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by

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1. Introduction

Risk aversion (or risk preference), as usually considered in the literature, characterizes a decisionmaker's response to a single period "fair gamble" or "actuarially neutral risk". The experiment determining the decisionmaker's audacity or caution generally takes the following form. Suppose the individual possesses wealth of value W . Confront him with a risk or gamble with an expected value of zero, that is, one in which the expectation involves no change from his present position. If the individual prefers his status quo to accepting the gamble he is a risk averter, while a preference for the gamble would show him to be a risk lover.

Numerous writers, dating back to Marshall,[1] have stated and discussed the relationship between an individual's attitude toward risk and the shape of his utility function. Taking the expected-utility hypothesis as the basis for behavior in a risk environment, it follows

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that the decisionmaker pursues that course of action which maximizes the expected value of his numerical-valued utility function $u(W)$. Then the individual is a risk averter if and only if $u''(W) \leqq 0$ and he is a risk lover if and only if $u''(W) \geqq 0$. That is, risk aversion is characterized by a concave utility function while risk loving is characterized by a convex utility function. More recently, Arrow and Pratt have refined the concept of risk aversion and have furthered our understanding of the implications of attitudes toward risk. In particular, as shall be discussed in detail in Section 2, Arrow and Pratt introduce a measure of absolute risk aversion and they investigate the implications of different shapes of the absolute risk aversion function.

The utility function whose shape is crucial in all of these definitions is a single-period utility-of-wealth function. "Strictly speaking," writes Pratt, "we are concerned with utility at a specified time (when a decision must be made) for money at a (possibly later) specified time."[2] But many decisions involving risk as a significant factor are not made in a one-period or point-input (decision made), point-output (money results) framework. The investment-planning decision or capital-budgeting decision of a firm is an example of a risky decision in which a decision taken now has a stream of future effects rather than a result at a <u>single</u> future (or present) point. The question naturally arises, then, as to whether the discussion of attitudes toward risk and statements about the behavioral implications of these attitudes can be extended to such a multiperiod context.

This paper is addressed to just that question--attitudes toward risk and their implications in the context of multiperiod planning decisions. In particular, one part of the paper will focus on a stylized version of a capital-budgeting problem in which the firm must make a planning decision today about which risky investments it will undertake over the course of the next several time periods. While the outlays on a project may or may not take the form of a stream of cash flows through time, the returns from the projects available will be assumed to occur at different points during the planning horizon. This capital-budgeting problem constitutes a point-input (decision), stream-output (returns flow) decision. Before discussing this particular capital-budgeting model, Section 2 reviews Pratt's discussion of risk aversion in the single-period case, and Section 3 discusses the question of decreasing absolute risk aversion in the multiperiod case. In Section 4, the particular and highly stylized capital-budgeting problem is described and the implications of decreasing risk aversion in a multiperiod sense are examined. The paper closes with several summary remarks and an indication of several open questions requiring further investigation.

2. Risk Aversion in the Single-Period Case[3]

Following Pratt, consider a decisionmaker with assets W and a (single-period) utility function $u(W)$. The utility function is unique up to a positive linear transformation, in accordance with Von Neumann-Morgenstern utility theory, and is bounded so that the expected-utility theorem may be used.[4] Moreover, $u(W)$ is taken to be at least twice differentiable with a postiive first derivative (positive marginal utility).

The individual is confronted with a risk or gamble that takes the form of the random variable $\tilde{z}$, the tilde indicating the stochastic nature of the variable. He will receive (or pay, depending on the sign of the realized value z) an amount which depends on the realization drawn from the distribution of the random variable $\tilde{z}$. As Pratt shows, one may--without loss of generality--restrict attention to actuarially neutral risks, risks for which $E(\tilde{z}) = 0$. In all of what follows in Sections 2 and 3, only such "fair gambles" will be considered.

The questions associated with risk aversion now center on the nature of the risk premium π , which depends on the level of assets and the distribution of the risk so that $\pi = \pi(W,\tilde{z})$. This risk premium is the deterministic sum such that the individual would be indifferent between receiving π dollars less than the actuarial value $E(\tilde{z})$-- hence, in our case receiving $-\pi$ dollars--and facing the risk $\tilde{z}$. In equation form, π is defined by

$$(2.1) \qquad u(W - \pi) = E\{u(W + \tilde{z})\} ,$$

where we write $u(W - \pi)$ instead of $u(W + E(\tilde{z}) - \pi)$ since $E(\tilde{z}) = 0$.

One might, alternatively, discuss the risk aversion or risk preference of a decisionmaker in terms of the amount of money he would pay an insurance company to assume his risk. This deterministic amount $\pi_I(W, \tilde{z})$, his insurance premium, would be defined by

$$(2.1') \qquad u(W - \pi_I) = E\{u(W + \tilde{z})\} .$$

A comparison of (2.1) and (2.1') shows that when $E(\tilde{z}) = 0$, π equals π_I . With actuarially neutral risks, then, the decisionmaker's risk premium equals his insurance premium.

Since the decisionmaker's utility function, $U(W)$, is continuous with positive marginal utility, the function $u(W - \pi)$ is a strictly decreasing, continuous function of π for a given W . But, then, the insurance premium or risk premium is uniquely defined by (2.1). For any given actuarially neutral risk $\tilde{z}$ and any given level of wealth or consumption income W , there is a single amount the decisionmaker would pay to avoid facing $\tilde{z}$.

In terms of these new definitions, a decisionmaker is a risk averter if and only if his insurance premium is nonnegative for all W and $\tilde{z}$: $\pi(W,\tilde{z}) \geqq 0$, all W and $\tilde{z}$. If $\pi(W, \tilde{z}) \leqq 0$ for all combinations of asset level and risk, the decisionmaker is a risk lover. Pratt then introduces a function $r(W)$, called the local risk aversion function, to measure the degree of aversion a utility function shows to small actuarially neutral risks. The function is defined as

$$(2.2) \qquad r(W) = -\frac{u''(W)}{u'(W)} \quad .$$

Since an individual is risk-averse if and only if his utility function is concave, that is, $u''(W) \leqq 0$, the risk premium $\pi(W,\tilde{z}) \geqq 0$ if and only if $r(W) \geqq 0$. The magnitude of $r(W)$ measures the extent of the decisionmaker's aversion to risks that are actuarially neutral and small in the sense of having small variances: $E(\tilde{z}) = 0$ and σ_z^2 infinitesimal.

Pratt goes on to introduce the concepts of increasing and decreasing absolute risk aversion.[5] A utility function is said to exhibit (strictly) decreasing absolute risk aversion in a global sense if $\pi(W, \tilde{z})$ is a (strictly) decreasing function of W for all $\tilde{z}$. Similarly, it is said to show (strictly) increasing absolute risk aversion in a global sense if $\pi(W, \tilde{z})$ is a (strictly) increasing function of W for all $\tilde{z}$. Pratt then proves that decreasing (increasing) local absolute risk aversion $r(W)$ is equivalent to decreasing (increasing) global absolute risk aversion. That is, the risk premium $\pi(W, \tilde{z})$ is a (strictly) decreasing function of W for all $\tilde{z}$ <u>if and only if</u> the local risk aversion function $r(W)$ is (strictly) decreasing, and similarly with "increasing" replacing "decreasing" in both of the relevant places. What is, in Pratt's words, "nontrivial" about this theorem is that $r(W)$ decreasing implies $\pi(W, \tilde{z})$ decreasing since $r(W)$ is a measure of risk aversion only for "small" risks. The theorem shows that if $r(W)$ is decreasing, that is, if $u'(W)u'''(W) \geqq [u''(W)]^2$, the insurance premium the decisionmaker would pay to protect himself against a given absolute risk $\tilde{z}$ --<u>no matter what its size</u>--decreases as his wealth increases.

Why should we be concerned with decreasing risk aversion? Pratt and Arrow offer somewhat different answers to this question. Pratt rests his case for being interested in whether a utility function shows decreasing, constant, or increasing absolute risk aversion on the insurance-premium implications of the concept. He writes

> These results have both descriptive and normative implications. Utility functions for which $r(W)$ is decreasing are logical candidates to use when trying to describe the behavior of people who, one feels, might generally pay less for insurance against a given risk the greater their assets....

> Normatively, it seems likely that many decision makers would feel they ought to pay less for insurance against a given risk the greater their assets. Such a decision maker will want to choose a utility function for which $r(W)$ is decreasing, adding this condition to the others he must already consider (consistency and probably concavity) in forging a satisfactory utility from more or less malleable preliminary preferences.[6]

Arrow, on the other hand, stresses the results of his investigation of a specific model of choice between risky and secure assets. In his model, the individual must allocate his initial wealth between <u>one</u> risky asset and <u>one</u> safe asset, which can be taken to be cash with no risk and no return or a perfectly safe bond with no risk but a positive return. The decisionmaker's goal is to maximize the expected utility of his final wealth at the end of the single period considered. That is, the individual wants to allocate his initial wealth, say $\bar{W}$, between an investment of amount A in an asset, whose net rate of return is described by the random variable $\tilde{z}$, and holding cash in amount $\bar{W} - A$ so as to maximize $E\{U(\bar{W} + \tilde{z}A)\}$, with $0 \leqq A \leqq \bar{W}$. In the context of this model, Arrow shows that "decreasing absolute risk aversion implies that...the amount of risky investment increases with wealth, as would be expected. In other words, risky investment is not an inferior good."[7] "If absolute risk aversion increased with wealth," he writes, "it would follow that as an individual became wealthier, he would actually decrease the amount of risky assets held."[8] "This result is empirically implausible" and "we must reject the hypothesis of increasing absolute risk aversion."[9]

In sum, Pratt argues that one ought to be interested in whether or not a utility function is decreasingly risk-averse because of the inherent plausibility of the property--that people will decrease the amount of insurance they buy to protect against a fixed risk when their wealth increases. Arrow's case for the importance of the concept of absolute risk aversion rests, in contrast, on a behavioral implication of decreasing absolute risk aversion--if utility functions possess the property, risky assets are a normal good. Two questions then arise. First, can the concept of decreasing absolute risk aversion be extended to situations involving multiperiod planning decisions and can Pratt's condition $u'(W)u'''(W) \geqq [u''(W)]^2$ for the single-period property be suitably generalized? Second, does decreasing absolute risk aversion in the multiperiod sense imply that risky investment is not an inferior good in such planning contexts? Section 3 indicates that the answer to the former question is yes while Section 4 shows that the answer to the latter question is no.

3. Decreasing Risk Aversion in the Multiperiod Case[10]

With the summary discussion of risk aversion and the single-period utility function completed, turn now to the more general case in which the decisionmaker's horizon extends beyond the present period. More precisely, attention is now focused on the case in which a decision made at a single point in time (now) gives rise to a whole stream of monetary effects (during future periods) rather than to a single monetary effect (in the present period or in some single future period).

The decisionmaker possesses a T-period horizon and a utility function $U = U(C)$ defined over that horizon where C is the vector

$(C_1, C_2, \ldots, C_T)$ describing the sequence of consumption incomes over the T periods. The utility function is again unique up to a positive linear transformation and is again assumed to be bounded. It is also assumed that $U(C)$ is at least twice continuously differentiable with respect to all elements, and that dollars in each period have positive marginal utility no matter what the time stream of consumption incomes, that is, $U_t = \frac{\partial U}{\partial C_t} > 0$ for all t and all C vectors. One further assumption--unnecessary in the single-period case--will be made about $U(C)$, or rather about the realtionship between incomes in different periods. It will be assumed that there exists the same diminishing marginal rate of substitution for substitutions in every direction among dollars in different periods as one finds in the theory of consumer behavior with many commodities. That is, we assume that $U(C)$ is a quasi-concave function.

While in the single-period case the decisionmaker was confronted with a single risk, $\tilde{z}$, in the present case the decisionmaker is confronted with a vector of risks (random variables) $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_T)$, one occurring in each period. The vector $\tilde{Z}$ is thus a T-dimensional random variable. The distribution of each random variable $\tilde{z}_t$ represents the decisionmaker's perception of the t^{th} risk he faces. The probability distribution may be objective or subjective in nature, but where an actual risk is specified to exist in the t^{th} period, it is assumed that $\tilde{z}_t$ is not constant with probability one. Once again without loss of generality, in defining decreasing risk aversion attention is restricted to vectors of "fair risks," that is, vectors $\tilde{Z}$ such that $E(\tilde{Z}) = 0$, or $E(\tilde{z}_t) = 0$ for all t .

In place of the single period's single risk premium π , in the multiperiod case one has a vector Π of risk premiums, one for each period: $\Pi = (\pi_1, \ldots, \pi_T)$. The vector Π represents a vector of money amounts such that the decisionmaker would be indifferent between facing the risks $\tilde{z}_t$ for $t = 1, \ldots, T$ and receiving the deterministic amounts $-\pi_t$ (that is, π_t less than the actuarial amount $E(\tilde{z}_t) = 0$) for $t = 1, \ldots, T$. A vector of risk premiums must therefore satisfy the condition

$$(3.1) \qquad U(C - \Pi) = E\{U(C + \tilde{Z})\} .$$

Whether a particular vector does or does not satisfy (3.1) will depend on the initial consumption-income stream and the vector of risks, that is, Π is a function of C and $\tilde{Z}$. That some vector Π does exist is guaranteed by the assumption that $E\{U(C + \tilde{Z})\}$ exists and is finite, and the assumption of positive first partial derivatives and continuous second partials for $U(C)$. Since attention is confined to a vector of actuarially neutral risks, a vector of risk premiums is also a vector of insurance premiums, just as was true for π and π_I in the single-period case. Thus, a vector Π_I is a vector of period-by-period insurance payments the decisionmaker would willingly make to avoid the risk vector $\tilde{Z}$ when his consumption incomes are given by C if and only if it satisfies (3.1).

