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On Disequilibrium Economic Dynamics. Part III. A Keynesian Theory of Money Wage Adjustment

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ON DISEQUILIBRIUM ECONOMIC DYNAMICS

PART III

A KEYNESIAN THEORY OF MONEY WAGE ADJUSTMENT

Katsuhito Iwai

November 14, 1975

ON DISEQUILIBRIUM ECONOMIC DYNAMICS

PART III

A KEYNESIAN THEORY OF MONEY WAGE ADJUSTMENT*

by

Katsuhito Iwai

1. Introduction

It is often argued that Keynesian economics is nothing but a special case of neo-classical economics and that only its ad hoc institutional assumption of "downward money wage rigidity" lends it a realistic flavor and makes it a useful special case. The purpose of the Keynesian disequilibrium dynamics to be developed in the present and subsequent articles is to reconstruct Keynesian economics firmly on the foundation of a coherent microdynamic theory of the firm and to demonstrate that it is indeed "the general theory" which is at the same time both realistic and useful.

The labor market is not a "bourse."¹ Money wages are never determined by symmetrical exchanges of biddings between approximately equal numbers of firms and workers circulating among each other. In the unionized labor market they are negotiated between the firm and the

*This is Part III of a series of papers on disequilibrium economic dynamics. This paper can be read, however, with little prior knowledge of Part I and Part II of the series circulated earlier; furthermore, its Mathematical Appendix (Steady-State Theorems for the Random Walk Model with Two Return Barriers) is an independent mathematical treatise on some elementary problems in the theory of random walk. A sequel: "Part IV: The Theory of Long-Run Phillips Curve" will appear also as a Cowles Foundation Discussion Paper. Research for this paper was in part supported by grants from the National Science Foundation and the Ford Foundation.

¹See Dunlop [6], Chapter 2.

trade union. In the non-unionized labor market they are unilaterally quoted by the firm on a "take-it-or-leave-it" basis. In the present series of papers we consider only the non-unionized labor market, simply because it is easier to investigate analytically. But most of the qualitative results obtained in this series would hold true even in the unionized labor market.

We observed in Part I of this series that the money wage, unilaterally quoted by a firm on a take-it-or-leave-it basis, can no longer be regarded as a mere exchange rate between labor-service and the means of payment, but as a "signal" informing potential employees of the possible job opportunities open in the firm quoting it.² Then, two fundamental problems confront the wage fixing firm. First, in order for the money wage to function as an information signal at all, it must be announced before workers make up their labor supply decisions and reveal their true labor supplies in the labor market. This implies that when the entrepreneur of the firm decides the level of money wage he is able to possess only an imperfect information about the magnitude of labor-services offered to his firm. The firm must, therefore, make his money wage decision under uncertainty. Second, the entrepreneur must determine the period of time between money wage changes, because the effectiveness of the money wage as an information signal depends crucially upon how frequently it is adjusted.

As a consequence of the first problem, which arises out of the role of money wage as an information signal, we can conclude that the entrepreneur's decision on money wage must be governed by his "expectations"

²See Iwai [10], Section 2.

of factors influencing the labor supply schedule to his firm. Obviously, all expectations are subject to errors. When he has overestimated the level of labor supply relative to his desired labor demand, his expectation error would materialize in the form of "unfilled vacancies"; and similarly when he has underestimated the labor supply relative to his desired labor demand, his expectation error would materialize in the form of "involuntary unemployment" of some workers willing to work at the going money wage.³ This simple observation is in fact the essence of our micro-economic and short-run theory of involuntary unemployment.

However, why the involuntary unemployment can spread over the whole economy and why it can persist even in the long-run are entirely different questions which demand entirely different answers. In order to explain the involuntary unemployment as a macroeconomic phenomenon, we need to elucidate the mechanism through which a majority of firms commit the same mistake--an underestimation of the labor supply relative to the labor demand--involuntarily but inevitably under certain reasonable circumstances. Part II of this series was devoted to the analysis of this fundamental macroeconomic mechanism.⁴

It is then the task of the present and subsequent articles to explain the involuntary unemployment as a long-run phenomenon as well. It goes without saying that the key to this "long-run theory of involuntary unemployment," so to speak, lies in the "downward rigidity of money wage." But the problem is not so trivial, because we must demonstrate

³ Here we are implicitly assuming that when there is an excess-supply of labor-services it is absorbed not by the uniform reduction of working hours of all the willing workers but by the reduction of the number of workers by a certain rationing scheme such as the first-come-first-serve rule.

⁴ Iwai [11].

that the downward rigidity of money wage has not only transient but also permanent influences on the determination of the real variables in the economy, in particular, the rate of involuntary unemployment. It is clear that the conventional Keynesian economics which typically formalizes the notion of downward money wage rigidity by postulating that there is a fixed money wage level at which the "actual" labor supply schedule becomes infinitely elastic is of no help for this end. We need the reformulation of the notion of downward rigidity itself.

If the labor market is not a bourse and if money wage is fixed by the entrepreneur of a firm on the basis of take-it-or-leave-it, the most natural way to formalize the notion of the rigidity of money wage is to suppose that his firm incurs certain adjustment costs whenever he changes the level of money wage. One determinant of the money wage adjustment costs is the loss of the value of money wage as a reliable information to potential employees when its level is adjusted. In this sense, our long-run theory of involuntary unemployment is related to the second problem created by the role of money wage as an information signal, mentioned earlier in this section. However, this information cost is only one of many determinants of money wage adjustment costs. Indeed, so little can be said on raisons d'être for money wage adjustment costs from the standpoint of pure economic theory. Therefore, it is only to the following more tractable problem that the present and subsequent articles will be addressed: if it is costly for the entrepreneur of a firm to adjust the level of money wage, then what would be the consequences? Accordingly, we start our Keynesian disequilibrium dynamics by developing a model of the individual firm whose entrepreneur must determine the level of money wage in a dynamic as well as stochastic labor market environment under

the condition that an adjustment of money wage is costly to him.

2. Money Wage Adjustment Rule

Let us consider the entrepreneur of a firm who must determine the level of money wage at the beginning of, say, the t^{th} period. In the present paper we shall assume for the sake of simplicity that all workers are homogeneous and can be regarded as a completely variable productive factor. Then, the entrepreneur has to pay the same money wage to all the workers, new and old. We shall then denote by w_t the logarithmic level of this single money wage to be quoted at the beginning of the t^{th} period. (Warning: in the following all the variables will be measured by logarithmic scale!) However, by interpreting w_t as "the marginal scale wage" --the wage paid to new workers in the typical job-category, we can apply our model of money wage adjustment equally well to the heterogeneous and immobile labor market in which the majority of workers remain with the same firm from one period to the next and a wide variety of jobs are open to workers with a wide variety of skills and experiences.⁵

Let us suppose that our firm is monopsonistically competing with other firms for a given level of aggregate labor supply in the economy-wide labor market.⁶ Therefore, our firm has its own labor market in the

⁵ See Hall [8]. This paper is an interesting attempt to develop a model of money wage adjustment in the heterogeneous and immobile labor market. However, because of the failure to introduce the adjustment cost of the scale wage he derives a conclusion which supports the so-called "natural rate theory of unemployment," in contrast with our theory which rather refutes this sophisticated revival of pre-Keynesian neo-classical theory of employment.

⁶ Hence, in this model we are assuming that the labor market is composed of a large number of small firms which behave as if there were no oligopolistic or strategic interdependence with each other. If the number of rival firms is small, this behavioral hypothesis breaks down.

sense that it can control the supply of labor to itself by adjusting the level of money wage relative to its expectations of other firms' money wages. The economy-wide labor market thus consists of numerous labor markets, one for each monopsonistically competitive firm. We also suppose that the level of money wage is the sole means of the monopsonistic competition in the short-run. Let us then denote by w_t^* the logarithmic level of money wage that would uniquely maximize the firm's short-run subjective expected profit in period t , were there no money wage adjustment costs. To ease the terminological burden, we shall simply call w_t^* logarithmic level of the "optimal" money wage in period t . Presumably, it depends upon the entrepreneur's subjective expectations of current labor market conditions as well as those of future product market conditions, upon various fixed productive factors endowed in the firm and upon available technology known to him in period t . This is indeed the case for our highly parameterized model of the firm developed in Part I of this series;⁷ but in this paper we need not confine ourselves to that special model.

If an adjustment of money wage is costless, the optimal money wage adjustment policy for our entrepreneur becomes trivial; it is only to set the logarithmic level of actual money wage w_t equal to that of the "optimal" money wage w_t^* in each period. However, if our entrepreneur incurs some costs whenever he adjusts the level of money wage, his optimal money wage adjustment policy becomes no longer trivial. Clearly, he does not always set w_t equal to w_t^* in every period, and his task becomes that of finding an optimal money wage adjustment policy that allows the actual money wage deviate from the "optimal" money wage in each period,

⁷ See equation (53) in Iwai [10].

taking a due account of the costly nature of the money wage adjustment activity itself. This is an inherently dynamic problem. Since the mathematical difficulty of determining a fully optimal money wage adjustment policy out of all the possible policies seems insurmountable, let us assume in this paper that our entrepreneur chooses the best policy from a restricted class of feasible rules whose functional forms are given a priori except for a few parameter values. We believe that our specialized formulation of the model of costly money wage adjustment is a reasonable first approximation which captures most of its essential features.⁸

A class of feasible money wage adjustment rules which is reasonable is of the following type:

$$(1) \quad \begin{cases} w_t = w_{t-1} & \text{if } \lambda_2 < w_t^* - w_{t-1} < \lambda_1, \\ w_t = w_t^* & \text{if } w_t^* - w_{t-1} \leq \lambda_2 \text{ or } w_t^* - w_{t-1} \geq \lambda_1; \end{cases}$$

where λ_1 and λ_2 are given parameters assumed to be positive and negative, respectively. If our entrepreneur adopted this adjustment rule, his money wage determination activity would proceed as follows. In both the product and labor markets he is constantly gathering and collating various information about their current as well as future conditions. At the beginning of every period, in the light of new information acquired through his actions and observations during the previous period, he revises

⁸This simplification of problem is clearly due to the "bounded rationality" of the present author, who unabashfully assumes that the entrepreneur of the firm in question has as poor computation capacities as he has. See Simon [19] for the notion of bounded rationality. Note that unlike the (S,s) policy of the mathematical inventory policy, there is no guarantee that the form of feasible adjustment rules presented below is the truly "optimal" form. (See Scarf [18] for the proof of the optimality of the (S,s) inventory policy.)

his subjective expectations of relevant random variables which he believes influencing the demand schedule for his product and the supply schedule of labor to his firm. He is then able to calculate the logarithmic level of the new "optimal" money wage w_t^* in this period and compare it with the logarithmic level of money wage w_{t-1} quoted in the previous period. If the deviation of w_t^* from w_{t-1} is in the tolerable range in the sense that $\lambda_2 < w_t^* - w_{t-1} < \lambda_1$, he defers the adjustment of money wage in this period and simply requotes the same money wage as before. However, whenever the deviation of w_t^* from w_{t-1} exceeds the ceiling λ_1 he immediately raises money wage in this period to the new "optimal" level and whenever the deviation gravitates below the floor λ_2 he immediately lowers it to w_t^* . Therefore, the ceiling parameter λ_1 and the floor parameter λ_2 , which completely specify the money wage adjustment rule given by (1), can be regarded as "thresholds" of our entrepreneur's money wage adjustment activity.

However, the above money wage adjustment rule has one obvious defect. In this Keynesian economy where money wage adjustment is costly, w_t^* is no longer optimal in the genuine sense of the word, and there is no reason why our entrepreneur should adjust his money wage to this "optimal" money wage level if he has decided to adjust it. Therefore, in the following we shall allow our entrepreneur to adjust w_t to the level different from w_t^* , namely, to the level equal to $w_t^* + \lambda_0$; where λ_0 is a constant parameter which can take on either positive or negative value. Then, the class of feasible money wage adjustment rules can be respecified as follows:

$$(1)' \quad \begin{cases} w_t = w_{t-1} & \text{if } \lambda_2 < w_t^* + \lambda_0 - w_{t-1} < \lambda_1, \\ w_t = w_t^* + \lambda_0 & \text{if } w_t^* + \lambda_0 - w_{t-1} \leq \lambda_2 \text{ or } w_t^* + \lambda_0 - w_{t-1} \geq \lambda_1; \end{cases}$$

where $\lambda_0 \geq 0$ and $\lambda_2 < 0 < \lambda_1$. Since a money wage adjustment rule of the above type is completely characterized by three constant parameters -- λ_0 , λ_1 and λ_2 , the determination of the optimal money wage adjustment policy by our entrepreneur has been reduced to a much simpler problem of choosing the optimal values of these three parameters. This is still a dynamic problem; but a tractable one.

Note in passing that if $\lambda_1 = \lambda_2 = \lambda_0 = 0$ then we have $w_t = w_t^*$ for all t and the above adjustment rule becomes equivalent to the optimal money wage adjustment policy in the economy where there are no money wage adjustment costs. In other words, the microdynamic model of the firm in the "Wicksellian" economy, discussed in Part I, is a very special case of the one in this truly Keynesian economy. If $\lambda_2 = -\infty$ and $\lambda_1 = +0$, the same adjustment rule dictates our entrepreneur to raise money wage whenever desirable but to refuse any wage cut no matter how low w_t^* falls below w_{t-1} . Money wage in this case is perfectly flexible in the upward direction but absolutely rigid in the downward direction. If $\lambda_2 = -\infty$ and $0 < \lambda_1 < \infty$, money wage becomes imperfectly flexible upwards but absolutely rigid downwards. If both λ_2 and λ_1 are non-zero and finite, money wage becomes imperfectly flexible in both directions.

Let us now denote by x_t the rate of deviation of the money wage level to be returned from the actual level of money wage, and by ξ_t the rate of change in the "optimal" money wage;⁹ that is, we put

$$(2) \quad x_t \equiv w_t^* + \lambda_0 - w_t ,$$

$$(3) \quad \xi_t \equiv w_t^* - w_{t-1}^* .$$

Note that the variable $x_t - \lambda_0 \equiv w_t^* - w_t$ can be interpreted as a subjective measure of disequilibrium in our entrepreneur's labor market.

In fact, in our special model of a firm developed in Part I of this series it can be shown that it is proportional to the rate of deviation of his subjective expectation of the rate of excess supply in his own labor market from what he believes to be a "normal rate" of excess labor supply.¹⁰

Substituting (2) and (3) into (1)' and rearranging terms, we can eliminate the parameter λ_0 and transform the money wage adjustment rule as follows

⁹ Since all the variables are measured by logarithmic scale, the difference between two variables can be regarded as approximately equal to the ratio of these variables measured by the standard non-logarithmic scale. In particular, the time-difference of a variable can be interpreted as the rate of change of that variable measured by the non-logarithmic scale.

¹⁰ We have

$${}_t \hat{h}_t - \hat{h}^* = \frac{\hat{\eta} + \hat{\epsilon}\{\hat{\eta}(1-\gamma) + \gamma\}}{\hat{\eta}(1-\gamma) + \gamma} (w_t - w_t^*) ;$$

where $\hat{\eta}$ and $\hat{\epsilon}$ are the firm's subjective elasticities of product demand and labor supply, γ is the labor input elasticity in production, ${}_t \hat{h}_t$ is the entrepreneur's subjective expectation of the rate of excess labor supply in period t conditional upon the information available to him at the beginning of period t , and \hat{h}^* is the constant normal rate of excess labor supply whose value is determined by $\hat{\eta}$, $\hat{\epsilon}$, γ and the firm's subjective specifications of the variability of the random variables influencing the product demand and labor supply schedules. It can be easily seen from the equation given above that the optimal money wage policy without adjustment costs in our Wicksellian economy developed in Part I can be characterized by the condition that ${}_t \hat{h}_t = \hat{h}^*$.

$$(4) \quad \begin{cases} x_t = x_{t-1} + \xi_t & \text{where } \lambda_2 < x_{t-1} + \xi_t < \lambda_1, \\ x_t = 0 & \text{where } x_{t-1} + \xi_t \leq \lambda_2 \text{ or } x_{t-1} + \xi_t \geq \lambda_1. \end{cases}$$

Once the values of λ_1 and λ_2 are chosen by our entrepreneur at the beginning of period, say, zero, his prediction of the dynamic motion of the new variable x_t in the future is governed by his own subjective specification of the dynamic motion of ξ_t . Now, the value of w_t^* and hence that of ξ_t in the future are unknown to him, because he will revise their values in later periods on the basis of new information available to him. So he must regard ξ_t as a random variable, and give a stochastic specification to it. We shall assume in the following that our entrepreneur believes $\xi_1, \xi_2, \dots, \xi_t, \dots$ are mutually independent random variables with a common subjective probability distribution $\hat{F}(\xi)$:

$$(5) \quad \hat{F}(\xi) \equiv \hat{\Pr}\{\xi_t \leq \xi\} \text{ for } t = 1, 2, \dots;^{11}$$

whose mean value is denoted by $\hat{\mu}$:

$$(6) \quad \hat{\mu} \equiv \int_{-\infty}^{\infty} \xi \cdot d\hat{F}(\xi).$$

In words, he believes that the "optimal" money wage will undergo a multiplicative random walk with the average growth rate $\hat{\mu}$. This is by no means an innocuous assumption, but we believe it is a useful first-order approximation. Note that the subjective probability distribution $\hat{F}(\xi)$

¹¹We denote by $\hat{P}_r\{Z\}$ the entrepreneur's subjective probability of an event Z .

summarizes not only the entrepreneur's view of the stochastic properties of the variables relevant to his calculation of the "optimal" money wage but also his own estimation of his possible measurement errors of these variables.

