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A Model of Output, Employment, Capital Formation and Inflation

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A MODEL OF OUTPUT, EMPLOYMENT, CAPITAL FORMATION
AND INFLATION

by

R.W. Bailey, V.B. Hall and P.C.B. Phillips

May, 1980

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1. INTRODUCTION

This paper reports the theoretical development of a small aggregative model of output, employment, capital formation and inflation. The model (which we subsequently refer to as BHP) is designed to explain medium term cyclical growth in a small open economy. It allows explicitly for disequilibrium in the markets for goods and labour services and has a wage-price sector in which the movements in these variables are specified to allow for intended price setting behaviour by firms while, in addition, responding to realisations which may differ from

* R.W.Bailey is at the University of Birmingham, England, V.B.Hall is at the University of Sydney, and P.C.B.Phillips is at Yale University. The authors wish to acknowledge fruitful discussion with Peter Ledingham during the preliminary stages of this work, and to thank Peter Jonson, Mervyn King and Grant Spencer for their comments on a preliminary draft of the paper's first two sections, circulated in September 1977. An earlier version of this paper was presented at the 49th ANZAAS Congress, Auckland, January 1979. Our thoughts have subsequently been stimulated by comments and discussion in seminars at the Universities of Auckland, Canterbury, New South Wales, Southampton, Sydney, the Australian Graduate School of Management, the Australian National University, the University of Lausanne, Monash University, and the Reserve Bank of Australia.

these intentions as well as responding to the effects of disequilibrium in the real sector. The model is formulated in continuous time as a system of non-linear differential equations and has a particular solution which corresponds to plausible steady state growth behaviour for the variables of the model. The properties of this particular solution are analysed directly, and solution trajectories for the variables corresponding to various initial values which deviate from the steady state growth paths are computed numerically and compared with the steady state growth paths. The model has been developed with a view to subsequent empirical application to a small open economy and, as a foundation for this later work, some econometric methodology for the treatment of non-linear differential equations is developed in the paper.¹

The closest precedent of the present model is the prototype model developed for the United Kingdom by Bergstrom and Wymer (1976) (we refer to their model subsequently as BW). BW is also a highly aggregative system and is characterised by the substantial role which economic theory plays in the specification of its equations leading to strong interdependencies in the system, including across equation parameter restrictions, and non-linearities in both the parameters and variables. With reference to its empirical application, BW has been successful as a medium term forecasting model of the UK economy with root

1. The present paper covers the model development and econometric methodology of a more comprehensive project involving the modelling of the New Zealand economy. Subsequent papers will concentrate on the estimation of the model's equations, the analysis and treatment of residual autocorrelation, and present numerical estimates of the model's parameters and an analysis of forecasting performance. The quarterly data base for this empirical work is being kindly supplied to us by the Research Department of the Reserve Bank of New Zealand.

mean squared errors of one per cent or less for a number of key variables like aggregate consumption, employment and prices in one-period forecasts over eight quarters (in 1969-1970) and root mean squared errors of less than four per cent for all variables except exports in multi-period forecasts over the same period (with, in addition, the forecast period commencing eight quarters subsequent to the end of the sample data). Moreover these forecasts were obtained from a model whose only exogenous variable is time. This latter point is of some importance in evaluating the forecast performance of the model since most macro-econometric models gain considerable leverage in ex-post forecasting exercises by the use of actual values of what is often a large number of exogenous variables.

The framework of our own model BHP has much in common with BW but involves a number of significant departures and innovations. These differences between the two models are highlighted in the succeeding paragraphs.

First, in several equations we have attempted to recognise explicitly the role of an effective supply constraint in certain markets, and have specified adjustment coefficients and other parameters as explicit functions of particular variables in ways which allow us to examine their effect on the solution trajectories and which will facilitate empirical testing of their significance. We argue that decision makers are planning to adjust (for instance, their rates of consumption or demand for labour) to certain desired levels at certain rates; but these plans can be disrupted by shortages in supply and by spillover effects from other markets (particularly, monetary effects, labour shortages and so on). Supply side elements and spillover effects in situations of disequilibrium have been emphasised in the work of Clower (1965), Barro and Grossman (1971, 1976), Malinvaud

(1977) and others. These authors stress the role that quantity rationing can play in market transactions when markets do not necessarily clear. Thus, if there is an excess supply of labour at the prevailing level of wages and the wage rate does not adjust fast enough to clear the market, the resulting unemployment reduces the effective demand for goods. In this way, the consumption function is quantity constrained and its formulation should depend quantitatively on the degree of "quantity rationing" that occurs in the labour market. The short side of the labour market, in this case the binding demand for labour, determines the level of employment; and the spillover effect of disequilibrium in this market on the goods market is characterised by the consumption function now being conditional on the employment level. This type of disequilibrium model has been receiving an increasing amount of attention in the theoretical literature. Most of this literature, together with the research that has been done on the econometric estimation of single markets in disequilibrium,¹ has assumed the existence of a switching regime where transactions always take place on the short side of the market; and, hence, on either the supply curve or demand curve.² This hypothesis is more realistic at the micro than at the macro level. In the latter case, we are essentially aggregating over a large number of different markets for different goods, occurring in different locations and, indeed, occurring at different times during the discrete time interval that corresponds with the interval of observation for our data. An interesting recent

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1. For example, Fair and Jaffee (1972), Maddala and Nelson (1974), Goldfeld and Quandt (1975), and Laffont and Garcia (1977).
 2. In the case of a multi-market model, these curves may be quantity constrained by other markets (and vice versa) and this leads naturally to the question under what conditions will there be a consistent set of transactions in both markets. This question has recently been tackled by Gourieroux, Laffont and Monfort (1977).

paper by Muellbauer (1978) attempts to deal with some of these problems caused by aggregation. Muellbauer allows for a cross section of micro markets that may be in different regimes. These differences are smoothed out by aggregation over markets resulting in behavioural equations which are smooth functions of the systematic component of excess demand (individual market demand and supply functions are characterised by deviations from mean demand and supply functions over all markets and these latter functions lead to a systematic measure of excess demand). While this approach to aggregating micro markets in different regimes is very promising, it is not easy to see how to generalise it to deal with the problems of temporal aggregation where even the same market may be in different regimes at different times during the sampling interval and the systematic components of the demand and supply functions will, naturally enough, be time varying since the sampling interval does not correspond with the decision making intervals used by the various economic agents at the micro level. As indicated at the beginning of this paragraph, the approach taken in the present paper is rather different. We assume that quantity rationing constraints do occur at the macroeconomic level and that these constraints typically take the form whereby they influence the rate at which adjustments can be made towards the partial equilibrium values of the relevant variables. In other words, certain adjustment rates in the present model are endogenously determined.

Supply side constraints are not, however, directly treated in the BW model nor in the several subsequent models of this

genre that have so far been developed and with which we are familiar.¹ Most of the equations in the BW model and its derivative models allow the actual levels of a variable to be determined by the adjustment towards a partial equilibrium (or desired) level of that variable at a constant rate. The following description of the main features of this type of specification is given by Jonson, Moses and Wymer (1977, pp.10-11), (we subsequently refer to their article and model as JMW):

"Effective demands for commodities and assets are modelled in two steps. Equilibrium or long run demands are determined subject to the relevant constraints and relative prices. Effective demands and equilibrium demands are assumed to be linked by a first order adjustment process which is intended to capture the net effects of lags in the adjustment of expectations, uncertainties and other real world frictions."

If this type of specification is to be interpreted as a way of modelling market disequilibrium, then it is important to distinguish its features from the conceptual framework introduced by Clower (1965). In the first place the use of the term "effective demand" by Jonson et.al. above does not correspond to the usage by Clower since it does not explicitly allow for quantity rationing and quantity rationed functions in the same sense. Note that this is true even of those equations in JMW where the additional "effects of disequilibrium in real money balances" is taken into account. To take the consumption function

1. See, for example, Knight and Wymer (1975, 1978), and Jonson, Moses and Wymer (1977).

of JMW (1977, equation (1) page 28) as an example, we see that actual expenditure is assumed to adjust by a first order process towards a partial equilibrium or desired expenditure function (determined by disposable income and interest rates) and that, in addition, the adjustment of consumption is influenced by an additive term reflecting "the gap between actual and desired money balances where desired money balances are a function of income and interest rates" (JMW (1977, p.16)). This equation assumes that actual expenditure is completely determined by the level of actual household demand (as distinct from the desired expenditure function of households) which emerges from this (differential) equation. Thus there is no mechanism in the equation to allow for the effect of disequilibrium in the market for goods on actual expenditure if the goods market does not clear. The same point has recently been made by Muellbauer (1978, pp.35-36) who argues, in addition, for the use of excess demand indicators in all behavioural equations of macro-models. Thus:

"Conventional employment equations are a good example. Typically, desired labour demand is specified as a function of some variables such as output and the capital stock (obtained by inverting the production function) and a partial adjustment mechanism relates employment to desired labour demand. This either assumes that employment equals actual (as opposed to desired) labour demand, or else proxies non-clearing labour markets by an ad hoc adjustment mechanism which has nothing to do with the state of excess demand."

In BHP we have allowed for variable rates of adjustment in the specification of a number of equations to take account of

supply side effects, spillover influences from other markets, asymmetrical responses and the state of excess demand, where we believe these various effects are most likely to be relevant. These speed of adjustment coefficients are determined as explicit functions of other variables in the system and, in this way, are endogenously determined. The theoretical work by Treadway (1971, 1974) lends some support to this type of specification. Treadway showed that, if a firm uses n quasi-fixed inputs measured by the vector x and \dot{x} denotes the rate of investment in these inputs, then a local solution to the optimal path which maximises the present value of profits is given by the multivariate adjustment process

$$\dot{x} = M(x^*, r) [x - x^*]$$

where x^* is a stationary equilibrium and M denotes a matrix of adjustment coefficients. These coefficients are seen to be functionally dependent on the target vector x^* and the rate of interest r . It seems reasonable to suppose that local solutions to the optimal control problems faced by other representative economic agents such as consumers will display similar characteristics. Of course, these arguments apply strictly to the microeconomic problems only. But, to the extent that we can work on the assumption of representative agent behaviour in a macroeconomic model, it is useful to be guided by the implications microeconomic analysis may have for the form of local solutions such as the above adjustment mechanism.

Unfortunately, economic theory does not yet provide any guidance concerning the functional dependence $M = M(x^*, r)$ and this is at present a matter for the empirical investigator. Our present philosophy is, therefore, to allow for variable adjustment rates in certain equations and select functional forms which seem to correspond to plausible economic behaviour. This type of specification enables us to test the significance of the supply and spillover effects that enter into the variable adjustment rates in our empirical application of the model to the New Zealand economy. Two supply side constraints are thought to be particularly important for the New Zealand economy: a supply of labour services constraint affecting the level of employment; and an internal funds constraint affecting real imports, and hence, the overall supply of goods and services.

Second, these variable adjustment coefficients are in some cases, specified as asymmetric functions; that is, the response time path may be different, depending on the direction of movement. The concept of asymmetry seems to have received relatively little attention in the recent macroeconomic model-building literature despite its obvious empirical importance. Suppose, for example, one were able to establish a significant quantitative empirical relationship between an increase in the real wage overhang and a subsequent increase in unemployment; there would seem to be many reasons why an exactly opposite quantitative response should not follow from a decrease in the

real wage overhang.¹

One example of an asymmetric response function in our model BHP is the adjustment rate in the employment equation, which is more severely affected in periods of labour shortage than otherwise; another is the adjustment rate of consumer expenditure to its partial equilibrium level, which may be curtailed as inventories are being run down but not significantly influenced when inventories are being built up.

The third way in which our model differs from the BW framework is with respect to the (desired) factor demand functions for capital and labour services. These are jointly determined through a cost minimisation process so that relative prices figure prominently in these functions.

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1. The relevance of asymmetric responses is, of course, not new. See, for example, in the ninth (variorum) edition of Marshall (1961, Appendix H, pp.807-8):

"...the list of demand prices which holds for the forward movement of the production of a commodity will seldom hold for the return movement, but will in general require to be raised.

Again, the list of supply prices may have fairly represented the actual fall in the supply price of the thing that takes place when the supply is being increased; but if the demand should fall off, or if for any other reason, the supply should have to be diminished, the supply price would not move back by the course by which it had come, but would take a lower course. The list of supply prices which had held for the forward movement would not hold for the backward movement, but would have to be replaced by a lower schedule. This is true whether the production of the commodity obeys the law of diminishing or increasing return; but it is of special importance in the latter case, because the fact that the production does obey this law, proves that its increase leads to great improvements in organization."

We are grateful to Peter Groenewegen for drawing our attention to this reference.

Fourth, we have moved to second order adjustment processes in a number of equations (BW use second order differential equations only for capital and money). We have specified, in addition, our wage and price equations as second order differential equations. Such a formulation seems necessary in view of the importance changes in the rates of wage and price inflation in recent years which require explanation. The movement to a higher order adjustment process allows us to marry different theories of price determination consistently in the same equation.

The plan of the paper is as follows. The model itself is specified and its main features discussed and compared with other models in Section 2. Section 3 sets out the particular solution of the model which corresponds with steady state growth behaviour for the variables of the model and analyses the sensitivity of these growth paths to some of the model's key parameters. In Section 4 we detail the general solution trajectories for various combinations of initial values and compare these with the steady state paths. And in the Appendix we develop some econometric methodology for treating systems of non-linear stochastic differential equations with discrete data. The methods we discuss there involve the treatment of variable non-linearity directly by the use of discrete non-linear approximations and by the fitting of general solution trajectories to the observed sample path. These methods go some way towards

meeting the important criticisms by Fisher (1976) of the number and type of linear approximations used in previous empirical work with continuous time models. Our approach should, therefore, be useful to other investigators working with non-linear continuous systems.

2. OUTLINE OF THE MODEL AND ITS MAIN FEATURES

2.1 Overview of the Model

The model presented below in deterministic form is for a small open economy whose single good is used for consumption or capital formation purposes, can be supplied either domestically or from foreign sources, and can be demanded either by the rest of the world or domestically by both the private and public sector. There is no explicit private income sector, the only explicit private sector financial asset is money, and there are therefore no significant private holdings of financial assets by the rest of the world.¹ Monetary activity influences are taken as exogenous.

The principal economic agents are households, firms, Central Bank and Central Government Authorities, and the rest of the world. As will become more apparent when each equation is more

1. In the larger version of this model that is currently under development, both money and bonds are explicitly available as financial assets. Net private capital inflow has not been significant for New Zealand and, therefore, remains an exogenous variable within the overall level of foreign reserves.

fully explained, the behaviour of these agents is assumed to be determined on the following basis. Households demand goods and services and real money balances, and supply labour. The extent to which the demand for goods and services is achieved depends on the stocks of goods and services available and on each household's supply of real money balances. The supply of labour services is assumed to vary, in the short run, with the real wage and with trend demographic factors and, in the long run, with further demographic factors and the rate of technological progress. Firm behaviour is consistent with supplying goods and services and demanding both capital and labour services.¹ A major constraint on this domestic production process is provided by the form of the firm's production function; and the extent to which plans for the hire of labour services are achieved is dependent both on the nature and the degree of disequilibrium in the market for labour. Neither households (either alone or through their agents, the trade unions) nor firms are assumed to have complete control over prices and wages. Firms adjust prices in an attempt to cover their marginal (wage) costs, but the extent to which prices are changing may, in addition, be influenced by disequilibrium in the market for goods and services. Wage changes are assumed to

1. It has been necessary to specify labour services in terms of persons only, as there are no appropriate quarterly data for hours worked. This is obviously less of a problem for our medium term growth model than it would be for a short term forecasting model, but it does mean that BHP has not incorporated the most rapidly adjusting part of Helliwell's (1977b, pp.253-4) three tier hierarchical factor adjustment procedure.

be dependent on the extent of labour market disequilibrium; but the rate of change of wage inflation may also be influenced directly by the rate of change in prices. The Central Bank and Central Government Authorities are assumed responsible for interest rates, the level of real government expenditure and taxation rates, specific direct controls on the trading banks and imports, the exchange rate, and for allowing movements in the aggregate money supply to accommodate the aggregate demand for money. Exports are determined by the level of demand from the rest of the world and imports respond, subject to the direct controls of the authorities, to the domestic demand by households for imported goods and services and by firms for imported capital goods.

