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PROBABILISTIC VALUES FOR GAMES

Robert James Weber

September, 1978

PROBABILISTIC VALUES FOR GAMES*

by

Robert James Weber

Yale University

1. Introduction

Much attention has been given to methods for measuring the "value" of playing a particular role in an n -person game. The study of various values is motivated by several considerations. One is to determine an equitable distribution of the wealth available to the players through their participation in the game. Another is to help an individual compare his prospects from participation in several games. A study of equitable distributions may shed light upon a player's prospects. However, a study of individual prospects need not yield any information concerning the relative fairness of various distributions of wealth.

The well-known Shapley value assigns to every n -person game an n -vector of payoffs. Since this value serves as a method for determining equitable distributions, it is natural that a defining property of the Shapley value is its "efficiency" (or "Pareto optimality"); that is, the sum of

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the individual payoffs is constrained to equal the payoff achieved through the cooperation of all of the players. However, when the players of a game individually assess their positions in the game, there is no reason to suppose that these assessments (which may depend on subjective or private information) will be jointly efficient. Indeed, conservative assessments may combine into a sub-efficient vector, while optimistic assessments may be super-efficient.

This paper presents an axiomatic development of values for games involving a fixed finite set of players. Our results will center around the class of "probabilistic" values, which are defined the next section. Since this class of values includes both the Shapley value and the also-familiar Banzhaf value, our work provides a suitable context for further study of both.

2. Definitions and Notation

For our purposes, we fix a particular set $N = \{1, 2, \dots, n\}$ of players. The collection of coalitions (subsets) in N is denoted by 2^N . A game on N is a real-valued function $v: 2^N \rightarrow \mathbb{R}$ which assigns a "worth" to each coalition, and which satisfies $v(\emptyset) = 0$. Let \mathcal{G} be the collection of all games on N (note that \mathcal{G} is a $(2^n - 1)$ -dimensional vector space), and let v be any game in \mathcal{G} . The game v is monotonic if $v(S) \geq v(T)$ for all $S \supset T$; v is superadditive if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. The class of all monotonic games is denoted by \mathcal{M} , and the class of all superadditive games by \mathcal{S} . For future reference, note that \mathcal{M} and \mathcal{S} are cones in \mathcal{G} ; that is, each is closed under addition, and under multiplication by nonnegative real numbers. Also note that neither

class contains the other.

(The zero-normalization of a game v is the game $v_{(z)}$, defined for all $T \subset N$ by $v_{(z)}(T) = v(T) - \sum_{i \in T} v(i)$. The game v is

zero-monotonic if $v_{(z)}$ is monotonic. The class of all zero-monotonic games is denoted by \mathcal{Z} . Every super-additive game is zero-monotonic; however, neither \mathcal{M} nor \mathcal{Z} contains the other. In this paper, a number of results are obtained for the class \mathcal{S} of super-additive games. All of these results can also be obtained, mutatis mutandis, for the class \mathcal{Z} .)

If the game v takes only the values 0 and 1, then v is simple. If $v(S) = 1$, then S is a winning coalition; otherwise S is a losing coalition. \mathcal{S}^* , \mathcal{M}^* , and \mathcal{S}^* denote, respectively, the class of all simple games on N , those which are monotonic, and those which are superadditive. For simple games, note that superadditivity implies monotonicity; hence, $\mathcal{M}^* \supset \mathcal{S}^*$. (Some authors prefer to restrict the term "simple game" to elements of \mathcal{M}^* ; the more general games \mathcal{S}^* are then called "0-1 games.")

Two special types of games will play an important role in our work. For any nonempty coalition T , let v_T be defined by $v_T(S) = 1$ if $S \supset T$, and 0 otherwise. Also, let \hat{v}_T be defined by $\hat{v}_T(S) = 1$ if $S \supsetneq T$, and 0 otherwise. Let $\mathcal{C} = \{v_T: \emptyset \neq T \subset N\}$, and $\hat{\mathcal{C}} = \{\hat{v}_T: \emptyset \neq T \subset N\}$; any game in \mathcal{C} is a carrier game. Observe that every game in \mathcal{C} or $\hat{\mathcal{C}}$ is monotonic, superadditive, and simple. We shall occasionally refer to the game \hat{v}_\emptyset defined by $\hat{v}_\emptyset(S) = 1$ for all nonempty coalitions S . This game is monotonic and simple, but is not superadditive.

For any collection $\mathcal{J} \subset \mathcal{N}$ of games, and for any player $i \in N$, a value for i on \mathcal{J} is a function $\phi_i: \mathcal{J} \rightarrow R$. As we have previously indicated, the value $\phi_i(v)$ of a particular game v represents an assessment by i of his prospects from playing the game. This definition stands somewhat in contrast to the more traditional definition of a "group value" $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ which associates an n -vector with each game. The construction of group values from our individual values will be treated later in this paper.

Recently, Blair [1] and Dubey [3] have discussed a family of values which arise from individual perceptions of the coalition-formation process.

(Earlier discussions of related matters appear in [4] and [7].) Fix a player i , and let $\{p_T^i: T \subset N \setminus i\}$ be a probability distribution over the collection of coalitions not containing i . (Incidentally, notice that we shall often omit the braces when writing one-player coalitions such as $\{i\}$.) A value ϕ_i for i on \mathcal{J} is a probabilistic value if, for every $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)] .$$

Let i view his participation in a game as consisting merely of joining some coalition S , and then receiving as a reward his marginal contribution $v(S \cup i) - v(S)$ to the coalition. If, for each $T \subset N \setminus i$, p_T^i is the (subjective) probability that he joins coalition T , then $\phi_i(v)$ is simply his expected payoff from the game.

Both the Shapley and Banzhaf values are instances of probabilistic values. The Banzhaf value (for an individual player i) arises from the subjective belief that the player is equally likely to join any coalition; that is, $p_T^i = 1/(2^{n-1})$ for all $T \subset N \setminus i$. The Shapley value arises from the belief that the coalition he joins is equally likely to be of any size t ($0 \leq t \leq n - 1$), and that all coalitions of size t are equally likely;

that is, $p_T^i = \frac{1}{n} \cdot \frac{1}{\binom{n-1}{t}} = \frac{t! \cdot (n-t-1)!}{n!}$ for all $T \subset N \setminus i$, where

$t = |T|$.

In the following sections, we shall investigate several reasonable conditions which a value might be expected to satisfy. We will find that the only values which satisfy these conditions are closely related to the probabilistic values.

