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A NOTE ON THE ROLE OF MONEY IN PROVIDING SUFFICIENT INTERMEDIATION

David Cass and Menahem E. Yaari

September 23, 1966

A NOTE ON THE ROLE OF MONEY IN PROVIDING SUFFICIENT INTERMEDIATION *

by D. Cass and M. E. Yaari

I. Introduction

In his 1958 article on consumption loans Samuelson [5] showed that the dynamic competitive equilibrium involving only trade may not be Pareto optimal. Later Diamond [4] extended Samuelson's result, and showed that the dynamic competitive equilibrium involving both production and trade may not even be efficient. Both Samuelson and Diamond took the position that, because their models assume that the economy continues indefinitely, their results must be intimately related to the presence of "infinity."

To the contrary, in two recent papers we [2,3] have argued that invoking the presence of "infinity" to explain such phenomena is, while perhaps not wrong, certainly misleading. Our central position rests on the following argument: Suppose we have a dynamic model which is infinite and in which the competitive equilibrium is inefficient (and, a fortiori, not Pareto optimal). If sufficient intermediation is now admitted into the model, then the difficulty disappears; efficiency and, indeed, Pareto optimality are assured. This point was in fact recognized by Samuelson. What neither he nor Diamond did recognize was that this situation is not peculiarly associated with the presence of "infinity." In particular, we have demonstrated that it is possible to construct an analogous static model which is finite and in which the competitive equilibrium is efficient.

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and Pareto optimal again only when there exists sufficient intermediation (see Section X of [2]). What can be concluded? Simply that a more pertinent interpretation has to do with the function of markets in intermediating indirect trade between individuals when direct trade is not possible or desirable, for example, by means of the use of money.

The purpose of the present paper is to explore further the role of money in providing sufficient intermediation. For this purpose we again utilize a model similar to the simplest of Samuelson's two, as it appears that further complications involving such things as technology, preferences or population do not significantly alter our main conclusions. These are first, that with a fiat money (i.e., a money which is accepted in trade by virtue of convention) even positive equilibrium prices of money may not be sufficient ^{1/}, while by contrast, second, that with a commodity money (i.e., a money which is accepted in trade by virtue of intrinsic value) simply mere existence is sufficient.

II. The Model

People live for two periods, earning a fixed income of one unit of output in their first period and nothing in their second. Output is just a good which thus appears and which can be either costlessly stored or consumed. The other good in the model is money, which exists in a given stock M and which can be bought and sold for output at the nonnegative price P_t at time t .

The group of people born at time t consists of $(1+n)^t$ members, where the nonnegative number n is thus the population growth rate. This group will be referred to as generation t . The members of generation t are

^{1/} We have previously noted this result in a more elaborate model (see Section VII of [3]).

identical in all respects. Let C_t^1 be first period consumption, C_t^2 be second period consumption, X_t be first period inventory of output and M_t be first period stock of money for a member of generation t . Then the opportunities open to him can be summarized in the budget equations and nonnegativity constraints

$$(1) \quad \begin{aligned} C_t^1 + X_t + P_t M_t &= 1 \\ C_t^2 &= X_t + P_{t+1} M_t \\ C_t^1 &\geq 0, \quad C_t^2 \geq 0, \quad X_t \geq 0, \quad M_t \geq 0. \end{aligned}$$

Here we introduce the assumption which carries the present discussion beyond those cited above. Namely, a member of generation t is assumed to evaluate his opportunities according to a utility function depending on his first and second period consumptions and his first period stock of money

$$(2) \quad U(C_t^1, C_t^2, M_t) \text{ defined for } C_t^1 \geq 0, C_t^2 \geq 0 \text{ and } M_t \geq 0.$$

The utility function U is common to all generations. In addition to being twice differentiable, it exhibits the following properties: a) Positive and finite marginal utility of consumption:

$$(3) \quad 0 < \frac{\partial U}{\partial C_t^i} < \infty \text{ for } 0 < C_t^i < \infty, \quad i = 1, 2;$$

b) Generalized diminishing marginal rate of substitution:

$$(4) \quad \left\{ C_t^1, C_t^2, M_t : U \geq \text{const.}, C_t^1 \geq 0, C_t^2 \geq 0, M_t \geq 0 \right\} \text{ is a strictly convex set}$$

c) Bounded rate of time discount:

(5) for some $-1 < \rho < \infty$, if $\frac{\partial U}{\partial c_t^1} / \frac{\partial U}{\partial c_t^2} = 1$, then $\frac{c_t^1}{c_t^2} - 1 \leq \rho$.^{2/}

Regarding the marginal utility of money, we will consider two polar situations, first, that in which money never has intrinsic value or is a fiat money,

(6) $\frac{\partial U}{\partial M_t} = 0$,

and second, that in which money always has intrinsic value or is a commodity money,

(7) for some $\mu > 0$ $\frac{\partial U}{\partial M_t} \geq \mu$ for $M_t \geq 0$.^{3/}

What we have in mind for the latter is a nonreproducible and completely durable consumption good (i.e., an idealization of residential land or rare minerals).

To complete the model we assume that a member of generation t acts so as to maximize his utility subject to his opportunities. Applying a variant of the Kuhn-Tucker theorem to this constrained maximization problem (see Arrow and Enthoven [1]), and simplifying the result, thus leads to the marginal conditions describing the behavior of a member of generation t (besides the budget equations and nonnegativity constraints (1)):

^{2/} Notice that our notion of the rate of time discount ($\frac{c_t^1}{c_t^2} - 1$ given $\frac{\partial U}{\partial c_t^1} / \frac{\partial U}{\partial c_t^2} = 1$) simply reverses the roles of consumption and marginal utility in the standard definition of the rate of time preference ($\frac{\partial U}{\partial c_t^1} / \frac{\partial U}{\partial c_t^2} - 1$ given $\frac{c_t^1}{c_t^2} = 1$).