2.2 Equations of the Model

The model BHP consists of 11 equations. Seven equations are behavioural and four equations are identities. There are ten exogenous variables. The total number of parameters is 33, consisting of 20 behavioural parameters (such as elasticities, propensities, efficiency and share parameters), 11 adjustment parameters and 2 growth rates.

Our notations for the variables are as follows:

Endogenous

C = real private consumption expenditure

Y = real net output or income

S = real stock of inventories

- k = proportional rate of change of real net
fixed capital
 K = stock of real net fixed capital
 W = wage rate
 P = price of output
 L = employment
 I = real imports of goods and services
 p = proportional rate of change of price
 w = proportional rate of change of wages
 L_s = labour supply
 MC = short-run marginal (wage) costs

Exogenous

- GB = real government expenditure on monetary
(social security) benefits
 TP = real personal income tax payments
 MBD = bank demand deposits, \$m
 J = interest rate
 t = time
 E = real exports of goods and services
 G = real government expenditure
 P_I = imports price level
 ZQI = synthetic variable representing the relative
strength of quantitative imports restrictions
 ZAT = synthetic variable representing official
monetary policy with respect to trading banks'
advances

Writing the differential operator as $D = d/dt$, the equations of the model are:

$$(1) \quad \frac{DC}{C} = \gamma_1 \ln \left[\frac{\hat{C}}{C} \right]$$

$$\hat{C} = \beta_1 (Y + GB - TP)$$

$$\gamma_1 = \beta_2 + \beta_3 \left\{ \frac{D(MBD)}{MDB} - \frac{DP}{P} \right\} + \beta_4 \exp \left\{ -\frac{DS}{S} \right\}$$

$$(\beta_4 < 0)$$

$$(2) \quad Dk = \gamma_2 \left[\gamma_3 \ln \left[\frac{\hat{K}}{K} \right] - k \right]$$

$$\hat{K} = \beta_5 Y \left\{ 1 + \beta_6 \frac{1}{1+\beta_7} e^{\left(\frac{\beta_7 \lambda_1}{1+\beta_7} \right) t} H^{\frac{\beta_7}{1+\beta_7}} \right\}^{\frac{1}{\beta_7}}$$

$$H = \frac{W}{P(\beta_8 + \beta_9 (J - DP/P))}$$

$$(3) \quad \frac{DL}{L} = \gamma_4 \ln \left[\frac{\hat{L}}{L} \right]$$

$$\hat{L} = \beta_5 \beta_6^{1/\beta_7} e^{-\lambda_1 t} Y \left\{ 1 + \beta_6 \frac{1}{1+\beta_7} e^{\left(\frac{\beta_7 \lambda_1}{1+\beta_7} \right) t} H^{-\frac{\beta_7}{1+\beta_7}} \right\}^{\frac{1}{\beta_7}}$$

$$\gamma_4 = \beta_{10} + \beta_{11} \ln \left[\frac{L_s}{\hat{L}} \right]$$

$$L_s = \beta_{12} e^{\lambda_2 t} \left(\frac{W/P}{e^{\lambda_1 t}} \right)^{\beta_{13}}$$

$$(4) \quad DS = Y + I - C - DK - E - G$$

$$(5) \quad \frac{DY}{Y} = \gamma_5 \ln \left[\frac{\hat{Y}}{Y} \right] + \gamma_6 \ln \left[\frac{\hat{S}}{S} \right]$$

$$\hat{Y} = (1 - \beta_{14}) (C + DK + E + G)$$

$$\hat{S} = \beta_{15} (C + DK + E + G)$$

$$\beta_{14} = 1/(1+e^\beta)$$

$$\beta = \beta_{16} + \beta_{17} (P/P_I) + \beta_{18} (ZQI), \quad \beta_{17} < 0$$

$$(6) \quad \frac{DI}{I} = \gamma_7 \ln \left[\frac{\hat{I}}{I} \right] + \gamma_8 \ln \left[\frac{\hat{S}}{S} \right]$$

$$\hat{I} = \beta_{14} (C + DK + E + G)$$

$$\gamma_7 = \beta_{19} + \beta_{20} ZAT$$

$$(7) \quad DP = \gamma_9 \left[\gamma_{10} \ln \left\{ \frac{(1+\beta_{21})MC}{P} \right\} - p \right] + \gamma_{11} \ln \left[\frac{\hat{Y}}{Y} \right]$$

$$MC = W \beta_5^{-\beta_7} \beta_6^{-1} e^{\beta_7 \lambda_1 t} (L/Y)^{1+\beta_7}$$

$$(8) \quad Dw = \gamma_{12} \left[\gamma_{13} \ln \left[\frac{\hat{L}}{L} \right] - w \right] + \gamma_{14} \frac{DP}{P}$$

$$(9) \quad k = \frac{DK}{K}$$

$$(10) \quad p = \frac{DP}{P}$$

$$(11) \quad w = \frac{DW}{W}$$

2.3 Equation (1): The Consumption Function

Equation (1) shows private consumer expenditure as being primarily demand determined but subject to important additional supply side influences including both monetary and goods market effects. It is assumed that the rate (of flow) of real private consumer expenditure adjusts to a desired or partial equilibrium rate \hat{C} in such a way that the greater the excess of \hat{C} over C the greater the proportional rate of change of C . \hat{C} is assumed to be dependent on the level of real private disposable income and the parameter β_1 is the relevant long run marginal propensity to consume. Clearly, many alternative specifications of \hat{C} are possible but it is interesting to note that not all of these are necessarily consistent with plausible long run steady state growth paths of the real variables in the system. For instance, an alternative specification is

$$(12) \quad \hat{C} = \alpha_1 (Y + GB - TP)^{\alpha_2} \left(\frac{M}{P} \right)^{\alpha_3}$$

which incorporates the additional real liquid assets variable M/P , where M is the aggregate volume of money. Such a specification is consistent with that of many aggregate consumption functions which have been used elsewhere (c.f. Mayer (1972)). However, if (12) is to be used in the context of our complete system of equations, then the parameter constraint

$$(13) \quad \alpha_2 + \alpha_3 = 1$$

is required if the real variables C and Y are to grow at the same constant proportional rate in the steady state (assuming an exponential growth path for M and the other exogenous variables in the system).

The formulation of equation (1) in terms of a proportional rate of adjustment towards some partial equilibrium value of the relevant variable is common in applied econometric work and has been used extensively in previous work with continuous time systems.¹ The use of a partial equilibrium value in discrete time, representing the value for which the variable would show no tendency to change given the value of other variables in the system seems justifiable and is well accepted. However, when time is treated as a continuous variable, a fixed structural representation of a partial equilibrium value such as \hat{C} in terms of other variables at each instant in time may appear less satisfactory in view of the high frequency oscillations that can be expected in such variables as the instantaneous rate of aggregate consumption and the rate of aggregate income generation (for example, there will be considerable variation in the rate of aggregate consumption on an hourly basis and between day-time and night-time²). The frequent interpretation

1. For example, Bergstrom and Wymer (1976), Knight and Wymer (1978) and Jonson et.al. (1977).

2. A further discussion of this point is given in Phillips and Wickens (1978, p.456).

of partial equilibrium values as desired values of a variable by certain representative economic agents reinforces this view indicating in the present case that there is a desired time path for the rate of aggregate consumption and this bears the same proportional relation to the rate of income generation at each instant in time. An obvious alternative approach is to argue that the systematic component of the equation is designed to explain low frequency rather than high frequency movements in these variables and that if, for example, δ represents the decision making interval of the representative consumer with respect to expenditure, then we may write

$$(14) \quad \hat{C}(s) = \beta \int_{t-\delta}^t \{Y(r) + GB(r) - TP(r)\} dr$$

for all values of $s \in (t-\delta, t)$. \hat{C} as defined above still defines the partial equilibrium value of the variable but all high frequency variation will now be absorbed into the residual or disturbance that drives the stochastic version of the equation. This formulation is more consistent with the notion of the systematic component explaining low frequency variation and also tends to reduce the amount of serial correlation that can be expected to occur in the stochastic specification of the continuous system. The econometric implications of the alternative formulation of \hat{C} are discussed in the Appendix.

The channel we have selected for a liquidity influence to work in equation (1) enters through the speed of adjustment parameter γ_1 . In the specification of γ_1 it is assumed that if, for example, real aggregate demand deposits, MBD/P , are falling

then the speed of adjustment of C to \hat{C} will be curtailed. In this way monetary behaviour can have a direct spillover effect on consumer expenditure. The importance of this effect is measured by the parameter β_3 , and the effect is symmetric for positive and negative values of $D(\ln(\text{MBD})/P)$ -- that is, the magnitude of the effect on γ_1 will be the same regardless of the sign of the change in $(\text{MBD})/P$. This channel of influence for liquidity was chosen in preference to the "disequilibrium money balances" approach of Knight and Wymer (1975, 1978) and Jonson, Moses and Wymer (1977). As will be apparent from the explicit monetary sector of the larger version of BHP being developed, the variable MBD is the residual in the private sector wealth constraint. If money is to take the role of a buffer stock and signalling device as argued by Jonson, Moses and Wymer (1977, p.11), then it is our view that MBD will be more appropriate in measuring the effect of such a role than the aggregate volume of money. Note that the form of our specification suggests that if the real purchasing power of the residual in the private sector wealth constraint is being run down then the rate at which the rate of consumption expenditure is being adjusted to its desired level \hat{C} will be curtailed. This type of specification preserves the recursive nature of the model, because, although $D\ln C$ is a first derivative, C is a flow and $D\ln C$ measures the proportional rate of change of the rate of consumption expenditure at a point in time. It is, therefore, perfectly consistent with the recursive nature of the model to have the adjustment coefficient γ_1 dependent on the proportional

rate of change $D \ln((\text{MBD})/P)$. The importance of the recursive structure in continuous time modelling in order to ensure meaningful solutions was stressed in the fundamental paper by A.W. Phillips (1959).

An additional supply side influence involving the real supply of goods and services is incorporated in γ_1 through the term $\beta_4 \exp(-DS/S)$. At the microeconomic level, supply side bottlenecks often mean that observations refer to the (short) supply side of the market; and, when there are no supply constraints, observations refer to the demand side of the market so that the method of switching regression regimes is appropriate for analysing this situation (c.f. Fair and Jaffee (1972)). At the macroeconomic level where we are aggregating across markets that may be expected to be in different regimes, the strict switching regressions technique seems less appropriate. Nevertheless, it seems reasonable to consider, even in aggregate, market behaviour where supply side constraints may be a dominating (if temporary) influence.¹ Our specification of γ_1 allows for this possibility, so that if stocks are being run down in aggregate ($DS < 0$) the speed of adjustment with which C can effectively adjust to \hat{C} is curtailed ($\beta_4 < 0$). The form of the function $\exp(-DS/S)$ has been selected so that the supply constraint is binding in the right direction and, thus, when stocks are being built up the effect of this term will be small. The importance

1. Thus, at any point in time, it seems reasonable to suppose that some proportion of all micro goods markets will be in excess demand. Weighting this proportion by the relative size of the markets, we might then be prepared to argue that supply constraints would be a dominating influence if the resulting weighted proportion exceeded one half.

of this supply side effect is measured through the parameter β_4 , which we would expect to have a negative sign as already indicated. In addition to these goods market and monetary supply influences on γ_1 there is assumed to be a constant component in the speed of adjustment measured by the parameter β_2 .

Our specification of the variable adjustment rate γ_1 in equation (1) is based on the hypothesis that, in general, $\hat{C} > C$ and this is considered to be the empirically relevant alternative. Since we would not expect real aggregate demand deposits or supply side constraints to influence the rate of adjustment of C down to \hat{C} when $C > \hat{C}$ the specification of γ_1 should strictly speaking involve multiplicative unit step functions to turn the effect on and off according to the regime $C \leq \hat{C}$ or $C > \hat{C}$. In practice, however, this additional non-linearity and complexity can be avoided by making the plausible assumption that C is, in general, being adjusted upwards towards \hat{C} . This argument is even more convincing when we smooth out high frequency oscillations in the representation of \hat{C} as indicated in (14) above.

2.4 Equations (2) and (3): The Investment and Employment Relations

The fundamental ideas underlying equations (2) and (3) can conveniently be explained together. The expressions \hat{K} and \hat{L} represent the desired demand for capital and labour services

respectively and result from a conventional cost minimisation process under the assumption of a CES production function of the form

$$(15) \quad Y = \beta_5^{-1} \left\{ K^{-\beta_7} + \beta_6 \left(e^{\lambda_1 t} L \right)^{-\beta_7} \right\}^{-1/\beta_7}$$

Thus, the representative firm is assumed to know its relative factor prices and its production function and to jointly determine its desired demands for factor services on the basis of cost minimisation. One innovation in equations (2) and (3) is our representation of the cost of capital as a linear function of the real rate of interest $(J-DP/P)$ multiplied by the general price level P . The use of the price variable P is consistent with the strict interpretation of the entire model as a single good model and if we were to distinguish consumption and capital goods in the theoretical structure of the model we would, of course, use the appropriate capital goods price variable instead. Our cost of capital variable $P(\beta_8 + \beta_9(J-DP/P))$ then has the same form as that normally used in models of investment behaviour¹ except that the rate of depreciation and tax variables do not appear explicitly in the expression.² Since these latter variables can vary in practice considerably across firms it

1. See, for instance, Nickell (1978, p.10).

2. Compare Jorgenson (1967), Brechling (1975) and Nickell (1978, particularly ch.9).

seems to us worthwhile to attempt to estimate an empirical measure of the cost of capital that is application to our "representative" firm by the introduction of the parameters β_8 and β_9 . Note that this allows the representative firm to introduce a proportional markup (through β_9) on the real rate of interest in its decision on the profitability of investment. This form also enables us to handle empirically observations for which the real rate of interest is negative¹ and yet the cost of capital still positive (since $\beta_8 > 0$).

Since $k = DK/K$ (equation (9)), (2) is a second order differential equation in K and describes the way in which the rate of capital formation is adjusted over time according to the extent to which the capital stock deviates from the desired level of capital \hat{K} , which in turn depends on the level of output, the prevailing wage rate and the cost of capital as well as the rate of technical progress. As in BW (but with a rather different functional form, due to the presence of \hat{K}), equation (2) assumes that there is a partial equilibrium proportional rate of increase in the stock of capital (measured by $\gamma_3 \ln(\hat{K}/K)$). It is further assumed that for a variety of reasons (including the costs associated with varying the rate of capital formation, delivery lags in the supply of capital goods and the availability of internally generated funds to finance investment) the rate of capital formation is not necessarily equal to this partial

1. This is of some importance because some common algebraic specifications exclude this possibility.

equilibrium rate. Instead the rate of change of the rate of capital formation is taken to depend on the excess of the partial equilibrium rate over the actual rate. Some consideration was given to the idea of specifying the adjustment rate γ_2 explicitly in terms of a costs of adjustment variable (Brechling (1975)) or a measure of the flow of internally generated funds (Nickell (1978, pp.262-263)); but this idea was rejected, at least for our present study of BHP, since (2) is already a heavily complicated equation.

The employment equation (3) differs considerably from that of BW in that it seeks not only to capture the major (and somewhat unique) features associated with New Zealand's market for labour services but also to tackle certain deficiencies of the BW and Knight and Wymer (1975)¹ formulations.

