5-1-1966

Some Econometric Problems in the Analysis of Inventory Cycles

Kenneth F. Wallis

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 209

SOME ECONOMETRIC PROBLEMS IN THE ANALYSIS OF INVENTORY CYCLES

Kenneth F. Wallis

May 9, 1966

SOME ECONOMETRIC PROBLEMS IN THE ANALYSIS OF INVENTORY CYCLES*

Kenneth F. Wallis

CHAPTER 1

INTRODUCTION

During the postwar period, inventory changes have made a significant contribution to fluctuations in the level of economic activity. The change in business inventory investment accounted for 68 per cent of the reduction in gross national product during the contractions of 1948-49, 1953-54, 1957-58, and 1960-61, while during the expansions of 1949-53, 1954-57, and 1958-60 the increase in investment in stocks represented 13 per cent of the increase in GNP. Stanback [33] found that the increase in inventory investment relative to the increase in GNP tends to be largest in the early stage of an expansion; during the first year of four postwar expansions the increase in business inventory investment represented 37 per cent of the increase in GNP. The data on which these calculations are based are presented in Table 1. The greater part of the changes in nonfarm inventory investment is accounted for by movements in manufacturers' inventory investment, and within this latter category durable goods inventory investment plays the major role.

These simple comparisons do not, however, demonstrate the whole influence of inventory behavior on cycles in aggregate economic activity since they ignore the multiplier effect of inventory investment upon other cyclically variable expenditures. The model of Klein and Popkin [19] was designed to study this problem, and in general to ask how much stabilization in the aggregate economy would result from the stabilization of inventory fluctuations by given amounts. On the

* This paper reports on research carried out under a grant from the National Science Foundation.

TABLE 1

Changes in gross national product and business inventory investment between peaks and troughs of postwar cycles in aggregate economic activity.

	Billions of 1954 dollars seasonally adjusted quarterly data at annual rates		Change in inventory investment as a percentage of change in GNP
	Change in gross national product	Change in nonfarm inventory investment	
Contractions:			
1948,IV - 1949,IV	- 4.3	- 8.4	195
1953,II - 1954,II	- 13.7	- 7.5	55
1957,III - 1958,I	- 18.0	- 6.8	38
1960,II - 1961,I	- 7.9	- 7.1	90
Total	- 43.9	- 29.8	68
Expansions:			
1949,IV - 1953,II	80.2	9.5	12
1954,II - 1957,III	51.5	4.7	9
1958,I - 1960,II	49.3	9.2	19
Total	181.0	23.4	13
First year of expansions:			
1949,IV - 1950,IV	38.6	19.9	52
1954,II - 1955,II	30.0	9.2	31
1958,I - 1959,I	28.7	11.0	38
1961,I - 1962,I	34.9	9.1	26
Total	132.2	49.2	37

Sources: U.S. Income and Output, 1958; Survey of Current Business, July 1962, and July 1964.

basis of their simulation studies, Klein and Popkin found that a 25 per cent reduction in the amplitude of inventory fluctuations produced readily discernible stabilization effects throughout the economy, while a 75 per cent reduction virtually eliminated the business cycle. They estimated that a 50 per cent reduction in inventory fluctuations would have a great effect on the aggregate economic cycle, and they concluded that "there is real justification in calling the 1953-54 and 1957-58 cycles 'inventory cycles'." [19, p. 76].

When analyzing the behavior of inventories in the economy, it seems natural to apply the distinction, often drawn in other areas of economics, between macro and micro. The macro approach considers inventories and inventory investment at the level of an industry or the whole economy, while the micro, normative, approach usually discusses the problems of inventory control for an individual firm.

The macro approach is typified by the theoretical work of Metzler [25] and by the empirical work of, on the one hand, those using the National Bureau of Economic Research methodology for the study of business cycles, such as Abramovitz [1] and Stanback [33], and, on the other hand, those using regression techniques, such as Darling [6] and Lovell [21]. The micro problem of determining the optimal inventory policy for an individual firm under various assumptions concerning demands and costs is discussed by Arrow, Karlin, and Scarf [3]; further examples may be found in the literature of operations research and management science. The two approaches have recently been combined by Mills [26], who first analyzes the inventory and production decisions of individual firms in different market situations, and then applies his models to empirical data at the industry level.

In this essay we study some problems surrounding the use of regression techniques in analyzing aggregate inventory behavior. These are, in particular, the distinction between production "to order" and production "to stock" and the problem of their aggregation, the formation of sales expectations, the treatment of seasonality in the data, and the development of an alternative estimation procedure, based upon the techniques of spectral analysis of economic time series.

Chapter 2 gives a survey of published regression studies, concentrating on the inventory-sales relationship, the role of unfilled orders, and the use of sales forecasts. In Chapter 3 we present some theoretical results concerning the behavior of the firm, distinguishing between production to stock and to order, and we develop a regression equation which we later estimate. The problems in, and advantages of using seasonally unadjusted data are discussed in Chapter 4, and a spectral analytic approach to regression estimation is described. In Chapter 5 we present the results of this estimation of our equations, and some concluding remarks are presented in Chapter 6.

CHAPTER 2

A SURVEY OF EARLIER CONTRIBUTIONS

This chapter gives a limited survey of the regression studies of manufacturers' inventory behavior. The aspects emphasized are the inventory-sales relationship, the role of unfilled orders, the problems of sales expectations and seasonality in the data, and the part played by the speculative motive for holding inventories. The chapter concludes with a brief comment on statistical methods.

2.1. The inventory-sales relationship

Much of the econometric analysis of inventory behavior stems from the theoretical work of Metzler [25]. The most general model developed by Metzler incorporated the assumptions that firms have some desired inventory level which is a function of sales, and that demand leads production, which is a function of expected sales, this in turn being a function of past sales. Metzler demonstrated that when this system receives an exogenous shock, such as an increase in non-induced investment, the inventory mechanism generates cycles through its interaction with production and sales.

The most basic version of this acceleration principle can be written

$$H_t = a + bS_t$$

where H_t represents the level of inventories at the end of period t and S_t represents the volume of sales during period t. An equation

with inventory investment as the dependent variable can be obtained by taking first differences of this equation, and Smyth [32] estimated the parameters of the following equation:

$$\Delta H_t = \alpha_0 + \alpha_1 \Delta Y_t + \alpha_2 t + e_t$$

where Y represents gross national product. Smyth used annual national income data for the United States and Australia; in neither case was the trend coefficient significant.

Metzler's formulation of the acceleration principle has subsequently been modified to incorporate the flexible accelerator of Goodwin [8]. This assumes that a firm's adjustment of its inventory to the desired level is subject to a lag, thus only a partial adjustment is achieved in any one period. In similar fashion, Darling [5] argued that the Metzler formulation, which consists of a target stock-sales ratio, a term which compares the actual inventory level with the target, and a lagged response to this deviation, must be modified to include a speed of adjustment parameter, which reflects the various technical and organizational factors governing the changing of inventory levels. Accordingly, Darling postulated the following relation:

$$H = a + b(rS - H)_{-\tau} + e$$

where r is the desired inventory-sales ratio, b expresses the rate at which actual inventories approach desired inventories, and τ is an unspecified response lag. After experimentation with different values of τ, Darling estimated the following equation using quarterly data on U.S. manufacturing industries:

$$\Delta H_t = \alpha_0 + \alpha_1 S_{t-1} + \alpha_2 H_{t-2} + e_t .$$

Both these models omit certain aspects of the Metzler formulation, for instance the role of the expected, or anticipated, sales. A more complete exposition of the Metzler-Goodwin model is given by Lovell [20]. Considering the case of inventories of finished goods, Lovell first assumed that the firm's desired, or equilibrium, end-of-period inventory level, I_t^d, is a linear function of anticipated sales, $\hat{S}_t$, these anticipations being formed before the start of time period t. (We reserve discussion of the formation of sales expectations for Section 2.3 below.) Therefore

$$I_t^d = a + b\hat{S}_t .$$

The lag-adjustment assumption then implies that

$$I_t^p = \delta I_t^d + (1 - \delta)I_{t-1} ,$$

where I_t^p is the planned inventory level, and δ is a reaction coefficient. This inventory plan will be accomplished only if expected sales and actual sales are equal; more generally we have

$$I_t = I_t^p + \hat{S}_t - S_t .$$

Hence we see that actual inventory investment, $I_t - I_{t-1}$, can be regarded as consisting of planned inventory investment, $I_t^p - I_{t-1}$, and unintended inventory investment, $\hat{S}_t - S_t$ (or disinvestment, if $S_t > \hat{S}_t$). Combining these three relations gives

$$I_t = \delta a + (1 + \delta b)\hat{S}_t + (1 - \delta)I_{t-1} - S_t ,$$

and Lovell estimated this equation using quarterly data on durable goods and nondurable goods manufacturing after making an assumption about $\hat{S}_t$ which is discussed below.

The breakdown of actual inventory investment into planned and unintended inventory investment is equivalent to the condition that production plans are exactly achieved. For the following identity holds between production, P_t, sales, and finished goods inventories:

$$P_t + I_{t-1} \equiv I_t + S_t ,$$

and a similar identity holds for the planned levels of the relevant variables:

$$P_t^p + I_{t-1} \equiv I_t^p + \hat{S}_t ,$$

and it is clear that

$$P_t = P_t^p \text{ if and only if } \qquad I_t = I_t^p + \hat{S}_t - S_t .$$

Lovell attempted to relax this constraint by assuming that production plans are partly flexible, and may be changed in response to a discrepancy between actual and anticipated sales. Unintended inventory investment is then only a fraction of this discrepancy. This may be written

$$I_t = I_t^p + \lambda(\hat{S}_t - S_t) .$$

The two identities written above then imply that

$$P_t = P_t^p - (1 - \lambda)(\hat{S}_t - S_t) ,$$

hence the reduction in unintended inventory investment is achieved by

permitting an "unintended" deviation between planned and realized production. If the production inflexibility coefficient, λ, is equal to zero then production plans are completely flexible, and $I_t = I_t^p$. However this would negate a commonly stated motive for holding inventories, namely to prevent fluctuations in demand from immediately causing fluctuations in production. Lovell was unable to estimate λ, since its inclusion resulted in an underidentified equation in that the number of estimated coefficients was one less than the number of structural parameters, so he set $\lambda = 1$. In a more recent paper [23], however, Lovell <u>has</u> been able to obtain estimates of this coefficient, by using an alternative specification of the model together with data on actual anticipations, discussed below (Section 2.3).

Nevertheless, the rationale for the introduction of this parameter is not sufficiently clear. A small value of λ implies that I_t may approach I_t^p more closely than would otherwise be the case, once it is realized by the firm that sales expectations are not being met. However, once sales expectations have been revised, I_t^p no longer represents the appropriate inventory plan. If sales expectations may be revised during the period for which plans are drawn up, and production plans may be revised, possibly to a lesser extent, then it is not reasonable to suppose that inventory plans go unaltered. The appropriate planned inventory level is now a function of the revised sales expectations. It <u>is</u> reasonable to infer that the relevant production planning period is shorter than had been supposed, and that plans concerning all three variables are drawn up much more frequently than had been supposed. Thus the real consideration concerns the time

interval to which the available observations correspond. The t subscript used in the theoretical considerations described above relates to the firm's decision period, yet when the derived relationship is estimated using available data, the t subscript assumes a new meaning, namely the observation period of these data. The fact that Lovell [23] obtains small values of λ using quarterly data implies that the whole planning process based on the relations given above is undertaken more frequently than once a quarter, rather than that the quarterly production plans are themselves flexible.

An equation in which the end-period inventory level is the dependent variable can easily be transformed into an equation for inventory investment. This can be achieved by taking first differences of all variables in the equation, but since this procedure introduces serial correlation into the residuals of the equation, it is preferable to simply subtract the beginning inventory level (H_{t-1} or I_{t-1}) from both sides of the equation. Equally, an equation explaining finished goods inventories or inventory investment can be transformed into an equation with production as the dependent variable by means of the identity stated above. This identity is used in such a fashion by Johnston [16] and Modigliani and Sauerlender [28].

Mills [26] derives a production equation directly by considering a model of the profit-maximizing behavior of the firm. The profit function depends on the levels of sales, production, and inventories of finished goods, the price charged by the firm, and the costs of production, storage, stock-out, and production change. Expected profit is then computed by calculating the expected levels of sales, etc. given the distribution function for the stochastic demand curve. This expected

profit is then maximized with respect to price and production in order to derive the firm's decision rules for these two variables. Finally, some linear approximations to the cost functions and the assumption of a rectangular distribution for the stochastic element in the demand function result in linear decision rules for price and production, that for production containing terms in P_{t-1}, $\hat{S}_t$ and I_{t-1}. The assumption of rational expectations (discussed in Section 2.3 below) introduces a random error term to the equations, and these regression equations are then estimated using industry data.

