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Abstract

The recent ellipsoidal method for solving linear programs due to Khachian and Shor is shown to process linear complementarity problems with positive semidefinite matrix. Suitable modifications of all lemmas are presented and it is shown that the algorithm operates in polynomial time of the same order as that required for linear programming. Thus quadratic programming problems are solvable in polynomial time.

Key Words: polynomial algorithm, linear complementarity problem, quadratic programming, computational complexity.

One of the most studied problems in operations research is the linear complementarity problem (LCP) which may be stated as follows. Given an $n \times n$ matrix M and a vector $q \in R^n$, find $z \in R^n$ that satisfies the following conditions:

$$\text{LCP} \qquad \begin{aligned} Mz + q &\geqq 0 \\ z &\geqq 0 \\ z^T(Mz+q) &\leqq 0 \,. \end{aligned}$$

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It has been shown that LCP is the fundamental mathematical problem associated with linear programming, bimatrix games, quadratic programming and a number of important problems arising in engineering science. See [2], [4], [5], [9] and the references cited there.

In this paper, we extend some results of [8] (as elucidated in [6]) and provide a polynomial algorithm for computing a solution to LCP, or verifying that none exists, in the case when M is positive semi-definite. This case is of particular importance because it provides a polynomial algorithm for solving convex quadratic programs. Consider the problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}x^T E x + c^T x \\ \text{subject to} \quad & Ax + b \geqq 0 \\ & x \geqq 0 \end{aligned}$$

where E is a positive semi-definite matrix. The associated Kuhn-Tucker conditions are:

$$\begin{aligned} Ex + c - A^T y &\geqq 0 \\ Ax + b &\geqq 0 \\ x &\geqq 0 \\ y &\geqq 0 \\ x^T(Ex+c) + b^T y &= 0 \; . \end{aligned}$$

This is a LCP with $z = \begin{bmatrix} x \\ y \end{bmatrix}$, $q = \begin{bmatrix} c \\ b \end{bmatrix}$ and

$$M = \begin{bmatrix} E & -A^T \\ A & 0 \end{bmatrix}$$

and it is easy to verify that M is positive semi-definite if and only if E is positive semi-definite. We remark that the general quadratic programming problem and the general LCP are NP complete (see [11], [12]). Thus, a polynomial algorithm for an important class of subproblems is very desirable.

Processing LCP by the Shor-Khachian method requires that we solve a system of convex non-linear inequalities of the form:

$$(1) \qquad \begin{aligned} x^T(Mx+q) &< b_0 \\ Ax &< b \end{aligned}$$

where A is a $4n \times n$ matrix, b is a $4n \times 1$ vector and b_0 is a positive scalar. System (1) represents a sufficiently small relaxation of the inequalities in LCP. Specifically,

$$b_0 = 2^{-3L}, \quad b = \begin{bmatrix} q + 2^{-4L}e \\ 2^{-4L}e \\ 2^{L}e - q \\ 2^{L}e \end{bmatrix}, \quad A = \begin{bmatrix} -M \\ -I \\ M \\ I \end{bmatrix}$$

where e is an n-vector of ones and $L = \sum_i \sum_j \log_2(|m_{ij}|+1) + \sum_j \log_2(|q_j|+1) + \log n + 1$ is the space needed to state the problem. As in [6], we outline the algorithm as follows. Define a sequence $\{x_n\}_{n=0}^{\infty}$ of vectors in R^n and a sequence $\{A_n\}_{n=0}^{\infty}$ of matrices recursively. Let $x_0 = 0$ and $A_0 = 2^k I$ where k is a sufficiently large integer so that the volume of an ellipsoid determined by A_0 and centered at the origin contains a certain part of the feasible region. As in [6], it suffices to let $k = 2^L$. Now assume (x_K, A_k) is defined and check if x_K is a solution of (1). If so, stop. If not pick any inequality which is violated and set

$$x_{k+1} = x_K - \frac{1}{n+1} \frac{A_K a_i}{\sqrt{a_i^T A_K a_i}}$$

$$A_{k+1} = \frac{n^2}{n^2 - 1}\left[A_K - \frac{2}{n+1} \frac{(A_K a_i)(A_K a_i)^T}{a_i^T A_K a_i}\right]$$

where a_i is the i^{th} row of the matrix A in (1) if $i = 1, \ldots, 4n$ and $a_i = 2Mx_K + q$ if $i = 0$. A sequence of ellipsoids of geometrically decreasing volume is generated each containing the feasible set of the system of inequalities described in (1). It remains to show that the feasible region has sufficiently large volume and includes a solution to LCP if one exists. We state the necessary lemmas and prove them in the appendix.

Lemma 1. If there exists a solution to LCP, then there exists a solution whose coordinates are rational numbers with numerator and denominator less than $2^L/n$.

Lemma 2. A solution to LCP exists if and only if the system (1) has a solution.

Lemma 3. If the region defined by (1) is non-empty, then its volume is at least 2^{-69nL}.

Theorem. The algorithm detects a feasible point in $99n(n+1)L$ steps.