Before specifying our entrepreneur's objective function and determining the optimal money wage adjustment policy, let us examine the nature of a given adjustment rule characterized by arbitrarily chosen parameter values of λ_1 and λ_2 .

The position of the random variable, defined by (4), fluctuates along an open interval $\overline{\lambda_2, \lambda_1}$ as a partial sum of independent random variables $\xi_1, \xi_2, \dots, \xi_{t-1}$ and of a given initial position x_0 in period zero. However, whenever the value of $x_{t-1} + \xi_t$ enters into either of a half-open interval $\overline{-\infty, \lambda_2}$ or a half-open interval $\overline{\lambda_1, \infty}$ in period t , the money wage is immediately adjusted and the system returns to the origin: $x_t = 0$ in that period. This random walk must start anew from the origin from then on. According to the theory of random walks, the two parameters λ_1 and λ_2 are called return barriers, and the sequence of random variables $\{x_t\}$ is said to constitute a random walk with two return barriers at λ_1 and λ_2 .

3. The Steady-State Theorem for a Money Wage Adjustment Rule

Let the probability distribution $\hat{\Pi}_t(x|x_0)$ summarize our entrepreneur's prediction of the position of the random variable x_t in period t on the basis of his information about its initial position x_0 ; that is, for $\lambda_2 < x < \lambda_1$ and $t = 1, 2, \dots$, we put

$$(7) \quad \hat{\Pi}_t(x|x_0) \equiv \hat{\Pr}\{x_t \leq x|x_0\}.$$

By convention we put $\hat{\Pi}_t(x|x_0) = 0$ for $x \leq \lambda_2$ and $\hat{\Pi}_t(x|x_0) = 1$ for $x \geq \lambda_1$. As time goes on, the money wage will be adjusted over and over again and the process of random walk will start from scratch over and over again. Our entrepreneur can therefore anticipate that after a sufficiently long time this stochastic process will settle down to a "stochastic steady-state" independently of the initial condition. In the Mathematical Appendix we are able to prove the following fundamental proposition that justifies this conjecture.¹²

Steady-State Theorem. If (i) both λ_1 and λ_2 are finite, or (ii) λ_1 is finite, $\lambda_2 = -\infty$ and $0 < \hat{\mu} < \infty$, or (iii) λ_2 is finite, $\lambda_1 = +\infty$ and $-\infty < \hat{\mu} < 0$, then as $t \rightarrow \infty$ the sequence of random variables $\{x_t\}$ converges to a steady-state random variable \underline{x} , in the sense that $\hat{\Pi}_t(x|x_0)$ converges to a steady-state distribution $\hat{\Pi}(x)$, independently of the initial condition x_0 ; that is, we have

$$(8) \quad \lim_{t \rightarrow \infty} \hat{\Pi}_t(x|x_0) = \hat{\Pi}(x) \equiv \hat{\Pr}\{\underline{x} \leq x\}.$$

It is easy to see that $\hat{\Pi}(x)$ is determined only by the two parameters λ_1 and λ_2 and by the random walk probability distribution $\hat{F}(\xi)$. The limit distribution $\hat{\Pi}(x)$ is called the steady-state distribution because it satisfies the following steady-state property:

$$(9) \quad \begin{aligned} \hat{\Pi}(x) &= \hat{\Pr}\{x_{t+t} \leq x | \hat{\Pr}\{x_t \leq x\} = \hat{\Pi}(x)\} \\ &= \int_{\lambda_2}^{\lambda_1} \hat{\Pi}_t(x|x_t) \cdot d\hat{\Pi}(x_t) \end{aligned}$$

¹²See section 4 of Mathematical Appendix.

for $t' = 1, 2, \dots$ and $t = 1, 2, \dots$. Once the position of the random variable x_t in period t is believed to be distributed according to this distribution, its positions in the entire future from period t on can be also predicted by the same probability distribution. It is, in other words, a "self-perpetuating" subjective probability distribution. Alternatively, the steady-state distribution can be interpreted as a description of the long-run average behavior of the random variable x_t . Because $\hat{\Pi}(x)$ also represents the average proportion of periods during which x_t is expected to spend in an interval $\overline{\lambda_2, x}$ during a very long period of time.¹³ Therefore, we shall interchangeably call $\hat{\Pi}(x)$ either the steady-state distribution or the long-run average distribution. The reader familiar with the mathematical theory of inventory management should have already noticed that our steady-state theorem is a generalization of Karlin's steady-state theorem for the well-known Arrow-Harris-Marschak inventory model.¹⁴ Their connections will be briefly discussed in the Mathematical Appendix.

We shall suppose in the following that one of the three conditions stated in our Steady-State Theorem is always satisfied.

Our interest in the steady-state behavior of the given money wage adjustment rule is two-fold. In the first place, because our Steady-State Theorem characterizes the long-run average behavior of an individual entrepreneur's money wage adjustment activity, it would certainly

¹³This second interpretation of $\hat{\Pi}(\cdot)$ can be rigorously justified by the so-called Mean-Ergodic Theorem. For this, see Billingsley [3], Chapter 1.

¹⁴Karlin [13]. See also Prabhu [16], pp. 177-178. The pioneering paper is, of course, Arrow, Harris and Marschak [1].

facilitate the determination of his optimal money wage adjustment policy. In the present paper, our attention will be focused upon this micro-economic application of our Steady-State Theorem. However, the second and at least equally important reason lies in the fact that if we consider the dynamic behavior of the economy as a whole in which numerous firms are making money wage decisions in a decentralized but interdependent manner the steady-state distribution $\hat{\mu}(x)$ could be given a novel interpretation as the cross-sectional description of the economy's stochastic macro-equilibrium--an equilibrium which is maintained by offsetting motions of a large number of firms perpetually thrown out of equilibrium by incessant intersectional disturbances of product demand, labor supply, capital accumulation and technology.¹⁵ Our Steady-State Theorem would enable us to analyze this fundamental equilibrium concept in macroeconomic dynamics within the most coherent theoretical framework. This topic will be taken up in Part IV of this series of papers.

4. Expected Rate of Money Wage Change in the Short-Run and in the Long-Run

It is clear from the specified form of the money wage adjustment rule (1)' that the motion of the rate of change in the actual money wage, $\Delta w_t \equiv w_t - w_{t-1}$, is inherently discrete. There is an upward jump by the rate equal to $(w_t^* + \lambda_0) - w_{t-1} = x_{t-1} + \xi_t$ when $x_{t-1} + \xi_t \geq \lambda_1$ in period t , and a downward jump when $x_{t-1} + \xi_t \leq \lambda_2$; otherwise

¹⁵ It is Tobin [22] who first introduced the concept of stochastic macro-equilibrium into macroeconomics literature. But we can find similar notions in Lipsey [12], Rees [17] and many other literature in the theory of Phillips curve relation.

there is no adjustment of the level of money wage at all. However, we can still acquire some useful insight into the nature of this money wage adjustment process by examining its short-run as well as long-run average behaviors.

We know that $x_0 - \lambda_0 \equiv w_0^* - w_0$ measures the size of disequilibrium perceived by our entrepreneur in his own labor market at the beginning of period zero. Then, the expected rate of money wage change in period t , predicted on the basis of his knowledge of $x_0 - \lambda_0$, can be calculated as follows

$$\begin{aligned}
 (10) \quad \hat{E}(\Delta w_t | x_0 - \lambda_0) &= 0 \cdot \Pr \{ \lambda_1 < x_{t-1} + \xi_t < \lambda_2 | x_0 - \lambda_0 \} \\
 &+ \hat{E} \left\{ \int_{\lambda_1 - x_{t-1}}^{\infty} (x_{t-1} + \xi_t) \cdot d\hat{F}(\xi_t) + \int_{\infty}^{\lambda_2 - x_{t-1}} (x_{t-1} + \xi_t) \cdot d\hat{F}(\xi_t) \mid x_0 \right\} \\
 &= \int_{-\infty}^{\infty} \int_{\lambda_1}^{\infty} \{ z \cdot d\hat{F}(z-y) + \int_{-\infty}^{\lambda_2} z \cdot d\hat{F}(z-y) \} \cdot d\hat{\Pi}_{t-1}(y | x_0) \\
 &= \hat{\mu} - \int_{-\infty}^{\infty} \int_{\lambda_2}^{\lambda_1} \{ z \cdot d\hat{F}(z-y) - y \} \cdot d\hat{\Pi}_{t-1}(y | x_0) ;
 \end{aligned}$$

in view of (4) and (7).¹⁶ In particular, the expected rate of change in money wage during one unit period can be expressed as

$$(11) \quad \hat{E}(\Delta w_1 | x_0 - \lambda_0) - \hat{\mu} = x_0 - \int_{\lambda_2}^{\lambda_1} z \cdot d\hat{F}(z - x_0) .$$

¹⁶ We denote by $\hat{E}\{z | I\}$ the entrepreneur's subjective expectation of a random variable z conditional upon the information I available to him.

Let the right-hand-side of the above equation be represented by a function $\vartheta(x_0 - \lambda_0)$. The shape of this function is determined by λ_0 , λ_1 , λ_2 and $\hat{F}(\xi)$. Though, in general, $\vartheta(\cdot)$ is not necessarily an increasing function, it is easy to show that it is a non-decreasing function if $\hat{F}(\xi)$ is the simplest Bernouilli trial distribution and that it is a strictly increasing function if $\hat{F}(\xi)$ is a mixture of two exponential distributions.¹⁷ Therefore, at least in these cases we can regard the equation (11) as our entrepreneur's perceived law of supply and demand, for it maintains that his subjective expectation of the rate of change of money wage in excess of that of the "optimal" money wage \hat{w} , is positively correlated with his perceived size of disequilibrium in his own labor market, $x_0 - \lambda_0$, which is proportional to the rate of deviation of his subjective expected rate of excess labor demand from its normal rate.

¹⁷ In the case of the Bernouilli trial distribution, $\vartheta(x_0 - \lambda_0) = -(1-q)\lambda_2 < 0$ when $x_0 = \lambda_2 + s$; $= 0$ when $x_0 = \lambda_2 + 2s, \lambda_2 + 3s, \dots, \lambda_1 - 2s$; and $= q \cdot \lambda_1 > 0$ when $x_0 = \lambda_1 - s$; where $s > 0$ is a unit step size and $0 < q < 1$ is the probability of a positive jump. In the case of a mixed exponential distribution where $F'(\xi) = \frac{1}{a+b} \exp(-\xi/a)$ for $\xi > 0$ and $F'(\xi) = \frac{1}{a+b} \exp(\xi/b)$ for $\xi < 0$, we have

$$\begin{aligned} \vartheta(x_0 - \lambda_0) &= x_0 - \int_{x_0}^{\lambda_1} \frac{z}{a+b} \exp\left(\frac{x_0 - z}{a}\right) dz - \int_{\lambda_1}^{x_0} \frac{z}{a+b} \exp\left(\frac{z - x_0}{b}\right) dz \\ &= b - a + \frac{a(a + \lambda_1)}{a+b} \exp\left(\frac{x_0 - \lambda_1}{a}\right) - \frac{b(b - \lambda_2)}{a+b} \exp\left(\frac{\lambda_2 - x_0}{b}\right). \end{aligned}$$

This is clearly a strictly increasing function of x_0 or $x_0 - \lambda_0$.

It is important to bear in mind, however, that this is only a "perceived" law for our entrepreneur, and ex post both the actual size of labor market disequilibrium and the actual rate of change in money wage may turn out to be different from their ex ante or expected values.¹⁸ It should be also noted that this perceived law of supply and demand does not necessarily have the desired property: $\phi(0) = \hat{E}(\Delta w_1 | x_0 - \lambda_0 = 0) - \hat{\mu} = 0$. Therefore, even if the entrepreneur perceives that his labor market is in equilibrium in the sense that $x_0 - \lambda_0 = 0$, he does not necessarily expect the rate of change in money wage to be equal to the average rate of change in the "optimal" money wage. It may exceed or fall short of $\hat{\mu}$, depending upon his specification of $\hat{F}(\xi)$ and his choice of the values of λ_0 , λ_1 and λ_2 .

In the long-run, however, this subjective law of supply and demand will evaporate. That is to say, as time goes on the influence of the perceived size of labor market disequilibrium in the initial period will gradually fade away, and finally as $t \rightarrow \infty$ the expected rate of change in money wage will converge to a constant long-run average value, independently of the initial perceived size of labor market disequilibrium. It is remarkable but after a second thought obvious that this long-run

¹⁸In Iwai [9], we demonstrated that under the assumption of no money wage adjustment costs the actual rate of money wage change is an increasing function of the realized rate of excess labor demand (in excess of its normal rate). This ex post law of supply and demand is based upon the observation that the latter measures the extent of the entrepreneur's overestimation of the labor supply relative to his labor demand, and that his downward revision of his expectations of the factors influencing the labor supply schedule, induced by this revelation of expectation-error, pushes up the level of money wage in the next period. In our Keynesian dynamics, however, the existence of money wage adjustment costs prevents the full working of this ex post law.

average rate of change in money wage turns out to be equal to the expected rate of change in the "optimal" money wage, $\hat{\mu}$. This important proposition can be confirmed simply as follows:

$$\begin{aligned}
 (12) \quad \hat{E}(\Delta \tilde{w}) &\equiv \lim_{t \rightarrow \infty} \hat{E}(\Delta w_t | x_0 - \lambda_0) \\
 &= \hat{\mu} - \int_{-\infty}^{\lambda_1} \left\{ \int_{\lambda_2}^{\lambda_1} z \cdot d\hat{F}(z-y) - y \right\} \cdot d\hat{\Pi}(y), \text{ by (19),} \\
 &= \hat{\mu} - \left[\int_{\lambda_2}^{\lambda_1} z \cdot d \left\{ \int_{-\infty}^{\infty} \hat{F}(z-y) d\hat{\Pi}(y) \right\} - \int_{-\infty}^{\infty} y \cdot d\hat{\Pi}(y) \right] \\
 &= \hat{\mu} - \left[\int_{\lambda_2}^{\lambda_1} z \cdot d\hat{\Pi}(z) - \int_{\lambda_2}^{\lambda_1} y \cdot d\hat{\Pi}(y) \right], \text{ by (8),} \\
 &= \hat{\mu}.
 \end{aligned}$$

It should be emphasized that this proposition is dependent neither on the choice of the parameter values of λ_1 , λ_2 and λ_0 , nor the specification of the probability distribution $\hat{F}(\xi)$, so long as one of the three conditions stated in the Steady-State Theorem is fulfilled.¹⁹

¹⁹ A model of price adjustment developed by Barro [2] can be regarded as a special case of our model of money wage adjustment if we replace his price variable by the logarithmic level of money wage. His model in fact assumes that the random walk probability distribution $\hat{F}(\xi)$ is a symmetric Bernoulli trial distribution, so that the drift $\hat{\mu}$ is equal to zero. He calculated the ratio of the expected rate of price change and the expected duration of time up to the first price adjustment as an approximation of the expected rate of price change per time up to the first price adjustment, and showed that it is an increasing function of the size of initial disequilibrium. Unfortunately, it is hard to give any meaningful economic interpretation to this result, although Barro himself seems to regard it as an aggregative approximation of the law of supply and demand that relates the (expected) rate of price change to the size of initial disequilibrium. However, as is clear from our discussion above, the expected rate of price change per unit period is easily calculable as $\hat{E}(\Delta p_t | x_0 - \lambda_0) \equiv \hat{\mu}(x_0 - \lambda_0)$ in the short-run or $\hat{E}\{(p_t - p_0)/t | x_0 - \lambda_0\} = \hat{E}\{\sum_{j=1}^t \Delta p_j / t | x_0 - \lambda_0\}$ in the medium-run. Neither of the above expressions

5. The Cost Structure of the Firm

So far our analysis has been confined to the entrepreneur's prediction of the long-run as well as short-run behaviors of the random variable x_t , induced by a given money wage adjustment rule (1)'. We have, in fact, deliberately separated this problem from the specification of his firm's cost-profit structure. We must now turn to the latter.

