12-1-1972

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James Tobin

Recommended Citation

Tobin, James, "Technological Development and Employment" (1972). *Cowles Foundation Discussion Papers*. 581.
https://elischolar.library.yale.edu/cowles-discussion-paper-series/581

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 348

TECHNOLOGICAL DEVELOPMENT AND EMPLOYMENT

James Tobin

December 13, 1972

TECHNOLOGICAL DEVELOPMENT AND EMPLOYMENT

by

James Tobin

For developing economies, the process of economic growth can be described as one of shifting resources from one known technology to another. Modern technologies used in advanced countries are more productive than traditional technologies. But their adoption generally requires saving to accumulate new forms of capital. The capital used in old technology cannot be trans-formed into modern capital, except possibly by the slow process of depre-ciation and replacement.

The newer technologies typically also require human capital not needed in the older processes, not only different manual and physical skills but also literacy and managerial capacity. But though they require more skilled labor, the modern technologies require much less unskilled labor, and less labor in toto, per unit of output. It is precisely this labor-saving that makes their adoption the eventual avenue to higher standards of living.

The other side of the coin, of course, is the structural unemployment that typically accompanies economic development. Concentration of available saving on the accumulation of modern capital often appears to accentuate the problem of surplus labor. Sometimes it appears that the allocation of

investment to older and more labor-intensive techniques would be more rational. Employment would be greater, and national output would benefit accordingly, even though the capital stock would be less productive than if investment had been otherwise directed. An important issue of development strategy is how to allocate available saving among alternative forms of capital accumulation.

The issue of investment strategy is the main subject of this paper. I come to it in Section 5.   . In the earlier sections I set up a simple model within which the problem can be discussed. There are two technologies, old and modern. Their difference is embodied in capital: that is, capital useable in one technology is not useable in the other. Both technologies involve fixed input-output coefficients. This makes unemployment due to capital shortage a logical possibility. At the same time, the modern technology requires skilled labor, trained either on the job or by educational investment. Unlike modern capital, skilled labor is not completely specific. Lacking skilled jobs, these workers are suitable for employment as unskilled labor on either technology. Typically skilled labor can command a premium wage, even when general labor is surplus.

The economy's propensity to save is assumed to be adequate to accumulate the physical and human capital required under the modern technology as rapidly as the total labor force is growing. Otherwise full development, a full shift of resources to modern technology, would not be a feasible long-run objective. Of course in many actual economies insufficiency of the propensity to save may be a fundamental obstacle to development. But here my interest is in a different question. Assuming the propensity to save is adequate for full development, is it possible "to get there from here," and if so, how?

The economy starts with a considerable portion of its capital and employment in the old technology, and possibly with unemployment. Generally it is feasible to achieve both full employment and a full shift of resources, although the optimal path may involve a detour during which investment is concentrated in the older sector until unemployment is eliminated. In some circumstances, however, there can be an impasse, with no feasible path to the ultimate goal--even though the position of full development and full employment would be sustainable once reached. The answer turns on the relative capital requirements of the two sectors. If the modern technology is the more capital-using, as well as the more labor-saving, there is always a feasible path. The impasse arises if the modern technology saves capital as well as labor.

I have assumed throughout this paper that the economy under discussion is closed, although I am aware that in reality development is always inter-twined with foreign trade and capital transactions. More precisely, the economy here need not be literally closed; the essential assumption is that it is dependent on its own saving. Although the capital goods needed for modern technology may be imported, they must be purchased with exports from either sector of the developing economy, at constant terms of trade. Foreign aid and foreign loans are not available. Realistically, they could provide an escape from the detours or impasses that, according to the analysis of the paper, might complicate the process of self-contained development.

1. The Technologies Available

There are two technologies, using different kinds of capital, but producing commensurable outputs. Both are characterized by fixed proportions. The amounts of inputs required for a unit of gross output are given in Table 1.

TABLE 1

	Technology:	1	2	Total supplies
i)	Capital type 1	α_1	0	k_1
ii)	Capital type 2	0	α_2	k_2
iii)	Skilled labor	β_{11}	0	ℓ_1
	[Unskilled labor	β_{21}	β_{22}]	
iv)	All labor	$\beta_1 \ (= \beta_{11} + \beta_{21})$	$\beta_2 \ (= \beta_{22})$.	ℓ
	Total output	y_1	y_2	y

Technology 1 is the modern technology. It is the more capital intensive in the sense that more physical capital is required per unit of gross output $(\alpha_1 > \alpha_2)$, and it requires less total labor per unit of output $(\beta_1 < \beta_2)$.

However, technology 1 requires some input (β_{11}) of skilled labor, and only a part of the labor force is skilled $(\ell_1 < \ell)$. Skilled labor can also perform in unskilled jobs, just as well as unskilled labor. Thus the supply of unskilled labor per se is not a resource constraint. The resource constraints are for skilled labor, line (iii) of the table, and

for all labor, line (iv) of the table. Here is a summary of the four resource constraints at any moment of time, given the supplies of the four resources:

(1.1)   i) $\alpha_1 y_1 \leqq k_1$

ii) $\alpha_2 y_2 \leqq k_2$

iii) $\beta_{11} y_1 \leqq \ell_1$

iv) $\beta_1 y_1 + \beta_2 y_2 \leqq \ell$.

Unless the four supplies stand in particular ratios to each other, not all of these four relations will hold with equality. Whatever the supplies of resources, an efficient use of them implies full employment of at least two of the resources. The most interesting regimes are:

I. General Capital Shortage. Constraints i) and ii) are binding, i.e. hold with equality. Resources iii) and iv), both kinds of labor, are in surplus; the corresponding relations are inequalities. The efficiency prices or shadow prices of the four resources are respectively $1/\alpha_1$, $1/\alpha_2$, 0, 0 . (These are found by solving the two breakeven equations for the technologies. Let r_1 and r_2 be the rents per time period for the use of the two types of capital. Let w be the wage rate of labor in unskilled jobs, and w_1 be the premium paid for skilled labor in jobs requiring skill--the total wage rate of such a worker is $w + w_1$. The two breakeven equations are:

(1.2) $1 = \alpha_1 r_1 + \beta_{11} w_1 + \beta_1 w$

$1 = \alpha_2 r_2 + \beta_2 w$

whenever a resource is slack, its marginal productivity--and hence its efficiency price--is zero. In the present case, then, $w_1 = w = 0$, and therefore $r_1 = 1/\alpha_1$, $r_2 = 1/\alpha_2$.)

II. <u>Shortage of Skilled Labor and Labor-Intensive Capital</u>. Constraints ii) and iii) are binding; the others are slack. The efficiency prices, calculated by the same procedure as above, are 0, $1/\alpha_2$, $1/\beta_{11}$, 0 . There is general unemployment of unskilled labor, because of a shortage of suitable capital for technology 2. Modern capital is not fully used, for lack of skilled labor to man it. Skilled labor commands a wage, even though general manpower is surplus.

III. <u>Shortage of Labor-Intensive Capital</u>. Constraints i), ii), and iii) are binding, but general labor is surplus. This requires that the supplies of modern capital and skilled labor are in balance: $k_1/\alpha_1 = \ell_1/\beta_{11}$. In this case the breakeven equations for prices are:

$$1 = \alpha_1 r_1 + \beta_{11} w_1$$

$$1 = \alpha_2 r_2$$

Here we know that the general wage $w = 0$, and that $r_2 = 1/\alpha_2$. However r_1 and w_1 are indeterminate. Since neither can be less than zero, we have $0 \leq r_1 \leq 1/\alpha_1$ and $0 \leq w_1 \leq 1/\beta_{11}$.

IV. <u>Full Employment of Labor, Surplus of Labor-Intensive Capital</u>. Constraints i), iii), iv) are binding. As in III, k_1 and ℓ_1 must be in balance and their prices are indeterminate. Old-technology capital is in excess supply, and its rent is zero. The wage of general labor is $1/\beta_2$.

V. <u>Full Employment of Labor, Only Modern Technology in Use</u>. Constraints i), iii), iv) are binding, and $\beta_1 y_1 = \ell$. There is no labor left over for technology 2, even though some capital for that technology is available. Since $\beta_1 < \beta_2$ it would clearly be inefficient to reduce y_1 in order to release labor for the older technology. In this case there is only one breakeven equation:

$$1 = \alpha_1 r_1 + \beta_{11} w_1 + \beta_1 w \ .$$

The other one is an inequality, showing that operation of technology 2 is unprofitable:

$$1 < \alpha_2 r_2 + \beta_2 w \ .$$

We know that $r_2 = 0$, given the excess supply of k_2 . Therefore, $w > 1/\beta_2$. At the same time $w \leqq 1/\beta_1$. The upper limit on r_1 is found by setting w_1 and w at their lower limits, 0 and $1/\beta_2$ respectively:

$1 = \alpha_1 \bar{r}_1 + \beta_1/\beta_2 \ , \quad \bar{r}_1 = \dfrac{1 - \beta_1/\beta_2}{\alpha_1}$. Similarly the upper limit on w_1 is

$\bar{w}_1 = \dfrac{1 - \beta_1/\beta_2}{\beta_{11}}$.

VI. <u>All Resources Fully Employed</u>. All the four constraints of (1.1) apply. Thus,

$$(1.3) \qquad \frac{k_1}{\ell_1} = \frac{\alpha_1}{\beta_{11}}$$

$$\text{and} \quad \frac{\beta_1 k_1}{\alpha_1} + \frac{\beta_2 k_2}{\alpha_2} = \ell \ .$$

Prices are indeterminate, within the constraints (1.2).

2. Depreciation, Accumulation, and Net Output

To complete the description of the technologies, I must specify the mechanisms of depreciation and accumulation of the stocks of physical capital k_1 and k_2, and of the stock of human capital ℓ_1.

