8-1-1967

A Noncooperative View of Oligopoly

James W. Friedman

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 234

A NONCOOPERATIVE VIEW OF OLIGOPOLY

J. W. Friedman

August 31, 1967

A NONCOOPERATIVE VIEW OF OLIGOPOLY

by

J. W. Friedman*

1. Introduction

There are two basic approaches to the theory of oligopoly which may be called the cooperative and the noncooperative. Typically a cooperative approach will utilize a bargaining model under which the several firms in the oligopoly are supposed to bargain among themselves in order to agree on some joint decision (say, a set of prices to be charged -- one per firm) which yields to the industry one of the outcomes which is Pareto optimal for them.

A noncooperative approach will, by contrast, involve each firm in isolated decision making. This is not to imply that each firm ignores the effects of its rivals' decisions on its own profit or of its own decisions on its rivals' behavior (and hence its own profits). Noncooperative formulations will generally assume the firms not to make decisions jointly. An equilibrium of a noncooperative nature typically features a set of strategies (say a strategy is a rule for choosing one's price), one for each firm, having the

---

* The author is Assistant Professor of Economics at Yale University and a staff member of the Cowles Foundation. The present paper was written during the author's tenure as a Yale Junior Faculty Fellow, while he was a guest at the Center for Research in Management Science at the University of California, Berkeley.

Research undertaken by the Cowles Commission for Research in Economics under Contract Nonr-3055(01) with the Office of Naval Research.

following property: For each firm, its equilibrium strategy gives at least as much profit as any other strategy it might choose, given the strategy choices of its rivals.

One possible method of finding a cooperative solution is to define a bargaining process, or a set of axioms which the solution must obey, which turns the process of determining a solution into a noncooperative game. The Nash cooperative solution [5] is a case in point. Here, each firm may be regarded as a) agreeing to abide by the Nash axioms to find the solution point which results from the "threat point" and b) choosing a "threat strategy." The several threat strategies determine a threat point, which determines a solution; while the choice of a threat strategy is dictated by the wish to arrive at the threat point whose associated solution gives one maximum profit.

In the literature some discussion may be found of the relative merits of cooperative and noncooperative theories. The former have appeal because it seems so obviously sensible and in the interest of all oligopolists to jointly exploit their market to the full. On the other hand, making joint agreements is difficult. The more firms there are, the more interests to reconcile; hence, the harder it is to come to agreement. In addition, where each firm has knowledge not shared by others (say knowledge of one's own profit function), it becomes even harder to bargain because no one knows the full alternatives open to the group and it may be in a firm's own interest to misrepresent when disclosing information.

Thus it is not clear that either type of theory could ever explain by itself the behavior of oligopolists, and as a result both lines should be actively pursued. The present effort is principally in the tradition of Cournot [4] and Chamberlin [2]. Cournot founded the noncooperative approach to oligopoly and also invented the reaction function, a device which gives the decision of a firm in a period as a function of decisions of all firms in the preceding period. The analysis to follow will make much use of reaction functions. It will be assumed that the oligopoly is Chamberlinian in the sense that the firms produce differentiated but fairly closely substitutable products, and the decision variable of each is its price.

The program followed below is quite modest and may be explained briefly as follows: Assume there are n firms, where n is any integer greater than one, and each firm has a profit function:

$$p_{it} = p_i(x_{1t}, x_{2t}, \dots, x_{nt}) = p_i(x_t) \qquad i = 1, \dots, n$$

where x_{it} is the price of the i^{th} firm in the t^{th} period and p_{it} is the profit of the i^{th} firm in the t^{th} period. Assume also that each firm chooses its price according to a reaction function:

$$x_{it} = \psi_i(x_{t-1}) .$$

The problem with which this paper is concerned is: Could there exist a set of reaction functions (one for each firm) that would be an "equilibrium" set in the sense that it would give a set of stable and self-perpetuating prices that would be profit maximizing for the firms.

This question can be stated in mathematical form as follows: Does there exist a set of reaction functions, $(\psi_1^*, \ldots, \psi_n^*)$ which has the following properties:

(a) The functions $\psi_1^*, \ldots, \psi_n^*$ have a unique fixed point x^*.

(b) x^* is a stable fixed point in the sense that if one starts with an arbitrary set of prices x_t and chooses x_{t+1} according to the reaction function (i.e., $x_{t+1} = (\psi_1^*(x_t), \ldots, \psi_n^*(x_t))$, $x_{t+\tau} \to x^*$ as $\tau \to \infty$.

(c) A set of reaction functions $(\psi_1^*, \ldots, \psi_{i-1}^*, \varphi_i, \psi_{i+1}^*, \ldots, \psi_n^*)$ exists which gives the same fixed point, x^*, as $(\psi_1^*, \ldots, \psi_n^*)$ where φ_i is optimal for any i^{th} firm. That is, given the profit function for the i^{th} firm and its discount parameter,

$\alpha_i (0 < \alpha_i < 1)$, if the i^{th} firm calculated the reaction functions φ_{iT} which maximize the discounted profit streams

$$\sum_{\tau=0}^{T} p_{i,t+\tau} \alpha_i^{\tau} , \qquad T = 0, 1, 2, \ldots,$$

knowing the reaction functions of the other firms, ψ_j^* $(j \neq i)$, the φ_{iT} would converge to φ_i as $T \to \infty$ and x^* would be a stable fixed point of $(\psi_1^*, \ldots \psi_{i-1}^*, \varphi_i, \psi_{i+1}^*, \ldots \psi_n^*)$.

The equilibrium set of reaction functions $(\psi_1^*, \ldots, \psi_n^*)$ is not unique. Regarding the reaction function of a firm as its strategy, an equilibrium set does not form a noncooperative equilibrium in the sense of Nash [6]. The latter would require $\varphi_i = \psi_i^*$ (for all i). In other words, ψ_i^* would have to maximize the discounted profit stream of the i^{th} firm, given the ψ_j^* $(j \neq i)$, rather than merely giving rise to the same fixed point as φ_i . One might call $\psi_1^*, \ldots, \psi_n^*$ an approximate Nash equilibrium, approximate in the sense noted above. The stress laid on the profit maximizing character of the ψ_i^* is, of course, crucial. While there are many sets of reaction functions having a fixed point, those which do not maximize some reasonable objective function would appear to have little bearing on the behavior of firms.