This assumption considerably simplifies the argument in the latter part of Section V. Though pretty innocuous as it stands, it is probably unnecessarily stringent.

^{3/} With a given stock of money and an indefinitely growing population, the intermediate situation in which money has intrinsic value only up to a point eventually reduces, for all practical purposes, to the second of these two polar situations.

$$(8) \quad \frac{\partial U}{\partial c_t^2} \leq \frac{\partial U}{\partial c_t^1}, \text{ with equality for } X_t > 0$$

and

$$(9) \quad \frac{\partial U}{\partial M_t} \leq P_t \frac{\partial U}{\partial c_t^1} - P_{t+1} \frac{\partial U}{\partial c_t^2}, \text{ with equality for } M_t > 0.$$

There are several possibilities for describing competitive equilibrium in the dynamic model just outlined, depending on the further specification of the beginning and the ending of the economy. Our purpose is best served by assuming that the economy has a definite beginning at period $t = 0$ but an indefinite ending with period $t \rightarrow \infty$. Then, at the start of period $t = 0$ we have an older generation -1 whose members have stored an inventory of output $X_{-1} \geq 0$ and have created (condition (6)) or collected (condition (7)) a stock of money $M_{-1} = (1+n)M$, and a younger generation 0 whose members have earned an income of output l . As the only market is that for money, while a member of generation -1 will sell his money at any nonnegative price, the equilibrium conditions in this economy are simply the market clearing relations,

$$(10) \quad (1+n)^t M_t \leq M, \text{ with equality for } P_t > 0,$$

for each period $t = 0, 1, \dots$.

We now turn to an investigation of the properties of this dynamic competitive equilibrium under the alternative assumptions of a fiat or a commodity money.

III. Equilibrium with a Fiat Money

To begin with suppose that the utility function (2) satisfies condition (6). The striking thing about equilibrium under this hypothesis is that, as noted in the introduction, it may be inefficient. This property follows from the fact that the necessary and sufficient condition for inefficiency in the economy

with fiat money is that

(11) for some $X > 0$ there is a $T < \infty$ such that $(1+n)X_t \geq X$ for $t \geq T$, an aggregate inventory of output be maintained indefinitely.^{4/} Thus, for example, it is easily seen that when the price of money is identically zero, there is an equilibrium which, because a member of generation t can save only by storing an inventory of output, is inefficient.

The central idea of this last example can be extended to cases where the price of money remains positive forever. Roughly speaking, the idea is simply that, with a given stock of fiat money, if the price of money is too low, then a member of generation t can enjoy an optimal lifetime consumption pattern only by storing both an inventory of output and a stock of money. For simplicity let us consider just the equilibria in which

$$(12) \quad P_t = P > 0 \text{ for } t = 0, 1, \dots,$$

the price of money is positive and constant. Faced with such prices, all generations behave identically, or the economy is stationary (so that we can ignore the time subscript). Moreover, by adding together the budget equations in (1), we see that a member of any generation can now consume only those combinations (c^1, c^2) satisfying the overall budget equation

$$(13) \quad c^1 + c^2 = 1 .$$

Hence, he will save precisely

$$(14) \quad 1 - c^1 > 0 ,$$

his second period consumption, and his optimality condition (8) must hold with equality, or, he will consume that combination $(c^1, 1 - c^1)$ for which

^{4/} A formal proof of this theorem is provided in the Appendix to [2].

$$(15) \quad \frac{\partial U(c^1, 1 - c^1)}{\partial c^1} = 0,$$

his marginal rate of substitution is exactly one. It follows directly, that, if

$$(16) \quad 0 < P M < 1 - c^1,$$

the price of money is such that a member of generation 0 must save in both forms, then

$$(17) \quad (1+n)^t X_t = (1+n)^t (1-c^1) - P M \geq 1 - c^1 - P M > 0 \quad \text{for } t = 0, 1, \dots,$$

an aggregate inventory of output is maintained forever. In other words, for any positive price of money satisfying condition (16), there is an equilibrium which is inefficient.

We remark for emphasis that none of the foregoing is peculiar to the present sort of model. For example, it is easy to construct a static model with fiat money (along the lines of the model analyzed in Section X of [2]) which is finite, but in which, even though the price of money (in terms of an arbitrarily chosen numeraire good) is positive, reasonable reserve constraints may entail that competitive equilibrium is inefficient.

To summarize the main point of this section : A fiat money may not provide sufficient intermediation to insure that competitive equilibrium yields an efficient allocation of resources. And, in particular, this result can occur even when the price of money in terms of goods is positive.

IV. Equilibrium with a Commodity Money

If we now drop condition (6) and assume instead condition (7) then an interesting result emerges. Namely, in the economy with commodity money the equilibrium must be efficient and, indeed, Pareto optimal.

To see this result, observe initially that such an economy will surely

be inefficient unless generation t holds the entire stock of money. However, provided it does so, condition (11) is again necessary and sufficient for inefficiency. Now, in equilibrium, generation t must in fact hold the entire stock of money (i.e., the market clearing relation (10) must hold with equality). For suppose that in some period t the price of money were zero. Then each member of generation t could demand an arbitrarily large amount of money, and would, by virtue of the assumption of condition (7), in general demand an amount greater than that available $(M/(1+n)^t)$. That is, there would be an excess demand for money, or, the money market would not be in equilibrium. Thus, we conclude that we need only concern ourselves with the second possibility for inefficiency of equilibrium.