3. The Linearity and Dummy Axioms

Given a game v , and any constant $c > 0$, consider the game cv defined by $(cv)(S) = c \cdot v(S)$ for all $S \subset N$. It seems reasonable to assume that such a rescaling of the original game would simply rescale a player's assessment of his prospects from playing the game. Similarly, let v and w be games, and consider the game $v + w$ defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$. A rational player, facing the latter game, might well consider his prospective gain to be the sum of his prospective gains from the two original games.

Consider a cone \mathcal{J} of games in \mathcal{H} . A linear function on \mathcal{J} is a function $f: \mathcal{J} \rightarrow \mathbb{R}$ satisfying $f(v + w) = f(v) + f(w)$ and $f(cv) = c \cdot f(v)$ for all $v, w \in \mathcal{J}$ and $c > 0$. Let ϕ_i be a value for i on \mathcal{J} . The preceding comments are reflected in the following criterion.

Linearity Axiom. ϕ_i is a linear function on \mathcal{J} .

Since \mathcal{H} , \mathcal{M} , and \mathcal{S} are all cones in \mathcal{H} , the following theorem applies to a value on any of these domains.

THEOREM 1. Let ϕ_i be a value for i on a cone \mathcal{J} of games. Assume that ϕ_i satisfies the linearity axiom. Then there is a collection of constants $\{a_T: T \subset N\}$ such that for all $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) .$$

Proof. ϕ_i has a unique linear extension to the linear subspace $\mathcal{L} \subset \mathcal{H}$ spanned by \mathcal{J} . This extension can in turn be extended to a linear function ϕ_i^{ext} on all of \mathcal{H} , by defining ϕ_i^{ext} arbitrarily on a basis of the orthogonal complement of \mathcal{L} .

For any nonempty $T \subset N$, define the game w_T by $w_T(S) = 1$ if $S = T$, and 0 otherwise. Then $\{w_T : \emptyset \neq T \subset N\}$ is a basis for \mathcal{H} , and ϕ_i^{ext} is uniquely determined by its values on this basis. Any $v \in \mathcal{H}$ can be written as $v = \sum_{\emptyset \neq T \subset N} v(T) \cdot w_T$; since ϕ_i^{ext} is linear,

$$\phi_i^{\text{ext}}(v) = \sum_{\emptyset \neq T \subset N} v(T) \cdot \phi_i^{\text{ext}}(w_T) .$$

However, ϕ_i is simply the restriction of ϕ_i^{ext} to \mathcal{J} . Therefore, upon taking $a_T = \phi_i^{\text{ext}}(w_T)$ for all nonempty $T \subset N$, and defining a_\emptyset arbitrarily, we obtain the desired result. \square

A player i is a dummy in the game v if $v(S \cup i) = v(S) + v(i)$ for every $S \subset N \setminus i$. This terminology derives from the observation that such a player has no meaningful strategic role in the game; no matter what the situation, he contributes precisely $v(i)$. Therefore, the following criterion seems reasonable. Let ϕ_i be a value for i on a collection \mathcal{J} of games.

Dummy Axiom. If i is a dummy in $v \in \mathcal{J}$, then $\phi_i(v) = v(i)$.

This axiom actually has two aspects. While specifying the prospective gain of a dummy in a game v , it implicitly states that ϕ_i and v are measured in common units, under a common normalization. These aspects are exploited separately in the proof of the following result. Recall that \mathcal{C} denotes the collection of carrier games.

THEOREM 2. Let ϕ_i be a value for i on a collection \mathcal{J} of games, defined by $\phi_i(v) = \sum_{T \subset N} a_T v(T)$ for every $v \in \mathcal{J}$. Assume that ϕ_i satisfies the dummy axiom, and that \mathcal{J} contains \mathcal{C} . Then there is a collection of constants $\{p_T: T \subset N \setminus i\}$ satisfying $\sum_{T \subset N \setminus i} p_T = 1$, such that for every $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] .$$

Proof. First, note that for any nonempty $T \subset N \setminus i$, player i is a dummy in $v_T \in \mathcal{C}$. Therefore, $\phi_i(v_T) = v_T(i) = 0$. It follows that $\phi_i(v_{N \setminus i}) = a_N + a_{N \setminus i} = 0$. For inductive purposes, assume it has been shown that $a_{T \cup i} + a_T = 0$ for every $T \subset N \setminus i$ with $|T| \geq k \geq 2$. (The case $k = n - 1$ has just been established.) Take any fixed $S \subset N \setminus i$ with $|S| = k - 1$. Then

$$\begin{aligned} \phi_i(v_S) &= \sum_{T \supset S} a_T = \left\{ \sum_{\substack{T \subset N \setminus i \\ T \supset S \\ T \neq S}} (a_{T \cup i} + a_T) \right\} + (a_{S \cup i} + a_S) \\ &= a_{S \cup i} + a_S = 0 ; \end{aligned}$$

the next-to-last equality follows from the induction hypothesis, and the last from the dummy axiom.

Therefore, $a_{T \cup i} + a_T = 0$ for all $T \subset N \setminus i$ with $0 < |T| \leq n - 1$. For every such T , define $p_T = a_{T \cup i} = -a_T$. Also, define $p_\emptyset = a_i$. Then for every $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] .$$

Consider $v_i \in \mathcal{C}$. Player i is a dummy in this game; indeed, every player is a dummy in v_i . Therefore, $\phi_i(v_i) = v_i(i) = 1$. But, since $v_i(T \cup i) - v_i(T) = 1$ for every $T \subset N \setminus i$, the expression in the preceding paragraph yields $\phi_i(v_i) = \sum_{T \subset N \setminus i} p_T$. \square

When this theorem is taken in conjunction with the preceding one, we obtain the following result.

THEOREM 3. Let ϕ_i be a value for i on \mathcal{S} , \mathcal{M} , or \mathcal{L} . Assume that ϕ_i satisfies the linearity and dummy axioms. Then there is a collection of constants $\{p_T: T \subset N \setminus i\}$ satisfying $\sum_{T \subset N \setminus i} p_T = 1$, such that for every game v in the domain of ϕ_i ,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)].$$

4. The Monotonicity Axiom

Let v be any monotonic game. A player i , facing the prospect of playing this game, may be uncertain concerning his eventual payoff. However, for every $T \subset N \setminus i$, $v(T \cup i) - v(T) \geq 0$; therefore player i knows, at the least, that his presence will never "hurt" a coalition. This motivates the following criterion. Let ϕ_i be a value for i on a collection \mathcal{J} of games.