The equilibrium reaction functions considered here $(\psi_1^*, \ldots, \psi_n^*)$ may be termed "asymptotically optimal." They are asymptotically optimal in the sense that, given an arbitrary initial price vector, use of the ψ_i^* will lead to a sequence of price vectors which, in the limit, converge to the fixed point of the ψ_i^*. The ψ_i^* are, in fact, optimal reaction functions at this point.

An equilibrium satisfying a), b) and c) is a natural and straightforward generalization of the classic Cournot solution, regarded as the fixed point of a set of Cournot reaction functions.[1] In Cournot's formulation, each firm has a reaction function:

$$x_{it} = \omega_i(x_{1,t-1}, \ldots, x_{i-1,t-1}, x_{i+1,t-1}, \ldots, x_{n,t-1}) .$$

Thus x_{it} depends upon the period $t - 1$ prices of every firm except the i^{th}. In addition ω_i is chosen to maximize current period profit, p_{it}, not a discounted stream; and, in determining ω_i, the i^{th} firm does not know the true reaction functions used by its rivals. Instead it makes the usually incorrect assumption that $x_{jt} = x_{j,t-1}$ $(j \neq i, \; j = 1, \ldots, n)$.

In contrast to the Cournot formulation, then, the present formulation will result in a reaction function for each firm which is asymptotically optimal given the true reaction functions used by the rivals.

2. The Characterization and Existence of Equilibria

2.1 The Model of the Firm[2]

Each firm is assumed to have a demand function, $f_i(x_t)$, giving its sales (equals output) as a function of all prices charged in the current period. Price vectors, x , are elements of the n-dimensional Euclidean space R^n . Let A be the subset of R^n containing all price vectors which have nonnegative components and which correspond to nonnegative sales for all firms:

$$A = \{x | x_i \geq 0,\ f_i(x) \geq 0,\ i = 1, \ldots, n\} \subset R^n .$$

Denote by $\mathring{A}$ the interior of A . The f_i are assumed to have continuous second partial derivatives at all points of $\mathring{A}$ and:

$$\left.\begin{array}{lll} \text{A1} & f_i^i(x) < 0 & \\ \text{A2} & f_i^j(x) > 0 & j \neq i , \quad j = 1, \ldots, n \end{array}\right\} \begin{array}{l} i = 1, \ldots, n \\ x \in \mathring{A} \end{array}$$

These assumptions embody the differentiated products notion: the sales of the i^{th} firm will fall when its own price rises, but its sales will rise as the price of any competing firm rises. Each firm has also a (twice differentiable) total cost function C_i , giving total cost as a function of sales:

$$\left.\begin{array}{lll} \text{A3} & C_i(0) \geq 0 & \\ \text{A4} & C_i'(f_i(x)) = \dfrac{dC_i}{df_i} \geq 0 & \end{array}\right\} \begin{array}{l} i = 1, \ldots, n \\ x \in \mathring{A} \end{array}$$

It is assumed here that marginal cost is nonnegative. Some additional assumptions are made on the f_i and C_i ; however, they are made in terms of properties of the profit function. The profit function is:

$$p_i(x) = x_i f_i(x) - C_i(f_i(x)) \qquad i = 1, \ldots, n \ , \quad x \in \mathring{A} \ .$$

The additional conditions on the p_i are:

A5 there is a point, $x^c \in \mathring{A}$, such that

$$p_i^i(x^c) = 0 \ , \qquad i = 1, \ldots, n$$

$$\left.\begin{array}{lll} \text{A6} & p_i^{ij}(x) \geq \epsilon \ , \quad \sum\limits_{j \neq i} p_i^{ij}(x) \leq R \quad j \neq i & \\ \text{A7} & \sum\limits_{j=1}^{n} p_i^{ij}(x) \leq -\epsilon & \end{array}\right\} \begin{array}{l} \epsilon > 0 \\ (n-1)\epsilon \leq R < \infty \\ x \in \mathring{A} \end{array}$$

$$\text{A8} \qquad |p_i^{jk}(x)| \leq M \qquad j \ , \ k \neq i \qquad 0 < M < \infty$$

Assumption A5 places the Cournot point, x^c , in the interior of A, and output levels for all firms. A6 and A7 insure the uniqueness of x^c.

The assumptions made on the p_i^{ij} insure that the rate at which p_i diminishes per unit increase in x_i falls with increases in x_i, rises with increases in $x_j (j \neq i)$, and falls if all prices are raised by equal amounts. A8 merely places a finite bound on the p_i^{jk} $(j, k \neq i)$. Of course, assumptions A6 and A7 imply $p_i^i(x) \leq -n\epsilon$.

It is clear that $p_i^i(x) \leq 0 \Longrightarrow p_i^j(x) > 0 \quad (j \neq i), \quad x \in \mathring{A}$.

$$p_i^j(x) = (x_i - C_i'(f_i))f_i^j(x), \qquad j \neq i$$

$$p_i^i(x) = (x_i - C_i'(f_i))f_i^i(x) + f_i.$$

$f_i > 0$ and $f_i^i < 0$; therefore $p_i^i \leq 0$ implies $(x_i - C_i') > 0$. It is known that $f_i^j > 0$; hence $p_i^j(x) > 0$.

2.2 Some Properties of the Reaction Functions

In this section the existence of sets of reaction functions with stable fixed points is shown. The latter are not necessarily optimal. Additional properties of the reaction functions are also found.

It will be useful at this point to establish some additional notation. The reaction function of the i^{th} firm is denoted ψ_i. It maps a point of R^n into R^1. Ψ will denote a mapping from R^n into R^n, whose i^{th} coordinate in the image space is ψ_i. I.e., given x, y such that Ψ: $x \to y$, $y_i = \psi_i(x)$ $(i = 1, \ldots, n)$. ${}_i\Psi$ will also denote a mapping from R^n into R^n; however, if ${}_i\Psi$ maps the point x onto the point y, the following is understood: $y_j = \psi_j(x)$ $(j \neq i)$, $y_i = x_i$.

The symbol C will be used to denote a closed cube in R^n and $\mathring{C}$ will be its interior. $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$ will be the origin of the cube and δ $(0 < \delta < \infty)$ will be the length of a side. Thus $C = \{x | \underline{x}_i \leq x_i \leq \underline{x}_i + \delta$, $i = 1, \ldots, n\}$. Let $\bar{x}$ denote $(\underline{x}_1 + \delta, \ldots, \underline{x}_n + \delta)$. Let C be contained in $\mathring{A}$. When necessary, additional properties of C will be specified.