Completely analogous to the single-period case, an individual is defined to be risk-averse if he prefers (or is indifferent between) his certain status quo position C to (and) the fair risky result determined by the realization of the vector random variable $C + \tilde{Z}$. For a risk averter and only for a risk averter, $U(C) \geqq E\{U(C + \tilde{Z})\}$ for any risk vector with $E(\tilde{Z}) = 0$. For a strict risk averter the weak inequality is replaced by $U(C) > E\{U(C + \tilde{Z})\}$. By extending Jensen's Inequality to functions of several variables, it can easily be shown that an individual is a risk averter if and only if his multiperiod utility function $U(C)$ is concave.[11]

There would appear, however, to be some difficulty in defining risk aversion in terms of insurance policies or vectors of risk premia in the multiperiod case. The crux of the problem is that the vector Π is not uniquely defined by equation (3.1). In the single-period case, the assumption of a continuous utility function with positive marginal utility implied that the risk premium π was unique; but more than one Π-vector can be found--in fact, an infinite number can be found--which satisfy (3.1) for a given C and a given $\tilde{Z}$. It is easy to see why this occurs. Given his asset vector C and the risk vector $\tilde{Z}$, the expected utility resulting from accepting the risks can be calculated. From the assumptions about $U(C)$, it follows that this expected utility $E\{U(C + \tilde{Z})\}$ exists and is finite, say it equals k. Equation (3.1) now leads us to determine all vectors Y such that $U(Y) = k$. Corresponding to each of these vectors is a vector of risk premiums $\Pi(C, \tilde{Z})$ which is found by substracting the vector Y from the initial consumption vector C. But in the multiperiod case an infinite number of consumption-income vectors generate the same utility level k. All such vectors lying on the indifference surface corresponding to $U(C) = k$ will have utility k and corresponding to each of them will be a vector Π satisfying (3.1).

Consider, for example, the situation of the decisionmaker in Figure 1. His initial position is $\bar{C} = (\bar{C}_1, \bar{C}_2)$ and he is confronted with the risks $\tilde{Z} = (\tilde{z}_1, \tilde{z}_2)$. We assume that he is a risk averter so that the indifference curve corresponding to utility level $E\{U(\bar{C} + \tilde{Z})\}$ lies below the indifference curve on which $\bar{C}$ lies. Corresponding to each point on indifference curve I_1 there is an insurance policy the decisionmaker would be willing to purchase to avoid the gamble. For any point on I_1, say (C_1', C_2'), the corresponding insurance policy is given by $(\pi_1, \pi_2) = (\bar{C}_1, \bar{C}_2) - (C_1', C_2')$. One particular insurance policy vector is shown in the figure, namely, the one going from $\bar{C}$ to $\bar{C} - \Pi$.

The question arises, then, as to whether one can determine if an individual is a risk averter or a risk lover solely by observing the "insurance policies" he would be willing to buy. The answer is yes. To do so, however, one must examine a specific subset of such policies. Consider an insurance-premium vector feasible for the $(C, \tilde{Z})$ pair in which the individual pays nothing in any period but the t^{th} one. That is, define the risk-permium vector $\Pi^t(C, \tilde{Z})$ such that $U(C - \Pi^t) = E\{U(C + \tilde{Z})\}$ and $\pi_i^t = 0$ for all i except $i = t$.

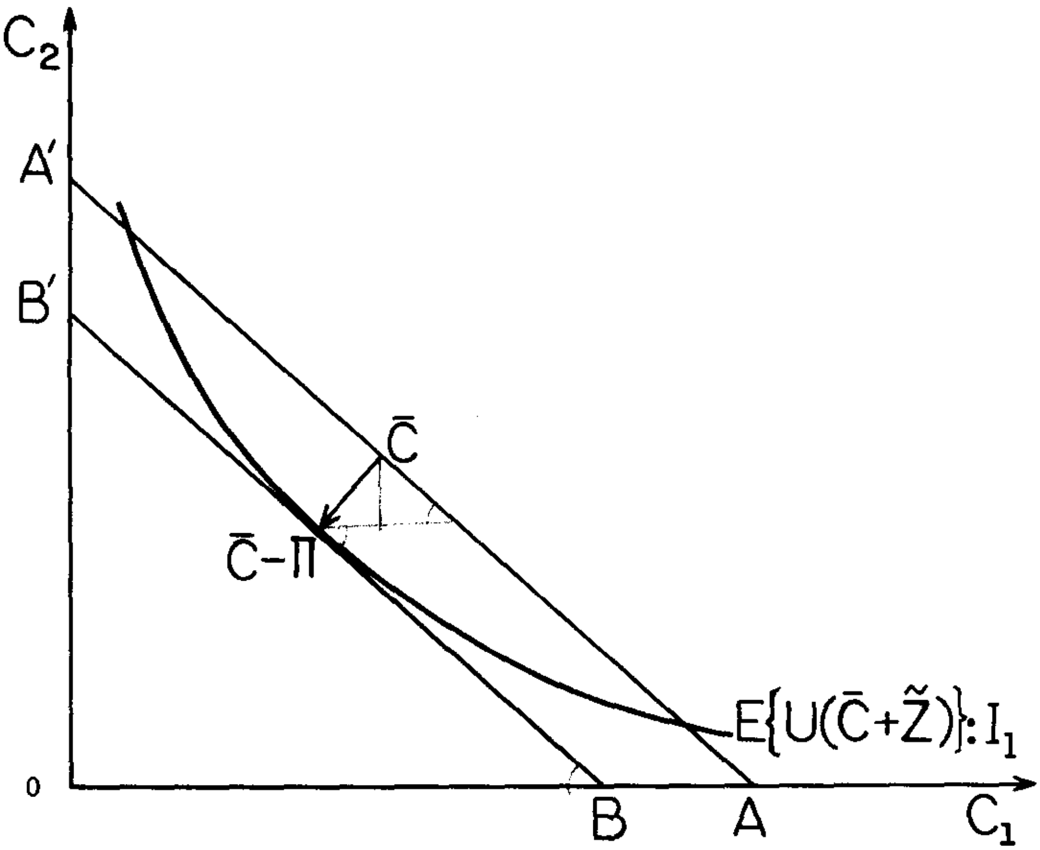

FIGURE 1

AN EXAMPLE OF AN INSURANCE POLICY

Since there are T periods in the planning horizon, there are T such "all-in-one-period" policies. The following theorem, which demonstrates the importance of these "all-in-one-period" policies in assessing a decisionmaker's attitudes toward risk, can be proven.[12]

> <u>THEOREM 1</u>. A decisionmaker is a strict risk averter if and only if for any C, $\tilde{Z}$ pair $\Pi^t(C, \tilde{Z})$ is a semi-positive vector for all t; that is, if and only if $\pi_t^t(C, \tilde{Z}) > 0$ while $\pi_i^t(C, \tilde{Z}) = 0$ for all $i \neq t$, for all t.

The theorem states that an individual is a strict risk averter if and only if he would buy one insurance policy in which a premium must be paid in the first and only in the first period, one in which the entire premium must be paid in the second period, and so on for $t = 1, 2, \ldots, T$.

As the individual decisionmaker's initial consumption-income vector changes one might well expect that the insurance policies he would be willing to purchase to avoid a given risk vector would also change. In particular, if one agreed with Pratt in the single-period case, decreasing risk aversion would seem to be a plausible property for a multiperiod utility function to possess if it is to be used in making decisions under risk. Intuitively, decreasing risk aversion has the same meaning in the multiperiod case as in the single-period model. As the individual's consumption incomes increase so that he becomes better off (as a result of the assumed positive marginal utility of dollars in any period at every point in consumption-income space), he will want to pay less for insurance against a given vector of risks.

Formalizing this intuitive definition is not a straightforward matter. The stumbling block, again, is the multiplicity of insurance policies a decisionmaker would be willing to purchase in order to avoid a given set of risks when his consumption incomes in the T periods are at given levels. The following line of reasoning, however, points the way to a most reasonable formal definition of decreasing risk aversion. Note that at any point in consumption-income space, C , the decisionmaker has a set of subjective rates of time preference which indicates his relative valuation of dollars in different periods. The rates of time preference at a point, C , are defined by the normal at C to the indifference surface on which C lies. In the two-dimensional case shown in Figure 1, the decisionmaker's time preference is simply given by the marginal rate of substitution of dollars in period 2 for dollars in period 1 . For example, for point $\bar{C} - \Pi$, his rate of time preference is the absolute value of the slope of BB' . If one evaluates the discounted present value (DPV) of the individual's insurance policy at these subjective discount factors, one then has that the DPV of Π , denoted Π_0 , equals $\sum_{t=1}^{T} \frac{U_t}{U_1} \Pi_t$. For example, in the figure, the DPV of the policy shown there is AB .

Denote $\frac{U_t}{U_1}$, the discount factor to be applied to t^{th} period consumption income to put it on an equal footing with today's (the first period's) consumption income, by D_t , and denote

$(D_1, D_2, \ldots, D_T)$ by D. It is then clear that every triple of vectors $(C, \tilde{Z}, D)$ defines an insurance policy, and the discounted present value of that policy, denoted Π^o, is $\Pi^o(\bar{C}, \tilde{Z}, D) = \sum_{t=1}^{T} D_t \pi_t(\bar{C}, \tilde{Z}, D)$. Decreasing absolute risk aversion in the multiperiod sense is then defined as follows. A decisionmaker with utility function $U(C)$ is decreasingly risk averse if as C increases--that is, at least one element of C increases and none decrease--the discounted present value of his insurance policy, $\Pi^o(C, \tilde{Z}, D) = \sum_{t=1}^{T} D_t \pi_t$ decreases for all risks $\tilde{Z}$ and all sets of positive discount factors $D = (D_1, D_2, \ldots, D_T)$. That is, a decisionmaker (or his utility function) is decreasingly risk averse in the multiperiod sense if for any risk vector $\tilde{Z}$ and any vector of discount factors D, $\Pi^o(\bar{C}', \tilde{Z}, D) \leqq \Pi^o(\bar{C}, \tilde{Z}, D)$ for all $\bar{C}' \geq \bar{C}$ (where $\geq$ means semipositively related), and this is true for all positive D-vectors, and all $\tilde{Z}$-vectors. In terms of Figure 1, this states that for the given $\tilde{Z}$ vector, as the point $\bar{C}$ is moved northeast, the length of segments corresponding to AB will decrease, and this will be true no matter what slope one gives to BB'. And, moreover, it states that this will remain true no matter what risk-vector $\tilde{Z}$ is chosen.

There are two basic reasons one can offer for the desirability of this definition. One line of argument runs in terms of the existence of a perfectly competitive capital market. This argument observes first that a perfectly competitive multiperiod capital market can be defined by a vector of single-period interest rates $(r_1, r_2, \ldots, r_T)$, where the decisionmaker can borrow or lend in period t at the fixed single-period interest rate r_t. Suppose the decisionmaker faces such a market, and consider a sequence of situations in which he begins at different $\bar{C}$ positions but always faces the same risk vector. For each $\bar{C}$ vector in the sequence, there will be a different equilibrium position, and each such equilibrium point will be the point on the relevant indifference curve, the one corresponding to the value $E\{U(\bar{C} + \tilde{Z})\}$, where the normal to the indifference surface is given by the D-vector.[13] The appropriate measure of the discounted present value of any policy Π in a perfect capital market of this kind is then found by applying the discount factors $D_1, D_2, \ldots, D_T$ where

$$D_t = \frac{1}{(1+r_1)(1+r_2)\ldots(1+r_{t-1})}$$ to the respective insurance premia

$\pi_1, \pi_2, \ldots, \pi_T$. The definition given here states that the decisionmaker is decreasingly risk averse in the multiperiod sense if for any arbitrary $\tilde{Z}$ vector, the value (DPV) of his insurance policy decreases as his endowment vector increases semipositively, <u>no matter what perfect capital market he faces</u>. (The only restriction on the market is that no interest rate be -1 so that each D_t is ensured of being finite.)

The second justification for the given definition runs solpy in terms of the decisionmaker's preferences and does not involve any appeal to a perfect capital market. It rests on the observation that the added difficulty that arises in discussing attitudes toward risk in the multiperiod case, as contrasted with the same discussion in the single-period case, derives from the fact that the multiperiod utility function is the repository of information about both (1) the decisionmaker's time preferences and (2) his attitudes toward risk. In order to isolate the latter--our primary concern here--one wants to focus on the changes in the decisionmaker's insurance purchases that result from given endowment changes, holding his time preference "constant." This is essentially what the measure presented here accomplishes. The positions the decisionmaker attains via insurance purchases that we examine to see whether or not he is decreasingly risk-averse are all marked by the same rates of time preference--the vector of discount factors, D , is the same at all of these points. Hence, any changes that occur in the value of his insurance policies as his consumption-income vector changes are the result of his risk preferences (or aversion) working alone. In order for a decisionmaker to be classified as decreasingly risk averse it is required that for all configurations of his time preference, as represented by all positive D-vectors, the DPV of his insurance policy decrease as his consumption-income endowment is augmented by a semipositive vector.

Having defined decreasing absolute risk aversion with regard to multiperiod situations, it is desirable to characterize utility functions that display this property. I have been unable to obtain a general set of necessary and sufficient conditions for a utility function to be decreasingly risk averse under this definition. What can and will be provided is 1) a set of sufficient conditions for a utility function to exhibit decreasing absolute risk aversion in the face of independent risks, 2) a statement about a necessary condition for decreasing absolute risk aversion in the multiperiod case, and 3) a statement of the necessary and sufficient condition for a particular class of multiperiod utility functions to be decreasingly risk averse in the face of independent risks.

Focusing attention on the class of independent risks means we are considering risk vectors $\tilde{Z}$ which can be written as

$$\tilde{Z} = \sum_{t=1}^{T} \tilde{Z}^t \tag{3.2}$$

where $\tilde{Z}^t$ is a vector each of whose elements is zero except the t^{th} which is the random variable $\tilde{z}_t$. The risk faced in a specific period is assumed to be independent of the risk faced in any other period so that the distribution of $\tilde{z}_t$ is unaffected by the realization of any other risk $\tilde{z}_s$ for $s \neq t$. The class of risk vectors considered has been restricted to such independent risks in order to derive meaningful results concerning multiperiod decreasing risk aversison.