Monotonicity Axiom. If $v \in \mathcal{J}$ is monotonic, then $\phi_i(v) \geq 0$.

The following proposition will be of value.

Proposition. Let ϕ_i be a value for i on a collection \mathcal{J} of games.

Assume that there is a collection of constants $\{p_T: T \subset N \setminus i\}$, such that for all $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] .$$

Further assume that \mathcal{J} contains the game \hat{v}_T , for some $T \subset N \setminus i$ (note that T may be empty), and assume that ϕ_i satisfies the monotonicity axiom. Then $p_T \geq 0$.

Proof. The game \hat{v}_T is monotonic. Therefore, $\phi_i(\hat{v}_T) = p_T \geq 0$. \square

The collections of games \mathcal{S} and \mathcal{M} each contain \hat{c} , and also contain \hat{v}_\emptyset . On the other hand, \mathcal{S} contains \hat{c} , but not \hat{v}_\emptyset . Therefore, we have the following theorems.

THEOREM 4. Let ϕ_i be a value for i on \mathcal{S} or \mathcal{M} . Assume that ϕ_i satisfies the linearity, dummy, and monotonicity axioms. Then ϕ_i is a probabilistic value. Furthermore, every probabilistic value on \mathcal{S} or \mathcal{M} satisfies these three axioms.

THEOREM 5. Let ϕ_i be a value for i on \mathcal{S} . Assume that ϕ_i satisfies the linearity, dummy, and monotonicity axioms. Then there is a collection of constants $\{p_T: T \subset N \setminus i\}$ satisfying $\sum_{T \subset N \setminus i} p_T = 1$, and $p_T \geq 0$ for all nonempty $T \subset N \setminus i$, such that for every game $v \in \mathcal{S}$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] .$$

Furthermore, every such value on \mathcal{S} satisfies these three axioms.

In the case of values on \mathcal{B} or \mathcal{M} , we thus have a natural axiomatic characterization of the probabilistic values. However, for values on \mathcal{S} we are unable to rule out the possibility that $p_\emptyset < 0$. This phenomenon is investigated in the next section.

5. Values for Superadditive Games

It is natural to seek an explanation of the preceding results. A value for a class of games yields a relative evaluation of one's prospects from playing the various games. If the class of games is sufficiently rich, the only evaluation functions satisfying certain reasonable criteria are the probabilistic values. Why, if one's consideration is restricted solely to superadditive games, does the class of reasonable evaluation functions broaden in the indicated manner? We shall attempt to provide a rationale.

Consider any particular game v . A player i , faced with the prospect of playing this game, may seek to determine the amount of gain which he is "guaranteed," in the sense that he contributes at least this amount to any coalition which he joins. In the case where v is superadditive, this "floor" to his expectation is precisely $v(i)$, since $v(T \cup i) - v(T) \geq v(i)$ for all $T \subset N \setminus i$ (and since, when $T = \emptyset$, his marginal contribution is exactly $v(i)$). Taking this amount as assured, the player will then strive to achieve as great a reward as he can in the new game $v^{(i)}$ defined by

$$v^{(i)}(S) = \begin{cases} v(S) & \text{if } i \notin S, \\ v(S) - v(i) & \text{otherwise.} \end{cases}$$

(This is the game that he perceives himself to be playing, after having mentally "withdrawn" the amount $v(i)$ from the game.) However, any gain from this new game is uncertain, and depends upon such factors as the bargaining ability of the player. Hence, the two amounts under consideration, $v(i)$ and his gain from playing $v^{(i)}$, are measured respectively in "certain" and "uncertain" units.

Assume that the player's attitude toward risk is such that one unit of uncertain gain is worth γ units of certain gain to him. (Hence, $\gamma < 1$ corresponds to risk-aversion, and $\gamma = 1$ to risk-neutrality.) Further assume that he evaluates his prospects, from any game v with $v(i) = 0$, in terms of a probabilistic value $\phi_i(v)$. Then, his evaluation of any superadditive game v , expressed in units of certain gain, will be

$$\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i).$$

One would expect an aversion to risk to limit a player's options. That such is the case is the impact of the following theorem. Let P be the set of probabilistic values on \mathcal{S} , and for any $\gamma \geq 0$ let $V(\gamma) = \{\xi_i : \xi_i \text{ is a value on } \mathcal{S}, \text{ and for some } \phi_i \in P, \xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i) \text{ for all } v \in \mathcal{S}\}$. This is the set of all evaluation functions on \mathcal{S} arising from the considerations discussed previously, when γ represents player i 's attitude toward uncertain gain.

THEOREM 6. A value ξ_i on \mathcal{S} satisfies the linearity, dummy, and monotonicity axioms if and only if $\xi_i \in V = \bigcup_{\gamma \geq 0} V(\gamma)$. If

$0 \leq \gamma' < \gamma$, then $V(\gamma') \not\subseteq V(\gamma)$. Furthermore, $V(1) = P$.

Proof. Let ξ_i satisfy the indicated axioms on \mathcal{S} . Then ξ_i is associated with a collection $\{p_T^i : T \subset N \setminus i\}$ of constants, as in Theorem 5. Let $\gamma = 1 - p_\emptyset^i \geq 0$. If $\gamma > 0$, define the probability distribution $\{q_T^i : T \subset N \setminus i\}$ by $q_T^i = p_T^i / \gamma$ if $T \neq \emptyset$, and $q_\emptyset^i = 0$; if $\gamma = 0$, take any probability distribution $\{q_T^i\}$. Then, if ϕ_i is the associated probabilistic value, $\xi_i(v) = \phi_i(v^{(i)}) + v(i)$ for all $v \in \mathcal{S}$. Hence, $\xi_i \in V(\gamma) \subset V$. Conversely, it is easily verified that any $\xi_i \in V$ satisfies the axioms on \mathcal{S} . (It is essential to this verification that, for every monotonic $v \in \mathcal{S}$, $v^{(i)}$ is a monotonic game; hence $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i) \geq v(i) \geq 0$.)