Let Ψ have the following properties:

A9 $\quad \psi_i^j(x) > 0$, $\quad i$, $j = 1, \ldots, n$

A10 $\quad \sum_{j=1}^{n} \psi_i^j(x) \leq \gamma < 1$, $\quad i = 1, \ldots, n$

(A9 and A10 hold for) x contained in the interior of the domain of Ψ.

A11 $\quad \psi_i^{jk}$ $(i$, j, $k = 1, \ldots, n)$ are continuous and bounded in absolute value on the interior of the domain of Ψ.

The assumptions A9 and A10 (which restrict attention to reaction functions for which an increase in price by one firm leads always to subsequent increase by the others $(\psi_i^j > 0)$ and for which a one unit increase by all firms leads to a subsequent increase of less than one unit by all firms $(\Sigma\, \psi_i^j \leq \gamma < 1)$) may leave out of account reaction functions which some may regard as very plausible; however, the class which is allowed is both moderately general and of some interest. It is to be hoped that later efforts will broaden the slope of the present results.

Proposition 1: Let $C \subset \overset{\circ}{A}$ be a closed cube with origin $\underline{x}$ and side δ, and let y^* be a point on the diagonal of C:

$$y^* = (\underline{x}_1 + \beta\ ,\ \underline{x}_2 + \beta,\ \ldots,\ \underline{x}_n + \beta) \qquad 0 < \beta < \delta\ .$$

If a transformation $\underline{\Psi}$ with domain C satisfies A9, A10, A11, and has y^* as a fixed point, then:

$$x \in C \Longrightarrow \underline{\Psi}(x) \in C\ .$$

Let $y = (y_1^* + \xi_1,\ \ldots,\ y_n^* + \xi_n)$ be any point in C $(-\beta \leq \xi_i \leq \delta - \beta$, all i). Consider the i^{th} coordinate of $\underline{\Psi}(y)$:

$\psi_i(y) \geq \psi_i(\underline{x})$ because $\psi_i^j > 0$ for all i, j.

$\psi_i(\underline{x}) = \psi_i(y^*) - \beta\ \Sigma_{j=1}^n\ \psi_i^j(\underline{z})$ for suitable chosen $\underline{z}$ on the line segment connecting $\underline{x}$ and y^*, from the mean value theorem.

$$\psi_i(y^*) - \beta\, \Sigma_{j=1}^{n}\, \psi_i^j(\underline{z}) \geq \underline{x}_i + \beta - \beta\gamma = \underline{x}_i + \beta(1-\gamma) > \underline{x}_i$$

because $\Sigma_{j=1}^{n}\, \psi_i^j \leq \gamma < 1$.

By similar reasoning

$$\psi_i(y) \leq \psi_i(\overline{x}) = \psi_i(y^*) + (\delta+\beta) \sum_{j=1}^{n} \psi_i^j(\overline{z}) \leq \underline{x}_i + \beta + (\delta-\beta)\gamma < \underline{x}_i + \delta \ .$$

Thus, for all i , $\underline{x}_i < \psi_i(y) < \underline{x}_i + \delta$, for $y \in C$. $\Psi(y) \in C$.

This completes the proof of Proposition 1.

Proposition 2: Let Ψ be a transformation, satisfying A9, A10 and A11, which maps points of C into C . For any scalar x_i^* such that $\underline{x}_i \leq x_i^* \leq \underline{x}_i + \delta$, there is a unique fixed point, x^* , of ${}_i\Psi$ such that $x^* = (x_1^*, \ldots, x_n^*) \in C$. If x_i^* and x_i^{**} are two such scalars, corresponding to fixed points x^* and x^{**} , respectively, then $x_i^* < x_i^{**} \Longrightarrow x_j^* < x_j^{**}$ $(j = 1, \ldots, n)$.

From conditions A9 and A10 it is clear that Ψ is of Lipschitz class with ratio $\eta \leq \gamma$. Define the distance from x to y as follows:

$$d(x,y) = \max_i |x_i - y_i| \ .$$

Clearly under this definition of distance $d(\Psi(x) \ , \ \Psi(y)) \leq \gamma \cdot d(x,y)$.

As $\gamma < 1$, Ψ is a contraction. A contraction which maps a set into itself has a unique fixed point in the set. Thus ${}_i\Psi$ has a unique fixed point for given x_i^* , and ${}_i\Psi$ is a contraction which maps C into C if Ψ is.[3]

It remains to show that if x^* and x^{**} are both fixed points of Ψ , then $x_i^* < x_i^{**}$ $x_j^* < x_j^{**}$ $(j = 1, \ldots, n)$. Let $x^1 = (x_1^*, \ldots, x_{i-1}^*, x_i^{**}, x_{i+1}^*, \ldots, x_n^*)$ and

$$x^{\ell} = {}_i\Psi(x^{\ell-1}) \quad \text{for } \ell = 2, \ldots .$$

Clearly:

$$x_j^{\ell} = \psi_j(x^{\ell-1}) > x_j^{\ell-1} \qquad j \neq i , \quad \ell = 2, \ldots$$

as a consequence of $\psi_j^k > 0$ j , $k = 1, \ldots, n$. Also, because ${}_i\Psi$ is a contraction, x^{ℓ} converges to x^{**} as $\ell \to \infty$.

Denote the set of fixed points of ${}_i\Psi$ in C by B_i . For every finite scalar β_i^* such that $\underline{x}_i \leq \beta_i^* \leq \underline{x}_i + \delta$, there is an element $\beta^* \in B_i$ such that $\beta^* = (\beta_1^*, \ldots, \beta_i^*, \ldots, B_n^*)$.