Physical capital. I assume that each capital stock k_i depreciates at the constant exponential rate d_i. Let $I_i(t)$ be the rate of gross investment in physical capital of type i at time t, and let the stock of such capital at time t be $k_i(t)$, its time rate of change $k_i'(t)$. (I will suppress the notation (t) whenever it is clear that the variable is a function of time.) The familiar expression for net accumulation is:

$$(2.1) \qquad k_i' = I_i - d_i k_i \qquad (i = 1, 2).$$

Since $I_i \geq 0$ --capital in existence cannot be converted into another type of capital or into consumable goods,

$$(2.2) \qquad k_i' \geq d_i k_i \qquad (i = 1, 2).$$

Skilled labor. Without new recruits, the supply of skilled labor would decline by attrition due to retirement, disability, and death. This attrition is analogous to the depreciation of physical capital. It is assumed to occur at an exponential rate of m_1 per year.

New recruits to the cadre of skilled labor occur in two ways. One is educational training. Let the rate of recruitment from this source be $n_1(t)$ per year, and the cost per recruit be e units of output. The implicit assumption is that education is performed by technology 1, with the same capital, skilled labor, and general labor requirements of that modern technology.

Thus training costs $en_1(t)$ are a claim on y_1 , the output of technology
1. A more elaborate model would, of course, specify a distinct educational
technology, and in addition allow for the lengthy gestation of skilled manpower
over successive grades of education.

The other source of skilled manpower is learning by doing, or more
accurately in this case learning by observation and contact. Upgrading of
unskilled personnel associated with modern technology is a normal method of
enlarging the availability of skilled labor, perhaps more important than
formal education off the job. Here it is assumed that a fraction μ of
the unskilled labor employed in technology 1 (ℓ_{21}) becomes skilled each
year. This on-the-job training is a special advantage of producing by modern
technology. Unskilled labor employed in technology 2 does not become skilled.

Taking account of attrition and of both sources of recruitment, we
have the growth of the supply of skilled labor:

$$(2.3) \qquad \ell_1'(t) = -m_1\ell_1(t) + n_1(t) + \mu\ell_{21}(t) \ .$$

We know that $\ell_{21} = \beta_{21}y_1(t)$. Likewise, provided skilled manpower is fully
employed, $\ell_1(t) = \beta_{11}y_1(t)$. In this case we can write (2.3) as:

$$(2.4) \qquad \ell_1'(t) = n_1(t) + y_1(t)[\mu\beta_{21} - m_1\beta_{11}] \ .$$

Conceivably the term in brackets, which represents the net of attrition and
upgrading is positive, i.e. the use of technology 1 generates human capital
faster than it depreciates.

The total capital requirement, human plus physical, in technology 1,
is $\alpha_1 + e\beta_{11}$ per unit of gross output. In later sections I shall find

it convenient to aggregate the two capital requirements of technology 1, on the assumption that within the sector balance between the two kinds of capital is maintained. I shall use $\hat{\alpha}_1$ to refer to the aggregated capital requirement:

$$(2.5) \qquad \hat{\alpha}_1 = \alpha_1 + e\beta_{11} \ .$$

The parameter, e , educational cost, converts the skilled labor requirement into units of output, commensurate with the physical capital requirement.

The <u>net</u> output of sector 1, $\bar{y}_1$ must allso for the depreciation of the two kinds of capital:

$$(2.6) \qquad \bar{y}_1 = y_1 + d_1 k_1 - e m_1 \ell_1 + e \mu \ell_{21} \ .$$

Assuming both kinds of capital are fully used, this can be written as:

$$(2.7) \qquad \bar{y}_1 = y_1 (1 - d_1 \alpha_1 - e m_1 \beta_{11} + e \mu \beta_{21}) \ .$$

In these circumstances, it is convenient to have a summary depreciation rate $\hat{d}_1$ for sector 1, defined so that

$$(2.8) \qquad \bar{y}_1 = y_1 (1 - \hat{d}_1 \hat{\alpha}_1) \ .$$

It follows that

$$(2.9) \qquad \hat{d}_1 = \frac{d_1 \alpha_1 + e m_1 \beta_{11} - e \mu \beta_{21}}{\alpha_1 + e \beta_{11}} \ .$$

These definitions would not be appropriate in regimes where either physical or human capital were in surplus. In that case no deduction for depreciation of the surplus stock should be made in going from gross to net output.

Sector 2 is less complicated. Net output is simply:

$$(2.10) \qquad \bar{y}_2 = y_2(1 - d_2\alpha_2) \ .$$

Once again, the deduction for depreciation is appropriate only when k_2 is fully employed.

To find the input coefficient for net output, it is only necessary to divide the coefficients in Table 1 by the appropriate ratio of net to gross output. When no capital resources are redundant, these ratios are respectively $1 - \hat{d}_1\hat{\alpha}_1$ and $1 - d_2\alpha_2$.

I have already assumed that the physical capital requirement per unit of gross output is larger in sector 1, $(\alpha_1 > \alpha_2)$ and in the same spirit I would assume also that the physical capital requirement per unit of net output is larger in sector 1 $\left(\dfrac{\alpha_1}{1 - \alpha_1 d_1} > \dfrac{\alpha_2}{1 - \alpha_2 d_2} \right)$. Otherwise I would be allowing greater durability of capital in technology 1 to offset its greater capital coefficient.

But the distinctive learning-by-doing capacity of technology 1 suggests keeping open the interesting possibility that technology 1 is not the more capital-using when human as well as physical capital is taken into account. I shall analyze both cases: sector 1 the more capital using $\left(\dfrac{\hat{\alpha}_1}{1 - \hat{\alpha}_1 d_1} > \dfrac{\alpha_2}{1 - \alpha_2 d_2} \right)$, and sector 1 the less capital using (the inequality reversed).

3. <u>Total Labor Supply</u>. The growth of total labor supply is taken to be exogenous, at a constant exponential rate μ .

It is possible to interpret this "natural rate" to include labor-augmenting technological progress as well as population growth proper, provided that such progress is costless and "disembodied," and applies impartially to all labor, wherever employed and whether skilled or unskilled. Under this interpretation labor supplies and employments ℓ , ℓ_1 , ℓ_{21} , ℓ_2 are numbers of effective manhours; the effective manhours associated with a natural manhour grows at the rate of technological progress. The education or training cost e , which is assumed to be constant, is then the cost of turning an effective unit of unskilled labor into an effective unit of skilled labor. Since educating one man converts more effective units this year than last, constancy of e implies that the education cost <u>per</u> <u>man</u> grows at the rate of general labor-augmenting progress. This is probably a reasonable assumption, more so than the more optimistic opposite assumption that education costs <u>per</u> <u>natural</u> <u>man</u> remain constant. Education is notoriously immune to the forces of progress that pervade other sectors. Morevoer, it makes highly intensive use of labor, skilled labor at that, at wages that rise along with economy-wide productivity.

In what follows we shall not make the distinction between natural and effective labor, counting on the reader to remember the alternative interpretations of the model.

4. Golden Ages

In a "golden age" the economy grows at its natural rate n. All outputs, capital stocks, labor supplies grow at this common rate, and all resources are fully utilized. There are two possible polar golden ages, one with exclusive use of technology 1 $(y_2 = 0)$, one with exclusive use of technology 2 $(y_1 = 0)$. Mixtures of these polar cases are also conceivable, as the proportion of y_2, in total output y varies from 0 to 1, and that of y_1 correspondingly from 1 to 0.

Consider first the golden age in which $y_1 = 0$ and technology 2 is used exclusively. Capital of type 2 must be sufficient to employ the whole labor force: $k_2 = \alpha_2 y_2 = \dfrac{\alpha_2 \ell}{\beta_2}$. The net saving required to make k_2 grow at the same rate as the labor force is

$$k_2'(t) = nk_2(t) = n\alpha_2 y_2(t) = n\alpha_2 y(t) .$$

Therefore the fraction of net income which must be saved, s, is given by

$$s(y_2(t) - d_2 k_2(t)) = s(1 - d_2\alpha_2)y_2(t) = n\alpha_2 y(t) , \quad \text{and}$$

$$(4.1) \qquad\qquad s = \frac{n\alpha_2}{1 - d_2\alpha_2} .$$

Consumption per worker in the golden age of old technology is:

$$(4.2) \qquad\qquad \frac{c}{\ell} = \frac{1 - \alpha_2(n + d_2)}{\beta_2} .$$

In the other polar case, the golden age of the modern technology, the required saving propensity is $s = \dfrac{n\hat{\alpha}_1}{1 - \hat{\alpha}\hat{d}_1}$, and consumption per worker is:

(4.3)
$$\frac{c}{\ell} = \frac{1 - \hat{\alpha}_1 (n + \hat{d}_1)}{\beta_1} \; .$$

I assume, of course, that consumption per worker is larger in (4.3) than in (4.2). Otherwise there is no reason for the economy to modernize. In other words,

(4.4)
$$\beta_2 (1 - \hat{\alpha}_1 \hat{d}_1) - \beta_1 (1 - \alpha_2 d_2) > n(\beta_2 \hat{\alpha}_1 - \beta_1 \alpha_2) > 0 \; .$$

Golden ages are also possible with both technologies in use. To examine them, let the ratio of employment in the modern sector, ℓ_1 , to the total labor force ℓ be x . On the assumption that full employment obtains in these golden ages, the fraction of the labor force employed with technology 2 is $\ell_2/\ell = 1-x$. For given x $(0 \leq x \leq 1)$:

$$c_1 = \frac{x\ell(1 - \hat{\alpha}_1 (n + \hat{d}_1))}{\beta_1}$$

(4.5)
$$c_2 = \frac{(1-\alpha)\ell(1 - \alpha_2 (n + d_2))}{\beta_2}$$

$$\frac{c}{\ell} = \frac{c_1 + c_2}{\ell} = x \, \frac{(1 - \hat{\alpha}_1 (n + \hat{d}_1))}{\beta_1} + (1-x) \, \frac{(1 - \alpha_2 (n + d_2))}{\beta_2} \; .$$

Evidently c/ℓ is a linear function of x , with a slope which I have already, in comparing (4.3) and (4.2), assumed to be positive.