The vector $\tilde{Z}^t$ thus removes all the risks the individual was originally supposed to face--as given by $\tilde{Z}$--except for the one occurring in the t^{th} period. Each $\tilde{Z}^t$ is a T-element vector with

$\tilde{z}_i^t = 0$ for all i except $i = t$ and $\tilde{z}_t^t = \tilde{z}_t$ where $\tilde{z}_t$ is the t^{th} element of the original risk vector $\tilde{Z}$. Similarly, define a set of consumption vectors, C^t for $t = 1, \ldots, T$, that are derived from C in the same way the $\tilde{Z}^t$ vectors are derived from $\tilde{Z}$. Each new vector C^t leaves the individual with the same amount of consumption income he originally had in the t^{th} period but with nothing in any other period. Hence, each C^t is a T-element vector with $C_i^t = 0$ for all $i \neq t$ and $C_t^t = C_t$ where C_t is the t^{th} element of the original consumption-income vector.

In order to state the set of sufficient conditions, it is necessary to introduce some further notational structure and a new set of random variables. Specifically, we introduce the following functional notation:

$U^{-1}[p, \bar{C}, t]$ is defined as the T-element vector C such that

(3.3) (i) $U(C) = p$ and (ii) C_i equals $\bar{C}_i$ for all i except $i = t$.

Hence, for example, the insurance policy which requires the individual to pay a positive amount in the t^{th} period and nothing in any other period--denoted $\Pi^t(C, \tilde{Z})$--can be written as

(3.4) $$\Pi^t(C, \tilde{Z}) = C - U^{-1}[E\{U(C + \tilde{Z})\}, C, t] .$$ [14]

As a last preliminary before the statement of Theorem 2, a new set of random variables is introduced. Let $\tilde{h} = U(C + \tilde{Z})$.

The risk vector $\tilde{Z}$ generates a vector of stochastic additions to the original consumption-income vector. This results in a new random variable, $C + \tilde{Z}$, which in turn yields a random variable describing the decisionmaker's utility. It is this latter random variable--the individual's (stochastic) utility level resulting from the risk vector $\tilde{Z}$--that we call $\tilde{h}$. The risk vector $\tilde{Z}$ maps into a unique distribution for $\tilde{h}$ but the converse is not true: $U^{-1}(\tilde{h})$ is a correspondence not a function.

Similarly, define $\tilde{h}_t = U(C^t + \tilde{Z}^t)$. This says the following. Suppose the decisionmaker has consumption income in the t^{th} period and only in the t^{th} period. Assume, moreover, that the level of consumption income in that period is the same as what he was allotted by the original consumption-income vector C . The decisionmaker is then confronted with a risk in the t^{th} and only the t^{th} period and that risk is identical with the t^{th} risk he faced in the original risk vector, $\tilde{Z}$. The vector $\tilde{Z}^t$ generates a stochastic addition to his t^{th} period consumption income and yields the new stochastic variable $C^t + \tilde{Z}^t$. This new stochastic variable then generates a distribution of the individual's utility level. It is this stochastic utility level that we denote $\tilde{h}_t$.

With C , and hence C^t, given the risk vector $\tilde{Z}^t$ maps into a unique distribution for $\tilde{h}_t$. In addition, since each period's consumption income is assumed to have continuous positive marginal utility, with C^t given each $\tilde{h}_t$ maps into a unique risk vector $\tilde{Z}^t$ since $\tilde{Z}^t$ contains only one possibly nonzero element. Hence, (3.5) $\tilde{h}_t = U(C^t + \tilde{Z}^t)$ and $C^t + \tilde{Z}^t = U^{-1}[\tilde{h}_t, C^t, t]$.

Similarly, with $\tilde{Z}^t$ given each C^t maps into a unique distribution for $\tilde{h}_t$ and by the assumed continuous positive marginal utility of each period's consumption income, each distribution of $\tilde{h}_t$ maps into a single C^t . Finally, if attention is restricted--as it is here--to actuarially neutral risks, $E(\tilde{Z}) = 0$, then there is a one-to-one mapping between distributions of $\tilde{h}_t$ and $(C^t, \tilde{Z}^t)$ pairs.

With these new definitions of variables and functions at hand, one can proceed to the statement of a set of sufficient conditions for multiperiod decreasing absolute risk aversion in the face of independent risks.[15]

THEOREM 2. If $U(C)$ is the utility function of a strictly risk-averting decisionmaker, then the decisionmaker is decreasingly risk-averse in the multiperiod sense with respect to independent risks if the following conditions are satisfied.

(3.6) $U(\sum_{t=1}^{T} U^{-1}[\tilde{h}_t , C^t ,t])$ is convex in the $\tilde{h}_t$ variables;

(3.7) $dU(\sum_{t=1}^{T} U^{-1}[\tilde{h}_t , C^t , t])$ is convex in the $\tilde{h}_t$ variables

where the differential is with respect to nonnegative increments in the T initial consumption incomes;

(3.8) setting all $C_i = 0$ except C_t in $U(C)$,

$$\frac{\partial^3 U}{\partial C_t^3} \cdot \frac{\partial U}{\partial C_t} \geqq \left(\frac{\partial^2 U}{\partial C_t^2}\right)^2 \quad \text{and this is true for all} \quad t \ .$$

In view of our objective of seeing whether the concept of single-period decreasing absolute risk aversion and Pratt's condition for the single-period property can be suitably generalized, it is important to note the relationship between the set of sufficient conditions given in (3.6) - (3.8) for decreasing risk aversion in the multiperiod sense and Pratt's result for the single-period case. As Pratt shows,[16] a single-stage utility function $u(W)$ is (strictly) decreasingly risk averse if and only if

$$u'(u^{-1}(\tilde{q})) \text{ is a (strictly) convex function of } \tilde{q} \text{ or,} \tag{3.9}$$
$$\text{equivalently, } \quad u'(W)u'''(W) \geqq [u''(W)]^2 \ ,$$

where $\tilde{q}$ is the random variable indicating the stochastic level of utility. If the horizon T is set equal to unity, the conditions in (3.6) - (3.8) reduce to Pratt's single-period condition given in (3.9). Let us verify this now.

Examining the conditions in the theorem in reverse order, we see that (3.8) simply requires that the decisionmaker be decreasingly risk averse in the single-period sense if he has consumption income in only one period and faces a single-period risk in the same period. Hence, with $T = 1$ condition (3.8) is exactly the second equivalent form of Pratt's condition in (3.9) . Condition (3.7) , on the other hand, is the multiperiod analogue of the first equivalent form of

Pratt's condition given above. With $T = 1$, condition (3.7) requires that $dU(U^{-1}[\tilde{h}_1, C^1, 1])$ be convex in $\tilde{h}_1$ where the differential is with respect to positive increments in the only initial consumption income C_1. Since the mapping in (3.5) is one-to-one and since the vector involved contains only one element, this is equivalent to requiring that $du(u^{-1}(\tilde{h}_1))$ be convex in $\tilde{h}_1$ for a positive differential change in C_1. But this means $u'(u^{-1}(\tilde{h}_1))dC_1$ convex in $\tilde{h}_1$ for $dC_1 > 0$ or, therefore, $u'(u^{-1}(\tilde{h}_1))$ is convex in $\tilde{h}_1$. This is precisely the condition in (3.9).

There is, however, no analogue to condition (3.6) in Pratt's statement of necessary and sufficient conditions for single-period decreasing absolute risk aversion. Some interpretation of (3.6) therefore seems in order. It can be shown[17] that this condition is equivalent to the statement

$$(3.10) \qquad U(C - \Pi^t(C, \tilde{Z})) \geqq U(C - \sum_{i=1}^{T} \Pi^i(C^i, \tilde{Z}^i)) \quad \text{for all } t ,$$

which has the following interpretation. It expresses the decisionmaker's preference for an insurance policy that allows him to consider simultaneously all the risks he faces in all periods and to decide on a premium or set of premiums to cover such risks rather than having to insure against each risk out of the consumption income of the particular period in which the risk must be confronted. A decisionmaker who shows such a preference will be called a "risk balancer over time," or in what follows simply a "risk balancer."

Assume there were two insurance companies from which the decisionmaker could buy a policy. Suppose the first company, Allrisk Incorporated, offered the individual a policy in which he paid a certain premium in the t^{th} period which insured him against his T risks. The second company, One-At-A-Time Incorporated, makes available a different type of insurance. It tells the decisionmaker that in insuring against a given period's risk it does not want to consider his consumption income in any other period or the other risks he faces. Instead, One-At-A-Time wants each risk insured against separately and strictly out of the consumption income the decisionmaker possesses in the period in which the risk occurs. If the individual in question is a risk balancer he will purchase his policy from Allrisk while if he is not a risk balancer, One-At-A-Time will have gained itself a customer. This pairing of customers with companies follows because Allrisk is offering the individual the policy $\Pi^t(C, \tilde{Z})$ while One-At-A-Time is offering him the policy $\sum_{i=1}^{T} \Pi^i(C^i, \tilde{Z}^i)$.

It should be noted that the risk-balancing property does not imply that the individual necessarily prefers an insurance policy under which he pays the whole premium in one period to a policy in which he pays a premium in each period. On the contrary, it can be shown[18] that for <u>any</u> strict risk averter there is <u>at least</u> one insurance policy (generally an infinite number of such policies if his utility function is smooth) he would purchase that involves payment of a positive premium in every period. Call this policy Π^* . Then since $U(C - \Pi^*(C, \tilde{Z})) = E\{U(C + \tilde{Z})\}$ and $U(C - \Pi^t(C, \tilde{Z})) = E\{U(C + \tilde{Z})\}$, one has, using (3.10),

$$(3.11) \qquad U(C - \Pi^*(C, \tilde{Z})) \geqq U(C - \sum_{i=1}^{T} \Pi^i(C^i , \tilde{Z}^i)) .$$

For a risk balancer, the policy $\Pi^*(C, \tilde{Z})$ is also preferred to the policy offered by the One-At-A-Time company. Thus the risk-balancing aspect of the decisionmaker's behavior does not mean he does not like to spread his premium payments over time. Rather, it means that he does not want to have to meet each risk from the consumption income of the particular period in which the risk occurs. This risk-balancing property would seem to be a natural property of rational decisionmaking under risk in a multiperiod setting.

Returning to the relationship between conditions (3.6) - (3.8) and Pratt's condition (3.9), it is clear that there is no single-period analogue to condition (3.6) since there is no meaning to risk balancing over time when there is only one period. Nevertheless, formally setting $T = 1$ in (3.6), the condition becomes that $U(U^{-1}[\tilde{h}_1, C^1, 1])$, with C^1 a single-element vector, must be convex in $\tilde{h}_1$. But $U(U^{-1}[\tilde{h}_1 , C^1 , 1]) = \tilde{h}_1$ as a result of the one-to-one nature of the mapping defined in (3.5) . Since $\tilde{h}_1$ is convex in itself the first condition of the theorem is met by any single-period utility function.

In summary, then, if there is in fact only one period to the decisionmaker's planning horizon, Pratt's condition (3.9) and the sufficient conditions (3.6) - (3.8) are identical. The results just described lead one to answer affirmatively the first question posed at the end of Section 2. The concept of decreasing absolute risk aversion defined in terms of insurance policies can be extended to multiperiod planning decision situations and a set of sufficient conditions for the property is a generalization of the Pratt condition for the single-period case.

While I have been unable to find a complete set of necessary conditions for multiperiod decreasing absolute risk aversion--and in particular have been unable to prove that conditions (3.6) - (3.8) constitute a set of necessary conditions--one such condition is clear. From Theorem 1 it follows that for any C vector and any $\tilde{Z}$ vector, there will exist T "all-in-one-period" insurance policies, Π^t, $t = 1, 2, \ldots, T$ with $\pi_t^t > 0$ and $\pi_i^t = 0$ for all $i \neq t$, that a strictly risk-averse individual would be willing to purchase. That is, for any consumption-income vector-risk vector pair, there will be an insurance policy a risk averter would purchase in which he would pay nothing in any period but the t^{th} one, and there will be one such policy for each of the T periods in his planning horizon.

Now consider a case in which the decisionmaker possesses an initial consumption income in only the t^{th} period and faces a risk in the t^{th} period alone. In this "corner"-type case multiperiod decreasing risk aversion requires that π_t^t, the only nonzero element of the t^{th} all-in-one-period insurance policy, decrease as C_t, the t^{th} element of his consumption-income vector, and only C_t increases in the face of the given risk. Decreasing absolute risk aversion in the multiperiod sense thus requires that if all C_i are set equal to zero except, say the t^{th}, the resulting single-period utility function must exhibit decreasing absolute risk aversion in the single-period sense. The utility function $U(0, 0, \ldots, 0, C_t, 0, \ldots, 0)$ must meet the Pratt condition (3.9) when considered as a function of C_t alone. Thus, condition (3.8) is also a necessary condition for decreasing absolute risk aversion in the multiperiod case.

Combining the necessity of (3.8) for multiperiod decreasing risk aversion with the sufficiency of (3.6) - (3.8), we can state the single necessary and sufficient condition for an additive multiperiod utility function to be decreasingly risk averse in the face of independent risks. Assume the utility function $U(C)$ is additive in the individual periods' utilities: $U(C) = \sum_{t=1}^{T} u_t(C_t)$, where the decisionmaker's rate of time preference is reflected in the relative magnitudes of the parameters of the different $u_t(C_t)$ functions. Condition (3.6) is then satisfied because we have

(i) $U(\sum_{t=1}^{T} U^{-1}[\tilde{h}_t, C^t, t]) = U(\sum_{t=1}^{T} [C^t + \tilde{Z}^t]) = U(C + \tilde{Z})$

from (3.5) and the definition of C^t and $\tilde{Z}^t$, and

(ii) $U(C + \tilde{Z}) = \sum_{t=1}^{T} U(C^t + \tilde{Z}^t)$ from the additivity property,

and finally

(iii) $\sum_{t=1}^{T} U(C^t + \tilde{Z}^t) = \sum_{t=1}^{T} \tilde{h}_t$ from (3.5).