If $0 \leq \gamma' < \gamma$, then any $\xi_i \in V(\gamma')$ corresponds to some $\phi'_i \in P$, which is in turn associated with a probability distribution $\{p_T: T \subset N \setminus i\}$. But then, let $\phi_i \in P$ be associated with the probability distribution $\{q_T: T \subset N \setminus i\}$, where $q_T = \frac{\gamma'}{\gamma} \cdot p_T$ for all nonempty $T \subset N \setminus i$, and $q_\emptyset = 1 - \sum_{T \neq \emptyset} q_T$. It follows that $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i)$ for all $v \in \mathcal{S}$, so $\xi_i \in V(\gamma)$. Hence, $V(\gamma') \subset V(\gamma)$.

Consider any probability distribution $\{p_T: T \subset N \setminus i\}$ such that $p_\emptyset = 0$. Then, if ϕ_i is the associated probabilistic value on \mathcal{S} , $\xi_i(v) = \gamma \cdot \phi_i(v^{(i)}) + v(i)$ defines a value $\xi_i \in V(\gamma)$ which is not in $V(\gamma')$ for any $\gamma' < \gamma$. Hence the indicated containment is strict.

Finally, observe that, when $\gamma = 1$, every value ξ_i in $V(\gamma) = V(1)$ is of the form

$$\begin{aligned} \xi_i(v) &= \phi_i(v^{(i)}) + v(i) \\ &= \left\{ \sum_{T \subset N \setminus i} p_T [(v(T \cup i) - v(i) - v(T))] \right\} + v(i) \\ &= \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] \\ &= \phi_i(v) \end{aligned}$$

so $V(1) = P$. \square

This theorem can be viewed in several different ways. One might ask whether the addition of some other natural axiom will lead to the conclusion that $p_\emptyset \geq 0$. For example, it has been suggested by Milnor [6] that it is unreasonable for any player $i \in N$ to hope to attain more than $b_i(v) = \max_{S \subset N \setminus i} [v(S \cup i) - v(S)]$. If we require that, for all $v \in \mathcal{A}$, $\phi_i(v) \leq b_i(v)$, then

$$\phi_i(\hat{v}_{\{i\}}) = \sum_{\emptyset \neq T \subset N \setminus i} p_T = 1 - p_\emptyset \leq b_i(\hat{v}_{\{i\}}) = 1.$$

Hence, $p_\emptyset \geq 0$.

Another point of view is the following. If a player wishes to evaluate his prospects from superadditive games, he can satisfy our criteria of rationality while still basing his evaluation in part on his posture toward risk. However, these same criteria, when

applied to the evaluation of broader classes of games, force the player into a posture of risk-neutrality. It would be of interest to learn of a behavioral justification for this consequence of risk-neutrality,

6. Values for Simple Games

Simple games, particularly those which are monotonic, are often used to represent political games. A value for a player may then indicate the player's perceived political power in various games. Under this interpretation, the dummy and monotonicity axioms remain reasonable. However, the linearity axiom does not seem to apply; indeed, the sum of simple games is generally not simple.

An alternative axiom has been suggested by Dubey [2]. For any games v and w , define $v \vee w$ by $(v \vee w)(S) = \max(v(S), w(S))$ and define $v \wedge w$ by $(v \wedge w)(S) = \min(v(S), w(S))$, for all $S \subset N$. If v and w are simple, then $v \vee w$ and $v \wedge w$ are also simple. A coalition is winning in $v \vee w$ if it wins in either v or w ; it is winning in $v \wedge w$ if it wins in both. Therefore, each coalition wins as often in v and w together as it does in $v \vee w$ and $v \wedge w$ together. Let ϕ_i be a value for i on a collection \mathcal{J} of games.

Transfer Axiom. If v , w , $v \vee w$, and $v \wedge w$ are all in \mathcal{J} , then

$$\phi_i(v) + \phi_i(w) = \phi_i(v \vee w) + \phi_i(v \wedge w).$$

The name of this axiom is motivated by the following observation. The game $v \wedge w$ arises from v when all of the coalitions which win only in v are made losing; $v \vee w$ arises from w when these same coalitions

are made winning. Hence, $v \wedge w$ and $v \vee w$ arise from v and w when winning coalitions are "transferred" from one game to the other.

We require several definitions. Let v be a simple game. A minimal winning coalition in v is a winning coalition with no proper subsets which are also winning; a hole in v is a losing coalition with a winning subset. Note that the monotonic simple games are precisely those without holes.

Let \mathcal{J} be a collection of simple games, and let v be any game in \mathcal{J} . We define two types of operations which can be performed on v . Let T be a minimal winning coalition in v . Define the game v^{-T} by $v^{-T}(S) = v(S)$ for all $S \neq T$, with $v^{-T}(T) = 0$; v^{-T} arises from v by the deletion of a minimal winning coalition. On the other hand, let T be a hole in v , and define the game v^{+T} by $v^{+T}(S) = v(S)$ for all $S \neq T$, with $v^{+T}(T) = 1$; v^{+T} arises from v by the insertion of a (new) winning coalition. The collection \mathcal{J} is closed under deletion and insertion if these operations, applied to any game in \mathcal{J} , give rise only to other games in \mathcal{J} . In particular, \mathcal{S}^* , \mathcal{M}^* , and \mathcal{B}^* are all closed under deletion and insertion.

The following result is an analogue of Theorem 1.

THEOREM 7. Let \mathcal{J} be a collection of simple games which contains \mathcal{C} and is closed under deletion and insertion. Let ϕ_i be a value for i on \mathcal{J} , and assume that $\phi_i(\hat{v}_N) = 0$. * Finally, assume that ϕ_i satisfies the

*Recall that the game \hat{v}_N is defined by $\hat{v}_N(S) = 0$ for all $S \subset N$. This game is contained in every nonempty collection of games which is closed under deletion, and every player in N is a dummy in the game.

transfer axiom. Then there is a collection of constants $\{a_T; T \subset N\}$ such that, for all games $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N} a_T v(T) .$$

Proof. We claim that ϕ_i is determined on all of \mathcal{J} by its values on \mathcal{C} . In order to verify this claim, first consider the collection \mathcal{J}_M of monotonic games in \mathcal{J} . This subcollection of \mathcal{J} is also closed under deletion and insertion, and contains \mathcal{C} . Since $v_N \in \mathcal{C}$, the claim is trivially true for this game. Assume that the claim has been verified for all games in \mathcal{J}_M which have at most k winning coalitions (the only game in \mathcal{J}_M with just one winning coalition is v_N), and let $v \in \mathcal{J}$ be any game with $k + 1$ winning coalitions. Let T be any minimal winning coalition in v , and consider the games v_T , v^{-T} , and $v_T \wedge v^{-T}$. The first is a carrier game, while the latter two are both in \mathcal{J}_M and have no more than k winning coalitions. Since $v_T \vee v^{-T} = v$, we have from the transfer axiom that $\phi_i(v) = \phi_i(v_T) + \phi_i(v^{-T}) - \phi_i(v_T \wedge v^{-T})$. It follows from the induction hypothesis that $\phi_i(v)$ depends only on the values of ϕ_i on \mathcal{C} . This verifies the claim throughout \mathcal{J}_M . (Observe that the game \hat{v}_N requires special treatment; since it has no winning coalitions, it is not covered by the induction.)