2.2. The role of unfilled orders

The backlog of unfilled orders, or changes in unfilled orders, have been used as explanatory variables in the inventory equations of Darling [5], Lovell [20], and Terleckyj [34]; in all three cases the unfilled orders term is highly important, apparently representing a considerable modification of the basic sales-inventory relationship. Although the unfilled orders term was included in equations for total inventories, it is generally agreed that the main influence of this variable is upon inventories of purchased materials and goods in process. This is demonstrated by Eisner and Strotz [7] in their discussion of Lovell's work. Lovell estimated an equation for inventories of purchased materials and goods in process, which contained the unfilled order backlog, an equation for finished goods inventories, which did not, and an equation for total inventories, formed by adding together the previous two equations. Eisner and Strotz point out that Lovell's comparison of the coefficients of the unfilled orders term in the first and third

of these equations amounts to a test of the importance of omitting this term from the second (finished goods inventory) equation. They then show that the estimated coefficients obtained in the first and third regressions are consistent with a zero coefficient of the unfilled orders term in the second equation, were the order backlog to be included in that equation. An exact test would, of course, require a re-estimation of Lovell's second equation, this time including the unfilled orders term.

The consideration underlying the use of unfilled orders as an explanatory variable can be described as follows. To the manufacturer, unfilled orders represent a known demand to be met from future production, and changes in the rate of production in response to changes in the order backlog will almost immediately cause changes in the volume of goods in process and necessitate changes in the stock of raw materials. In addition Stanback, in his business cycle study [33], argues that the unfilled orders series reflects the tightness of the markets in which producers purchase raw materials, tight supply conditions leading firms to increase stocks of raw materials as far as possible so as to avoid any disruptions of production which shortages would cause. Clearly, the orders received by some firms are the orders placed by other firms, but, in Stanback's words, "the logic is apparent: high or rising unfilled order backlogs bring assurance of a high level of operation to the seller, but to the buyer they bring delays in delivery and problems of procurement. As a result, both seller and buyer find justification for high levels of purchased-materials stocks." [33, p. 48]

Zarnowitz [41] distinguishes between (a) firms (or industries) which produce to stock, where fluctuations in demand are borne by the

finished goods inventory and no backlog of unfilled orders exists, and (b) firms which produce to order, where fluctuations in demand are reflected in the volume of unfilled orders, and there is no inventory of finished goods, although stocks of raw materials will usually be present. Production to order will occur when the costs of meeting demand from future production are less than the costs of storing the finished product. The costs and risks attached to storage of finished goods will be high when one or more of the following conditions apply:

(i) the manufacturer is unable to predict the specific requirements of the customer, for example in the machine tools industry

(ii) the finished product is physically or economically perishable, for example unstorable chemical explosives, or women's fashion-wear

(iii) the demand for the product is highly unstable or sporadic, for example rails, which are sold infrequently in widely varying quantities to a few companies [41, p. 372].

The multiproduct firm will lie between the extremes of pure stock-production and pure order-production, and may also shift from one category to the other from time to time, although the nature of the costs involved will make such behavior relatively infrequent. In general, however, Zarnowitz characterizes durable goods manufacturing as an order-industry, and nondurable goods manufacturing as a stock-industry.

This distinction seems implicit in Mack's comment [24], pointing out that the total unfilled order series is dominated by the

machinery and transportation equipment industries, which constitute over 70 per cent of total outstanding orders and which also dominate rates of change. Given this consideration, Mack questions whether the order backlog is a meaningful explanatory variable in equations fitted to data aggregated over all manufacturing industries, or whether it is acting as a surrogate for other omitted effects, two possible candidates being market conditions in general (as discussed above) and the effect of stock feedback on the economy. To account fully for the feedback of inventory investment on the volume of sales would require a system of simultaneous equations; in this case the standard argument is that the least squares estimates of the coefficients of a single regression are subject to simultaneous equations bias.

Darling's original introduction of unfilled orders into his regression resulted from the conjecture that the inventory-sales ratio, r , presented in Section 2.1 above, is not constant but variable. A volume of new orders in excess of sales would lead to an increase in r , and Darling made r a function of the ratio of the change in unfilled orders to sales, lagged one period [5]. Darling subsequently concentrated on the production to order considerations presented above, since in durable goods manufacturing, characterized by Zarnowitz as an order-industry, "inventory investment is more closely associated in time with the receipt of the order, or more accurately with changes in the unfilled order backlog, than with the delivery (sale) of the goods to the buyers" [6, p.30]. The introduction of unfilled orders and their rate of change into a regression explaining total manufacturing inventory investment resulted in a reduction of the coefficients of the sales variable to

less than one half of their prior values.

In Chapter 3 we shall adopt the distinction between production to order and production to stock, and the problem of aggregating these two types of production into a single relation will be discussed in Section 3.3. Here we simply describe the difficulty presented by this aggregation.

In general, the data which are used for estimation purposes are published for a small number of quite broadly defined industries. For all industries within the durable goods sector and for certain non-durable goods industries, data are given on both unfilled orders and finished goods inventories. However, when an equation with finished goods inventory as the dependent variable is estimated, the sales data used relate not only to the stock-production sector but also to the order-production sector, which carries no finished goods inventories, accepting Zarnowitz's distinction. For example, Lovell's equation has sales, current and lagged, and lagged finished goods inventories as regressors, but it is not clear from the theory to what extent the sales of production to order firms can explain the finished goods inventories held by production to stock firms. Since sales data disaggregated in this fashion are not available, in Chapter 3 we shall use a "homogeneous" dependent variable, namely production, and give an explicit treatment of this aggregation problem.

2.3. Expectations and seasonality

The problem of obtaining data from which to estimate the parameters of an inventory-sales relationship which contains a term

in expected sales, which is not observed, has been treated in essentially three different ways. First, some work has been carried out using the results of questionnaire surveys of business anticipations. Modigliani and Sauerlender [28] used data from the quarterly forecasts of freight carloadings published by the Shippers' Advisory Boards of the Association of American Railroads, the Fortune Magazine survey of business intentions, and the Dun and Bradstreet business anticipations survey. In general these surveys gave incomplete coverage and poor forecasts of sales, the shippers' survey forecasts performing worse than a simple extrapolation of the recent past would have done. Recently, however, the United States Department of Commerce, Office of Business Economics, has made available data on anticipated sales and planned inventory investment for both a one-quarter and a two-quarter horizon obtained from a quarterly survey of manufacturers. In [23], Lovell gives a preliminary report of his researches with these data; in particular he suggests that firms are able to forecast their sales much more accurately than is usually assumed by inventory cycle theorists. The available series are relatively short, however, and Lovell's regressions are based on only 18 observations.

An alternative to the use of anticipations data is, without making assumptions about the precise way in which forecasts are made, to set up hypotheses concerning the results of the forecasting procedure. Lovell [22] hypothesized that the resulting forecast is a weighted average of the previous period's sales and actual developments, together with a random error term:

$$\hat{S}_t = \rho S_{t-1} + (1 - \rho)S_t + e_t , \qquad Ee_t = 0 .$$

This equation contains earlier assumptions as special cases. In Lovell's earlier paper, [20], it was assumed that $e_t = 0$ for all t. If, in addition to this, the "coefficient of expectations," ρ, is equal to one then we have the case known as naive expectations. With $e_t = 0$ and $\rho = 0$ we are in effect making no assumptions about forecasts, but simply using actual sales as a proxy variable for forecast sales. A value of ρ between zero and one is in accordance with Theil's empirical studies [36], which suggest that expectations have a systematic tendency to underestimate actual changes; this effect has been called the "regressiveness" of expectations.

When the random error term, e_t, is not identically zero then it is possible to assume that $\hat{S}_t$, as written above, is the source of the stochastic residual in a regression equation. In this case it is also necessary to assume that e_t is distributed independently of S_t and S_{t-1}. With $\rho = 0$ we have the unbiased, or "rational," expectations of Mills [26], and the above equation reduces to

$$\hat{S}_t = S_t + e_t ,$$

the assumption of the independence of e_t and S_t avoiding the errors-in-variables problem. (This last problem would exist if we were to write the observed variable, S_t, as the sum of the unobserved variable, $\hat{S}_t$, and the random error, e_t, assumed independent of $\hat{S}_t$.)

As a second alternative to the use of sales anticipations survey data, it is possible to make a specific assumption concerning the mechanism by which sales forecasts are generated. This procedure was adopted by Johnston [16], using quarterly, seasonally unadjusted

data. Johnston assumed that sales forecasts are generated by the following formula:

$$\hat{S}_t = S_{t-4} \left\{ 1 + \gamma \frac{S_{t-1} - S_{t-5}}{S_{t-5}} \right\}$$

thus the forecast for a given quarter is based on the same quarter of the previous year, together with the most recent estimate of the (long-run) change over the year. Johnston expected the γ-coefficient to be less than one, in accordance with the regressive effect in expectations noted above. This formula is a particular example of the general approach of predicting a variable using its observed past, and Johnston subsequently used a more general model, having obtained results from the use of the above formula in relations fitted to separate quarters which were not very satisfactory. This second approach was based on the method of exponentially weighted moving averages, or adaptive expectations, and necessitated the construction of an artificial $\hat{S}_t$ series, since the simple recursive formula for $\hat{S}_t$ involves S_{t-1} and $\hat{S}_{t-1}$ and hence cannot be substituted directly into a regression equation. Johnston used a form of exponentially weighted moving average which also permits trend adjustment and seasonal adjustment ratios to be employed.

The method of fitting a separate regression for each quarter has also been employed by Modigliani and Sauerlender [28]. An alternative procedure, which equally implies that the form of the estimated relationship varies from season to season, is to introduce seasonal dummy variables into a single equation fitted to the complete time series; this approach was also used by Johnston [16].

With the exception of Smyth [32], who used annual data, all the authors whose work we have described used data for time-periods shorter than a year in estimating their regressions. Lovell [20, 21, 22,], Darling [5, 6], and Terleckyj [34] used quarterly data which were seasonally adjusted; Stanback's business cycle study [33] also employed seasonally adjusted data. The general methodology which we apply to the problem of seasonality in regressions will be discussed in Chapter 4; here we finally note the position taken by Mills [26], which contrasts with that of Johnston and Modigliani and Sauerlender, mentioned in the previous paragraph. Mills estimated a single equation, without dummy variables, using seasonally unadjusted data, arguing that there is no reason why the form of the firm's decision rule should be changed when seasonal movements are present and that, from a decision theory point of view, it is not clear why seasonal fluctuations should be treated differently from fluctuations resulting from other factors [26, p. 194].

2.4. Price speculation

It is usual, when discussing business motives for inventory holding, to include price speculation as such a motive (see, for example, Arrow [3, Chapter 1], Eisner and Strotz [7], and Modigliani [27]). The basic reasoning is that if purchased materials prices, or production costs, or finished product prices are expected to rise at a rate sufficiently high to cover storage costs, then firms have an incentive to increase inventories of, in the first case, raw materials and, in the other two cases, finished goods. Equally, an expected fall in these costs or prices would lead to a reduction of inventories.

The main problem in obtaining evidence of price speculation is that of measuring price expectations. Since survey data on anticipated price changes are not available, actual price changes have been used in some studies on the grounds that profitable speculation results from accurate anticipation of actual changes, but these have not in general achieved significant results. For example, Lovell, summarizing the results of his paper [20], states that he finds "no evidence in support of the hypothesis that the magnitude of manufacturing inventories is influenced by speculation in stocks of purchased materials in advance of anticipated price changes." [21, p. 123] Similarly, Stanback [33] finds that price changes, though significant in themselves, have relatively little influence on inventory behavior. This result may occur, however, because firms change the composition of their inventories rather than their total stocks, as Lovell points out. Equally, firms may just be unsuccessful in anticipating price changes.

Nevertheless, it seems that we should not expect to find evidence of price speculation in aggregate data, simply because speculation is an individual phenomenon. Speculation is successful when an individual manages to outwit the crowd, in Keynes' phrase, and profits "from knowing better than the market what the future will bring forth" [17, p. 170]. To expect firms in the aggregate to exhibit evidence of speculative activity may simply represent an example of the fallacy of composition. For if all firms in an industry predict a rise in raw materials prices in the future, and hence increase their current purchases in advance of the expected price rise, then they will find that the current price will rise, thus removing the anticipated profits of advance buying. It may be that the predicted price is "accurate"

in the sense that this future price would actually obtain if firms acted in ignorance of the prediction. Whether or not this is so, the speculative activity of firms based on the predicted price will in general prevent the predicted price from being realized. We could, however, follow Grunberg and Modigliani [12] in applying Brouwer's Fixed Point Theorem to show that if the accurate prediction is public knowledge and hence is also known to the suppliers, who base their actions upon it, then it is conceptually possible for a predicted price to be realized. But this would require each firm to have complete knowledge of demand and supply relations and expectations functions, and our main point remains unchanged. This is that in generalizing from the behavior of a firm to the behavior of an industry, assumptions concerning the actions of an individual firm in competition with others in the same industry do not carry over. Thus it is not surprising that the observed prices for an industry, which is assumed by the aggregation process to consist of identically acting firms, do not show evidence of speculation.

2.5. Statistical methods

In this section are stated the standard results concerning the method of estimation used in almost all of the above-mentioned studies, namely ordinary least squares.