A larger allocation of labor to sector 1--a higher value of--also means a larger capital stock per worker, as follows:

$$(4.6) \qquad \frac{k}{\ell} = x \, \frac{\hat{\alpha}_1}{\beta_1} + (1-x) \, \frac{\alpha_2}{\beta_2} \, .$$

The gross marginal product of capital is equal to the increment of gross output associated with an increase in x, divided by the increment in capital stock associated with an increase in x :

$$(4.7) \qquad \frac{\partial\left(\dfrac{y_1 + y_2}{\ell}\right)}{\partial(k/\ell)} = \frac{\beta_2 - \beta_1}{\beta_2 \hat{\alpha}_1 - \beta_1 \alpha_2} \qquad (0 \leq x \leq 1)$$

The net marginal product must allow for the increase in depreciation and training requirements associated with an increment in the capital stock

$$(4.8) \qquad \frac{\partial\left(\dfrac{\overline{y}_1 + \overline{y}_2}{\ell}\right)}{\partial(k/\ell)} = \frac{\beta_2(1 - \hat{\alpha}_1 \hat{d}_1) - \beta_1(1 - \alpha_2 d_2)}{\beta_2 \hat{\alpha}_1 - \beta_1 \alpha_2} \, .$$

Normally we expect a net marginal product to be less than the gross marginal product. Here, however, it is conceivable that the reverse is true; $\beta_2 \hat{\alpha}_1 \hat{d}_1$ may be smaller than $\beta_1 \alpha_2 d_2$.

Neo-classical growth theory tells us that a shift to a more capital-intensive golden age will increase consumption per worker if and only if the net marginal product exceeds the rate of growth n . As is evident from (4.4) and (4.8), this is another way of expressing the assumption that an increase in x augments consumption per worker.

Alternative golden ages are summarized graphically in Figures 1a and 1b. These give the standard diagrams of growth theory for the two cases. In Figure 1a, the normal case, technology 1 is the more capital-using technology in all the senses discussed above. In Figure 1b technology 1 requires less total capital per unit of net output. This leads to a number of anomalies: the net marginal product of capital is higher than the gross marginal product, and also higher than the net average product. In the upper panel of each diagram a dashed broken line represents the net saving generated by a saving propensity $\hat{s}$ adequate to support the maximum-consumption golden age, the golden age which corresponds to the allocation of all productive resources to the new technology. In the normal case, Figure 1a, this propensity to save is more than adequate for any less capital-intensive golden age. In the other case, Figure 1b, it is not; the maximum-consumption golden age is the only one it will support. I will return to this theme in section 5.

5. Development Strategy: Saving and Its Allocation

I assume now that the economy is not initially in a golden age, and that its ultimate objective is to reach the maximum-consumption golden age, with all labor and capital allocated to modern technology. This objective is feasible only if the economy's propensity to save is sufficient to maintain that golden age, and accordingly I assume such a propensity to save. This is defined as a suitable constant ratio $\hat{s}$ of net saving to net income. Gross saving also includes allowance for depreciation of capital not in surplus.

FIGURE 1. Alternative Golden Ages

1a. Technology 1 The More Capital-Using

$\dfrac{1}{\beta_1}$

$\dfrac{1}{\beta_2}$

output and
consumption
per worker

$x = 0$
$k_1 = 0$

excess
labor

$0 \le x \le 1$

← full empl. of labor and capital →

$x = 1$
$k_2 = 0$

excess
capital

$\dfrac{Y}{\ell}$ gross
output
per worker

depreciation

$\dfrac{\overline{Y}}{\ell}$ net output
per worker

$\dfrac{c}{\ell}$, cons. per wkr.

max $\dfrac{c}{\ell}$

growth req.
$n \dfrac{k}{\ell}$

net saving
$\hat{s} \dfrac{\overline{Y}}{\ell}$

$\hat{\alpha}_1 / \beta_1$

0

α_2 / β_2

$\dfrac{k}{\ell}$, capital/labor ratio

gross and
net marginal
productivity
of capital

$\dfrac{1}{\alpha_2}$ GMPK

$\dfrac{1}{\alpha_2} - d_2$ NMPK

GMPK

NMPK

$\dfrac{1}{\hat{\alpha}_1}$

$\dfrac{1 - \hat{\alpha}_1 d_1}{\hat{\alpha}_1}$

n

natural growth
rate

0

GMPK

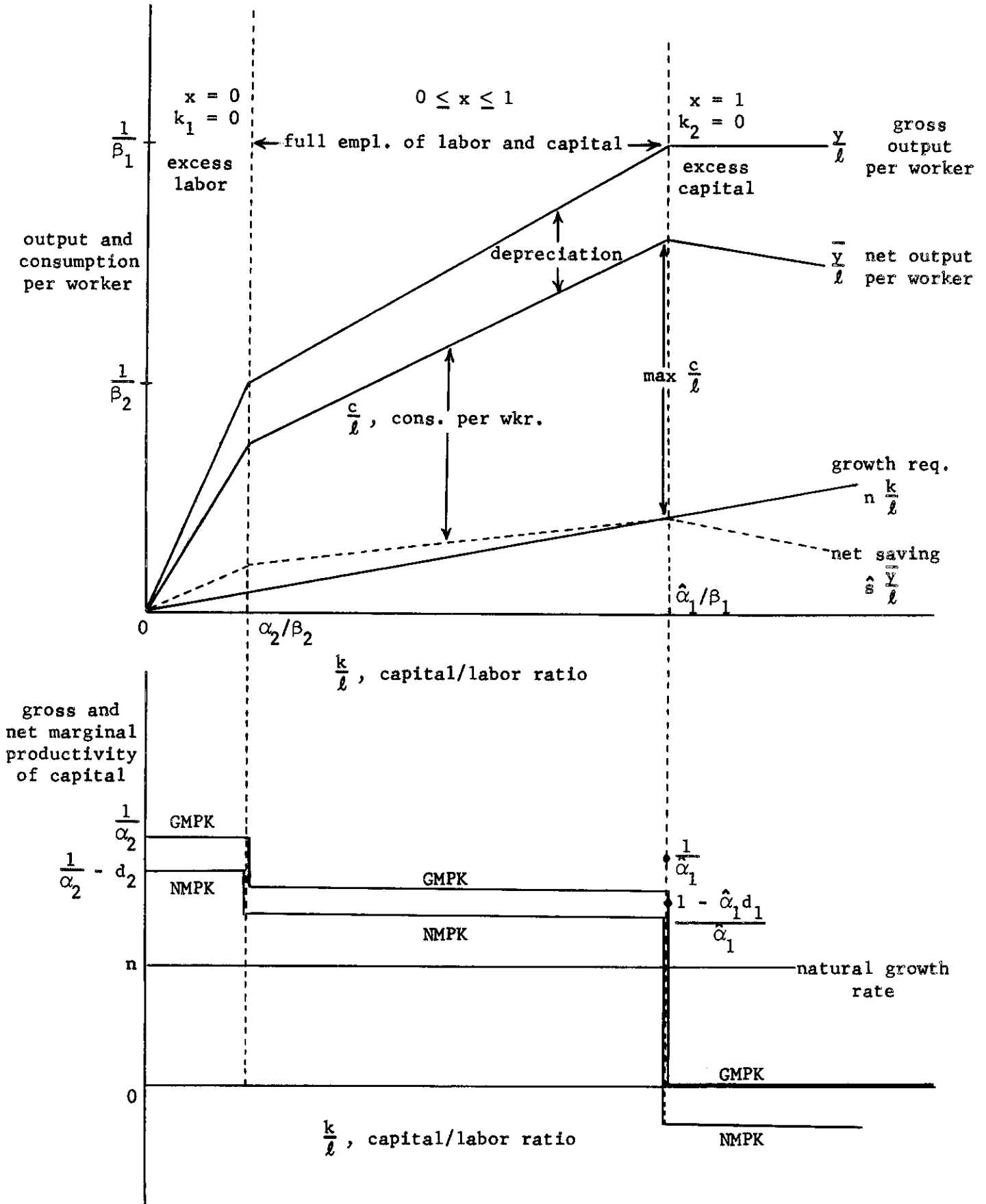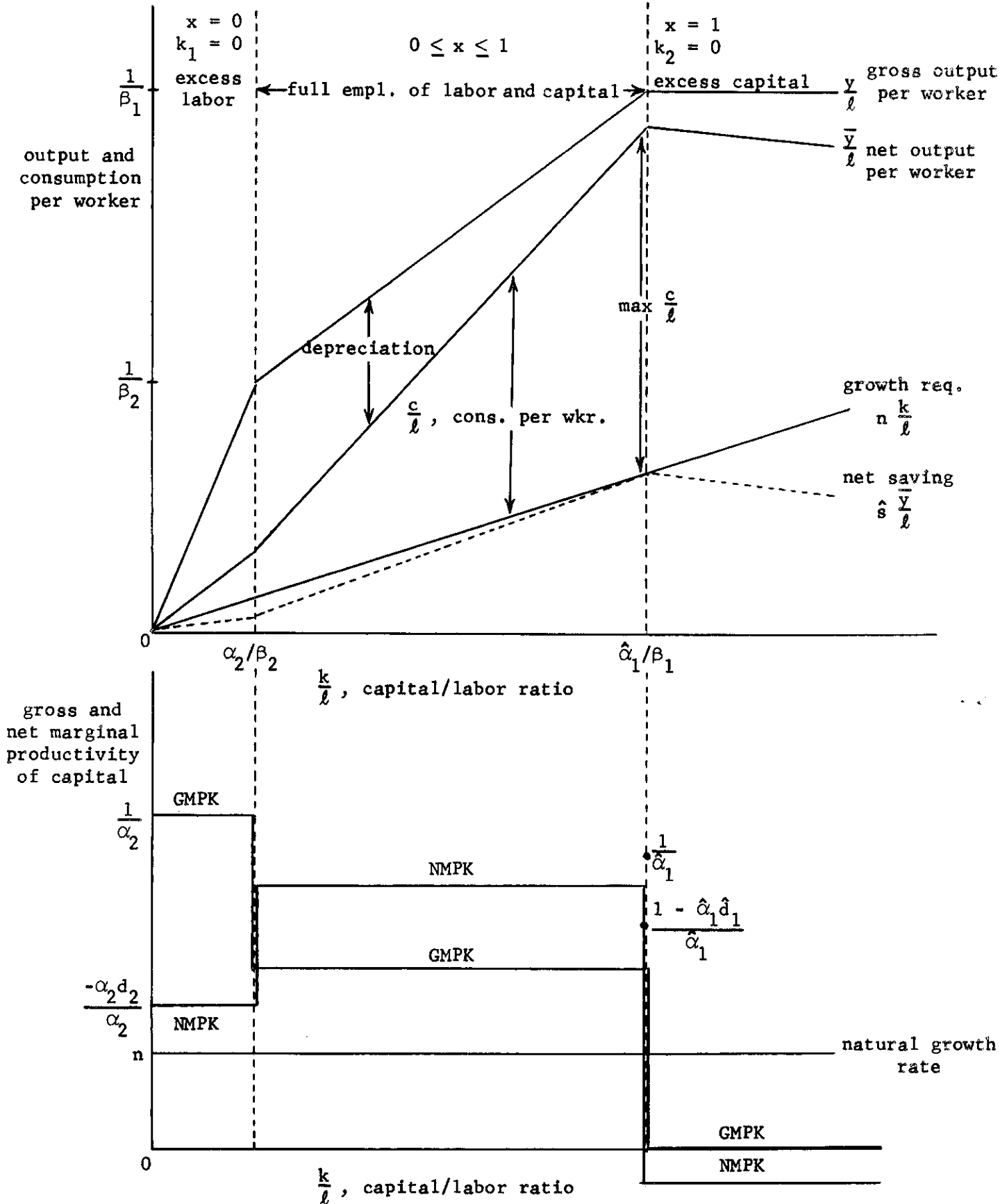NMPK