Next, assume that the claim holds for all games in \mathcal{J} which have at most k holes (the case $k = 0$ has just been treated), and let $v \in \mathcal{J}$ be a game with $k + 1$ holes. Let T be any hole of maximum cardinality, and consider the games v_T , $v \wedge v_T = \hat{v}_T$, and $v \vee v_T = v^{+T}$. The first of these is in \mathcal{C} , the second is in \mathcal{J}_M , and the third is in \mathcal{J}

and has only k holes. Since $\phi_i(v) = \phi_i(v \vee v_T) + \phi_i(v \wedge v_T) - \phi_i(v_T)$, it follows (by induction) that $\phi_i(v)$ depends only on the values of ϕ_i on \mathcal{C} . This completes the verification of the claim.

We have just seen that ϕ_i is determined by its values on \mathcal{C} . Since* \mathcal{C} is a basis for \mathcal{H} , there is a unique linear function ϕ_i^{lin} on \mathcal{H} which coincides with ϕ_i on \mathcal{C} . This linear function must satisfy the transfer axiom, because $(v \vee w) + (v \wedge w) = v + w$ for all v and w in \mathcal{H} . Therefore, ϕ_i^{lin} and ϕ_i must coincide on \mathcal{J} . Since ϕ_i^{lin} can be expressed in terms of its values on the basis $\{w_T: \emptyset \neq T \subset N\}$ of \mathcal{H} (see the proof of Theorem 1), it follows that ϕ_i has the desired form. \square

We can now invoke Theorem 2 and the proposition concerning monotonicity, in order to obtain analogues of Theorems 4 and 5.

THEOREM 8. Let ϕ_i be a value for i on \mathcal{H}^* or \mathcal{M}^* . Assume that ϕ_i satisfies the transfer, dummy, and monotonicity axioms. Then ϕ_i is a probabilistic value. Furthermore, every probabilistic value on \mathcal{H}^* or \mathcal{M}^* satisfies these three axioms.

THEOREM 9. Let ϕ_i be a value for i on \mathcal{S}^* . Assume that ϕ_i satisfies the transfer, dummy, and monotonicity axioms. Then there is a collection of constants $\{p_T: T \subset N \setminus i\}$ satisfying $\sum_{T \subset N \setminus i} p_T = 1$, and

*Assume that $\sum_{\emptyset \neq S \subset N} c_S v_S = 0$. Then for any nonempty $T \subset N$, $\sum_{\emptyset \neq S \subset T} c_S = 0$.

Solving this system of equations successively for $|T| = 1, 2, \dots, n$ yields $c_T = 0$ for all $T \subset N$. Hence the $2^n - 1$ games v_T are linearly independent in \mathcal{H} .

$p_T \geq 0$ for all nonempty $T \subset N \setminus i$, such that for every game $v \in \mathcal{S}^*$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T [v(T \cup i) - v(T)] .$$

Furthermore, every such value on \mathcal{S}^* satisfies these three axioms.

The discussion of the previous section, interpreting the class of values on \mathcal{S} , applies with equal strength to \mathcal{S}^* .

7. Symmetric Probabilistic Values

A probabilistic value assesses the relative desirability of being a particular player in various games. At times, one might also want to compare the desirability of playing various roles within a particular game. Such comparisons can be facilitated by the use of a collection $\phi = (\phi_1, \dots, \phi_n)$ of values, with $\phi_i(v)$ representing the value of being player i in game v . Such a collection is a group value.

Let $\pi = (\pi(1), \dots, \pi(n))$ be any permutation of N . For any $S \subset N$, define $\pi S = \{\pi(i) : i \in S\}$. The game πv is defined by $(\pi v)(\pi S) = v(S)$ for all $S \subset N$. (πv arises upon the re-labelling of the players $1, \dots, n$ with the labels $\pi(1), \dots, \pi(n)$.) Let \mathcal{J} be a collection of games with the property that, if $v \in \mathcal{J}$, then every $\pi v \in \mathcal{J}$; such a collection is symmetric.

Let $\phi = (\phi_1, \dots, \phi_n)$ be a group value on \mathcal{J} . For the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game, but not on the labels of the players.

Symmetry Axiom. For every $v \in \mathcal{J}$ and every permutation π of N , and for every $i \in N$, $\phi_i(v) = \phi_{\pi(i)}(\pi v)$.

Observe that each of the classes $\mathcal{S}, \mathcal{m}, \mathcal{S}, \mathcal{S}^*, \mathcal{m}^*$, and \mathcal{S}^* contains both \mathcal{C} and $\hat{\mathcal{C}}$; furthermore, each of these classes is symmetric. Therefore, the following theorem applies to values on any of these classes.

THEOREM 10. Let \mathcal{J} be a symmetric collection of games, containing \mathcal{C} and $\hat{\mathcal{C}}$. Let $\phi = (\phi_1, \dots, \phi_n)$ be a group value on \mathcal{J} , such that for each $i \in N$ and $v \in \mathcal{J}$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)] .$$

Assume that ϕ satisfies the symmetry axiom. Then there are constants $\{p_t^i\}_{t=0}^{n-1}$ such that for all $i \in N$ and $T \subset N \setminus i$, $p_T^i = p_{|T|}$.

Proof. For any $i \in N$, let T_1 and T_2 be any two coalitions in $N \setminus i$ satisfying $0 < |T_1| = |T_2| < n - 1$. Consider a permutation π of N , which takes T_1 into T_2 while leaving i fixed. Then $p_{T_1}^i = \phi_i(\hat{v}_{T_1}) = \phi_i(\hat{v}_{T_2}) = p_{T_2}^i$, where the central equality is a consequence of the symmetry axiom.