Proposition 1 has shown that any set of reaction functions having a fixed point on the diagonal of the cube C , and satisfying assumptions A9-A11, necessarily maps C into itself. This means that, given such a set of reaction functions, if the price vector, x_0, chosen by the firms in time zero is in C, then every succeeding price vector, x_t $(t > 0)$, will also be in C; hence an initial price vector in C $(\subset \mathring{A})$ will lead to future price vectors which are in $\mathring{A}$. From Proposition 2 it is seen that when Ψ maps C into itself, ${}_i\Psi$ (for any given i) has a set of fixed points, B_i. Each element x^* of B_i corresponds to a different value of x_i^* $(\underline{x}_i \leq x_i^* \leq \underline{x}_i + \delta)$ and if a particular coordinate of $x^* \in B_i$ is larger than the corresponding coordinate of $x^{**} \in B_i$, it is larger in all coordinates. This property of the elements of B_i will prove useful later.

2.3 Optimality for a Single Firm

This section is devoted to showing that a set of reaction functions exists which has a stable fixed point and for which the fixed point is a profit maximizing point for the i^{th} firm. The central result is Proposition 5. Indeed it is the central result of the paper. Proposition 3 merely establishes an open interval $(0, \alpha_i^*)$, which is non-empty, over which the discount parameter of the firm may vary. When the discount parameter is within this range, Proposition 5 will hold. Proposition 4 is not readily interpretable; however, it facilitates the part of Proposition 5 which shows that (given the price vector of the preceding period, x_{t-1}

the reaction functions of the other firms, ${}_i\Psi$, the firms own profit function, p_i , and discount parameter, α_i) the longer is the time horizon over which the firm seeks to maximize profits, the higher will be the price it will choose in period t (item c)).

It will be useful now to introduce some functions of the derivatives of the profit functions. Let:

$$\left.\begin{aligned} G_{is}(x_t) &= p_i^i(x_t) + \alpha_i \sum_{j\neq i} p_i^j(x_{t+1})\psi_j^i(x_t) \qquad & i=1,\dots,n \\ G_{i1}(x_t) &= p_i^i(x_t) & s=2,\dots \end{aligned}\right\}$$

where

$$x_t = (\psi_1(x_{t-1}), \dots, \psi_{i-1}(x_{t-1}), x_{it}, \psi_{i+1}(x_{t-1}), \dots, \psi_n(x_{t-1}))$$

$$x_{t+1} = (\psi_1(x_t), \dots, \psi_{i-1}(x_t), \varphi_{i,s-1}(x_t), \psi_{i+1}(x_t), \dots, \psi_n(x_t))$$

$x_{it} = \varphi_{i1}(x_{t-1})$ is defined by the condition $G_{i1}(x_t) = 0$, and $x_{it} = \varphi_{is}(x_{t-1})$ is defined by the condition $G_{is}(x_t) = 0$. It will be understood that:

$$G_{is}^k(x_t) = \frac{\partial G_{is}}{\partial x_{kt}} = p_i^{ik}(x_t)$$

$$+ \alpha_i \sum_{j\neq i}\left[\psi_j^i(x_t)\left(p_i^{ji}(x_{t+1})\varphi_{i,s-1}^k(x_t) + \sum_{\ell\neq i} p_i^{j\ell}(x_{t+1})\psi_\ell^k(x_t)\right) + p_i^j(x_{t+1})\psi_j^{ik}(x_t)\right.$$

$$i\ ,\quad k = 1, \dots, n, \quad s \geq 2$$

$$G_{i1}^{k}(x_t) = p_i^{ik}(x_t) .$$

It will be seen in the proof of Proposition 5 that G_{is} and φ_{is} have very natural interpretations. $G_{is}(x_t)$ is the first derivative of

$$\sum_{\tau=1}^{s} p_{i,t-1+\tau}\alpha_i^{\tau-1}$$

with the side conditions $G_{i\tau}(x_{t+s-\tau}) = 0$ $(\tau = 1, \ldots, s-1)$. Hence, $G_{is}(x_t) = 0$ is the first order condition for profit maximization when a) the firm wishes to choose its current price, x_{it}, so as to maximize profits over an horizon of s periods and b) it assumes in the next period it will seek to maximize over a $s-1$ period horizon, in two periods hence over a $s-2$ period horizon, etc.

The functions $x_{it} = \varphi_{is}(x_{t-1})$ are merely the profit maximizing reaction functions for the i^{th} firm when its horizon is s periods. $x_{it} = \varphi_{is}(x_{t-1})$ is another way of expressing the condition $G_{is}(x_t) = 0$. As $G_{is}(x_t) = 0 = G_{is}(\psi(x_{t-1}), \ldots, \psi_{i-1}(x_{t-1}), x_{it}, \psi_{i+1}(x_{t-1}), \ldots, \psi_n(x_{t-1}))$, the latter expression may be inverted to give $x_{it} = \varphi_{is}(x_{t-1})$ (when $G_{is}^{i} \neq 0$). It is seen below that $G_{is}^{i} < 0$ everywhere on $\overset{\circ}{C}$.

In the remainder of the paper, the cube, C, will be assumed to have the properties which have previously been attributed to it (i.e., origin $\underline{x}$, side δ and $C \subset \mathring{A}$). In addition, C will be chosen so that:

$$p_i^i(\underline{x}) > 0 \qquad\qquad i = 1, \ldots, n$$

$$p_i^i(\underline{x}_1 + \delta, \ldots, \underline{x}_n + \delta) < 0 \qquad\qquad i = 1, \ldots, n .$$

The new characteristics of C merely mean the Cournot point, x^c, is interior to it. A consequence of the choice of C is the following:

Let: $y(x_i)$ be a price vector $(y_1, \ldots, y_n)$ with components fixed as follows:

$$y_j = y_j^* \qquad \underline{x}_j \leq y_j^* \leq \underline{x}_j + \delta \qquad j \neq i$$

$$y_i = x_i .$$

There is a number x_i^* such that $\underline{x}_i < x_i^* < \underline{x}_i + \delta$ and $p_i^i(y(x_i^*)) = 0$, and $p_i^i(y(x_i)) < 0$ for $x_i^* < x_i \leq \underline{x}_i + \delta$. The x_i which satisfy this condition may be written as follows:

$$x_i = h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) .$$

The implicit function theorem applies, as $p_i^{ii}(x) < 0$ for any $x \in \mathring{A}$

such that $p_i^i(x) = 0$. With time subscripts returned:

$$x_{it} = h_i(x_{1t}, \ldots, x_{i-1,t}, x_{i+1,t}, \ldots, x_{nt}) .$$

The functions φ_{i1} may be written as follows:

$$x_{it} = h_i(\psi_1(x_{t-1}), \ldots, \psi_{i-1}(x_{t-1}), \psi_{i+1}(x_{t-1}), \ldots, \psi_n(x_{t-1}))$$

$$= \varphi_{i1}(x_{t-1}) .$$

It is apparent from the definition of the φ_{is} that

$$\varphi_{is}^k(x_{t-1}) = \frac{\sum_{j \neq i} G_{is}^j(x_t) \cdot \psi_j^k(x_{t-1})}{-G_{is}^i(x_t)} \qquad \begin{array}{l} i, \ k = 1, \ldots, n \\ s = 1, \ldots . \end{array}$$