$\dfrac{k}{\ell}$, capital/labor ratio

FIGURE 1. Alternative Golden Ages
1b. Technology 1 The Less Capital-Using

To simplify the analysis, I assume that sector 1 is kept in balance as between physical capital and skilled labor by appropriate allocation of gross investment between capital investment and training expenditure. This enables me to treat sector 1 symmetrically with sector 2, each dependent on two factors, capital and labor. The "capital" of sector 1 is an amalgam of physical and human capital, with the input and depreciation parameters $\hat{\alpha}_1$ and $\hat{d}_1$ defined in (2.5) and (2.9). The assumption that the propensity to save is adequate to support a golden age with all resources in sector 1 means simply:

$$(5.1) \qquad \hat{s}\bar{y} = \frac{n\hat{\alpha}_1}{1 - \hat{d}_1\hat{\alpha}_1} \, \bar{y} \, .$$

I recall the other assumptions made about the technological and behavioral parameters of the system:

$$(5.2) \qquad \begin{cases} \hat{\alpha}_1 > \alpha_2 \, , \quad \beta_1 < \beta_2 \, , \quad \hat{\alpha}_1\beta_2 - \alpha_2\beta_1 > 0 \\[2mm] \beta_2(1 - \hat{d}_1\hat{\alpha}_1) - \beta_1(1 - d_2\alpha_2) > n(\hat{\alpha}_1\beta_2 - \alpha_2\beta_1) > 0 \, . \end{cases}$$

The last inequality, repeated from (4.4), can be written

$$(5.3) \qquad \beta_2 - \gamma\beta_1 > 0 \quad \text{where} \quad \gamma = \frac{1 - d_2\alpha_2}{1 - \hat{d}_1\hat{\alpha}_1} \, .$$

Normally, γ would be expected to be greater than 1. But for reasons already stated, I do not exclude the possibility that it is less than 1, and indeed less than $\alpha_2/\hat{\alpha}_1$.

The problem of development strategy is how to allocate the available saving between investments in the two sectors. The proper strategy for reaching the maximum-consumption golden age depends on the technological parameters just discussed and also on the initial position of the economy. I will consider three cases: (A) Initially there is full employment, and the stock of capital in sector 2, k_2 , is just adequate to provide jobs for everyone who cannot be employed in sector 1. This corresponds to Regime VI in section 1 above. (B) Initially there is full employment, and the stock of type 2 capital is actually redundant (Regime IV). (c) Initially there is surplus labor. The stock of type 2 capital k_2 is insufficient (Regime III). In all three cases, I assume that sector 1 is initially balanced and remains so. The third case is the one of greatest interest, but it is instructive to discuss the other cases first.

(A) Full Employment of All Recourses. Consider the constraints on the increments of gross output in the two sectors, $y_1'(t)$ and $y_2'(t)$. There are two constraints. The first is the labor force. Outputs cannot increase faster than the growth of the labor force permits. The second is capital. Output cannot grow faster than the overall propensity to save allows the two capital stocks to grow. The two constraints are as follows:

(5.4)   Labor $\beta_1 y_1' + \beta_2 y_2' \leqq n\beta_1 y_1 + n\beta_2 y_2$

Saving $\hat{\alpha}_1 y_1' + \alpha_2 y_2' = \hat{s}(1 - \hat{d}_1\hat{\alpha}_1)y_1 + \hat{s}(1 - d_2\alpha_2)y_2$,   or

(5.5) $\hat{\alpha}_1 y_1' + \alpha_2 y_2' \leqq n\hat{\alpha}_1 y_1 + n\hat{\alpha}_1 \gamma y_2$.

If both constraints are satisfied with equality, I can solve the two simultaneous equations in y_1' and y_2' in terms of y_1 and y_2 , as follows:

$$(5.6) \qquad y_1' = ny_1 + ny_2 \frac{\beta_2(\hat{\alpha}_1\gamma - \alpha_2)}{\beta_2\hat{\alpha}_1 - \beta_1\alpha_2}$$

$$(5.7) \qquad y_2' = \qquad ny_2 \frac{\hat{\alpha}_1(\beta_2 - \gamma\beta_1)}{\beta_2\hat{\alpha}_1 - \beta_1\alpha_2} .$$

The constraints are pictured diagrammatically in Figure 2. Here y_2' is plotted vertically and y_1' horizontally. If y_1' is less than the indicated point $-\hat{d}_1 y_1$, capital in sector 1 becomes redundant, because this represents the maximum speed at which type 1 capital can be consumed. The point $-d_2 y_2$ on the vertical axis has an analogous interpretation. The line LL , with slope of $-\frac{\beta_1}{\beta_2}$, less than 1 in absolute value, is the labor constraint (5.4). The line SS , with slope of $-\frac{\alpha_1}{\alpha_2}$, greater than 1 in absolute value, is the saving constraint (5.5). Point E represents the solution (5.6) and (5.7). The ray through the origin ZZ is the line $y_2' = \frac{y_2}{y_1} y_1'$. A point on the line means that the relative rates of growth of the two outputs are the same, so that y_2/y_1 remains unchanged. A point below the line, like E in Figure 2, means that y_2/y_1 is declining, while a point above the line means that y_2 is growing faster than y_1 . The point (ny_1, ny_2) , represented by N in Figure 2, is common to the lines LL and ZZ . In the normal variant, depicted in Figure 2a, γ exceeds 1 and a fortiori exceeds $\alpha_2/\hat{\alpha}_1$. Then (5.6) and (5.7) say that the solution value of y_1' exceeds ny_1 while that of y_2' is positive but less than ny_2 .