Suppose then that, in equilibrium, aggregate inventories of output satisfy condition (11). Then it must be the case that

$$(18) \quad P_t > P_{t+1} \quad \text{for } t = T, T+1, \dots,$$

after period $t = T$ the price of money is always falling, as otherwise in some period $t \geq T$ the real rate of return on money $(P_{t+1}/P_t - 1)$ would be at least as large as the real rate of return on output (0) , and a member of that period's generation would clearly prefer to hold only the former because it also has intrinsic value. But as the decreasing sequence (18) is bounded below by zero, the price of money must converge to some nonnegative value P ,

$$(19) \quad \lim_{t \rightarrow \infty} P_t = P \geq 0.$$

Now consider what must characterize the optimal lifetime allocation of a member of generation $t \geq T$ given that (11), (18) and (19) prevail in equilibrium. On the one hand, (11) and (18) imply that the marginal conditions (8) and (9) describing his optimum are satisfied with equality, or, substituting from (8) into

(9), that

$$(20) \quad \frac{\partial U}{\partial M_t} = (P_t - P_{t+1}) \frac{\partial U}{\partial C_t^1} = (P_t - P_{t+1}) \frac{\partial U}{\partial C_t^2} .$$

On the other hand, (19) implies that, as the generation becomes sufficiently distant, his marginal utility of money (20) becomes arbitrarily small, or, explicitly applying (19) to (20), that

$$(21) \quad \lim_{t \rightarrow \infty} \frac{\partial U}{\partial M_t} = 0 . \quad \text{5/}$$

But our conclusion (21) is inconsistent with the assumption of condition (7), and we see that equilibrium in which an aggregate inventory of output is maintained forever is also impossible.

The contradictory result (21) can in fact be deduced from a weaker statement of condition (11), namely that

$$(22) \quad \text{for some } T < \infty \quad (1+n)^t X_t > 0 \quad \text{for } t \geq T ,$$

an aggregate inventory of output be maintained eventually, though perhaps not indefinitely. (It should be clear that this condition likewise entails that after period $t = T$ the price of money is always falling, which is the basic idea of the preceding paragraph.) What this means is simply that, in equilibrium, there is a sequence of periods, say, t_i for $i = 0, 1, \dots$ in which

$$(23) \quad (1+n)^{t_i} X_{t_i} = 0 ,$$

5/ (19) implies directly that

$$\lim_{t \rightarrow \infty} (P_t - P_{t+1}) = 0 .$$

By adding together the budget constraints in (1) we see that this in turn implies that

$$\lim_{t \rightarrow \infty} (C_t^1 + C_t^2) = 1 .$$

Hence, (19) also implies indirectly via assumption (3) that

$$\lim_{t \rightarrow \infty} \frac{\partial U}{\partial C_t^1} < \infty \quad \left(\lim_{t \rightarrow \infty} \frac{\partial U}{\partial C_t^2} < \infty \right) .$$

the aggregate inventory of output is actually zero. We shall utilize this property to demonstrate that equilibrium in the economy with commodity money, in addition to being efficient, is also Pareto optimal.

V. Pareto Optimality with either Fiat or Commodity Money

In pursuit of the latter objective, it is worthwhile expanding the discussion to include all feasible distribution schemes -- rather than just the competitive distribution schemes considered thus far -- in an economy with either fiat or commodity money. A distribution scheme is still described simply by 1) the second period consumption C_{t-1}^2 allocated to each member of generation $t-1$ in period t , 2) the first period consumption C_t^1 and stock of money M_t allocated to each member of generation t in period t , and 3) the inventory of output X_t per member of generation t carried over from period t to period $t+1$, for $t = 0, 1, \dots$. It is now said to be feasible if it satisfies the recursive inventory equation

$$(24) \quad X_t = 1 + \frac{X_{t-1}}{1+n} - C_t^1 - \frac{C_{t-1}^2}{1+n}, \text{ with } X_{-1} \geq 0 \text{ given,}$$

the money supply equation

$$(25) \quad M_t = \frac{M}{(1+n)^t},$$

and the nonnegativity requirements

$$(26) \quad C_t^1 \geq 0, C_t^2 \geq 0, X_t \geq 0.$$

Notice, in particular, that we have maintained the assumption that each member of generation t is treated identically, in accordance with our primary concern with competitive distribution schemes. This means, among other things, that only the distribution of output, and not the distribution of money -- whether fiat or

commodity -- is of direct consequence in the ensuing discussion.

Our basic approach in analyzing Pareto optimality of the distribution of output is to think of the economy as if, over time, it transforms output (the "capital") into individual utility (the "final product"), and to ask under what conditions it does so efficiently. Now, we know that for many economic models the answer to such a question can best be formulated in terms of appropriate prices. This suggests that for the present model we might well look for a similar result. As we shall see, such an approach turns out to be quite fruitful.

What are the appropriate prices here? Well, noting again that "production" takes place over time, while the "final product" is individual utility, one should not be surprised to find that the relevant price in period t is

$$(27) \quad r_t = \frac{\partial U}{\partial c_{t-1}^1} / \frac{\partial U}{\partial c_{t-1}^2} - 1 ,$$

the marginal rate of substitution of a member of generation $t-1$ less one, say, the utility rate of interest.^{6/} That is, given a feasible distribution scheme, it is simply the terms on which individual utility today can be traded for individual utility tomorrow at the margin.

To verify that the presence of Pareto optimality can be described in terms of the utility rate of interest, consider first just the intragenerational

^{6/} Besides this notion of the rate of interest, there are at least two others which may be useful in the present model. One, the apparent rate of interest $c_{t-1}^2 / (1 - c_{t-1}^1) - 1$, we have utilized previously in [2]. The other, the real rate of interest $P_t / P_{t-1} - 1$, we have encountered earlier in Section IV.

Even given a competitive distribution scheme the three are not necessarily equivalent with commodity money (though they are with fiat money). For example, see the inequality (41) below.

(i.e., between a generation's first and second periods) distribution of output. Specifically, suppose that in some period $t+1$ $r_{t+1} < 0$. Then clearly, with the same output at his disposal, each member of generation t can improve his welfare by consuming less in his first period and more in his second (as costless storage of output is feasible). This possibility is depicted by the movement from point A on indifference curve I to point B on indifference curve II in Figure 1. Hence, we see that one necessary condition for Pareto optimality of the intragenerational distribution of output is that

$$(28) \quad r_{t+1} \geq 0 \text{ for } t = 0, 1, \dots, \text{I}$$

the utility rate of interest be nonnegative.