In regressions using time series data, the first consideration is whether the residuals are serially correlated. If they are, then ordinary least squares estimates, while still unbiased and consistent, are no longer efficient. Moreover, the estimates of the variance-covariance matrix of the estimated coefficients, and hence the

estimated standard errors of these coefficients, are biased. Watson [38] comments that "it is commonly believed that presence of serial correlation makes the variance estimates deceptively small." He then shows that while this underestimation is indeed the stronger tendency, it need not always apply, the determining factor being the relationship between the regression vectors and the characteristic vectors of the true variance-covariance matrix of residuals, $E \underline{e}\,\underline{e}'$, which we write as Γ. (The use of ordinary least squares implies the assumption that $E \underline{e}\,\underline{e}' = \sigma^2 I$, where I is the identity matrix.)

The usual test for serial correlation in the residuals, namely the Durbin-Watson statistic, tests for the presence of first order serial correlation. If the serial correlation is first order, then estimates which are asymptotically efficient can be obtained by applying least squares to the transformed variables $x_t' = x_t - \hat{\rho} x_{t-1}$ where $\hat{\rho}$ is the estimated first order serial correlation coefficient. Lovell [22] utilizes this procedure, in the single example which led to the use of the phrase "almost all" in the first paragraph. The estimated coefficients were "moderately affected by the transformation." The Durbin-Watson statistic does not test for serial correlation of higher orders, such as the fourth order serial correlation which might be expected if seasonally unadjusted quarterly data are employed. In such a case, estimation of the structure of the serial correlation is required in order to achieve efficient estimates. Essentially, we need to apply Aitken's generalized least squares, using an estimate $\hat{\Gamma}$ of the variance-covariance matrix of residuals.

Once the regression equation contains the lagged dependent variable among the regressors, however, (as do equations estimated by

Lovell, Darling and Mills) these conclusions do not hold. In the absence of serial correlation least squares estimates, although consistent, are subject to a small-sample bias. When the residuals are serially correlated, least squares estimates are inconsistent, thus not even the weakest of the desirable properties of estimators is achieved. Moreover, the Durbin-Watson statistic is not appropriate to this situation, as is discussed by Nerlove and Wallis [31]. An alternative estimation procedure, which overcomes some of these difficulties, will be presented in Chapter 4.

CHAPTER 3

RELATIONS BETWEEN PRODUCTION, FINISHED GOODS INVENTORIES, NEW ORDERS, AND UNFILLED ORDERS

We first discuss the firm's behavior with regard to production and finished goods inventories, considering the case of production to stock separately from the case of production to order. The aggregation of the two sectors is considered in Section 3.3. An hypothesis about firms' expectations of the future is formulated in Section 3.4, and the data used in estimation are described in Section 3.5.

3.1. Production and finished goods inventories in the case of production to stock

In the case of production to stock the firm's output is immediately transferred to the finished goods inventory, from which all demands are met. There is no backlog of unfilled orders, thus the volume of new orders is equal to the volume of shipments; we refer to both as "sales" in what follows. We have the basic identity that the end-period inventory of finished goods, I_t, is equal to the beginning inventory level, I_{t-1}, plus production during the period, P_t, less the volume of sales made from inventory during the period, S_t. Thus we have

$$(1) \qquad I_t \equiv P_t + I_{t-1} - S_t .$$

Our first assumption is the familiar Metzler hypothesis, a version of the acceleration principle, that the firm has a desired, or

equilibrium, level of inventories, I_t^*, which is a linear function of sales volume. Since the sales volume for a given period is not known when the firm's activities during that period are being planned, we substitute expected, or forecast, sales $\hat{S}_t$ into this relation. Throughout the following analysis we shall suppress constant terms, hence we may simply write

$$I_t^* = a\hat{S}_t \quad . \tag{2}$$

The formation of expectations is discussed in Section 3.4 below.

The implied volume of production necessary to achieve this inventory level now follows from the identity

$$P_t^* + I_{t-1} \equiv \hat{S}_t + I_t^* \quad . \tag{3}$$

We now assume that the firm finds it desirable to keep its production fairly stable, i.e. that it is interested in "production smoothing." There will be costs attached to the process of changing the rate of production and changing the size of the work force, which the firm will be interested in minimizing. Indeed, one reason for holding inventories at all is to prevent fluctuations in sales from immediately causing fluctuations in output. Thus the firm does not immediately adjust the volume of output to the "desired" P_t^*, rather its planned production P_t^p, envisages only a partial adjustment in any given period, as follows:

$$P_t^p = bP_t^* + (1-b)P_{t-1} \quad . \tag{4}$$

We postulate that b lies between 0 and 1, a larger value of b implying a greater flexibility in production plans.

Finally, we assume that the planned production is not changed during the period, but that factors not explicitly introduced into the above discussion will cause actual production to deviate from planned production, hence

$$P_t = P_t^p + e_t \,, \tag{5}$$

where e_t is a random error term. Substitution from equations (2), (3), and (4) now gives

$$P_t = \alpha_1 P_{t-1} + \alpha_2 \hat{S}_t + \alpha_3 I_{t-1} + e_t \tag{6}$$

which is the form used for eventual estimation. We have the following relations between the coefficients of (6) and the parameters of our model:

$$\begin{aligned} \alpha_1 &= 1 - b \\ \alpha_2 &= (a+1)b \\ \alpha_3 &= - b \,. \end{aligned} \tag{7}$$

Equation (6) cannot be estimated since data broken down into stock vs. order industries is not available; below we shall combine (6) with a relation concerning the production-to-order case. We see from (7), however, that if we were able to estimate the coefficients of equation (6), then the parameters a and b would be overidentified. It is unlikely, for example, that the estimated coefficients would be such that $\alpha_1 - \alpha_3 = 1$, although this restriction can be imposed by forming a single regression variable from an appropriate combination of P_{t-1} and I_{t-1}, a procedure which we adopt in Section 5.2.

In this context, it is interesting to note that (6) is of the same form as the estimating equation used by Mills [26], the derivation of which was described in Section 2.1, (p. 10 above). However, this derivation led to greater restrictions on the coefficients of (6), namely the requirement that

$$\alpha_1 = -\alpha_2 = 1 - \alpha_3 \ .$$

These three coefficients are functions of six parameters of Mills' model (four cost parameters, the price, and the parameter of the rectangular distribution) yet the derivation is such that the above restriction is imposed, leaving only one degree of freedom.

3.2. Production and unfilled order backlogs in the case of production to order

The firm which produces to order carries no inventory of finished goods, provided that we ignore cancellation of orders. Goods are produced in response to previously received orders, and are shipped immediately upon completion, thus the volume of shipments in a given period is equal to the production in that period. Fluctuations in the volume of new orders received, reflecting fluctuations in demand, cause variations in the volume of production and/or variations in the size of the unfilled order backlog. Hence, we have the basic identity that the end-period order backlog, U_t, is equal to the beginning order backlog, U_{t-1}, plus new orders received during the period, N_t, less production, or shipments, during the period, P_t, as follows

$$(1) \qquad U_t \equiv U_{t-1} + N_t - P_t \ .$$

We now hypothesize that the firm's desired volume of output during a given period is a function of the level of the order backlog at the beginning of the period and the anticipated volume of new orders to be received during the period. We postulate that these two effects have different weights, hence

$$P_t^* = a_1 U_{t-1} + a_2 \hat{N}_t \ . \tag{2}$$

Note that if it were the case that $a_1 = a_2$, then we could use (1) and (2) to imply a relationship between a desired end-period order backlog and production; however our formulation of the decision process places the primary emphasis on production planning.

Our second hypothesis is identical with that of Section 3.1, namely that the firm finds it worthwhile to prevent production levels from fluctuating too wildly from period to period. Hence we have

$$P_t^p = bP_t^* + (1-b)P_{t-1} \ . \tag{3}$$

Again, our final assumption is that actual output deviates from planned output only by virtue of random influences, thus

$$P_t = P_t^p + e_t \ , \tag{4}$$

and substitution from (2) and (3) now gives

$$P_t = \beta_1 P_{t-1} + \beta_2 \hat{N}_t + \beta_3 U_{t-1} + e_t \ . \tag{5}$$

In this case the parameters of our model are identified in the sense that if we were able to estimate β_1, β_2, and β_3 then from these estimated coefficients unambiguous estimates of a_1, a_2, and b could be derived.

3.3. Application to industry data

In general, the industry classifications for which data are published are neither pure production to stock industries nor pure production to order industries, but contain firms belonging to both categories. Indeed, given the diversification of the activities of individual manufacturers, many firms will undertake some production to stock and some production to order. We now combine the two relations derived above into a single relation with aggregate industry production as the dependent variable.

We recall that in the case of production to stock the volume of shipments, or sales, is identical with the volume of new orders in a given period. Hence we rewrite (3.1.7) as

$$(1) \qquad P_{s,t} = \alpha_1 P_{s,t-1} + \alpha_2 \hat{N}_{s,t} + \alpha_3 I_{t-1} + e_{s,t} .$$

The subscript s denotes the stock-production sector, and is omitted from I_t since this variable is in fact an industry aggregate, finished goods inventories being zero in the order-production sector. Similarly we rewrite (3.2.5) for the order-production sector, omitting the o-subscript from the unfilled order backlog:

$$(2) \qquad P_{o,t} = \beta_1 P_{o,t-1} + \beta_2 \hat{N}_{o,t} + \beta_3 U_{t-1} + e_{o,t} .$$

Given that these equations are correctly specified, in that variables appearing on the right-hand side of equation (1) do not appear in (2) and vice versa, then the true specification for the aggregate variable $P_t = P_{s,t} + P_{o,t}$ is as follows:

$$(3) \quad P_t = \alpha_1 P_{s,t-1} + \beta_1 P_{o,t-1} + \alpha_2 \hat{N}_{s,t} + \beta_2 \hat{N}_{o,t} + \alpha_3 I_{t-1} + \beta_3 U_{t-1} + e_t .$$

However, aggregation amongst the predetermined variables implies that we use the following equation for estimation purposes:

$$(4) \qquad P_t = \gamma_1 P_{t-1} + \gamma_2 \hat{N}_t + \gamma_3 I_{t-1} + \gamma_4 U_{t-1} + e_t$$

where $\hat{N}_t = \hat{N}_{s,t} + \hat{N}_{o,t}$.

We now consider the implications of this aggregation for the relation between the estimated γ-coefficients and the "true" α's and β's , following Theil [35].

Writing $\underset{\sim}{X}$ for the Tx4 matrix $[\underline{P}_{-1}, \hat{\underline{N}}, \underline{I}, \underline{U}]$ where $\underline{P}_{-1}$, $\hat{\underline{N}}$, $\underline{I}$, and $\underline{U}$ are column vectors of T observations on the predetermined variables, and $\underset{\sim}{\gamma}$ for the column vector of regression coefficients, then ordinary least squares estimates are given by

$$\hat{\underset{\sim}{\gamma}} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underline{P} \quad .$$

Considering the expected value of these estimates, we have

$$(5) \quad E\,\hat{\underset{\sim}{\gamma}} = E\left\{(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\,\underset{\sim}{X}'[\underline{P}_{s,-1},\underline{P}_{o,-1},\hat{\underline{N}}_s,\hat{\underline{N}}_o,\underline{I},\underline{U}][\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3]'\right\}$$

$$= \underset{\sim}{D}[\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3]'$$

where $\underset{\sim}{D}$ is a 4x6 matrix $[d_{ij}]$ whose columns are the expected values of the estimated coefficients in regressions of, respectively, $\underline{P}_{s,-1}$, $\underline{P}_{o,-1}$, $\hat{\underline{N}}_s$, $\hat{\underline{N}}_o$, $\underline{I}$, and $\underline{U}$ on "independent" variables $\underline{P}_{-1}$, $\hat{\underline{N}}$, $\underline{I}$, and $\underline{U}$, thus:

$$P_{s,t-1} = d_{11} P_{t-1} + d_{21} \hat{N}_t + d_{31} I_{t-1} + d_{41} U_{t-1} + \text{residual}$$

$$P_{o,t-1} = d_{12} P_{t-1} + d_{22} \hat{N}_t + d_{32} I_{t-1} + d_{42} U_{t-1} + \quad "$$

$$N_{s,t} = d_{13} P_{t-1} + d_{23} \hat{N}_t + d_{33} I_{t-1} + d_{43} U_{t-1} + \quad "$$

$$N_{o,t} = d_{14} P_{t-1} + d_{24} \hat{N}_t + d_{34} I_{t-1} + d_{44} U_{t-1} + \quad "$$

$$I_{t-1} = d_{15} P_{t-1} + d_{25} \hat{N}_t + d_{35} I_{t-1} + d_{45} U_{t-1} + \quad "$$

$$U_{t-1} = d_{16} P_{t-1} + d_{26} \hat{N}_t + d_{36} I_{t-1} + d_{46} U_{t-1} + \quad "$$

In the last two equations we must have

$$d_{15} = d_{16} = d_{25} = d_{26} = d_{45} = d_{36} = 0 \quad \text{and} \quad d_{35} = d_{46} = 1 .$$