FIGURE 2

2a. Normal Variant
Technology 1 More Capital-Using
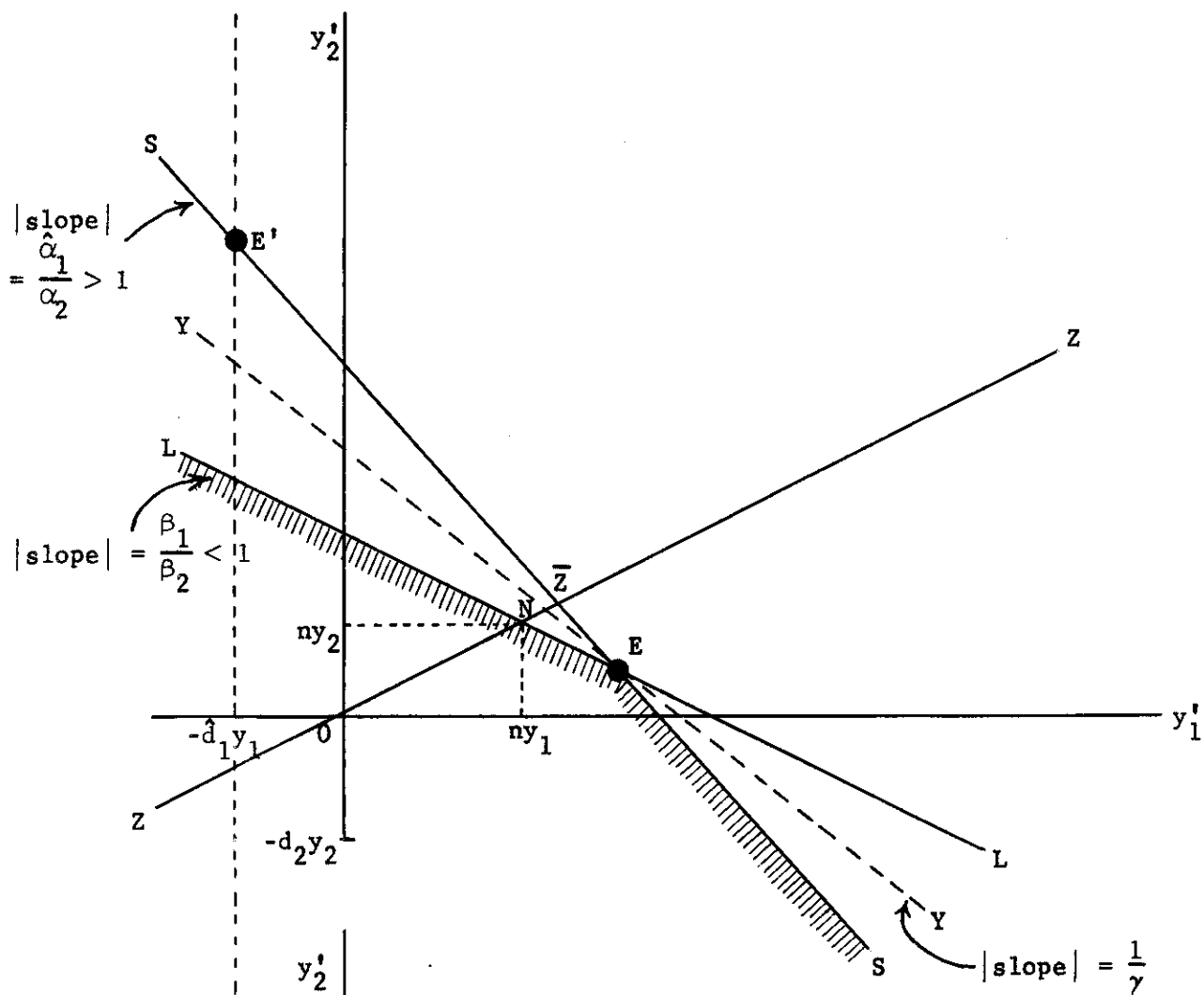
2b. Abnormal Variant
Technology 1 Less Capital-Using
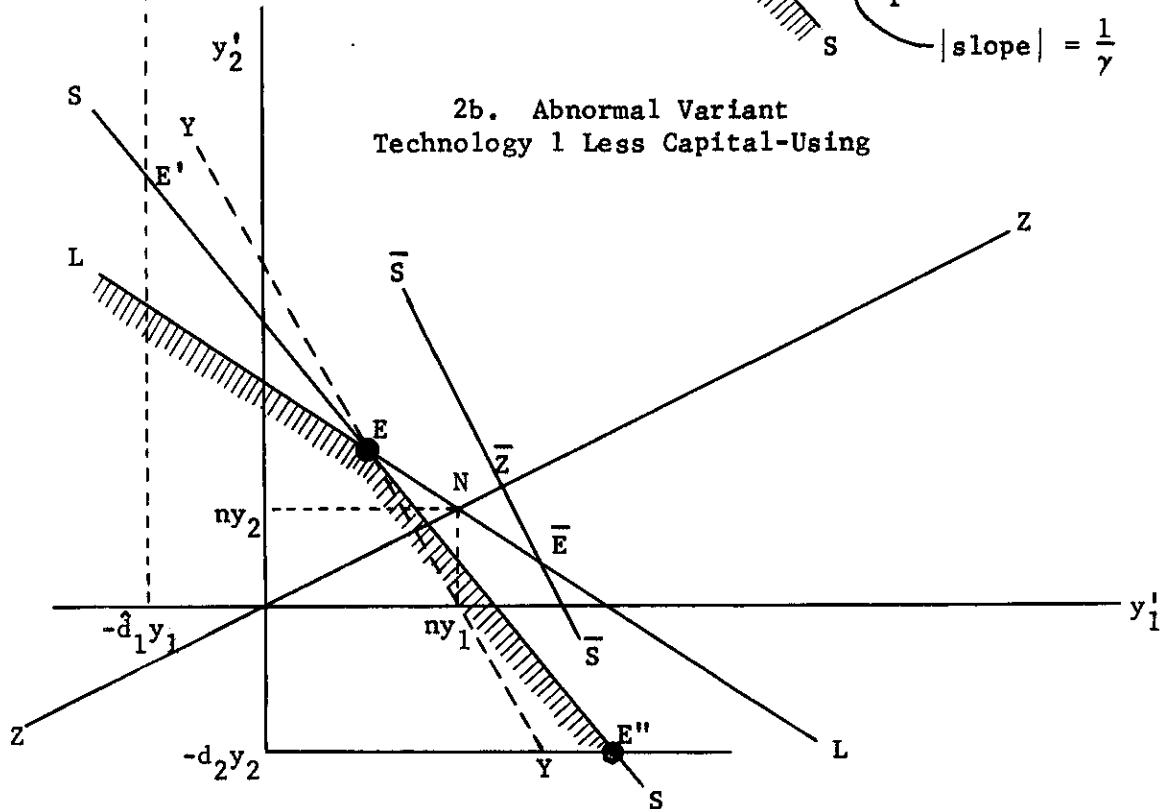
Consider also the growth of aggregate net output $\bar{y}$, defined as $\bar{y}_1 + \bar{y}_2 = y_1(1 - \hat{\alpha}_1\hat{d}_1) + y_2(1 - \alpha_2 d_2)$. The combinations (y_1', y_2') that yield any given increment in total net output are given by:

$$(5.8) \qquad y_1'(1 - \hat{\alpha}_1\hat{d}_1) + y_2'(1 - \alpha_2 d_2) = \bar{y}' .$$

One such line, the one through E, is shown in Figure 2a as YY. Its slope is $-\frac{1}{\gamma}$, always greater in absolute value than β_1/β_2, by (5.2) but normally less than one in absolute value. A faster growth of net income would be indicated by a line parallel to, but above YY'.

Figure 2a makes two things clear. First, departure from point E by increasing y_2' could not increase $\bar{y}'$ without violating the labor constraint. To the left of E, LL lies below YY. Likewise, second, departure from point E to the right, increasing y_1', could not increase $\bar{y}'$ without violating the capital constraint. In other words, the answer to the question, "Would it be possible to grow faster by shoving all saving into the modern sector even though it produces unemployment?", is no. Point E, which maintains full employment, is also the best that can be done with respect to increasing net income.

Note that everything said about aggregate net income applies also to the more fundamental variable consumption, which is a constant proportion $1-\hat{s}$ of net income.

The abnormal variant, where γ is smaller than $\alpha_2/\hat{\alpha}_1$, and $1/\gamma$ is larger than $\hat{\alpha}_1/\alpha_2$, is shown in Figure 2b. Now the solution point E is northwest of point N. But it is also clear that, so far as short-run growth of net income and consumption is concerned, a better strategy can

be found. Consider point E" , where all possible gross saving, including depreciation allowances generated in sector 2, is invested in sector 1. Compared to point E , point E" will increase the growth of income. But it will also create unemployment, as indicated by the fact that E" is well inside the labor constraint. Here there is a conflict, at least in the short run, between employment and income growth. Note also that if E" , or any other point to the right of E , is chosen, the economy shifts from full employment, case (A), to unemployment, case (C). Further analysis is deferred to the discussion of that case.

(B) Full Employment of Labor, Capital Surplus. If type 2 capital is in surplus, there is no reason to invest any gross saving in sector 2. The two constraints become:

(5.9)   Labor $\beta_1 y_1' + \beta_2 y_2' \leq n\beta_1 y_1 + n\beta_2 y_2$

(5.10)   Saving $\hat{\alpha}_1 y_1' \leq n\hat{\alpha}_1 y_1 + \dfrac{n\hat{\alpha}_1 y_2}{1 - \hat{\alpha}_1 \hat{d}_1}$.

The Saving constraint differs from (5.5) in allowing for zero gross investment in type 2 capital. This alters the definition of net income. At the same time, since type 2 capital is redundant, no allowance is made for its depreciation, so this source of gross saving is lost.

When the constraints are both met with equality, the solutions are:

(5.11) $y_1' = ny_1 + ny_2 \dfrac{1}{1 - \hat{\alpha}_1 \hat{d}_1}$

(5.12) $y_2' = ny_2 \left(1 - \dfrac{\beta_1}{\beta_2 (1 - \hat{\alpha}_1 \hat{d}_1)} \right)$.

The normal variant is depicted in Figure 3a, constructed in the same fashion as Figure 2a. The Saving constraint SS is now a vertical line, since capital growth places no limit on the growth of output in sector 2. Point E represents the solution given above. It will be southeast of point N , so that y_2/y_1 will be declining. In Figure 3a the solution for y_2' is shown as positive, i.e. $\dfrac{\beta_2}{\beta_1} > \dfrac{1}{1 - \hat{\alpha}_1 \hat{d}_1}$. This would be our normal expectation, but (5.3) does not permit us to exclude the contrary. If y_2' is positive, or indeed even if it is negative but declining less rapidly than the depreciation of the capital used in its production, the redundancy of capital in sector 2 will be only temporary. With the passage of time case (B) will evolve into case (A).

In case (B) the condition for constant increment to net income is

$$(5.13) \qquad\qquad y_1'(1 - \hat{\alpha}_1 \hat{d}_1) + y_2' = \bar{y}' \ .$$

This differs from (5.8) by omitting any deduction for the depreciation of k_2 . The slope of such a line, YY , is $-(1 - \hat{\alpha}_1 \hat{d}_1)$. In the normal variant illustrated in Figure 3a the slope of YY exceeds β_1/β_2 , the slope of LL , in absolute value. There is no way to make net income grow faster than at point E .

Should point E lie below the horizontal axis, it will also be true that YY is flatter than LL , as illustrated in Figure 3b. Here net income could grow faster by staying inside the Saving constraint SS and shifting as much later to sector 2 as the capital stock in that sector permits. But then it would no longer be proper to ignore the depreciation of sector 2 capital, and the situation would be immediately transformed into the normal variant of case (A), depicted in Figure 2a.