A traditional labour services formulation requires that the supply of labour services be infinitely elastic or be able to react through the appropriate adjustment procedures to fully satisfy the desired factor demand, because the supply of labour services is always at least that demanded. In New Zealand until very recently, however, the usual situation has been a significant shortage of labour (alleviated to some extent in the shorter run by the working of extra hours of overtime, and in the longer run by net immigration) with the supply of labour services consistently failing to satisfy demand (in the notation of the model $\hat{L} > L = L_s$). Since L and L_s are smooth

1. In a subsequent version of their model (Knight and Wymer (1978)), the labour supply equation is formulated as in our equation (3).

functions of other variables, this situation typically leads to an employment function for L with a discontinuity in the first derivative, which is difficult to model directly. Our own approach in equation (3) is based on the hypothesis that employment, L , adjusts towards the desired level \hat{L} but that the rate of adjustment explicitly depends on the supply of labour services, L_s , the extent of the gap between L_s and \hat{L} and, indeed the direction of the inequalities $\hat{L} \leq L_s$ or $\hat{L} > L_s$ (the latter being the more usual situation in New Zealand). More specifically, we take into account two separate factors in the formulation of γ_4 :

- (i) The speed of adjustment parameter γ_4 depends on the extent of labour market disequilibrium and does so in a non-linear way. The use of the logarithmic specification enables us to capture the asymmetry of a more powerful (negative) effect on the adjustment speed of employment when $L_s < \hat{L}$ than the (positive) effect on the adjustment speed when $L_s > \hat{L}$. Thus, although we cannot capture the full force of the inequality $L \leq L_s$, it is hoped that this specification will go some way towards modelling the labour supply constraint on movements in employment.¹
- (ii) The labour supply function, L_s , extends the earlier formulations used by BW, KW and JMW. In BW there seems to be

1. In empirical applications the parameters β_{10} , β_{11} and the constant term occurring in the specification of L_s are not all identifiable. We can then collapse the adjustment rate into the form

$$\gamma_4 = \gamma \ln (L_s / \hat{L})$$

for applied econometric work.

no allowance for direct short run adjustment of the labour supply (or \hat{L}) to any change in the real wage rate. KW introduce a real wage rate influence on the labour supply but need, in addition, to remove this influence for their model to have a steady state solution. Our own formulation of L_s in the model above is intended to tackle both these difficulties. It is a simplification of the following functional form:

$$(16) \quad L_s = \alpha_4 e^{\lambda_2 t} \left[\frac{W/P}{\alpha_5 e^{\lambda_1 t}} \right]^{\alpha_6}$$

where λ_2 = rate of growth of the labour force due to population growth and immigration and λ_1 = rate of technical progress.¹ In equation (16) it is assumed that the labour force is growing at the constant proportional rate λ_1 and in later work we hope to include separate influences for population growth and immigration (which has been subject to some fluctuation in recent years). We distinguish from the labour force, the labour supply which measures the number of persons available for employment at the ruling level of wages and prices. Although the parameters α_4 and α_5 in equation (16) are not separately identifiable (and, hence, become the composite parameter β_{12} in equation (3)

1. No specific allowance is made for the influence of changing effective income tax rates as these are probably important only at the margin.

of the model), α_5 has an important economic interpretation. It is assumed that at each point in time there will be level of the real wage (here $\alpha_5 e^{\lambda_1 t}$) for which the labour supply would be equal to the labour force. That is, at this level of the real wage the entire labour force would be available for employment; and, if wages are not at this level, then $L_s < \alpha_4 e^{\lambda_2 t}$. We also allow for the level of real wages for which the whole labour force would seek employment to change over time in accordance with the increasing efficiency of labour resulting from technical progress taking place at the rate λ_1 . In the steady state, $W/P = (W/P)^* e^{\lambda_1 t}$. $(W/P)^*$ is the equilibrium growth path constant given in section 3, so that we have

$$L_s = \alpha_4 \left(\frac{(W/P)^*}{\alpha_5} \right)^{\alpha_6} e^{\lambda_2 t}$$

which allows for some proportion of the labour force to be unemployed in the steady state; but this is not necessarily so, as α_5 is an unknown parameter and could equal $(W/P)^*$. Thus, the model encompasses the case where, even in the steady state, there may be some fraction of the potential labour force who are not willing to work at the prevailing real wage rate.

2.5 Equations (4), (5) and (6): Inventories, Output and Imports

The definitional equation (4) takes actual inventory investment (which of course incorporates unanticipated inventory

investment) as residually determined.

Equations (5) and (6) are similar in concept to the real output and real imports supply equations of BW, and are therefore largely demand determined; but they do have important additional supply constraints introduced through functional forms for the parameters β_{14} and γ_7 . This is because, in the short run, constraints operating through the variables ZQI and ZAT are considered to be more binding than the production function technology underlying the domestic production process. Thus, it is not assumed that the economy is always operating on its production function. But the technology represented by this function (see (15) above) plays an important role in the model, particularly in the specification of the factor demand equations and the equations of the wage price sector, and in the determination of the long run steady state growth paths of the model's variables.

Both the rate of increase of home production and of imports are assumed to depend on excess demand for inventories through the term $\ln(\hat{S}/S)$; the former depends also on excess demand for home production, $\ln(\hat{Y}/Y)$, and the latter depends also on excess demand for imports, $\ln(\hat{I}/I)$. The adjustment of actual real output, Y , to the partial equilibrium rate of home production \hat{Y} is assumed to take place at a constant rate γ_5 . A more sophisticated and realistic specification of γ_5 would allow this rate of adjustment to vary according to effective constraints on employment arising from the labour market.

The adjustment rate γ_7 , of actual real imports, I , to the desired rate of real imports, \hat{I} , is assumed to be influenced by monetary conditions as reflected by the availability of trading bank credit through the synthetic variable ZAT. The parameter β_{14} represents the proportion of total sales that is desired as imports; it varies not only with the relative prices of home produced and imported goods but also with the degree of severity of import quotas as measured by the synthetic variable ZQI. Postulating this latter influence on β_{14} implies that importers are aware of current and impending policy decisions on import quotas and do, in fact, incorporate this information in their decisions about the desired level of imports (and, hence, about the desired rate of home production). Our formulation of β_{14} ensures that $0 < \beta_{14} < 1$ as required.

2.6 Equations (7) and (8): The Price and Wage Sector

Equations (7) and (8) both feature adjustment processes of the second order to assist in modelling the important empirical phenomena of recent years of fluctuations in the rate of change of prices and wages.¹ Each equation also captures certain of the more traditional hypotheses about the inflationary mechanism.

1. These equations were specified before we became aware of the related arguments recently put forward by Flemming (1976) for higher order adjustment mechanisms in price equations.

The price equation (7) incorporates both cost and goods market excess demand influences within its second order dynamic specification. It is assumed that firms attempt to set prices on the basis of some markup (dependent on the degree of imperfection in the market for goods, here measured by the parameter β_{21}) on the marginal cost of labour.¹ More specifically, in the first part of the equation (noting from equation (10) that p is the proportional rate of change of prices, DP/P) it is assumed that firms would like prices to adjust according to the excess of $(1+\beta_{21})MC$ over the present level of prices, where MC is the marginal cost of labour obtained from the production function. However, if prices are not increasing at the desired proportional rate as given by the expression

$$\gamma_{10} \ln\left\{(1 + \beta_{21})MC/P\right\}$$

then inflationary pressure builds up and this pressure causes the rate of inflation, p , to change: the rate of change of p is then assumed to be greater, the greater the excess of this desired proportional rate over the actual current rate of inflation p . The final term on the right side of (7) measures the extent to which excess demand in the market for goods and services, \hat{Y}/Y , contributes directly to the inflationary process. If $\gamma_{11} > 0$, then excess demand in

1. As in BW we assume a fixed capital stock for the purpose of price setting, not only for simplicity but also because labour costs will be by far the more dominant variable of the two.

the goods market will cause prices to change at a rate which is faster than it would otherwise have been. Some of the reasons we have given for using an explicit second order differential equation for prices can be seen as an alternative to incorporating price expectations directly in the equation. It could be argued, for instance, that firms are expecting to adjust prices in an attempt to cover their marginal wage costs and that, in addition, this behaviour on the part of firms is expected by other economic agents in the system. Such a view would certainly play an important role in the general formation of price expectations. Our equation then assumes that these expectations are not necessarily realised and that, as a result, the actual rate of inflation will differ from the partial equilibrium rate (which would prevail if these expectations were realised) and that, moreover, this difference directly influences the acceleration of price changes.

Our specification of equation (7) can also be seen as an alternative to the procedure adopted in JMW of using a disequilibrium real money balances variable to proxy expectations in the price equation (note that our hypothesis does not assume that price expectations are formed solely on the basis of past observations of prices). It also avoids the problem of having to obtain a measure of the price expectations variable if it were to be included directly.

The wage equation (8), with the wage rate W referring to the nominal wage rate, is also specified as a second order differential equation. The primary idea underlying the equation is that

the wage rate adjusts (i.e. $w = DW/W$) according to the excess of desired demand for labour, \hat{L} , over the present level of employment, L . However, if wages are not adjusting at the desired rate, which is measured in the equation by $\gamma_{13} \ln(\hat{L}/L)$, then there is a change in the rate at which wages are changing. Thus, the rate of change of wage inflation is sensitive to deviations of the actual rate of adjustment in wages from the desired rate.¹ The final term of equation (8) is intended to measure the direct influence of the rate of price inflation on the rate of change of wage inflation. The specification implies that the rate of change of wages is dependent with a distributed time lag on present and past rates of inflation. More directly, if $\gamma_{14} > 0$, the rate of change of wage inflation is greater the greater the current rate of price inflation. The presence of this last term in equation (8) can also be seen as an alternative to including price expectations directly into the equation.

The remaining equations (9), (10) and (11) are definitions necessary to express the whole model as a system of first order differential equations.

1. Incomes policy may well affect what is considered to be a realistic rate of adjustment in wages and we could specify γ_{13} as a function of an incomes policy dummy. Incomes policy may also lead to a build up of inflationary pressure measured by the deviation of the actual rate of changes of wages, w , from the desired rate; and this pressure will, according to our specification, be released through a change in the rate of change of wages, i.e. through the second derivative. To be accurate in our timing of these effects according to what was legally permissible during incomes policy periods we would need an additional (multiplicative) dummy variable to switch the effect on and off or perhaps control its magnitude. For simplicity, we have decided to see how well the present specification performs without introducing extra dummy variables. However, it is worth noting that incomes policy dummy variables are already available for the New Zealand economy, such as those tested in Hall and King (1976).

3. THE STEADY STATE SOLUTION OF THE MODEL

The structural equations (1)-(11) of BHP given in Section 2 form a system of non-linear differential equations. This system involves nine exogenous variables (other than time) and the time paths generated for the system's endogenous variables from certain initial conditions will clearly depend on the paths taken by these exogenous variables. Under the explicit assumptions that the exogenous variables grow at constant proportional rates and that these growth rates are consistent for real variables and prices, we find a particular solution of the system in which each endogenous variable also grows at a constant proportional rate. As in BW, this particular solution displays the usual characteristics of a steady state (Solow (1970)) and for the real variables appearing as components of output the growth rate is the same as the equilibrium growth rate of output and capital in the simple neoclassical growth model.

Specifically, we assume that the following explicit paths for the exogenous variables:

$$(17) \quad GB = GB^0 e^{(\lambda_1 + \lambda_2)t}$$

$$(18) \quad TP = TP^0 e^{(\lambda_1 + \lambda_2)t}$$

$$(19) \quad MBD = MBD^0 e^{mt}$$

$$(20) \quad J = J^0$$

$$(21) \quad E = E^0 e^{(\lambda_1 + \lambda_2)t}$$

$$(22) \quad G = G^0 e^{(\lambda_1 + \lambda_2)t}$$

$$(23) \quad P_I = P_I^0 e^{\{m - (\lambda_1 + \lambda_2)\}t}$$

$$(24) \quad ZQI = ZQI^0$$

$$(25) \quad ZAT = ZAT^0 .$$

The steady state solution of (1)-(11) is then

$$(26) \quad C = C^* e^{(\lambda_1 + \lambda_2)t}$$

$$(27) \quad Y = Y^* e^{(\lambda_1 + \lambda_2)t}$$

$$(28) \quad K = K^* e^{(\lambda_1 + \lambda_2)t}$$

$$(29) \quad L = L^* e^{\lambda_2 t}$$

$$(30) \quad S = S^* e^{(\lambda_1 + \lambda_2)t}$$

$$(31) \quad I = I^* e^{(\lambda_1 + \lambda_2)t}$$

$$(32) \quad P = P^* e^{\{m - (\lambda_1 + \lambda_2)\}t}$$

$$(33) \quad W = W^* e^{(m - \lambda_2)t}$$

$$(34) \quad p = m - (\lambda_1 + \lambda_2)$$

$$(35) \quad w = m - \lambda_2$$

$$(36) \quad k = \lambda_1 + \lambda_2 .$$

According to (29), steady state employment grows at the rate λ_2 , the same rate as the rate of growth of the labor force due to population growth and immigration. Since W/P grows at the rate λ_1 in the steady

state solution, it follows from (29) and (16) that the ratio of employment to the supply of labor is constant in the steady state. The growth rate of the real variables C , Y , K , S and I equals $\lambda_1 + \lambda_2$, the same as the equilibrium growth rate of output and capital in the simple neoclassical growth model in which the supply of labor is exogenous and growing at the rate λ_2 and in which there is disembodied technical progress at the rate λ_1 in the production function. The steady state growth rates for P and W are not uniquely determined in this particular solution as a function of the model's parameters. In fact, any growth rate for P and W for which the growth rate of W/P is λ_1 constitutes a particular solution. This is because the model does not fully describe the relationship between money, income and prices. In fact, we would expect the steady state growth rate of prices to depend on the growth rate of the money supply relative to that of real output; and the indeterminacy results from the fact that the monetary sector is not endogenous in BHP. In the larger version of BHP under construction, the monetary sector is more fully developed and our specifications imply that, if M is the volume of money, then M/P and Y grow at the same rate in the steady state. We now suppose that $M = M^*e^{mt}$ in the steady state (this gives a growth rate of m , consistent with that assumed for the exogenous variable MBD which is a component of M) and then the growth rate of P becomes $m - (\lambda_1 + \lambda_2)$ as shown in (31).