In general, the firm's expected profit to be earned from its entrepreneur's one cycle activity starting from his recruiting activity in period t is a function of the actual money wage w_t and the time t . However, it seems reasonable to specify this short-run expected profit function in the following way:

$$(13) \quad \hat{\beta}(w_t^* - w_t) \cdot \exp(\hat{v}_t) .$$

Indeed, it is easy to check that the expected profit in the model of a firm developed in Part I of this series is precisely of this special form;²⁰

can be approximated by "the expected rate of price change per time up to the first price adjustment." Moreover, in the long-run, by the mean ergodic theorem mentioned in footnote 12, the rate of price change per unit

period: $\sum_{j=1}^t \Delta p_j / t$ approaches the steady-state expected rate of price change: $\lim_{t \rightarrow \infty} \hat{E}(\Delta p_t | x_0 - \lambda_0)$. But, according to the proposition we proved

in (12), the latter is equal to zero in the model of Barro that assumes $\hat{\beta} = 0$. In other words, contrary to his assertion, the rate of price change in his model is zero, regardless of the size of initial disequilibrium, if we average it over long time horizon! A correct way to derive the long-run average law of supply and demand, in addition to our ex post short-run law mentioned in footnote 18 and our ex ante short-run law discussed in this section, will be presented in Part IV of this series.

²⁰This can be easily seen if we substitute the equation in footnote 10 into equation (48) in Part I of this series.

but it is obviously not the only case that generates such an expected profit function. In any case, in (13) we have decomposed the expected profit into two multiplicative components-- $\hat{p}(w_t^* - w_t)$ and $\exp(\hat{v}_t)$. The latter exponential function can be regarded as the trend level of this firm's expected profit whose motion can be controlled only by its long-term policies such as fixed investment, R&D investment and marketing policies, and is assumed to be totally independent of his short-term money wage adjustment policy. On the other hand, we have specified the former component of (10) as a time-independent function of the rate of the deviation of the "optimal" from actual money wage, $(w_t^* - w_t)$, which is nothing but the perceived size of labor market disequilibrium, $x_t - \lambda_0$. We shall call this the trend-free expected profit function. Since we defined, at the outset of this paper, w_t^* as the logarithmic level of money wage that would uniquely maximize the short-run expected profit were there not any money wage adjustment costs, this trend-free expected profit function must attain the unique maximum at $w_t^* - w_t \equiv x_t - \lambda_0 = 0$; that is, we have

$$(14) \quad \hat{p}(0) > \hat{p}(w_t^* - w_t) = \hat{p}(x_t - \lambda_0) \text{ for any } w_t^* - w_t \equiv x_t - \lambda_0 \neq 0 .$$

If $\hat{p}(\cdot)$ is twice-differentiable, this property can be restated as follows

$$(15-a) \quad \hat{p}'(0) = 0 ,$$

$$(15-b) \quad \hat{p}''(0) < 0 .$$

Equation (15-a) is nothing but the first-order condition and inequality (15-b) is the second-order condition for the maximum, respectively. The more w_t deviates from w_t^* in either direction the less expected short-run profit our entrepreneur can earn. Therefore, the difference between

the maximum attainable trend-free expected profit and the actual expected profit generated by a given money wage level or by a given size of labor market disequilibrium: $\hat{\beta}(0) - \hat{\beta}(w_t^* - w_t) \equiv \hat{\beta}(0) - \hat{\beta}(x_t - \lambda_0)$ can be interpreted unambiguously as the opportunity cost of failing to set w_t equal to w_t^* or simply as the cost of labor market disequilibrium $x_t - \lambda_0$ in period t . Later in our determination of the optimal money wage adjustment policy we shall rely exclusively on its quadratic approximation, which can be expressed as:

$$(16) \quad - \frac{\hat{\beta}''(0)}{2} (w_t^* - w_t)^2 \equiv - \frac{\hat{\beta}''(0)}{2} (x_t - \lambda_0)^2 ;$$

where we have used (15-a). The proportionality factor, $-\hat{\beta}''(0)/2$, in the above quadratic disequilibrium cost function is positive by (15-b). In words, we have shown that the disequilibrium cost is approximately proportional to the square value of the rate of the deviation of the actual from "optimal" money wage or of the perceived size of labor market disequilibrium.

Against the disequilibrium cost (16) the cost of money wage adjustment must be weighed. Clearly, there are various kinds of costs a money wage adjustment would give rise to. Its administration may require some direct costs; or, as was briefly indicated in the introduction, it may depreciate the value of money wage as a reliable information signal to workers job-hunting in the labor market and hence affect the current as well as future labor supply schedules to the firm unfavorably; or it may trigger the internal equity of wage structure within a firm and create troublesome labor management problems;²¹ or it may give rise to a labor

²¹ Doeringer and Piore [5].

dispute because, as Keynes observed in his "General Theory," workers are concerned more with relative than absolute wages and tend to oppose any cut in money wages which is believed to lower their wages relative to wages elsewhere;²² and so on. We shall, however, leave the more systematic study of raisons d'être of money wage adjustment costs for another occasion, and concentrate on the analysis of their microeconomic implications in this paper.

We know from (4) that if $x_{t-1} + \xi_t \geq \lambda_1$ our entrepreneur cuts the level of money wage by the rate equal to $x_{t-1} + \xi_t$ and if $x_{t-1} + \xi_t \leq \lambda_2$ he increases it by the rate whose absolute value is equal to $-(x_{t-1} + \xi_t)$. Let us suppose then that the adjustment cost for a wage increase in period t equals the sum of the lump-sum adjustment cost: C_{1t} and the proportional adjustment cost: $C'_{1t}(x_{t-1} + \xi_t)$ and the adjustment cost for a wage cut in period t again equals the lump-sum adjustment cost: C_{2t} and the proportional adjustment cost: $-C'_{2t} \cdot (x_{t-1} + \xi_t)$.²³

The lump-sum costs, C_{1t} and C_{2t} , and the coefficients of the proportional costs, C'_{1t} and C'_{2t} , are all assumed to be independent of the rate of money wage adjustment. However, they must be in some way or another related to the firm's cost-profit structure. Accordingly, we shall suppose in the present paper that all C_{1t} , C_{2t} , C'_{1t} and C'_{2t} are proportional to the trend factor of the expected profit, given by $\exp(\hat{v}_t)$;

²² Keynes [14], pp. 13-15.

²³ It is easy to generalize this lump-sum-cum-proportional money wage adjustment cost equation to the one which is an arbitrary polynomial function of the rate of money wage change. We can thus approximate any form of money wage adjustment equation to any desired degree of accuracy. However, to simplify the analysis we shall not seek this generalization in the present paper.

that is, we put

$$C_{1t} \equiv c_1 \cdot \exp(\hat{v}_t) , \quad C_{2t} \equiv c_2 \cdot \exp(\hat{v}_t) ,$$

$$C'_{1t} \equiv c'_1 \cdot \exp(\hat{v}_t) , \quad C'_{2t} \equiv c'_2 \cdot \exp(\hat{v}_t) ;$$

where the trend-free lump-sum adjustment costs, c_1 and c_2 , and the trend-free coefficients of proportional adjustment costs, c'_1 and c'_2 , are all assumed to be invariant over time. This is only one of many possible specifications of money wage adjustment costs, and should be regarded as a rough approximation to the more realistic ones; but for the sake of analytical simplicity we shall stick to this in the following.

The firm's total loss is the sum of the disequilibrium cost and the adjustment cost. Therefore, the expected trend-free total loss in period t can be calculated as

$$(17) \quad \hat{\Lambda}_t \equiv \hat{E} \left\{ \frac{\hat{\rho}''(0)}{2} (x_t - \lambda_0)^2 | x_0 \right\} + c_1 \cdot \hat{\Pr} \{ x_{t-1} + \varepsilon_t \geq \lambda_1 | x_0 \}$$

$$+ c_2 \cdot \hat{\Pr} \{ x_{t-1} + \varepsilon_t \leq \lambda_2 | x_0 \} + \hat{E} \{ c'_1 (x_{t-1} + \varepsilon_t) | x_{t-1} + \varepsilon_t \geq \lambda_1; x_0 \}$$

$$- \hat{E} \{ c'_2 (x_{t-1} + \varepsilon_t) | x_{t-1} + \varepsilon_t \leq \lambda_2; x_0 \}$$

$$= - \frac{\hat{\rho}''(0)}{2} \int_{\lambda_2}^{\lambda_1} (x - \lambda_0)^2 \cdot d\hat{\Pi}_t(x | x_0) + c_1 \cdot \hat{\pi}_{1t} + c_2 \cdot \hat{\pi}_{2t} + c'_1 \cdot \hat{j}_{1t} - c'_2 \cdot \hat{j}_{2t} ;$$

where the probability of a money wage increase $\hat{\pi}_{1t}$, the probability of a money wage cut $\hat{\pi}_{2t}$, the expected rate of a money wage increase \hat{j}_{1t} and the expected rate of a money wage cut \hat{j}_{2t} in period t are given by

$$(18) \quad \hat{\pi}_{1t} \equiv \hat{\Pr}\{x_{t-1} + \varepsilon_t \geq \lambda_1 | x_0\} = \int_{-\infty}^{\infty} \{1 - \hat{F}(\lambda_1 - x)\} \cdot d\hat{\pi}_{t-1}(x|x_0),$$

$$\hat{\pi}_{2t} \equiv \hat{\Pr}\{x_{t-1} + \varepsilon_t \leq \lambda_2 | x_0\} = \int_{-\infty}^{\infty} \hat{F}(\lambda_2 - x) \cdot d\hat{\pi}_{t-1}(x|x_0);$$

$$(19) \quad \hat{j}_{1t} \equiv \hat{E}\{x_{t-1} + \varepsilon_t | x_{t-1} + \varepsilon_t \geq \lambda_1; x_0\} = \int_{-\infty}^{\infty} \int_{\lambda_1 - x}^{\infty} (x + \varepsilon) d\hat{F}(\varepsilon) d\hat{\pi}_{t-1}(x|x_0),$$

$$\hat{j}_{2t} \equiv \hat{E}\{x_{t-1} + \varepsilon_t | x_{t-1} + \varepsilon_t \leq \lambda_2; x_0\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\lambda_2 - x} (x + \varepsilon) d\hat{F}(\varepsilon) d\hat{\pi}_{t-1}(x|x_0).$$

It follows from the Steady-State Theorem that, as $t \rightarrow \infty$, $\hat{\pi}_{t-1}(x|x_0)$ converges to the steady-state distribution $\hat{\pi}(x)$ and hence $\hat{\pi}_{1t}$, $\hat{\pi}_{2t}$, \hat{j}_{1t} and \hat{j}_{2t} , defined above, all converge to constant steady-state values, $\hat{\pi}_1$, $\hat{\pi}_2$, \hat{j}_1 and \hat{j}_2 , given by

$$(20) \quad \hat{\pi}_1 \equiv \int_{-\infty}^{\infty} \{1 - \hat{F}(\lambda_1 - x)\} \cdot d\hat{\pi}(x), \quad \hat{\pi}_2 \equiv \int_{-\infty}^{\infty} \hat{F}(\lambda_2 - x) \cdot d\hat{\pi}(x);$$

$$(21) \quad \hat{j}_1 \equiv \int_{-\infty}^{\infty} \int_{\lambda_1 - x}^{\infty} (x + \varepsilon) d\hat{F}(\varepsilon) d\hat{\pi}(x), \quad \hat{j}_2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\lambda_2 - x} (x + \varepsilon) d\hat{F}(\varepsilon) d\hat{\pi}(x).$$

(The more complete characterizations of these steady-state parameters are given in Corollaries 1 and 2 of the Mathematical Appendix.) Therefore, as $t \rightarrow \infty$, we can conclude that the expected trend-free loss $\hat{\Lambda}_t$ converges to the limit value $\hat{\Lambda}$, given by

$$(22) \quad \hat{\Lambda} \equiv - \frac{\beta''(0)}{2} \int_{\lambda_2}^{\lambda_1} (x - \lambda_0)^2 d\hat{\pi}(x) + c_1 \cdot \hat{\pi}_1 + c_1 \cdot \hat{\pi}_2 + c_1' \cdot \hat{j}_1 - c_2' \cdot \hat{j}_2,$$

independently of the initial condition x_0 . Note that by construction $\hat{\pi}(x)$, $\hat{\pi}_1$, $\hat{\pi}_2$, \hat{j}_1 and \hat{j}_2 are all independent of the return parameter λ_0 .

In the present paper we suppose that our entrepreneur is concerned only with the trend-free profit and costs and has a very long planning horizon. In other words, we suppose that the entrepreneur seeks only to minimize the long-run average of the trend-free loss per unit period, given by (22). The determination of the optimal money wage adjustment policy under the more general objective function would be an interesting but difficult topic for the future research. Since the long-run average trend-free loss $\hat{\Lambda}$ can be regarded as a function of the three parameters, λ_0 , λ_1 and λ_2 , we have succeeded in reducing the entrepreneur's determination of the optimal money wage adjustment policy to a simpler problem of choosing the values of these parameters that would minimize the value of $\hat{\Lambda}$.

6. Optimal Money Wage Adjustment Policy

First of all, we want to minimize the long-run average trend-free loss $\hat{\Lambda}$, given in (22), with respect to the parameter λ_0 . Noting that $\hat{\Pi}(x)$, $\hat{\mu}_1$, $\hat{\mu}_2$, \hat{L}_1 and \hat{L}_2 are all independent of the value of λ_0 , a differentiation of $\hat{\Lambda}$, which is in turn equivalent to a differentiation of $\int_{\lambda_2}^{\lambda_1} (x - \lambda_0)^2 \cdot d\hat{\Pi}(x)$, with respect to λ_0 leads to the following simple optimal condition:

$$(23) \quad \hat{\lambda}_0^* = \int_{\lambda_2}^{\lambda_1} x \cdot d\hat{\Pi}(x) \equiv \hat{E}(x) .$$

In words, the optimal value of λ_0 , given by $\hat{\lambda}_0^*$, is equal to the expected value of the steady-state random variable \underline{x} . Moreover, the above condition can be rewritten in the following useful manner:

$$(24) \quad \lim_{t \rightarrow \infty} \hat{E}(w_t^* - w_t | x_0 - \hat{\lambda}_0^*) = \lim_{t \rightarrow \infty} \hat{E}(x_t - \hat{\lambda}_0^* | x_0 - \hat{\lambda}_0^*) = 0 .$$

In other words, when the entrepreneur chooses the value of λ_0 optimally at the beginning of his planning period, he can predict that the expected rate of the deviation of the actual from "optimal" money wage or equivalently the expected perceived size of disequilibrium in his labor market will vanish in the long-run, independently of the initial size of labor market disequi-

librium. Note that the validity of this useful result hinges upon our quadratic approximation of the disequilibrium cost function (17).

The proposition (24), along with the proposition (12), has established an important conclusion that even in the Keynesian economy in which money wage adjustment is costly, disequilibrium in the individual labor market is averaged out to zero in the long-run and that the "optimal" money wage in the "Wicksellian" economy in which money wage adjustment is costless reestablishes its position as the long-run average optimal money wage! Does this mean that in the long-run no trace of disequilibrium will be left in the labor market even under the assumption of non-negligible money wage adjustment costs? The answer is clearly "no"; but, we must examine the characteristics of the optimal money wage adjustment policy in more depth in order to give a satisfactory justification to this negative answer.

Let us substitute (23) into (24); then we can rewrite $\hat{\Lambda}$ as a function only of λ_1 and λ_2 :

$$(25) \quad \hat{\Lambda} = -\frac{\hat{p}''(0)}{2} \cdot \hat{\text{Var}}(\underline{x}) + \hat{c}_1 \cdot \hat{\pi}_1 + \hat{c}_2 \cdot \hat{\pi}_2 + \hat{c}_1' \cdot \hat{j}_1 - \hat{c}_2' \cdot \hat{j}_2 ,$$

where we denote by $\hat{\text{Var}}(\underline{x})$ the variance of the steady-state random variable \underline{x} :

$$(26) \quad \hat{\text{Var}}(\underline{x}) \equiv \int_{\lambda_2}^{\lambda_1} \{x - \hat{E}(\underline{x})\}^2 \cdot d\hat{\Pi}(x) .$$

Thus, the disequilibrium cost becomes, under the optimal choice of the value of λ_0 , proportional to the steady-state variance of the random variable x_t . Our entrepreneur is now faced with the following trade-

off in his determination of the optimal values of the parameters λ_1 and λ_2 . If the width between the ceiling λ_1 and the floor λ_2 is set narrow, the steady-state variance $\hat{\text{Var}}(\underline{x})$ and hence the disequilibrium cost become small. However, in this situation, the money wage is adjusted frequently, and the large lump-sum adjustment cost can be expected. If, on the other hand, the width between λ_1 and λ_2 is widened, the disequilibrium cost will increase whereas the lump-sum adjustment cost will diminish. Therefore, the optimal values of λ_1 and λ_2 , to be denoted by $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$, must be chosen by balancing between these two conflicting costs.

Unfortunately, the general characterization of the optimal money wage adjustment policy seems difficult. Therefore, in order to shed more light on its nature, we must turn to the analysis of a special case.

7. A Special Case

Our special model assumes that our entrepreneur believes the "optimal" money wage level fluctuating over time according to the Bernoulli trial multiplicative random walk model.²⁴ The Bernoulli random walk model is the discrete-time, discrete-state analogue of the celebrated Wiener process.²⁵ Then, the (subjective) probability distribution of

²⁴In the Mathematical Appendix we also discuss another special case in which the subjective probability distribution $\hat{F}(\xi)$ is a mixture of two exponential distributions concentrated on $0, \infty$ and $-\infty, 0$, respectively. However, because relatively little new information can be obtained from this special model, we shall not examine it in the text; the derivation of the optimal money wage adjustment policy for this case is left as an exercise for the interested reader.