In the first pair of equations, aggregation implies that

$$d_{11} + d_{12} = 1 , \quad \text{and} \quad d_{21} + d_{22} = d_{31} + d_{32} = d_{41} + d_{42} = 0 ,$$

a similar result holding for the second pair of equations. Therefore,

$$\underset{\sim}{D} = \begin{bmatrix} d_{11} & 1 - d_{11} & d_{13} & - d_{13} & 0 & 0 \\ d_{21} & - d_{21} & d_{23} & 1 - d_{23} & 0 & 0 \\ d_{31} & - d_{31} & d_{33} & - d_{33} & 1 & 0 \\ d_{41} & - d_{41} & d_{43} & - d_{43} & 0 & 1 \end{bmatrix}$$

and

$$(6) \qquad \begin{cases} E\,\hat{\gamma}_1 = d_{11}\alpha_1 + (1 - d_{11})\beta_1 + d_{13}(\alpha_2 - \beta_2) \\ E\,\hat{\gamma}_2 = d_{23}\alpha_2 + (1 - d_{23})\beta_2 + d_{21}(\alpha_1 - \beta_1) \\ E\,\hat{\gamma}_3 = \alpha_3 + d_{31}(\alpha_1 - \beta_1) + d_{33}(\alpha_2 - \beta_2) \\ E\,\hat{\gamma}_4 = \beta_3 + d_{41}(\alpha_1 - \beta_1) + d_{43}(\alpha_2 - \beta_2) \end{cases}$$

where

$$(7)\quad \begin{cases} P_{s,t-1} = d_{11} P_{t-1} + d_{21} \hat{N}_t + d_{31} I_{t-1} + d_{41} U_{t-1} + \text{residual} \\ P_{o,t-1} = (1-d_{11}) P_{t-1} - d_{21} \hat{N}_t - d_{31} I_{t-1} - d_{41} U_{t-1} + \text{"} \\ N_{s,t} = d_{13} P_{t-1} + d_{23} \hat{N}_t + d_{33} I_{t-1} + d_{43} U_{t-1} + \text{"} \\ N_{o,t} = -d_{13} P_{t-1} + (1-d_{23}) \hat{N}_t - d_{33} I_{t-1} - d_{43} U_{t-1} + \text{"} \end{cases}$$

This analysis is presented in terms of ordinary least squares estimates, although in Chapter 4 we present an alternative estimation procedure which takes account of the presence of the lagged dependent variable among the regressors and of serially correlated residuals. The analysis proceeds in identical fashion for any estimation procedure, provided that the procedure used to estimate $\underset{\sim}{\gamma}$ is also used to calculate the d-coefficients. We note that if it were not for the presence of the lagged dependent variable, the expectation operator on the right-hand side of (5) could be suppressed, and $\underset{\sim}{D}$ would then be a matrix of estimated coefficients, rather than their expected values.

The following two special cases are now of interest.

(i) In the case where $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ then it is clear from (3) that aggregation presents no problem, and (6) reduces to

$$E\,\hat{\underset{\sim}{\gamma}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_3 \end{bmatrix} .$$

However, this would imply that $(a+1)$, where a is as given in equation (3.1.2), is equal to a_2, as given in equation (3.2.2), and hence that $\hat{N}_{o,t}$ enters the desired production equation for the order sector with a coefficient $a_2 > 1$.

(ii) In the case where the sector variables are constant proportions of the corresponding aggregate variables, then the aggregation is consistent. In equations (7) we then have

$$d_{21} = d_{31} = d_{41} = 0$$

and

$$d_{13} = d_{33} = d_{43} = 0$$

hence equations (6) reduce to

$$E\,\hat{\gamma}_1 = d_{11}\,\alpha_1 + (1-d_{11})\beta_1$$

$$E\,\hat{\gamma}_2 = d_{23}\,\alpha_2 + (1-d_{23})\beta_2$$

$$E\,\hat{\gamma}_3 = \alpha_3$$

$$E\,\hat{\gamma}_4 = \beta_3 \ .$$

Thus there is zero aggregation bias in the estimates of γ_3 and γ_4. The strict proportionality assumption removes the effect of the non-corresponding coefficients from $\hat{\gamma}_1$ and $\hat{\gamma}_2$; these aggregate coefficients are now weighted averages of the corresponding sector coefficients. The weights d_{11} and d_{23} in these averages are the ratios of the production to stock sector to total manufacturing for the production and new orders series respectively. In the absence of any long run accumulation or decumulation of orders or inventories, we would expect that $d_{11} = d_{23}$.

One might suppose that the condition of constant proportionality between the production to order and production to stock sectors would be approximately met, since patterns of industrial organization and

practice at this level of aggregation do not change very quickly. It is difficult to obtain a clear picture, however, due to the lack of sector data. In an attempt to study this question, we calculated annual average ratios of unfilled orders to finished goods inventories for each of our five categories over the twelve years 1953-1964 (the data are described in Section 3.5 below). Broadly speaking, the overall ratios should give a general indication of the relative prevalence of production to order in each of the five categories, without necessarily giving a direct indication of the actual proportions in which the two types of production occur. In all five cases, however, the ratios show a decline over the twelve year period, in general decreasing by a factor of two or three. Nevertheless, while casting some doubt on the constant proportionality assumption, this does not amount to an absolute rejection, due to the difficulty in moving from these calculated ratios to estimates of the actual proportions of production for the two sectors. The unanswered questions are how big a change in the proportionate share in production of one sector is represented by a halving of the unfilled orders to inventory ratio, and to what extent have changes in the practices of firms with respect to these two variables changed over the twelve year period. In the absence of further data, the constant proportionality argument remains inconclusive.

3.4. Expectations

It was noted in Section 2.3 that in the absence of data on business expectations for future sales we may proceed in one of two ways. Either we may construct hypotheses concerning the mechanism by which sales forecasts are generated, or we may simply consider hypotheses concerning the outcome of the forecasting procedure and the nature of the forecast errors. Here we adopt the former course, and attempt to approximate a firm's sales forecasting procedure.

The specific formulation is that forecasts $\hat{N}_t$ are generated by the following scheme:

$$(1) \qquad \hat{N}_t = N_{t-12} + q(N_{t-1} - N_{t-13}) \ .$$

The forecast for a particular month consists of the actual figure for the same month of the previous year, modified by the most recent estimate of the change over the last year. This procedure is similar to that assumed by Johnston [16] and described in Section 2.3 above. We now give some further discussion of this assumption.

Consider a time series $\{x_t\}$ which is assumed to be covariance stationary (i.e. its autocovariance function $Ex_t x_{t+\ell}$ is a function only of the lag ℓ, and not of t), and purely nondeterministic (i.e. it is not possible to predict x_t with zero error by a linear combination of $x_{t-1}, x_{t-2}, \ldots$). Then, from a result of Wold [40, Theorem 7], $\{x_t\}$ can be represented as a moving average process,

$$x_t = \sum_{0}^{\infty} b_j \epsilon_{t-j}$$

where the ϵ_t are independent, identically distributed random variables with zero mean, commonly known as "white noise."

Equivalently, the moving average representation may be inverted to give the autoregression

$$\sum_{o}^{\infty} a_j x_{t-j} = \epsilon_t$$

which is the form generally used for estimation purposes, since the ϵ_t are not observed. Clearly, if the autoregressive structure is not of finite order, then we use a finite approximation

$$\sum_{o}^{p} a_j x_{t-j} = \epsilon_t \quad . \tag{2}$$

We also normalize by setting $a_o = 1$. Rewriting this expression then gives

$$x_t = - \sum_{1}^{p} a_j x_{t-j} + \epsilon_t .$$

The minimum mean square error predictor of x_t given observations $x_{t-1}, x_{t-2}, \ldots$ is then simply obtained as

$$\hat{x}_t = - \sum_{1}^{p} a_j x_{t-j} \quad .$$

Thus $\hat{x}_t$ is a function of the last p observed values alone. In effect, we use our knowledge of the autoregressive structure, and set ϵ_t equal to its expected value, namely zero, to obtain the predicted $\hat{x}_t$. A proof of these statements is given by Whittle [39].

We now draw on some earlier work to support the above formulation for $\hat{N}_t$. In [37], we reported the results of fitting an autoregressive scheme, as in equation (2), to eight series of monthly, seasonally unadjusted data on retail sales. In four of these cases, the significant coefficients were a_1, a_{12} and a_{13}, and in a fifth

case a_{23} was, in addition, significant. In two further cases the only significant coefficient was a_{12}, while in the remaining case a_1 and a_3 were significant. Further, in the first four cases, the coefficients were such as to suggest that the autoregression

$$x_t + a_1 x_{t-1} + a_{12} x_{t-12} + a_{13} x_{t-13} = \epsilon_t$$

could be replaced by the model

$$(3) \qquad (1 - \alpha U)(1 - \beta U^{12}) x_t = \epsilon_t$$

with $0 < \alpha < 1$, $0 < \beta < 1$, where U represents the backward shift operator. The two cases in which a_{12} was the only significant coefficient would then correspond to this model with α set equal to zero.

Now returning to the prediction problem, we see that if the time series $\{x_t\}$ obeys the autoregressive model (3), then the minimum mean-square error predictor of x_t based on $x_{t-1}, x_{t-2}, \ldots$ is

$$\hat{x}_t = \alpha x_{t-1} + \beta x_{t-12} - \alpha\beta x_{t-13}$$

Setting $\alpha = q$ and $\beta = 1$ then corresponds to the forecasting scheme (1). Inserting this forecasting scheme into the aggregated regression model (3.3.4) gives the regression equation which we later estimate:

$$(4) \qquad P_t = \gamma_1 P_{t-1} + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 I_{t-1} + \gamma_4 U_{t-1} + e_t .$$

3.5. The data

The published data used in this study are monthly series

on manufacturers' shipments, new orders, unfilled orders, and finished goods inventories published in the monthly Current Industrial Reports series of the Bureau of the Census. Most, though not all, of these series are also published in the Survey of Current Business. A full description of the series, together with data through July 1963, is given in [4].

These data are available from January 1953 to the present time in millions of current dollars for the following five categories:

Durable goods industries, total

Primary metals

Machinery (electrical and nonelectrical)

Transportation equipment

Nondurable goods industries, total.

We use observations up to March 1965, giving a total of 147 observations. Although these series are currently published for approximately 55 detailed industry categories, involving a much finer classification than that just given, it is only for these five categories that continuous monthly data on both unfilled orders and finished goods inventories, extending back to 1953, are available.

The basic source of these data is a monthly survey of companies engaged in manufacturing undertaken by the Industry Division of the Bureau of the Census. This is a complete survey of manufacturing companies with 1,000 or more manufacturing employees, and smaller companies are sampled with probabilities proportional to their employment size. The Annual Survey of Manufactures is then used as a benchmark for the monthly series on shipments and inventories, but no such

annual benchmark is available for the unfilled orders series.

In order that the shipments, new orders, and unfilled orders series should be consistent, only two of the three series can be estimated independently. Accordingly the monthly survey does not collect data on new orders, rather this is derived by adding the change in unfilled orders between the current and previous month to the shipments figure, i.e.,

$$N_t \equiv S_t + U_t - U_{t-1} .$$

This new orders figure is thus net of cancellations.

For some industries where backlog records are not maintained and where total backlogs are insignificant in relation to shipments, the Census Bureau assumes that unfilled orders are zero and hence new orders are equal to shipments. This would apply exactly in the case of pure production to stock, and most of these industries are, in fact, in the nondurable goods area. The notable exception is the motor vehicles and parts group, which is a subsection of our transportation equipment category. Here the Census Bureau is following the practice of the industry, which is "not to maintain any unfilled orders file even when there exists a substantial backlog of demand" [4, p. 13].

These series are available both with and without seasonal adjustment. For reasons outlined below (Section 4.1) the seasonally unadjusted data are used in this study; these are, however, adjusted for trading-day and calendar month variation before publication.

From the published data, we construct our own series for production by adding the change in finished goods inventories between the current and previous month to the shipments figure, i.e.,

$$P_t \equiv S_t + I_t - I_{t-1} .$$

Thus we now have the four series required for estimation of the regression equation derived above, viz. P_t, N_t, U_t and I_t .

Finally, these four series are deflated, using the Bureau of Labor Statistics Wholesale Price Index (1957-59=100) for the five relevant groups. This index is currently published in the Survey of Current Business; monthly data for the period 1953-1958 on the revised base, 1957-59=100, are available from the Bureau of Labor Statistics. Hence the data finally used in the analysis are measured in millions of constant (1957-59) dollars.

CHAPTER 4

STATISTICAL METHODOLOGY

The spectrum of a time series is first introduced, prior to a discussion of seasonality in economic time series and in regression problems. We then present an estimation procedure which employs the techniques of cross-spectral analysis, giving a general discussion of the procedure in Section 4.2 and details of the specific application to our model in Section 4.3.