The constant coefficients (or levels) of the steady state growth paths that appear in the steady state solutions (26) to (36) satisfy the equations:

$$\lambda_1 + \lambda_2 = \gamma_1 \ln \left[\frac{\beta_1 (Y^* + GB^0 - TP^0)}{C^*} \right]$$

$$\gamma_1 = \beta_2 + \beta_3(\lambda_1 + \lambda_2) + \beta_4 e^{-(\lambda_1 + \lambda_2)}$$

$$0 = \gamma_2 \gamma_3 \ln \left[\frac{\beta_5 Y^* \left\{ 1 + \beta_6 \frac{1}{1+\beta_7} \frac{\beta_7}{h} \right\}}{K^*} \right]^{\frac{1}{\beta_7}} - \gamma_2 (\lambda_1 + \lambda_2)$$

$$h = \frac{W^*}{P^* \{ \beta_8 + \beta_9 (J^0 - m + \lambda_1 + \lambda_2) \}}$$

$$\lambda_2 = \gamma_4 \ln \left[\frac{\beta_5 \beta_6 Y^* \left\{ 1 + \beta_6 \left(-\frac{1}{1+\beta_7} - \frac{7}{1+\beta_7} \right) \right\}}{L^*} \right]^{\frac{1}{\beta_7}}$$

$$\gamma_4 = \beta_{10} + \beta_{11} \ln \left[\frac{\beta_{12} (W^*/P^*)^{\beta_{13}}}{\beta_5 \beta_6 Y^* \left\{ 1 + \beta_6 \left(-\frac{1}{1+\beta_7} - \frac{\beta_7}{1+\beta_7} \right) \right\}} \right]^{\frac{1}{\beta_7}}$$

$$(\lambda_1 + \lambda_2) S^* = Y^* + I^* - C^* - (\lambda_1 + \lambda_2) K^* - E^0 - G^0$$

$$\lambda_1 + \lambda_2 = \gamma_5 \ln \left[\frac{(1 - \beta_{14}) (C^* + (\lambda_1 + \lambda_2) K^* + E^0 + G^0)}{Y^*} \right] + \gamma_6 \ln \left[\frac{\beta_{15} (C^* + (\lambda_1 + \lambda_2) K^* + E^0 + G^0)}{S^*} \right]$$

$$\beta_{14} = 1/(1 + e^\beta)$$

$$\beta = \beta_{16} + \beta_{17} (P^*/P_I) + \beta_{18} (ZQI^0)$$

$$\lambda_1 + \lambda_2 = \gamma_7 \ln \left[\frac{\beta_{14} (C^* + (\lambda_1 + \lambda_2)K^* + E^0 + G^0)}{I^*} \right] \\ + \gamma_8 \ln \left[\frac{\beta_{15} (C^* + (\lambda_1 + \lambda_2)K^* + E^0 + G^0)}{S^*} \right]$$

$$\gamma_7 = \beta_{19} + \beta_{20} ZAT^0$$

$$0 = \gamma_9 \gamma_{10} \ln \left[\frac{(1 + \beta_{21}) W^* \beta_5^{-\beta_7} \beta_6^{-1} (L^*/Y^*)^{1+\beta_7}}{P^*} \right] - \gamma_9 (m - (\lambda_1 + \lambda_2)) \\ + \gamma_{11} \ln \left[\frac{(1 - \beta_{14}) (C^* + (\lambda_1 + \lambda_2)K^* + E^0 + G^0)}{Y^*} \right]$$

$$0 = \gamma_{12} \gamma_{13} \ln \left[\frac{\frac{1}{\beta_5 \beta_6} Y^* \left\{ 1 + \beta_6 \left(-\frac{1}{1+\beta_7} - \frac{\beta_7}{1+\beta_7} \right) \right\}}{L^*} \right]^{\frac{1}{\beta_7}} \\ - \gamma_{12} (m - \lambda_2) + \gamma_{14} (m - (\lambda_1 + \lambda_2)) .$$

These equations can be solved to find the values of the steady state levels of the endogenous variables for given values of the model's parameters and the given coefficients of the exogenous variable time paths in (17) to (25). After some manipulations, we obtain explicitly

$$(37) \quad C^* = \beta_1 [\mu_3 h^{\beta_{13}} \ell_2^{-1/\beta_7} + GB^0 - TP^0] e^{-(\lambda_1 + \lambda_2)/\mu_1}$$

$$(38) \quad L^* = \mu_3 \mu_6 e^{-\mu_2} h^{\beta_{13}}$$

$$(39) \quad Y^* = \mu_3 h^{\beta_{13}} \ell_2^{-1/\beta_7}$$

$$(40) \quad K^* = \beta_5 \mu_3 e^{-(\lambda_1 + \lambda_2)/\gamma_3} h^{\beta_{13}} (\ell_3/\ell_2)^{1/\beta_7}$$

$$(41) \quad S^* = (\lambda_1 + \lambda_2)^{-1} (Y^* + I^* - d)$$

$$(42) \quad P^* = \beta_{17}^{-1} P_I^0 [\ln\{(1-g)/g\} - \beta_{16} - \beta_{18} ZQI^0]$$

$$(43) \quad W^* = h \ell_1 P^*$$

and g , h and I^* satisfy the following three non-linear simultaneous equations

$$(44) \quad 0 = \mu_4 + \gamma_9 \gamma_{10} \ln(h) + \gamma_9 \gamma_{10} \left[\frac{1 + \beta_7}{\beta_7} \right] \ln(\ell_2) + \gamma_{11} \ln \left[\left(\frac{1-g}{\mu_3} \right) dh^{-\beta_{13}} \ell_2^{1/\beta_7} \right]$$

$$(45) \quad \mu_5 = \gamma_5 \gamma_8 \ln \left[\left(\frac{1-g}{\mu_3} \right) dh^{-\beta_{13}} \ell_2^{1/\beta_7} \right] - \gamma_6 \gamma_7 \ln(gd/I^*)$$

and

$$(46) \quad I^* = g \beta_{15}^{\gamma_8/\gamma_7} e^{-(\lambda_1 + \lambda_2)/\gamma_7} d^{(\gamma_7 + \gamma_8)/\gamma_7} [(\lambda_1 + \lambda_2)^{-1} (Y^* + I^* - d)]^{-\gamma_8/\gamma_7}$$

where

$$(47) \quad \ell_1 = \beta_8 + \beta_9 (J^0 - m + \lambda_1 + \lambda_2)$$

$$(48) \quad \ell_2 = 1 + \beta_6 h^{-1/(1+\beta_7)} h^{-\beta_7/(1+\beta_7)}$$

$$(49) \quad l_3 = 1 + \beta_6 \frac{1/(1+\beta_7)}{h} \frac{\beta_7/(1+\beta_7)}{h}$$

$$(50) \quad d = C^* + (\lambda_1 + \lambda_2)K^* + E^0 + G^0$$

$$(51) \quad \gamma_7 = \beta_{19} + \beta_{20}ZAT^0$$

and the additional parameter functions are defined by

$$\mu_1 = \beta_2 + \beta_3(\lambda_1 + \lambda_2) + \beta_4 e^{-(\lambda_1 + \lambda_2)}$$

$$\mu_2 = \frac{m - \lambda_2}{\gamma_{13}} - \frac{\gamma_{14}}{\gamma_{12}\gamma_{13}} \{m - (\lambda_1 + \lambda_2)\}$$

$$\mu_3 = \beta_{12} e^{\beta_{10}/\beta_{11}} e^{-\lambda_2/(\beta_{11}\mu_2)} \mu_6^{-1} l_1^{\beta_{13}}$$

$$\begin{aligned} \mu_4 = & \gamma_9 \gamma_{10} \ln[(1 + \beta_{21})^{\beta_5} \beta_6^{-1} l_1] + \gamma_9 \gamma_{10} (1 + \beta_7) \ln(\mu_6) \\ & - \mu_2 \gamma_9 \gamma_{10} (1 + \beta_7) - \gamma_9 \{m - (\lambda_1 + \lambda_2)\} \end{aligned}$$

$$\mu_5 = (\gamma_8 - \gamma_6)(\lambda_1 + \lambda_2)$$

$$\mu_6 = \beta_5 \beta_6^{1/\beta_7}$$

In order to solve (44)-(46) for g , h and I^* we first substitute in these equations the expressions for Y^* , d , C^* , K^* , l_2 and l_3 given respectively by (39), (50), (37), (40), (48) and (49) in terms of h . The resulting equations can then be solved by numerical methods to obtain the values of g , h and I^* corresponding to given values of

the parameters and the levels of the exogenous variables in (17)-(25). To determine the impact on the steady state levels (37)-(43) of changes in the parameters or exogenous variable levels we differentiate these expressions and write the derivatives in terms of the corresponding derivatives of g , h and I^* with respect to the same parameters. The values of the latter can be found by solving the system of linear equations that obtain from the implicit differentiation of equations (44)-(46).

When the parameter $\gamma_6 = 0$ we can solve (44) and (45) directly for h , giving

$$(52) \quad h = \left\{ e^{\beta_7 \mu_7 / (1 + \beta_7)} - \frac{1}{\beta_6 (1 + \beta_7)} \right\}^{(1 + \beta_7) / \beta_7}$$

where

$$\mu_7 = -(\mu_4 + \gamma_{11} \mu_5 / \gamma_5 \gamma_8) / \gamma_9 \gamma_{10}$$

and, now,

$$\mu_5 = \gamma_8 (\lambda_1 + \lambda_2) .$$

Under $\gamma_6 = 0$, we see from equations (5) and (6) that excess stocks do not directly influence the rate of increase in domestic production but still influence the proportional rate of change in imports, provided $\gamma_8 \neq 0$. Under this restriction, any stock deficiency in the economy is assumed to be removed entirely by importing.¹ Note that, if we set both

¹In the empirical application of their model to the United Kingdom, BW restricted to zero the value of the parameter in their model which corresponds to γ_6 .

$\gamma_6 = 0$ and $\gamma_8 = 0$, equation (45) disappears and we cannot solve uniquely for the steady state levels of the system's endogenous variables. We, therefore, assume that $\gamma_8 \neq 0$.

Using (52) we can now consider the effect of changes in the parameters on the steady state paths of some of the variables. We consider the steady state level of employment L^* in particular. This will illustrate some of the most important effects of parameter changes on the long run properties of the model and the role of the parameters that appear in the non-linear adjustment mechanism for employment and the second order wage and price equations will be of special interest. We have the following conditions:

$$\frac{\partial L^*}{\partial \beta_6} < 0, \quad \frac{\partial L^*}{\partial \beta_{10}} > 0$$

$$\frac{\partial L^*}{\partial \beta_{11}} = -L^* \left(\frac{\beta_{10} - \lambda_2/\mu_2}{\beta_{11}^2} \right)$$

$$\frac{\partial \ln L^*}{\partial \ln \beta_{12}} = 1, \quad \frac{\partial L^*}{\partial \beta_{13}} > 0,$$

$$\frac{\partial L^*}{\partial \beta_{21}} < 0, \quad \frac{\partial L^*}{\partial \gamma_{11}} < 0,$$

$$\frac{\partial (L^* e^{\lambda_2 t} / L_s)}{\partial m} \begin{matrix} > \\ < \end{matrix} 0 \text{ iff } \gamma_{12} \begin{matrix} > \\ < \end{matrix} \gamma_{14}.$$

Since the parameter β_6 measures the capital intensity of the production function, the condition $\partial L^*/\partial \beta_6 < 0$ implies that the steady state level of employment is lower, the more capital intensive is the production function. The condition $\partial L^*/\partial \beta_{10} > 0$ implies that the steady state level of employment will be greater, the greater is the constant component β_{10} in the rate of adjustment of employment to its partial equilibrium level.

The parameter β_{11} measures the influence of labor market disequilibrium on the rate of adjustment of employment (in equation (3) of the model). The expression given above for $\partial L^*/\partial \beta_{11}$ shows that the sign of the partial derivative depends on the sign of $\beta_{10} - \lambda_2/\mu_2$. For most plausible values of the parameters this will be positive, so that we will normally expect $\partial L^*/\partial \beta_{11} < 0$. When there is disequilibrium in the labor market, an increase in β_{11} slows down (speeds up) the rate of adjustment of employment in conditions of excess demand (excess supply) of labor. The sign of the partial derivative $\partial L^*/\partial \beta_{11}$ suggests that we can normally expect this slowing down effect on the adjustment of employment (under excess demand) to lower the steady state level of employment as well. For less normal values of the parameters the sign of $\partial L^*/\partial \beta_{11}$ may change. The relevant parameters are the adjustment rates γ_{12} , γ_{13} and γ_{14} in the wage equation (8). One interesting case is as follows. First we note that, unless γ_{14} is large relative to γ_{12} , μ_2 is positive and becomes smaller as γ_{13} increases (for example, when $\gamma_{12} = \gamma_{14}$, $\mu_2 = \lambda_1/\gamma_{13}$). Then for large γ_{13} , $\beta_{10} - \lambda_2/\mu_2$ is negative and the partial derivative $\partial L^*/\partial \beta_{11} > 0$. This can be explained by the fact that, when γ_{13} is large, wages adjust very quickly in response to excess demand for labor through equation (8). With the rise in wages, the supply of labor also increases. This alleviates the excess demand for labor and allows employment to rise through equation (3). The condition $\partial L^*/\partial \beta_{11} > 0$ in this case shows that the steady state level of employment also rises when wage responses are sufficiently fast.

The condition $\partial \ln L^*/\partial \ln \beta_{12} = 1$ implies that an exogenous increase in the labor supply through the parameter β_{12} will cause an equal proportionate increase in the steady state level of employment. The condition

$\partial L^*/\partial \beta_{13} > 0$ implies the more elastic is the labor supply with respect to the real wage then the higher is the steady state level of employment. This is explained, as earlier in the case of the partial derivative $\partial L^*/\partial \beta_{11} > 0$, by the alleviating effect of a rise in the labor supply on the adjustment rate of employment in equation (3).

The conditions $\partial L^*/\partial \beta_{21} < 0$ and $\partial L^*/\partial \gamma_{11} < 0$ imply that the steady state level of employment is lower the greater the degree of imperfect competition in the economy and the greater the effect of excess demand for goods on the rate of price inflation. The first of these is explained by the lower level of production in the steady state resulting from an increase in β_{21} (we also have $\partial Y^*/\partial \beta_{21} < 0$); and the second follows from the effect of the additional price inflation from equation (7) on the real wage and hence the supply of labor, leading to a curtailment in the adjustment rate of employment via equation (3).

The final condition given above involves the sign of the partial derivative of the steady state employment to labor supply ratio with respect to m , the proportional rate of increase in the money supply. We find that, in the steady state,

$$\frac{L^* e^{\lambda t}}{L_s} = e^{(\beta_{10}^{-\lambda_2 / \mu_2}) / \beta_{11}}$$

and, since $\partial \mu_2 / \partial m \stackrel{\geq}{<} 0$ iff $\gamma_{12} \stackrel{\geq}{<} \gamma_{14}$, we have $\partial (L^* e^{\lambda t} / L_s) / \partial m \stackrel{\geq}{<} 0$ iff $\gamma_{12} \stackrel{\geq}{<} \gamma_{14}$ as given. This may be interpreted as implying that, when the money supply grows at a faster proportional rate, the unemployment ratio will be smaller in the steady state provided the inflationary effects of rising prices on changes in the wage rate are not too strong. On the other hand, when $\gamma_{14} > \gamma_{12}$, these inflationary effects become too strong and the resulting acceleration in wages is so great as to make

labor less competitive in the long run and through the factor demand relations the steady state employment ratio is reduced. Thus, the proportion of the labor force employed in the steady state is an increasing function of the rate of increase in the money supply provided the inflationary effects of rising prices on changes in the wage rate are not too strong.

4. GENERAL SOLUTION TRAJECTORIES

In the previous section a particular solution of the structural differential equations was obtained that demonstrated the model is capable of producing plausible long run behavior consistent with steady state growth of the variables at constant proportional rates. The exact time paths of the variables will not follow this particular steady state solution unless the initial conditions correspond with the steady state levels of the variables. When the initial conditions differ from these steady state levels, the exact paths of the variables are determined by the general solution of the differential equation system. In the present section, solution trajectories for the variables are obtained for various combinations of initial values and compared with the steady state paths. We also examine the effect on these paths of changes in the parameters involved in the variable adjustment rates of the consumption and employment equations.

The complete system of structural differential equations (1) to (11) is at present in the form

$$(53) \quad Dx(t) = f\{Dx(t), x(t), Dz(t), z(t); \theta\}$$

where $x(t)$ denotes the vector of endogenous variables, $z(t)$ denotes the vector of exogenous variables and θ is the vector of parameters in

the system. With some elementary substitution and redefinition of the exogenous variables the model can be rewritten more simply as

$$(54) \quad Dx(t) = f\{x(t), z(t); \theta\} .$$

Given initial conditions for $x(0)$, time paths for the exogenous variables and values of the parameters in θ , solution trajectories for $x(t)$, $t > 0$, can be obtained by numerical solution of (54). The routine employed was a variable-order Adams method based on Krogh's algorithm (Krogh (1973)) which is currently available in the NAG library and recommended for use in large problems and over large ranges.