²⁵See, for example, Feller [7] for an excellent exposition of the Bernoulli random walk model. See also Cox and Miller [4].

ξ_t can be specified as follows:

$$(27) \quad \hat{\Pr}\{\xi_t = +s\} = q, \quad \text{and} \quad \hat{\Pr}\{\xi_t = -s\} = 1-q;$$

where $s > 0$ is a step size and $0 < q < 1$ is the probability of a positive jump. Then, the mean $\hat{\mu}$ and variance of a jump $\hat{\text{Var}}(\xi_t)$ are given by

$$(28-a) \quad \hat{E}(\xi_t) \equiv \hat{\mu} = (2q - 1) \cdot s; \quad -s \leq \hat{\mu} \leq s;$$

$$(28-b) \quad \hat{\text{Var}}(\xi_t) = s^2 - \hat{\mu}^2.$$

In this random walk model, the position of the random variable x_t as well as that of λ_1 and λ_2 can take only the discrete values of $0, \pm s, \pm 2s, \dots$.

Let $\hat{\pi}(x)$ denote the steady-state probability of $x = x$, for $x = \lambda_1, \lambda_2 + s, \dots, \lambda_1 - s$ and λ_1 , in this Bernouilli random walk model with two return barriers at λ_1 and λ_2 . Then, we are able to show in the Mathematical Appendix that $\hat{\pi}(x)$ can be expressed as

$$(29) \quad \hat{\pi}(x) = \frac{s \cdot (1 - Q^{-\lambda_1}) \cdot (Q^{x - \lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} \quad \text{for } x = \lambda_2, \lambda_2 + s, \dots, 0,$$

$$= \frac{s \cdot (Q^{-\lambda_2} - 1) \cdot (1 - Q^{x - \lambda_1})}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} \quad \text{for } x = 0, s, \dots, \lambda_1;$$

where $Q \equiv \{q/(1-q)\}^{1/s} \equiv \{(\hat{\mu}+s)/(\hat{\mu}-s)\}^{1/s}$ if $\hat{\mu} \neq 0$. (If $\hat{\mu} = 0$ but

$s^2 > 0$, the expression of $\pi(x)$ can be obtained by applying l'Hopital rule to (29).²⁶ In the following we shall not distinguish this special case.) We can also record the explicit expressions of $\hat{E}(x)$, $\hat{Var}(x)$, $\hat{\pi}_1$, $\hat{\pi}_2$, \hat{j}_1 and \hat{j}_2 , which are obtained in the Mathematical Appendix, as follows:

$$(30-a) \quad \hat{E}(x) = \frac{1}{2} \left\{ \frac{\lambda_2^2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1^2 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} - \frac{s^2}{\hat{\mu}} \right\},$$

$$(30-b) \quad \hat{Var}(x) = \frac{1}{3} \left\{ \frac{\lambda_2^3 \cdot (1 - Q^{-\lambda_1}) + \lambda_1^3 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} - s^2 \right\} - \frac{1}{4} \left[\left\{ \frac{\lambda_2^2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1^2 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} \right\}^2 - \left(\frac{s^2}{\hat{\mu}} \right)^2 \right],$$

$$(30-c) \quad \hat{\pi}_1 = \frac{\hat{\mu} \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)},$$

$$(30-d) \quad \hat{\pi}_2 = \frac{\hat{\mu} \cdot (1 - Q^{-\lambda_1})}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)},$$

$$(30-e) \quad \hat{j}_1 = \frac{\hat{\mu} \cdot \lambda_1 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)},$$

²⁶See equation (43-b) in the Mathematical Appendix for the explicit formula.

$$(30-f) \quad \hat{\lambda}_2 = \frac{\hat{\mu} \cdot \lambda_2 (1 - Q^{-\lambda_1})}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} .$$

Substituting these expressions into (25), we can explicitly calculate the long-run average trend-free loss:

$$(31) \quad \hat{\Lambda} = - \frac{\hat{\beta}''(0)}{6} \left\{ \frac{\lambda_2^3 \cdot (1 - Q^{-\lambda_1}) + \lambda_1^3 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} - s^2 \right\} \\ + \frac{\hat{\beta}''(0)}{8} \left[\left\{ \frac{\lambda_2^2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1^2 \cdot (Q^{-\lambda_2} - 1)}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} \right\}^2 - \left(\frac{s}{\hat{\mu}} \right)^2 \right] \\ + \frac{\hat{\mu} \{ (c_1 + c_1' \cdot \lambda_1) (Q^{-\lambda_2} - 1) + (c_2 - c_2' \cdot \lambda_2) (1 - Q^{-\lambda_1}) \}}{\lambda_2 \cdot (1 - Q^{-\lambda_1}) + \lambda_1 \cdot (Q^{-\lambda_2} - 1)} .$$

By minimizing this with respect to $\lambda_1 \geq s$ and $\lambda_2 \leq -s$, we can get the optimal values of λ_1 and λ_2 as functions of the basic parameters of our model, $-\hat{\beta}''(0)$, $\hat{\mu}$, s , c_1 , c_1' , c_2 and c_2' . Then, by substituting them into (30-a) and noting that $\hat{\lambda}_0^* = \hat{\xi}(\underline{x})$, we can also express the optimal value of λ_0 as a function of the same set of the basic parameters.

Unfortunately, the expression of $\hat{\Lambda}$, we have just given in (31), is still so complicated that we have not been able to obtain the closed-form solutions of the optimal values of the parameters. (We are planning to examine their properties by the computer analysis.) However, there is at least one important special case for which the required computations

of the optimal values of the parameters can be done by pencils and papers. It is the case in which the lump-sum adjustment cost c_2 of a money wage cut is prohibitively high. Clearly, we have in this case

$$(32) \quad \hat{\lambda}_2^* = -\infty,$$

but the optimal value of the ceiling $\hat{\lambda}_1^*$ is yet to be determined. Money wage in this case is imperfectly flexible upwards and absolutely rigid downwards. Note that if the average growth rate of the "optimal" money wage $\hat{\mu}$ is non-positive x_t will drift to $-\infty$. Hence, in order to make the problem non-trivial we must assume $0 < \hat{\mu} \leq s < \infty$. Now if we let $\lambda_2 \rightarrow -\infty$ in (30-a)-(30-f) we can simplify their expressions as

$$(33-a) \quad \hat{e}(\tilde{x}) = \frac{1}{2} \left(\lambda_1 - \frac{s^2}{\hat{\mu}} \right)$$

$$(33-b) \quad \hat{\text{Var}}(\tilde{x}) = \frac{1}{12}\lambda_1^2 - \frac{1}{3}s^2 + \frac{1}{4} \left(\frac{s^2}{\hat{\mu}} \right)^2,$$

$$(33-c) \quad \hat{\mu}_1 = \frac{\hat{\mu}}{\lambda_1},$$

$$(33-d) \quad \hat{\mu}_2 = 0,$$

$$(33-e) \quad \hat{\lambda}_1 = \hat{\mu},$$

$$(33-f) \quad \hat{\lambda}_2 = 0$$

Then, the long-run average trend-free loss becomes

$$(34) \quad \hat{\Lambda} = -\frac{\hat{\rho}''(0)}{2} \left\{ \frac{1}{12}\lambda_1^2 - \frac{1}{3}s^2 + \frac{1}{4} \left(\frac{s^2}{\hat{\mu}} \right)^2 \right\} + \frac{c_1 \cdot \hat{\mu}}{\lambda_1} + c_1' \cdot \hat{\mu},$$

which is a simple convex function of λ_1 .

A minimization of $\hat{\lambda}$ with respect to $\lambda_1 \geq s$ leads to the optimal value of the ceiling threshold $\hat{\lambda}_1^*$ as

$$(35) \quad \hat{\lambda}_1^* = \begin{cases} \left\{ \frac{12\hat{\mu}c_1}{-\hat{\beta}''(0)} \right\}^{1/3}, & \text{when } s \geq \hat{\mu} \geq -\frac{s^3 \hat{\beta}''(0)}{12c_1}, \\ = s & \text{when } -\frac{s^3 \hat{\beta}''(0)}{12c_1} \geq \hat{\mu} > 0. \end{cases}$$

It should be noted here that in this special case the optimal ceiling threshold $\hat{\lambda}_1^*$ is independent of the coefficient of the proportional money wage adjustment cost c_1 . Now the equation (35) states that when $-s^3 \cdot \hat{\beta}''(0)/12 \cdot c_1 \leq \hat{\mu} \leq s$ the higher the expected rate of change in the "optimal" money wage $\hat{\mu}$ or the higher the lump-sum adjustment cost of a money wage increase, relative to the coefficient $-\hat{\beta}''(0)/2$ of the disequilibrium cost, the higher the optimal value of the ceiling threshold $\hat{\lambda}_1^*$. Because this would induce the entrepreneur to save the expected adjustment cost at the expense of the relatively lowered disequilibrium cost. When, on the other hand, $\hat{\mu} \leq -s^3 \cdot \hat{\beta}''(0)/12 \cdot c_1$, $\hat{\lambda}_1^*$ is anchored to the minimum step level s , independently of the parameter values.

We can also calculate the optimal value of $\hat{\lambda}_0^*$, which was shown to be equal to $\hat{E}(x)$ in (23):

$$(36) \quad \hat{\lambda}_0^* = \hat{E}(x) = \frac{1}{2} \left\{ \left(\frac{12 \cdot \hat{\mu} \cdot c_1}{-\hat{\beta}''(0)} \right)^{1/3} - \left(\frac{s^2}{\hat{\mu}} \right)^2 \right\}, \quad \text{when } s \geq \hat{\mu} \geq \frac{-s^3 \cdot \hat{\beta}''(0)}{12 \cdot c_1}, \\ = \frac{1}{2} \left(s - \frac{s^2}{\hat{\mu}} \right) \leq 0, \quad \text{when } \frac{-s^3 \cdot \hat{\beta}''(0)}{12 \cdot c_1} \geq \hat{\mu} > 0.$$

Therefore, when $s \geq \hat{u} \geq -s^3 \hat{\beta}''(0)/12c_1$, $\hat{\lambda}_0^*$ is increasing in \hat{u} and c_1 , and decreasing in $-\hat{\beta}''(0)$ and s^2 ; and, when $-s^3 \hat{\beta}''(0)/12c_1 \geq \hat{u} > 0$, it is increasing in \hat{u} and decreasing in s^2 but independent of $-\hat{\beta}''(0)$ and c_1 .

It is also very useful to obtain the explicit expression of the steady-state variance $\hat{\text{Var}}(\underline{x})$ induced by the optimal money wage adjustment policy in this special case. Substituting (35) into (33-b), we get

$$(37) \quad \hat{\text{Var}}(\underline{x}) = 48^{-1/3} \cdot \left(\frac{2\hat{u}c_1}{-\hat{\beta}''(0)} \right)^{2/3} - \frac{1}{3}s^2 + \frac{1}{4} \left(\frac{s^2}{\hat{u}} \right)^2, \quad \text{when } s \geq \hat{u} \geq \frac{-s^3 \hat{\beta}''(0)}{12c_1}$$

$$= \frac{s^2}{4} \left(\frac{s^2}{\hat{u}^2} - 1 \right), \quad \text{when } \frac{-s^3 \hat{\beta}''(0)}{12c_1} \geq \hat{u} > 0.$$

The function $(s^2/4)(s^2/\hat{u}^2 - 1)$ is monotonically increasing in s and monotonically decreasing in \hat{u} , while the function

$48^{-1/3}(-2\hat{u} \cdot c_1/\hat{\beta}''(0))^{2/3} - s^2/3 + (s^2/\hat{u})^2/4$ is monotonically increasing in c_1 and s , monotonically decreasing in $\hat{\beta}''(0)$, but decreasing in \hat{u} for $0 < \hat{u} \leq (s^3/3)^{1/2} \cdot (-\hat{\beta}''(0)/c_1)^{1/4}$ and increasing in \hat{u} for $\hat{u} > (s^3/3)^{1/2} \cdot (-\hat{\beta}''(0)/c_1)^{1/4}$. We can thus summarize the possible cases of the relation between \hat{u} and $\hat{\text{Var}}(\underline{x})$ as follows. (i) When

$s \geq (-12c_1/\hat{\beta}''(0))^{1/2}$, $\hat{\lambda}_1^*$ equals s for any possible values of \hat{u} , (i.e. $0 < \hat{u} \leq s$), and consequently $\hat{\text{Var}}(\underline{x})$ is monotonically decreasing

in \hat{u} for any \hat{u} such that $0 < \hat{u} \leq s$. (ii) If

$(-12c_1/\hat{\beta}''(0))^{1/2} > s \geq (9c_1/\hat{\beta}''(0))^{1/2}$; $\hat{\lambda}_1^* = s$ for $0 < \hat{u} \leq -s^3 \hat{\beta}''(0)/12c_1$ and $\hat{\lambda}_1^* = (-12\hat{u}c_1/\hat{\beta}''(0))^{1/3} > s$ for $-s^3 \hat{\beta}''(0)/12c_1 < \hat{u} \leq s$; but $\hat{\text{Var}}(\underline{x})$

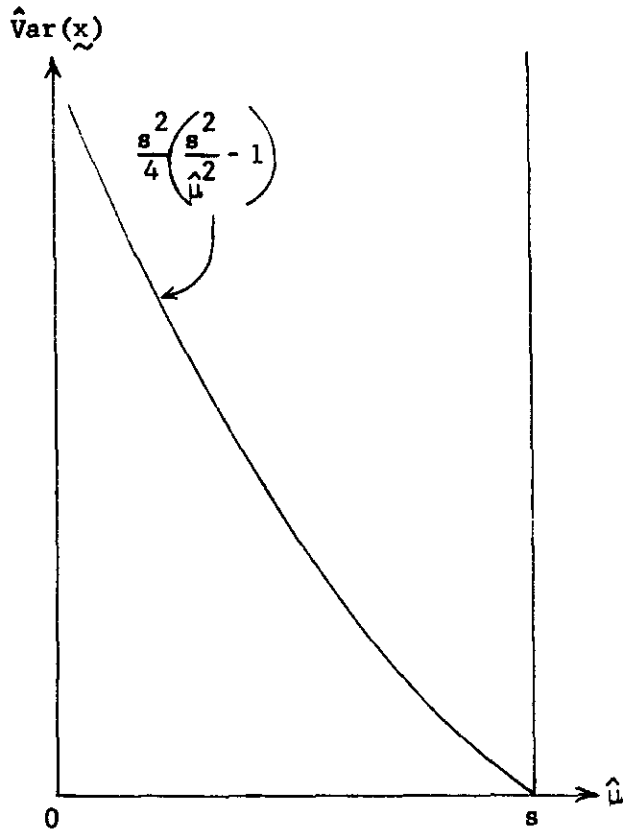
is still monotonically decreasing in \hat{u} . (iii) If $0 < s < (9c_1/2\hat{\beta}''(0))^{1/2}$, $\hat{\text{Var}}(\underline{x})$ is monotonically decreasing in \hat{u} for $0 < \hat{u} \leq (s^3/3)^{1/2} (-\hat{\beta}''(0)/c_1)^{1/4}$

but becomes monotonically increasing in \hat{u} for $(s^3/3)^{1/2}(-\hat{\beta}''(0)/c_1)^{1/4} < \hat{u} \leq s$. The following diagram summarizes these three possible cases.

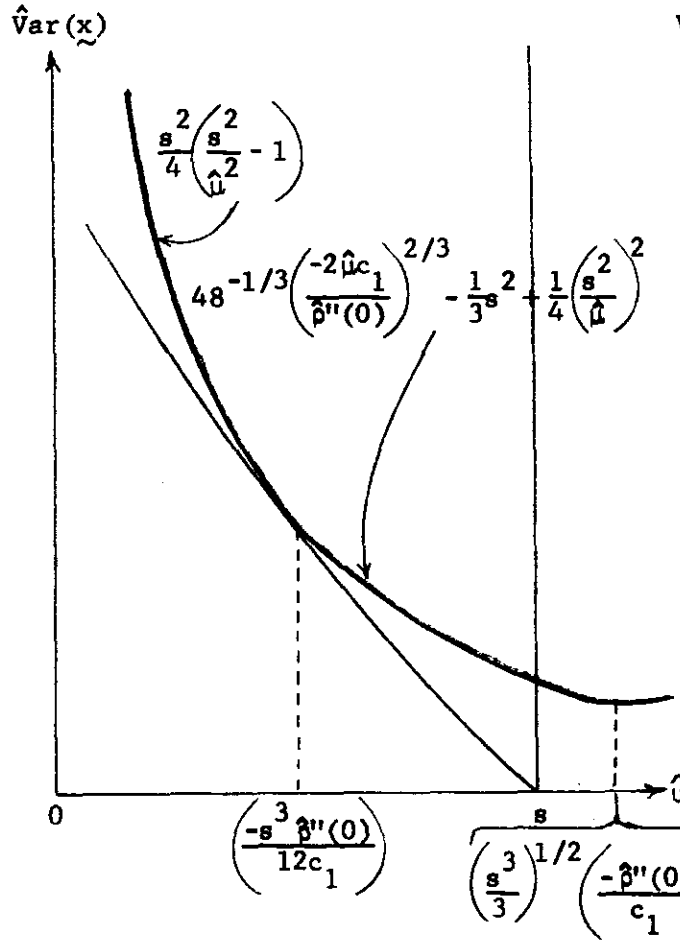
Even though we showed in the previous section that the entrepreneur's perceived size of disequilibrium in his own labor market, $x_t - \hat{\lambda}_0^*$, will average out to be zero in the long-run, the stochastic steady-state to which the entrepreneur's money wage adjustment activity is expected to converge in the long-run is far from a state of "tranquillity." The stochastic steady-state is a state in which the perceived size of labor market disequilibrium is fluctuating widely between the state of positive disequilibrium and the state of negative disequilibrium. What we showed in the previous section only means that the long-run average of positive disequilibria over time and that of negative disequilibria over time tend to balance with each other. In fact, our analysis of the special case in this section has demonstrated that the fluctuation of disequilibrium over time will never shrink to zero as long as the cost of money wage adjustment is non-negligible. We have also exhibited by examining the properties of the steady-state variance $\hat{\text{Var}}(x)$ that the "structure" of the stochastic steady-state is determined by the basic structural parameters of the model, in particular, by the expected rate of change of the "optimal" money wage \hat{u} .