Hence, if $U(C) = \sum_{t=1}^{T} u_t(C_t)$, $U(\sum_{t=1}^{T} U^{-1}[\tilde{h}_t, C^t, t]) = \sum_{t=1}^{T} \tilde{h}_t$ which is convex in the $\tilde{h}_t$ variables since it is the sum of them. In addition, with $U(C) = \sum_{t=1}^{T} u_t(C_t)$, it can be proven that condition (3.7) will be satisfied if each $u_t(C_t)$ is decreasingly risk averse in the single-period sense; that is, if each $u_t(C_t)$ meets condition (3.9).[19] But in the case of an additive utility function, condition (3.8)--which we have just seen to be a necessary condition for multiperiod decreasing risk aversion--is precisely the requirement that $u_t(C_t)$ be decreasingly risk averse in the single-period sense for each t.

The substance of these remarks can be summarized in the form of the following theorem.

<u>THEOREM 3</u>. If $U(C)$ is an additive utility function so that $U(C) = \sum_{t=1}^{T} u_t(C_t)$, then $U(C)$ is decreasingly risk averse in the multiperiod sense with respect to independent risks if and only if each $u_t(C_t)$ satisfies the Pratt condition (3.9).

4. Multiperiod Decreasing Risk Aversion and "A Capital-Budgeting Problem"

A. The "Capital-Budgeting" Model

Having answered the first of the two questions raised at the close of Section 2 in the affirmative by showing that the insurance-policy interpretation of decreasing absolute risk aversion could be extended to the multiperiod case and that a set of sufficient conditions for the property represented a generalization of the Pratt condition, let us now turn to the second question raised there. Specifically, does decreasing absolute risk aversion in the multiperiod sense imply that risky assets are a normal good in a multiperiod planning context? As a decreasingly risk-averse decisionmaker becomes wealthier, does the amount of risky assets he purchases increase? This question will be discussed in the context of a highly stylized version of a firm's capital-budgeting decision.[20]

The decisionmaking unit of the model is an existing firm planning an investment program for the next T periods.[21] The resources the firm controls before the investment program starts (that is, at $t = 0$) are expected to generate a stream of returns over the firm's investment-planning horizon. Specifically, in each of the first $T-1$ periods the firm anticipates a nonstochastic cash throw-off of X_t dollars, $t = 1, \ldots, T-1$, from operations apart from the investment program's returns. In addition, the enterprise has on hand X_0 dollars at the start of its investment-planning horizon. These are internal funds that have been generated prior to the start of the investment plan and that are available for use for the capital program in period 1.

The firm's investment program is not limited, however, by the cash throw-offs generated by the assets it owns at $t = 0$. Instead, the firm faces a "perfect" capital market in which it can enter into one-period contracts, which are de facto renewable, as either a lender or a borrower. If the firm lends L_t (> 0) dollars in period t, where L_t denotes the net amount loaned in period t, it makes this amount available to the borrower at the start of the t^{th} period. At the end of period t, it receives from the borrower the amount $(1+r)L_t$, where r is the single-period riskless rate of interest assumed to be constant over the course of the T-period horizon.[22] This sum, $(1+r)L_t$, is then available for use by the firm in period $t+1$. Since the planning horizon extends over only T periods, the firm will make such loans only in periods $1,2,\ldots, T-1$ because the benefits of any such loans made in the T^{th} period would go unrecorded.

Similarly, if the firm borrows $-L_t$ (> 0) dollars in period t, it receives this amount at the start of period t. On the last day of the period, when it is also receiving the t^{th} period cash throw-off from its original assets, X_t, the firm must pay its creditors $(1+r)(-L_t)$ dollars. Since it must repay the principal and interest on the last day of period t, the firm reduces the amount it has available for investment in period $t+1$ by the amount $(1+r)(-L_t)$. It repays the principal and interest on a t^{th} period loan out of the cash throw-off X_t. Since the model stops at the horizon period T , if the firm were allowed to borrow in period T it could then borrow unlimited quantities which it would not have to consider repaying. Hence, the option of borrowing in the horizon period is also completely removed.

There are two constraints imposed on the firm's borrowing and lending strategies. The first is that the capital market in which the firm operates--for example, the bank from which it borrows and through which it lends--requires that the firm state at $t=0$ how much it will borrow and lend in each period of the planning horizon. That is, the firm must not only set out a plan of its investments for its creditors and owners but it must also set out a plan of borrowing and lending; it must specify all T-1 elements of the vector $\{L_t\}$. The second constraint requires that the firm be solvent with probability one at the end of the planning horizon. During the course of the first T-1 periods for which investments are being planned the firm may be in debt as often and to any extent it thinks is necessary to effect the "best" investment plan. But at the start of period T , at which point it is also assumed the income-earning assets possessed by the firm at the start of the planning horizon disintegrate or evaporate, the firm must be solvent with probability one.

Within the setting of the capital market just described, the firm solves its capital-budgeting problem. Taking account of its anticipated cash throw-offs and the capital-market options available to it, the firm chooses the optimal subset of investments to be made from among the opportunities available over the planning horizon. It remains for us to describe the investment possibilities available to the firm and the way in which the firm measures whether one subset of projects is better than another. It will be assumed that projects are perfectly divisible so that the decisionmaker must decide not only which projects to accept and which to reject but also on what scale to undertake each accepted proposal.

Each project will be assumed to require a gross cash outlay of funds in at least one period, and some projects may require funds in several periods. Let s denote the last period in which a particular project requires a cash outlay by the firm. The scale of the project will then be defined so that 1 unit of the project involves a cash outlay which is the discounted-present-value equivalent of one dollar spent in period s. That is, one unit of a project whose last gross cash requirement occurs in period s will be that amount of the project that could be purchased by a stream of cash outlays equal to $(1+r)^{-s}$ dollars. It will also be assumed that a project does not begin providing gross cash returns to the firm until, at the earliest, the period in which its last gross cash outlay is made. Finally, it will be assumed that when the gross return from a project is made available for withdrawal from the firm, for example, in the form of a payment to stockholders, the project is terminated. That is, each project's gross return can be made available for withdrawal from the firm in one and only one period. Hence a project can be characterized by the two important periods in its life--the last period in which it requires an outlay of corporate funds, s , and the period in which its gross returns are made available for withdrawal from the firm, t .

Since there may be more than one project whose last outpayment occurs in period s and whose gross return is withdrawn in period t , one more index will be needed to fully characterize a project. Let I_{st} be the set of projects whose last cash outlays occur in period s and whose gross returns are withdrawn in period t . The projects in this set will be indexed by the triple ist . Denote by A_{ist} the scale on which the i^{th} project in the set I_{st} is accepted so that $A_{ist} \geqq 0$, and denote by the random variable g_{ist} the net return factor of that project measured <u>in terms of period t dollars</u>. Hence, when project ist is terminated in period t the amount available for withdrawal is given by the random variable $(1 + g_{ist})A_{ist}$.

The only restrictions placed on the return factors are: (1) all returns are finite so that $g_{ist} < \infty$ for all i , s , and t , and (2) the worst the firm can do on any project is lose its investment and the riskless return it could have earned on the amount invested in the asset so that $g_{ist} \geqq -1$ for all i, s, and t . Except for these two restrictions the vector of random variables $\{g_{ist}\}$ may possess any arbitrary probability distribution.

While this description of the projects available is obviously not perfectly general, it is, in fact, flexible enough to admit a wide range of possibilities. It may be worthwhile to indicate briefly, and by example, how wide this range is so that the model is not thought to be overly restrictive. The key feature of the description is that no specification is made about what happens between the last period in which an outlay is made on the project and the period in which its return is used for financing withdrawals from the firm. Any arbitrary set of events may occur between those two points in time. In particular, a "project" is not necessarily the same thing as, or coterminous with, a single physical investment. Instead, a project simply represents a definite sequence of events with the last planned outlay needed to support the sequence occurring in period s and with the set of events ending with the availability of funds for withdrawal from the firm in period t. The probability distribution of $1 + g_{ist}$ is thus the unconditional distribution of the funds that project ist representing this sequence of events makes available for disbursal in period t.

For example, a project might begin with the purchase of a new machine. The returns from the initial physical investment could be withdrawn immediately or they could be reinvested in another risky asset (be it a physical investment of the same firm or a risky security of another firm) or they could be reinvested in the riskless asset until the decisionmaker wants to use the returns to finance cash withdrawals from the firm. In combination with the original purchase of the machine, each of these alternative uses of the machine's returns would constitute a different investment project in the terminology being used here. The returns from a particular physical investment can be a stream through time. All that is required here is that the return from a project be concentrated at one point in time.

It should be noted, in particular, that the definition of a project used here does not require that the entire stream of returns accruing from a particular physical asset be withdrawn in the same period even if it is undesirable to incur any further risk with those funds. Suppose, for example, that the purchase of a machine i in period s leads to returns in only period t_0. The firm need not withdraw all the funds in t_0 but it can, instead, define a set of projects ist_0, ist_1, ist_2, and so on, with project ist_j representing purchase of the machine in period s and reinvestment of the returns accruing in period t_0 in the riskless asset for j periods. The sum of the A_{ist_j} over j would then indicate how much of a machine was purchased while the value of each A_{ist_j} would indicate how much of the machine's return was being used for withdrawal purposes in period t_j.

It is hoped that this brief discussion has served to indicate the asserted flexibility of the model. Of course, the present definition of a project implies that the firm will face an enormous number of projects from which to choose. But this paper is concerned with some characteristics of the total capital budget chosen, not with the problem of choosing particular assets. The great difficulty any firm would face in actually computing the solution to the capital-budgeting problem posed here is not of concern to us in the present paper.

In addition to using the cash throw-off it has available in a given period for outlays on projects and for loans, the firm can also distribute part of these internally generated funds to its owners. It can also borrow in a given period in order to increase the amount available for nonstochastic disbursement within that period. At the beginning of each period, then, the firm will set aside a non-negative sum of money, W_t dollars, for removal or withdrawal from the firm.

The question that remains to be answered is: Given the set of opportunities described, how does the firm decide which subset of projects to accept, how much to borrow or lend in each of the T periods for which it is planning, and how much to withdraw from the firm nonstochastically in each of the T periods? Stated in other terms, what is the objective function of the capital-budgeting problem? I would argue--and I have argued[23]--that the appropriate objective function for a corporation budgeting capital in the presence of risk is the maximization of expected utility. The utility function to be used is management's perception of the owners' utility defined over the owners' consumption alternatives during the budget's T-period horizon. The arguments of the utility function are taken to be the amounts available for withdrawal from the firm in the several periods, with these amounts serving as proxy variables for the true but non-measurable increases in stockholders' consumption possibilities that the firm's plan makes possible.

While I would argue strongly for this position, it is not possible to present the argument in full in this paper, and there remains much room for debate on this question. Since some readers may disagree with this position, let us simplify the discussion by supposing that the firm is owned by a single individual whose entire livelihood is derived from this firm. This simplifying assumption does not reduce the importance of the results obtained here since the problem under consideration remains a bona fide multiperiod planning decision even if there exists only one owner. The objective function of the capital-budgeting problem described is thus to maximize the expected utility of the consumption stream the firm provides to its single owner over the course of the planning horizon, that is,

(4.1) Maximize $E\{U(C_1, C_2, \ldots, C_T)\}$,

where C_t is the consumption income the firm makes available in period t.

This consumption income C_t is equal to the sum of the nonstochastic withdrawal the decisionmaker plans to make in period t, W_t, and the returns his risky assets provide for consumption purposes in period t. In terms of timing, it is assumed that all cash outlays in a period--for the firm's ongoing operations or its investment program or the nonstochastic withdrawal--are made at the start of the period while all cash inflows are assumed to occur at the end of the period. These inflows include the cash throw-offs generated by the assets owned by the firm at time $t = 0$ and the returns from the risky assets.

Hence, the owner's t^{th} period consumption income is

$$(4.2) \qquad C_t = W_t + \sum_{i=1}^{n_{st}} \sum_{s=1}^{t-1} (1 + g_{ist})A_{ist} ,$$

where it is assumed that the number of risky projects whose last cash outlay occurs in period s and whose gross return is used for withdrawals in period t is n_{st} .

Given this "story" about the timing of returns, projects undertaken in the T^{th} period are beyond the preview of the current planning-period decision. That is not to say that the returns on investments made during the planning period which accrue in periods $T+1$, $T+2$, and onward are ignored. On the contrary, the value of these returns, discounted to the horizon T at the market rate of interest, is included in the return factor of the project which corresponds to this physical asset and terminates in period T . Hence, such post-horizon returns are fully taken into account in the T^{th} consumption-income argument of the utility function. All that we exclude from consideration are projects "originating"--in the sense of having their last cash outlay--in the T^{th} period and beyond.

It is assumed that the decisionmaker is a strict risk averter and that he is, in fact, decreasingly risk averse in the multiperiod sense. The assumption of strict risk aversion means that $U(C_1, C_2, \ldots, C_T)$ is a strictly concave function and, hence, since nonnegative linear combinations of strictly concave functions are strictly concave

$E\{U(C_1, C_2, \ldots, C_T)\}$ is strictly concave. The capital-budgeting problem being discussed can thus be represented by the following non-linear programming model with strictly concave objective function and linear constraints.

$$(4.3)\begin{cases} \text{Maximize} \quad E\{U(C_1, C_2, \ldots, C_T)\} \\ \text{Subject to:} \quad W_1 + L_1 + \sum_{i=1}^{n_{1t}} \sum_{t=2}^{T} A_{i1t} = X_0 \\ W_t + L_t - (1+r)L_{t-1} + \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} A_{ist} = X_{t-1} \\ \qquad t = 2, \ldots, T-1 \\ W_T - (1+r)L_{T-1} = X_{T-1} \\ C_t = W_t + \sum_{i=1}^{n_{st}} \sum_{s=1}^{t-1} (1 + g_{ist})A_{ist} \quad t = 1, \ldots, T \\ W_t \geqq 0 \text{ all } t \ ; \quad A_{ist} \geqq 0 \text{ all } i, s, t . \end{cases}$$

Several comments are in order. First, the nonnegativity requirement on W_T represents the solvency constraint in this model. This follows because $W_T \geqq 0$ means that $X_{T-1} + (1+r)L_{T-1} \geqq 0$ so that at least in the final period the decisionmaker is able to cover his previous borrowings using the funds he is certain to have available. Second, the budget constraints are written as equations rather than inequalities because with riskless lending and withdrawal for consumption purposes always available options, and with each period's consumption income having positive marginal utility, the decisionmaker will always use all funds at his disposal in a particular period.