FIGURE 3

3a. Normal Variant
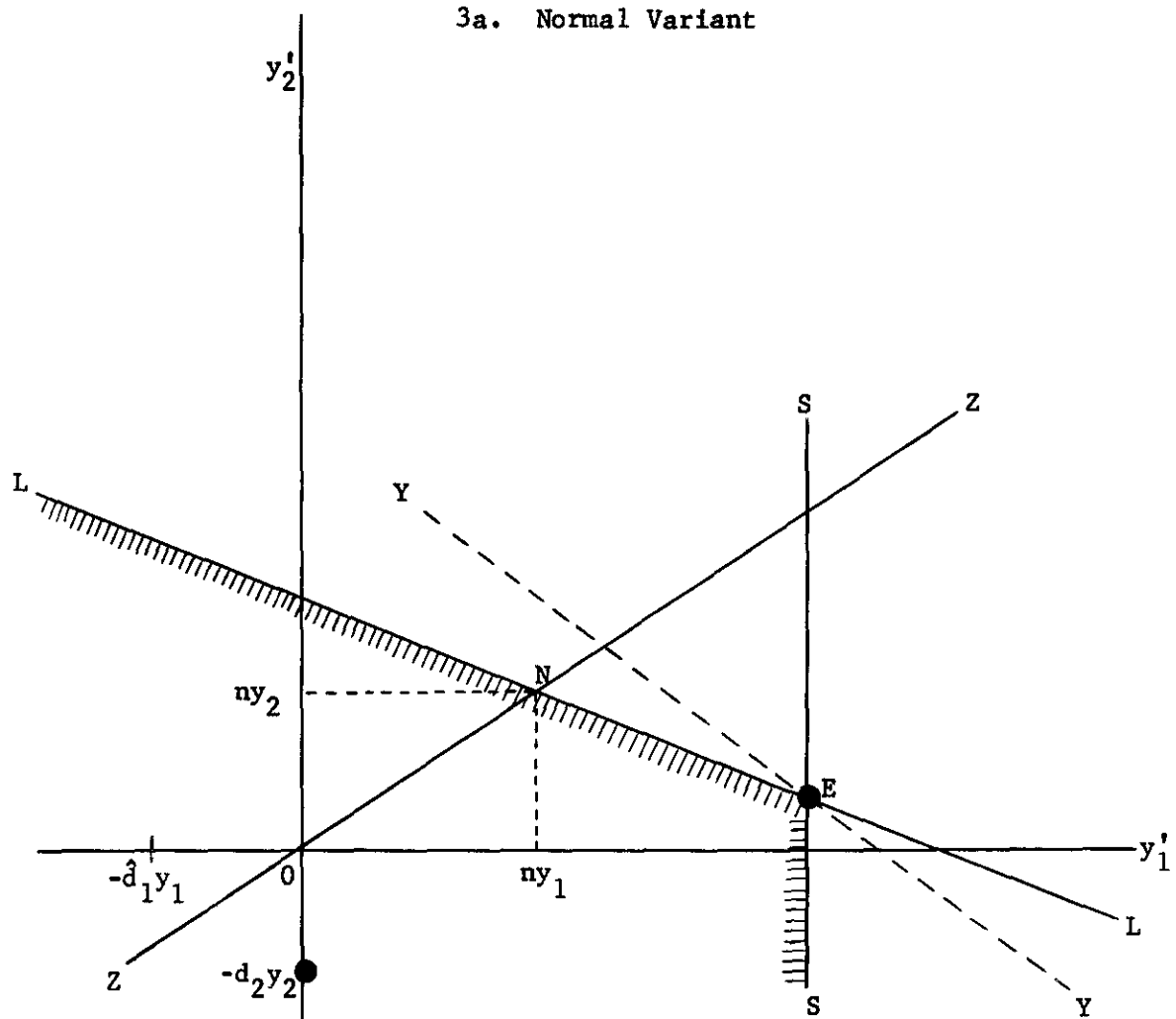
3b. Abnormal Variant
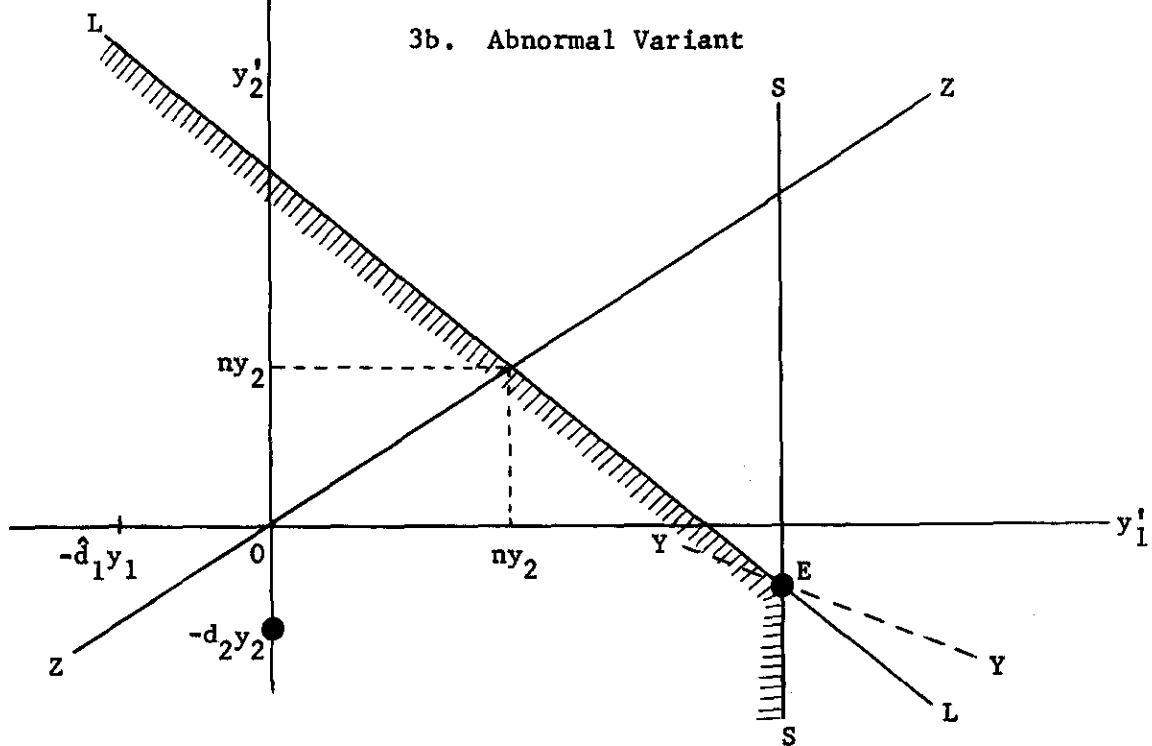
To avoid confusion, I should point out that the distinction between normal and abnormal variants of case (B) is not the same distinction as in the other cases. In case (B) it does not turn on the question which technology is the more capital using. That question is irrelevant when capital is in surplus.

(C) Unemployment of Labor and Capital Shortage: Normal Variant. Given initial unemployment, capital and its accumulation are the binding constraints on production and its growth. Constraint (5.5) applies, but (5.4) does not. In Figure 2, LL can be ignored. Points above it are not unattainable. They simply indicate that unemployment is declining.

In the normal variant of Figure 2a, it is clear that the short run growth of net income and consumption, as well as the growth of employment, will be maximized by placing all gross saving in sector 2 and letting sector 1 decline at its rate of depreciation (Point E') . As y_2 grows relative to y_1 , the line ZZ will rotate counter-clockwise. With the passage of time, this strategy will run out of surplus labor, and the conditions for case (A), Figure 2a, will be established. With the achievement of full employment, the allocation of investment will shift (to a point like E in Figure 2a), and sector 1 will from that time on grow faster than sector 2.

Some re-interpretation of Figure 2a is required for a strategy like E' which puts zero gross investment into sector 1. This strains my simplifying assumption that the two kinds of capital, physical and human, can always be kept balanced by shuffling of funds within sector 1. The spirit of the model is that depreciation is more rapid for physical capital, that

$d_1 > \hat{d}_1 > m_1 - \mu \dfrac{\beta_{21}}{\beta_{11}}$. When sector 1 is getting no investment funds, it

is unreasonable to expect that human capital can be converted into physical

capital. Consequently sector output will dwindle at the rate of depreciation

of physical capital, d_1 , rather than $\hat{d}_1$. Skilled labor will be surplus.

The Saving constraint (5.5) becomes

$$(5.5)' \qquad \alpha_1 y_1' + \alpha_2 y_2' \leq \frac{n\hat{\alpha}_1}{1 - d_1\hat{\alpha}_1} (1 - d_1\alpha_1)y_1 + n\hat{\alpha}_1\gamma y_2 \ .$$

In Figure 2a, SS moves in and has a gentler slope. Likewise the lower

limit to y_1' becomes $-d_1 y_1$, lower than the limit indicated in Figure 2a.

These amendments do not change the essentials of the argument.

Once full employment is attained by strategy E' , investment begins

again in sector 1. At first all such investment will be in the physical

capital of sector 1. Only when the stock of physical capital catches up

will investment in human capital become necessary.

But is it really correct to let sector 1 dwindle and then built it

up again? One answer, of course, is that only mistaken investment allocation

permitted it to become so large in the first place, while the shortage of

saving in sector 2 was creating unemployment. But we must let bygones be

bygones and compare the strategy indicated by E' in Figure 2a with alter-

natives.

First, note that among the points on SS , it takes one to the left

of E to reduce unemployment at all. And unless full employment is achieved,

the maximum-consumption golden age can never be reached. Indeed full employ-

ment must be achieved before sector 2 is phased out. Otherwise a pseudo

golden age of technology 1 will be reached with surplus labor, and with a propensity to save that creates enough capital to expand employment at the same rate as the labor force is growing but never enough to keep unemployment from growing at that same rate also.

Second, note that an intermediate strategy, between E and E', sacrifices net income and consumption in the short run, while reaching full employment later than strategy E' and with a lower ratio y_2/y_1. For example, the policy indicated by $\bar{Z}$ would maintain unchanged the initial ratio y_2/y_1 while slowly diminishing unemployment. I show now that path I, the path of net income, and therefore of consumption, under initial strategy E' dominates the path corresponding to any intermediate strategy. That is, path I is sometimes higher and never lower than the alternative, call it path II. I have already argued that path I is higher throughout the time corresponding to its unemployment phase. If path II ever overtakes path I their intersection occurs during the second or full employment phase of path I.

At the time of any such overtaking, Path II must have no higher a ratio y_2/y_1 than Path I. If the alleged overtaking occurs when both Paths are at full employment, they can show the same total net income only with an identical mixture of y_2 and y_1. If it occurs while Path II still displays unemployment, the ratio y_2/y_1 must be smaller for Path II. They can have the same total net income only if greater weight of the high productivity sector 1 in path II makes up for its unemployment. Now if path II has the same or lower ratio y_2/y_1 and the same total income, it must have at least as large a total capital stock as path I. Since the two paths

started with the same capital stocks, this means an equal or larger aggregate accumulation along Path II. But this is impossible, because in the inter=vening period net income, to which net saving is proportional, has always been higher along Path I than Path II. Thus it is proved that Path I--the strategy indicated initially by point E' --dominates all alternatives.

The key point is that none of the accumulation of type 2 capital during the phase of diminishing unemployment ever becomes redundant. This is clear from (5.7) and point E in Figure 2a, which indicate that sector 2 continues to expand in absolute size even during the full employment phase of develop=ment. Its continuing expansion, of course, in no way contradicts the fact that in the limit its relative share of the economy approaches zero.

