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AN EXAMPLE OF A MULTI-OBJECT AUCTION GAME

Richard Engelbrecht-Wiggans

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# AN EXAMPLE OF A MULTI-OBJECT AUCTION GAME\*

by

Richard Engelbrecht-Wiggans\*\*

## Abstract

Multi-object auctions are traditionally analyzed as if they were a number of simultaneous independent single object auctions. Such an approximation may be very crude if bidders have budget restrictions, capacity constraints, or, in general, have non-linear utility functions. This paper presents a very simple multi-object auction for which explicit equilibrium strategies can be calculated; these equilibrium strategies have several qualitative characteristics arising from the multi-object nature of the example and therefore not present in typical single object auctions.

## Introduction

Auctions are a common market mechanism. Public agencies, private institutions, and individuals alike often procure, sell, or allocate goods, services and resources through auctions. Each year there are many billions of dollars of transactions in auctions.

A typical buyer or seller is involved with the simultaneous auctioning of several items. Offshore oil lease sales, treasury bond auctions,

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the letting of defense contracts, and the procurement of supplies are a few examples of multi-object auctions. In a similar vein, an individual may participate in several unrelated but simultaneous auctions for different goods and services.

It is traditional to treat simultaneous and multi-object auctions as multiple simultaneous independent auctions; indeed, of approximately 350 papers listed by Stark and Rothkopf (1977) in their bibliography of research into auctions and competitive bidding, fewer than a dozen are explicitly concerned with multi-object auction models. However, such an approach may be inappropriate if the bidders have non-linear utility functions such as might, for example, be associated with budget and capacity constraints. In such cases, the value of "winning" a particular object depends on what other objects are also won. Thus, in general, the value for a set of objects is not simply the sum of the values of the individual objects.

It is proposed to study the effects of non-additivity on the equilibrium bidding strategies and on the efficiency of auctions. A very simple example of a multi-object auction game with quite severe capacity constraints is examined. Equilibrium strategies are calculated for this example.

The equilibrium strategies illustrate several possible aspects of multi-object auctions. The final allocation resulting from simultaneous independent single object auctions may, on the average, have only a small fraction of the maximum possible social value, where social value is defined as the sum of all bidders' profits and all sellers' revenues; the social value is maximized by any Pareto optimal allocation of the objects. Under equilibrium bidding in simultaneous independent single object auctions, bidders may bid more aggressively for some objects than for other objects of equal value; a bidder "wants" a certain amount of objects, but is willing

to take additional objects if they come at bargain prices. Thus, it may not be appropriate to analyze the bidding on each of a number of simultaneously auctioned objects independent of the remaining objects.

#### Example of Multi-Object Auction

There are  $n$  husband-wife couples each in the market for a used dresser. Each couple wants only one dresser; each couple is willing to pay \$100 for any one dresser and considers additional dressers to be worthless. Next weekend, there will be garage sales at  $n$  different locations; at each there will be one dresser for sale. Assume that the dresser will be sold according to a common progressive auction.

How should a couple go about shopping for a dresser? It could go to one randomly selected garage sale, participate in the auction there, hope to win the dresser...but take the chance of returning home without a dresser. Alternatively, the couple could split and participate in two (or, with the aid of children or other representatives, in more than two) garage sales, thereby hoping to increase the chance of coming home with at least one dresser...but taking the chance of coming home with more than one, all but the first of which is considered worthless. Although the capacity constraints need not be quite as sharp, analogous questions apply to individuals bidding for oil leases and government contracts.

#### Equilibrium Strategies

If each couple goes to a randomly selected sale, then the Nash equilibrium strategy is for each couple to start bidding as low as possible (say, at zero) and be willing to raise their bid all the way up to \$100. Thus, a particular dresser may remain unsold (if no one shows up at that

sale), be sold for an arbitrarily small amount (if exactly one couple shows up), or be sold for \$100 (if more than one couple shows up).

Alternatively, let each husband-wife couple split and go to a randomly chosen pair of sales, where the husband has instructions to bid up to some amount  $\$100p$  at his auction and the wife has instructions to bid up to some amount  $\$100q$  at her auction. In the limit as the number of couples (which is equal to the number of garage sales) becomes large, the Nash equilibrium strategy is for  $p$  to be chosen according to the probability distribution function

$$F(x) = \Pr(p \leq x) = -1 - \ln(1-x) \quad \text{for } 1 - 1/e \leq x \leq 1 - 1/e^2,$$

and to let  $q$  be determined by

$$q = 1 - 1/(e^2(1-p)) \quad (\text{for } p \text{ as restricted above, } 0 \leq q \leq 1 - 1/e).$$

(That this is indeed a symmetric equilibrium strategy may be verified through the following steps: 1. Observe that the joint range of  $p$  and  $q$  is the interval from zero to  $1 - 1/e^2$ ; 2. Since ties will occur with probability zero, letting  $p$  be  $1 - 1/e^2$  is strategically equivalent to any larger value for  $p$ ; 3. For any fixed value of one bid, the second should be equal to \$100 times the probability that the first bid will not win an object; 4. If the first bid is in the range associated with  $p$  then the second bid will be consistent with the specified  $q$ ; 5. If the first bid is the smaller, then by using the probability distribution implicitly determined by the specified relation between  $p$  and  $q$ , the  $p$  associated with the second bid must satisfy the specified relation between  $p$  and  $q$ .)