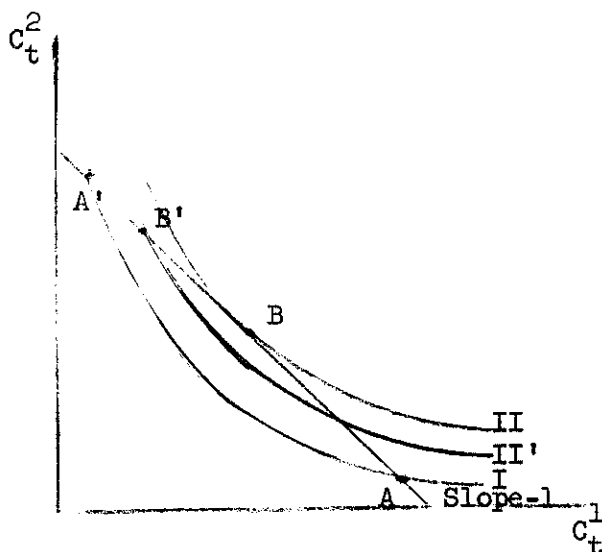


Figure 1

I/ Recall that we are assuming that first period consumption by generation -1 has already taken place. Thus the shift described above is not possible for that generation. A similar comment applies for the second necessary condition (29).

Now suppose that in some period $t+1$ $r_{t+1} > 0$ but $(1+n)^t X_t > 0$.

Here again each member of generation t can improve his welfare given the same output at his disposal, though now by consuming more in his first period and less in his second (as a reduction in the aggregate inventory of output is feasible). This possibility is depicted by the movement from point A' on indifference curve I to point B' on indifference curve II' in Figure 1. Thus, we find that a second necessary condition for Pareto optimality of the intragenerational distribution of output is that

$$(29) \quad \text{if } (1+n)^t X_t > 0, \text{ then } r_{t+1} = 0 \text{ for } t = 0, 1, \dots,$$

the utility rate of interest be zero when the aggregate inventory of output is positive.

It is easily seen that conditions (28) and (29) together are also sufficient; if these conditions are satisfied, then neither a shift from first period to second period consumption nor the opposite can increase (and, indeed, such shifts must decrease) the welfare of a member of generation t unless there is a simultaneous increase in the output at his disposal. These results can also be read off Figure 1. Notice too that conditions (28) and (29) together are equivalent to the marginal condition (8); individual maximizing behavior insures, at a minimum, Pareto optimality of the intragenerational distribution of output.

With respect to the intergenerational (i.e., between generations) distribution of output, matters are less obvious. What we would like to have is a condition, in terms of the utility rate of interest, which tells us when the following sort of redistribution is impossible: Assume that the original distribution scheme does not satisfy condition (11) (distribution of output is

efficient) while it does satisfy conditions (28) and (29) (intragenerational distribution of output is Pareto optimal). Also let the output left over for use by generation t after consumption by generation $t-1$, say, the intergenerational inventory of output in period t be

$$\begin{aligned}
 (30) \quad (1+n)^t Y_t &= (1+n)^t \left(1 + \frac{X_{t-1} - C_{t-1}^2}{1+n} \right) \\
 &= (1+n)^t \left(1 + \frac{Y_{t-1} - C_{t-1}^1 - C_{t-1}^2}{1+n} \right), \text{ substituting from equation (24)} \\
 &= Y_0 + \frac{(1+n)((1+n)^{t-1})}{n} - \sum_{s=0}^{t-1} (1+n)^s (C_s^1 + C_s^2), \text{ solving explicitly.}
 \end{aligned}$$

(Foregoing symmetry, we will continue to refer to the quantity $(1+n)^t X_t$ as the aggregate inventory of output in period t .) Now suppose that we increase the second period consumption and thereby the welfare of each member of generation -1 . This obviously requires a reduction in the intergenerational inventory of output in period $t = 0$, and therefore, everything else the same, in each period $t > 0$ by virtue of the definition (30). But suppose that we want to maintain the welfare of each member of generation 0 and all succeeding generations. If we attempt to do so by in fact continuing with everything else the same, then, because the original distribution of output was efficient, we find that

$$(31) \quad (1+n)^t X_t = (1+n)^t (Y_t - C_t^1) < 0 \text{ for some } t < \infty,$$

the aggregate inventory of output will become negative eventually, which is infeasible. In particular, this argument tells us that we will have to decrease the first period and increase the second period consumption of each member of generation t whenever the aggregate inventory of output in period t was

originally zero. Finally though, suppose that by such shifts in consumption for each member of some generations $t = t_j$ for $j = 0, 1, \dots, J \leq \infty$ we can actually maintain the welfare of all generations $t \geq 0$.^{8/} By supposition this redistribution is clearly better than the original distribution scheme.

Observe further, however, that because the original distribution of output satisfied condition (28), this redistribution requires an increase in the output at the disposal of each member of generation t_j just to maintain his welfare. That is, it requires an additional reduction in the intergenerational inventory of output in period $t_j + 1$. Thus, the essential question really is, when is it infeasible to raise this "extra" output, or to lower intergenerational inventories of output so? One useful answer is provided by the following theorem: Let the present value (i.e., value to a member of generation -1) of a unit of output in period t be

$$\begin{aligned}
 (32) \quad R_t^{-1} &= \frac{\partial U}{\partial c_{-1}^2} \prod_{s=1}^t (1+r_s)^{-1} \\
 &= \frac{\partial U}{\partial c_{-1}^2} \prod_{s=0}^{t-1} \left(\frac{\partial U}{\partial c_s^2} / \frac{\partial U}{\partial c_s^1} \right), \text{ substituting from the definition (27).}
 \end{aligned}$$

Then, given Pareto optimality of the intragenerational distribution of output, a sufficient condition for Pareto optimality of the intergenerational distribution of output as well is that

^{8/}It should be clear that the opposite shift is pointless except possibly when $r_{t+1} \geq n$. But, because the original distribution of output satisfied condition (29), if $r_{t+1} < n$, then the opposite shift is infeasible.

for some finite number $R < \infty$ there is a sequence of periods t_{j+1} for $j = 0, 1, \dots$.

$$(33) \quad \text{in which } R^{-1} r_{t_{j+1}} (1+n)^{t_{j+1}} \leq R,$$

the present value of a unit of output per member of generation t be occasionally bounded.