Proof. Let E_K be the current ellipsoid determined by (x_K, A_K) and $\lambda(E_K)$ be its volume. Then we know from [6], that the feasible region of (1) is contained in E_K for all K and has volume at least 2^{-69nL} according to Lemma 3. Furthermore

$$\lambda(E_K) < e^{\frac{-K}{2(n+1)}}\lambda(E_0) .$$

If the algorithm does not terminate in $\bar{K} = 99n(n+1)L$ steps, then

$$\begin{aligned} \lambda(E_{\bar{K}}) &< e^{\frac{-\bar{K}}{2(n+1)}}\lambda(E_0) \\ &= e^{\frac{-99n(n+1)L}{2(n+1)}} \cdot 2^{2nL} \\ &< 2^{-69nL} \end{aligned}$$

which is a contradiction.

It is interesting to note that the order of the algorithm for the positive semi-definite LCP and linear programming is the same. Linear programming might appear to be an easier, special case of quadratic programming but the existence of a non-zero quadratic term does not make the problem any harder.

Several remarks are in order. First, the positive semi-definite case that we have studied is not properly included in that class of LCP's solvable as linear programs (thus solvable by a polynomial algorithm) as studied in [3] and [10].

Second, our results are more general than those in [7] where a different method of proof is used to deal with the positive definite case. Finally, we indicate that our procedure can be used to solve the symmetric dual quadratic programs that are studied in [1].

APPENDIX

Lemma 1. If there exists a solution to the LCP, then there exists a solution whose coordinates are rational numbers with numerator and denominator less than $2^L/n$.

Proof. It is a well-known fact that the LCP has solutions which are vertices of the polyhedron of linear constraints. The lemma follows exactly as in [6 , Lemma 1].

Lemma 2. A solution exists to the LCP if and only if the system

$$(**) \qquad \left.\begin{aligned} x^T(Mx+q) &< 2^{-3L} \\ M_i x + q_i &> -2^{-4L}e \\ x_i &> -2^{-4L}e \\ M_i x + q_i &< 2^L e \\ x_i &< 2^L e \end{aligned}\right\} \; i = 1, \ldots, n$$

has a solution.

Proof. Any vertex solution to the LCP clearly satisfies (**), since Lemma 1 insures that the last two sets of inequalities are satisfied. Let x^0 be a solution to (**). First note that for each $i = 1, \ldots, n$,

$$x_i^0(M_i x^0 + q_i) > -2^{-4L} \cdot 2^L = 2^{-3L}$$

and so from the quadratic inequality

$$x_i^0(M_ix^0+q_i) < 2^{-3L} + \sum_{i=1}^{n} x_i^0(M_ix^0+q_i)$$

$$< 2^{-3L} + n2^{-3L} \leqq 2^{-2L} .$$

Thus $|x_i^0(M_ix^0+q_i)| < 2^{-2L}$, which means that for each $i = 1, \ldots, n$, either $x_i^0 < 2^{-L}$ or $M_ix^0 + q_i < 2^{-L}$. Set

$$a_i = \begin{cases} M_i , & i = 1, \ldots, n \\ (n-i)^{th} \text{ unit vector}, & i = n+1, \ldots, 2 , \end{cases}$$

$$b_i = \begin{cases} -q_i , & i = 1, \ldots, n \\ 0 , & i = n+1, \ldots, 2n \end{cases}$$

and

$$\theta_i(x) = a_ix - b_i . \quad i = 1, \ldots, 2n .$$

We have

Claim. There exists an $x^1 \in R^n$ such that

(1) $\theta_i(x^1) \geq \min(2^{-L}, \theta_i(x^0))$, $i = 1, \ldots, 2n$.

(2) The vectors $\{a_i : \theta_i(x^1) \leqq 2^{-L}\}$ span every other a_i .

The proof of the claim is almost identical to that of [6, Claim 1 of Lemma 1], and we omit it here.

Suppose, then, that x^1 satisfies the claim. Renumber the inequalities so that $\theta_i(x^1) \leqq 2^{-L}$, $i = 1, \ldots, k$, and so that $a_1, \ldots, a_r$ are linearly independent and span $a_{r+1}, \ldots, a_{2n}$, $r \leq k$. Let z be any solution to

$$a_i^T x = b_i , \quad i = 1, \ldots, r .$$

We know that for all $i = 1, \ldots, 2n$

$$a_i = \sum_{j=1}^{r} \lambda_j a_j$$

where $\lambda_j = D_j/D$ and D_j and D are integers of size at most $2^L/n$ and $D > 0$. Now

$$\begin{aligned} D(a_i^T z - b_i) &= \sum_{j=1}^{r} D_j a_i^T z - Db_i \\ &= \sum_{j=1}^{r} D_j b_j - Db_i \\ &= \sum_{j=1}^{r} D_j (a_j^T x^1 - \theta_j(x^1)) - D(a_i^T x^1 - \theta_i(x^1)) \\ &= D \cdot \theta_i(x^1) - \sum_{j=1}^{r} D_j \theta_j(x^1) \\ &\geq -D\, 2^{-3L} - \sum_{j=1}^{r} |D_j| 2^{-L} \\ &> -1 . \end{aligned}$$

Since $\sum_{j=1}^{r} D_j b_j - Db_i$ is integer, then $D(a_i^T z - b_i) \geqq 0$, and so $a_i^T z - b_i \geqq 0$, $i = 1, \ldots, 2n$.