8. Concluding Remarks

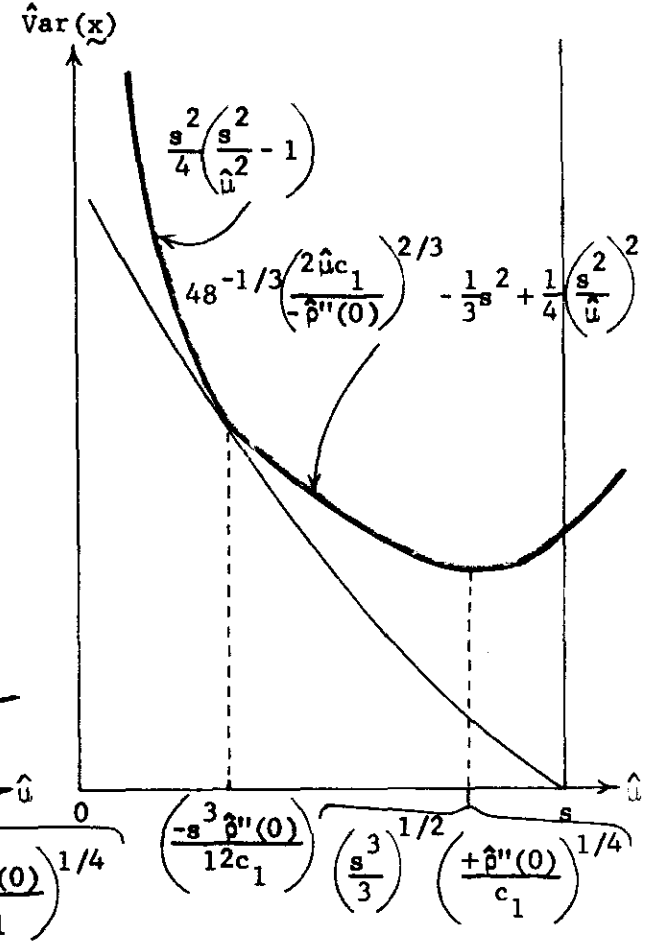
(1) Although the present paper has been confined to the analysis of an individual firm's wage adjustment, the formal method developed here can be applied to any kind of optimal adjustment problem with lump-sum adjustment costs. A list of such problems are: price, inventory and



(i) $s \geq (-12c_1/\hat{\rho}''(0))^{1/2}$



(ii) $(-12c_1/\hat{\rho}''(0))^{1/2} > s \geq (-9c_1/\hat{\rho}''(0))^{1/2}$



(iii) $0 < s < (-9c_1/\hat{\rho}''(0))^{1/2}$

DIAGRAM: Three Possible Cases of the Relation between $\hat{\text{Var}}(x)$ and \hat{u}

labor adjustment problems of the firm; cash management problem of the firm or the household; foreign reserve management of the government; and so on. In many cases, a simple reinterpretation of symbols would prove sufficient.²⁷

(2) It is well-known both theoretically and empirically that the average rate of unemployment is determined not only by the average size of labor market disequilibrium but also by its dispersion. In fact, it can be easily shown that in our special model of the firm developed in Part I of this series that the long-run average rate of unemployment is greater than what is called its "normal" rate by the magnitude approximately proportional to the steady-state variance of labor market disequilibrium. Therefore, our proposition that the average size of labor market disequilibrium tends to vanish in the long-run by no means implies that the long-run average rate of unemployment would also approach its normal rate, because we also showed in the present paper that the existence of money wage adjustment costs prevents the variance of labor market disequilibrium from shrinking to nil even in the stochastic steady-state. In particular, under the assumption of complete downward rigidity of money wage this steady-state variance was shown to be negatively correlated with the average growth rate of the "optimal" money wage at least for the relevant range of the values of the latter, which in turn was proved to be equal to the long-run average growth rate of actual money wage. Therefore, we have in fact established the following fundamental proposition: the higher the long-run average growth rate of money wage the lower the long-run average rate of unemployment. The theory of "long-run Phillips curve"

²⁷ An interesting paper by Barro [2] on the theory of price adjustment was already discussed in footnote 19. The cash management problem has been fairly extensively explored in recent years. See, for example, Miller and Orr [15].

we shall develop in Part IV of this series is nothing but the translation of this microeconomic proposition into the law of macroeconomics.

(3) The reader might have recognized a certain analogy between our model of money wage adjustment and the "satisficing model" advocated by Herbert Simon.²⁸ Our firm does not attempt to maximize its short-run expected profit every period but is satisfied so long as the rate of the deviation of the level of actual money wage from its short-run-profit-maximizing level (i.e., the subjective measure of labor market disequilibrium) does not exceed the upper "threshold" nor fall short of the lower "threshold." A change in the level of money wage is induced only when the rate of deviation (or the size of disequilibrium) strays from the "range of satisfaction" bound by these two thresholds. Like the "aspiration level" of the satisficing model, the levels of these thresholds are considered to be fixed in the short-run. But in the long-run they will gradually adjust themselves upwards or downwards on the basis of experiences accumulated over time. This adjustment of the threshold levels would not, however, drive the firm to behave like a firm in neo-classical economic theory even in the long-run equilibrium. In fact, we succeeded in demonstrating that the firm would only approach a "stochastic steady-state" in which its daily behavior shows no tendency to assimilate with that of the short-run profit-maximizing firm. Although there may exist a wide gap between the satisficing model which was proposed as an alternative to the notion of economic man (or economic agent) as a maximizing or minimizing animal and our model of money wage adjustment which is based upon the hypothesis of the firm as a minimizer of "long-run" average loss but with an explicit

²⁸See Simon [19, 20, 21].

introduction of adjustment costs, both have much richer implications for macroeconomics than the model of the firm as a short-run-profit-maximizer without any adjustment costs. This is simply because both theories are devised primarily for the analysis of disequilibrium situations, which is what macroeconomics is all about.

(4) The micro-foundations of our Keynesian disequilibrium dynamics have been laid. In the sequel we shall embark on a macroeconomic exploration.

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MATHEMATICAL APPENDIX:

SOME STEADY-STATE THEOREMS FOR THE RANDOM WALK MODEL

WITH TWO RETURN BARRIERS

A-1. Introduction

This Mathematical Appendix will be devoted to the study of a stochastic process, called the random walk model with two return barriers. Although this study was motivated by our Keynesian theory of the firm's money wage adjustment developed in the text, it can be read as an independent mathematical treatise.

After presenting our problem in this introduction, we shall give a brief exposition of some elements of the random walk theory which will be utilized in this Mathematical Appendix. (Our exposition will follow closely that of Feller [5].) This will make our study self-contained and at the same time facilitate the understanding of the reader who is not familiar with the random walk theory. But, in the first place we must explain the mathematical notations used in this Appendix.

We denote by $\overline{a,b}$, $\overline{a,b}$, $\overline{a,b}$ and $\overline{a,b}$ an open interval $a < z < b$, a closed interval $a \leq z \leq b$, a half-open interval $a < z \leq b$ and another half-open interval $a \leq z < b$, respectively. The limiting case where either a or b or both are replaced by $\pm\infty$ is admitted; in particular, the whole line is the interval $\overline{-\infty, \infty}$. An arbitrary interval is represented by I . If I is, for instance, $\overline{a,b}$ and y is a real number, $I-y$ means $\overline{a-y, b-y}$. $\Phi\{I\}$ denotes the probability that a probability distribution (or a probability measure) Φ assigns to an interval I . When Φ is a discrete distribution, $\Phi\{y\}$

represents the probability atom at a point y . The (cumulative) probability distribution function $\Phi(x)$, used in the text to represent a probability distribution Φ , can be defined as $\Phi\{\overline{-\infty, x}\}$. The Lebesgue-Stieltjes integral of a function with respect to a probability distribution Φ over an interval $\overline{a, b}$ is written as $\int_a^b u(x) \cdot \Phi\{dx\}$.

Let $\xi_1, \xi_2, \xi_3, \dots$, denote mutually independent random variables with a common probability distribution F , and let the sequence of random variables $\{x_t\}$ be defined by the rule:

$$(1) \quad \begin{cases} x_t = 0 & \text{when } x_{t-1} + \xi_t \leq \lambda_2 \text{ or } x_{t-1} + \xi_t \geq \lambda_1 \\ x_t = x_{t-1} + \xi_t & \text{when } \lambda_2 < x_{t-1} + \xi_t < \lambda_1, \end{cases}$$

where $\lambda_2 < 0 < \lambda_1$. If we start from a given initial position x_0 , the position of x_t fluctuates along an open interval $\overline{\lambda_2, \lambda_1}$ as a partial sum of $\xi_1, \xi_2, \dots, \xi_t$ and x_0 . However, when $x_{t-1} + \xi_t$ enters into either of a half-open interval $\overline{-\infty, \lambda_2}$ or another half-open interval $\overline{\lambda_1, +\infty}$, the system returns instantaneously to the origin $x_t = 0$, and the process starts anew from the origin. The two numbers λ_1 and λ_2 are called return barriers and the sequence $\{x_t\}$ is said to constitute a random walk with return barriers at λ_1 and λ_2 .

We denote by Π_t the probability distribution of x_t ; that is, we put for an interval $I \subset \overline{\lambda_2, \lambda_1}$ and $t = 1, 2, \dots$

$$(2) \quad \Pi_t\{I\} \equiv \Pr\{x_t \in I\}.$$

The probability distribution Π_0 is concentrated on the initial position $x_0 \in \overline{\lambda_2, \lambda_1}$. By convention we put $\Pi_t\{I\} = 0$ for $I \subset \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}$.

Our main concern in this Mathematical Appendix is to show that under very weak conditions, as $t \rightarrow +\infty$, x_t converges to a steady-state random variable \tilde{x} in the sense that Π_t converges to a steady-state distribution $\tilde{\Pi}$ independently of the initial condition x_0 . We shall also give a complete characterization of this steady-state distribution.

A-2. The Random Walk Model with Two Absorbing Barriers

The study of the random walk with two return barriers, given by (1), is intimately connected with the study of the random walk model with two absorbing barriers at λ_1 and λ_2 . Our exposition of this model in the following closely follows that of Feller [5], Chapters XII and XVIII. Let

$$(3) \quad S_0 = 0, \text{ and } S_t \equiv \xi_1 + \xi_2 + \dots + \xi_t, \quad t = 1, 2, \dots;$$

then the sequence $\{S_t\}$ constitutes the random walk generated by F which starts from the origin. However, if

$$(4) \quad S_1 \in \overline{\lambda_2, \lambda_1}, \dots, S_{t-1} \in \overline{\lambda_2, \lambda_1} \text{ and } S_t \in I \text{ where } I \subset \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty},$$

then we say that either of the absorbing intervals, $\overline{-\infty, \lambda_2}$ and $\overline{\lambda_1, \infty}$ is entered for the first time in period t and at a point of I . The process terminates whenever the event (4) occurs. The absorption period T is defined as the period of the first entry into one of the absorbing intervals, and the absorption point is defined accordingly by S_T . In other words, T is given by

$$(5) \quad T = \{t \mid \text{The event (4) takes place for the first time in period } t\}.$$

Clearly, both T and S_T are random variables. But if the event (4) does not take place at all, they become "defective" random variables. (A probability distribution, say, Φ is called "defective" if $\Phi\{\overline{-\infty, \infty}\} < 1$, whereas it is called "proper" if $\Phi\{\overline{-\infty, \infty}\} = 1$.) For the joint distribution of the pair (T, S_T) we write

$$(6) \quad H_t\{I\} \equiv \Pr\{T = t \text{ and } S_T \in I\}, \quad t = 1, 2, \dots$$

By convention we set $H_t\{I\} = 0$ for $I \in \overline{\lambda_2, \lambda_1}$. The marginal distributions of H_t are given by

$$(7) \quad \Pr\{T = t\} = H_t\{\overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}\} = H_t\{\overline{-\infty, \infty}\}, \quad t = 1, 2, \dots,$$

$$(8) \quad \Pr\{S_T \in I\} = \sum_{t=1}^{\infty} H_t\{I\} \equiv H\{I\}.$$

The random variables T and S_T are proper if and only if $H\{\overline{-\infty, \infty}\} = 1$.

Let $G_t\{I\}$ denote the probability distribution of the event that in period t a point of $I \subset \overline{\lambda_2, \lambda_1}$ is reached and up to period t no entry into the absorbing intervals took place; that is, for $I \subset \overline{\lambda_2, \lambda_1}$ and $t = 1, 2, \dots$ we put

$$(9) \quad G_t\{I\} \equiv \Pr\{S_1 \in \overline{\lambda_2, \lambda_1}, \dots, S_{t-1} \in \overline{\lambda_2, \lambda_1} \text{ and } S_t \in I\}.$$

We can extend the above definition to all the intervals on the line by letting $G_t\{I\} = 0$ for $I \subset \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}$. As a convention we shall denote by G_0 the atomic distribution with a unit jump at the origin:

$$(10) \quad G_0\{I\} = 0 \text{ for } I \not\ni 0 \text{ and } G_0\{I\} = 1 \text{ for } I \ni 0.$$

Summing G_t over t we obtain

$$(11) \quad G\{I\} \equiv \sum_{t=0}^{\infty} G_t\{I\},$$

if the series converges. We can interpret $G\{I\}$ as the expected number that the random variable x_t will visit an interval $I \subset \overline{\lambda_2, \lambda_1}$ prior to the first entry into the absorbing intervals.

Note that by definition we have

$$(12) \quad G_t\{\overline{-\infty, \infty}\} = G_t\{\overline{\lambda_2, \lambda_1}\} = \Pr\{T > t\}.$$

Summing over t we therefore obtain

$$(13) \quad G\{\overline{-\infty, \infty}\} \equiv \sum_{t=0}^{\infty} G_t\{\overline{-\infty, \infty}\} = \sum_{t=0}^{\infty} \Pr\{T > t\} = \varepsilon(T),$$

if the series converges; where $\varepsilon(T)$ is the expected absorption time.

It is easy to derive the recurrent relations for H_t and G_t by examining their definitions (6) and (9). We have for $I \subset \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}$,

$$(14) \quad H_{t+1}\{I\} = \int_{-\infty}^{\infty} G_t\{I-\xi\} \cdot F\{d\xi\}, \quad t = 1, 2, \dots,$$

and for $I \subset \overline{\lambda_2, \lambda_1}$,

$$(15) \quad G_{t+1}\{I\} = \int_{-\infty}^{\infty} G_t\{I-\xi\} \cdot F\{d\xi\}, \quad t = 1, 2, \dots$$

If we sum these equations over t , we can get the relations

for H and G ; that is, for $I \subset \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}$,

$$(16) \quad H\{I\} = \int_{-\infty}^{\infty} G\{I-\xi\} \cdot F\{d\xi\},$$

and for $I \in \overline{\lambda_2, \lambda_1}$,

$$(17) \quad G\{I\} - G_0\{I\} = \int_{-\infty}^{\infty} G\{I-\xi\} \cdot F\{d\xi\};$$

if the series converges. These relations can be regarded as integral equations determining the unknown probability distributions H and G , respectively. Many interesting results can be derived from these integral equations, but we shall not consider them in depth in the present note.

One of the key results in the theory of random walk with absorbing barriers is the following lemma which spells out the conditions for the "properness" of the random variables T and S_T as well as for the existence of their moments. The proof can be found, for instance, in Feller [5], pp. 380-381 and pp. 566-567.

Lemma 1. (i) If both λ_1 and λ_2 are finite, both the random variables T and S_T are proper and T has finite moments of all orders. S_T has a finite expectation and the equation:

$$(18) \quad \varepsilon(S_T) = \varepsilon(\xi_t) \cdot \varepsilon(T)$$

holds if and only if F has a finite expectation $-\infty < \mu \equiv \varepsilon(\xi_t) < \infty$.