Notice that in this case, there is no longer a "pure" Nash equilibrium

strategy; the equilibrium strategy requires randomization. It is conjectured that the above strategy is the unique equilibrium strategy in this case of the example. Again, some dressers may remain unsold, some sold for a pittance, and some sold for a substantially larger amount; a couple may end up going home with no dressers, exactly one dresser, or perhaps more than one dresser.

### Efficiency of the Allocation

A number of efficiency measures may be considered for auctions. For illustrative purposes, we will use "social value"; the social value being defined as the sum of all buyers' profits and all sellers' revenues. The social value is maximized by a Pareto optimal allocation; thus, social profit is a measure of how close the allocation is to being Pareto optimal.

In the example, the social value is mathematically equal to \$100 times the number of dressers times the probability that a couple wins at least one dresser. Thus, the Pareto optimal allocation, where each of the couples receives exactly one dresser, has a social value of  $\$100n$ . If each couple goes to a randomly chosen sale, then under the aforementioned equilibrium strategy, the probability that a couple receives a dresser is about two thirds (more precisely, the probability goes to  $1 - 1/e$  as  $n$  becomes large). The efficiency of this alternative is therefore approximately two thirds. With each couple participating in two randomly selected sales, the probability of a couple (between the two of them) winning at least one dresser increases to about four fifths (or, more precisely, tends to  $1 - 2/e^2$  in the limit). Not only is the second case more efficient, but (ignoring any additional cost of having the couple participate in two auctions rather than just one) both the expected profits and the average selling price of

a dresser are slightly larger than in the first case.

The example suggests that similar inefficiencies may exist with respect to other measures and in rather general multi-object auctions conducted as independent simultaneous single object auctions. Although the example assumes that each bidder knows the true value of each object and that each object is sold at a price just barely greater than the second highest bid on the object, it seems unlikely that these assumptions are the source of the inefficiencies. Indeed, there is an active after-market for offshore oil leases; a Congressional study (1976) notes that in OCS sale #40, Conoco apparently won substantially more leases than desired and promptly resold a fraction of them to Gulf.

#### Strategic Variance

The vast majority of published research considers only single object auction models, and apparently, at least implicitly, assumes that if an individual is participating in several simultaneous but independent identical auctions then he will use the same bidding strategy in each auction. The equilibrium strategies for the example indicate that this assumption is not in general correct. The existence of capacity constraints or budget restrictions may result in different strategies for different auctions even though the objects sold in the different auctions are identical.

In this particular example, the equilibrium strategies for the case of husband and wife attending separate auctions have a simple interpretation. Each couple desires a dresser, and thus bids aggressively on one dresser. However, to cover themselves in case they do not win with their aggressive bid, they also submit a second, less aggressive bid. By bidding on two objects, the chances of winning at least one are improved; if both the couple



receives two dressers, then the second one is obtained at a relatively low price.

Since everyone knows the value of each object precisely and since all the objects have the same value, there should be no variation among bids for any uncertainty or variation in values. Some variation in bids arises from the randomization necessary to obtain an equilibrium strategy. However, in addition to such variation, there is an additional "strategic variance" among bids due to the different levels of aggression with which one might bid; wives will tend to win objects at lower prices than husbands because wives submit only to bargain seeking bids to cover for the possibility that the husband comes home empty handed.

If one adopts a suggestion of Vickrey (1961) and models the progressive auctions used in this example as sealed bid auctions in which the object is sold to a highest bidder at a price equal to the second highest bid, then there are two distinct classes of bids. There are bids ranging from zero to  $1 - 1/e$  associated with the  $q$ 's and bids ranging from  $1 - 1/e$  to  $1 - 1/e^2$  associated with the  $p$ 's. Thus, there is considerably more variation among the bids than can be simply accounted for by the randomization necessary to obtain an equilibrium strategy.

