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Part VII: The Nonsymmetric Game: The Generalized Beat-the-Average Solution

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by

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1. The Symmetric Game: Beat-the-Average

In previous papers we have considered an especially violent form of market behavior which in two-person games has been called the "maximize the difference" solution and in general n-person games the "beat-the-average" solution. Suppose that there are n players, each player i with a payoff function $P_i(s_1, s_2, \dots s_n)$. Limiting ourselves to $n = 2$ we may consider each player as trying to maximize the difference between his payoff and that of his competitor. This yields:

$$\max_{s_1} \left\{P_1(s_1, s_2) - P_2(s_1, s_2)\right\}$$

$$\max_{s_2} \left\{P_2(s_1, s_2) - P_1(s_1, s_2)\right\}$$

or

$$\max_{s_1} \min_{s_2} \left\{P_1(s_1, s_2) - P_2(s_1, s_2)\right\} . \tag{1}$$

The natural generalization of this behavior to situations involving more than two players is given by:

$$\underset{s_i}{\text{maximize}} \left\{P_i(s_1, s_2, \dots, s_n) - \frac{1}{n-1} \sum_{j \neq i} P_j(s_1, s_2, \dots, s_n)\right\} . \tag{2}$$

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This is the "beat-the-average" solution.

If the game is symmetric and the players are highly competitive this type of behavior is conceivable. In experimental gaming exercises when monetary or other stakes are not sufficiently high, the players may convert the game into one of beat-the-average. When the game is a symmetric market game it can be shown that the beat-the-average and the efficient production solutions coincide. When the game is not symmetric this is no longer the case.

If the game is not symmetric, unbridled beat-the-average behavior does not appear to be reasonable, especially when some firms may be inherently at such a disadvantage that this type of behavior would not only be hostile but suicidal.

There is a natural extension of beat-the-average to the non-symmetric case which in terms of interfirm comparison may serve as a benchmark for extreme competition. This solution we call "maximize the share of industry profits," or:

$$\text{(3)} \qquad \underset{s_i}{\text{maximize}} \frac{P_i}{\sum_j P_j}$$

It follows immediately from inspection of (2) and (3) that when all of the P_i are the same we have:

$$\text{(2')} \qquad \underset{s_i}{\text{maximize}}\ P(1 - \frac{n-1}{n-1})$$

which implies a symmetric maximum with each firm obtaining $\frac{1}{n}$ of the profits, which is the same as (3').

(3')
$$\underset{s_i}{\text{maximize}} \; \frac{P}{nP}$$

2. The Maximum Profit Share Solution

In conformity with the notation in previous papers 1/ let the profit of the i^{th} firm be given by:

(4)
$$\Pi_i = \beta w_i(V - p_i - \gamma(p_i-\bar{p}))(p_i-c_i) \;.$$

We omit advertising at this point.

w_i is the market share for the i^{th} firm,

c_i is the average cost of production for the i^{th} firm

and $\bar{p}$ is the weighted average of prices.

Call

(5)
$$G_i = \frac{\Pi_i}{\Sigma\Pi_j} \;.$$

Taking the derivative of G_i with respect to p_i we obtain:

(6)
$$\frac{\partial G_i}{\partial p_i} = \frac{\Sigma\Pi_j \dfrac{\partial\Pi_i}{\partial p_i} - \Pi_i \sum_j{}' \dfrac{\partial\Pi_j}{\partial p_i}}{(\Sigma\Pi_j)^2} = 0$$

or

(7)
$$\frac{\partial\Pi_i}{\partial p_i} - \frac{\Pi_i}{\Sigma\Pi_j}\left(\Sigma \frac{\partial\Pi_j}{\partial p_i}\right) = 0 \;.$$

We see that when $\Pi_i = \Pi$ for the symmetric case $\frac{\Pi_i}{\Sigma\Pi_j} = \frac{1}{n}$ which is the equivalent of the beat-the-average solution. Before we can finish our analysis of this solution we must digress and solve a related problem.

2.1. The Cooperation Parameter Model

If the game described above were symmetric we have noted that the factor $\frac{\Pi_i}{\Sigma\Pi_j}$ can be replaced by the constant $\frac{1}{n}$. We can formalize a closely related game which has an interpretation in terms of different levels of cooperation. Suppose we represent Player i's attitude towards average behavior by a parameter ρ_i . We may regard him as trying to maximize:

$$\Pi_i - \rho_i \, \Sigma\Pi_i \, , \qquad i = 1, \ldots, n \, . \tag{8}$$

When $\rho_i = -\infty$ this is the equivalent of joint maximization. When $\rho_i = 0$ this gives the noncooperative equilibrium. This solution by itself is worth investigating as it may be used to try to measure and explain the observed behavior. Its relation to the maximize market share will come about in the obtaining of an algorithm based upon treating the expression

$$\frac{\Pi_i}{\Sigma\Pi_j} = \rho_i(p_1, p_2 \cdots p_n) \tag{9}$$

as a constant, solving the resultant model and then iterating with new values for the ρ_i . We now must demonstrate that we can solve the model given by equations (8).