4.1. Seasonality

Consider a time series $\{x_t\}$ which is assumed to be covariance stationary, that is, its autocovariance function, $c_{xx}(\ell)$, defined by

$$c_{xx}(\ell) = E\, x_t x_{t+\ell} \,, \tag{1}$$

is not a function of t. The spectral density, or spectrum, of the time series is defined as the Fourier transform of the autocovariance function,

$$f_{xx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{xx}(\ell)\, e^{-i\ell\theta} \,. \tag{2}$$

In general, the inverse Fourier transform can be written

$$c_{xx}(\ell) = \int_{-\pi}^{\pi} f_{xx}(\theta)\, e^{i\ell\theta}\, d\theta \,,$$

and, in particular,

$$c_{xx}(0) = \text{var } x = \int_{-\pi}^{\pi} f_{xx}(\theta)\, d\theta \,.$$

Thus, if we regard the time series as consisting of a number of orthogonal components each associated with a particular frequency, then the spectrum may be described as a dissection of the variance of the time series into portions associated with the various frequency components. A fuller introduction to the theory and estimation of spectra is given in Appendix I.

In practice, given a sample $x_1, \ldots\ldots, x_T$, we estimate the spectrum as

$$(3) \qquad \hat{f}_{xx}(\theta_k) = \frac{1}{2\pi} \sum_{\ell=-m}^{m} \hat{c}_{xx}(\ell)\, e^{-i\ell\theta_k}\, W(\ell)$$

$$= \frac{1}{2\pi} \{\hat{c}_{xx}(0) + 2 \sum_{\ell=1}^{m} \hat{c}_{xx}(\ell) \cos \ell\theta_k\, W(\ell)\} ,$$

for $\theta_k = \frac{\pi k}{m}$, $k = -m+1, \ldots\ldots, m$. Since the spectrum is symmetric about the origin, it is usual to consider $\hat{f}_{xx}(\theta_k)$ only for $0 \leq \theta_k \leq \pi$, i.e., for $k = 0, 1, \ldots, m$. $W(\ell)$ is the Parzen window, and we estimate the lag autocovariance as

$$(4) \qquad c_{xx}(\ell) = \frac{1}{T} \sum_{t=1}^{T-\ell} x_t x_{t+\ell} , \qquad \ell = 0, 1, \ldots m ,$$

m being referred to as the "maximal lag" or "truncation point."

Estimated spectra of the production series for the five categories discussed in this essay are shown in Figures 1 - 5. The series were prewhitened by means of a linear filter, and the estimated spectra, with $m = 36$, were subsequently recolored. A full discussion of these estimation procedures is given in Appendix I. The estimated spectra are plotted on a semi-logarithmic scale, the same scale being used throughout for purposes of comparison. Differences in the overall