To interpret this condition in terms of the preceding discussion, note initially that it implies that there is some subsequence of periods $t_{j_k} + 1$ for $k = 0, 1, \dots$ in which

$$(34) \quad \text{both } R^{-1} r_{t_{j_k} + 1} (1+n)^{t_{j_k} + 1} \leq R \text{ and } r_{t_{j_k} + 1} \geq n.$$

(To see this, suppose otherwise, that is, that there is some $j' < \infty$ such that $r_{t_{j+1}} < n$ for $j \geq j'$.) Without loss of generality we can assume that the

subsequence coincides with the original sequence. Then, by virtue of condition (29), the condition (33) also implies that in each period $t = t_j$

$$(35) \quad (1+n)^t X_{t_j} = 0,$$

the aggregate inventory of output is zero. (In a moment we will argue that for competitive distribution schemes the implication also runs the other way, or more specifically, that because of the property (23), equilibrium in the economy with commodity money must be Pareto optimal.) Hence, when it is satisfied, any attempt at the sort of redistribution described above requires, roughly speaking, a measurable increase in the output at the disposal of each member of an arbitrarily

large number of generations. But, as is at least suggested by the last expression in (30), this in turn means that the intergenerational inventory of output will become negative eventually, which is infeasible. A precise formulation of this result is contained in the Appendix.^{2/}

Applying the foregoing particularly to competitive distribution schemes, consider first equilibrium in the economy with commodity money. We want to show that the properties of such an equilibrium, especially property (23), imply the sufficient condition for Pareto optimality (33) (recalling that we have already noted the equivalence of the marginal condition (8) with (28) and (29)). Thus we will concentrate attention on the behavior of equilibrium in the periods $t = t_1$ and $t = t_1 + 1$.

Now, in every period $t = t_1$, and for an arbitrary fraction $0 < \lambda < 1$, we have either $c_{t_1}^1 \geq \lambda$ or $c_{t_1}^1 < \lambda$. On the one hand, suppose that

$$(36) \quad c_{t_1}^1 \geq \lambda > 0.$$

^{2/} Another, though less accurate way of looking at the condition (33) is to assume that it is not satisfied, i.e., that

$$\lim_{t \rightarrow \infty} R_t^{-1} = \infty.$$

One should find plausible that in such a situation it actually may be possible to trade from the indefinite future to the present at an infinite price ("to get something for nothing").

We mention too that the condition that the present value of a unit of "capital" per head be occasionally bounded appears to be a fairly general guarantee that the "production" of "final product" is intertemporally efficient (given that there is a diminishing marginal rate of transformation between "final product" today and the same tomorrow). In particular, we have also derived this result within the context of the standard neoclassical growth model in [3].

We know from the assumption (5), the marginal condition (8) and the supposition (36) that

$$(37) \quad c_{t_i}^2 \geq \frac{1}{1+\rho} c_{t_i}^1 \geq \frac{\lambda}{1+\rho},$$

from the second period budget equation in (1) and the equilibrium property (23) that

$$(38) \quad c_{t_i}^2 = P_{t_i+1} M_{t_i},$$

and from the early part of the discussion in Section IV that

$$(39) \quad M_{t_i} = \frac{M}{(1+n)^{t_i}}.$$

Combining these results we find

$$(40) \quad \frac{(1+n)^{t_i}}{P_{t_i+1}} \leq \frac{M(1+\rho)}{\lambda} < \infty.$$

Furthermore, from the assumption (7) and the marginal condition (9) we see that

$$0 < \frac{\partial U}{\partial c_{t_i}^1} P_{t_i} - \frac{\partial U}{\partial c_{t_i}^2} P_{t_i+1}$$

or

$$(41) \quad \frac{\partial U}{\partial c_{t_i}^2} / \frac{\partial U}{\partial c_{t_i}^1} < P_{t_i} / P_{t_i+1}.$$

Utilizing (40) and (41) along with the definition (32), we can immediately derive the series of relationships

$$(42) \quad R_{t_i+1}^{-1} (1+n)^{t_i+1} = \frac{\partial U}{\partial c_{-1}^2} \prod_{s=0}^{t_i} \left(\frac{\partial U}{\partial c_s^2} / \frac{\partial U}{\partial c_s^1} \right) (1+n)^{t_i+1} <$$

$$\frac{\partial U}{\partial c_{-1}^2} \prod_{s=0}^{t_1} (P_s/P_{s+1}) (1+n)^{t_{i+1}} = \frac{\partial U}{\partial c_{-1}^2} \frac{P_0(1+n)^{t_{i+1}}}{P_{t_i+1}} \leq$$

$$\frac{\partial U}{\partial c_{-1}^2} \frac{P_0 M (1+n) (1+\rho)}{\lambda} < \infty .$$