Thus, if it could be shown that $G_{is}^j > 0$ $(j \neq i)$, $G_{is}^i < 0$ and $\Sigma_{j=1}^n G_{is}^j < 0$, it would be known that $\psi_j^k > 0$ $(j , k = 1, \ldots, n)$ and $\Sigma_{k=1}^n \psi_j^k < 1$ imply

$$\varphi_{is}^k > 0 \quad \text{and} \quad \sum_{k=1}^n \varphi_{is}^k < 1 .$$

<u>Proposition 3</u>: Given: a) Ψ which satisfies A9, A10, and A11, and which maps C into C , b) $\varphi_{i,s-1}(x)$ such that $\varphi_{i1}(x) \leq \varphi_{i,s-1}(x) \leq \underline{x}_i + \delta$ $(x \in C)$, and c) $\varphi_{i,s-1}^j(x) > 0$,

$\Sigma_{j=1}^{n} \varphi_{i,s-1}^{j}(x) \leq \gamma < 1 \quad (x < \overset{\circ}{C})$. There is a number $\alpha_i^* > 0$ such that if $\alpha_i < \alpha_i^*$, then $G_{is}^{i}(x) < 0$, $G_{is}^{j}(x) > 0 \quad (j \neq i)$, $\Sigma_{j=1}^{n} G_{is}^{j}(x) < 0$ for $x \in \overset{\circ}{C}$ and $x_i \geq h_i \quad (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Denote by N the bound on $|\psi_i^{jk}(x)| \quad (i , j , k = 1, \ldots,)$, $x \in \overset{\circ}{A}$ and by Q ,

$$\max_{\substack{x \in C \\ i = 1, \ldots, n}} \sum_{j \neq i} p_i^j(x) .$$

Q is positive, and, of course, is finite because the derivatives of the p_i^j are bounded and C is compact. The proposition is an almost trivial consequence of the bounds on the sundry derivatives:

$$G_{is}^{k}(x_t) = p_i^{ik}(x_t)$$

$$+ \alpha_i \sum_{j \neq i} \left[\psi_j^i(x_t) \left(p_i^{ji}(x_{t+1}) \varphi_{i,s-1}^{k}(x_t) + \sum_{\ell \neq i} p_i^{j\ell}(x_{t+1}) \psi_\ell^k(x_t) \right) + p_i^j(x_{t+1}) \psi_j^{ik}(x_t) \right]$$

$$\geq \epsilon + \alpha_i [-(n-1)^2 \gamma^2 M - NQ] \geq 0 \Longrightarrow \alpha_i \leq \alpha_i^1 = \frac{\epsilon}{(n-1)^2 \gamma^2 M + NQ} \qquad k \neq i$$

$$G_{is}^{i}(x_t) \leq -n\epsilon + \alpha_i [\gamma^2 R + (n-1)^2 \gamma^2 M + NQ \leq 0 \Longrightarrow \alpha_i \leq \alpha_i^2 = \frac{n\epsilon}{\gamma^2 R + (n-1)^2 \gamma^2 M + NQ}$$

$$\sum_{k=1}^{n} G_{is}^{k} \leq -\epsilon + \alpha_i [\gamma^2 R + (n-1)^2 \gamma^2 M + nNQ] \leq 0 \Longrightarrow \alpha_i \leq \alpha_i^3 = \frac{\epsilon}{\gamma^2 R + (n-1)^2 \gamma^2 M + nNQ} .$$

Clearly, as ϵ , γ , R , n-1 , M , N , and Q are finite and strictly positive, $0 < \alpha_i^3 < \alpha_i^2$, α_i^1 ; hence, any value of α_i^* such that $0 < \alpha_i^* \leq \alpha_i^3$ will satisfy the prescribed conditions.

Thus Proposition 3 shows that when a) the s-1 period horizon profit maximizing reaction function, $\varphi_{i,s-1}$, has positive first partial derivatives which sum to no more than γ (< 1) , b) $\varphi_{i,s-1}$ lies above the one period reaction function $(s > 1$, $\varphi_{i,s-1}(x) > \varphi_{i1}(x)$, $x \in C$) , and c) the discount parameter is suitably restricted in its range, then $G_{is}^i(x) < 0$, $G_{is}^j(x) > 0$ ($j \neq i$ and $\sum_{j=1}^{n} G_{is}^j(x) < 0$ $(x \in \mathring{C})$. These conditions on the G_{is}^j will be seen in the proof of Proposition 5 to guarantee that φ_{is} will have positive first derivatives which sum to no more than γ .

<u>Proposition 4</u>: Given Ψ which satisfies A9, A10 and A11, which maps C into C ; $G_{is}(x)$, a function whose range is in R^1 and for which $G_{is}^i(x) < 0$, $G_{is}^j(x) > 0$ $(j \neq i)$, $\Sigma_{j=1}^n G_{is}^j(x) < 0$, $(x \in C)$ and B_i , the set of fixed points of ${}_i\Psi$ contained in C . $G_{is}(x)$ is strictly monotone decreasing along elements of B_i . (I.e., given β , $\beta' \in B_i$, $\beta_i < \beta_i'$, then $G_{is}(\beta) > G_{is}(\beta'))$.