Figure 1

Spectrum of production series, total durable goods

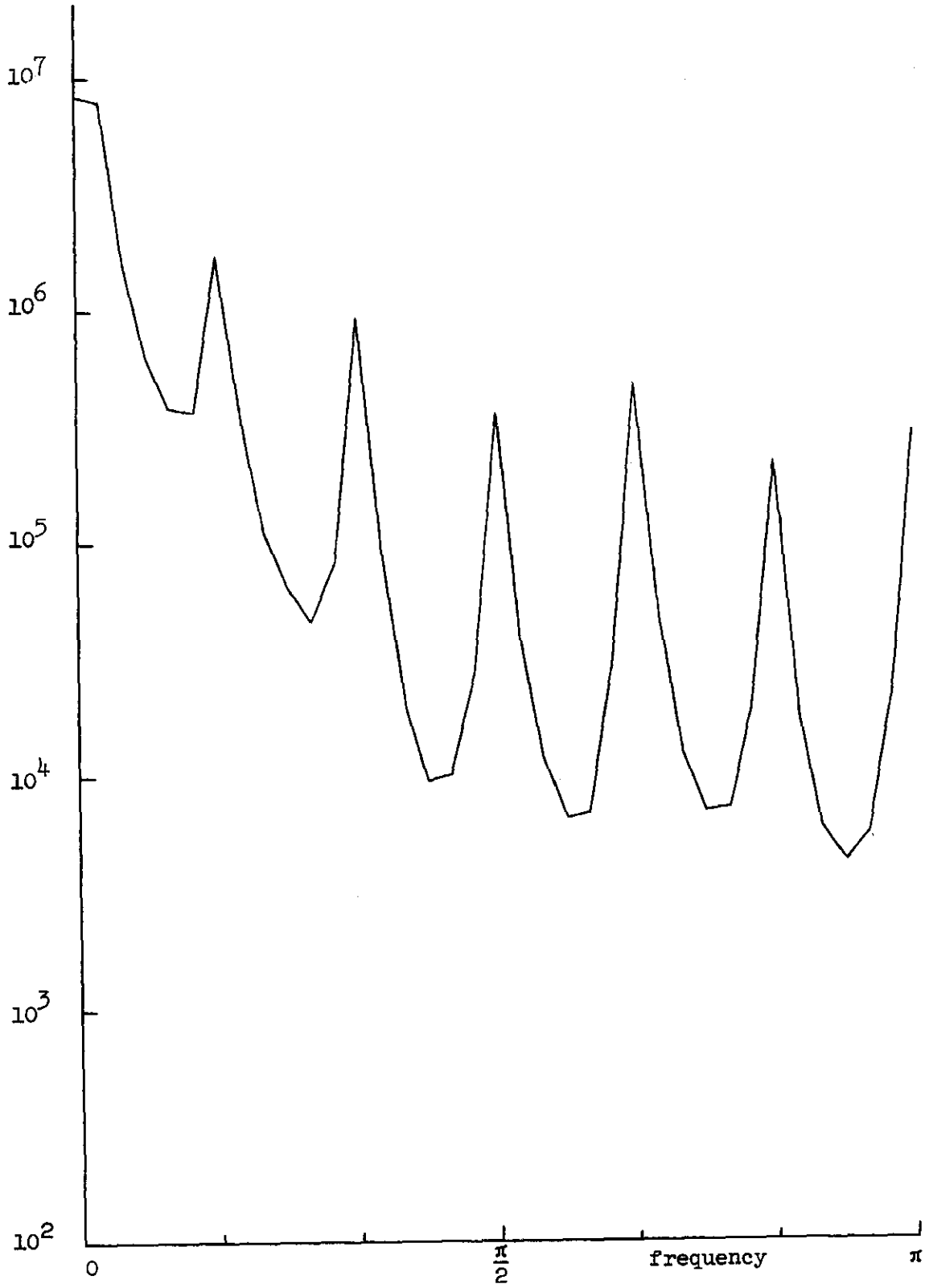


Figure 2

Spectrum of production series, primary metals

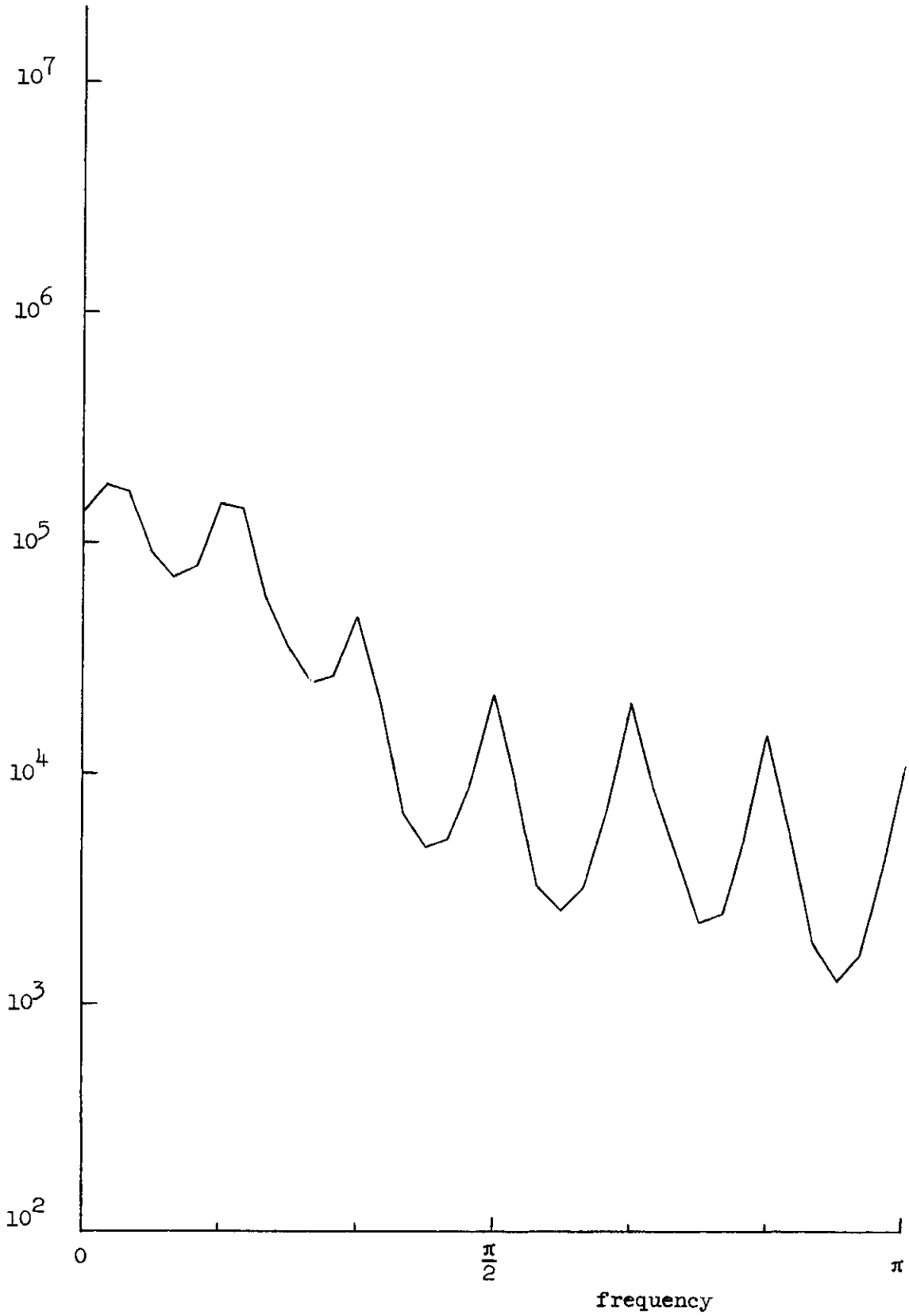


Figure 3

Spectrum of production series, machinery

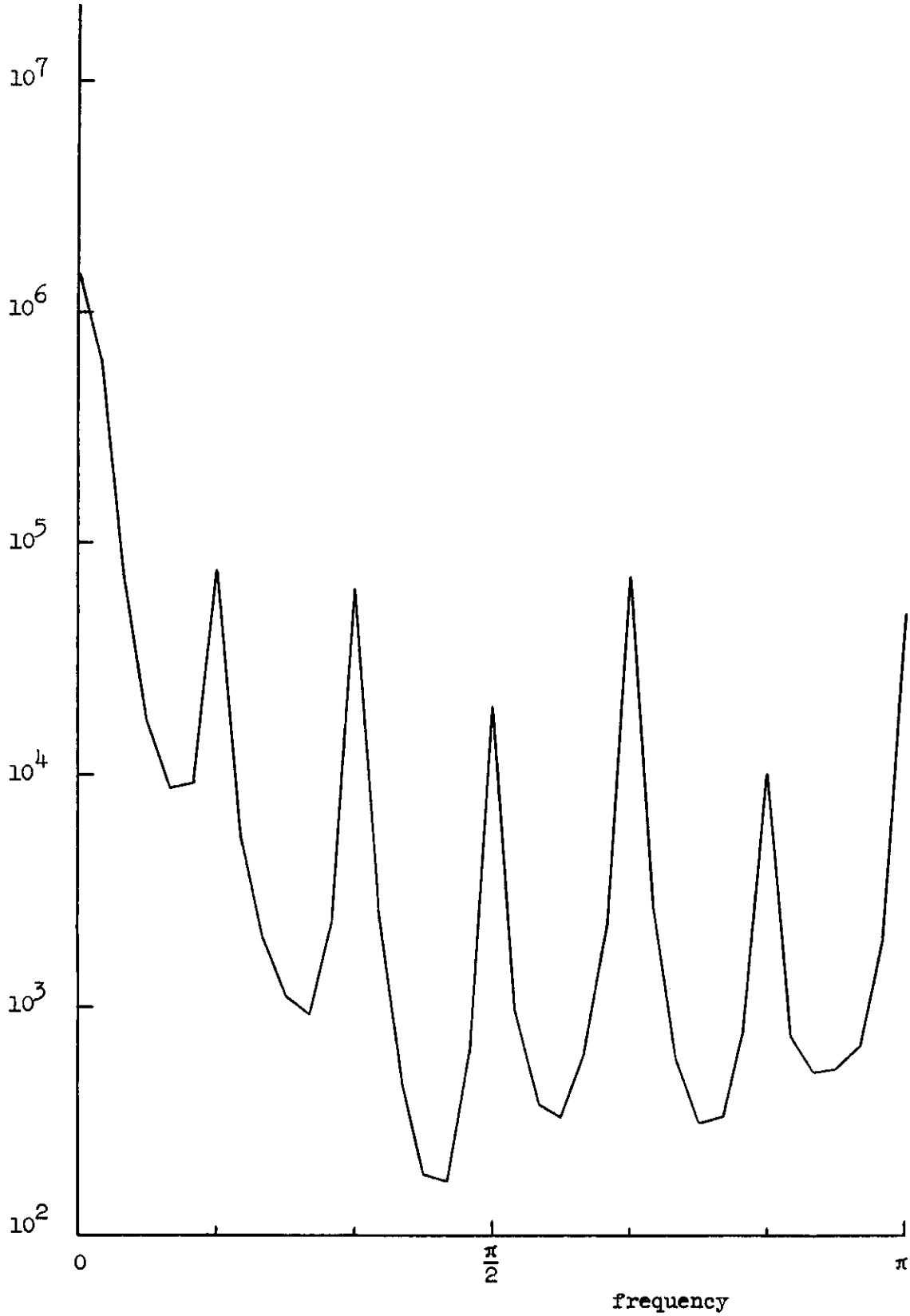


Figure 4

Spectrum of production series, transportation equipment

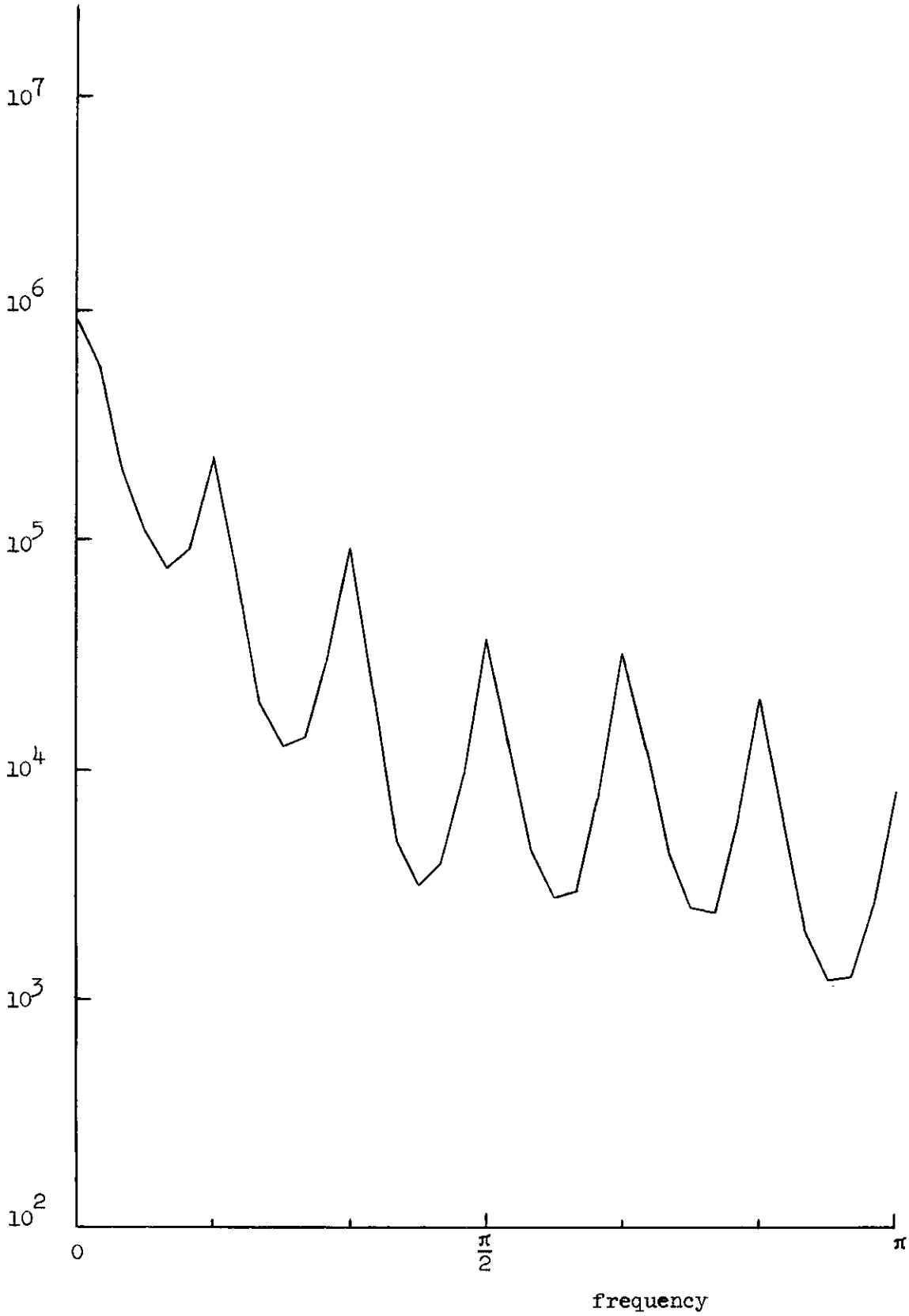
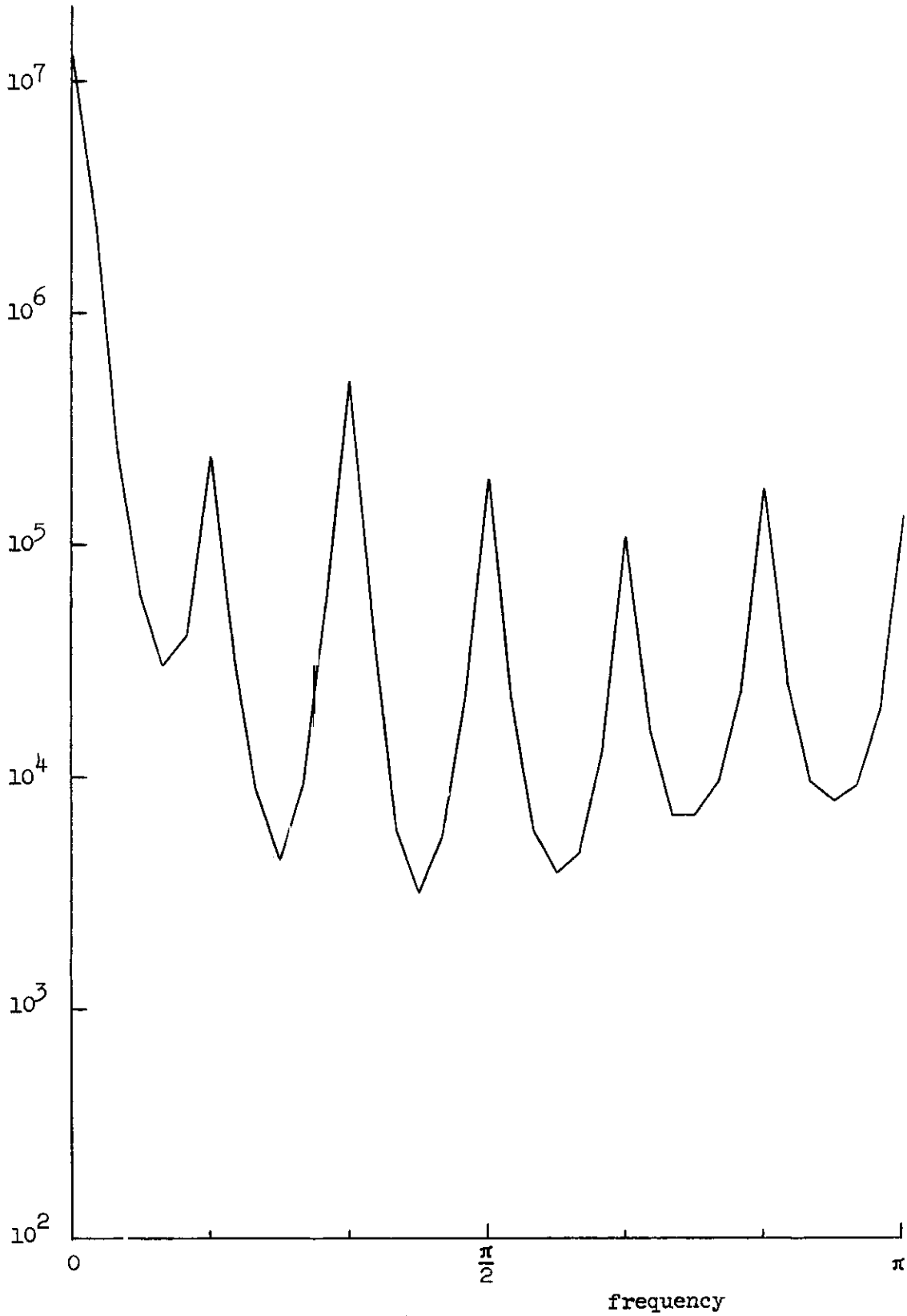


Figure 5

Spectrum of production series, total nondurable goods



height of the spectra simply reflect differences in the total variance of the time series.

Apart from the existence of a maximum at the origin, which is often interpreted as representing the existence of trend in the series, the prominent feature of all these spectra is the sequence of peaks at frequencies $\theta_k = \frac{\pi k}{36}$ for $k = 6, 12, 18, 24, 30, 36$. These frequencies are those of cycles with periods 12, 6, 4, 3, 2.4, and 2 months respectively, i.e., cycles which are completed 1, 2, 3, 4, 5, and 6 times per year respectively. Since an arbitrary twelve month pattern can be represented as a sum of sinusoidal variations with these six frequencies, these frequencies are referred to as seasonal frequencies, and the peaks at these frequencies are accepted as evidence of a strong seasonal pattern in the original series. We note that Nerlove defines seasonality as "that characteristic of a time series that gives rise to spectral peaks at seasomal frequencies" [29, p. 262]; the seasonality is relatively weak in the primary metals and transportation equipment industries. If we were able to eliminate the seasonality from the production series in some way, then these spectra would conform quite closely to Granger's "typical spectral shape of an economic variable" [9]. The only case in which the maximum is not at zero is that of primary metals (Figure 2); the small low-frequency peak which might otherwise be suggestive of business-cycle effects is, however, not significant.

The use of seasonally unadjusted data in regression problems has two main implications. Firstly, complications caused by seasonal adjustment procedures, which operate on individual series, are avoided. As Nerlove [29] has argued, some of the adjustment procedures currently in use eliminate from the series far more than can properly be called seasonal. Working with various unemployment series, Nerlove found that

when he compared spectra of seasonally adjusted and unadjusted series, the spectrum of the adjusted series had lower values than the spectrum of the unadjusted series at all but the very low frequencies. Moreover, at the seasonal frequencies, the spectra of the adjusted series often had dips, indicating that the seasonal peaks of the original spectra had been more than removed.

Furthermore, the extent to which adjustment procedures operating independently on each one of a number of series change the relationships between the series is not known. It is known that these procedures do not in general preserve sums. We noted earlier that the new orders series is derived by the Census Bureau from the shipments and unfilled orders series by using the identity $N_t \equiv S_t + \Delta U_t$; the three series are seasonally adjusted separately, however, and this identity does not hold between the seasonally adjusted series.

Secondly, the use of seasonally unadjusted data implies acceptance of the view that seasonality in the dependent variable is something which we should attempt to explain in our model. Seasonal variations, along with other types of variations, have causes which we should endeavor to identify. Seasonality in one economic variable is not an isolated phenomenon, but is related to seasonal changes in other economic variables with which it interacts. The comparative amplitudes of seasonal fluctuations in the various series then constitute useful information about the economic processes of which our model is a representation, and the use of seasonally adjusted data amounts to throwing away this information. We therefore use seasonally unadjusted data, and assume that the seasonal variation in the dependent

variable may be divided into a part explained by seasonal variation in the independent variables, and a remainder which can be assigned to the error term.

The usual approach to the estimation of regression equations from seasonally unadjusted data is to introduce "dummy" or seasonal interaction variables into the equation. The introduction of dummy variables, one less than the number of seasons, into the constant term of an equation implies that the regression relation makes parallel shifts from season to season, and is equivalent to the introduction of sine and cosine terms at appropriate frequencies. With monthly data, for example, an arbitrary but fixed seasonal pattern may be represented either as a series of 12 constants (11 dummy variables and the mean) or as linear combinations of sines and cosines with frequencies $\frac{2\pi k}{12}$, $k = 1, \ldots, 12$. We may also handle seasonal variation in the parameters of the equation by introducing dummy variables which interact with the dependent variable, implying that the slope of the equation changes from season to season.

This approach to the problem is discussed by Klein, Ball, Hazlewood and Vandome [18], who point out the disadvantage of being forced into a definite parametric treatment of the problem. Seasonal fluctuations are not perfectly regular, for example the specific climatic pattern changes from year to year. Nerlove [29] argues that it is this approximate but not perfect regularity which makes it so difficult to give a precise definition of seasonality, and it is equally difficult to give a precise parametric treatment of seasonality in the context of regression problems. A second disadvantage, especially to

the user of monthly data, lies in the loss of degrees of freedom; for example, to introduce explicit seasonal variables into the constant term of an equation and also into a single regression coefficient loses 22 degrees of freedom when monthly data are employed. If, in this case, there is only one independent variable, then this procedure is equivalent to the estimation of twelve separate regressions, one for each month.