On the other hand, suppose that

$$(36') \quad c_{t_i}^1 < \lambda < 1 .$$

Then, simply by using the first period rather than the second period budget constraint in (1), the argument which before yielded (40) now yields

$$(40') \quad \frac{(1+n)^{t_1}}{P_{t_1}} < \frac{M}{1-\lambda} < \infty .$$

Hence, utilizing in addition the marginal condition (8), the derivation in (42) becomes

$$(42') \quad R_{t_i+1}^{-1} (1+n)^{t_{i+1}} = \frac{\partial U}{\partial c_{-1}^2} \prod_{s=0}^{t_1} \left(\frac{\partial U}{\partial c_s^2} / \frac{\partial U}{\partial c_s^1} \right) (1+n)^{t_{i+1}} <$$

$$\frac{\partial U}{\partial c_{-1}^2} \prod_{s=0}^{t_1-1} (P_s/P_{s+1}) \left(\frac{\partial U}{\partial c_{t_i}^2} / \frac{\partial U}{\partial c_{t_i}^1} \right) (1+n)^{t_{i+1}} \leq \frac{\partial U}{\partial c_{-1}^2} \frac{P_0(1+n)^{t_{i+1}}}{P_{t_i}} <$$

$$\frac{\partial U}{\partial c_{-1}^2} \frac{P_0 M (1+n)}{1-\lambda} < \infty .$$

Finally, we see that if we let

$$(43) \quad R = \max \left[\frac{\partial U}{\partial C_{-1}^2} \frac{P_o M (1+n)(1+\rho)}{\lambda}, \frac{\partial U}{\partial C_{-1}^2} \frac{P_o M (1+n)}{1-\lambda} \right],$$

then (42) and (42') together imply the sufficient condition for Pareto optimality (33).

As emphasized previously, the latter conclusion leans heavily on the property (23). But it is easily seen that if this same property holds for equilibrium in the economy with fiat money, then it too must be Pareto optimal. Indeed, the only change necessary in the foregoing argument is the substitution of equality for inequality in (41), corresponding to the effect on the marginal condition (9) of the substitution of assumption (6) for assumption (7).^{10/} Hence, we have some idea of what, at a maximum, is required of fiat money in providing sufficient intermediation to insure Pareto optimality: The price of money should be "high enough" so that, occasionally, money alone will satisfy generation t's desire to save for its second period of life (a notion we shall formulate more precisely, given an additional assumption about behavior, shortly).

This last point prompts us to emphasize the obvious, namely, that the reason why a commodity but not a fiat money always provides sufficient intermediation is simply because the former but not the latter is intrinsically valuable to all. Thus, as we have argued earlier, now speaking heuristically, if the price of commodity money were too low (i.e., satisfied the inequalities

^{10/} Perhaps note worthy too is the fact that assumption (5) rules out the extreme case in which the property (23) holds, but the price of money is identically zero. In relation to this, see the argument at the beginning of the next section.

(18)), then it would be bid up because the commodity money is desired for its own sake, while if the price of fiat money is too low (e.g., satisfies the equation (12) and the inequality (16)), then it will just remain so.

VI. Classical Saving Behavior and the Price Level

In equilibrium in the economy with fiat money, if the price of money is ever positive, then it is always positive, and nondecreasing as well. (To see this, simply consider the maximizing behavior of a member of generation t faced with the prices either $0 = P_t < P_{t+1}$ or $P_t > P_{t+1} \geq 0$.) Nevertheless we know that the price of money may not be high enough to insure even efficiency, much less Pareto optimality. In light of the preceding results this naturally leads to the sort of question, when in fact is the price of money just high enough so that the property (23) holds, or, more specifically, so that money alone suffices for desired real saving in period t ?

Unfortunately, given only the assumptions about behavior made thus far, there is no unambiguous answer to this question; desired real saving, and therefore, say, the minimum required price of money in period t may increase or decrease when the real rate of interest increases. However, assume that the maximizing behavior of a member of generation t exhibits the further property that

$$(44) \quad \frac{\partial(1 - C_t^1)}{\partial\left(\frac{P_{t+1}}{P_t} - 1\right)} = - \frac{\partial C_t^1}{\partial\left(\frac{P_{t+1}}{P_t}\right)} \geq 0 ,$$

his desired real saving never decreases when the real rate of interest increases.

(This assumption, which expresses the saving behavior often associated with classical

macroeconomic theory, is illustrated in Figure 2.) Then a straightforward derivation of the minimum required price of money in period t is possible.^{11/}

Suppose that, in equilibrium, a member of generation t holds a zero inventory of output, or, from his first period budget equation in (1) and the market clearing condition (10), that his real saving is

$$(45) \quad 1 - c_t^1 = P_t M_t = P_t \frac{M}{(1+n)^t} .$$

Under this hypothesis we know, by virtue of the marginal conditions (8) and (9), that in period $t+1$ the utility rate of interest must be equal to the real rate of interest and greater than or equal to zero

$$(46) \quad r_{t+1} = \frac{P_{t+1}}{P_t} - 1 \geq 0 .$$

But this means, because of the assumption (44), that the real saving of a member of generation t (45) cannot be less than it would be if the real rate of interest were actually zero (14),

$$(47) \quad 1 - c_t^1 \geq 1 - c^1 ,$$

^{11/}The same result follows from the weaker assumption that the maximizing behavior of a member of generation t implies that

$$\frac{\partial c_t^2}{\partial \left(\frac{P_{t+1}}{P_t} \right)} \geq 0 ,$$

his second period consumption is never a Giffen good.

as is shown in Figure 2. Substituting from (46) into (44) we see immediately

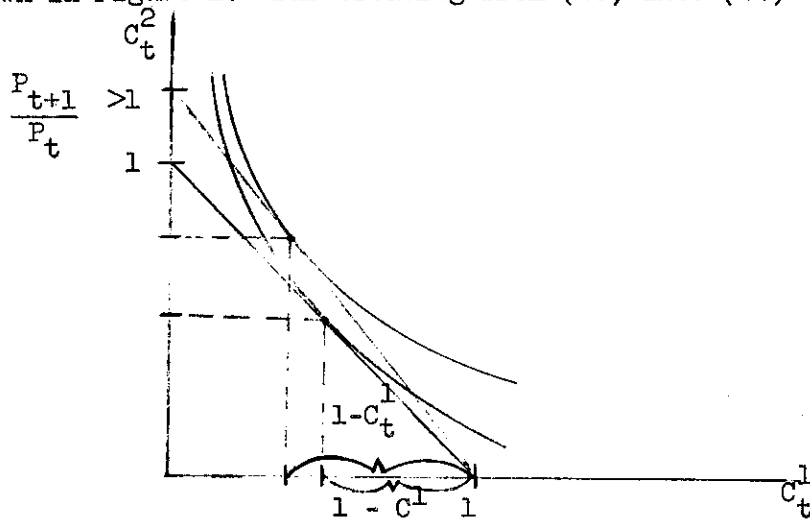


Figure 2

that a necessary condition for a zero aggregate inventory of output in period t is that