Let β and $\beta + \xi$ be elements of B_i . From the mean value theorem it is known that:

$$\psi_j(\beta + \xi) - \psi_j(\beta) = \sum_{k=1}^{n} \psi_j^k(z_j)\xi_k , \qquad j \neq i ,$$

where the z_j are appropriately chosen points on the line segment joining β and $\beta + \xi$. Then:

$$\xi_j = \sum_{k=1}^{n} \psi_j^k(z_j)\xi_k \qquad j \neq i .$$

It is known that if $\xi_1 > 0$, then $\xi_k > 0$ $(k = 2, \ldots, n)$, and similarly if $\xi_1 < 0$. In addition, $|\xi_k| < |\xi_i|$. This may be seen readily by assuming $\max_{j \neq i} |\xi_j| = |\xi_{j'}| \geq |\xi_i|$.

$$\sum_{k=1}^{n} \psi_{j'}^k(z_{j'})|\xi_k| \leq |\xi_{j'}| \sum_{k=1}^{n} \psi_{j'}^k(z_{j'}) \leq \gamma |\xi_{j'}| < |\xi_{j'}| .$$

A contradiction results, thus, $|\xi_j| < |\xi_i|$ $(j \neq i)$. Assume $\xi_j > 0$. The proof may now be concluded directly.

$$G_{is}(\beta + \xi) - G_{is}(\beta) = \sum_{j=1}^{n} G_{is}^j(y)\xi_j < \xi_i \sum_{j=1}^{n} G_{is}^j(y) < 0 .$$

With $\xi_j < 0$, the above inequalities are reversed. This completes the proof.

Let:

$$G_i(x) = p_i^i(x) + \alpha_i \sum_{j \neq i} p_i^j(x)\psi_j^i(x) .$$

Given, α_i (where $0 < \alpha_i < \alpha_i^*$) it is obviously possible to select a Ψ such that

$$G_i(y^*) > 0 \qquad i = 1, \ldots, n$$

where y^* is the fixed point of Ψ. One need only choose y^* close enough to x^c, given the values of the $\psi_j^i(y^*)$.

<u>Proposition 5</u>: Given Ψ, p_i and α_i' such that:

1. $\Psi(y^*) = y^* = (\underline{x}_1 + \beta, \underline{x}_2 + \beta, \ldots, \underline{x}_n + \beta) \quad 0 < \beta < \delta$

2. $G_i(y^*) = p_i^i(y^*) + \alpha_i' \sum_{j \neq i} p_i^j(y^*)\psi_j^i(y^*) = 0 \quad (0 < \alpha_i' < \alpha_i^*)$

3. Φ_{is}, a transformation which takes $x \to y$ such that

$$y_j = \psi_j(x) \qquad j \neq i$$
$$y_i = \varphi_{is}(x) .$$

Then the following are true:

a) φ_{is} is the reaction function which maximizes $\sum_{\tau=1}^{s} \alpha_i'^{\tau-1} p_{i,t+\tau-1}$.

b) $\varphi_{is}^k(x) > 0$, $\Sigma_{k=1}^n \varphi_{is}^k(x) \leq \gamma$, $x \in \mathring{C}$, $s = 1, \ldots$.

c) $\varphi_{is}(x) > \varphi_{i,s-1}(x)$, $s = 2, \ldots,$ $x \in C$.

Denote by φ_i the limit of the φ_{is} and by Φ_i the limit of the Φ_{is}.

d) y^* is the unique fixed point of Φ_i.

e) Φ_{is} maps C into C .

f) The sequence φ_{is} converges as $s \to \infty$.

g) φ_i is monotone, continuous and of Lipschitz class with ratio $\eta \leq \gamma$.

To show a) one need merely turn to the maximization process itself. When $s=1$, the firm seeks to maximize present period profits only:

$$\max_{x_{it}} p_i(x_t) = p_i(\psi_i(x_{t-1}), \ldots, \psi_{i-1}(x_{t-1}), x_{it}, \psi_{i+1}(x_{t-1}), \ldots, \psi_n(x_{t-1})) .$$

The first-order condition is

$$G_{i1}(x_t) = p_i^i(x_t) = 0 = p_i^i(\psi_1(x_{t-1}), \ldots, \psi_{t-1}(x_{t-1}), x_{it}, \psi_{i+1}(x_{t-1}), \ldots, \psi_n(x_{t-1}))$$

which is a maximum, as $p_i^{ii}(x_t) < 0$ where $p_i^i(x_t) = 0$.

$x_{it} = \varphi_i(x_{t-1})$ is defined implicitly by the condition $p_i^i = 0$.

When $s = 2$, the firm seeks to maximize as follows:

$$\max_{x_{it}} p_i(x_t) + \alpha_i' p_i(x_{t+1}) \ , \quad \text{where} \quad x_{t+1} = \Phi_{i1}(x_t) \ ,$$

for the nature of optimal behavior for the firm in the <u>last</u> period of its horizon is independent of whether the last period is the t^{th} or some other. The first-order condition is:

$$G_{i2}(x_t) = p_i^i(x_t) + \alpha_i' p_i^i(\Phi_{i1}(x_t))\varphi_{i1}^i(x_t) + \alpha_i' \sum_{j\neq i} p_i^j(\Phi_{i1}(x_t))\psi_j^i(x_t) = 0$$

$$= p_i^i(x_t) + \alpha_i' G_{i1}(\Phi_{i1})\varphi_{i1}^i(x_t) + \alpha_i' \sum_{j\neq i} p_i^j(\Phi_{i1})\psi_j^i(x_t) = 0 \ .$$

However, as Φ_{i1} has been defined by the condition $G_{i1}(\Phi_{i1}) = 0$:

$$G_{i2}(x_t) = p_i^i(x_t) + \alpha_i' \sum_{j\neq i} p_i^j(\Phi_{i1})\psi_j^i(x_t) = 0 \ .$$

$G_{i2} = 0$ implies $p_i^i < 0$, for $\alpha_i' \Sigma_{j\neq i}\, p_i^i \psi_j^i$ is necessarily positive; hence, $G_{i2}^i < 0$ and $G_{i2} = 0$ corresponds to a maximum.

In general:

$$G_{is}(x_t) = p_i^i(x_t) + \alpha_i' G_{i,s-1}(\Phi_{i,s-1})\varphi_{i,s-1}^i(x_t) + \alpha_i' \sum_{j\neq i} p_i^j(\Phi_{i,s-1})\psi_j^i(x_t)$$

$$= p_i^i(x_t) + \alpha_i' \sum_{j\neq i} p_i^j(\Phi_{i,s-1})\psi_j^i(x_t) \ .$$

Again, $G_{is} = 0 \Longrightarrow p_i^i(x_t) < 0 \Longrightarrow G_{is}^i < 0 \Longrightarrow G_{is} = 0$ is a maximum.