Next, let i and j be distinct players in N , and let T be a nonempty coalition in $N \setminus \{i, j\}$. Consider the permutation π which interchanges i and j while leaving the remaining players fixed. Then $\pi \hat{v}_T = \hat{v}_T$, and $p_T^i = \phi_i(\hat{v}_T) = \phi_j(\hat{v}_T) = p_T^j$, where the central equality is again a consequence of the symmetry axiom. Combining this with the

previous result, we find that for every $0 < t < n - 1$ there is a p_t such that $p_T^i = p_t$ for every $i \in N$ and $T \subset N \setminus i$ with $|T| = t$.

Again, for distinct players i and j , let π interchange i and j while leaving the remaining players fixed. Then $p_{N \setminus i}^i = \phi_i(v_N) = \phi_j(v_N) = p_{N \setminus j}^j$. Let p_{n-1} be this common value. Then for all $i \in N$, $p_{N \setminus i}^i = p_{n-1}$.

Finally, for each $i \in N$,

$$p_{\emptyset}^i = 1 - \sum_{\substack{T \subset N \setminus i \\ T \neq \emptyset}} p_T^i = 1 - \sum_{t=1}^{n-1} \binom{n-1}{t} p_t ;$$

this last expression is independent of i .

Therefore, $p_{\emptyset}^i = p_{\emptyset}^j$ for all $i, j \in N$. Letting p_0 be this common value completes the proof of the theorem. \square

We shall return to this result later in the paper, when we briefly consider the Shapley value.

8. Efficiency without Symmetry: Random-order Values

Consider a collection $\phi = (\phi_1, \dots, \phi_n)$ of values, all on the domain \mathcal{J} , one for each player in N . Depending on the game v under consideration, the players' assessments, as a group, of their individual prospects may be either optimistic or pessimistic; that is, $\sum_{i \in N} \phi_i(v)$ may be either greater than or less than $v(N)$. However, if the group assessment is neither optimistic nor pessimistic, the payoff vector $\phi(v) = (\phi_1(v), \dots, \phi_n(v))$ may be taken as an equitable distribution of the resources available to the grand coalition N . Therefore, it is of interest to study those collections of values $\phi = (\phi_1, \dots, \phi_n)$ which meet the following criterion.

Efficiency Axiom. For every $v \in \mathcal{J}$, $\sum_{i \in N} \phi_i(v) = v(N)$.

A group value satisfying this axiom is said to be efficient.

Any efficient group value ϕ provides a fair distribution scheme for the games in \mathcal{J} . The following theorem characterizes all such group values.

THEOREM 11. Let $\phi = (\phi_1, \dots, \phi_n)$ be a group value on \mathcal{J} , defined for all $i \in N$ and all $v \in \mathcal{J}$ by $\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)]$. Assume that \mathcal{J} contains \mathcal{C} and $\hat{\mathcal{E}}$. Then ϕ satisfies the efficiency axiom if and only if $\sum_{i \in N} p_{N \setminus i}^i = 1$, and $\sum_{i \in T} p_{T \setminus i}^i = \sum_{j \notin T} p_T^j$ for every nonempty $T \subsetneq N$.

Proof. For any $v \in \mathcal{J}$, let $\phi_N(v) = \sum_{i \in N} \phi_i(v)$. Then

$$\begin{aligned} \phi_N(v) &= \sum_{i \in N} \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)] \\ &= \sum_{T \subset N} v(T) \left(\sum_{i \in T} p_{T \setminus i}^i - \sum_{j \notin T} p_T^j \right). \end{aligned}$$

It is immediately clear that any ϕ which satisfies the conditions of the theorem is efficient; that is, $\phi_N(v) = v(N)$.

For any nonempty $T \subset N$, consider the games v_T and \hat{v}_T . Since $v_T(S) = \hat{v}_T(S)$ for all $S \neq T$, and $v_T(T) = 1$ while $\hat{v}_T(T) = 0$, it follows from the preceding equation that

$$\phi_N(v_T) - \phi_N(\hat{v}_T) = \sum_{i \in T} p_{T \setminus i}^i - \sum_{j \notin T} p_T^j.$$

However, $v_T(N) - \hat{v}_T(N)$ is 1 if $T = N$, and is 0 otherwise. Therefore, if ϕ satisfies the efficiency axiom, then the indicated conditions must also hold. \square

It is conceivable that the efficiency of a group value is an artifact, existing in spite of the fact that the players have grossly different views of the world. However, we can define a family of group values, each of which arises from a viewpoint common to all of the players. Let $\{r_\pi: \pi \in \Pi\}$ be a probability distribution over the set Π of $n!$ orderings of N ; r_π is the probability associated with the ordering $\pi = (i_1, \dots, i_n)$ in which the k -th player is player i_k . For any ordering $\pi = (i_1, \dots, i_n)$, let $\pi^{i_k} = \{i_1, \dots, i_{k-1}\}$ be the set of predecessors of i_k in π . A random-order group value $\xi = (\xi_1, \dots, \xi_n)$ on \mathcal{J} is defined by

$$\xi_i(v) = \sum_{\pi \in \Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)] ,$$

for all $i \in N$ and all $v \in \mathcal{J}$.

An interpretation of this definition can be given. Assume that the players have as their goal the eventual formation of the grand coalition, N . Further assume that they see coalition-formation as a sequential process: given any ordering π of the players, each player i joins with his predecessors in π , making the marginal contribution $v(\pi^i \cup i) - v(\pi^i)$ in the game v . Then, if the players share a common perception $\{r_\pi: \pi \in \Pi\}$ of the likelihood of the various orderings, the expected marginal contribution of a player is precisely his component of the random-order group value.

THEOREM 12. Let $\xi = (\xi_1, \dots, \xi_n)$ be a random-order group value on \mathcal{J} , associated with with the probability distribution $\{r_\pi: \pi \in \Pi\}$. There exists a collection $\phi = (\phi_1, \dots, \phi_n)$ of probabilistic values on \mathcal{J} , such that $\phi_i(v) = \xi_i(v)$ for all $i \in N$ and all $v \in \mathcal{J}$. Furthermore, ϕ satisfies the efficiency axiom.