$$(48) \quad P_t \geq \frac{(1-C^1)(1+n)^t}{M} .$$

Suppose now that, in equilibrium, a member of generation t holds a positive inventory of output. Then, by reasoning along the lines of the preceding paragraph, we see that his real saving must be

$$1 - C^1 = X_t + P_t \frac{M}{(1+n)^t} > P_t \frac{M}{(1+n)^t} ,$$

or, that the price of money must satisfy the relationship

$$(49) \quad P_t < \frac{(1-C^1)(1+n)^t}{M} .$$

That is, we see that the condition (48) is also sufficient for a zero aggregate inventory of output in period t . Thus, given the assumption (44), we find

that the minimum required price of money in period t is simply

$$(50) \quad \frac{(1-C^1)(1+n)^t}{M},$$

the price which would equate the real money supply to desired real saving at a zero real rate of interest. Or, to put this result more generally, with classical saving behavior equilibrium in the economy with fiat money will be Pareto optimal if there is a sequence of periods, again say, t_i for $i=0, 1, \dots$ in which

$$(51) \quad P_{t_i} \geq \frac{(1-C^1)(1+n)^{t_i}}{M},$$

the price of money is at least as large as its minimum required price in period t (50).^{12/}

It may be of some interest to note briefly one implication of the condition (51). Now, nothing substantive is altered in the economy with fiat money if the money supply increases at some perfectly foreseen rate (for example, by means of a "government" transfer to the older generation $t-1$ in each period t). Moreover, what we have been referring to as the price of money is nothing more or less than the reciprocal of the more familiar notion of the price level. Hence, we see that (51) implies that a sufficient condition for both a roughly constant price level and Pareto optimality in distribution is that the money supply increase at the same rate as the population. But this is in fact only a specific formulation of the proposal for monetary policy usually associated with Milton Friedman (though here, we can't resist adding, with a

^{12/}As the condition (50) is equivalent to the property (23), it is a sufficient, though probably not a necessary condition for Pareto optimality.

rationale other than instinctive distrust of discretionary government power).

VII. A Final Comment

The model we have been discussing here drastically simplifies in order to exaggerate the role of money, and emphasize the contrast between the effects of a fiat money and a commodity money. However, at least one simplifying assumption does require further comment, namely, the assumption that commodity money has intrinsic value only during the first period of life. This could be relaxed, but only at the expense of considerable added complication. We have in mind, in particular, the fact that if the members of each generation desire to hold commodity money during their second period of life as well, then, at a minimum, some extra-market relations between generations are necessary (e.g., in the form of bequests). More importantly, the logic of our main point about commodity money does not appear to depend crucially on our present simplified treatment. But this is a worthwhile area for further investigation.

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Appendix

This appendix is devoted entirely to the proof of the following theorem:

THEOREM. Given any feasible distribution scheme described by (24), (25) and (26), if in each period $t+1 = 1, 2, \dots$

$$(52) \quad r_{t+1} \geq 0, \text{ with equality for } (1+n)^t X_t > 0,$$

and for some finite number $R < \infty$ there is a sequence of periods $t_j + 1$ for $j = 0, 1, \dots$ in which

$$(53) \quad R_{t_{j+1}}^{-1} (1+n)^{t_j+1} \leq R,$$

then that scheme is Pareto optimal.

Proof. As mentioned in the text, the essential idea of the proof is to demonstrate the contradiction that if there were a better feasible distribution scheme, then the intergenerational inventory of output for that scheme would become negative eventually.

Suppose then that the scheme which satisfies the hypotheses of the theorem is not Pareto optimal, and denote any better scheme with tilded variables. Then, by this supposition, we have the inequalities

$$(54) \quad U(\tilde{C}_t^1, \tilde{C}_t^2, \tilde{M}_t) - U(C_t^1, C_t^2, M_t) \geq 0 \text{ for } t = -1, 0, \dots, \text{ with strict inequality for some } t,$$

which, because of the assumptions of nonsatiation (3) and generalized diminishing marginal rate of substitution (4), imply the inequalities

$$(55) \quad (\tilde{c}_t^1 - c_t^1) + (1+r_t)^{-1}(\tilde{c}_t^2 - c_t^2) \geq 0 \quad \text{for } t=-1, 0, \dots, \text{ with strict inequality for some } t, \underline{13/}$$

The determination of the latter from the former, which is central to our argument, is illustrated in Figure 3.