One might view the problem from the sellers' side. Suppose that they are unaware that the buyers consider all dressers equally valuable and that all the auctions are actually conducted as second bid price sealed auctions. Any seller receiving at least two bids could plot the distribution of the sizes of bids received. To achieve a common scale, each seller might divide all bids on his dresser by the average bid on his dresser; the seller, unaware of the relative values of the dresser, might consider the average bid an appropriate indication. Finally, the sellers might pool

their resulting data to obtain a composite distribution (with an undetermined scale factor) of what fraction the typical bid is of the estimated value. The average composite distribution appears as Figure 1, which also contains the graph of a lognormal density function with mean and variance (of the underlying normal distribution) of zero and 0.09 respectively; the similarity between the two distributions might be of interest to anyone contemplating using the observation that bid fractions tend to be lognormally distributed to infer that true value (or bidders' estimates) are lognormally distributed!

Note that although the example is described in terms of identical couples, each using a mixed equilibrium strategy, it could be modified to an example with different types of bidders each using a pure equilibrium strategy. In particular, a variety of combinations of  $p$  and  $q$  are possible;  $p$  and  $q$  may be equal,  $q$  may be zero and  $p$  be equal to  $1 - 1/e^2$ , or  $p$  and  $q$  may each have an intermediate value. The different cases might correspond to couples with different bidding philosophies (perhaps arising from secondary considerations). Some couple may desire to have  $p$  and  $q$  relatively equal (and therefore both result in moderately competitive bids), whereas others might prefer the more extreme cases (in which one bid is very competitive and the other is much less so). If the mixture of couples is that specified by the probability distribution function on  $p$ , then a couple's pure equilibrium strategy would be for them to bid \$100 times their  $p$  and \$100 times the corresponding  $q$ . Thus, it is possible to construct an asymmetric auction game for the distribution of bids is identical to that of the symmetric example.

### Conclusion

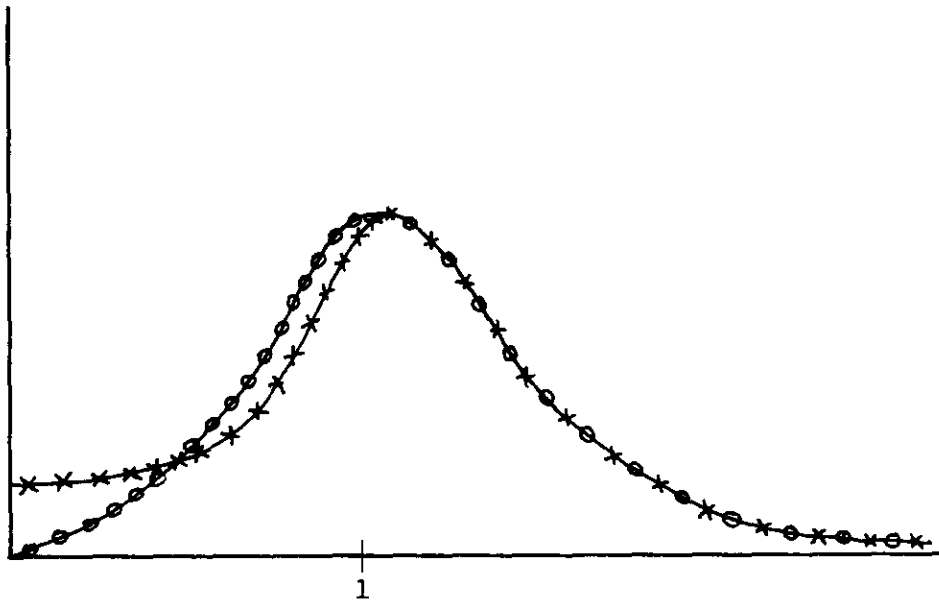
In this paper, several aspects of auctions unique to multi-object auctions have been examined by considering a specific example. It appears that conducting multi-object auctions as independent simultaneous single object auctions may in general be quite inefficient. In addition, analyzing simultaneous independent single object auctions as symmetric replicas of one single object auction ignores the possibility that individuals may bid differently in different auctions; even if identical objects are for sale in different auctions, different bids can arise from bidders considering their capacity constraints, budget restrictions, or other non-linearities in their utility functions. Although the example studied is quite specific and uses second bid price sealed auction mechanisms, the phenomena discussed appear to be somewhat independent of the specific example and should be of concern whenever bidders have non-linear utilities and more than one object is being sold in simultaneous independent auctions.

If the bidders are not participating in independent auctions (e.g., if husband and wife couples may communicate via telephone), then a more efficient allocation might result. Alternative auction methods, however, may be more costly to implement. It appears that further research into multi-object auctions and alternative schemes is warranted.

## REFERENCES

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FIGURE 1



—x— average distribution of the ratio between a bid and the average bid on the corresponding object.

—●— lognormal distribution with a mean = 0 and variance = 0.09 for the underlying normal distribution.