This last fact enables us to simplify the constraint set enormously. (Indeed, since the decisionmaker faces a perfect capital market, one would expect that only the discounted present values of the relevant cash flows should matter.) The fact that all the budget constraints hold as equations enables us to reduce the entire set of T budget constraints to one discounted-present-value financial constraint. Beginning with the first budget constraint, use the t^{th} budget constraint as the definition of L_t and substitute this definition into the $t+1^{st}$ constraint. This process of forward substitution reduces the T budget constraints to

$$(4.4')\quad \sum_{s=1}^{T}(1+r)^{T-s}X_{s-1} - \sum_{s=1}^{T-1}(1+r)^{T-s}\left[\sum_{i=1}^{n_{st}}\sum_{t=s+1}^{T} A_{ist} + W_s\right] = W_T \geqq 0$$

Constraint (4.4') states the financial restriction on the firm in terms of the horizon value (the value at time T) of the set of planned expenditures, loans, and withdrawals. Multiplying (4.4') by $(1+r)^{-T}$, the constraint is stated in terms of discounted present values as:

$$(4.4)\quad \sum_{s=1}^{T}(1+r)^{-s}X_{s-1} - \sum_{s=1}^{T}(1+r)^{-s}W_s - \sum_{s=1}^{T-1}\sum_{i=1}^{n_{st}}\sum_{t=s+1}^{T}(1+r)^{-s}A_{ist} = 0 ,$$

$$W_T \geqq 0 .$$

Before restating the capital-budgeting problem in its final form, let us introduce the last assumptions to be made about the decisionmaker. First, it will be assumed that his utility function is additive in the several periods' utilities, so that

$$(4.5) \qquad U(C_1, C_2, \ldots, C_T) = \sum_{t=1}^{T} u_t(C_t) ,$$

where, again, the decisionmaker's rate of time preference is reflected in the relative magnitudes of the parameters of the different $u_t(C_t)$ functions. This assumption enables us to draw upon the result in Theorem 3 concerning the necessary and sufficient condition for multi-period decreasing risk aversion with additive utility functions. Second, it will be assumed that the decisionmaker sets aside part of the available cash throw-off for nonstochastic withdrawal in each and every period. That is, the solution to (4.3) is assumed to be interior with respect to $W_1, W_2, \ldots, W_T$ so that $W_t > 0$ for all $t = 1, \ldots, T$. This requires that the individual have some positive consumption with probability one in each period, no matter how his risky investments fare.

The capital-budgeting problem faced by the firm of the model presented can thus be written as (4.6) .

$$(4.6)\quad \begin{cases} \text{Maximize} & E\{U(C_1, C_2, \ldots, C_T)\} \\ \text{Subject to:} & \sum_{s=1}^{T} (1+r)^{-s}[X_{s-1} - W_s] - \sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist} = 0 \\ & C_t = W_t + \sum_{i=1}^{n_{st}} \sum_{s=1}^{t-1} (1 + g_{ist}) A_{ist} \quad t = 1, 2, \ldots, T \\ & W_t \geqq 0 \text{ all } t \;;\; A_{ist} \geqq 0 \text{ all } i, s, t . \end{cases}$$

The Lagrangian form for problem (4.6) is then

$$(4.7)\quad \begin{cases} \text{Max}\, \mathcal{L} = E\{U(W_1, W_2 + \sum_{i=1}^{n_{12}} [1 + g_{i12}] A_{i12}, W_3 + \sum_{i=1}^{n_{s3}} \sum_{s=1}^{2} [1+g_{is3}] A_{is3}, \\ \qquad \ldots, W_T + \sum_{i=1}^{n_{sT}} \sum_{s=1}^{T-1} [1 + g_{ist}] A_{ist})\} \\ \qquad + \lambda \left[\sum_{s=1}^{T} (1+r)^{-s} (X_{s-1} - W_s) - \sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist} \right]. \\ W_t \geqq 0 \text{ all } t \;;\; A_{ist} \geqq 0 \text{ all } i, s, t . \end{cases}$$

As noted earlier, since the decisionmaker is a strict risk averter, $E\{U(C_1, C_2, \ldots, C_T)\}$ is a strictly concave function of the C_t variables. Since each C_t is, in turn, a linear function of the W_t and A_{ist} variables, it follows that $E\{U(C_1, C_2, \ldots, C_T)\}$ is also concave in the W_t and A_{ist} variables. If, moreover, no project used for withdrawals in period t is perfectly correlated with the other projects used for withdrawals from the firm in period t, and if this is true for all $t = 1,\ldots, T$, then $E\{U(C_1, C_2,\ldots, C_T)\}$ is strictly concave in the W_t and A_{ist} variables.[24] <u>From this point on, we shall assume that this condition of imperfect correlation is met.</u> It follows, then, that the problem whose Lagrangian appears in (4.7) calls for the maximization of a strictly concave function of the W_t and A_{ist} variables subject to a linear constraint and nonnegativity constraints on these variables.

From the standard theory of nonlinear programming[25] it follows that the Kuhn-Tucker conditions are necessary and sufficient for a maximum of (4.6), the problem whose Lagrangian form is given in (4.7). For the capital-budgeting problem in (4.6) these conditions are (4.8) - (4.13), where we make use of the fact that $W_t > 0$ for all t, by assumption.

(4.8) $$\frac{\partial \mathcal{L}}{\partial W_t} = E\{U_t\} - \lambda(1+r)^{-t} = 0 \quad \text{for} \quad t = 1, 2, \ldots, T .$$

(4.9) $$\frac{\partial \mathcal{L}}{\partial A_{ist}} = E\{U_t(1 + g_{ist})\} - \lambda(1+r)^{-s} \leqq 0 \text{ for all } i, s, t$$

such that $i \in I_{st}$, $s < t$.

(4.10) $$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{s=1}^{T} (1+r)^{-s}[X_{s-1} - W_s] - \sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist} = 0 .$$

(4.11) $$(A_{ist})\left(\frac{\partial \mathcal{L}}{\partial A_{ist}}\right) = 0 , \quad A_{ist} \geqq 0 \text{ all } i, s, t \text{ such that}$$

$i \in I_{st}$, $s < t$.

(4.12) $W_t > 0$ for $t = 1, \ldots, T$.

(4.13) $\lambda \geqq 0$.

The optimal solution value of λ , denoted λ^o , is the marginal expected utility of the discounted present value of the firm's nonstochastic cash throw-offs at the global optimum; that is,

$$(4.14) \qquad \lambda^o = \frac{\partial[E\{U\}]^o}{\partial(\sum_{s=1}^{T} (1+r)^{-s} X_{s-1})} ,$$

where superscript ^{o} indicates optimal values.

Our interest in this model concerns the changes that occur in the decisionmaker's purchase of risky assets as his wealth increases. Given the perfect capital market the decisionmaker of the model faces, this question becomes: What happens to the discounted present value of the decisionmaker's purchases of risky projects,

$\sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist}$, as the discounted present value of his endowments, $\sum_{s=1}^{T} (1+r)^{-s} X_{s-1}$, increases? To investigate this question totally differentiate the system of first-order conditions with respect to a change in the stream of nonstochastic cash throw-offs which increases the discounted present value of those throw-offs; that is, the X_s's change but r and the g_{ist}'s do not. Recall that since

$U(C_1, C_2, \ldots, C_T) = \sum_{t=1}^{T} u_t(C_t)$, $U_{st} = 0$ for all $s \neq t$. One obtains

$$(4.15) \quad E\{U_{tt}[dW_t + \sum_{i=1}^{n_{st}} \sum_{s=1}^{t-1} (1 + g_{ist})dA_{ist}]\} - (1+r)^{-t}d\lambda = 0$$

$$t = 1, 2, \ldots, T .$$

$$(4.16) \quad E\{U_{tt}(1 + g_{ist})[dW_t + \sum_{i=1}^{n_{st}} \sum_{s=1}^{t-1} (1 + g_{ist})dA_{ist}]\} - (1+r)^{-s}d\lambda$$

$$= d\left(\frac{\partial \mathcal{L}}{\partial A_{ist}}\right)$$

for all i , s , t such that $i \in I_{st}$, $s < t$.

$$(4.17) \quad \sum_{s=1}^{T} (1+r)^{-s}dW_s + \sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s}dA_{ist} = \sum_{s=1}^{T} (1+r)^{-s}dX_{s-1} .$$

$$(4.18) \quad \frac{\partial \mathcal{L}}{\partial A_{ist}}dA_{ist} + A_{ist}d\left(\frac{\partial \mathcal{L}}{\partial A_{ist}}\right) = 0 \quad \text{all } i , s , t \text{ such that}$$

$$i \in I_{st} , \quad s < t .$$

From (4.15) - (4.18), it is clear that some prior information about the sign of $d\lambda$ would be very helpful in trying to determine qualitative changes in the holding of risky assets. The inequality in (4.13) provides us with no such information since all it states is that λ must stay nonnegative: $d\lambda$ may be positive, negative, or zero if $\lambda^o > 0$. But the interpretation of λ^o in (4.14) in conjunction with the following well-known result does provide some definite information about $d\lambda$.[26]

<u>LEMMA</u>. Given the maximization problem

Maximize $f(y)$ Subject to: $h(y) \leqq b$, $y \geqq 0$,

the function $\varphi(b) = \max_{\substack{h(y) \leqq b \\ y \geqq 0}} f(y)$

is a (strictly) concave function of b if $f(y)$ is a (strictly) concave function of y and $h(y)$ is a convex function of y.

The budget constraint in (4.6) can be rewritten as

$$(4.6a) \quad \sum_{s=1}^{T} (1+r)^{-s} W_s + \sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist} = \sum_{s=1}^{T} (1+r)^{-s} X_{s-1} ,$$

which is convex in the variables $\{W_s\}$, $\{A_{ist}\}$ since it is linear in them, while the objective function in (4.6) is strictly concave. For a given r and a given vector of returns on risky assets $\{g_{ist}\}$, the maximum value of expected utility, $[E\{U\}]^o$, is a function of $\sum_{s=1}^{T} (1+r)^{-s} X_{s-1}$ alone. Hence, the problem in (4.6) satisfies the conditions of the Lemma with $b = \sum_{s=1}^{T} (1+r)^{-s} X_{s-1}$, and it therefore follows from the conclusion of the Lemma that for $\{g_{ist}\}$ and r fixed, $[E\{U\}]^o$ is a strictly concave function of $\sum_{s=1}^{T} (1+r)^{-s} X_{s-1}$:

$$(4.19) \qquad \frac{\partial^2 [E\{U\}]^o}{\partial \left(\sum_{s=1}^{T} (1+r)^{-s} X_{s-1} \right)^2} < 0 .$$

Combining (4.14) with (4.19), one therefore obtains:

$$(4.20) \qquad \frac{\partial\lambda^{o}}{\partial\left(\sum_{s=1}^{T}(1+r)^{-s}X_{s-1}\right)} < 0$$

so that the sign of $d\lambda$ in (4.15) - (4.18) is negative. This is exactly what one would expect: as the value of the decisionmaker's endowments increases, the marginal expected utility of the endowments at the global optimum decreases.

B. <u>A Counterexample to the Normal-Good Character of Risky Assets</u>

A counterexample will now demonstrate that in the context of the capital-budgeting problem under discussion, an increase in the discounted present value of the firm's period-by-period endowments--its cash throw-offs--does not necessarily lead a decreasingly risk-averse decisionmaker to increase the discounted present value of his risky investments. That is, Arrow's single-period result cannot be generalized to the multiperiod case; decreasing absolute risk aversion does not imply that risky assets are always a normal good.

To see this, consider a firm with a three-period planning horizon. Suppose the firm has only one project available in each of the first two periods of its horizon and that each of these projects terminates in the third period. Let g_1 denote the net return factor for the project coming available in period 1, g_2 denote the net return factor for the project coming available in period 2, and let A_1 and A_2 denote the scale on which each project is undertaken, respectively. The Lagrangian in (4.7) thus simplifies to

$$\text{Max } \mathcal{L} = E\{U(W_1, W_2, W_3 + (1+g_1)A_1 + (1+g_2)A_2)\}$$

$$(4.21) \qquad + \lambda \left[\sum_{s=1}^{3} (1+r)^{-s}(X_{s-1} - W_s) - (1+r)^{-1}A_1 - (1+r)^{-2}A_2 \right] ,$$

and the conditions for an optimum become:

$$(4.22) \qquad E\{U_t\} - \lambda(1+r)^{-t} = 0 \qquad t = 1, 2, 3 .$$

$$(4.23) \qquad E\{U_3(1+g_1)\} - \lambda(1+r)^{-1} \leqq 0 .$$

$$(4.24) \qquad E\{U_3(1+g_2)\} - \lambda(1+r)^{-2} \leqq 0 .$$

$$(4.25) \qquad \sum_{s=1}^{3} (1+r)^{-s}(X_{s-1} - W_s) - (1+r)^{-1}A_1 - (1+r)^{-2}A_2 = 0 .$$

$$(4.26) \qquad A_1 \cdot \frac{\partial \mathcal{L}}{\partial A_1} = 0 ; \quad A_2 \cdot \frac{\partial \mathcal{L}}{\partial A_2} = 0 .$$

$$(4.27) \qquad W_t > 0 \text{ for } t = 1, 2, 3 .$$

$$(4.28) \qquad \lambda \geqq 0 .$$

Since the utility function is additive in the several periods' individual utilities, it is clear from (4.15) - (4.18) that the changes in the amounts of risky assets purchased can be determined <u>as a function of dλ alone</u> from the total differentials of the equilibrium conditions for the third period. The reason is really quite clear: since the utility function is additive, $U(C_1, C_2, C_3) = \sum_{t=1}^{3} u_t(C_t)$, A_1 and A_2 just do not enter the equilibrium conditions for W_1 and W_2 . Assume (and this assumption will presently be justified) that at its initial optimum the firm undertakes both projects, so that (4.23) and (4.24) hold as equations. Totally differentiating (4.22) with t = 3 , and (4.23) and (4.24), one obtains the following system describing the changes in the acceptance of risky projects and the nonstochastic withdrawal in period 3 for small variations $\sum_{s=1}^{3} (1+r)^{-s} X_{s-1}$.