It is suggested that spectral techniques be used to study the causes of seasonality in the dependent variable of a regression equation which is estimated using seasonally unadjusted data. We fit a single relation to the data under the assumption, discussed by Mills [26, p. 194], that the form of the decision rule is unchanged from one month to the next. Thus seasonal fluctuations are treated no differently from fluctuations caused by other factors, and the disadvantages of seasonally adjusted data, described above, are avoided. Then, by simply comparing the spectra of the dependent variable, the regression estimate, and the regression residual it should be possible to assign the major seasonal influence either to the independent variables or to factors not explicitly introduced into the equation. This amounts to a spectral analogue of the partition of the variance of the dependent variable into a part associated with the regression estimate ("explained variance," or R^2) and a remainder, which is usually carried out following ordinary least squares estimation.

This comparison will be applicable whatever estimation procedure has been used, but in the next section we suggest an estimation procedure more appropriate to seasonally unadjusted data, which itself employs spectral techniques.

4.2 Spectral techniques of regression analysis

In Section 2.5 we noted that the use of the Durbin-Watson statistic to test for the serial independence of regression residuals is inappropriate when the regression equation contains the lagged dependent variable. Furthermore, with seasonally unadjusted data, we might expect serial correlation to be fourth order (quarterly data) or twelfth order (monthly data), whereas the Durbin-Watson statistic tests only for first order serial correlation. In such cases it might then be more prudent to assume that the residuals are serially correlated than to assume otherwise. This conclusion is strengthened when data for as short a time period as a month are being used, for reducing the period between observations on a time series will in general increase the serial correlation. In spectral analysis, lack of serial independence is a central feature rather than an unfortunate complication. By taking any lack of serial independence into account when estimating regression coefficients we shall essentially improve the efficiency of our coefficient estimates.

In similar fashion to equations (4.1.1) - (4.1.4) we define the cross-spectrum between two time series $\{y_t\}$ and $\{x_t\}$ as

$$(1) \qquad f_{yx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{yx}(\ell)\, e^{-i\ell\theta}$$

where $c_{yx}(\ell)$ is the lagged covariance between $\{y_t\}$ and $\{x_t\}$

$$(2) \qquad c_{yx}(\ell) = E\, y_t x_{t+\ell} = c_{xy}(-\ell) \ .$$

The real and negative imaginary parts of $f_{yx}(\theta)$ are called the

co-spectrum and quadrature spectrum, thus

$$f_{yx}(\theta) = C_{yx}(\theta) - iQ_{yx}(\theta)$$

and

$$(3) \qquad C_{yx}(\theta) = \frac{1}{2\pi}\left\{c_{yx}(0) + \sum_{\ell=1}^{\infty}\left(c_{yx}(\ell) + c_{xy}(\ell)\right)\cos \ell\theta\right\}$$

$$Q_{yx}(\theta) = \frac{1}{2\pi}\sum_{\ell=1}^{\infty}\left(c_{yx}(\ell) - c_{xy}(\ell)\right)\sin \ell\theta \ .$$

We may interpret $C_{yx}(\theta)$ as the covariance of the in-phase components of $\{y_t\}$ and $\{x_t\}$, and $Q_{yx}(\theta)$ as the covariance of components which are 90 degrees out of phase or in quadrature. Again, given samples $x_1, \ldots x_T$, and $y_1, \ldots y_T$, we estimate the cross-spectrum as

$$(4) \qquad \hat{f}_{yx}(\theta_k) = \frac{1}{2\pi}\sum_{\ell=-m}^{m}\hat{c}_{yx}(\ell)\, e^{-i\ell\theta_k} W(\ell)$$

for $\theta_k = \frac{\pi k}{m}$, $k = -m, -m+1, \ldots, m$, where the lagged covariance is estimated as

$$(5) \qquad \hat{c}_{yx}(\ell) = \frac{1}{T}\sum_{t=1}^{T-\ell} y_t x_{t+\ell} \qquad \ell = 0, 1, \ldots m \ .$$

Again, a fuller description is given in Appendix I.

The development of spectral techniques of regression analysis is due to Hannan, and the following discussion is based on [15].

Consider the simple regression

$$(6) \qquad y_t = \beta x_t + e_t \ .$$

We assume that x_t and e_s are independent for all t and s, and

that e_t is a linear process of the form $e_t = \sum_0^\infty \varphi_j \epsilon_{t-j}$, $\sum_0^\infty |\varphi_j| < \infty$, where the ϵ_t are independent identically distributed random variables with zero mean and finite moments through the fourth order. We also require conditions on x_t which permit generalized harmonic analysis (Grenander and Rosenblatt [11, p. 233]); these could be more than obtained by assuming that x_t is a linear process of the form described above.

From (6) we may write

$$y_t x_{t+\ell} = \beta x_t x_{t+\ell} + e_t x_{t+\ell}$$

and taking expectations gives

$$c_{yx}(\ell) = \beta c_{xx}(\ell) \quad . \tag{7}$$

Substituting sample covariances gives the instrumental variable estimator

$$\hat{\beta} = \frac{\hat{c}_{yx}(\ell)}{\hat{c}_{xx}(\ell)} \quad , \tag{8}$$

$x_{t+\ell}$ being the instrumental variable. This provides a consistent estimate of β, the case $\ell = 0$ corresponding to ordinary least squares. However, taking the Fourier transform of both sides of equation (7) gives

$$f_{yx}(\theta) = \beta f_{xx}(\theta) \quad , \tag{9}$$

and the analogy with (7) and (8) implies that

$$\hat{\beta}(k) = \frac{\hat{f}_{yx}(\theta_k)}{\hat{f}_{xx}(\theta_k)} \tag{10}$$

be used as an estimator of β, given estimates of the spectral densities. We now have a $\hat{\beta}(k)$ for each value of k, and require a method of combining these into a single estimate. The variance of (10) is proportional to $f_{ee}(\theta_k)/f_{xx}(\theta_k)$, which is sometimes called the noise-to-signal ratio for y. When we require a minimum-variance weighted average of a series of independent random variables, the optimal weights are the reciprocals of the individual variances, thus neglecting the covariance between the $\hat{\beta}(k)$ gives

$$\hat{\beta} = \frac{\sum_k \hat{\beta}(k) \, [\text{var } \hat{\beta}(k)]^{-1}}{\sum_k [\text{var } \hat{\beta}(k)]^{-1}} \tag{11}$$

as the best estimate of β, the presence of the denominator retaining consistency. We calculate $\hat{\beta}$ as

$$\hat{\beta} = \left\{ \sum_k \frac{\hat{f}_{xx}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\}^{-1} \left\{ \sum_k \frac{\hat{f}_{yx}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\} . \tag{12}$$

The asymptotic variance of $\hat{\beta}$ is

$$\left\{ \frac{T}{2\pi} \int_{-\pi}^{\pi} f_{xx}(\theta) \, f_{ee}^{-1}(\theta) \, d\theta \right\}^{-1} , \tag{13}$$

which is the same as that calculated for the best linear unbiased estimator by Grenander and Rosenblatt [11], and it is estimated by

$$\left\{ \frac{T}{2m} \sum_k \frac{\hat{f}_{xx}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\}^{-1} . \tag{14}$$

These results may be generalized to the case of multiple regression. To estimate the coefficients of the equation

(15) $$y_t = \underline{z}_t' \underline{\beta} + e_t$$

where $\underline{z}_t'$ is a row vector with n components $(z_{1t}, \ldots, z_{nt})$ and $\underline{\beta}$ is $n \times 1$, we use the following generalization of equation (12):

(16) $$\hat{\underline{\beta}} = \left\{ \sum_k \frac{\hat{\underline{\underline{F}}}_{zz}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\}^{-1} \left\{ \sum_k \frac{\hat{\underline{f}}_{yz}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\} .$$

Here $\hat{\underline{f}}_{yz}(\theta_k)$ is an $n \times 1$ vector with i^{th} element $\hat{f}_{yz_i}(\theta_k)$, and $\hat{\underline{\underline{F}}}_{zz}(\theta_k)$ is an $n \times n$ matrix with $(i,j)^{th}$ element $\hat{f}_{z_i z_j}(\theta_k)$. The variance-covariance matrix of $\hat{\underline{\beta}}$ is estimated as

(17) $$\left\{ \frac{T}{2m} \sum_k \frac{\hat{\underline{\underline{F}}}_{zz}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\}^{-1} .$$

As indicated in the discussion of these techniques by Amemiya and Fuller [2, p. 4], if f_{ee} is assumed known, then $\hat{\beta}$ is approximately Aitken's generalized least squares estimate with the covariance matrix $\underline{E}\underline{e}\underline{e}' = \Gamma$ assumed known. In general f_{ee} will not be known, and Hannan suggests that it be estimated as the spectrum of the calculated residuals $\{\hat{e}_t\}$ in a regression using some consistent estimate of β, such as the least squares estimate. $\hat{\beta}$ is then approximately Aitken's least squares estimate, using a consistent estimate of Γ, and hence has the general properties of Aitken's estimate, namely consistency and efficiency.

It is suggested that the spectral approach will be advantageous in cases where the regression residuals should be assumed to be serially correlated, either because the period of observation is short, or

because the usual test for serial correlation is not appropriate. But it is clear that the advantages of a non-parametric treatment of serial correlation in the residuals can equally be claimed for Aitken's least squares procedure, provided that a consistent estimate of Γ is available. However, as Amemiya and Fuller [2] point out in discussing the analogy between these two approaches, in the spectral approach we are essentially estimating Γ using estimated covariances up to some maximal lag m weighted by $W(\ell)$, the lag window, and considerable discussion of the optimality of these procedures is contained in the literature of spectral analysis. On the other hand, such estimation of Γ is not required if the underlying mechanism generating the error terms can be exactly specified.

Secondly, as indicated by equation (11), these methods recognize variations in signal-to-noise ratio with frequency. Hamon and Hannan [13], in discussing this point, consider that the spectral approach will be more useful for time series which exhibit more marked variation of signal-to-noise ratio with frequency; our experience suggests that this will be the case whenever seasonally unadjusted data are used.

We are also enabled, by means of this approach, to study more fully the validity of the model employed. For example, if (6) is valid, then β should be the same at all frequencies, hence a study of the variation of $\hat{\beta}(k)$ with k will shed light on this question. An example of the analysis of variation in $\hat{\beta}(k)$ with k is contained in Hamon and Hannan [13]. Furthermore, while $\hat{\beta}(k)$ as calculated in (10) is apparently a complex number, this should not be so (apart from sampling errors) if the model is correctly specified. For the fact that lagged values of y or x do not appear in the equation implies that

y_t and x_t are either in phase (if $\beta > 0$) or 180 degrees out of phase (if $\beta < 0$) and in such cases $Q_{yx}(\theta)$ will be zero. Hamon and Hannan [13] also discuss testing the validity of this neglect of phase differences.

4.3. Application to our model

We now consider the application of the method, described in the previous section and specified in equation (4.2.16), to the equation derived in Chapter 3, namely

$$(1) \quad P_t = \gamma_1 P_{t-1} + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 I_{t-1} + \gamma_4 U_{t-1} + e_t .$$

Remembering that we have no estimate of $f_{ee}(\theta)$ to begin with, our first suggested estimation procedure is as follows:

(i) apply (4.2.16) to equation (1) above, under the assumption that $f_{ee}(\theta) = 1$, and obtain estimates χ^*

(ii) enter these estimated coefficients into (1), calculate residuals $\hat{e}_t$ and their spectrum $\hat{f}_{ee}(\theta_k)$

(iii) reapply (4.2.16) using this estimate of $f_{ee}(\theta)$, and obtain final estimates $\hat{\chi}$.

This procedure has two defects. Firstly, (i) is analogous to ordinary least squares, for the assumption that $f_{ee}(\theta) = 1$ is identical with the assumption that $\{e_t\}$ is white noise with variance 2π, i.e. that $\underline{E}\underline{e}\underline{e}' = 2\pi I$. As indicated in Section 2.5, when ordinary least squares is applied to a model containing the lagged dependent variable and serially correlated residuals, the resultant estimates are not consistent. Thus the estimated $\hat{f}_{ee}(\theta_k)$ obtained in stage (ii) is not

consistent. Secondly, even if it were, the stage (iii) estimate $\hat{\gamma}$ would not have the property of asymptotic efficiency. It was noted in Section 4.2 that stage (iii) is analogous to Aitken's generalized least squares using an estimate $\hat{\Gamma}$ of Eee' and, as shown in Appendix II, this method does not give asymptotically efficient estimates when the lagged dependent variable is among the regressors.

The first of these difficulties can be overcome by the use of an instrumental variable for P_{t-1} in stage (i). This will achieve consistency; the cost of this achievement is measured in terms of a loss of efficiency if the residuals actually are serially independent.