Figure 4 shows schematically the fastest development path in the case under discussion, compared with a typical alternative.

Unemployment of Labor and Capital Shortage: Abnormal Variant. In the abnormal variant pictured in Figure 2b, there is a conflict between growth of income and consumption and growth of employment. With initial unemploy=ment, the labor constraint LL in Figure 2b does not apply. Points on SS to the left of E are feasible, and they will reduce unemployment. Points to the right of E will increase income faster, but actually increase unemployment.

Indeed, "you can't get there from here." Getting there means two things--achieving full employment, and shifting all resources to the modern sector. In this case, these two developments are contradictory. As Figure 2b makes clear, if unemployment is to be reduced or even kept from rising, sector 2 has to grow relative to sector 1. On the other hand, if resources

FIGURE 4
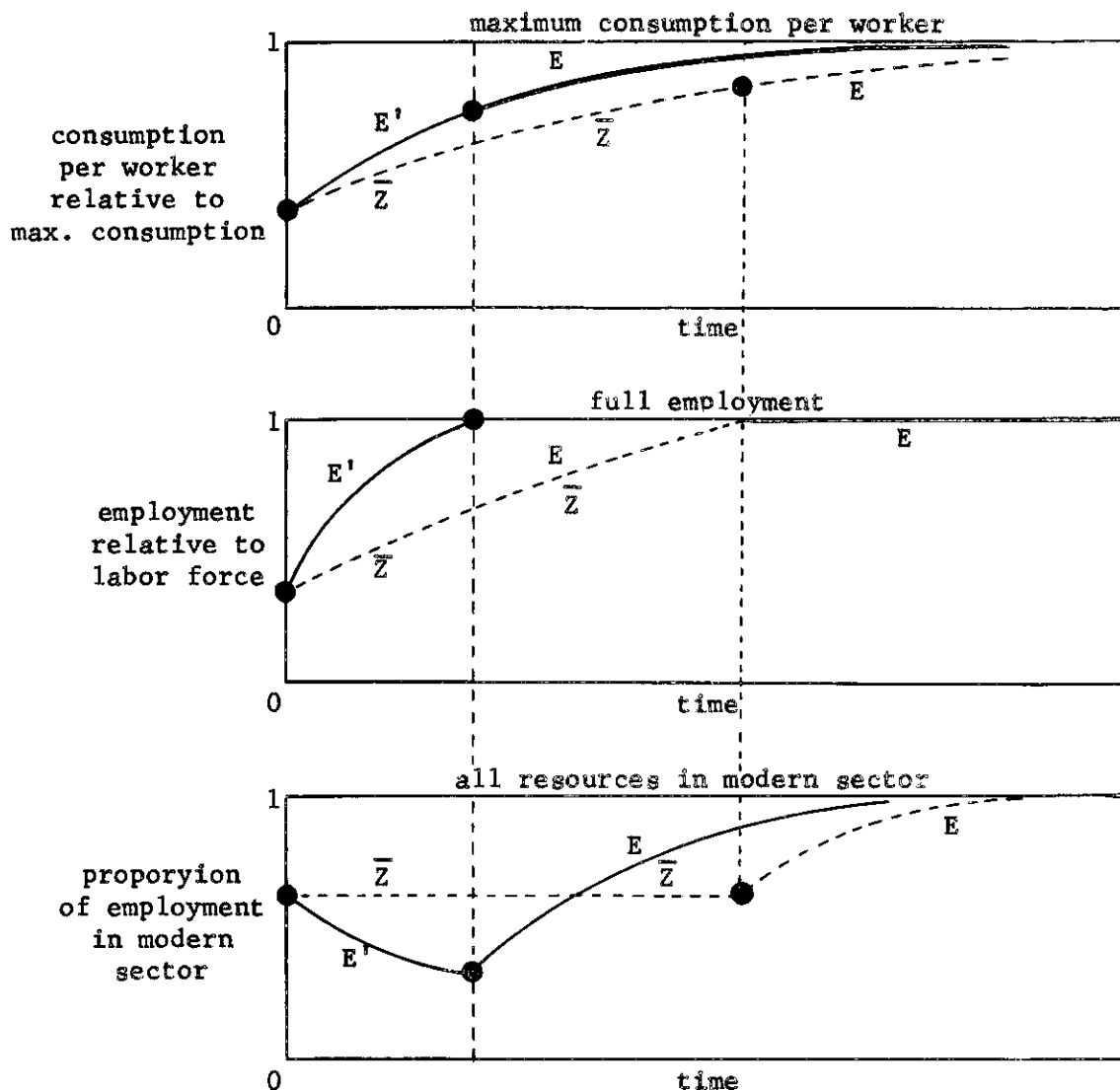
Optimal path (solid line), following strategy E' (Fig. 2a) until full employment, then strategy E .

Alternative path (dashed line), following strategy $\bar{Z}$ (Fig. 2a) until full employment, then strategy E .
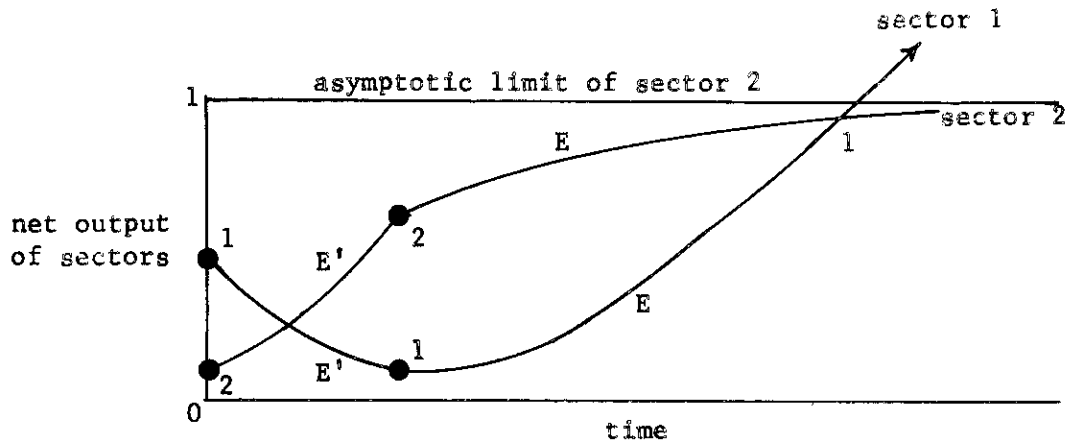
are to be shifted to the modern sector, saving is insufficient to keep up
with the growth of the labor force, much less to reduce unemployment.

The essential problem was already clear in Figure 1b. The dashed
broken line in that figure represents net saving generated by the propensity
to save I have been assuming. The saving propensity is adequate to sustain
a maximum-consumption golden age once established. (In Figure 2b ZZ would
coincide with the horizontal axis, and points E , N , and E" would all
converge at the point ny_1 on that axis.) From any initial position in
which sector is operating, the assumed saving propensity generates inadequate
saving. This is because sector 2 is more capital-using than sector 1, in
the sense already discussed. From such a starting point, the economy needs
a temporary burst of saving to make up for the fact that its permanent pro-
pensity to save yields too little from sector 2.

In these circumstances, the concentration of investment in sector 2
is not even a permanent solution to unemployment. For as sector 1 vanishes,
the saving generated by sector 2 alone is not enough to keep its capital
stock growing at the pace of the labor force. This is true because the
propensity to save is just adequate to give sector 1 a rate of growth of
n , not high enough to give sector 2 a growth rate of n . If the Saving
constraint (5.5) is solved for y_2' on the assumption that $y_1' = -d_1 y_1$
--all gross investment is diverted to sector 2--we have

$$(5.14) \qquad y_2' = (n + \hat{d}_1) \frac{\hat{\alpha}_1}{\alpha_2} y_1 + n \frac{\hat{\alpha}_1}{\alpha_2} \gamma y_2 \ .$$

Since $\dfrac{\hat{\alpha}_1}{\alpha_2}\gamma$ is less than 1 in the variant under discussion, $\dfrac{y_2'}{y_2}$ is less than n when $y_1 = 0$ and even when y_1 is small. (Graphically, as ZZ swings to the vertical axis in Figure 2b, point N becomes point ny_2 on that axis but SS is still below N .)

A higher propensity to save would escape the impasse. Suppose, for example, that the propensity to save $\hat{s}$ were sufficient to support steady golden-age growth of sector 2, the more capital-using sector in the case under discussion, namely $\hat{s} = \dfrac{n\alpha_2}{1 - d_2\alpha_2}$. Then the Saving constraint (5.5) would be

$$(5.15) \qquad \hat{\alpha}_1 y_1' + \alpha_2 y_2' = \frac{n\alpha_2}{\gamma} y_1 + n\alpha_2 y_2 \ .$$

With $\gamma < \alpha_2/\hat{\alpha}_1$ this is a less confining constraint than (5.5). Now the line SS in Figure 2b would cross ZZ above and to the right of N , like $\overline{S}\,\overline{S}$. This can be seen by noting that if $y_2' = ny_2$,

$y_1' = \dfrac{\alpha_2}{\gamma\hat{\alpha}_1} ny_1 > ny_1$. Consequently point E moves to the southeast of point

N , like $\overline{E}$, indicating that it is possible to shift resources to the modern sector while maintaining or even diminishing the unemployment rate.

Consider two possible strategies for exploiting the higher propensity to save. One is to abolish unemployment first, and then shift resources-- follow the strategy indicated by point $\overline{Z}$ until the labor constraint becomes binding, and then shift to $\overline{E}$. The other is to shift resources first-- point $\overline{E}$ from the very beginning. As resources shift, ZZ rotates clockwise

and $\bar{E}$ ends up on the horizontal axis to the right of ny_1. So unemploy-
ment will continue to fall after sector 2 is phased out. Once full employ-
ment is restored, a lower saving rate-- $\hat{s} = \dfrac{n\hat{\alpha}_1}{1 - \hat{\alpha}_1 \hat{d}_1}$ will suffice to main-
tain it. As between the two strategies the second makes aggregate net
income and consumption grow faster; given the slope of YY , income growth
is greater at $\bar{E}$ than at $\bar{Z}$. But the first one might be preferred none-
theless if reduction of unemployment is valued per se, independently of its
contribution to consumable output.