Our main objective in these numerical exercises is to investigate how the adjustment path of the variables is affected by different combinations of initial values and by changes in important parameters of the model. As a basis of comparison we will use the steady state growth paths of the variables which we may compute from (37)-(46) using only the values of the parameters and the initial values of the exogenous variables. We specify these as follows:

$$\beta_1 = 0.8, \quad \beta_2 = 0.4, \quad \beta_3 = 0.02, \quad \beta_4 = -0.075, \quad \beta_5 = 0.5$$

$$\beta_6 = 2.0, \quad \beta_7 = 1.0, \quad \beta_8 = 0.075, \quad \beta_9 = 1.01, \quad \beta_{10} = 0.1$$

$$\beta_{11} = 0.05, \quad \beta_{12} = 172.0, \quad \beta_{13} = 1.4, \quad \beta_{14} = *, \quad \beta_{15} = 0.25$$

$$\beta_{16} = 2.0, \quad \beta_{17} = -0.5, \quad \beta_{18} = 0.05, \quad \beta_{19} = 0.4, \quad \beta_{20} = 0.1,$$

$$\beta_{21} = 0.01;$$

$$\gamma_1 = *, \quad \gamma_2 = 0.042, \quad \gamma_3 = 0.05, \quad \gamma_4 = *, \quad \gamma_5 = 1.0$$

$$\gamma_6 = 0.0 , \quad \gamma_7 = * , \quad \gamma_8 = 0.5 , \quad \gamma_9 = 0.2 , \quad \gamma_{10} = 0.1$$

$$\gamma_{11} = 0.2 , \quad \gamma_{12} = 0.3 , \quad \gamma_{13} = 0.1 , \quad \gamma_{14} = 0.1 ;$$

$$\lambda_1 = 0.005 , \quad \lambda_2 = 0.005 ;$$

$$m = 0.03 ;$$

$$GB^0 = 64 , \quad TP^0 = 84 , \quad D\ln(MBD) = 0.03 , \quad J^0 = 0.055 ,$$

$$E^0 = 160 , \quad G^0 = 141 , \quad P_I^0 = 1.005 , \quad ZQI^0 = 0.0 , \quad ZAT^0 = 1.0 ;$$

where * = variable parameter or speed of adjustment. These numerical values are intended to correspond to an underlying time unit of a quarter and are based as far as possible on the values of previous empirical work with the New Zealand economy (with which our ultimate applied econometric work will be concerned). Some empirical knowledge concerning plausible adjustment rates is also available from the recent application of the BW model to the UK economy referenced earlier and this has been used in the setting of the above parameter values. Moreover, a check of the overall consistency of these parameter values is obtained by computing the steady state levels of the variables from equations (37)-(46) above. These turn out to be

$$C^* = 477.833 , \quad Y^* = 635.901 , \quad K^* = 1392.528 , \quad L^* = 651.095 ,$$

$$S^* = 184.187 , \quad I^* = 158.699 , \quad P^* = 1.203 , \quad W^* = 1.256$$

and are based on the given initial values of the exogenous variables.

The latter were obtained from actual quarterly data for the New Zealand economy in the year 1960. The calculated steady state levels above turn

out to be quite compatible with the observed pattern in the data, although the capital output ratio is somewhat lower than that observed.

The calculated solution trajectories for a variety of initial values and some changes in the values of key parameters are graphed for output, consumption, employment, the rate of inflation, DP/P , and the proportional rate of change of wages, DW/W , in Figures 1-5. In each case the unbroken curve represents the steady state path of the variable, representing an exponential growth path for output, consumption and employment and a constant level for the proportional rates of price and wage inflation.¹ The broken curves give the trajectories for initial values based on some fixed proportion of the known steady state levels at time $t = 0$ and for changes in the variable adjustment rate parameters β_4 (for the consumption trajectory) and β_{11} (for the employment trajectory).

The adjustment paths for real output are shown in Figure 1. Against the steady state path (a) we graph the paths of output when: (b), consumption expenditure alone is 20% below its steady state value; (c), all real variables are 10% below their steady state values; (d), all variables including the price level and wage level (but excluding the rates of price and wage inflation) are 10% below their steady state values. In each case, the resulting adjustment paths for output involve mild oscillations about the steady state path. In case (b), output falls sharply in the first two quarters because of the effect of excess supply in equation (5) on the rate of change of output. Excess supply of domestic goods also lowers the

¹In fact, the steady state paths were calculated as the solution trajectories corresponding to initial values equal to the steady state levels obtained from formulae (37)-(46) in Section 3. This provided a useful check on the validity of the steady state formulae.

rate of acceleration in prices through equation (7), causing the rate of inflation to fall (in fact, by 2.5% in the first quarter).¹ Since import prices are growing at a constant proportional rate (equation (23)) domestic goods now become more competitive (through the influence of the price relative P/P_I in the proportion, β_{14} , of total sales desired as imports in equations (5) and (6)). This removes the excess supply on the domestic market and real output rises. It takes seven quarters before real output regains its steady state level. Mild oscillations follow, with a long period of approximately 30 years about the steady state path. The oscillations appear to be damped but only slightly.

This case gives us an interesting example where, when we are off the steady state path and there is excess supply in one market (here for goods), quantities adjust more quickly than prices but subsequent changes in prices tend to remove the excess supply, so that demand more closely follows supply. An important element in this process in the present model is the second order price equation--whereas quantities (here DY/Y) start changing immediately in response to excess supply, price levels do not, but the rate of inflation does change and, in particular, falls so that prices do not subsequently increase as fast (or, in extreme cases, may fall). This brings about a change in relative prices, raising demand and, ultimately, removing excess supply.

When all real variables in the model have initial values 10% below their steady state levels, the adjustment path (c) in Figure 1 shows that output rises steadily but at a declining rate up to the steady state path

¹In our discussion of the graphs in Figure 1-5 we will make use, as at this point, of numerical information given in the full output of the solution trajectories.

within 20 quarters, runs close to this path for 30 quarters and then moves into a long and marked downward cycle. A very similar path in this case is followed by real consumption expenditure as shown by curve (b) in Figure 2. The proportional rate of increase in output, $D \ln Y$, is at its highest level in the first quarter at 5.1% per quarter, it then declines and reaches its lowest level of 0.75% per quarter around $t = 65$ quarters and then begins to rise until it reaches 1.2% per quarter at $t = 120$ quarters.¹ Thus, although output is increasing throughout the whole 120 quarters and is never more than 10% from the steady state path, there is considerable variation in the rate at which output is growing. The fast growth rate in output in the first few quarters is explained by the fact that supply is 10% below its steady state level at $t = 0$, while the investment, exports and government expenditure components of aggregate sales are all consistent with the steady state. This leads to a high level of excess demand in the goods market and, hence, through equation (5) the fast growth rate in output in the first few quarters. A further consequence of this excess demand is to raise the rate of inflation through equation (7) from the steady state level of 2% a quarter at $t = 0$ to 2.43% in quarter $t = 2$ as shown in graph (c) of Figure 4. Although output responds quickly to excess demand, employment adjusts quite slowly in comparison. This is partly explained by the differences in the constant adjustment rates in equations (3) and (5) viz $\beta_{10} = 0.1$ and $\gamma_5 = 1.0$; but it is also explained by the non-linear constraint on the adjustment of employment involving the parameter β_{11} , which prevents employment from adjusting at the constant rate β_{10} until the labor supply has increased sufficiently

¹The steady state rate of increase in output is 1% per quarter since $\lambda_1 + \lambda_2 = 0.01$.

through demographic or real wage effects to alleviate the excess demand for labor (in the present case, caused by the rapid rise in output in the first few quarters).

The differences in the adjustment paths of output and employment causes output per man to rise by 7.8% between $t = 0$ and $t = 6$. The increase in efficiency reduces the marginal cost of output and the rate of inflation then falls, through the operation of the adjustment mechanism in equation (7), below the steady state rate of inflation, as shown in curve (c) of Figure 4 between quarters $t = 6$ and $t = 22$. The following period is one of sustained inflation above the steady state level until quarter 81. The effect of this sustained inflation is to reduce the competitiveness of domestic output relative to imports and, through the operation of the price relative in β_{14} and the output adjustment mechanism of equation (5), to lead to a marked downward swing in the output path compared with the steady state as shown by curve (c) in Figure 1.

Turning to Figure 5 we see from curve (c) that with the excess demand for goods, rising rate of price inflation and increased labor efficiency, wage inflation also rises above its steady state path. But not as far nor as fast as price inflation. Moreover, the rise in wage inflation is abruptly shut off once the rate of price inflation falls below its steady state path around the sixth quarter. This is explained by the role played by the rate of price inflation in the wage bargaining process and represented by the final term of equation (8). Partly because of the continued excess demand for labor, and partly because of the sustained rate of inflation above the steady state level during quarters $t = 23$ through to $t = 81$ the rate of wage inflation remains buoyant above the steady state level of 2.5% per quarter until quarter $t = 56$. Wage inflation then goes into a strong downward cycle until the final quarter $t = 120$.

It is important to emphasize that prices and wages are rising throughout the entire period $t = 0$ to $t = 120$. It is then clear that our solution trajectories are compatible with rising prices, output and employment as well as a world in which prices are still rising but in which the output and employment gains appear to be lost (in our case, as evidenced in Figures 1, 3 and 4 the rate of inflation can be above the steady state level while output and employment are below their steady state levels and on a marked downward cycle).

Many of the explanations given above apply to the behavior of the other curves in Figures 1-5, not already discussed. We will complete our discussion in this section by considering those curves which represent the outcome of parameter changes rather than changes in initial conditions.

The first of these occurs in Figure 2 with the change in value of parameter β_4 in curves (c) and (d). As discussed in Section 2, this parameter measures the effect of supply side constraints on the rate of adjustment in consumption expenditure. The form of the function embodying this constraint implies that β_4 is negative and the force of the constraint on expenditure adjustment increases as β_4 decreases. From our results in Section 3 we deduce that $\partial C^*/\partial \beta_4 > 0$, so that as β_4 decreases in value (becomes larger in absolute value, but remains negative) the steady state level of consumption is reduced. Thus, as supply side constraints become more binding, the level of consumption expenditure in the steady state is reduced. We would also expect some short run effects of changes in β_4 on the adjustment path of consumption. Curve (c) in Figure 2 shows the time path for consumption when $\beta_4 = -0.30$ against the steady state path (curve (a)) where $\beta_4 = -0.075$. For curve (c) we are in the regime where all real variables (including consumption) are 10% below the steady

state paths based on the original value of β_4 . As already discussed, in this regime there is a high level of excess demand for goods in the first few quarters. Although output responds quickly, there is a substantial run down in stocks; in the first quarter, this amounts to -17% and, in the second quarter, -3%. As shown in curve (c) of Figure 2, the effect of this initial run down in stocks is to limit the rate of adjustment in consumption expenditure (in contrast to curve (b) where, in the same regime, consumption expenditure rises rapidly). In succeeding quarters, the time path of consumption seems to be a smooth adjustment to a new lower level --as expected from the sign of the partial derivative $\partial C^*/\partial \beta_4 > 0$. It is interesting to note that the following conditions also apply in the steady state

$$\frac{\partial Y^*}{\partial \beta_4} = \frac{\partial L^*}{\partial \beta_4} = \frac{\partial K^*}{\partial \beta_4} = 0$$

so that the long run effect of the supply side constraint on the consumption function is to leave the steady state level of real output, employment and capital unchanged. Thus the long run effect of β_4 on consumption expenditure is absorbed through a reduction in imports.¹

Curve (d) in Figure 2 shows the time path for consumption expenditure when β_4 is increased, almost sixfold, to -0.013. As is clear from the graph, the difference between this and the reference path (b) in this

¹This is exemplified in the values of output, employment, and capital in the solution trajectories. After 120 quarters with $\beta_4 = -0.30$, $Y = 1940.6$, $L = 1092.0$, $K = 3987.9$ while with $\beta_4 = -0.075$, $Y = 1998.2$, $L = 1110.9$, $K = 4064.2$; the corresponding values for imports are 459.4 and 522.6 respectively.

regime is negligible. This is a feature of the non-linear way in which the supply side constraint enters equation (1), so that changes in the parameter β_4 can be expected to lead to effects of different orders depending on the region of the parameter space. In the present case, as β_4 increases in absolute value the effect on the consumption path of the squeeze on consumption expenditure when there are supply shortages is greater than the effect on the consumption path of the same conditions as β_4 decreases in absolute value.

The second curve representing the outcome of a major parameter change is curve (d) in Figure 3. This curve shows the effect on the employment path when the parameter β_{11} rises by 50% to 0.075. As discussed in Section 2, this parameter controls the influence of labor market disequilibrium on the adjustment rate of employment. We know from our analysis of the steady state in Section 3 that normally $\partial L^*/\partial \beta_{11} < 0$ and by evaluation we can check that this sign is correct in the present case. Thus, the steady state path relevant to curve (d) in Figure 3 is lower than that given by curve (a). From curve (d), we see that the effect of labor market disequilibrium with $\hat{L} > L_s$ is so strong as to prevent employment from rising significantly for almost 20 quarters. It is only after the competition for labor has bid up the wage rate sufficiently that the labor supply expands to alleviate the pressure of demand. Then, employment begins to rise steadily, ignoring small oscillations in the rate of growth, for 40 quarters. The rate of growth in employment falls significantly around quarter 60 but employment still continues to rise slowly (again with minor oscillations in the rate) until quarter 110. During most of this period, prices grow faster than wages through demand pressure in the market for goods and this reduces the growth rate of the labor supply relative to

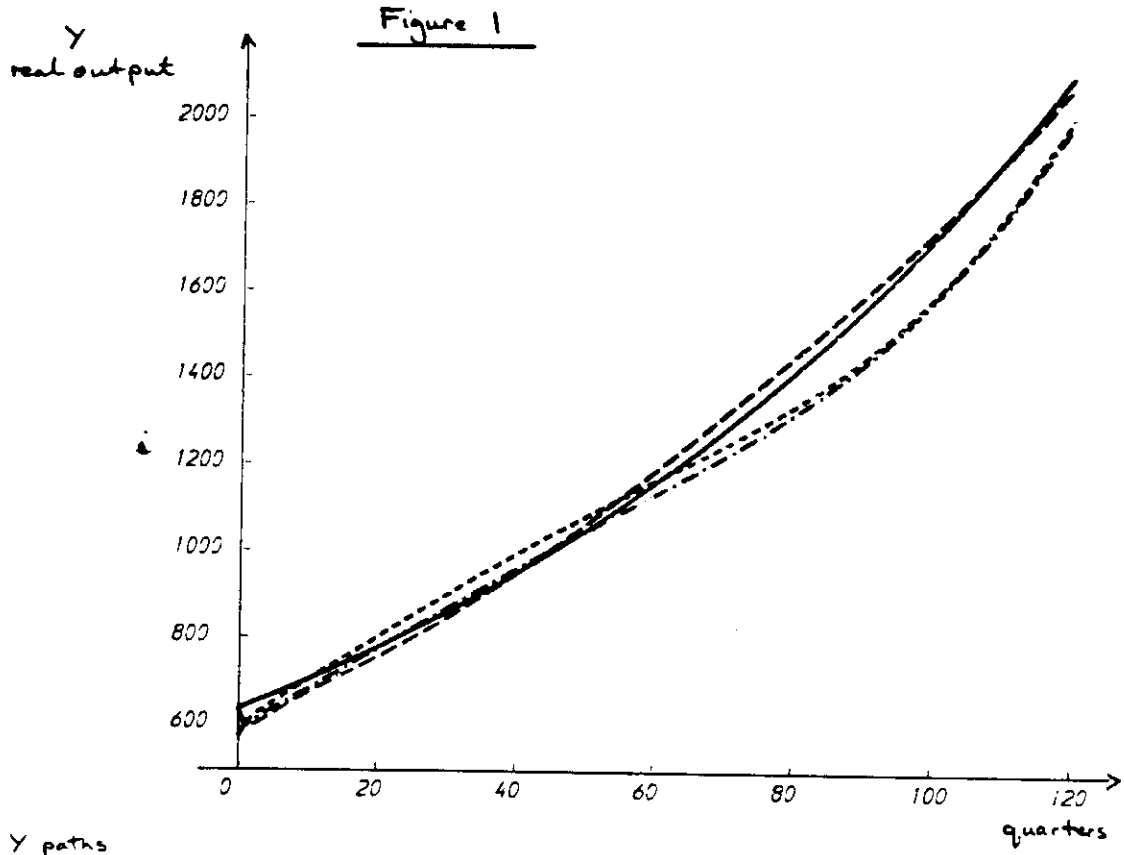
that of labor demand. Thus, towards the end of the period, around $t = 110$, we find the effect of disequilibrium once again becoming strong and at this point actually reducing the level of employment quite sharply.