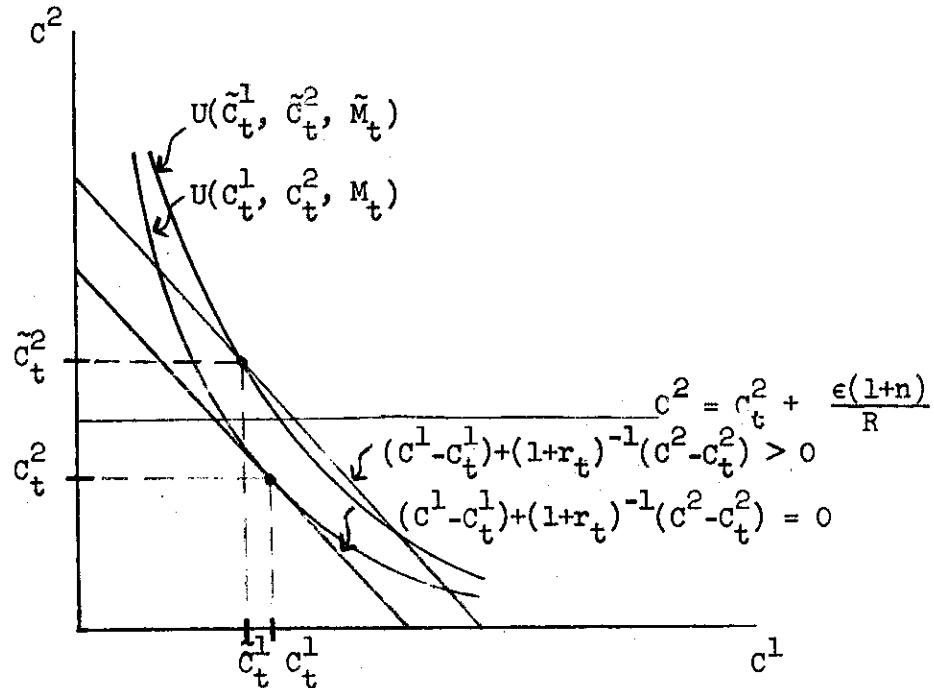


Figure 3

We now proceed to show that this supposition together with the hypotheses of the theorem implies that

13/

Recall that we assume that every feasible distribution scheme satisfies (25). This means that in the derivation of (55) from (54) the term $\frac{\partial U}{\partial M_t} (\tilde{M}_t - M_t)$ equals zero under both assumptions (6) and (7), and can thus be omitted.

$$(56) \quad \lim_{j \rightarrow \infty} \tilde{Y}_{t_{j+1}} = -\infty,$$

which clearly entails our desired conclusion. For this purpose we consider the present value of the difference between the intergenerational inventories of output in period t_{j+1} for the two schemes

$$(57) \quad \tilde{V}_{t_{j+1}} = R_{t_{j+1}}^{-1} (1+n)^{t_{j+1}} (\tilde{Y}_{t_{j+1}} - Y_{t_{j+1}}).$$

By utilizing the relationship (24), this present value can be rewritten

$$(58) \quad \begin{aligned} \tilde{V}_{t_{j+1}} &= \tilde{V}_{t_{j+1}} + \sum_{s=0}^{t_j} R_s^{-1} (1+n)^s \left(\left[1 + \frac{\tilde{X}_{s-1}}{1+n} - \tilde{C}_s^1 - \frac{\tilde{C}_{s-1}^2}{1+n} - \tilde{X}_s \right] - \left[1 + \frac{X_{s-1}}{1+n} - C_s^1 - \frac{C_{s-1}^2}{1+n} - X_s \right] \right) \\ &= - \sum_{s=-1}^{t_j} R_s^{-1} (1+n)^s \left([\tilde{C}_s^1 - C_s^1] + [1+r_{s+1}]^{-1} [\tilde{C}_s^2 - C_s^2] + r_{s+1} [\tilde{X}_s - X_s] \right) \\ &\leq - \sum_{i=0}^j R_{t_i}^{-1} (1+n)^{t_i} \left([\tilde{C}_{t_i}^1 - C_{t_i}^1] + [1+r_{t_i+1}]^{-1} [\tilde{C}_{t_i}^2 - C_{t_i}^2] \right) \leq 0, \end{aligned}$$

where the inequalities follow from (52) and (55). It is easily seen that from the expressions (57) and (58) we can derive the following bounds: a) By virtue of (26), (35) and (53)^{14/} that

$$(59) \quad -R_{t_{j+1}}^{-1} (1+n)^{t_{j+1}} \leq \tilde{V}_{t_{j+1}}$$

^{14/} In particular, (26) implies that

$$\tilde{Y}_{t_{j+1}} \geq \tilde{Y}_{t_{j+1}} - \tilde{C}_{t_{j+1}}^1 = \tilde{X}_{t_{j+1}} \geq 0$$

while (35) implies that

$$Y_{t_{j+1}} = 1 - \frac{C_{t_j}^2}{1+n} \leq 1.$$

and

$$(60) \quad R(\tilde{Y}_{t_{j+1}} - 1) \leq R(\tilde{Y}_{t_{j+1}} - Y_{t_{j+1}}) \leq \tilde{V}_{t_{j+1}} ,$$

and (b) by virtue of (55) that

$$(61) \quad \text{for some } \epsilon > 0 \text{ there is a } J < \infty \text{ such that } \tilde{V}_{t_{j+1}} \leq -\epsilon \text{ for } j \geq J.$$

But (59), (60) and (61) in turn imply the bounds

$$(62) \quad R_{t_j}^{-1} (1+n)^{t_j} \geq R_{t_{j+1}}^{-1} (1+n)^{t_j} \geq \frac{\epsilon}{1+n} \quad \text{for } j \geq J ,$$

$$(63) \quad \tilde{C}_{t_j}^2 - C_{t_j}^2 \geq (\tilde{C}_{t_j}^2 - \tilde{X}_{t_j}) - C_{t_j}^2 = (1+n)(Y_{t_{j+1}} - \tilde{Y}_{t_{j+1}}) \geq \frac{\epsilon(1+n)}{R} \quad \text{for } j \geq J ,$$

and

$$(64) \quad \tilde{Y}_{t_{j+1}} \leq \frac{1}{R} \tilde{V}_{t_{j+1}} .$$

Hence, finally, substituting from (58) and (62) into (64) and taking limits we find that

$$(65) \quad \lim_{j \rightarrow \infty} \tilde{Y}_{t_{j+1}} \leq -\frac{\epsilon}{R(1+n)} \lim_{j \rightarrow \infty} \sum_{i=J}^j ([\tilde{C}_{t_i}^1 - C_{t_i}^1] + [1+r_{t_{i+1}}]^{-1} [\tilde{C}_{t_i}^2 - C_{t_i}^2]) = -\infty ,$$

where the second limit follows directly from (63) and the relationships shown in Figure 3.