6. Prices and Rates of Return Along Development Paths

I return now to the price calculations of section 1. These gave
wage rates and capital rents in various regimes. The capital rents can
be converted into net rates of return by subtracting depreciation per unit
of capital. Net rates of return should be equal in the two sectors when
non-zero gross investments are being made in both. Along paths where gross
investment is being made only in one sector, the rate of return for that
technology should exceed the other one.

Consider first case (A) of section 5 (which corresponds to regime VI
in section 1), with full employment of all resources. With the skilled
labor or sector 1 amalgamated into capital, the basic price equations (1.2)
become

$$1 = \hat{\alpha}_1 \hat{r}_1 \qquad + \beta_1 w$$
(6.1)
$$1 = \qquad \alpha_2 r_2 + \beta_2 w \;.$$

Let the rate of return on type 1 capital be $\hat{\rho}_1 = \hat{r}_1 - \hat{d}_1$, and on type 2 capital $\rho_2 = r_2 - d_2$. The two equations of (6.1) become:

(6.2)

$$1 - \hat{\alpha}_1 \hat{d}_1 = \hat{\alpha}_1 \hat{\rho}_1 + \beta_1 w$$

$$1 - \alpha_2 d_2 = \alpha_2 \rho_2 + \beta_2 w .$$

If $\hat{\rho}_1 = \rho_2 = \rho$, the solutions for ρ and w are:

(6.3)

$$\rho = \frac{\beta_2 (1 - \hat{\alpha}_1 \hat{d}_1) - \beta_1 (1 - \alpha_2 d_2)}{\hat{\alpha}_1 \beta_2 - \beta_1 \alpha_2}$$

$$w = \frac{\alpha_1 (1 - \alpha_2 d_2) - \alpha_2 (1 - \hat{\alpha}_1 \hat{d}_1)}{\hat{\alpha}_1 \beta_2 - \beta_1 \alpha_2} .$$

The solution for ρ is the net marginal product of capital in Figure 1a and 1b, for the range of golden ages $(0 \leq x \leq 1)$. It is always positive, and greater than n, by (4.4). In the normal case--sector 1 the more capital-using--the solution for w is also positive. A positive wage and equal rates of return are consistent with the discussion of the normal variant of case (A), where it turned out that the optimal path involved positive investment in both sectors and also maintained full employment of labor.

In the abnormal case--sector 1 the less capital-using--(6.3) gives a nonsense negative solution for w. If w is set at zero, (6.2) tells us that $\hat{\rho}_1$ and ρ_2 must be different, each equal to the average net product of capital in its sector. Since that is greater in technology 1, all invest- ment should be allocated to that sector. This is precisely the conclusion of the analysis of section 5. In the abnormal case, concentration of invest- ment in modern capital maximizes income growth, while at the same time making labor increasingly redundant.

If the economy begins with unemployment--case (C) of section 5 above --the wage rate will be zero and (6.2) will give divergent values of $\hat{p}_1$ and p_2 . In the normal case, p_2 will be the larger. As already concluded, growth is maximized by concentrating investment in that sector.

A word is in order regarding the interpretation of $\hat{p}_1$. If both kinds of capital in sector 1 are binding constraints and are receiving non-zero gross investment, they must have equal rates of return:

$$(6.4) \qquad \hat{p}_1 = r_1 - d_1 = \frac{w_1}{\ell} - m_1 + \mu \frac{\beta_{21}}{\beta_{11}} .$$

It follows then from the definitions of $\hat{\alpha}_1$ and $\hat{d}_1$, (2.5) and (2.9) that:

$$(6.5) \qquad 1 - \beta_1 w = \alpha_1 r_1 + \beta_{11} w_1 = \hat{\alpha}_1 (\hat{p}_1 + \hat{d}_1) .$$

When skilled labor is in excess supply, w_1 is zero. The rate of return on physical capital in sector 1 can then be found by substituting α_1 , d_1 and p_1 for $\hat{\alpha}_1$, $\hat{d}_1$ and $\hat{p}_1$ in the first equation of (6.2).

Conclusions

In the normal case, when the modern technology is the more capital-using, the proper strategy is to eliminate unemployment first and only then to attend to the shift of resources to the modern sector. The elimination of unemployment requires channeling of all gross saving to the old technology, temporarily reducing the size of the modern sector. There is nevertheless no conflict of objectives. The strategy that eliminates unemployment most rapidly also maximizes the growth of net income and consumption. Once full

employment is achieved, it can be maintained while the relative allocation

of resources shifts in favor of the modern sector. But none of the interim

investment in old-technology capital is lost; the older sector grows absolutely

even while it declines relatively.

In the abnormal case, when the modern sector is the less capital-

using, an impasse arises. The propensity to save is geared to the eventual

low capital requirements of that sector and is inadequate to expand the

capital stock as fast as the labor force is growing when the older technology

is still operating. The impasse can be escaped only by a temporary burst

of saving. If this is available, there is a conflict between rapid reduction

of unemployment and rapid growth of income and consumption. Channeling saving

predominantly to the modern sector is the way to maximize income growth,

but it postpones the day when unemployment is eliminated.

The essential conclusions of the paper can be obtained without finding

explicit expressions for the various development paths discussed. But these

expressions can be obtained, and the method is described in the Appendix

for interested readers.

APPENDIX

Along the development paths discussed in the paper, the exponential rate of growth of each sectoral output is a linear function of the ratio of the two outputs:

$$(A.1) \qquad \begin{cases} \dfrac{y_1'}{y_1} = b_1 + c_1 \dfrac{y_2}{y_1} \\[2em] \dfrac{y_2'}{y_2} = b_2 + c_2 \dfrac{y_2}{y_1} \end{cases} \qquad (y_1 \neq 0)$$

The parameters b_i and c_i vary with the path under discussion. In at least one case the two rates of growth are linear functions of y_1/y_2 ($y_2 \neq 0$) instead of y_2/y_1 . But the same method of solution applies symmetrically.

Let $y_2/y_1 = z$, $b_2 - b_1 = b$, and $c_2 - c_1 = c$. Then:

$$(A.2) \qquad \begin{cases} y_1 = y_1(0) e^{b_1 t} e^{c_1 \int_0^t z(\tau)d\tau} \\[2em] y_2 = y_2(0) e^{b_2 t} e^{c_2 \int_0^t z(\tau)d\tau} \\[2em] z = z(0) e^{bt} e^{c \int_0^t z(\tau)d\tau} \end{cases} .$$

The last equation may be written:

$$(A.3) \qquad \ln z = \ln z(0) + bt + c \int_0^t z(\tau)d\tau .$$

Differentiating (A.3) with respect to time t gives:

(A.4)
$$z' = bz + cz^2 .$$

The solution to the differential equation (A.4) is:

(A.5)
$$z = \frac{z(0)e^{bt}}{1 + \frac{c}{b} z(0) - \frac{c}{b} z(0)e^{bt}} = \frac{1}{\left(\frac{1}{z(0)} + \frac{c}{b}\right)e^{-bt} - \frac{c}{b}} .$$

Moreover,

(A.6)
$$\int_0^t z(\tau)d\tau = -\frac{tb}{c} - \frac{1}{c} \ln\left(\left(\frac{1}{z(0)} + \frac{c}{b}\right)e^{-bt} - \frac{c}{b}\right) + \frac{1}{c} \ln \frac{1}{z(0)}$$

(A.7)
$$e^{c_1 \int_0^t z(\tau)d\tau} = \left(\frac{c}{b} z(0)(1 - e^{bt}) + 1\right)^{-\frac{c_1}{c}} .$$

These results can be used to obtain explicit descriptions of paths discussed in section 5 of the text. Consider, for example, case (A), full employment of all resources, with the solutions for y_1' and y_2' given by (5.6) and (5.7). These may be put in the form of (A.1) with:

(A.8)
$$\begin{cases} b_1 = n & c_1 = n \frac{\beta_2(\hat{\alpha}_1\gamma - \alpha_2)}{\beta_2\hat{\alpha}_1 - \beta_1\alpha_2} \\[3mm] b_2 = 0 & c_2 = n \frac{\hat{\alpha}_1(\beta_2 - \gamma\beta_1)}{\beta_2\hat{\alpha}_1 - \beta_1\alpha_2} \\[3mm] b = -n & c = n \frac{\hat{\alpha}_1(\beta_2 - \gamma\beta_1) - \beta_2(\hat{\alpha}_1\gamma - \alpha_2)}{\beta_2\hat{\alpha}_1 - \beta_1\alpha_2} \end{cases}$$

From (A.2) and (A.7) may be obtained solutions for y_1, y_2, and z in terms of basic parameters and initial conditions. From these, of course, the paths for $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_1 + \bar{y}_2$, and for employment variables, sectoral and aggregate, may be derived.

For another example, take the case where the labor force constraint does not apply and the strategy is to place all investment in sector 1.

$$(A.9) \quad \begin{cases} b_1 = n & c_1 = n\gamma + \dfrac{\alpha_2}{\alpha_1} d_2 \\[2ex] b_2 = -d_2 & c_2 = 0 \\[2ex] b = -(n + d_2) & c = -\left(n\gamma + \dfrac{\alpha_2}{\alpha_1} d_2 \right) \end{cases}$$

If the saving constraint is (5.15) instead of (5.5), the parameters are different:

$$(A.10) \quad \begin{cases} b_1 = \dfrac{n\alpha_2}{\gamma} & c_1 = \dfrac{\alpha_2}{\alpha_1} (n + d_2) \\[2ex] b_2 = -d_2 & c_2 = 0 \\[2ex] b = -\left(\dfrac{n\alpha_2}{\gamma} + d_2 \right) & c = \dfrac{\alpha_2}{\alpha_1} (n + d_2) \ . \end{cases}$$

Finally, for the path along which all saving is allocated to sector 2--discussed under (C) in section 5--the rates of growth are linear functions of $1/z$ instead of z. Essentially the same set-up as (A.10) applies, but with the roles of the y_1 and y_2 interchanged.