Differentiating (8) with respect to p_i we obtain:

$$\frac{\partial \Pi_i}{\partial p_i} = \rho_i \Sigma \frac{\partial \Pi_j}{\partial p_i} \tag{10}$$

for $i = j$

$$\frac{\partial \Pi_i}{\partial p_i} = \beta w_i(V - p_i - \gamma(p_i - \bar{p}) - (p_i - c_i)(1+\gamma(1-w_i)) \tag{11}$$

$$= \beta w_i(V + c_i(1+\gamma(1-w_i)) - (2(1+\gamma) - \gamma w_i)p_i + \gamma\bar{p})$$

for $i \neq j$

$$\frac{\partial \Pi_j}{\partial p_i} = \beta w_j(p_j - c_j)\gamma w_i \tag{12}$$

Summing we obtain:

$$\Sigma \frac{\partial \Pi_j}{\partial p_i} = \gamma\beta w_i \sum_{j\neq i} w_j(p_j - c_j) + \beta w_i \left[\gamma w_i(p_i - c_i) + V + (1+\gamma)c_i - 2(1+\gamma)p_i + \gamma\bar{p}\right] \tag{13}$$

$$= \gamma\beta w_i \, \Sigma w_j(p_j - c_j) + \beta \left[V + (1+\gamma)c_i - 2(1+\gamma)p_i + \gamma\bar{p}\right]$$

$$= \gamma\beta w_i(\bar{p} - \bar{c}) + \beta \left[V + (1+\gamma)c_i - 2(1+\gamma)p_i + \gamma\bar{p}\right]$$

Substituting this in (10) we may factor out βw_i and obtain

$$V + ((1+\gamma) - \gamma w_i)c_i - (2(1+\gamma) - \gamma w_i)p_i + \gamma\bar{p} \tag{14}$$

$$= \rho_i(V + (1+\gamma)c_i - 2(1+\gamma)p_i + 2\gamma\bar{p} - \bar{c})$$

Writing (14) more generally in matrix form, collecting terms and solving for ρ we have:

(15) $$\rho = [2(1+\gamma)(I-R) - \gamma W - \gamma(I-2R)SW]^{-1} [V(I-R)\hat{1} + ((1+\gamma)(I-R) - \gamma(I-RS)W)C]$$

call the first term on the right X^{-1} hence:

(16) $$X = \gamma(I-2R)\left[(I-2R)^{-1}\left\{\frac{2(1+\gamma)}{\gamma}(I-R) - W\right\}W^{-1} - S\right]W .$$

From the general relation we have

(17) $$(Y^{-1}-S)^{-1} = (Y + \left(\frac{1}{1-\Sigma y_i}\right) YSY) .$$

Define (18) $$Y^{-1} = (I-2R)^{-1}\left\{\frac{2(1+\gamma)}{\gamma}(I-R) - W\right\}W^{-1}$$

then (19) $$X^{-1} = \frac{1}{\gamma}W^{-1}(Y + \frac{1}{1-\Sigma y_i} YSY)(I-2R)^{-1} .$$

call (20) $$Z = \frac{2(1+\gamma)}{\gamma}(I-R) - W ;$$

then (21) $$X^{-1} = \frac{1}{\gamma}W^{-1}(WZ^{-1}(I-2R) + \frac{1}{1-2\gamma} WZ^{-1}(I-2R)SWZ^{-1}(I-2R))(I-2R)^{-1}$$

$$= \frac{1}{\gamma}Z^{-1} + \frac{1}{\gamma(1-\Sigma y_i)} Z^{-1}(I-2R)SWZ^{-1}$$

where $$z_i^{-1} = \frac{\gamma}{2(1+\gamma)(1-\rho_i) - \gamma w_i} ,$$

and $$y_i = \frac{\gamma w_i(1-2\rho_i)}{2(1+\gamma)(1-\rho_i) - \gamma w_i} .$$

The typical element of X^{-1} may be written as

(22) $$x_{i,j}^{-1} = \frac{\delta_{ij}}{2(1+\gamma)(1-\rho_i) - \gamma w_i} + \left(\frac{1}{1-\underset{k}{\Sigma} y_k}\right)\frac{\gamma(1-2\rho_i)w_j}{(2(1+\gamma)(1-\rho_i) - \gamma w_i)(2(1+\gamma)(1-\rho_j) - \gamma w_j)}$$

where δ_{ij} is the Kronecker delta.

Substituting X^{-1} in (15) we obtain the general solution for ρ. This does not have a particularly convenient form, but does provide a closed form expression for the evaluation of any particular case.