Proof. For any $i \in N$ and $v \in \mathcal{J}$,

$$\begin{aligned} \xi_i(v) &= \sum_{\pi \in \Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)] \\ &= \sum_{T \subset N \setminus i} \left\{ \sum_{\{\pi \in \Pi: \pi^i = T\}} r_\pi \right\} [v(T \cup i) - v(T)] . \end{aligned}$$

Define, for all $i \in N$ and all $T \subset N \setminus i$,

$$p_T^i = \sum_{\{\pi \in \Pi: \pi^i = T\}} r_\pi ,$$

and let $\phi = (\phi_1, \dots, \phi_n)$ be the associated collection of probabilistic values. (It is easily verified that, for each $i \in N$, $\{p_T^i: T \subset N \setminus i\}$ is a probability distribution.) Clearly, $\phi = \xi$.

Observe that, for any $v \in \mathcal{J}$,

$$\begin{aligned} \sum_{i \in N} \xi_i(v) &= \sum_{i \in N} \sum_{\pi \in \Pi} r_\pi [v(\pi^i \cup i) - v(\pi^i)] \\ &= \sum_{\pi \in \Pi} r_\pi \sum_{i \in N} [v(\pi^i \cup i) - v(\pi^i)] \\ &= \sum_{\pi \in \Pi} r_\pi \cdot v(N) = v(N) . \end{aligned}$$

Therefore, since $\phi = \xi$, it follows that ϕ satisfies the efficiency axiom. \square

The preceding theorem shows that every random-order value is an efficient probabilistic (group) value. The converse result also holds.

THEOREM 13. Let $\phi = (\phi_1, \dots, \phi_n)$ be a collection of probabilistic values on \mathcal{J} . Assume that \mathcal{J} contains \mathcal{C} and $\hat{\mathcal{C}}$, and that ϕ satisfies the efficiency axiom. Then there is a random-order value $\xi = (\xi_1, \dots, \xi_n)$ on \mathcal{J} , such that $\xi_i(v) = \phi_i(v)$ for all $i \in N$ and $v \in \mathcal{J}$.

Proof. Let ϕ be defined for all $i \in N$ and all $v \in \mathcal{J}$ by $\phi_i(v) = \sum_{T \subset N \setminus i} p_T^i [v(T \cup i) - v(T)]$. For any $i \in N$ and $T \subset N \setminus i$, define

$$A^d(T) = \sum_{j \notin T} p_T^j, \quad \text{and} \quad A(i; T) = p_T^i / A^d(T). \quad \text{Consider any ordering}$$

$\pi = (i_1, \dots, i_n) \in \Pi$, and define

$$r_\pi = p_{\emptyset}^{i_1} \cdot A(i_2; \{i_1\}) \cdot A(i_3; \{i_1, i_2\}) \cdots A(i_n; \{i_1, \dots, i_{n-1}\}) .$$

It is easily verified, by repeated summation, that

$$\sum_{\pi \in \Pi} r_\pi = \sum_{i_1=1}^n \sum_{i_2 \notin \{i_1\}} \sum_{i_3 \notin \{i_1, i_2\}} \cdots \sum_{i_n \notin \{i_1, \dots, i_{n-1}\}} r_{(i_1, \dots, i_n)} = 1 .$$

so $\{r_\pi : \pi \in \Pi\}$ is a probability distribution.

Let ξ be the random-order value associated with $\{r_\pi : \pi \in \Pi\}$.

Since

$$\xi_i(v) = \sum_{T \subset N \setminus i} \left[\sum_{\{\pi \in \Pi : \pi^i = T\}} r_\pi \right] [v(T \cup i) - v(T)] ,$$

it will suffice to show that for all $i \in N$ and $T \subset N \setminus i$,

$$p_T^i = \sum_{\{\pi: \pi^i = T\}} r_\pi .$$

Observe that

$$\begin{aligned} \sum_{\{\pi: \pi^i = T\}} r_\pi &= \sum_{i_t \in T} \sum_{i_{t-1} \in T \setminus \{i_t\}} \dots \sum_{i_1 \in T \setminus \{i_t, \dots, i_2\}} \\ &\quad \sum_{i_{t+2} \notin TU\{i\}} \sum_{i_{t+3} \notin TU\{i, i_{t+2}\}} \dots \sum_{i_n \notin TU\{i, i_{t+2}, \dots, i_{n-1}\}} r_{(i_1, \dots, i_n)} \\ &= \frac{p_T^i}{A^d(T)} \sum_{i_t \in T} \frac{p_{T \setminus \{i_t\}}^{i_t}}{A^d(T \setminus \{i_t\})} \sum_{i_{t-1} \in T \setminus \{i_t\}} \frac{p_{T \setminus \{i_t, i_{t-1}\}}^{i_{t-1}}}{A^d(T \setminus \{i_t, i_{t-1}\})} \\ &\quad \dots \sum_{i_1 \in T \setminus \{i_t, \dots, i_2\}} \frac{p_\emptyset^{i_1}}{A^d(\emptyset)} \sum_{i_{t+2} \notin TU\{i\}} A(i_{t+2}; TU\{i\}) \\ &\quad \dots \sum_{i_n \notin TU\{i, i_{t+2}, \dots, i_{n-1}\}} A(i_n; T \cup \{i, i_{t+2}, \dots, i_{n-1}\}) . \end{aligned}$$

This summation can be carried out explicitly. Proceeding from right to left, the first $n - (t + 1)$ sums each, in turn, have value 1. Continuing

inductively, each term of the form $\sum_{i_k \in T_k} \frac{p_{T_k \setminus i_k}^{i_k}}{A^d(T_k)}$ is preceded by a factor

with denominator $A^d(T_k) = \sum_{j \in T_k} p_{T_k}^j$.

Each two such sums are equal; this follows from the hypotheses of the theorem and from Theorem 11. Therefore, the entire expression simplifies to p_T^i , as desired. \square

Combining the preceding results, we obtain an interesting observation. A collection of individual probabilistic values is efficient for all games in its domain precisely when the players' probabilistic views of the world are consistent; that is, only when the various $\{p_T^i : T \subset N \setminus \{i\}\}$ arise from a single distribution $\{r_\pi : \pi \in \Pi\}$.

The family of random-order values associates a set of imputations (that is, efficient group allocations) with each game. This set clearly contains the Shapley value of the game; in addition, it can be shown that it contains the core of the game.

For any finite set K , let Π_K be the set of all one-to-one functions from K to $\{1, 2, \dots, |K|\}$. Given $i \in K$ and $\pi \in \Pi_K$, define $\pi^i = \{j \in K : \pi(j) < \pi(i)\}$. (Π_K is the set of orderings of K , and π^i is the set of predecessors of i in the ordering π). If v is a game on the player set N , and if $\pi \in \Pi_N$, then we define the marginal worth vector $a^\pi(v)$ as the imputation satisfying $a_i^\pi(v) = v(\pi^i \cup \{i\}) - v(\pi^i)$ for all $i \in N$. Let $W(v)$ be the convex hull of the set $\{a^\pi(v) : \pi \in \Pi_N\}$; $W(v)$ is the set of all imputations which are associated with v by some random-order value.