Item b) is a consequence of Proposition 3. It is known that if $\varphi_{i,s-1}^j(x) > 0$ and $\Sigma_{j=1}^n \varphi_{i,s-1}^j(x) \leq \gamma < 1$ $(x \in \mathring{C}_0)$, then $G_{is}^i(x) < 0$, $G_{is}^j(x) > 0$, $\Sigma_{j=1}^n\, G_{is}^j(x) < 0$ $(x \in \mathring{C}_0$ and $x_i \geq h_i(x_1, \ldots, x_{i+1}, x_{i+1}, \ldots, x_n))$. That $G_{is} = 0$ $(s > 1)$ implies $x_i \geq h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. It is known that the conditions on the G^i_{is} hold for $s = 1$. If these conditions hold for s, then the conditions on the φ^j_{is} hold for s:

$$\varphi^j_{is}(x_{t-1}) = \frac{\sum_{k \neq i} G^k_{is}(x_t)\psi^j_k(x_{t-1})}{-G^i_{is}(x_t)} .$$

Thus, by induction, b) holds for all s.

Consider $G_{i,s+1}$:

$$G_{i,s+1}(x_t) = p^i_i(x_t) + \alpha'_i \sum_{j \neq i} p^j_i(\psi_1, \ldots, \psi_{i-1}, \varphi_{is}, \psi_{i+1}, \ldots, \psi_n)\psi^i_j(x_t)$$

where the argument of the ψ_j and φ_{is} is x_t and

$$x_t = (\psi_1(x_{t-1}), \ldots, \psi_{i-1}(x_{t-1}), x_{it}, \psi_{i+1}(x_{t-1}), \ldots, \psi_n(x_{t-1})) .$$

Choose x_{t-1} arbitrarily in $\overset{\circ}{C}$ and choose $x_{it} = \varphi_{i,s-1}(x_{t-1})$ (i.e., so that $G_{i,s-1}(x_t) = 0$). Surely if $\varphi_{is}(x_{t-1}) > \varphi_{i,s-1}(x_{t-1})$, then values of x_{it} and x_{t-1} such that $G_{is} = 0$ imply $G_{i,s+1} > 0$ (because $p^{ji}_i > 0$); and if x^*_{it} is chosen so that $G_{i,s+1} = 0$ (given x_{t-1}), then $x^*_{it} > x_{it}$ (because $G^i_i < 0$). Thus if $\varphi_{is}(x_{t-1}) > \varphi_{i,s-1}(x_{t-1})$ then $\varphi_{i,s+1}(x_{t-1}) > \varphi_{i,s}(x_{t-1})$. It was seen earlier that $\varphi_{is}(x_{t-1}) > \varphi_{i1}(x_{t-1})$ $(s = 2, \ldots)$, so c) is proved by induction.

Item d) is easily shown. The fixed point of Φ_{i1} is in B_i. Let z be that point on the diagonal of C for which $p_i^i(z) = 0$. By continuity of p_i^i it exists because $p_i^i(\underline{x}) > 0$ and $p_i^i(y^*) < 0$. It is known that $\psi_j(z) > z_j$ $j \neq i$ because z is below y^* on the diagonal of C; hence $\varphi_{i1}(z) > z_i$. On the other hand, $\varphi_{i1}(y^*) < y_i^*$ (because $p_i^i(y^*) < 0$). Thus the cube $C^* \subset C$, whose origin is z and side is $y_i^* - z_i$, is mapped into itself by Φ_{i1} and therefore contains the unique fixed point of Φ_{i1}. If the Φ_{is} have fixed points in C (i.e., in B_i), then they are successively higher. If β_s is the fixed point of Φ_{is}, then $\beta_s \ll \beta_{s+1}$ ($\beta_{js} < \beta_{j,s+1}$ for all j). It is seen that:

$$G_{is}(y^*) < 0 .$$

Consider

$$G_{i2}(y^*) = p_i^i(y^*) + \alpha_i' \sum_{j \neq i} p_i^j(y_1^*, \ldots, y_{i-1}^*, \varphi_{i1}(y^*), y_{i+1}^*, \ldots, y_n^*)\psi_j^i(y^*) .$$

$G_{i2}(y^*) < 0$ because $\varphi_{i1}(y^*) < y_i^*$. The latter holds because G_{i1} declines along elements of B_i, $G_{i1}(y^i) = 0$ and $y^i \ll y^*$, $(y^i = \beta_1)$. Thus $\beta_{s-1} \ll y^*$ implies $G_{is}(y^*) < 0$ which implies $\beta_s \ll y^*$. Clearly y^* is the limit of the β_s. It is obviously an upper bound; however, given a $\beta_s \ll y^*$, its successor, β_{s+1} , will be $\beta_s \ll \beta_{s+1} \ll y^*$. Hence for any element, β^* , of B_i such that $\beta_1 \ll \beta^* \ll y^*$ there is an $s > 0$ such that $\beta^* \ll \beta_s$.

Knowing that $\varphi^j_{is} > 0$, $\Sigma^n_{j=1} \varphi^j_{is} \leq \gamma$, $\varphi_{i,s-1}(x) < \varphi_{i,s}(x)$ and that $\varphi_{is}(y^*) < y^*_i$ establish that Φ_{is} maps C into C . It also establishes the convergence of φ_{is} . The conditions b) on the φ^k_{is} imply that the φ_{is} are of Lipschitz class with ratio $\eta \leq \gamma$ (i.e., $|\varphi_{is}(x) - \varphi_{is}(y)| \leq \eta d(x,y)$ for $x, y \in C$). In addition the limit of the φ_{is} coincides with the upper envelope of the φ_{is} because they are a montone sequence. The limit, φ_i , is finite for at least one point, $\varphi_i(y^*) = y^*_i$. A proposition in Choquet [3, page 136] shows that under these conditions φ_i is everywhere finite and of Lipschitz class $\eta \leq \gamma$ (which means φ_i is also continuous). It remains now only to show that φ_i is monotone. Let x , $y \in C$ and $x_i \leq y_i$ for all i . It must be proved that $\varphi_i(x) \leq \varphi_i(y)$. It is known that $\varphi_{is}(x) \leq \varphi_{is}(y) < \varphi_i(y)$ for all y ; therefore, $\varphi_i(x) \leq \varphi_i(y)$.