(4.29)

$$E\{U_{33}\}dW_3 + E\{U_{33}(1+g_1)\}dA_1 + E\{U_{33}(1+g_2)\}dA_2 = (1+r)^{-3}d\lambda$$

(4.30)

$$E\{U_{33}(1+g_1)\}dW_3 + E\{U_{33}(1+g_1)^2\}dA_1 + E\{U_{33}(1+g_1)(1+g_2)\}dA_2 = (1+r)^{-1}d\lambda$$

(4.31)

$$E\{U_{33}(1+g_2)\}dW_3 + E\{U_{33}(1+g_1)(1+g_2)\}dA_1 + E\{U_{33}(1+g_2)^2\}dA_2 = (1+r)^{-2}d\lambda .$$

The relevant information about the returns from the two risky ventures, the interest rate, and the decisionmaker's utility function is presented in Table 1. The market rate of interest is .098114 in each period. There are three states of nature, θ_1 , θ_2 , and θ_3 , which are equally probable as indicated by the fact that the probability of a particular state θ_i occuring, namely $\pi(\theta_i)$, is 1/3 for each of the three states. The first risky project yields a third-period net return per first-period dollar invested of -1 in state θ_1 , and + 1/2 in each of states θ_2 and θ_3 , while the second risky project yields third-period net returns per second-period dollar invested of +2 in the first state of nature, +1 in state θ_2, and -1 in state θ_3.

It is assumed, as indicated in Table 1, that the individual's asset holdings--that is, the quantities A_1 and A_2--are such that $C(\theta_1) > C(\theta_2) > C(\theta_3)$. It is then assumed that the rate of change of marginal utility in the state with the lowest consumption income, state θ_3, is -30 ; the rate of change of marginal utility in θ_2 is assumed to be -6 ; and the rate of change of marginal utility in state θ_1, the best state of nature from the point of view of total consumption income, is -2. The marginal utility of consumption income in the several states of nature is taken to be 20.0045 in state θ_1, 26.0045 in state θ_2, and 56.0045 in state θ_3. The ratio $-\frac{U_{33}}{U_3}$ shows the value of Pratt's local risk aversion function in each state of nature.

Essentially all we have done is choose a structure of returns for two assets, a rate of interest, and three points on a utility function. Two important facts must be verified. First, it must be shown that one can fit a decreasingly risk-averse utility function to the three given points with the ordering of states of nature in terms of consumption income being $C(\theta_1) > C(\theta_2) > C(\theta_3)$. Second, it must be shown that a decisionmaker whose utility function fits

TABLE 1

THE STRUCTURE OF RETURNS AND THE DECISIONMAKER'S UTILITY FUNCTION

	θ_1	θ_2	θ_3
$1 + g_1$	0	$\frac{3}{2}$	$\frac{3}{2}$
$1 + g_2$	3	2	0
$1 + r$	1.098114	1.098114	1.098114
U_{33}	-2	-6	-30
U_3	20.0045	26.0045	56.0045
$-\frac{U_{33}}{U_3}$	.09998	.23073	.53567
$\pi(\theta_i)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$C_3(\theta_i)$	$W_3 + 3A_2$	$W_3 + \frac{3}{2}A_1 + 2A_2$	$W_3 + \frac{3}{2}A_1$

A_1 and A_2 are such that $C_3(\theta_1) > C_3(\theta_2) > C_3(\theta_3)$

these three points would find himself purchasing both available risky projects at his equilibrium position, that is, that (4.23) and (4.24) obtain as equations. Consider these two points in order.

As a result of Theorem 3, in verifying that the decisionmaker in question is decreasingly risk-averse in the multiperiod sense, all one needs to do is show that each of his single-period utility functions satisfies the Pratt condition (3.9) which is equivalent to

$$(3.9') \quad -\frac{u_t''(C_t)}{u_t'(C_t)} = -\frac{d}{dC_t}\,[\ln u_t'(C_t)] \quad \text{is a decreasing function of } C_t \,.$$

Since one can essentially choose any single-period utility functions for periods 1 and 2, with the consumption incomes in those periods depending only on the respective nonstochastic withdrawals, there is no difficulty in choosing arbitrary decreasingly risk-averse utility functions for those periods. What must be verified is that the three points shown in Table 1 could be points on a decreasingly risk-averse single-period utility function.

It is clear that the utility function whose points appear in Table 1 belongs to a risk averter since $U_{33}(\theta_i)$ is negative for all i. To see what requirements must be met for a decreasingly risk-averse function consider Figure 1 where the consumption income in period 3 appears on the horizontal axis and the negative of the natural logarithm of the marginal utility of consumption in period 3 appears on the vertical axis. Table 1 specifies three triples (C_3, U_3, U_{33}). In

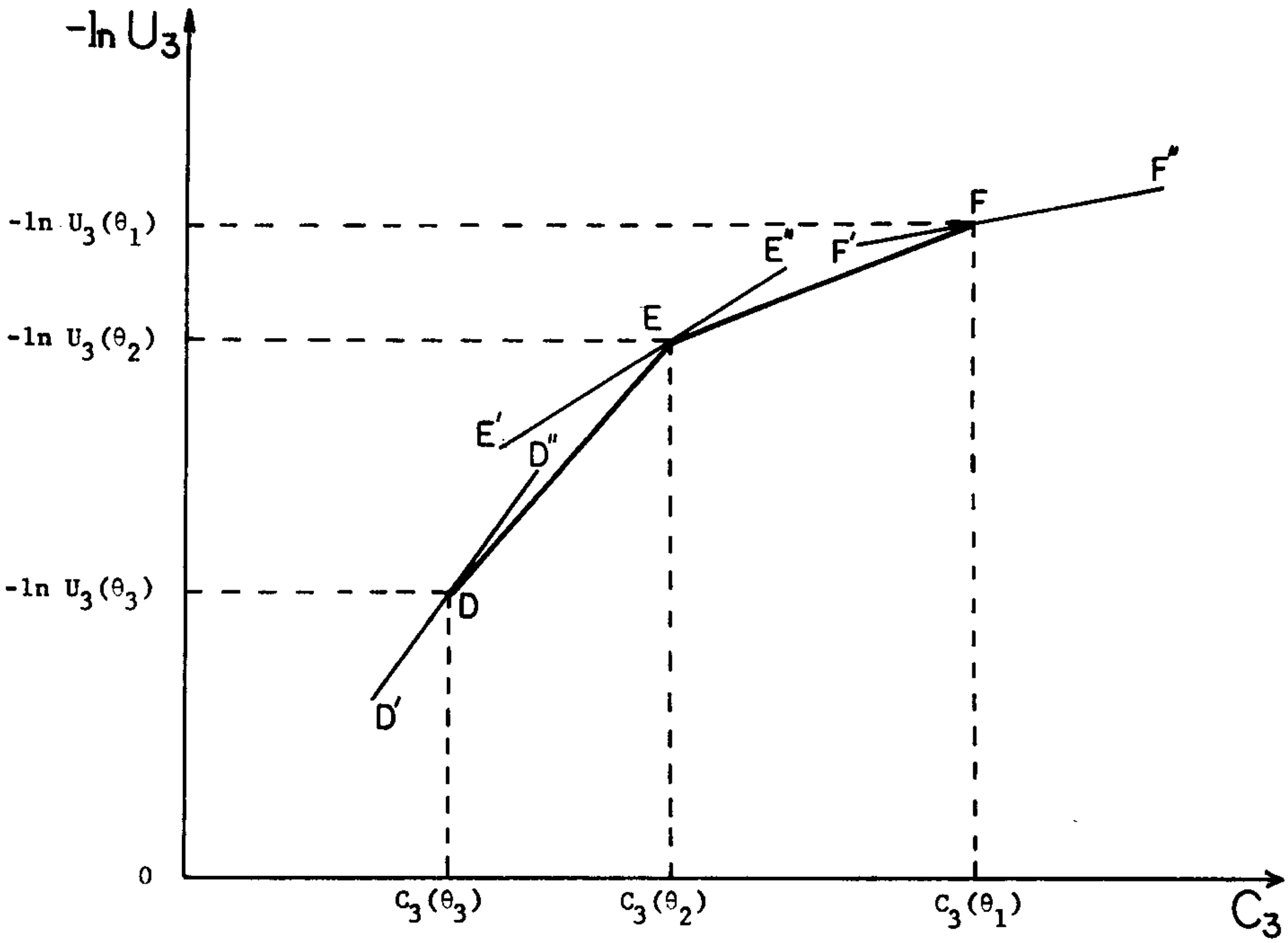

FIGURE 2

REQUIREMENTS FOR FITTING A DECREASINGLY RISK-AVERSE UTILITY FUNCTION TO THE THREE GIVEN (C_3, U_3, U_{33}) POINTS other words, we have specified, in terms of W_3 , A_1 , and A_2 , the abscissa $C_3(\theta_3)$, the ordinate $-\ln U_3(\theta_3)$ or the height of point D, and the value of the absolute risk aversion function = $\left(-\frac{U_{33}}{U_3}\right)_{\theta_3}$ $=\left(-\frac{d\ln U_3}{dC_3}\right)_{\theta_3}$ or the slope of D'D" ; and similarly the abscissa $C_3(\theta_2)$, the height of point E , and the slope of E'E" ; and finally the abscissa $C_3(\theta_1)$, the height of point F , and the slope of F'F" . In order to be able to fit a smooth curve to the three given points D, E, and F with the tangents at these points having the indicated slopes D'D" , E'E" , and F'F" , respectively, it must be possible to find W_3 , A_1 , and A_2 values such that the implied values of $C_3(\theta_1)$, $C_3(\theta_2)$, and $C_3(\theta_3)$ satisfy the conditions:

slope D'D" > slope DE > slope E'E" > slope EF > slope F'F" .

In sum, in order to be able to fit a utility function with decreasing absolute risk aversion to the three points indicated in Table 1, it must be possible to satisfy the following set of inequalities:

$$\left(-\frac{U_{33}}{U_3}\right)_{\theta_3} > \frac{-\ln U_3(\theta_2) - [-\ln U_3(\theta_3)]}{C_3(\theta_2) - C_3(\theta_3)} > \left(-\frac{U_{33}}{U_3}\right)_{\theta_2}$$

$$(4.32) \qquad > \frac{-\ln U_3(\theta_1) - [-\ln U_3(\theta_2)]}{C_3(\theta_1) - C_3(\theta_2)} > \left(-\frac{U_{33}}{U_3}\right)_{\theta_1} .$$

Given the data of Table 1, this set of inequalities becomes:

$$(4.33) \qquad .53567 > \frac{.7673}{2A_2} > .23073 > \frac{.2624}{A_2 - \frac{3}{2}A_1} > .09998 \ .$$

The constraints implied on A_1 and A_2 are then

$$(4.34) \begin{cases} \text{(i)} \ 1.4325 < 2A_2 < 3.3255 \quad \text{or} \quad .71625 < A_2 < 1.66275 \\ \text{(ii)} \ 1.1373 < A_2 - \frac{3}{2}A_1 < 2.6246 \ . \end{cases}$$

Clearly, there are positive values of A_1 and A_2 satisfying these constraints. For example, choose any A_2 such that $1.1374 < A_2 < 1.66275$ and then set $A_1 > 0$ such that $A_2 - \frac{3}{2}A_1 > 1.1373$. We conclude, then, that the information Table 1 presents concerning the decisionmaker's utility function could be generated by a <u>bona fide</u> single-period decreasingly risk-averse utility function.

Turning to the second point requiring verification, it must be shown that in equilibrium the firm will be undertaking both risky projects. Examining the three relevant equilibrium conditions, it is clear that conditions (4.23) and (4.24) will hold as equations if the following two conditions are met:

$$(4.35) \begin{cases} \text{(i)} \quad E\{U_3(g_2 - r)\} = 0 \\ \text{(ii)} \quad \dfrac{E\{U_3(1 + g_1)\}}{E\{U_3(1 + g_2)\}} = 1+r \ . \end{cases}$$

If (4.35)(i) holds, then $E\{U_3(1 + g_2)\} = E\{U_3(1 + r)\}$ or $(1 + r)E\{U_3\}$

$= E\{U_3(1 + g_2)\}$. Since $W_3 > 0$ implies $E\{U_3\} = \lambda(1 + r)^{-3}$, this means we also have $E\{U_3(1 + g_2)\} = \lambda(1 + r)^{-2}$ which is (4.24) as an equation. And, if (4.35)(ii) obtains, then $E\{U_3(1 + g_1)\} = (1 + r)E\{U_3(1 + g_2)\} = \lambda(1 + r)^{-1}$ which is (4.23) as an equation. Given the data of Table 1, one finds that $E\{U_3 g_2\} = 10.0090$ and $E\{U_3 r\} = 10.0090$ so that (i) holds, while $E\{U_3(1 + g_1)\} = 123.0135$ and $E\{U_3(1 + g_2)\} = 112.0225$ so that $\frac{E\{U_3(1 + g_1)\}}{E\{U_3(1 + g_2)\}} = 1.098114$ and (ii) also obtains. Hence, given the utility-function properties and the investment opportunities in Table 1, the decisionmaker would be in equilibrium pursuing both risky projects.