Recall that the core of a game v with player set N is the set $C(v) = \{x \in R^N : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\}$.

THEOREM 14. Let v be any game on N . Then $W(v) \supset C(v)$.

Proof. We proceed by induction on n , the number of players in N . The theorem is easily seen to hold for the cases $n = 1, 2$. For $n = 3$, the diagram illustrates the situation for a game with a non-empty core; for the sake of completeness, the situation when the core is empty is also illustrated. Assume that the theorem is true for all games with fewer than n players.

Since the core of a game is convex, it will suffice to show that all points in the boundary of $C(v)$ are members of $W(v)$. Let x be a boundary point of $C(v)$. Then, for some non-empty $S \subsetneq N$, $x(S) = v(S)$. Define the game u on S by $u(T) = v(T)$ for all $T \subset S$; define w on $N \setminus S$ by $w(T) = v(T \cup S) - v(S)$ for $T \subset N \setminus S$. Clearly $x^S \in C(u)$. Furthermore, for any $T \subset N \setminus S$, $x(T) = x(T \cup S) - x(S) \geq v(T \cup S) - v(S) = w(T)$; hence, $x^{N \setminus S} \in C(w)$.

Express $x^S = \sum \alpha_\sigma a^\sigma(u)$ as a convex combination of marginal worth vectors in $\{a^\sigma(u) : \sigma \in \Pi_S\}$. Similarly express $x^{N \setminus S} = \sum \beta_\tau a^\tau(w)$ as a convex combination of vectors in $\{a^\tau(w) : \tau \in \Pi_{N \setminus S}\}$. For any $\sigma \in \Pi_S$ and $\tau \in \Pi_{N \setminus S}$, write (σ, τ) for the ordering $\pi \in \Pi_N$ defined by $\pi(i) = \sigma(i)$ if $i \in S$, $\pi(j) = |S| + \tau(j)$ if $j \in N \setminus S$. Then $x = \sum (\alpha_\sigma \cdot \beta_\tau) a^{(\sigma, \tau)}(v)$, and hence $x \in W(v)$, as claimed. \square

This theorem bears upon several well-known results. For example,

if a game v is convex (that is, if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subset N$), then $W(v) = C(v)$. (This result is due to Shapley [9]; the converse has recently been noted by Ichiishi [5].) Further attributes of the set $W(v)$ are currently being investigated.

9. The Shapley Value

A standard characterization of the Shapley (group) value is as the only value which satisfies the linearity, dummy, symmetry, and efficiency axioms [8]. From our previous results, we can quickly prove the uniqueness of the Shapley value, and simultaneously obtain a simple derivation of the explicit formula for the Shapley value. Traditional proofs center around a consideration of the carrier games in \mathcal{C} . It appears that our consideration, as well, of the games in $\hat{\mathcal{C}}$ simplifies matters.

THEOREM 15. Let $\phi = (\phi_1, \dots, \phi_n)$ be a group value on \mathcal{G} , \mathcal{M} , or \mathcal{L} . Assume that each ϕ_i satisfies the linearity and dummy axioms, and that ϕ satisfies the symmetry and efficiency axioms. Then for every v in the domain of ϕ , and every $i \in N$,

$$\phi_i(v) = \sum_{T \subset N \setminus i} \frac{t!(n-t-1)!}{n!} [v(T \cup i) - v(T)] ,$$

where t generically denotes the cardinality of T .

Proof. From Theorems 3 and 10, it follows that there is a sequence $\{p_t\}_{t=0}^{n-1}$, such that each $\phi_i(v) = \sum_{T \subset N \setminus i} p_t [v(T \cup i) - v(T)]$. Specializing Theorem 11 to the symmetric case, we must have $\sum_{i \in N} p_{N \setminus i}^i = n p_{n-1} = 1$, and

$$\sum_{i \in T} p_{T \setminus i}^i = t p_{t-1} = \sum_{j \notin T} p_T^j = (n - t) p_t \quad \text{for all nonempty } T \subsetneq N .$$

Consequently,

$$p_{n-1} = \binom{n-1}{n-1} p_{n-1} = \frac{1}{n} ,$$

and

$$\binom{n-1}{t} p_t = \binom{n-1}{t-1} p_{t-1}$$

for all $1 \leq t \leq n - 1$. It follows that, for each t , $\binom{n-1}{t} p_t = \frac{1}{n}$,

and therefore, $p_t = \frac{t!(n-t-1)!}{n!}$. \square

It may be noted that, upon replacement of the linearity axiom with the transfer axiom, we obtain a similar theorem characterizing the Shapley value on \mathcal{G}^* , \mathcal{M}^* , or \mathcal{S}^* .

10. Remark

Throughout this paper, we have studied values of games on a fixed finite set of players. Along similar lines, one may consider values defined for all finite-player games in an infinite universe of players, or values of infinite-player (non-atomic) games, or the asymptotic connection between these two types of values. Such considerations will be presented in a series of papers written by various subsets of {Pradeep Dubey, Abraham Neyman, the author} .

REFERENCES

- [1] D. Blair, "Essays in Social Choice Theory," Ph.D. thesis, Yale University, 1976
- [2] P. Dubey, "On the Uniqueness of the Shapley Value," International Journal of Game Theory, 4 (1975), 131-139
- [3] P. Dubey, "Probabilistic Generalizations of the Shapley Value," Mathematics of Operations Research, to appear
- [4] P. Dubey and L.S. Shapley, "Some Properties of the Banzhaf Power Index," RAND Report P6016, October 1977
- [5] T. Ichiishi, "The Core and the Marginal Worth Vectors," manuscript
- [6] J. Milnor, "Reasonable Outcomes for n-Person Games," RAND Memorandum RM-916, August 1952
- [7] G. Owen, "Multilinear Extensions of Games," Management Science, 18, Part 2 (1972), 64-79
- [8] L. Shapley, "A Value for n-Person Games," Annals of Mathematics Study, 28 (1953), 307-318
- [9] L. Shapley, "Cores of Convex Games," International Journal of Game Theory, 1 (1971), 11-26

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