Curve (d) represents the outcome of a substantial increase in an important parameter of the model. The resulting time path of employment is quite dramatic, with abrupt changes in direction and rates of growth that are not among the possible solution paths of linear models. Since one of the features of reality a disequilibrium model should be capable of confronting is the abrupt movement in quantities transacted when markets are in disequilibrium, we consider it to be a useful aspect of the present model that it is consistent with such behavior. The importance of such effects in practice will be measured according to the significance of empirical estimates of parameters such as β_{11} .

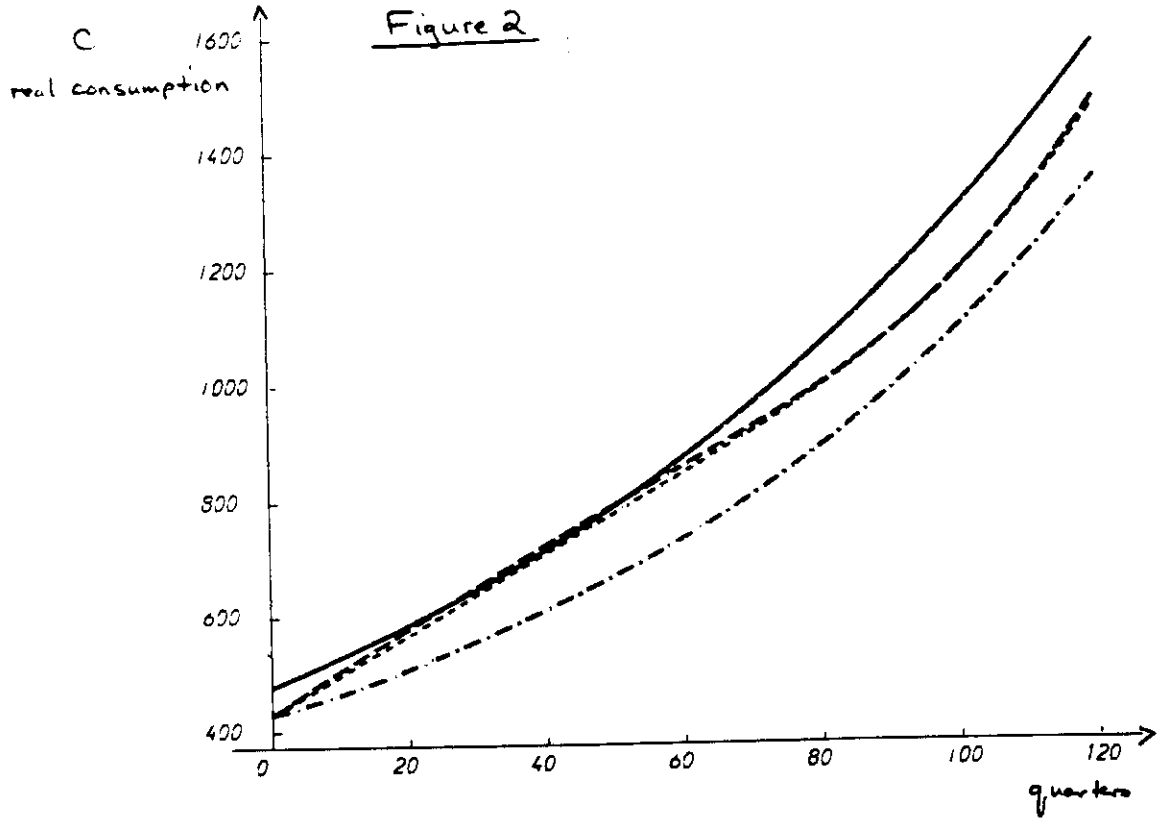
5. CONCLUSION

The model developed and analyzed in this paper is designed to explain movements in aggregate, output consumption expenditure, employment, capital formation and the inflationary process. It has been constructed with a view to subsequent empirical application to the New Zealand economy and, as a result, incorporates some explicit features of this economy in the specification of its equations; but it is hoped that the new features and innovations in specification that it embodies will be found useful in other macroeconometric models. In this respect, our treatment of variable rates of adjustment in the specification of a number of equations and the resulting non-linearities may be of special interest. The effect of endogenously determined adjustment coefficients have been analyzed, in the long run,

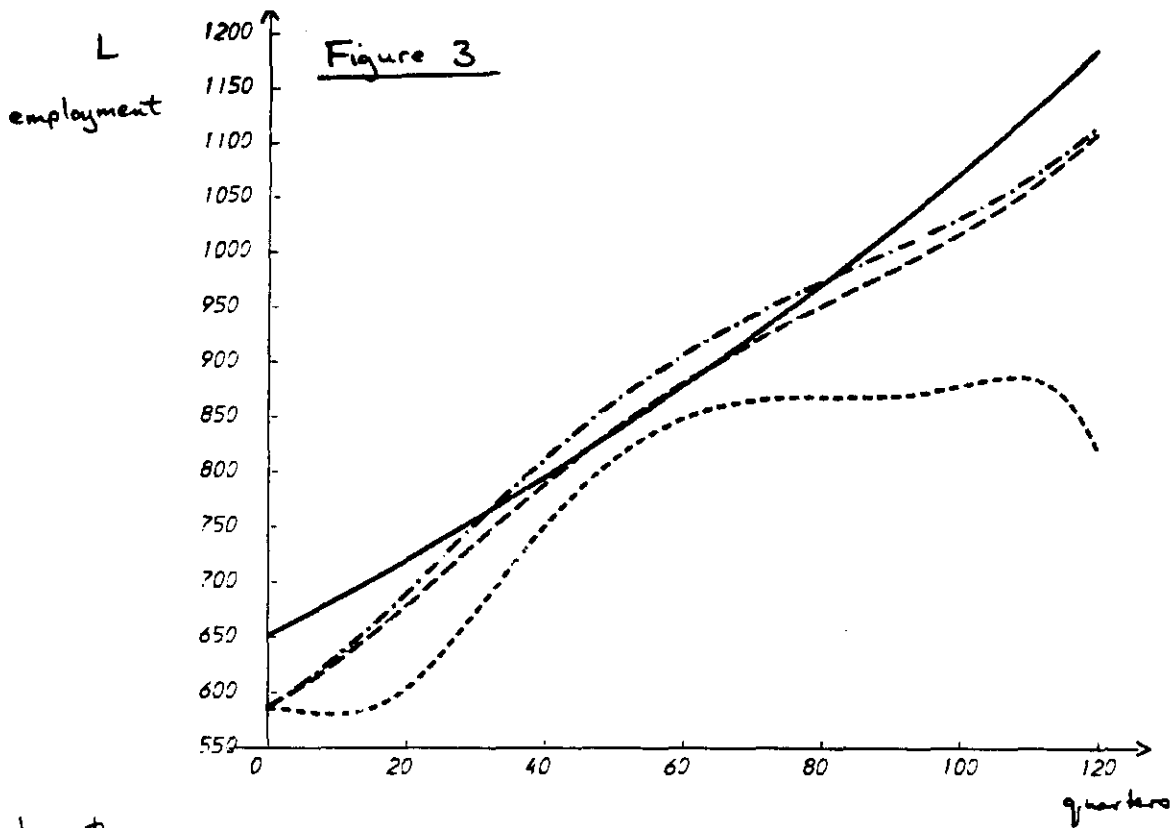
through the impact of the relevant parameters on the steady state solution and, on the adjustment paths themselves, through the general solution trajectories obtained under various initial conditions by numerical methods. The analysis has shown that the presence of these non-linearities in the adjustment mechanism enriches the possible set of response patterns and makes the model more flexible in its ability to capture the complex dynamic process behind the determination of macroeconomic data. Work on the estimation of the model using some of the methods outlined in the Appendix is currently under way and will be reported in a later paper.



- Y paths
- (a) steady state path
 - (b) when $C(0)$ is 20% below its steady state value
 - · - · - (c) when $C(0), Y(0), K(0), L(0), S(0), I(0)$ are 10% below their steady state values
 - (d) when $C(0), Y(0), K(0), L(0), S(0), I(0), P(0), W(0)$ are 10% below their steady state values

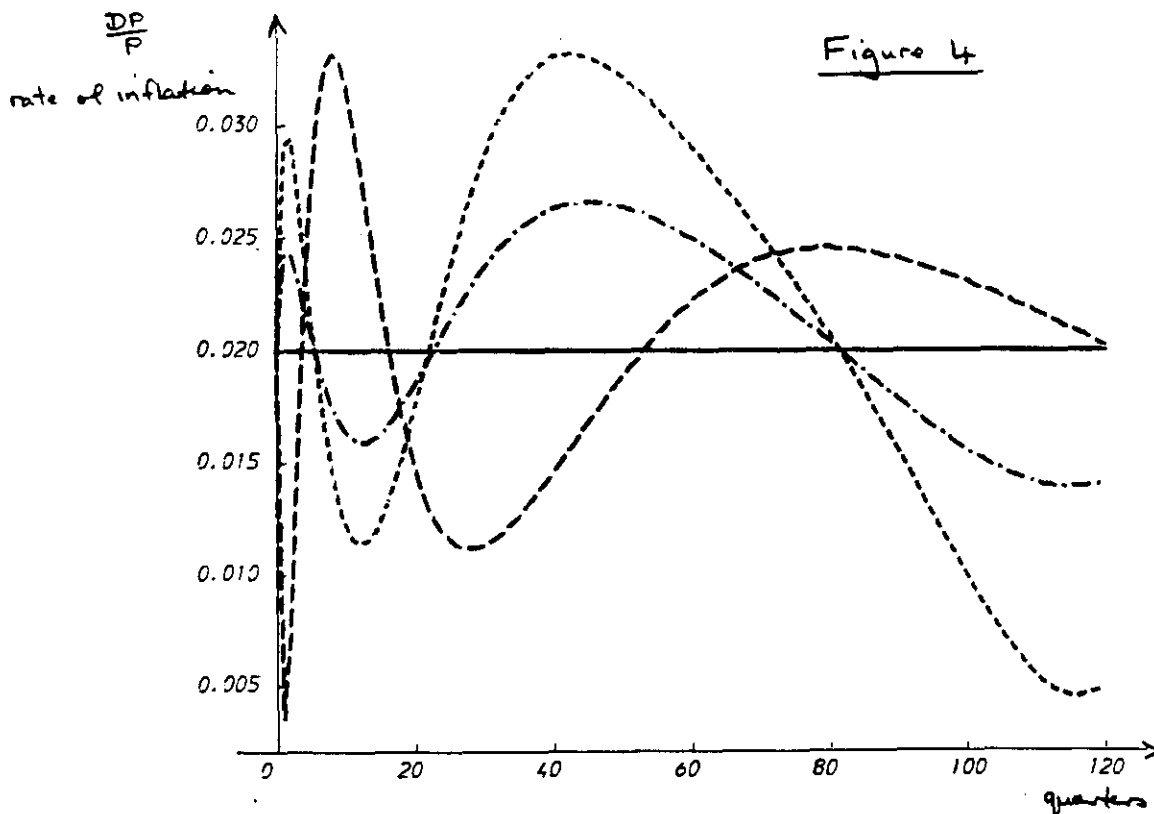


- C paths
- (a) steady state path
 - (b) when $C(0), Y(0), K(0), L(0), S(0), I(0)$ are 10% below their steady state values
 - · - · - (c) as in (b) but with $\beta_4 = -0.30$
 - (d) as in (b) but with $\beta_4 = -0.013$



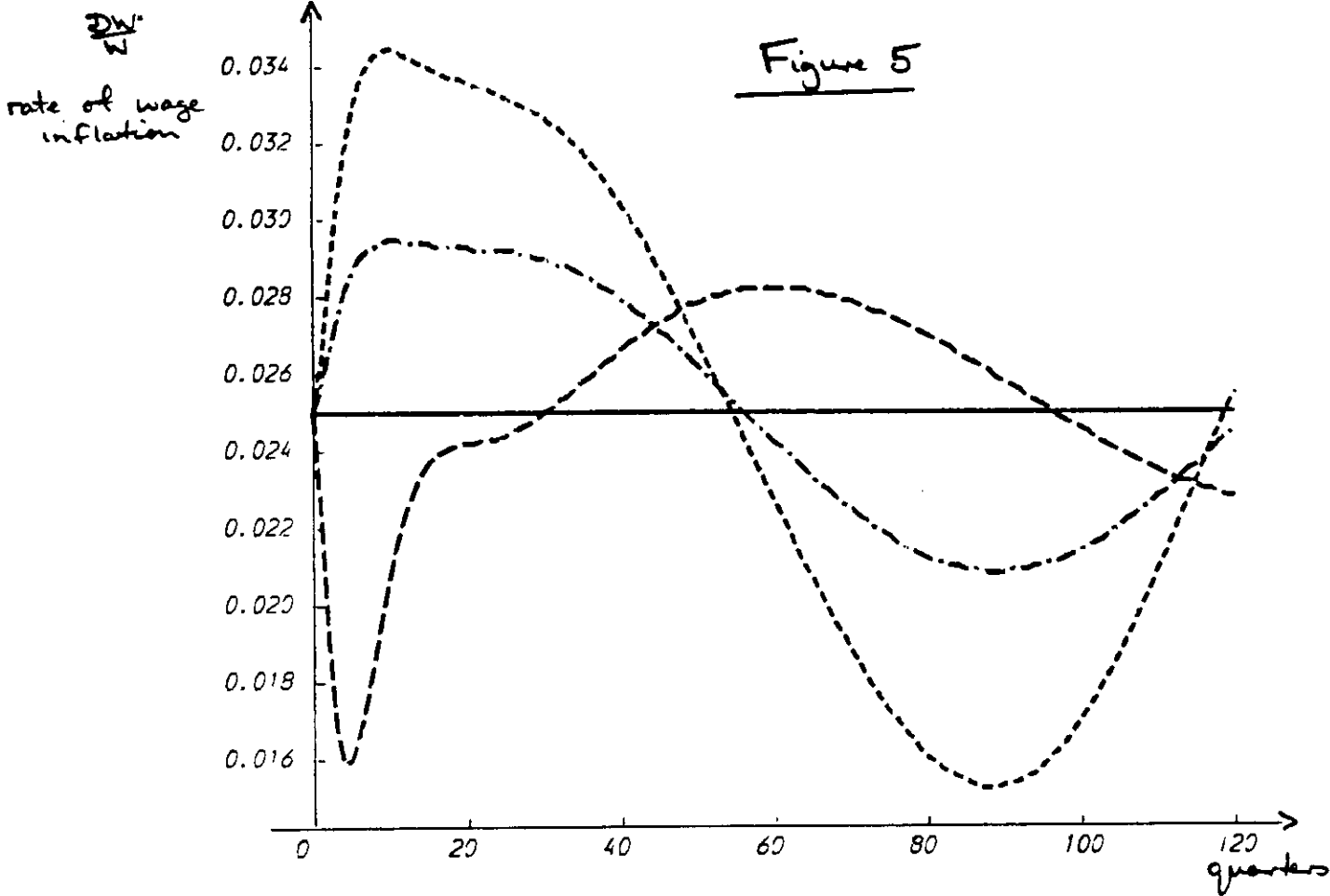
L paths

- (a) steady state path
- - - (b) when $C(0), Y(0), K(0), L(0), S(0), I(0)$ are 10% below their steady state values
- · - · (c) when $C(0), Y(0), K(0), L(0), S(0), I(0), P(0), W(0)$ are 10% below their steady state values
- · · (d) as in (b) but with $p_{11} = 0.075$.