With this assurance that the individual whose utility function possesses the points shown in Table 1 is decreasingly risk averse and that equilibrium conditions (4.23) and (4.24) would hold as equations for him, the data of Table 1 can be used in the system of total differentials (4.29) - (4.31) to determine the changes in asset purchases resulting from small variations in $\sum_{s=1}^{3} (1+r)^{-s} X_{s-1}$. Letting $d\lambda' = 3(1+r)^{-1} d\lambda$, system (4.29) - (4.31) becomes

(4.36) $\quad 38dW_3 + 54dA_1 + 18dA_2 = -.82928d\lambda'$

$$(4.37) \qquad 54dW_3 + 81dA_1 + 18dA_2 = -d\lambda'$$

$$(4.38) \qquad 18dW_3 + 18dA_1 + 42dA_2 = -.91065d\lambda'$$

for the decisionmaker whose situation is presented in Table 1 . The solution to this system of equations is:

$$(4.39) \qquad \begin{cases} dW_3 \approx -.051219\ d\lambda' > 0 \\ dA_1 \approx \ .024029\ d\lambda' < 0 \\ dA_2 \approx -.010029\ d\lambda' > 0 , \end{cases}$$

where the signs of the changes are based on the result derived earlier that $d\lambda$, and hence $d\lambda'$, is negative. From (4.39) it follows that the change in the discounted present value of risky assets purchased is:

$$.024029\ d\lambda' + (1+r)^{-1}(-.010029)\ d\lambda' \quad \text{or}$$

$$[(.024029) - (.91065)(.010029)]\ d\lambda' = .014896\ d\lambda' < 0 .$$

Hence, the discounted present value of the risky assets purchased by the firm pictured in Table 1 decreases as the discounted present value of its stream of cash throw-offs increases, ceteris paribus. In fact, since it is A_1 that decreases and A_2 that increases and $|dA_1| > dA_2$, even the undiscounted present value of the portfolio of risky assets decreases as, ceteris paribus, the discounted present value of the endowments stream increases!

C. A Set of Sufficient Conditions for Risky Investment to Be A Normal Good in the Multiperiod Case

This counterexample leads one to ask the question: Under what conditions will the purchase of risky assets increase as the endowments stream increases in the multiperiod case? What further restrictions on the utility function, beyond multiperiod decreasing absolute risk aversion, or on the risky projects available are necessary and/or sufficient to ensure that $\sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist}$ increases as $\sum_{s=1}^{T} (1+r)^{-s} X_{s-1}$ increases? In this section a particular set of sufficient conditions is presented. Denoting by I_t the set of all risky projects whose returns are made available for withdrawal in period t, the main result of this section is summarized in the following theorem.

<u>THEOREM 4</u>. If (a) $U(C)$ is an additive multiperiod utility function which is decreasingly risk averse in the multiperiod sense, (b) the stochastic part of each period's consumption income is derived from one <u>and</u> only one risky asset so that $A_{ist} > 0$ for only one element of I_t, and (c) the solution to (4.6) is nondegenerate, then as the discounted present value of the firm's cash throw-offs increases, the discounted present value of the investment in risky projects increases; in fact, the scale on which each accepted risky project is pursued increases.

The theorem states that if $U(C) = \sum_{t=1}^{T} u_t(C_t)$ and each $u_t(C_t)$ satisfies the Pratt condition (3.9) or (3.9'); if $A_{ist} > 0$ for one and only one member of the set I_t ; and if not both $A_{ist} = 0$

and $\frac{\partial \mathcal{L}}{\partial A_{ist}} = 0$ obtain in (4.11) then $\frac{d\left(\sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist}\right)}{d\left(\sum_{s=1}^{T} (1+r)^{-s} X_{s-1}\right)} > 0$.

The firm may be basing each period's stochastic consumption on only one risky project's return either because each set I_t contains only one element or because the optimal solution to (4.7) finds only one $A_{ist} > 0$ for all $A_{ist} \in I_t$. Given the additive nature of the utility function and the fact that the risky part of each period's consumption income comes from only one project, the resulting situation is obviously close to the simple portfolio model discussed by Arrow. But the two models are not the same nor is the present model simply a sum of Arrow one-period models. First, the several periods' investments are not independent of one another. They are, instead, tied together by the discounted-present-value budget constraint in (4.4). There is no analogue to this in Arrow's model nor would there be such an interrelationship if the present model were just a sum of independently made Arrow single-period decisions. Second, the firm may purchase more than one risky project in a particular period. Of course, given the fact that there are only as many projects accepted as there are periods, if more than one project were accepted in a particular period, there would have to be at least one period in which no projects were accepted. Nevertheless, the option of pursuing more than one risky project in a particular period, while present in the model under discussion, is not open to the firm in Arrow's model or in a series of Arrow models extending over the set of T periods.

We now turn to the proof of Theorem 4 .

Proof: Denote by $s(t)$ the period in which the project providing the stochastic part of period t's consumption is purchased. For notational convenience, assume that the accepted member of the set $I_{s(t)t}$ -- that is, all projects available for purchase in period $s(t)$ whose returns accrue in period t -- has the index $i=1$. Notationally, then $A_{1s(t)t} > 0$ for $t = 2, \ldots, T$ and $A_{ist} = 0$ for all other i , s , t triples. (Recall that in the model being discussed no risky project's returns become available in period 1: I_1 is the empty set.) Equilibrium condition (4.11) states that

$$(4.11) \quad A_{ist} \cdot \frac{\partial \mathscr{L}}{\partial A_{ist}} = 0 \, , \; A_{ist} \geqq 0 \quad \text{for all } i, s, t \text{ such that}$$

$$i \in I_{st} \, , \quad s < t \, ,$$

while equilibrium condition (4.9) states that

$$(4.9) \quad \frac{\partial \mathscr{L}}{\partial A_{ist}} \leqq 0 \quad \text{for all } i, s, t \text{ such that } i \in I_{st} \, , \quad s < t \, .$$

From (4.11) it follows that $\frac{\partial \mathscr{L}}{\partial A_{1s(t)t}} = 0$, and if the optimal solution to problem (4.6) is not degenerate so that not both

$A_{ist} = 0$ <u>and</u> $\frac{\partial \mathcal{L}}{\partial A_{ist}} = 0$, it follows from (4.9) that $\frac{\partial \mathcal{L}}{\partial A_{ist}} < 0$ for all $[i,s,t] \neq [1,s(t),t]$ for $t = 2,\ldots, T$. From this point on we make use of the assumption in Theorem 4 that the solution to (4.6) is nondegenerate.[27] The total differential of (4.11), given in equation (4.18), then implies that

$$(4.40) \qquad d\left(\frac{\partial \mathcal{L}}{\partial A_{1s(t)t}}\right) = 0 \quad \text{for} \quad t = 2, \ldots, T \quad \text{and}$$

$$(4.41) \qquad dA_{ist} = 0 \quad \text{for all } [i, s, t] \neq [1, s(t), t] \quad \text{for} \quad t = 2,\ldots, T .$$

Using (4.40) and (4.41) in (4.15) and (4.16) for the triple $[1, s(t), t]$, one obtains (4.42) and (4.43), respectively

$$(4.42) \quad E\{U_{tt}[dW_t + (1 + g_{1s(t)t})dA_{1s(t)t}]\} = (1+r)^{-t}d\lambda \quad \text{for all } t ,$$

$$(4.43) \quad E\{U_{tt}(1 + g_{1s(t)t})[dW_t + (1 + g_{1s(t)t})dA_{1s(t)t}]\} = (1+r)^{-s(t)}d\lambda$$

$$\text{for all} \quad t \geqq 2 .$$

Multiplying (4.42) by $(1+r)^{t-s(t)}$ and subtracting from (4.43) one obtains

$$(4.44) \qquad E\{U_{tt}[(1 + g_{1s(t)t}) - (1+r)^{t-s(t)}][dW_t + (1 + g_{1s(t)t})dA_{1s(t)t}]\}=0$$

$$\text{for all} \quad t \geqq 2 .$$

Denote $(1+r)^{t-s(t)}$ by $1+r_{s(t)t}$ so that $r_{s(t)t}$ is the net compound riskless rate of return on one dollar loaned period by period from period s(t) to period t . Equation (4.44) can then be rewritten as

$$(4.44') \quad E\{U_{tt}(g_{1s(t)t} - r_{s(t)t})[dW_t + (1 + g_{1s(t)t})dA_{1s(t)t}]\} = 0$$

for all $t \geqq 2$.

Since there is only one return factor, $g_{1s(t)t}$, and one net compound riskless rate of return, $r_{s(t)t}$, involved in what follows, in order to simplify notation denote $g_{1s(t)t}$ by g^* and $r_{s(t)t}$ by r^* . Solving (4.44') for dW_t , we have

$$(4.45) \quad dW_t = -dA_{1s(t)t} - \frac{E\{U_{tt}(g^* - r^*)g^*dA_{1s(t)t}\}}{E\{U_{tt}(g^* - r^*)\}} \quad \text{for all } t \geqq 2 .$$

Substituting the expression for dW_t into (4.42), we obtain:

$$(4.46) \quad E\left\{U_{tt}\left[g^*dA_{1s(t)t} - \frac{E\{U_{tt}(g^* - r^*)g^*\}dA_{1s(t)t}}{E\{U_{tt}(g^* - r^*)\}}\right]\right\} = (1+r)^{-t}d\lambda$$

which one finds reduces to:

$$(4.47) \quad \frac{[E\{U_{tt}g^*\}]^2 - E\{U_{tt}\}E\{U_{tt}(g^*)^2\}}{E\{U_{tt}(g^* - r^*)\}} dA_{1s(t)t} = (1+r)^{-t}d\lambda$$

for all $t \geqq 2$.

But $(1+r)^{-t} > 0$ and from (4.20) it follows that since only $\sum_{s=1}^{T} (1+r)^{-s}X_{s-1}$ is changing exogenously, $d\lambda < 0$. Hence, the left-hand side of (4.47) must be negative, and it follows that

$$(4.48)\quad \text{sign}\,(dA_{1s(t)t}) = \text{sign}\left(\frac{E\{U_{tt}\}E\{U_{tt}(g^*)^2\} - [E\{U_{tt}g^*\}]^2}{E\{U_{tt}(g^* - r^*)\}}\right)$$

and this is true for $t = 2, \ldots, T$. It remains only to determine the sign of the fraction on the right-hand side of (4.48).

It follows from the Schwarz Inequality that the numerator on the right-hand side of (4.48) is positive. Schwarz's Inequality states that

(4.49) $[E\{f\cdot h\}]^2 \leqq E(f^2)E(h^2)$ where f and h are functions of random variables with finite variances. Letting $f = (-U_{tt})^{1/2}$, $h = (-U_{tt})^{1/2}(-g^*)$, (4.49) yields

$$(4.50)\quad E\{U_{tt}\}E\{U_{tt}(g^*)^2\} - (E\{U_{tt}g^*\})^2 \geqq 0 \quad \text{for all} \quad t \geqq 2 .$$

In fact, the strict inequality obtains in (4.50) for all $t \geqq 2$ because the equality $[E\{f\cdot h\}]^2 = E\{f^2\}E\{h^2\}$ obtains if and only if $h = \alpha f$ with probability one; that is, if and only if h is proportional to f with probability one. But for this to be true in the present problem, one would have to have

$(-U_{tt})^{1/2}(-g^*) = \alpha(-U_{tt})^{1/2}$ or $-g^* = \alpha$ with probability one.

This would mean g^* is a degenerate random variable, which is impossible in our model. Therefore,

$$(4.51)\quad E\{U_{tt}\}E\{U_{tt}(g^*)^2\} - (E\{U_{tt}g^*\})^2 > 0 \quad \text{for all} \quad t \geqq 2 .$$

It follows from the fact that the decisionmaker is decreasingly risk averse in the multiperiod sense that the denominator of the right-hand side of (4.48) is also positive.[28] As argued earlier, condition (3.8) or its equivalent,

$$(3.8') \begin{cases} \text{Setting all } C_i = 0 \text{ except } C_t \text{ in } U(C)\ , \\ -\dfrac{U_{tt}}{U_t} \quad \text{is a decreasing function of } C_t \text{ and this is true} \\ \qquad \text{for all } t\ , \end{cases}$$

is a necessary condition for decreasing absolute risk aversion in the multiperiod sense. Since the decisionmaker's utility function is assumed by the hypothesis of the theorem to be additive in the individual periods' utilities, $-\frac{U_{tt}}{U_t}$ is the same for all $C = \{C_1, C_2, \ldots, C_T\}$ vectors. Therefore, condition (3.8') requires in the present case of an additive utility function that $\left(-\frac{U_{tt}}{U_t}\right)_{C_t}$ be a decreasing function of C_t for all t, all C vectors. Recalling that under hypothesis (b) of the theorem $C_t = W_t + (1+g_{1s(t)t})A_{1s(t)t}$ for $t \geqq 2$, this requirement implies that

$$(4.52) \quad \left(-\frac{U_{tt}}{U_t}\right)_{C_t} < \left(-\frac{U_{tt}}{U_t}\right)_{W_t + (1 + r^*)A_{1s(t)t}} \qquad \text{if} \quad g^* - r^* > 0$$

while

$$(4.53)\quad \left(-\frac{U_{tt}}{U_t}\right)_{C_t} > \left(-\frac{U_{tt}}{U_t}\right)_{W_t + (1 + r^*)A_{1s(t)t}} \qquad \text{if} \quad g^* - r^* < 0 .$$

At the same time since $U_t > 0$ for all C_t,

$$(4.54)\qquad (g^* - r^*)(U_t)_{C_t} > 0 \quad \text{for} \quad g^* - r^* > 0$$

while

$$(4.55)\qquad (g^* - r^*)(U_t)_{C_t} < 0 \quad \text{for} \quad g^* - r^* < 0$$

Combining (4.52) with (4.54) and (4.53) with (4.55), one obtains:

$$(4.56)\quad \left(\frac{U_{tt}}{\cancel{U_t}}\right)_{C_t} \cancel{(U_t)_{C_t}} (g^* - r^*) > \left(\frac{U_{tt}}{U_t}\right)_{W_t + (1 + r^*)A_{1s(t)t}} (U_t)_{C_t}(g^* - r^*)$$

for all $(g^* - r^*) \neq 0$.

