E$ satisfying simultaneously $e(a)^2 \not\simeq 0$, $\succ_a \in {}^*P$, ${}^\circ\|e(a)\| < \infty$, ${}^\circ\|f(a)\| < \infty$, $p\cdot f(a) \simeq p\cdot e(a) \simeq \inf\{p\cdot x : x \succ_a f(a)\}$. $p\cdot e(a) \geq p^2 e(a)^2 \not\simeq 0$. Hence, $p\cdot f(a) \not\simeq 0$, so there exists j such that $p^j \not\simeq 0$, $f(a)^j \not\simeq 0$. Let $x = f(a) + (1, 0, \ldots, 0)$, y the j^{th} basis vector.

Since P is equi-convex, there exists $\lambda > 0$, $\lambda \in R$ such that if $z = f(a) + (1/2, 0, \ldots, 0)$, $B({}^\circ z, \lambda) \succ_a B({}^\circ x, \lambda)$ or $B({}^\circ z, \lambda) \succ_a B({}^\circ f(a), \lambda)$. It can't be the first alternative, since that would imply ${}^\circ x + (\lambda/2, \ldots, \lambda/2) >> {}^\circ z \succ_a {}^\circ x + (\lambda/2, \ldots, \lambda/2)$, a contradiction of free disposal and irreflexivity. Thus, $B({}^\circ z, \lambda) \succ_a B({}^\circ f(a), \lambda)$. If $\delta = \frac{1}{2}\min\{\lambda, f(a)^j\}$, $z - \delta y \succ_a f(a)$, $p\cdot(z - \delta y) = p\cdot f(a) + p^1/2 - \delta p^j \lesssim p\cdot f(a)$, contradicting $p\cdot f(a) \simeq \inf\{p\cdot x : x \succ_a f(a)\}$. This shows ${}^\circ p >> 0$, and establishes Lemma 4 by Transfer.

Therefore, $D(p,a)$ is a unique vector for all $a \in A$ by transferring Theorem 1. μ-almost all $a \in E$ satisfy $p\cdot f(a) \simeq p\cdot e(a) \simeq \inf\{p\cdot x : x \succ_a f(a)\}$, ${}^\circ\|e(a)\| < \infty$, and ${}^\circ\|f(a)\| < \infty$, as shown above. Consider any such a.

If $e(a) \simeq 0$, $p\cdot f(a) \simeq 0$, so $f(a) \simeq 0$. $p\cdot D(p,a) \simeq 0$, so $D(p,a) \simeq 0 \simeq f(a)$.

If $e(a) \not\simeq 0$, $p\cdot e(a) \not\simeq 0$. If $f(a) \not\simeq D(p,a)$,

$$B\left({}^\circ\left[\frac{f(a)+D(p,a)}{2}\right],\ \lambda\right) \succ_a B({}^\circ f(a),\ \lambda) \quad \text{or} \quad B\left({}^\circ\left[\frac{f(a)+D(p,a)}{2}\right],\lambda\right) \succ_a B({}^\circ D(p,a),\ \lambda)\ ,$$

for some $\lambda \not\simeq 0$. The second alternative violates the maximality of demand in the budget set. In the first case, $\exists\ x \in B\left({}^\circ\left[\frac{f(a)+D(p,a)}{2}\right],\ \lambda\right)$ (and so $x \succ_a f(a)$) such that $p\cdot x \lessapprox p\cdot f(a)$. This contradicts $p\cdot f(a) \simeq \inf\{p\cdot x : x \succ_a f(a)\}$.

We have shown that for μ-almost all $a \in E$, ${}^\circ f(a) \simeq D(p,a)$. For any $\gamma \in R$, $\gamma > 0$, $\nu(\{a \in E : \|f(a) - D(p,a)\| < \gamma\}) \simeq 0$. Transferring, we get conclusion 1).

Suppose now that assumption (vi) holds, so that the endowments are uniformly integrable. By Anderson [3, Section 6], e is SL^1 with respect to ν . For any internal S with $\nu(S) \simeq 0$, $p\cdot\int_S f d\nu = \int_S p\cdot f d\nu$ $\leq \int_S p\cdot e d\nu + 2M_\varepsilon\ \omega/|E| \simeq \int_S p\cdot e d\nu \simeq 0$, since e is S-integrable. Hence $\int_S f d\nu \simeq 0$, so f is S-integrable. $p\cdot D(p,a) \leq p\cdot e(a)$, so D is S-integrable. Hence $\int_E \|D(p,a) - f(a)\| d\nu \simeq \int_E {}^\circ\|D(p,a) - f(a)\| d\mu = 0$, since the integrand is 0 almost everywhere. In other words,

$$\sum_{a\in E} \|D(p,a) - f(a)\|/\omega \simeq 0\ ,$$

so

$$\sum_{a\in E_n} \|D(p,a) - f(a)\|/|A_n| \to 0 \text{ by Transfer.}$$

BIBLIOGRAPHY

[1] Robert M. Anderson, "Star-Finite Probability Theory," Yale Ph.D. Dissertation, May 1977.

[2] _________, "An Elementary Core Equivalence Theorem," Econometrica, 46 (1978), 1483-1487.

[3] _________, "Star-Finite Representations of Measure Spaces," Trans. Amer. Math. Soc. (to appear).

[4] Donald J. Brown and Abraham Robinson, "The Cores of Large Standard Exchange Economies," J. Economic Theory 9 (1974), 245-254.

[5] Gerard Debreu, "Smooth Preferences," Econometrica 40 (1972), 603-615.

[6] _________, "The Rate of Convergence of the Core of an Economy," J. Math. Econ. 2 (1975), 1-7.

[7] Egbert Dierker, "Gains and Losses at Core Allocations," J. Math. Econ. 2 (1975), 119-128.

[8] Werner Hildenbrand, Core and Equilibria of a Large Economy, Princeton University Press, 1974.

[9] _________, "Core of an Economy," in Kenneth Arrow and Michael Intriligator, eds., Handbook of Mathematical Economics, to appear.

[10] M. Ali Khan, "Oligopoly in Markets with a Continuum of Traders: An Asymptotic Interpretation," J. Econ. Theory 12 (1976), 273-297.

[11] _________ and Salim Rashid, "Limit Theorems on Cores with Costs of Coalition Formation," preprint, May 1976.

[12] Peter A. Loeb, "Conversion from Nonstandard to Standard Measure Spaces with Applications in Probability Theory," Trans. Amer. Math. Soc. 211 (1975), 113-122.

[13] Andreu Mas-Colell, "An Equilibrium Existence Theorem without Complete or Transitive Preferences," J. Math. Econ. 1 (1974), 237-246.

[14] Salim Rashid, "Economies with Infinitely Many Traders," Yale Ph.D. Dissertation, May 1976.

[15] Hugo F. Sonnenschein, "Demand Theory without Transitive Preferences, with Applications to the Theory of Competitive Equilibrium," in John S. Chipman, et al., eds., Preferences, Utility and Demand, Harcourt Brace Jovanovich, 1971.

[16] Walter Trockel, "A Limit Theorem on the Core," J. Math. Econ. 3 (1976), 247-264.

[17] Jean-Phillipe Vial, "Sphero-Convex Sets," Discussion Paper 7810, Center for Operations Research and Econometrics, Université Catholique de Louvain.