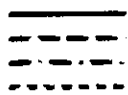


$\frac{DP}{P}$ paths

- (a) steady state path
- - - (b) when $C(0)$ is 30% below its steady state value
- · - · (c) when $C(0), Y(0), K(0), L(0), S(0), I(0)$ are 10% below their steady state values
- · · (d) when $C(0), Y(0), W(0), L(0), S(0), I(0)$ are 20% below their steady state values



$\frac{Dw}{w}$ paths



- (a) steady state path
 (b) when $Z(t)$ is 30% below its steady state value
 (c) when $C(t), Y(t), K(t), S(t), L(t), I(t)$ are 10% below their steady state values
 (d) when $C(t), Y(t), K(t), S(t), L(t), I(t)$ are 20% below their steady state values

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APPENDIX
ECONOMETRIC METHODS FOR NON-LINEAR DIFFERENTIAL EQUATIONS
IN STOCK AND FLOW VARIABLES

a.1. Existing Procedures and Non-linear Differential Equations

The general form of the system of structural differential equations developed in the body of the paper can be written as in equation (53) and complete with additive random disturbances this system takes the form

$$(55) \quad Dx(t) = f\{Dx(t), x(t), Dz(t), z(t); \theta\} + \zeta(t)$$

where the elements of the vector $\zeta(t)$ are random disturbance functions, some of which may be zero corresponding to equations of (55) which are identities in the model.

Equation (55) is a system of non-linear stochastic differential equations. Procedures which have recently been developed for the estimation of stochastic differential equations from a sample of discrete observations¹ are applicable only in the context of models that are linear in the variables. These procedures cannot, therefore, be applied directly to a model such as (55). The approach initiated by Bergstrom and Wymer (1976) for dealing with this difficulty in practice is to form a linear approximation to the right hand side of (55) evaluated at the sample means of the variables. This preserves any parameter non-linearity in the system, but allows the use of estimation procedures developed for models that are linear in the variables. This approach has now become fairly standard

¹The papers contained in the recent volume on continuous time econometric models edited by Bergstrom (1976) provide a fairly complete coverage of these procedures.

practice amongst empirical investigators working in this area (for example, Knight and Wymer (1976) and Jonson, Moses and Wymer (1977)). In his comments on Knight and Wymer (1976), Fisher (1976) raised the criticism that to linearize a non-linear continuous time model and then estimate an approximate discrete analogue of the linear model¹ is to throw away an important element of assumed behavior that is implicit in the original non-linear formulation. In our view, this is an important criticism which deserves the attention of model builders and researchers in this area. It may also be worthy of mention that the solution set for non-linear differential equations is, in general, much wider (including, for example, the possibility of limit cycles) than that for linear systems. Preserving the variable non-linearity in the structure during estimation, prediction and simulation would, therefore, seem to be an important element in capturing the behavioral implications of variable non-linearity. This consideration is thought to be particularly relevant in the present case, where in a number of equations of our model the adjustment rates are endogenously determined, leading directly to a variable non-linearity; and the analysis in Sections 3 and 4 of the body of the paper show that the behavioral implications of this variable non-linearity are significant both in the long-run steady state solution and in the transient adjustment paths.

¹Note that we are unable to estimate directly the exact discrete analogue, as suggested by Fisher (1976, p. 167), when there are exogenous variables in the model. See Phillips (1974 and 1976).

a.2. Non-Linear Discrete Approximations

Our own approach to this problem is to extract a non-linear discrete approximation directly from the structural system (55). Integrating (55) over the interval $(t-h, t)$, where h denotes the time interval between observations, we have

$$(56) \quad x(t) - x(t-h) = \int_{t-h}^t f(w(\tau); \theta) d\tau + \int_{t-h}^t \zeta(\tau) d\tau$$

where $w(t)' = (Dx(t)', x(t)', Dz(t)', z(t)')$. There are now several problems that need to be overcome before (55) can be used for estimation purposes.

The first problem is to represent the integral $\int_{t-h}^t f(w(\tau); \theta) d\tau$ as a function of observable variables. An apparent way of achieving this is to employ the approximation

$$(57) \quad \int_{t-h}^t f_i(w(\tau); \theta) d\tau \sim \frac{h}{2} \{f_i(w(t); \theta) + f_i(w(t-h); \theta)\}.$$

This will work for some elements of f which do not depend on the derivatives $Dx(t)$ and $Dz(t)$. For the remaining elements, we can separate the variables and write

$$f_j(w(t); \theta) = b_j(\dot{x}(t), \dot{z}(t); \theta) g_j(x(t), z(t); \theta)$$

as the product of two distinct functions, where the dot over a variable denotes differentiation with respect to t . We can then employ the approximation

$$(58) \quad \int_{t-h}^t f_j(w(\tau); \theta) d\tau \sim b_j(x(t) - x(t-h), z(t) - z(t-h); \theta) \\ \cdot \frac{h}{2} \{g_j(x(t), z(t); \theta) + g_j(x(t-h), z(t-h); \theta)\} .$$

We can justify (58) by arguing that, from the mean value theorem,

$$\int_{t-h}^t b_j(\dot{x}(\tau), \dot{z}(\tau); \theta) g_j(x(\tau), z(\tau); \theta) d\tau \\ = b_j(\dot{x}(\tilde{t}), \dot{z}(\tilde{t}); \theta) \int_{t-h}^t g_j(x(\tau), z(\tau); \theta) d\tau$$

where \tilde{t} lies in the interval $(t-h, t)$. We then approximate

$$b_j(\dot{x}(\tilde{t}), \dot{z}(\tilde{t}); \theta) \sim b_j(x(t) - x(t-h), z(t) - z(t-h); \theta)$$

and

$$(59) \quad \int_{t-h}^t g_j(x(\tau), z(\tau); \theta) d\tau \sim \frac{h}{2} \{g_j(x(t), z(t); \theta) + g_j(x(t-h), z(t-h); \theta)\} .$$

Note that an alternative to the approximations (57) and (58) which are based on the trapezoidal rule, is to use the mean value theorem again and write, in the case of (57),

$$(60) \quad \int_{t-h}^t f_1(w(\tau); \theta) d\tau = hf_1(w(\tilde{t}); \theta) \\ \sim hf_1\left\{\frac{1}{2}(w(t) + w(t-h)); \theta\right\} .$$

If $w(t)$ is a sufficiently smooth function with bounded derivatives as $h \rightarrow 0$ then it is easy to demonstrate by Taylor series expansions that the error in both types of approximations is of $O(h^2)$ as the sampling

interval $h > 0$. Thus, the specification error implicit in the approximations (57), (58) and (60) is of $O(h^2)$ and this suggests, as in Sargan (1976), that the asymptotic bias of simultaneous equations estimators derived from the corresponding approximate model will be of the same order. Note that since the approximations in (57), (58) and (60) involve current period values of the endogenous variables that appear as elements in $w(t)$, the resulting discrete approximation is a non-linear in variables and parameters simultaneous equations model. The use of the non-linear discrete approximation based on either (57) and (58) or (60) therefore reduces to the treatment of this type of simultaneous equations model. The estimation of such systems has recently been a topic of interest in the theoretical literature (notably Amemiya (1974, 1975, 1977), Hausman et al. (1975), Jorgenson and Laffont (1975), Gallant (1976) and Gallant and Holly (1980)); and practical empirical work with a system of this type involving 26 stochastic equations has recently been undertaken by Fair (1976).

a.3. The Combined Treatment of Stock and Flow Variables

The second problem that arises in preparing (56) for estimation is to deal with those variables (such as C , Y and I in the model of the paper) which are flows and not instantaneously observable. A theoretical treatment of this problem for linear models involving only flow variables has been given in Phillips (1978). The present model in the body of the paper involves both flow and stock (including prices and other instantaneously observable) variables so that the procedure for dealing with flow variables as outlined in that paper is not strictly applicable. Moreover, in the theoretical treatment of Phillips (1978) it is assumed that the flow variables appear directly in levels form and not in logarithmic form as they do in

the present model. This latter point is of some importance and it is interesting to observe in this connection some of the implications of the treatment of flow variables in Bergstrom and Wymer (1976) which has been followed by later investigators such as Jonson, Moses and Wymer (1977) and others. This treatment suggests that, since $x(t)$ is not measurable for a flow variable, whereas the integral $\int_{t-h}^t x(s)ds$ is, a model such as (56) should be integrated again over the sampling interval to convert the variables into measurable flows over a distinct time period like a quarter or a year. When the components of $x(s)$ are logarithms of the underlying variables, this procedure is more difficult to justify. If we take the variable $C(t)$, for example, which measures the (unobservable) rate of consumption taking place at the moment in time t , we see that the procedure involves the approximation

$$(61) \quad \int_{t-h}^t \ln(C(s))ds \sim \ln\left\{\int_{t-h}^t C(s)ds\right\} = \ln(C_t)$$

where C_t denotes the measurable flow of consumption over the interval $(t-h, t)$. As mentioned earlier, this treatment of flow variables involves a second integration of the underlying equation which induces a moving average in the resulting disturbance on the equation. A first approximation¹ to this moving average is given by $(1 + 0.268L)\xi(t)$ where

$$\xi(t) = \int_{t-h}^t \zeta(\tau)d\tau \text{ in (56), say. BW remove this moving average by inversion,}$$

replace $(1 + 0.268L)^{-1}$ by the first four terms in its expansion and apply

¹A second order approximation involving the parameters of the model is derived in Phillips (1978). This applies when working with the exact discrete model corresponding to a linear differential equation system.

the resulting transformation to the variables of the model. This transformation is now popularly known as COSMA (see Norton (1977)). In summary, this treatment of flow variables involves three different approximations: first, the approximation implicit in (61); then a first order approximation to the induced moving average disturbance; and a further approximation resulting from the truncation of the expansion of the inverted moving average operator.

Clearly, the proper treatment of flow variables in the derivation of a useable discrete approximation is a delicate task. An obvious alternative to that above is to try to overcome the problem by an appropriate centering of variables. This approach has also been suggested in the RBA conference proceedings (Norton (1977)), although not in the context of an alternative to the COSMA transformation described above but rather as a timing adjustment to ensure that stock and flow variables refer to a common point each period (Bacon and Johnston (1977),¹ p. 120).

The approach we now suggest can be explained as follows. Consider the i^{th} equation of the system (55), which we write as

$$(62) \quad Dx_i(t) = f_i(w(t); \theta) + \zeta_i(t)$$

where $f_i(w(t); \theta) = b_i(\dot{x}, \dot{z}; \theta)g_i(x, z; \theta)$, $x_i(t) = \ln(y_i(t))$ and $y_i(t)$ is a flow variable. We observe, not $y_i(t)$, but the time integrals

¹Bacon and Johnston suggest an averaging of stocks to synchronize with the timing of flow variables at the mid-point of the observation period. Helliwell (1977a) in his commentary points out (p. 329) that the information content in the data may be blurred by this type of successive averaging, in spite of the fact that the formulation of the discrete approximation may appear more consistent.

$$(63) \quad y_{it} = \int_{t-1}^t y_i(s) ds, \quad t = 1, \dots, T.$$

Since the sample path of $y_i(t)$ is continuous in the mean square sense¹ (Bartlett (1965)) it follows that for some $\mu \in (t-1, t)$ and some $\eta \in (t-2, t-1)$ we have, in the mean square sense,

$$(64) \quad y_{it} = \int_{t-1}^t y_i(s) ds = y_i(\mu)$$

$$(65) \quad y_{it-1} = \int_{t-2}^{t-1} y_i(s) ds = y_i(\eta).$$

If we now integrate (62) over the interval (η, μ) we obtain

$$\begin{aligned} x_i(\mu) - x_i(\eta) &= \ln y_{it} - \ln y_{it-1} \\ &= \int_{\eta}^{\mu} b_i(\dot{x}, \dot{z}; \theta) g_i(x, z; \theta) ds + \int_{\eta}^{\mu} \zeta_i(s) ds \\ (66) \quad &\sim b_i(\dot{x}(\tilde{\mu}), \dot{z}(\tilde{\mu}); \theta) \left\{ \frac{\mu - \eta}{2} \right\} \{ g_i(x(\mu), z(\mu); \theta) + g_i(x(\eta), z(\eta); \theta) \} + \int_{\eta}^{\mu} \zeta_i(s) ds \end{aligned}$$

where $\tilde{\mu}$ lies in the interval (η, μ) . In order for (66) to become a useable discrete approximation we need the arguments of the functions $b_i(\cdot, \cdot; \theta)$ and $g_i(\cdot, \cdot; \theta)$ to be specified in terms of measurable variables. Since μ and η lie in the intervals $(t-1, t)$ and $(t-2, t-1)$ we can further approximate (66) by arguing that, on average over the sample period, we can set μ and η equal to the mid-point of these intervals, viz. $t - \frac{1}{2}$

¹Note that this is true even when $\zeta_i(t)$ is pure noise. In this case $y_i(t)$ is continuous but not differentiable in the mean square sense.

and $t - \frac{3}{2}$. By the same argument, we can set $\tilde{\mu}$ at the mid-point of $\left(t - \frac{3}{2}, t - \frac{1}{2}\right)$, that is $t-1$. Moreover, when the component variables of $x(t)$ and $z(t)$ are flows we can employ the mean value representations (64) and (65) on the argument once again that, on average over the sample period, this will be an acceptable approximation. The only outstanding problem in converting (66) into a useable discrete approximation is to represent the derivatives $\dot{x}(t-1)$ and $\dot{z}(t-1)$ in terms of measurable quantities. One approach here might be to use first differences as a simple approximation. For flow variables these may then be considered to be appropriately centered at the moment $t-1$ as required. However, for stocks the simple use of first differences will inevitably result in some mistiming unless the differences themselves are averaged. This then leads to approximations such as $\dot{x}(t-1) \sim \frac{1}{2}(x(t) - x(t-2))$, which is the numerical first derivative of x at $t-1$ based on a three point Lagrange interpolation of $x(s)$ in the interval $[t-2, t]$ and passing through the points $x(t-2)$, $x(t-1)$ and $x(t)$ (see, for instance, Ralston (1965), p. 81). The error in this approximation is of $O(h^2)$ as $h \rightarrow 0$ when h is the sampling interval. The order of magnitude of this error is consistent with that of the error in the integral approximations (57), (58) and (60).

a.4. The Treatment of Integral Partial Equilibrium Formulations

As argued in the paper in Section 2.3, there are reasons why we may prefer integral rather than instantaneous partial equilibrium specifications. The example given in (14) is the partial equilibrium rate of consumption which we write here as

$$(67) \quad \hat{c}(s) = \beta \int_{t-\delta}^t Y(r) dr \quad \text{for all } s \in (t-\delta, t)$$

where we neglect the variables GB and TP that occur in (14) for algebraic simplicity, but these cause no real complication. In (67), δ is the decision making interval of the representative consumer.¹ Unless there is particular reason for thinking otherwise or some particular economic importance to the parameter δ which we wish to retain, it seems acceptable to replace the interval $(t-\delta, t)$, on average across all individuals involved in consumption decisions, by the sampling interval $(t-1, t)$. Now when we come to integrate the differential equation for $C(t)$ according to the procedure of the previous section we need to treat an integral of the form

$$\int_{t-3/2}^{t-1/2} \ln \hat{C}(s) ds .$$

This integral reduces to the sum of two integrals of constant functions as

$$\begin{aligned} & \int_{t-1}^{t-1/2} \ln \hat{C}(s) ds + \int_{t-3/2}^{t-1} \ln \hat{C}(s) ds \\ &= \int_{t-1}^{t-1/2} \ln(Y_t) ds + \int_{t-3/2}^{t-1} \ln(\beta Y_{t-1}) ds \\ (68) \quad &= \ln \beta + \frac{1}{2} \{ \ln Y_t + \ln Y_{t-1} \} \end{aligned}$$

where $Y_t = \int_{t-1}^t Y(s) ds .$

¹Of course, many alternatives to (67) are possible but the example given is only illustrative of the implications of this type of formulation for the discrete approximation.