(ii) If λ_1 is finite but $\lambda_2 = -\infty$, we have the following three possibilities. (a) If μ is finite and positive, then T and S_T are proper, have finite expectations and (14) holds. (b) If

$\mu = 0$, then T and S_T are proper, and $E(T) = \infty$. (c) Otherwise either the random walk drifts to $-\infty$ (in which case T and S_T are defective), or else $E(S_T) = \infty$ and $E(T) = \infty$.

(iii) The case where λ_2 is finite but $\lambda_1 = +\infty$ can be treated in the same way as the case (ii).

The equation (18) is called Wald's equation. (See Wald [10].)

Note in passing that if a system of the random walk with absorbing intervals, $[-\infty, \lambda_2]$ and $[\lambda_1, \infty]$, starts at some arbitrary but finite initial condition $S_0 = x_0$, where $x_0 \in (\lambda_2, \lambda_1)$, then we can transform it into the system of the random walk with absorbing intervals, $[-\infty, \lambda_2 - x_0]$ and $[\lambda_1 - x_0, +\infty]$, which starts from the origin. We can then apply the results obtained above to this case without any modification.

A-3. Recurrent Cycles and the Induced Renewal Process

Let us go back to our original random walk model, defined by (1). Starting at a given initial condition x_0 (which is not necessarily the origin), the system returns instantaneously to the origin in period t , whenever

$$(19) \quad x_{t-1} + \xi_t \leq \lambda_2 \quad \text{or} \quad x_{t-1} + \xi_t \geq \lambda_1,$$

and the process starts anew from the origin from then on. We define the k^{th} return period T_k as the period of k^{th} entry into either of the two intervals, $[-\infty, \lambda_2]$ and $[\lambda_1, +\infty]$; that is, we put for $k = 1, 2, \dots$

$$(20) \quad T_k \equiv \{t \mid \text{The event (19) takes place for the } k^{\text{th}} \text{ time in period } t\}.$$

The characteristic nature of our random walk model with two return barriers is that the section of the random walk following the k^{th} return period T_k is a "probabilistic replica" of the random walk after the occurrence of the first return to the origin in period T_1 . Therefore, the sequence of the numbers of periods between two successive returns to the origin, $T_2 - T_1, T_3 - T_2, \dots$, are mutually independent, positive and integer-valued random variables with a (possibly defective) common probability distribution. These random variables are called recurrence cycles. It is clear from (5) that this common recurrence cycle distribution, which we shall denote by Θ , is identical with the probability distribution of the absorption period T in the random walk model with two absorbing barriers at λ_1 and λ_2 . Hence, we have

$$(21) \quad \Theta\{\tau\} \equiv \Pr\{T_k - T_{k-1} = \tau\} \\ = \Pr\{T = \tau\} = H_\tau\{\overline{-\infty, \infty}\} = G_{\tau-1}\{\overline{-\infty, \infty}\} - G_\tau\{\overline{-\infty, \infty}\};$$

where equations (7) and (12) have been employed. It is also clear that the probability distribution of the first return period T_1 , to be denoted by Θ^1 , is identical with that of the absorption period in the random walk model with absorbing barriers at $\lambda_1 - x_0$ and $\lambda_2 - x_0$; that is, we have

$$(22) \quad \Theta^1\{\tau\} \equiv \Pr\{T_1 = \tau\} \\ = \Pr\{t = \tau \mid S_1 + x_0 \in \overline{\lambda_2, \lambda_1}, \dots, S_{t-1} + x_0 \in \overline{\lambda_2, \lambda_1} \\ \text{and } S_t + x_0 \in \overline{-\infty, \lambda_2} \cup \overline{\lambda_1, \infty}\} \\ = \Pr\{t = \tau \mid S_1 \in \overline{\lambda_2 - x_0, \lambda_1 - x_0}, \dots, S_{t-1} \in \overline{\lambda_2 - x_0, \lambda_1 - x_0} \\ \text{and } S_t \in \overline{-\infty, \lambda_2 - x_0} \cup \overline{\lambda_1 - x_0, \infty}\}.$$

If $x_0 = 0$, then Θ^1 coincides with Θ .

Therefore, the sequence of the k^{th} return periods $\{T_k\}$ constitutes a delayed renewal process generated by the common recurrence cycle distribution Θ and the initial recurrence cycle distribution Θ^1 .

(An excellent exposition of the integer-valued renewal theory can be found in Chapter 13 of Feller [4].) A delayed renewal process is said to be persistent if both Θ and Θ^1 are proper. (If one of Θ and Θ^1 is defective, it is said to be transient.) Lemma 1, which states the conditions for the properness of the first absorption period, turns out directly applicable to the classification of our delayed renewal process $\{T_k\}$.

We thus obtain

Lemma 2. (i) If both λ_1 and λ_2 are finite, $\{T_k\}$ is always persistent. (ii) If λ_1 is finite and $\lambda_2 = -\infty$, $\{T_k\}$ is persistent if and only if μ is finite and non-negative. (iii) If λ_2 is finite and $\lambda_1 = +\infty$, $\{T_k\}$ is persistent if and only if μ is finite and non-positive.

Our concern with the classification of the delayed renewal process $\{T_k\}$ would be justified by the next Lemma, often referred as the renewal theorem. This is the fundamental "ergodic" result in the renewal theory and its proof can be found, for example, in Feller [4], p. 318.

Lemma 3. (Renewal Theorem) Let $\{T_k\}$ be a persistent, integer-valued renewal process, then as $t \rightarrow \infty$

$$(23) \quad \theta_t \equiv \Pr\{\text{The renewal event (19) occurs in period } t\} \\ \rightarrow \frac{1}{E(T)} ;$$

if $E(T) = 0$, then $\theta_t \rightarrow 0$.

A-4. The Steady-State Theorem

We are now in a position of proving the steady-state theorem for our random walk model with two return barriers at λ_1 and λ_2 .

Theorem 1. (The Steady-State Theorem)

If either (i) both λ_1 and λ_2 are finite, or (ii) λ_1 is finite, $\lambda_2 = -\infty$ and $0 < \mu < \infty$, or (iii) λ_2 is finite, $\lambda_1 = +\infty$ and $-\infty < \mu < 0$, then as $t \rightarrow \infty$ the random variable x_t converges to a steady-state random variable \underline{x} in the sense that

$$(24) \quad \Pi_t\{I\} \rightarrow \underline{\Pi}\{I\}, \text{ for any } I,$$

independently of the initial position. The steady-state distribution $\underline{\Pi}$ is characterized by the equation:

$$(25) \quad \underline{\Pi}\{I\} = \frac{G\{I\}}{\underline{E}(T)};$$

where G and $\underline{E}(T)$ are given by (11) and (13).

(Proof) It is easy to see from (1) and (23) that for $I \subset \overline{\lambda_2, 0}$ we have

$$\Pi_t\{I\} = \Pr\{x_{t-1} + \varepsilon_t \in I\},$$

and for $I \subset \overline{0, \lambda_1}$ we have

$$\begin{aligned} \Pi_t\{I\} &= \Pr\{x_{t-1} + \varepsilon_t \leq \lambda_2 \text{ or } x_{t-1} + \varepsilon_t \geq \lambda_1\} + \Pr\{x_{t-1} + \varepsilon_t \in I\} \\ &= \theta_t + \Pr\{x_{t-1} + \varepsilon_t \in I\}. \end{aligned}$$

Now it is also easy to see that for $I \in \overline{\lambda_2, \lambda_1}$

$$\begin{aligned}
\Pr\{x_{t-1} + \varepsilon_t \in I\} &= \Pr\{x_0 + S_1 \in \overline{\lambda_2, \lambda_1}, \dots, x_0 + S_t \in I \subset \overline{\lambda_2, \lambda_1}\} \\
&\quad + \sum_{\tau=1}^{t-1} \Pr\{\text{The event (15) occurs in period } \tau\} \\
&\quad \cdot \Pr\{S_1 \in \overline{\lambda_2, \lambda_1}, \dots, S_{t-\tau-1} \in I \subset \overline{\lambda_2, \lambda_1}\} \\
&= G_t^1\{I\} + \sum_{\tau=1}^{t-1} \theta_{t-\tau} \cdot G_{t-\tau}\{I\};
\end{aligned}$$

where $G_t^1\{I\} \equiv \Pr\{x_0 + S_1 \in \overline{\lambda_2, \lambda_1}, \dots, x_0 + S_t \in I \subset \overline{\lambda_2, \lambda_1}\}$, just as $G_t\{I\}$ is defined by (9). Let $t' < t$, then we can rewrite the above equation as

$$\begin{aligned}
\sum_{\tau=1}^{t-1} \theta_{t-\tau} \cdot G_{t-\tau}\{I\} + G_t^1\{I\} &= \sum_{\tau=1}^{t-1} \theta_{t-\tau} \cdot G_{\tau}\{I\} + G_t^1\{I\} \\
&= \sum_{\tau=1}^{t'} \theta_{t-\tau} \cdot G_{\tau}\{I\} + \sum_{\tau=t'+1}^{t-1} \theta_{t-\tau} \cdot G_{\tau}\{I\} + G_t^1\{I\}.
\end{aligned}$$

Now it follows from Lemma 2 that under the conditions stated above the delayed renewal process $\{T_k\}$ is persistent, so that we can apply Lemma 3 (Renewal Theorem) and assert that as $t \rightarrow \infty$

$$\theta_t \rightarrow 1/\varepsilon(T).$$

Therefore, if we let $t \rightarrow \infty$ first and then let $t' \rightarrow \infty$, we have

$$\sum_{\tau=1}^{t'} \theta_{t-\tau} \cdot G_{\tau}\{I\} \rightarrow \frac{1}{\varepsilon(T)} \sum_{\tau=1}^{t'} G_{\tau}\{I\} \rightarrow \frac{G\{I\} - G_0\{I\}}{\varepsilon(T)}.$$

Similarly, if we let $t \rightarrow \infty$ first and then let $t' \rightarrow \infty$, we have

$$0 \leq \sum_{\tau=t'+1}^{t-1} \theta_{t-\tau} G_{\tau}\{I\} \leq \sum_{\tau=t'+1}^{t-1} G_{\tau}\{I\} \leq \sum_{\tau=t'+1}^{\infty} G_{\tau}\{\overline{-\infty, \infty}\} \rightarrow 0 ;$$

where the second inequality is assured by $0 \leq \theta_t \leq 1$ and the last series converges to zero because by Lemma 1 the conditions given above imply that

$$\varepsilon(T) = \sum_{\tau=0}^{\infty} G_{\tau}\{\overline{-\infty, \infty}\} < \infty . \text{ Finally, if we let } t \rightarrow \tau ,$$

$0 \leq G_t^1\{I\} \leq G_t^1\{\overline{-\infty, \infty}\} \rightarrow 0$, because again by Lemma 1 the stated conditions imply that the expected absorption period for the random walk model with absorbing barriers at $\lambda_1 - x_0$ and $\lambda_2 - x_0$ is finite, so that the series $\sum_{t=0}^{\infty} G_t^1\{\overline{-\infty, \infty}\}$ is convergent.

In consequence, we can conclude that as $t \rightarrow \infty$ for $I \subset \overline{\lambda_2, 0}$

$$\Pi_t\{I\} \rightarrow \frac{G\{I\} - G_0\{I\}}{\varepsilon(T)} = \frac{G\{I\}}{\varepsilon(T)} ,$$

and for $I \subset \overline{0, \lambda_1}$

$$\Pi_t\{I\} \rightarrow \frac{G\{I\} - G_0\{I\}}{\varepsilon(T)} + \frac{1}{\varepsilon(T)} = \frac{G\{I\}}{\varepsilon(T)} ;$$

where we have used the definition of G_0 , given by (8), that $G_0\{I\} = 0$ for $I \not\supset 0$ and $G_0\{I\} = 1$ for $I \ni 0$. (Q.E.D.)

The above proof is similar to that of the steady-state theorem for (S,s) inventory policy in the mathematical inventory theory. (See Prabhu [8], pp. 176-177. For the general reference, see, for example, Arrow-Harris-Marschak [1], Arrow-Karlin-Scarf [2] and Scarf-Gilford-Shelly [9].) In the mathematical inventory theory, one has only to consider the random walk probability distribution $F(\xi)$ concentrated on the non-positive

half-line $[-\infty, 0]$ because the level of inventory is necessarily decumulating until it is restocked by an order, so that only the renewal theory is necessary for the proof of the steady-state theorem. Thus, our steady-state theorem includes that of mathematical inventory theory as a special case in the sense that our random walk probability distribution $F(\xi)$ is completely general and hence not only the floor barrier (s in the inventory theory and λ_2 in our model) but also the ceiling barrier (λ_1) must be considered.

In the above theorem, the steady-state distribution $\tilde{\Pi}\{I\}$ has been shown to equal the ratio between the expected number of visits to an interval $I \subset [\lambda_2, \lambda_1]$ prior to the absorption, given by $G\{I\}$, and the expected absorption period, given by $E(T)$, in the random walk model with absorbing barriers at λ_1 and λ_2 . $G\{I\}$ can be computed to any desired degree of accuracy by applying the recurrence relation (15) step by step from $t = 0$ and summing $G_t\{I\}$'s over t . Or if we are lucky enough, the relation (17) can be directly solved to get the explicit form of $G\{I\}$. In this sense, we can claim that the steady-state distribution $\tilde{\Pi}$ is completely characterized by the equation (25).

A-5. Two Corollaries

Let π_{1t} and π_{2t} be the probabilities that $x_{t-1} + \xi_t$ enters into the upper absorption interval $[\lambda_1, \infty)$ and into the lower absorption interval $[-\infty, \lambda_2]$, respectively. They can be easily calculated as

$$(26-a) \quad \pi_{1t} = \Pr\{x_{t-1} + \xi_t \in [\lambda_1, \infty)\} = \int_{-\infty}^{\infty} \pi_{t-1}([\lambda_1 - \xi, \infty)) \cdot F\{d\xi\},$$

$$(26-b) \quad \pi_{2t} \equiv \Pr \{x_{t-1} + \xi_t \in \overline{-\infty, \lambda_2}\} = \int_{-\infty}^{\infty} \pi_{t-1} \overline{-\infty, \lambda_2 - \xi} \cdot F\{d\xi\} .$$

Then, as an application of Theorem 1, we can establish

Corollary 1. Under the same conditions as stated in Theorem 1, as $t \rightarrow \infty$,

π_{1t} and π_{2t} converge to their steady-state probabilities π_1 and π_2 , respectively, which can be characterized as

$$(27) \quad \pi_1 = \frac{H\{\lambda_1, \infty\}}{\varepsilon(T)} \equiv \frac{\Pr \{S_T \in \overline{\lambda_1, \infty}\}}{\varepsilon(T)}, \quad \pi_2 = \frac{H\{-\infty, \lambda_2\}}{\varepsilon(T)} \equiv \frac{\Pr \{S_T \in \overline{-\infty, \lambda_2}\}}{\varepsilon(T)} .$$

(Proof) It follows from Theorem 1 that, as $t \rightarrow \infty$, we have

$$\begin{aligned} \pi_{1t} \rightarrow \pi_1 &\equiv \int_{-\infty}^{\infty} \pi \overline{\lambda_1 - \xi, \infty} \cdot F\{d\xi\} = \int_{-\infty}^{\infty} \frac{G\{\lambda_1 - \xi, \infty\}}{\varepsilon(T)} \cdot F\{d\xi\} \\ \pi_{2t} \rightarrow \pi_2 &\equiv \int_{-\infty}^{\infty} \pi \overline{-\infty, \lambda_2 - \xi} \cdot F\{d\xi\} = \int_{-\infty}^{\infty} \frac{G\{-\infty, \lambda_2 - \xi\}}{\varepsilon(T)} \cdot F\{d\xi\} . \end{aligned}$$

If we apply the relation (16) to the above two equations, then we can immediately obtain the desired results. (Q.E.D.)

Note that the overall probability of the return to the origin in the steady-state, $\pi_1 + \pi_2$, is simply the reciprocal of the expected absorption time $\varepsilon(T)$; that is, we have

$$(28) \quad \pi_1 + \pi_2 = \frac{H\{\lambda_1, \infty\} + H\{-\infty, \lambda_2\}}{\varepsilon(T)} = \frac{H\{-\infty, \infty\}}{\varepsilon(T)} = \frac{1}{\varepsilon(T)},$$

because T is proper under the stated conditions in Theorem 1.