Hence, applying the expectations operator to both sides of (4.56), we have

$$(4.57)\quad E\{(U_{tt})_{C_t}(g^* - r^*)\} > \left(\frac{U_{tt}}{U_t}\right)_{W_t + (1 + r^*)A_{1s(t)t}} E\{(U_t)_{C_t}(g^* - r^*)\} .$$

If $A_{1s(t)t} > 0$, however, it follows from (4.11) that $\frac{\partial \mathcal{L}}{\partial A_{1s(t)t}} = 0$ so that (4.9) becomes for $[i, s, t] = [1, s(t), t]$ and $t = 2, \ldots, T$,

$$(4.58)\quad E\{U_t(1 + g^*)\} = \lambda(1+r)^{-s(t)} \quad \text{for each} \quad t \geqq 2 .$$

At the same time from (4.8), one has $E\{U_t\} = \lambda(1+r)^{-t}$, so that multiplying (4.8) by $(1+r)^{t-s(t)}$ and subtracting the result from (4.58), one finds that

$$E\{U_t[(1 + g^*) - (1 + r)^{t-s(t)}]\} = 0 \quad \text{or}$$

(4.59) $$E\{U_t(g^* - r^*)\} = 0$$

since $1+r^* \equiv (1+r)^{t-s(t)}$. But then the right-hand side of (4.57) is zero and one has

(4.60) $E\{U_{tt}(g^* - r^*)\} > 0$ for each $t \geqq 2$.

Combining the result of the Schwarz Inequality in (4.51) with the implication of multiperiod decreasing risk aversion in (4.60), we find that the fraction on the right-hand side of (4.48) is positive. Hence,

(4.61) $dA_{1s(t)t} > 0$ for each $t \geqq 2$.

The scale on which each accepted risky project is pursued increases. But from (4.41) we also know that $dA_{ist} = 0$ for all $[i, s, t] \neq [1, s(t), t]$, $t = 2, \ldots, T$, and combining this with (4.61), we have:

(4.62) $$d\left(\sum_{s=1}^{T-1} \sum_{i=1}^{n_{st}} \sum_{t=s+1}^{T} (1+r)^{-s} A_{ist}\right) > 0 .$$

The discounted present value of the investment in risky projects increases.

The proof of Theorem 4 is now complete. For a decisionmaker with an additive multiperiod utility function, this theorem delineates a set of conditions ensuring that an increase in the discounted present value of the firm's cash throw-offs will produce an increase in the discounted present value of the risky projects pursued.

5. Concluding Comments

In summary we have seen that the insurance-policy concept of decreasing absolute risk aversion can be generalized to the context of multiperiod planning problems and that a set of sufficient conditions for the property to obtain is a generalization of the Pratt condition for the single-period case. It was also shown that in the case of an additive multiperiod utility function, the necessary and sufficient condition for multiperiod decreasing absolute risk aversion is that the Pratt condition obtain for each individual period's utility function, $u_t(C_t)$. On the other hand, it was proven by counterexample that in at least one multiperiod planning context, Arrow's result that decreasing risk aversion implies the "normal" character of risky asset purchases does not obtain. A set of conditions sufficient for obtaining Arrow's result in the particular "capital-budgeting problem" considered was then presented in the form of Theorem 4. The nature of these conditions and the nature of the counterexample suggests that the real difficulty in generalizing Arrow's result arises from the multiplicity of assets yielding stochastic returns in the same period and not from the inherent multiperiod nature of the model. This conjecture is supported by some recent work of D. Cass and J.E. Stiglitz on single-period portfolio problems involving many risky assets. In the context of such problems, they have found cases in which the total investment in risky assets decreases as wealth increases despite the fact that the decisionmaker's utility function is decreasingly risk averse.

Several interesting questions remain open for further investigation. First, it would be desirable to find a set of necessary and sufficient conditions for multiperiod decreasing risk aversion for the case of non-additive multiperiod utility functions. Second, it would be desirable to determine whether there exist necessary and sufficient conditions on the decisionmaker's utility function alone--that is, independent of the structure of returns--that ensure that risky assets are not an inferior good. The results presented in this paper indicate that if one's concern in choosing utility functions for use in normative and positive models of decisionmaking in multiperiod planning contexts ends with risk aversion and the decisionmaker's attitude toward insurance against risks of a given size as his endowments are altered, then multiperiod decreasing risk aversion is a sufficient condition to impose on the function chosen. If, on the other hand, one is concerned with ensuring that risky assets will be a "normal" good for the decisionmaker whose behavior one is analyzing, the results presented here indicate that multiperiod decreasing risk aversion will not suffice. It would be desirable to know what conditions are necessary and sufficient for this latter property to obtain.

FOOTNOTES

[1] Marshall (1920, p. 843); Arrow (1963, p. 26; 1965, p. 31); Friedman and Savage (1948, pp. 72-76); Markowitz (1952, pp. 151-152; 1959, pp. 215-218); Pratt (1964, pp. 124-127).

[2] Pratt (1964, p. 123).

[3] The review of the single-period results is basically a summary of parts of Pratt's article, with some notational changes.

[4] Pratt does not require that the utility function be bounded. The need for boundedness of the utility function if one is to apply the expected-utility hypothesis was first observed by Menger in Menger (1934). Arrow (1963, 1965) also emphasizes the requirement that the utility function be bounded if the expected-utility result is to be used.

[5] The same terminological convenience used by Pratt is employed in what follows. Namely, "decreasing" is used in place of the cumbersome "nonincreasing" and "increasing" is used instead of "nondecreasing."

[6] Pratt (1964, pp. 122-123).

[7] Arrow (1965, p. 43).

[8] Ibid. (1965, p. 35).

[9] Arrow (1963, p. 26).

[10] This section is primarily a summary without proofs of the material in the revised version of Chapter 2, Sections 2.3 through 2.5 of my doctoral dissertation, Capital Budgeting Under Risk: A Mathematical-Programming Approach. The revised version defines the concept of decreasing absolute risk aversion in terms of the discounted present value of the stream of insurance premiums and states the main theorem of the chapter--the one that provides a set of sufficient conditions for decreasing risk aversion in the multi-period sense--in terms of this new definition. The original version of the chapter had stated the concept and the theorem in terms of the set of "all-in-one-period" insurance policies--policies in which the individual pays an insurance premium in one and only one period, there being one such policy corresponding to each period of the decisionmaker's horizon. Only minor changes are required to prove the theorem using the new definition.

[11] Jensen's Inequality in the single-variable case states that if $f(x)$ is a concave (convex) function of x, then if x is a random variable $E\{f(x)\} \leqq f(E[x])$ $(E\{f(x)\} \geqq f(E[x]))$.

See Feller (1966, pp. 151-152) for a discussion of Jensen's Inequality in the case of functions of one variable.

[12] Klevorick (1967, pp. 34-38).

[13] It is assumed the solution to the optimization problem occurs in the interior of consumption-income space.

[14] It is important to note that with C and t given, there is a one-to-one mapping between the values $E\{U(C + \tilde{Z})\}$ and the vectors $U^{-1}[E\{U(C + \tilde{Z})\}, C, t]$. Hence, the notation in (3.4) informs us, as it should, that $\Pi^t(C, \tilde{Z})$ is a function not a correspondence: with each $(C, \tilde{Z})$ pair there is associated one and only one risk-premium vector that has zeros everywhere but in the t^{th} element.

[15] The proof of the sufficiency of these conditions is almost identical with the proof of Theorem 2.2 in Klevorick (1967, pp. 51-56).

[16] Pratt (1964, pp. 130-131).

[17] Klevorick (1967, Lemma 2.3, pp. 47-49).

[18] Ibid., Corollary 2.2, p. 38.

[19] Ibid., Theorem 7.1, pp. 253-259; and pp. 275-276.

[20] Many aspects of actual capital-budgeting decisions--for example, indivisibilities of projects, physical and cash-flow relationships among projects, borrowing limitations, leverage considerations in the raising of new funds, and so on--are ignored in the model that follows. These features are, indeed, fundamental to the nature of real-world capital-budgeting problems, but they can be safely ignored for the purpose of answering the question at hand: the implications of multiperiod decreasing absolute risk aversion. For a discussion of the problems raised by these other aspects of the capital-budgeting problem, see, for example, Weingartner (1963, 1966), Klevorick (1967).

The model to be presented has an alternative interpretation which the reader may find more appealing. It can be viewed as a description of the portfolio problem of an individual investor. His decision process is basically a sequential one as he changes his portfolio's composition from one planning period to the next. But transactions costs (which will be important if his total investment is small and transactions costs are approximately at real-world levels) and the costs of gathering and processing reliable information force him to revise his portfolio's composition only after stated intervals of time. The number of time periods covered in the horizon of the model that follows--namely, T --then represents the length of time over which he expects his portfolio to remain fixed. Clearly, the model becomes of greater interest the longer the period for which the individual is locked into his portfolio. The discussion in the text will consider the model as a capital-budgeting problem, but the reader who prefers to may reinterpret it at every step as a portfolio problem of the type just described.

[21] In choosing the horizon, one would like to follow certain guidelines. Specifically, the decisionmaker would like to set the horizon T at a point in time "such that the set of accepted projects having outlays or revenues in year T or sooner are exactly the same whether the model makes use of an infinite horizon or a horizon set at T" or a point in time "such that the decisions which call for implementation before this date will be exactly the same, whether or not events past that moment are treated explicitly or implicitly..." (Weingartner (1963, p. 153)) Unfortunately, as Weingartner goes on to tell us, "In dynamic models in general such a horizon does not necessarily exist, or there may be many of them. If there are several, the earliest having this property may be designated as the preferred one" (Ibid., pp. 153-154) because of the problems entailed in collecting data about prospective investments. In the case of a multiplicity of potential horizon periods, Weingartner's choice seems sound. Clearly, the real difficulty arises when no period T satisfies the desiderata for a horizon cut-off point. The present paper, however, is not concerned with this matter. Instead, it circumvents the problem by assuming that a horizon period T has been given by the decisionmaking unit, whether or not the horizon so chosen satisfies the conditions set out above.

[22] It would be possible to allow the riskless rate to vary from period to period, but no insights are gained and much notational simplicity is lost. There is no loss in generality involved in assuming the riskless rate of interest is the same in every period.

[23] Klevorick (1967, pp. 80-113, 235-252). Although the argument for maximizing utility presented in the first set of pages referred to rested upon imperfections of the capital market, that argument applies equally well to the case in which the capital market is perfect but the environment in which the firm and its owners live and make decisions is one of risk or uncertainty.

[24] See Pye (1967, pp. 111-112) for a proof of this statement. Pye's proof is actually for a single-period utility function, but the argument is completely analogous for the multiperiod function in use here. This assumption of imperfect correlation between the return factors of projects used for withdrawals in the same period is not inconsistent with our earlier assumption that the returns from the same initial physical investment may be used for withdrawal in the same period but reach that period via different routes. All the assumption of imperfect correlation says is if the two resulting

return factors for period t withdrawals are perfectly correlated, there is really no point to considering both projects: either one proposal will dominate the other or they will be identical. In the former case, only the better proposal need be considered while in the latter case, there is really only one distinct project.

[25] Arrow, Hurwicz, and Uzawa (1961); Arrow and Enthoven (1961).

[26] This Lemma states a well-known result. Since I know of no reference in which it is proven, a brief proof is offered here.

<u>Proof</u>: Suppose y_1 and y_2 are vectors such that

(i) $\varphi(b_1) = f(y_1)$ and (ii) $\varphi(b_2) = f(y_2)$.

Consider $\varphi(\mu b_1 + (1-\mu)b_2)$ with $0 \leqq \mu \leqq 1$, that is,

$$\text{(iii)} \qquad \varphi(\mu b_1 + (1-\mu)b_2) = \max_{\substack{h(y) \leqq \mu b_1 + (1-\mu)b_2 \\ y \geqq 0}} f(y) .$$

But from (i) and (ii), $y_1 \geqq 0$, $y_2 \geqq 0$, $h(y_1) \leqq b_1$, $h(y_2) \leqq b_2$,

so that

$$\text{(iv)} \qquad \mu h(y_1) + (1-\mu)h(y_2) \leqq \mu b_1 + (1-\mu)b_2 .$$

Since $h(y)$ is convex in y , it follows from (iv) that

$$\text{(v)} \qquad h(\mu y_1 + (1-\mu)y_2) \leqq \mu b_1 + (1-\mu)b_2 .$$

That is, $\mu y_1 + (1-\mu)y_2$ is feasible for the problem in (iii). But then we have

$$\text{(vi)} \qquad \varphi(\mu b_1 + (1-\mu)b_2) \geqq f(\mu y_1 + (1-\mu)y_2) .$$

Since $f(y)$ is a concave function of y , we have, using (i) and (ii),

$$\text{(vii)} \quad f(\mu y_1 + (1-\mu)y_2) \geqq \mu f(y_1) + (1-\mu)f(y_2) = \mu\varphi(b_1) + (1-\mu)\varphi(b_2) .$$

It follows from (vi) and (vii) that

$$\text{(viii)} \quad \varphi(\mu b_1 + (1-\mu)b_2) \geqq \mu\varphi(b_1) + (1-\mu)\varphi(b_2) \quad \text{for } 0 \leqq \mu \leqq 1 .$$

But (viii) states that $\varphi(b)$ is a concave function of b .

Note that if $f(y)$ is strictly concave in y , then the inequality in (vii) is a strict one and as a result so is the one in (viii): that is, $\varphi(b)$ is then strictly concave in b .

[27] This assumption is trivially fulfilled if I_t contains only one project. Indeed, if this were the case, the entire discussion of dA_{ist} for all $[i, s, t] \neq [1, s(t), t]$, $t = 2, \ldots, T$ would be unnecessary and one could proceed immediately from the observation that $d\left(\frac{\partial \mathcal{L}}{\partial A_{1s(t)t}}\right) = 0$ to equations (4.42) and (4.43).

[28] The fact that decreasing risk aversion implies that the denominator of the right-hand side of (4.48) is positive is essentially the same result as that referred to by Arrow (1965, p. 43). The proof presented here is a slight variation of the one Arrow presents in his unpublished lecture notes on liquidity preference for the case of a single risky asset and a single safe asset with zero return.

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