We consider a special case with symmetric $W = \frac{1}{n} I$ and $R = \rho I$. This means that the firms are symmetric in market share and in their views of the profit of the market as a whole. This gives:

(23) $$y_i = \frac{(\gamma/n)(1-2\rho)}{2(1+\gamma)(1-\rho) - \gamma/n} = \frac{\gamma(1-2\rho)}{2n(1+\gamma)(1-\rho) - \gamma}$$

and (24) $$\frac{1}{1-\Sigma y_i} = \frac{1}{1 - \frac{n\gamma(1-2\rho)}{2n(1+\gamma)(1-\rho) - \gamma}} = \frac{2n(1+\gamma)(1-\rho) - \gamma}{2n(1-\rho) + \gamma(n-1)} .$$

Thus (25) $$X^{-1} = \frac{n}{2n(1+\gamma)(1-\rho) - \gamma} I + \left(\frac{1}{1-\Sigma y_k}\right)\frac{n(1-2\rho)\gamma}{(2n(1+\gamma)(1-\rho) - \gamma)^2} S .$$

or (26) $$X^{-1} = \frac{n}{2n(1+\gamma)(1-\rho) - \gamma} I + \frac{n(1-2\rho)\gamma}{(2n(1-\rho) + \gamma(n-1))(2n(1+\gamma)(1-\rho) - \gamma)} S$$

Returning to the second term of (15), it may be written as:

(27) $$V(1-\rho)\hat{1} + ((1+\gamma)(1-\rho)c - \frac{\gamma}{n} c + \gamma\rho\bar{c}\hat{1})$$

$$= V(1-\rho)\hat{1} + \frac{1}{n}((n(1+\gamma)(1-\rho) - \gamma)c + \gamma\rho n\bar{c}\hat{1}) .$$

The value of ρ may now be calculated

(28) $$\rho_i = \frac{n}{2n(1+\gamma)(1-\rho) - \gamma}\left\{V(1-\rho)\left(1 + \frac{(1-2\rho)\gamma n}{2n(1-\rho) + \gamma(n-1)}\right) + \frac{n(1+\gamma)(1-\rho) - \gamma}{n} c_i\right.$$

$$\left. + \frac{[n(1+\gamma)(1-\rho) - \gamma](1-2\rho)\gamma}{2n(1-\rho) + \gamma(n-1)} \bar{c} + \gamma\rho\bar{c}\left(1 + \frac{(1-2\rho)\gamma n}{2n(1-\rho) + \gamma(n-1)}\right)\right\} .$$

or (29)

$$p_i = \frac{n(1-\rho)V}{2n(1-\rho) + (n-1)\gamma} + \frac{n(1+\gamma)(1-\rho) - \gamma}{2n(1+\gamma)(1-\rho) - \gamma}\, c_i + \frac{n\gamma\bar{c}}{2n(1-\rho) + (n-1)\gamma}\left[\frac{n + (n-1)\gamma}{2n(1+\gamma)(1-\rho) - \gamma}\right].$$

Equation (29) can be rewritten as:

(30)

$$\boxed{\rho_i = \frac{nV}{2n + \frac{n-1}{1-\rho}\gamma} + \frac{1}{2n(1+\gamma) - \frac{\gamma}{1-\rho}}\left[\left(n(1+\gamma) - \frac{\gamma}{1-\rho}\right) c_i + \frac{n\gamma(n + (n-1)\gamma)\bar{c}}{2n(1-\rho) + (n-1)\gamma}\right]}$$

We may verify the special cases:

$\rho = -\infty$	or joint maximum
$= 0$	noncooperative equilibrium
$= \frac{1}{n}$	beat-the-average.

As $\rho \to -\infty$ we have

(31)
$$\rho_i \to \frac{nV}{2n} + \frac{1}{2n(1+\gamma)}\left[n(1+\gamma)c_i + 0\right] = \tfrac{1}{2}(V + c_i) .$$

For $\rho = 0$ we have:

(32)
$$\rho_i = \frac{nV}{2n + (n-1)\gamma} + \frac{1}{2n(1+\gamma) - \gamma}\left[(n(1+\gamma) - \gamma)c_i + \frac{n\gamma(n + (n-1)\gamma)\bar{c}}{2n + (n-1)\gamma}\right]$$

$$= \frac{V}{2 + \frac{n-1}{n}\gamma} + \frac{1 + \frac{n-1}{n}\gamma}{2(1+\gamma) - \frac{\gamma}{n}}\left[c_i + \frac{\gamma\bar{c}}{2 + \frac{n-1}{n}\gamma}\right]$$

which is the noncooperative equilibrium for nonsymmetric costs.

Finally setting $\rho = \frac{1}{n}$ we obtain

$$p_i = \frac{nV}{2n + \frac{n-1}{1-\frac{1}{n}}\gamma} + \frac{1}{2n(1+\gamma) - \frac{n\gamma}{n-1}}\left[(n(1+\gamma) - \frac{n}{n-1}\gamma)c_i + \frac{n\gamma(n + (n-1)\gamma\bar{c}}{2(n-1) + (n-1)\gamma}\right]$$

$$= \frac{V}{2+\gamma} + \frac{1}{2(n-1)(1+\gamma) - \gamma}\left[((n-1)(1+\gamma) - \gamma)c_i + \frac{\gamma(n + (n-1)\gamma)\bar{c}}{2+\gamma}\right].$$

Setting $\bar{c} = c_i = c$ we obtain

$$\rho_i = \frac{1}{2+\gamma}\ [V + (1+\gamma)\bar{c}]$$

The solution of the cooperation parameter model gives the necessary computation method for the iteration procedure to calculate the maximum profit share solution. This is done in a computer program written by one of the authors.