Let j_{1t} and j_{2t} denote the expected values of $x_{t-1} + \xi_t$ when

it enters in $\overline{\lambda_1, \infty}$ and in $\overline{-\infty, \lambda_2}$, respectively. Then, they can be calculated as

$$(29-a) \quad j_{1t} \equiv \int_{-\infty}^{\infty} \int_{\lambda_1 - x_{t-1}}^{\infty} (x_{t-1} + \xi_t) \cdot F\{d\xi_t\} \cdot \Pi_{t-1}\{dx_{t-1}\} \\ = \int_{-\infty}^{\infty} \int_{\lambda_1}^{\infty} \xi \cdot F\{d\xi - x\} \cdot \Pi_{t-1}\{dx\},$$

$$(29-b) \quad j_{2t} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\lambda_2} \xi \cdot F\{d\xi - x\} \cdot \Pi_{t-1}\{dx\}.$$

Then, as a second application of Theorem 1, we can establish

Corollary 2. Under the same conditions as stated in Theorem 1, as $t \rightarrow \infty$, j_{1t} and j_{2t} converge to their steady-state values j_1 and j_2 , respectively, which can be characterized as

$$(30) \quad j_1 = E(S_T | S_T \in \overline{\lambda_1, \infty}) \cdot \pi_1, \quad j_2 = E(S_T | S_T \in \overline{-\infty, \lambda_2}) \cdot \pi_2.$$

(Proof) It follows from Theorem 1 that as $t \rightarrow \infty$, we have

$$j_{1t} \rightarrow j_1 = \int_{-\infty}^{\infty} \int_{\lambda_1}^{\infty} \xi \cdot F\{d\xi - x\} \cdot \Pi\{dx\} = \int_{-\infty}^{\infty} \int_{\lambda_1}^{\infty} \xi \cdot F\{d\xi - x\} \frac{G\{dx\}}{E(T)} \\ = \frac{1}{E(T)} \int_{\lambda_1}^{\infty} \xi \int_{-\infty}^{\infty} F\{d\xi - x\} G\{dx\} = \frac{1}{E(T)} \int_{\lambda_1}^{\infty} \xi \cdot H\{d\xi\}, \quad \text{by (16)} \\ = \frac{E(S_T | S_T \in \overline{\lambda_1, \infty}) \cdot \Pr(S_T \in \overline{\lambda_1, \infty})}{E(T)} = E(S_T | S_T \in \overline{\lambda_1, \infty}) \cdot \pi_1, \quad \text{by (27)}$$

$$j_{2t} \rightarrow j_2 = E(S_T | S_T \in \overline{-\infty, \lambda_2}) \cdot \pi_2. \quad (\text{Q.E.D.})$$

It should be noted that the sum of j_1 and j_2 is equal to μ , because by Wald's equation (18) we have

$$(31) \quad j_1 + j_2 = \frac{\mathcal{E}\{S_T | S_T \in \overline{\lambda_1, \infty}\} \Pr\{S_T \in \overline{\lambda_1, \infty}\} + \mathcal{E}\{S_T | S_T \in \overline{-\infty, \lambda_2}\} \Pr\{S_T \in \overline{-\infty, \lambda_2}\}}{\mathcal{E}(T)}$$

$$= \frac{\mathcal{E}(S_T)}{\mathcal{E}(T)} = \mu .$$

A-6. Calculations of Steady-State Mean and Variance

If we are interested only in the mean and variance (and the higher moments) of the steady-state random variable \tilde{x} , there is a method which enables us to calculate them directly without having recourse to the prior computation of the steady-state distribution. To this end, let us define the joint expression of the generating function of T and the characteristic function of S_T as follows:

$$(32) \quad \kappa(\omega, k) \equiv \mathcal{E}\{\exp(i\omega \cdot S_T) \cdot k^T\}$$

$$= \sum_{t=1}^{\infty} k^t \left[\int_{-\infty}^{\lambda_2} \exp(i\omega z) \cdot H_t \{dz\} + \int_{\lambda_1}^{\infty} \exp(i\omega z) \cdot H_t \{dz\} \right] ,$$

where $i \equiv \sqrt{-1}$. Let us also define the generating-cum-characteristic function of G_t as follows

$$(33) \quad \gamma(\omega, k) \equiv \sum_{t=1}^{\infty} k^t \int_{\lambda_2}^{\lambda_1} \exp(i\omega z) \cdot G_t \{dz\} .$$

Finally, let us define the characteristic function of F by

$$(34) \quad \varphi(u) \equiv \int_{-\infty}^{\infty} \exp(iuz) \cdot F\{dz\} .$$

Then, it is easy to derive the following relation from the recurrence relations (14) and (15):

$$(35) \quad \kappa(u, k) = 1 - \{1 - k \cdot \varphi(u)\} \cdot \gamma(u, k)$$

for all u for which $\varphi(u)$ exists and for all k for which the two series (32) and (33) converge. This relation is called the generalized Wald's identity whose properties are extensively discussed in Cox-Miller [3], Miller [7], Kemperman [6], or Feller [5].

If the expectations of T , S_T and ξ_t exist, then they can be expressed in terms of the generating-cum-characteristic functions as $\varepsilon(T) = G\{\overline{-\infty, \infty}\} = \gamma(0, 1)$, $i \cdot \varepsilon(S_T) = \partial \kappa(0, 1) / \partial u$ and $i \cdot \varepsilon(\xi_t) \equiv i \cdot \mu = \varphi'(0)$. In this case, differentiating the generalized Wald's identity with respect to u and evaluating the derivative at $u = 0$ and $k = 1$, we have

$$(36) \quad \frac{\partial \kappa(0, 1)}{\partial u} = \varphi'(0) \cdot \gamma(0, 1) .$$

This is nothing but the Wald's equation stated by (18). If we keep differentiating the generalized Wald's identity with respect to u , we obtain

$$(37-a) \quad \frac{\partial^2 \kappa(0, 1)}{\partial u^2} = 2 \cdot \varphi'(0) \cdot \frac{\partial \gamma(0, 1)}{\partial u} + \varphi''(0) \cdot \gamma(0, 1)$$

$$(37-b) \quad \frac{\partial^3 \kappa(0, 1)}{\partial u^3} = 3 \cdot \varphi'(0) \cdot \frac{\partial^2 \gamma(0, 1)}{\partial u^2} + 3 \cdot \varphi''(0) \cdot \frac{\partial \gamma(0, 1)}{\partial u} + \varphi'''(0) \cdot \gamma(0, 1) ,$$

$$(37-c) \quad \frac{\partial^4 \kappa(0, 1)}{\partial u^4} = 4 \cdot \varphi'(0) \cdot \frac{\partial^3 \gamma(0, 1)}{\partial u^3} + 6 \cdot \varphi''(0) \cdot \frac{\partial^2 \gamma(0, 1)}{\partial u^2} \\ + 4 \cdot \varphi'''(0) \cdot \frac{\partial \gamma(0, 1)}{\partial u} + \varphi''''(0) \cdot \gamma(0, 1) ,$$

and so on. If we substitute the relations: $i^{n \cdot \omega^{(n)}}(0) = \mathcal{E}(\xi_t^n)$, $i^{n \cdot \partial^{(n)}}\kappa(0,1)/\partial \omega^n = \mathcal{E}(S_T^n)$, $i^{n \cdot \partial^{(n)}}\gamma(0,1)/\partial \omega^n = \int_{\lambda_2}^{\lambda_1} z^n \cdot dG(z)$ and $\gamma(0,1) = G\{-\infty, \infty\} = \mathcal{E}(T)$ into the above equations, and rearrange terms, we can easily establish the next theorem.

Theorem 2. (The Steady-State Mean and Variance)

If $\mu \equiv \mathcal{E}(\xi_t) \neq 0$, the mean and variance of the steady-state random variable \underline{x} , if any, can be expressed as

$$(38) \quad \mathcal{E}(\underline{x}) \equiv \int_{\lambda_2}^{\lambda_1} z \cdot \Pi\{dz\} = \int_{\lambda_2}^{\lambda_1} z \frac{G\{dz\}}{\mathcal{E}(T)}$$

$$= \frac{1}{2} \left\{ \frac{\mathcal{E}(S_T^2)}{\mathcal{E}(S_T)} - \frac{\mathcal{E}(\xi_t^2)}{\mathcal{E}(\xi_t)} \right\}$$

$$(39) \quad \text{Var}(\underline{x}) = \int_{\lambda_2}^{\lambda_1} \{z - \mathcal{E}(\underline{x})\}^2 \cdot \Pi\{dz\} = \int_{\lambda_2}^{\lambda_1} \{z - \mathcal{E}(\underline{x})\}^2 \cdot \frac{G\{dz\}}{\mathcal{E}(T)}$$

$$= \frac{1}{3} \left\{ \frac{\mathcal{E}(S_T^3)}{\mathcal{E}(S_T)} - \frac{\mathcal{E}(\xi_t^3)}{\mathcal{E}(\xi_t)} \right\} - \frac{1}{4} \left[\left\{ \frac{\mathcal{E}(S_T^2)}{\mathcal{E}(S_T)} \right\}^2 - \left\{ \frac{\mathcal{E}(\xi_t^2)}{\mathcal{E}(\xi_t)} \right\}^2 \right].$$

If $\mu = 0$ and $\text{Var}(\xi_t) > 0$, they can be expressed as

$$(40) \quad \mathcal{E}(\underline{x}) = \frac{1}{3} \left\{ \frac{\mathcal{E}(S_T^3)}{\mathcal{E}(S_T^2)} - \frac{\mathcal{E}(\xi_t^3)}{\mathcal{E}(\xi_t^2)} \right\},$$

$$(41) \quad \text{Var}(\underline{x}) = \frac{1}{6} \left\{ \frac{\mathcal{E}(S_T^4)}{\mathcal{E}(S_T^2)} - \frac{\mathcal{E}(\xi_t^4)}{\mathcal{E}(\xi_t^2)} \right\} - \frac{1}{9} \left[\left\{ \frac{\mathcal{E}(S_T^3)}{\mathcal{E}(S_T^2)} \right\}^2 - \left\{ \frac{\mathcal{E}(\xi_t^3)}{\mathcal{E}(\xi_t^2)} \right\}^2 \right].$$

The moments of S_T are often easy to calculate or easy to approximate.

A-7. Two Special Cases

In this section we examine two special examples for which we can explicitly calculate the steady-state distribution or at least its mean and variance. The first example is the well-known Bernoulli random walk model and the second one is the random walk in which the distribution of each step is a mixture of two exponential distributions. The former is the discrete-time, discrete-state approximation of the celebrated Wiener process and the latter can be regarded as a discrete-time, continuous-state analogue of the Birth-and-Death process.

Let us first consider the Bernoulli trial random walk model, in which the probability distribution F of ξ_t is characterized by the atomistic distribution:

$$(42) \quad F\{s\} \equiv \Pr\{\xi_t = s\} = q, \quad \text{and} \quad F\{-s\} = \Pr\{\xi_t = -s\} = 1-q,$$

where $s > 0$ is a single-period step and $0 < q < 1$ is the probability of a positive jump. The mean and variance of ξ_t are given by

$$(43) \quad \mathcal{E}(\xi_t) \equiv \mu^2 = s \cdot q - s \cdot (1-q) = (2q-1) \cdot s,$$

$$\text{Var}(\xi_t) = \mathcal{E}(\xi_t^2) - \mu^2 = s^2 - \mu^2.$$

Note that the position of x_t as well as that of λ_1 and λ_2 can take only the values of $0, \pm s, \pm 2s, \dots$.

Let $g(x) \equiv G\{x\}$ for $x = \lambda_2 + s, \lambda_2 + 2s, \dots, \lambda_1 - s$, where G is defined by (11) in the random walk model with two absorbing barriers at λ_1 and λ_2 . We can interpret $g(x)$ here as the expected number that x_t visits the position x prior to the first entry into either

of the two absorbing intervals. It is easy to see from our elementary consideration that in the case of the Bernouilli random walk the relation (17) can be transformed into:

$$(44) \quad g(x) = q \cdot g(x-s) + (1-q) \cdot g(x+s)$$

for $x = \lambda_2 + s, \lambda_2 + 2s, \dots, -s, s, \dots, \lambda_1 - s$, with $g(\lambda_2) = g(\lambda_1) = 0$;
and for $x = 0$

$$g(0) - 1 = q \cdot g(-s) + (1-q) \cdot g(s) .$$

When $\mu \neq 0$, we can solve the above difference equations to get

$$(45-a) \quad g(x) = \frac{s \cdot (1-Q^{-\lambda_1}) \cdot (Q^{x-\lambda_2} - 1)}{\mu \cdot (Q^{-\lambda_2} - Q^{-\lambda_1})} \quad \text{for } x = \lambda_2, \lambda_2 + s, \dots, 0 ,$$

$$= \frac{s \cdot (Q^{-\lambda_2} - 1) \cdot (1 - Q^{x-\lambda_1})}{\mu \cdot (Q^{-\lambda_2} - Q^{-\lambda_1})} \quad \text{for } x = 0, s, \dots, \lambda_1 ,$$

where $Q \equiv \{q/(1-q)\}^{1/s}$. When $\mu = 0$, we can solve it to get

$$(45-b) \quad g(x) = 2 \frac{x - \lambda_2}{\lambda_1 - \lambda_2} \quad \text{for } x = \lambda_2, \lambda_2 + s, \dots, 0 ,$$

$$= 2 \frac{\lambda_1 - x}{\lambda_1 - \lambda_2} \quad \text{for } x = 0, s, \dots, \lambda_1 .$$

In the Bernouilli random walk model, an absorption can occur only at λ_1 or λ_2 . Let us then denote by $h(\lambda_2)$ the probability that $S_T = \lambda_2$ and by $h(\lambda_1)$ the probability that $S_T = \lambda_1$. Obviously,
 $h(\lambda_2) \equiv H\{\lambda_2\} = H\{\overline{-\infty, \lambda_2}\}$, $h(\lambda_1) \equiv H\{\lambda_1\} = H\{\overline{\lambda_1, \infty}\}$ and $h(\lambda_1) + h(\lambda_2) = 1$.

When $\mu \neq 0$, these two probabilities can be easily calculated as

$$(46-a) \quad h(\lambda_2) = (1-q) \cdot g(\lambda_2 + s) = (1-q) \frac{s \cdot (1 - Q^{-\lambda_1}) \cdot (Q^s - 1)}{\mu \cdot (Q^{-\lambda_2} - Q^{-\lambda_1})}$$

$$= \frac{1 - Q^{-\lambda_1}}{Q^{-\lambda_2} - Q^{-\lambda_1}},$$

$$h(\lambda_1) = q \cdot g(\lambda_1 - s) = \frac{Q^{-\lambda_2} - 1}{Q^{-\lambda_2} - Q^{-\lambda_1}};$$

and when $\mu = 0$ they can be calculated as

$$(46-b) \quad h(\lambda_2) = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \quad h(\lambda_1) = \frac{-\lambda_2}{\lambda_1 - \lambda_2}.$$

Using these expressions, we can obtain for the moments of the absorption point S_T ,

$$(47) \quad \mathcal{E}(S_T^n) = \lambda_2^n \cdot h(\lambda_2) + \lambda_1^n \cdot h(\lambda_1)$$

$$= \frac{\lambda_2^n (1 - Q^{-\lambda_1}) + \lambda_1^n (Q^{-\lambda_2} - 1)}{Q^{-\lambda_2} - Q^{-\lambda_1}}, \quad \text{when } \mu \neq 0,$$

$$= \frac{\lambda_2^n \cdot \lambda_1 - \lambda_1^n \cdot \lambda_2}{\lambda_1 - \lambda_2}, \quad \text{when } \mu = 0,$$

for $n = 1, 2, \dots$. Then, the explicit expression of the expected absorption period $\mathcal{E}(T)$ can be derived from Wald's equation (18) when $\mu \neq 0$,

$$(48-a) \quad \varepsilon(T) = \frac{\varepsilon(S_T)}{\mu} = \frac{\lambda_2(1 - Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2} - 1)}{\mu \cdot (Q^{-\lambda_2} - Q^{-\lambda_1})},$$

and from (37-a) when $\mu = 0$ and $s^2 \neq 0$

$$(48-b) \quad \varepsilon(T) = \frac{\varepsilon(S_T^2)}{\varepsilon(\xi_T^2)} = \frac{\lambda_2^2 \lambda_1 - \lambda_1^2 \lambda_2}{s^2 \cdot (\lambda_1 - \lambda_2)} = -\frac{\lambda_1 \lambda_2}{s^2}.$$

Let $\tilde{\pi}(x) \equiv \tilde{\Pi}\{x\}$, $x = \lambda_2, \lambda_2 + s, \dots, \lambda_1$, be the steady-state probability of $\tilde{x} = x$ in the random walk model with two return barriers at λ_1 and λ_2 . Then, its explicit formula can be deduced from (25) in Theorem 1. Thus, when $\mu \neq 0$, we have

$$(49-a) \quad \tilde{\pi}(x) = \frac{g(x)}{\varepsilon(T)} = \frac{s \cdot (1 - Q^{-\lambda_1}) \cdot (Q^{x-\lambda_1} - 1)}{\lambda_2(1 - Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2} - 1)} \quad \text{for } x = \lambda_2, \lambda_2 + s, \dots, 0,$$

$$= \frac{s \cdot (Q^{-\lambda_2} - 1) \cdot (1 - Q^{x-\lambda_2})}{\lambda_2(1 - Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_1} - 1)} \quad \text{for } x = 0, s, \dots, \lambda_1;$$

and, when $\mu = 0$ and $s^2 \neq 0$, we have

$$(49-b) \quad \tilde{\pi}(x) = -2s^2 \frac{x - \lambda_2}{\lambda_1 \lambda_2} \quad \text{for } x = \lambda_2, \lambda_2 + s, \dots, 0,$$

$$= -2s^2 \frac{\lambda_1 - x}{\lambda_1 \lambda_2} \quad \text{for } x = 0, s, \dots, \lambda_1.$$