The expression (68) is exact. In addition, it corresponds precisely with the recommended approximation (66) together with the recommended treatment of flow variables. For (66) becomes in this case just

$$\int_{\eta}^{\mu} \ln \hat{C}(s) ds \sim \left(\frac{\mu - \eta}{2} \right) \{ \ln \hat{C}(\mu) + \ln \hat{C}(\eta) \}$$

and, with the instantaneous formulation of the partial equilibrium level as $\hat{C}(s) = \beta Y(s)$ rather than (67), this becomes

$$(69) \quad \left(\frac{\mu - \eta}{2} \right) \{ \ln(\beta Y_t) + \ln(\beta Y_{t-1}) \}$$

using (64) and (65). Then, arguing as in the previous section that the interval (η, μ) has the same length, on average over the sample period, as the sampling interval, we see that (69) is equivalent to (68).

It is important to note that while (69) has been derived on the basis of the integral being approximated according to the method (66), (68) has been derived with the underlying integral being given exactly. Thus, the treatment of integral partial equilibrium formulations remains the same as in the previous section; but whereas the earlier derivation was based on the principle of the underlying integrals' being approximated, they now reduce to the same discrete time formulation exactly.

a.5. An Application of the Procedure for Discrete Approximation

We can illustrate the procedure for extracting the discrete approximation in the presence of variable non-linearities and stocks and flows by taking the consumption function, (1), of our model in the body of the paper. Writing this out explicitly in terms of the defined variables, we have

$$D \ln C(t) = \gamma_1 \ln \left[\frac{\beta_1 \{Y(t) + GB(t) - TP(t)\}}{C(t)} \right] + \zeta_1(t)$$

where

$$\gamma_1 = \beta_2 + \beta_3 \{D \ln MBD(t) - p(t)\} + \beta_4 \exp\{-D \ln S(t)\} .$$

Multiplying out the variable adjustment rate, we obtain

$$\begin{aligned} D \ln C(t) &= \beta_2 \ln \beta_1 + \beta_2 \ln\{Y(t) + GB(t) - TP(t)\} \\ &\quad - \beta_2 \ln C(t) + \beta_3 \ln \beta_1 \{D \ln MBD(t) - p(t)\} \\ &\quad + \beta_3 \{D \ln MBD(t) - p(t)\} \ln\{Y(t) + GB(t) - TP(t)\} \\ &\quad - \beta_3 \{D \ln MBD(t) - p(t)\} \ln C(t) \\ &\quad + \beta_4 \ln \beta_1 \exp\{-D \ln S(t)\} + \beta_4 \exp\{-D \ln S(t)\} \ln\{Y(t) + GB(t) - TP(t)\} \\ &\quad - \beta_4 \exp\{-D \ln S(t)\} \ln C(t) + \zeta_1(t) . \end{aligned}$$

Since $C(t)$ is a flow, the suggested approach is to integrate this equation over the interval (η, μ) or, as we have argued above, on average in the sample over the interval $\left[t - \frac{3}{2}, t - \frac{1}{2} \right]$. The following is the discrete approximation we obtain:

$$\begin{aligned}
(70) \quad \ln C_t - \ln C_{t-1} = & \beta_2 \ln \beta_1 + \beta_2 \frac{1}{2} \left\{ \ln(Y_t + GB_t - TP_t) + \ln(Y_{t-1} + GB_{t-1} - TP_{t-1}) \right\} \\
& - \beta_2 \frac{1}{2} (\ln C_t + \ln C_{t-1}) + \beta_3 \ln \beta_1 \left[\ln \text{MBD} \left(t - \frac{1}{2} \right) - \ln \text{MBD} \left(t - \frac{3}{2} \right) - \frac{1}{2} \left\{ p \left(t - \frac{1}{2} \right) + p \left(t - \frac{3}{2} \right) \right\} \right] \\
& + \beta_3 \left[\frac{1}{2} (\ln \text{MBD}(t) - \ln \text{MBD}(t-2)) - p(t-1) \right] \frac{1}{2} (\ln(Y_t + GB_t - TP_t) + \ln(Y_{t-1} + GB_{t-1} - TP_{t-1})) \\
& - \beta_3 \left[\frac{1}{2} (\ln \text{MBD}(t) - \ln \text{MBD}(t-2)) - p(t-1) \right] \frac{1}{2} (\ln C_t + \ln C_{t-1}) \\
& + \beta_4 \ln \beta_1 \exp[-(\ln S_t - \ln S_{t-1})] \\
& + \beta_4 \exp[-(\ln S_t - \ln S_{t-1})] \frac{1}{2} (\ln(T_t + GB_t - TP_t) + \ln(Y_{t-1} + GB_{t-1} - TP_{t-1})) \\
& - \beta_4 \exp[-(\ln S_t - \ln S_{t-1})] \frac{1}{2} (\ln C_t + \ln C_{t-1}) + \int_{t-3/2}^{t-1/2} \zeta_1(s) ds
\end{aligned}$$

where we use a subscript t on a variable to represent a measured flow over a unit time period as in (63); thus, for example $GB_t = \int_{t-1}^t GB(s) ds$.

Note that (70) is a non-linear difference equation involving non-integral differences in some of the variables. The simplest way to treat this is to replace the non-integral differences by integral differences so that, for instance, $\ln \text{MBD} \left(t - \frac{1}{2} \right) - \ln \text{MBD} \left(t - \frac{3}{2} \right)$ is replaced by $\ln \text{MBD}(t) - \ln \text{MBD}(t-1)$. A more sophisticated alternative would be to use the three point quadratic approximation in Phillips (1974). If $w(t)$ is the variable function for which observations are available at integral points in time we then use the approximation

$$\begin{aligned}
\hat{w}(t-s) = & w(t) - \frac{1}{2}s\{w(t-2) - 4w(t-1) + 3w(t)\} \\
& + \frac{1}{2}s^2\{w(t) - 2w(t-1) + w(t-2)\}
\end{aligned}$$

(equation (5) in Phillips (1974)). The error $w(t-s) - \hat{w}(t-s)$ is of $O(h^3)$ as the sampling interval $h \rightarrow 0$, so that this approximation will not affect the misspecification bias in the discrete approximation.

Note also that the additive error on (70) is just a single integral of the continuous time disturbance function $\zeta_1(s)$. If this is specified to be a pure noise process as in Bergstrom and Wymer (1976), the disturbance on (70) is a serially uncorrelated process. However, it is important to point out that those equations such as (62) which refer to stock variables in the model will be integrated over the interval $(t-1, t)$ rather than $\left(t - \frac{3}{2}, t - \frac{1}{2}\right)$. The disturbances on the resulting discrete approximations will also be serially uncorrelated. But, in view of the overlap between the intervals $(t-2, t-1)$ and $\left(t - \frac{3}{2}, t - \frac{1}{2}\right)$, it is clear that there will be, in general, across equation disturbance serial correlation. This should be taken into account in any systems estimation of the model. Of course, the assumption of pure noise disturbances in (55) is already a strong assumption and one we should wish to test in empirical applications.

a.6. Trajectory Fitting Methods

One obvious procedure for estimating the parameters of a dynamic model such as (55) is to minimize the distance, defined in some specific way, between the solution trajectories as obtained for example in Section 2.4 of the paper and the observed sample path. This is a general approach to estimating dynamic models, both continuous and discrete, and has been in the folklore of econometrics for many years. Klein is often attributed with the original suggestion. It is occasionally mentioned as a possible technique in seminar discussion but rarely discussed in written work. One exception is Fair (1976) who calls the method, in the discrete time

context, full information dynamic (FDYN) estimation. Another is Luterbacher (1979) who is currently using the method to estimate the parameters of a continuous time model of the arms race.

To fix ideas in the present case we let $\hat{x}(t; \theta, x_0)$, $0 < t < T$, be the vector of solution trajectories conditional on a given parameter vector θ and set of initial values x_0 . Assuming $x(t)$ is observable at discrete points in time, we then minimize the determinant of the residual covariance matrix or

$$(71) \quad \ln \det \left\{ T^{-1} \sum_{t=1}^T (\hat{x}(t; \theta, x_0) - x_t)(\hat{x}(t; \theta, x_0) - x_t)' \right\}$$

with respect to the parameter vector θ and the initial value vector x_0 . In cases where, as with flow variables, the observed sample paths x_t ($t = 1, \dots, T$) represent integrals such as (63), we may replace the appropriate elements, say \hat{x}_{it} , of $\hat{x}(t; \theta, x_0)$ by $\hat{x}_{it} = \int_{t-1}^t \hat{x}(s; \theta, x_0) ds$ by a numerical integration of the solution trajectory before use in the optimization routine to minimize (71). Thus, the procedure is quite flexible in enabling us to convert the solution trajectories into a form where they may most appropriately confront the actual data.

An insight into the properties of estimates obtained by fitting the solution trajectories in this way can be gained by examining the implications of this procedure in a linear model such as

$$(72) \quad Dx(t) = A(\theta)x(t) + \zeta(t)$$

where the eigenvalues of the coefficient matrix A are taken to lie in the left half plane for stability. The solution of (72) in mean square from

initial condition $x(0)$ at $t = 0$ is

$$(73) \quad x(t) = e^{tA}x(0) + \int_0^t e^{sA}\zeta(t-s)ds \\ = e^{tA}x(0) + \xi(t), \quad \text{say.}$$

The solution trajectories of the deterministic system conditional on θ and x_0 are

$$\hat{x}(t; \theta, x_0) = e^{tA}x_0$$

and the procedure is now to minimize $\log \det V_T$, as in (71), where now

$$V_T = T^{-1} \sum_{t=1}^T (e^{tA}x_0 - x_t)(e^{tA}x_0 - x_t)'$$

We use the notation x_t here to represent observed data at discrete in time and ignore the additional complexity of flow variables. The vector x_0 is really a vector of nuisance parameters. It is tempting to set $x_0 = x(0)$ the actual observed value of $x(t)$ at $t = 0$. If we do so, then it is easier to analyze the statistical properties of the resulting estimates. In fact, as the following result shows for the scalar model

$$(74) \quad Dx(t) = ax(t) + \varepsilon(t); \quad a < 0, \quad x(0) = \text{constant}$$

the estimates obtained by this procedure are not consistent.

Proposition

The estimator \hat{a} of a in the model (74) obtained by minimising $V_T = T^{-1} \sum_{t=1}^T (e^{ta}x(0) - x(t))^2$ with respect to a cannot be consistent.

Proof

This is based on Example 1 in Malinvaud (1969, p. 957). We suppose the opposite is true, viz that \hat{a} does not converge in probability to a . We note that \hat{a} satisfies the condition

$$\sum_{t=1}^T t e^{\hat{a}t} x(0) (e^{\hat{a}t} x(0) - x(t)) = 0$$

which we rewrite in the form

$$(75) \quad \sum_{t=1}^T t e^{at} x(0) \xi(t) + \sum_{t=1}^T t (e^{\hat{a}t} - e^{at}) x(0) \xi(t) \\ + \sum_{t=1}^T t e^{\hat{a}t} x(0)^2 (e^{at} - e^{\hat{a}t}) = 0.$$

Now, if \hat{a} converges in probability to a , it follows that the second and third terms of (75) converge in probability to zero. On the other hand, the first term of (75) has a limiting distribution with a finite non-zero variance. To see this we note that the variance of $\sum_{t=1}^T t e^{at} \xi(t)$ is given by

$$(76) \quad \sum_{t=1}^T \sum_{s=1}^T t s e^{at} e^{as} \int_0^{\min(t,s)} e^{2ra} \sigma^2 dr$$

where $\sigma^2 = \text{var}(\int_{t-1}^t \xi(r) dr)$. Now (76) is a double series of positive terms that is majorised by

$$(77) \quad \sigma^2 \int_0^{\infty} e^{2ra} dr \sum_{t=1}^T \sum_{s=1}^T t s e^{at} e^{as} = \frac{\sigma^2}{-2a} \sum_{t=1}^T \sum_{s=1}^T t s e^{at} e^{as},$$

and, as $T \rightarrow \infty$, (77) converges to

$$\frac{\sigma^2}{-2a} \left\{ \frac{1}{1-e^a} + \frac{e^a}{(1-e^a)^2} \right\}^2$$

so it follows by the comparison test that (76) also converges and the limit is positive since it is a series of positive terms. Hence, the left side of (75) has a limit distribution with positive variance as $T \rightarrow \infty$ and this contradicts the supposition that \hat{a} converges in probability to the constant a . It follows that \hat{a} cannot be consistent.

The interpretation of this result is that the procedure of fitting a trajectory from a fixed initial value neglects the important information about the parameter a (or strictly e^a) in neighbouring observations. Hence, as $T \rightarrow \infty$ this method does not give enough leverage from the observations to yield consistent estimates. It is formally equivalent to running a non-linear regression on the model

$$x(t) = e^{at} x(0) + \xi(t)$$

to estimate a . This is analagous to the example referred to in Malinvaud (1969), the difference being that $\xi(t)$ is by definition serially dependent whereas in Malinvaud's example $\xi(t)$ is white noise.

To overcome this difficulty the obvious procedure is to fit the trajectory using the latest observation as initial value. In our general notation we can then let $\hat{x}(t; \theta, x_{t-1})$ be the vector of solution trajectories conditional on a given parameter vector θ and the latest vector of observations x_{t-1} as initial values. We would then minimise the determinant of the residual covariance matrix or

$$\ln \det \left[T^{-1} \sum_{t=1}^T \{ \hat{x}(t; \theta, x_{t-1}) - x_t \} \{ \hat{x}(t; \theta, x_{t-1}) - x_t \}' \right]$$

with respect to the parameter vector θ . In the context of the linear model (72),

$$\hat{x}(t; \theta, x_{t-1}) = e^{A} x_{t-1}$$

and the procedure is to minimise $\ln \det V_T$ where now

$$V_T = T^{-1} \sum_{t=1}^T (e^{A} x_{t-1} - x_t) (e^{A} x_{t-1} - x_t)'$$

This procedure is then equivalent to quasi-maximum likelihood applied to the exact discrete model

$$(78) \quad x(t) = e^{A} x(t-1) + \int_{t-1}^t e^{sA} \zeta(t-s) ds$$

corresponding to (72) (Phillips (1972)), at least when $x(0)$ is a fixed constant, and asymptotically equivalent when the real initial conditions are in the infinite past.

Since the latter method is known to produce consistent and asymptotically efficient estimators of the underlying parameters under fairly general conditions, this provides some basis for recommending the use of this type of trajectory fitting method in more general models than (72). The problems to be faced in analyzing the properties of the procedure in more general models arise mainly from the fact that the dependence of the process on its initial conditions, disturbances and, where present, exogenous processes is more complicated than (73) and (78). These complications will be further increased by any integration necessary to convert the trajectories into flow data format and by any interpolation needed for the exogenous series in the computation of the trajectories. Nevertheless, in view of the close link between the trajectory fitting method and the use of the exact discrete model for estimation in linear models, we feel this method is worthy of consideration. On the computational side, all that is required is an accurate solution algorithm to compute the tra-

jectories appropriately tailored into a suitable optimization routine. Luterbacher (1979) and Wymer (1980) already have such software developed and working for small models. For, the model of the present paper, solution trajectories covering 120 periods for 11 variables and using nine exogenous variables were computed in seven seconds of CPU time on a 1906A computer. This makes the estimation of the complete system of equations (1) - (11) almost feasible (i.e., within half an hour of CPU time) on the same machine.