Therefore z satisfies the linear inequalities for LCP. Further, for $i = r+1, \ldots, k$, we have

$$\begin{aligned} D\theta_i(x^1) - \sum_{j=1}^{r} D_j \theta_j(x^1) &\leqq D2^{-L} + \sum_{j=1}^{r} |D_j| 2^{-L} \\ &< 1 . \end{aligned}$$

Here the integer $D(a_i^T z - b_i) \leqq 0$, so that $a_i^T z - b_i = 0$, $i = 1, \ldots, k$. But the choice of x^0 and the claim insure that, for every $i = 1, \ldots, n$, at least one of z_i and $M_i z + q_i$ is 0, and hence z satisfies the complementary portion of the LCP. This completes the lemma.

Lemma 3. If the region defined by (**) is non-empty, then its volume is at least 2^{-69nL} .

Proof. By Lemma 2, if (**) is non-empty, then it contains a vertex solution u_0 to the LCP. Thus u_0 has coordinates $u_{0j} = D_{0j}/D_0$, where D_{0j} and D_0 are integers at most 2^L . Further, u_0 is in the interior of

$$\text{conv}\{v_0, \ldots, v_n\}$$

where $v_0, \ldots, v_n$ are vertices of the polyhedron defined by

$$-2^{-4L} \leqq M_i x + q_i \leqq 2^L$$

$$-2^{-4L} \leqq x_i \leqq 2^L .$$

By multiplying through by 2^{4L} and applying Cramer's rule, we get that the v_i have coordinates of the form

$$v_{ij} = \frac{2^{4(n-1)L} D_{ij}}{2^{4nL} D_i} = \frac{D_{ij}}{2^{4L} D_i}$$

where D_i is the determinant of a matrix of original coefficients, and D_{ij} is a matrix of original coefficients plus the right hand side column. Thus $2^{4L} D_i < 2^{5L}$ and $D_{ij} < 2^{6L}$. Further the vectors $v_1 - u_0, \ldots, v_n - u_0$ are linearly independent. For $i = 1, \ldots, n$ choose

λ_i^* to be the maximum λ_i satisfying

$$[u_0 + \lambda_i(v_i - u_0)]^T\{M[u_0 + \lambda_i(v_i - u_0)] + q\} \leq 2^{-2L} \ , \quad 0 \leq \lambda_i \leq 1 \ .$$

Then $\lambda_i^* > 0$, since u_0 satisfies the quadratic constraint with equality. In fact, either $\lambda_i^* = 1$ or λ_i^* is the unique positive root to the quadratic solved at equality. In particular, if we set

$$A = (v_i - u_0)^T M(v_i - u_0)$$

$$B = [u_0^T(M + M^T) + q^T](v_i - u_0)$$

$$C = (u_0^T M + q^T)u_0^T - 2^{-3L} \ .$$

Then λ_i^* satisfies

$$A(\lambda_i^*)^2 + B\lambda_i^* + C = 0$$

and so

$$\lambda_i^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \ .$$

Multiplying top and bottom by common denominator $D_0^2 \cdot D_i^2 \cdot 2^{3L}$, we can assume that A, B , and C are integers at most

$$D_0^2 \cdot D_i^2 \cdot 2^{3L} \cdot 2^L \cdot (\max_j D_{ij})^2 \leq 2^{28L} \ .$$

Using the inequality

$$|a - \sqrt{b}| \geq \frac{|a^2 - b|}{|a| + \sqrt{b}}$$

when $b > 0$, we obtain

$$\lambda_i^* > [2A(|B| + \sqrt{B^2 - 4AC})]^{-1}$$
$$> 2^{-57L} .$$

Now the points

$$\hat{v}_0 = u_0 , \quad \hat{v}_1 = u_0 + 2^{-57L}(v_1 - u_0), \ \ldots, \ \hat{v}_n = u_0 + 2^{-57}(v_n - u_0)$$

are affinely independent, and so the simplex

$$\Delta = \operatorname{conv}\{\hat{v}_0, \ldots, \hat{v}_n\}$$

is non-empty and contained entirely in the region defined by (**). The volume of this simplex is

$$\lambda(\Delta) = \frac{1}{n!}\det(\hat{v}_0, \ldots, \hat{v}_n)$$

$$= \frac{1}{n!} \frac{1}{|D_0|} \frac{2^{-57L}}{2^4|D_1 D_0|} \cdots \frac{2^{-57L}}{2^4|D_n D_0|} \det\begin{pmatrix} 2^L & D_1 & \cdots & D_n \\ \hat{v}_0' & \hat{v}_1' & \cdots & \hat{v}_n' \end{pmatrix}$$

where the matrix on the right has integer coefficients. Thus

$$\lambda(\Delta) \geq \frac{1}{n!} \frac{2^{-57nL}}{2^{(n-1)L+4nL+5nL}} \geq 2^{-69nL}$$

and this completes Lemma 3.

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