The steady-state mean and variance, $\varepsilon(\tilde{x})$ and $\text{Var}(\tilde{x})$, can be also calculated either from the explicit expressions of $\tilde{\pi}(z)$ given above or from equations (38)-(41) in Theorem 2. In any case, when $\mu \neq 0$,

we obtain

$$(50-a) \quad \varepsilon(\tilde{x}) = \frac{1}{2} \left\{ \frac{\lambda_2^2(1-Q^{-\lambda_1}) + \lambda_1^2(Q^{-\lambda_2}-1)}{\lambda_2(1-Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2}-1)} - \frac{s^2}{\mu} \right\},$$

$$(51-a) \quad \text{Var}(\tilde{x}) = \frac{1}{3} \left\{ \frac{\lambda_2^3(1-Q^{-\lambda_1}) + \lambda_1^3(Q^{-\lambda_2}-1)}{\lambda_2(1-Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2}-1)} - s^2 \right\} \\ - \frac{1}{4} \left[\left\{ \frac{\lambda_2^2(1-Q^{-\lambda_1}) + \lambda_1^2(Q^{-\lambda_2}-1)}{\lambda_2(1-Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2}-1)} \right\}^2 - \left(\frac{s^2}{\mu} \right)^2 \right];$$

and, when $\mu = 0$ and $s^2 \neq 0$, we have

$$(50-b) \quad \varepsilon(\tilde{x}) = \frac{1}{3}(\lambda_1 + \lambda_2),$$

$$(51-b) \quad \text{Var}(\tilde{x}) = \frac{1}{18}(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) - \frac{1}{9}s^2.$$

We can also calculate π_1 , π_2 , j_1 and j_2 by substituting (46) and (48) into (27) in Corollary 1 and (30) in Corollary 2. When $\mu \neq 0$, we have

$$(52-a) \quad \pi_1 = \frac{H\{\lambda_1, \infty\}}{\varepsilon(T)} = \frac{h(\lambda_1)}{\varepsilon(T)} = \frac{\mu \cdot (Q^{-\lambda_2} - 1)}{\lambda_2(1-Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2} - 1)},$$

$$\pi_2 = \frac{H\{-\infty, \lambda_2\}}{\varepsilon(T)} = \frac{h(\lambda_2)}{\varepsilon(T)} = \frac{\mu \cdot (1 - Q^{-\lambda_1})}{\lambda_2(1-Q^{-\lambda_1}) + \lambda_1(Q^{-\lambda_2} - 1)},$$

$$(53-a) \quad \begin{aligned} \underline{j}_1 &= E\{S_T | S_T \geq \lambda_1\} \cdot \underline{\pi}_1 = \frac{\mu \cdot \lambda_1 (Q^{-\lambda_2} - 1)}{\lambda_2 (1 - Q^{-\lambda_1}) + \lambda_1 (Q^{-\lambda_2} - 1)}, \\ \underline{j}_2 &= E\{S_T | S_T \leq \lambda_2\} \cdot \underline{\pi}_2 = \frac{\mu \cdot \lambda_2 (1 - Q^{-\lambda_1})}{\lambda_2 (1 - Q^{-\lambda_1}) + \lambda_1 (Q^{-\lambda_2} - 1)}. \end{aligned}$$

When $\mu = 0$ and $s^2 \neq 0$, we have

$$(52-b) \quad \underline{\pi}_1 = \frac{s^2 \lambda_1}{\lambda_1 - \lambda_2}, \quad \underline{\pi}_2 = \frac{-s^2 \lambda_2}{\lambda_1 - \lambda_2},$$

$$(53-b) \quad \underline{j}_1 = \frac{s^2 \lambda_1^2}{\lambda_1 - \lambda_2}, \quad \underline{j}_2 = \frac{-s^2 \lambda_2^2}{\lambda_1 - \lambda_2}.$$

If one of the barriers is infinite, then the above formulae for various characteristics of the steady-state random variable \underline{x} can be much simplified. If $\lambda_2 = -\infty$ and $0 < \mu < \infty$, we have by applying l'Hopital rule to (50)-(52):

$$(53) \quad E(T) = \frac{\lambda_1}{\mu};$$

$$(54) \quad \begin{aligned} \underline{\pi}(x) &= \frac{s(Q^{\lambda_1} - 1)Q^x}{\lambda_1 Q} \quad \text{for } x = 0, -s, -2s, \dots, \\ &= \frac{s(Q^{\lambda_1} - Q^x)}{\lambda_1 Q} \quad \text{for } x = 0, s, 2s, \dots, \lambda_1; \end{aligned}$$

$$(55) \quad E(\underline{x}) = \frac{1}{2} \left(\lambda_1 - \frac{s^2}{\mu} \right),$$

$$(56) \quad \text{Var}(\tilde{x}) = \frac{1}{12}\lambda_1^2 - \frac{1}{3}s^2 + \frac{1}{4}\left(\frac{s^2}{u}\right)^2 ;$$

$$(57) \quad \tilde{\mu}_1 = \frac{u}{\lambda_1} = \frac{1}{\varepsilon(T)} , \quad \tilde{\mu}_2 = 0 ;$$

$$(58) \quad \tilde{\lambda}_1 = u , \quad \tilde{\lambda}_2 = 0 .$$

We can do similar calculations for the case where $\lambda_1 = \infty$ and $-\infty < u < 0$, but they will be left for the reader's finger exercise.

Next let us turn to the second example in which the probability distribution F of each step ξ_t has the density

$$(59) \quad f(\xi) = \frac{1}{a+b} \exp\left(-\frac{\xi}{a}\right) , \quad \text{for } \xi > 0 , \\ = \frac{1}{a+b} \exp\left(\frac{\xi}{b}\right) , \quad \text{for } \xi < 0 ,$$

where $a > 0$ and $b > 0$. The mean and variance of ξ_t are given by

$$(60) \quad \varepsilon(\xi_t) = \mu = a-b , \\ \text{Var}(\xi_t) = a^2 + b^2 .$$

In this example, $f(\xi)$ is the convolution of two exponential densities concentrated on $\overline{0, \infty}$ and $\overline{-\infty, 0}$, respectively. Alternatively, each step ξ_t is the difference of two positive exponentially distributed random variables, one having a density $(1/a) \cdot \exp(-\xi/a)$ and the other having a density $(1/b) \cdot \exp(\xi/b)$.

Although we have not been able to derive the explicit expression of the steady-state probability distribution $\tilde{\mu}$ in this example, we can at least calculate its mean and variance as well as its other characteristics

relevant for our calculation of the long-run trend-free average loss $\hat{\Lambda}$ in the text.

To this end, let us consider the random walk model with two absorption barriers at λ_1 and λ_2 , induced by this mixed exponential distribution. If an absorption occurs in, say, the upper absorption interval $\overline{\lambda_1, \infty}$ in period T , then the step ξ_T which carries S_T over λ_1 must be generated by the positive part of the exponential densities, i.e., by $(1/a) \cdot \exp(-\xi/a)$. The excess of the absorption point over the upper barrier, $S_T - \lambda_1$, is the excess of the random variable ξ_T over $\lambda_1 - S_{T-1}$ conditional upon $\xi_T \geq \lambda_1 - S_{T-1}$ or $S_T \geq \lambda_1$. Because of the well-known "lack-of-memory" property of the exponential distribution (see, for example, Feller [5], p. 8), $S_T - \lambda_1$ conditional upon $S_T \geq \lambda_1$ has the same exponential distribution as that of the positive component of ξ_T , independently of the absorption period T . Similar remarks can be applied to the occurrence of an absorption in the lower absorption interval, $\overline{-\infty, \lambda_2}$. Hence, we can show that

$$(61) \quad \begin{aligned} E\{\exp(i\omega S_T) | S_T \geq \lambda_1\} &= \frac{\exp(i\omega \lambda_1)}{1 + i\omega/a}, \\ E\{\exp(i\omega S_T) | S_T \leq \lambda_2\} &= \frac{\exp(i\omega \lambda_2)}{1 - i\omega/b}. \end{aligned}$$

Keeping these remarks in mind, let us now substitute $k = \varphi^{-1}(\omega)$ into the generalized Wald's identity (35) and obtain

$$(62) \quad \begin{aligned} E[\exp(i\omega S_T) \{\varphi^{-1}(\omega)\}^T] &= h_1 \cdot E[\exp(i\omega S_T) \{\varphi^{-1}(\omega)\}^T | S_T \geq \lambda_1] \\ &\quad + h_2 \cdot E[\exp(i\omega S_T) \{\varphi^{-1}(\omega)\}^T | S_T \leq \lambda_2] \\ &= 1, \end{aligned}$$

where $h_1 \equiv H\{\lambda_1, \infty\} \equiv \Pr\{S_T \geq \lambda_1\}$, $h_2 \equiv H\{-\infty, \lambda_2\} \equiv \Pr\{S_T \leq \lambda_2\}$ and $h_1 + h_2 = 1$. This is called Wald's fundamental identity of sequential analysis. (See Wald [10].) Since in our example $\varphi(\omega)$ can be shown to be

$$\varphi(\omega) = 1/(1 + i\omega\mu) \cdot (1 - i\omega\mu),$$

if $\mu = a - b \neq 0$, by substituting $\omega = i(1/a - 1/b)$ into (59) and employing (58), we get

$$\begin{aligned} h_1 \cdot \mathcal{E}[\exp\{(1/a - 1/b)S_T\} | S_T \geq \lambda_1] + h_2 \cdot \mathcal{E}[\exp\{(1/a - 1/b)S_T\} | S_T \leq \lambda_2] \\ = h_1 \left(\frac{b}{a}\right)^R R^{-\lambda_1} + h_2 \left(\frac{a}{b}\right)^R R^{-\lambda_2} = 1, \end{aligned}$$

where $R \equiv \exp(1/b - 1/a)$. Noting that $h_1 + h_2 = 1$, we can solve this equation for h_2 and get for $\mu = a - b \neq 0$

$$(63-a) \quad h_2 = 1 - h_1 = \frac{1 - (b/a)^R R^{-\lambda_1}}{(a/b)^R R^{-\lambda_2} - (b/a)^R R^{-\lambda_1}}.$$

The explicit expression of $h_2 = 1 - h_1$ for the case where $\mu = a - b = 0$ can be obtained by applying l'Hopital's rule to (63-a). Thus, we get for $\mu = a - b = 0$

$$(63-b) \quad h_2 = 1 - h_1 = \frac{\lambda_1 + a}{\lambda_1 - \lambda_2 + 2a}.$$

Because of the remarks given in the paragraph above, we can also calculate the moments of the absorption point S_T by

$$E(S_T^n) = E(S_T^n | S_T \geq \lambda_1) \cdot h_1 + E(S_T^n | S_T \leq \lambda_2) \cdot h_2 ,$$

for $n = 1, 2, \dots$. Hence, we have

$$(64-a) \quad E(S_T) = (\lambda_1 + a) \cdot h_1 + (\lambda_2 - b) \cdot h_2 ,$$

$$(64-b) \quad E(S_T^2) = (\lambda_1^2 + 2a\lambda_1 + 2a^2) \cdot h_1 + (\lambda_2^2 - 2b\lambda_2 + 2b^2) \cdot h_2 ,$$

$$(64-c) \quad E(S_T^3) = (\lambda_1^3 + 3a\lambda_1^2 + 6a^2\lambda_1 + 6a^3) \cdot h_1 + (\lambda_2^3 - 3b\lambda_2^2 + 6b^2\lambda_2 - 6b^3) \cdot h_2 ;$$

and so on. Then, the explicit expression of $E(T)$ can be derived from Wald's equation (18) when $\mu = a-b \neq 0$,

$$(65-a) \quad E(T) = \frac{(\lambda_2 - b)\{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a)\{(a/b)R^{-\lambda_2} - 1\}}{\mu\{(a/b)R^{-\lambda_2} - (b/a)R^{-\lambda_1}\}} ;$$

and when $\mu = a-b = 0$ it can be derived from (35-a),

$$(65-b) \quad E(T) = \frac{(\lambda_2^2 - 2b\lambda_2 + 2b^2)(\lambda_1 + a) + (\lambda_1^2 + 2a\lambda_1 + 2a^2)(\lambda_2 - b)}{2a^2(\lambda_1 - \lambda_2 + 2a)} .$$

If we substitute (64) into (38)-(41) in Theorem 2, we can obtain the explicit expressions of $E(\underline{x})$ and $\text{Var}(\underline{x})$. When $\mu = a-b \neq 0$, we have

$$(66-a) \quad E(\underline{x}) = \frac{1}{2} \left\{ \frac{(\lambda_2^2 - 2b\lambda_2 + b^2)\{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1^2 + 2a\lambda_1 + a^2)\{(a/b)R^{-\lambda_2} - 1\}}{(\lambda_2 - b)\{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a)\{(a/b)R^{-\lambda_2} - 1\}} - \frac{2(a^2 + b^2 - ab)}{(a-b)} \right\} ,$$

$$\begin{aligned}
 (67-a) \quad \text{Var}(\tilde{x}) = & \frac{1}{3} \left[\frac{(\lambda_2^3 - 3b\lambda_2^2 + 6b^2\lambda_2 - 6b^3) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1^3 + 3a\lambda_1^2 + 6a^2\lambda_1 + 6a^3) \{(a/b)R^{-\lambda_2} - 1\}}{(\lambda_2 - b) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a) \{(a/b)R^{-\lambda_2} - 1\}} \right. \\
 & \left. - 6(a^2 + b^2) \right] \\
 & - \frac{1}{4} \left\{ \frac{(\lambda_2^2 - 2b\lambda_2 + b^2) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1^2 + 2a\lambda_1 + a^2) \{(a/b)R^{-\lambda_2} - 1\}}{(\lambda_2 - b) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a) \{(a/b)R^{-\lambda_2} - 1\}} \right\}^2 \\
 & - \left\{ \frac{2(a^2 + b^2 - ab)}{(a-b)} \right\}^2 \Bigg] ;
 \end{aligned}$$

when $\mu = a - b = 0$ and $a^2 + b^2 \neq 0$, we have

(66-b)

(67-b)

We can also calculate π_1 , π_2 , j_1 and j_2 by substituting (63) and (65) into (27) and (30). When $\mu = a-b \neq 0$, we have

$$(68-a) \quad \pi_1 = 1 - \pi_2 = \frac{h_1}{\varepsilon(T)} = \frac{\mu \{(a/b)R^{-\lambda_2} - 1\}}{(\lambda_2 - b) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a) \{(a/b)R^{-\lambda_2} - 1\}} ;$$

$$(69-a) \quad j_1 = \mu - j_2 = \varepsilon(S_T | S_T \geq \lambda_1) \cdot \pi_1 \\ = \frac{\mu(\lambda_1 + a) \{(a/b)R^{-\lambda_2} - 1\}}{(\lambda_2 - b) \{1 - (b/a)R^{-\lambda_1}\} + (\lambda_1 + a) \{(a/b)R^{-\lambda_2} - 1\}} .$$

When $\mu = a-b = 0$, we have

$$(68-b) \quad \pi_1 = 1 - \pi_2 = \frac{2a^2(\lambda_2 - a)}{(\lambda_2 - a)(\lambda_1^2 + 2a\lambda_1 + 2a^2) + (\lambda_1 + a)(\lambda_2^2 - 2a\lambda_2 + 2a^2)}$$

$$(69-b) \quad j_1 = -j_2 = \frac{2a^2(\lambda_1 + a)(\lambda_2 - a)}{(\lambda_2 - a)(\lambda_1^2 + 2a\lambda_1 + 2a^2) + (\lambda_1 + a)(\lambda_2^2 - 2a\lambda_2 + 2a^2)} .$$

If $\lambda_2 = -\infty$ and $0 < \mu = a-b < \infty$, then the above steady-state formulae become much simpler. We have in this case

$$(70) \quad \varepsilon(T) = \frac{\lambda_1 + a}{\mu} ;$$

$$(71) \quad \varepsilon(x) = \frac{1}{2} \left\{ \frac{\lambda_1^2 + 2a\lambda_1 + 2a^2}{\lambda_1 + a} - \frac{2(a^2 + b^2 - ab)}{a-b} \right\} ;$$

$$(72) \quad \text{var}(\tilde{x}) = \frac{1}{3} \left\{ \frac{\lambda_1^3 + 3a\lambda_1^2 + 6a^2\lambda_1 + 6a^3}{\lambda_1 + a} - 6(a^2 + b^2) \right\} \\ - \frac{1}{4} \left\{ \left(\frac{\lambda_1^2 + 2a\lambda_1 + 2a^2}{\lambda_1 + a} \right)^2 - \frac{4(a^2 + b^2 - ab)^2}{(a-b)^2} \right\}$$

$$(73) \quad \tilde{\mu}_1 = \frac{\mu}{\lambda_1 + a} = \frac{1}{\varepsilon(T)}, \quad \tilde{\mu}_2 = 0$$

$$(74) \quad \tilde{\lambda}_1 = \mu, \quad \tilde{\lambda}_2 = 0.$$

We can do similar calculations for the case where $\lambda_1 = \infty$ and $-\infty < \mu = a-b < 0$.

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