It is tempting to call φ_i the reaction function for an infinite horizon; however, it would not be quite correct to do so. The method of calculating the φ_{is} is one of working backward from a finitely distant terminal period. It is obviously a legitimate technique for finite s, and the convergence question is important and interesting; for intuition suggests that while behavior today might well be affected by the length of the planning horizon, the effect of extending the horizon by one period should surely diminish as the horizon lengthens, and become arbitrarily small as the horizon becomes arbitrarily large. Intuition also suggests that behavior with an infinite horizon should coincide with the limit of behavior for a finite horizon, s, as $s \to \infty$. Very likely φ_i does coincide with the reaction function for an infinite horizon; however, this point is not proved in the present paper.

2.4 Optimality for the Industry

If one had a transformation Ψ which fulfilled all of the conditions assumed in Proposition 5 and for which $G_i(y^*) = 0$ for all i, ψ_i would be optimal for the i^{th} firm (for all i), in the sense of Proposition 5 (alluded to in the first section): If one calculates the φ_{is}, which are profit maximizing reaction functions for an horizon of s periods, it is seen that as $s \to \infty$, φ_{is} tends to a limit, φ_i. This limit, φ_i, with ${}_i\Psi$, gives rise to the same fixed point, y^*, as does Ψ and φ_i is monotone in each argument, continuous and of Lipschitz class with ratio $\eta \leq \gamma$. Therefore ψ_i has the approximate optimality property:

It is optimal at y^*. It has been shown that there exist transformations, Ψ, which satisfy the assumed condition for a single firm, but not for all firms simultaneously.

Proposition 6: There exist transformations, Ψ, which are equilibria in the sense of Proposition 5 for each firm simultaneously.

Choose the cube C so that x^c, the Cournot point, is on the diagonal (i.e., $x_i^c - \underline{x}_i = x_j^c - \underline{x}_j$ for all i, j). Let $Z = \{z\}$ be the set of diagonal points of C in the closed interval $[x^c, \bar{x}]$.

Let Ψ^* be a transformation satisfying assumptions A9, A10 and A11, whose domain is the cube with origin $(\underline{x}_1 - \delta, \ldots, \underline{x}_n - \delta)$ and side 2δ, with fixed point x^c. There is a least upper bound $z^* \in Z$ such that:

$$G_i(z) = p_i^i(z) + \alpha_i^* \sum_{j \neq i} p_i^j(z)\psi_j^{*i}(x^c) > 0$$

for $i = 1, \ldots, n$ and all $z \in Z$ such that $z_i < z_i^*$. The α_i^* are the values determined according to Proposition 3.

Now choose an arbitrary $y^* \in Z$ for which $x_i^c < y_i^* < z_i^*$. Let Ψ be defined as follows:

$$\Psi(x) = \Psi(x_1 - \beta, x_2 - \beta, \ldots, x_n - \beta) \quad \beta = y_i^* - x_i^c \quad x \in C .$$

Finally, choose α_i' :

$$\alpha_i' = \frac{-p_i^i(y^*)}{\sum_{j \neq i} p_i^j(y^*)\psi_j^i(y^*)} \qquad i = 1, \ldots, n$$

Ψ and the α_i' meet all the conditions of Proposition 5 for all i ; hence Ψ is an equilibrium set of reaction functions.

3. Concluding Comments

The result of this paper is the demonstration of conditions under which one may prove the existence of a sort of equilibrium. The equilibrium is described by a transformation Ψ which is a set of reaction functions, one of each firm, and a point y^* which is the fixed point of Ψ . y^* is a stable noncooperative equilibrium in the sense that a) Ψ is a contraction, therefore $x_t = (\Psi(x_{t-1})) \to y^*$ as $t \to \infty$, irrespective of the initial vector x_0 , b) if the i^{th} firm calculates φ_i , the limit of the φ_{is} (φ_{is} is optimal for the firm when its horizon is s periods), then $\Phi_i = (\psi_1, \ldots, \psi_{i-1}, \varphi_i, \psi_{i+1}, \ldots, \psi_n)$ is a contraction which also has y^* as a unique fixed point.

This equilibrium overcomes several objections frequently levied at formulations of the oligopoly problem: a) the firms maximize a discounted profit stream, rather than current period profits, b) optimal behavior for the firm can be found, in equilibrium, without forcing the firms to make incorrect assumptions about rival behavior and c) within the limits of price variation oligopoly, the demand, cost and reaction functions are fairly general in form.

This paper leaves untouched many very important problems. a) It does not propose, or give the properties of, a dynamic adjustment process for firms which start with no notion of rival behavior patterns, and which periodically estimate these patterns from available data. b) No attempt is made to incorporate uncertainty about rival reaction functions or the firm's own profit function. c) The analysis does not account for additional variables. Firms could make price and output decisions and carry inventories. Advertising and product design might be introduced. d) There is also the large, important question of the objective function. If the remuneration of the decision makers depends on, say, both sales and profits, firms might act so as to maximize a discounted stream of a weighted sum of both.

REFERENCES

1. Bartle, Robert G., The Elements of Real Analysis, Wiley (New York, 1964).

2. Chamberlin, Edward H., The Theory of Monopolistic Competition, 7^{th} ed., Harvard (Cambridge, 1956).

3. Choquet, Gustave, Topology, Academic Press (New York, 1966).

4. Cournot, Augustin A., Researches into the Mathematical Principles of the Theory of Wealth, translated by N. T. Bacon, Kelley (New York, 1960).

5. Nash, John F., "Two Person Cooperative Games," Econometrica, XXI, 1953, pp. 128-140.

6. ________, "Non-cooperative Games," Annals of Mathematics, LIV, 1951, pp. 286-295.

FOOTNOTES

[1] It will be neglected that Cournot applied his concept to a quantity model. The spirit of his accomplishment is equally applicable to a differentiated products price model, which is the context in which it shall be applied here.

[2] In the interest of concise notation, partial derivatives will be denoted by subscripts, as follows:

$$\partial p_i/\partial x_j = p_i^j , \qquad \partial^2 p_i/\partial x_j \partial x_k = p_i^{jk} , \qquad \text{etc.}$$

The indices j and k are integers denoting the j^{th} and the k^{th} arguments of p_i, respectively. Where no confusion is likely to arise, time subscripts will be omitted.

[3] The definitions and results of analysis upon which this paragraph draws may be found in a standard reference such as Bartle [1].