We describe the procedure by considering the simple model

$$y_t = \alpha_1 y_{t-1} + \alpha_2 x_t + e_t \ , \tag{2}$$

where $\{e_t\}$ is serially correlated. The use of an instrumental variable z_t for y_{t-1} implies that we estimate α_1 and α_2 as follows:

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \Sigma\, z_t y_{t-1} & \Sigma\, z_t x_t \\ \Sigma\, x_t y_{t-1} & \Sigma\, x_t^2 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma\, z_t y_t \\ \Sigma\, x_t y_t \end{pmatrix} .$$

In general, the selected instrumental variable z_t should be independent of e_t and strongly correlated with y_{t-1}. In this case we revert to equation (2), and note that y_t may be written as a linear combination of past x's

$$(3) \qquad y_t = \alpha_2 \sum_{j=0}^{\infty} \alpha_1^j x_{t-j} + u_t \ ,$$

where

$$u_t = \sum_{j=0}^{\infty} \alpha_1^j e_{t-j} \ .$$

Assuming that $|\alpha_1| < 1$, which is required for stationarity, we may approximate y_t by the first few terms of the summation in (3), and to avoid losing too many observations, we take only the first two terms. Then estimating the coefficients of the regression

$$(4) \qquad y_t = \beta_1 x_t + \beta_2 x_{t-1} + \text{residual}$$

will give the linear combination of x_t and x_{t-1} most highly correlated with y_t. We now lag this linear combination by one period to obtain our instrumental variable. Thus, we use z_t as an instrumental variable for y_{t-1}, where

$$z_t = \hat{\beta}_1 x_{t-1} + \hat{\beta}_2 x_{t-2} \ .$$

Since x_t and e_s are independent for all t and s, z_t is independent of e_t, and z_t is highly correlated with y_{t-1}, this correlation being measured by the R^2 achieved when equation (4) is estimated.

Applying this procedure to equation (1), we first regress P_t on the exogenous variables, current and lagged:

$$(5) \qquad P_t = \beta_1 N_{t-12} + \beta_2 (N_{t-1} - N_{t-13}) + \beta_3 I_{t-1} + \beta_4 U_{t-1}$$
$$+ \beta_5 N_{t-13} + \beta_6 (N_{t-2} - N_{t-14}) + \beta_7 I_{t-2} + \beta_8 U_{t-2} + \text{residual}.$$

From the estimated coefficients, we form the estimated dependent variable, lagged one period, $\hat{P}_{t-1}$. This is then used as an instrumental variable in equation (1), giving ordinary least squares estimates of γ which are now consistent. Hence the first difficulty outlined above is removed, and a consistent estimate of $f_{ee}(\theta)$ is now available for use in stage (iii).

Nevertheless the second difficulty described above, i.e. lack of asymptotic efficiency, still remains. In the absence of an alternative, this procedure is adopted, and the resulting consistent but asymptotically inefficient estimates are presented in Chapter 5. We see in Appendix II that the only way in which Aitken's generalized least squares estimates could achieve asymptotic efficiency in the presence of the lagged dependent variable would be for the residual covariance matrix, $\underline{Eee}' = \Gamma$, or equivalently the spectrum of residuals, $f_{ee}(\theta)$, to be known exactly.

A full example of the computations, containing the estimates derived at each stage of the procedure, is given in Appendix III. Summarizing, the procedure is as follows:

(i) regress P_t on the exogenous variables, current and lagged, as in equation (5).

(ii) from these estimated coefficients, form the regression estimate $\hat{P}_t$, and lag this one period.

(iii) use this $\hat{P}_{t-1}$ as an instrumental variable for P_{t-1} in an ordinary least squares estimation of equation (1).

(iv) calculate the estimated residuals from this regression, $\hat{e}_t$, and their spectrum $\hat{f}_{ee}(\theta_k)$.

(v) apply (4.2.16) to equation (1) using this estimate of $f_{ee}(\theta)$, giving a spectral analogue of Aitken's generalized least squares, and obtain final estimates $\hat{\gamma}$.

(vi) for the purposes of the comparison discussed in Section 4.1, compute spectra of the final regression estimate and regression residuals.

CHAPTER 5

EMPIRICAL RESULTS

We first present estimates of the coefficients of the production equation. Then, in Section 5.2, a restriction is placed upon this equation in order to achieve identification of the parameters of the model. The resulting coefficient estimates are presented in Section 5.3. The remaining sections are concerned with the goodness of fit of the regression; Section 5.4 with respect to the conventional variance-explained concept, and Section 5.5 with respect to seasonality in the dependent variable.

5.1. The production equation

In Table 2 are presented the results of the application of the above estimation procedure to the model

$$P_t = \gamma_1 P_{t-1} + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 U_{t-1} + \gamma_4 I_{t-1} + e_t .$$

The numbers in parentheses are the estimated standard errors of the respective coefficients.

The table also gives information with respect to goodness of fit. If the equation had been estimated by ordinary least squares, then we would have the result that $\sum_t \hat{P}_t \hat{e}_t = 0$, which would allow us to partition exactly the variance of the dependent variable into "explained" (var $\hat{P}_t$) and "unexplained" (var $\hat{e}_t$) variance. The above identity no longer holds, however, when Aitken's generalized least squares estimation is applied, hence the table presents var $\hat{P}_t$, var $\hat{e}_t$, and

TABLE 2

Estimates of the production equation

	Total durables	Primary metals	Machinery	Transport equipment	Total nondurable
P_{t-1}	0.562 (0.054)	0.467 (0.066)	0.507 (0.068)	0.682 (0.066)	0.134 (0.036)
N_{t-12}	0.455 (0.067)	0.096 (0.054)	0.235 (0.044)	0.044 (0.045)	0.820 (0.036)
$(N_{t-1}-N_{t-13})$	0.254 (0.041)	0.150 (0.036)	0.117 (0.039)	-0.002 (0.034)	0.748 (0.041)
U_{t-1}	-0.030 (0.010)	0.057 (0.017)	-0.015 (0.020)	0.005 (0.010)	-0.039 (0.064)
I_{t-1}	-0.068 (0.089)	0.266 (0.125)	0.108 (0.107)	0.370 (0.356)	0.030 (0.041)
constant	1965	517	987	774	630
estimated q	0.56	1.56	0.50	-0.04	0.91
$\frac{\text{var } \hat{P}_t}{\text{var } P_t}$	0.89	0.54	0.56	0.52	0.89
$\frac{\text{var } \hat{e}_t}{\text{var } P_t}$	0.21	0.37	0.23	0.32	0.03
$\frac{2 \text{ cov } \hat{P}_t \hat{e}_t}{\text{var } P_t}$	-0.10	0.09	0.20	0.16	0.08

2 x covariance $\hat{P}_t \hat{e}_t$, in each case expressed as a proportion of the variance of P_t .

The estimated coefficients of P_{t-1} are all significantly different from 0 and 1 . Interpreting these coefficients as 1 - b , where b is the "production adjustment coefficient," gives estimates of b of 0.438, 0.533, 0.493, 0.318, and 0.866 respectively. These values imply that the greatest flexibility in production planning is enjoyed by manufacturers of nondurable goods, while manufacturers of transportation equipment operate under the greatest pressures for production smoothing.

With regard to the terms in new orders, an estimate of q in the relation

$$\hat{N}_t = N_{t-12} + q(N_{t-1} - N_{t-13})$$

is obtained by dividing the coefficient of $(N_{t-1} - N_{t-13})$ by that of N_{t-12} . For transportation equipment, the former coefficient can be taken to be zero, while the latter is no larger than its standard error, hence the estimated q-value merits little attention. In the case of primary metals, the estimated q is greater than 1 , but the coefficient of N_{t-12} is not significant. A true value of q which was greater than 1 would cast doubt on the applicability of the model used for obtaining predictions. For referring to equation (3.4.3), and considering only the first-difference operator in this equation, we see that a value of 1.56 for q , or α , would imply that the series x_t is generated by the following explosive Markov process:

$$x_t = 1.56\, x_{t-1} + \epsilon_t ,$$

which is not plausible. In the remaining three manufacturing categories, both new orders coefficients are significant, and the values of q lie between 0 and 1, in accordance with the regressiveness of expectations.

The major unsatisfactory feature in these results concerns the coefficient of I_{t-1}. The hypotheses formulated in Section 3.1 imply, first, that this coefficient should be negative, and this turns out to be the case only for total durables and this coefficient is not significantly different from zero. Second, these hypotheses imply that, in the stock-production sector, the difference between the coefficient of P_{t-1} and that of I_{t-1} should be 1, and in the only case where the coefficient of I_{t-1} is negative, this relation does not hold. It was noted in Section 3.1, however, (see page 26) that the equation is over-identified in the sense that the two parameters of the model are estimated by three regression coefficients (equation (3.1.7)). We now seek to overcome this difficulty.

5.2 A restricted equation

Rewriting equation (3.1.6) in terms of the parameters of the model, and adding a subscript s to denote the stock-production sector, gives

$$(1) \qquad P_{s,t} = (1-b_s)P_{s,t-1} + (a+1)b_s\hat{N}_{s,t} - b_sI_{t-1} + e_{s,t} .$$

The restriction on the coefficients of $P_{s,t-1}$ and I_{t-1} may now be imposed, and identification achieved, by writing

$$(2) \qquad \Delta P_{s,t} = - b_s(P_{s,t-1} + I_{t-1}) + (a+1)b_s\hat{N}_{s,t} + e_{s,t}$$

where

$$\Delta P_{s,t} = P_{s,t} - P_{s,t-1} \cdot$$

For the production to order sector we have, from equation (3.2.5),

$$(3) \qquad P_{o,t} = (1-b_o)P_{o,t-1} + a_2 b_o \hat{N}_{o,t} + a_1 b_o U_{t-1} + e_{o,t} \cdot$$

An equation for production change, analogous to (2) above, may thus be written

$$(4) \qquad \Delta P_{o,t} = - b_o P_{o,t-1} + a_2 b_o \hat{N}_{o,t} + a b_o U_{t-1} + e_{o,t} \cdot$$

Equations (2) and (4) imply that the true specification for the aggregate dependent variable $\Delta P_t = \Delta P_{s,t} + \Delta P_{o,t}$ is

$$(5) \quad \Delta P_t = - b_s(P_{s,t-1} + I_{t-1}) - b_o P_{o,t-1} + (a+1)b_s \hat{N}_{s,t} + a_2 b_o \hat{N}_{o,t} + a_1 b_o U_{t-1} + e_t$$

whereas aggregation among the regressors implies that the following equation be estimated:

$$(6) \qquad \Delta P_t = \gamma_1(P_{t-1} + I_{t-1}) + \gamma_2 \hat{N}_t + \gamma_3 U_{t-1} + e_t \cdot$$

By an argument similar to that presented in Section 3.3, the relation between the estimated γ-coefficients and the coefficients of the two sector equations (2) and (4) is given by

$$E\,\hat{\gamma}_1 = d_{11}(- b_s) + (1-d_{11})(- b_o) + d_{13}[(a+1)b_s - a_2 b_o]$$

$$E\,\hat{\gamma}_2 = d_{23}(a+1)b_s + (1-d_{23})a_2 b_o + d_{21}[- b_s + b_o]$$

$$E\,\hat{\gamma}_3 = a_1 b_o + d_{31}[- b_s + b_o] + d_{33}[(a+1)b_s - a_2 b_o]$$

where the d's are estimated coefficients in regressions of the sector variables on the aggregate variables:

$$P_{s,t-1} + I_{t-1} = d_{11}(P_{t-1} + I_{t-1}) + d_{21}\hat{N}_t + d_{31}U_{t-1} + \text{residual}$$

$$P_{o,t-1} = (1-d_{11})(P_{t-1} + I_{t-1}) - d_{21}\hat{N}_t - d_{31}U_{t-1} + \text{"}$$

$$\hat{N}_{s,t} = d_{13}(P_{t-1} + I_{t-1}) + d_{23}\hat{N}_t + d_{33}U_{t-1} + \text{"}$$

$$\hat{N}_{o,t} = -d_{13}(P_{t-1} + I_{t-1}) + (1-d_{23})\hat{N}_t - d_{33}U_{t-1} + \text{"}$$

Again there is no aggregation bias if $b_s = b_o$ and $(a+1)b_s = a_2 b_o$. However the assumption of strict proportionality does not necessarily give consistent aggregation, for the assumption that $P_{s,t} = \lambda P_t$, $P_{o,t} = (1-\lambda)P_t$ does not imply that $(P_{s,t-1} + I_{t-1})$ and $(P_{t-1} + I_{t-1})$ are strictly proportional.

In estimating equation (6), the estimated coefficients will differ from those presented in Section 5.1 only by virtue of the restriction imposed, whereas the "variance explained" may be expected to drop further, by virtue of the change in dependent variable. This may be illustrated by a simple example.

Consider the equation

$$y_t = (1-b)y_{t-1} + ax_t + e_t \,, \tag{7}$$

which may equivalently be written

$$\Delta y_t = -by_{t-1} + ax_t + e_t \,. \tag{8}$$

It is clear that if both equations are estimated by the same procedure,

say ordinary least squares, then identical estimates of a and b will result. This implies that the estimated residuals, $\hat{e}_t$, will be the same in both cases. Writing R_1^2 and R_2^2 for the proportion of variance explained in the first and second equations respectively, we have

$$R_1^2 = 1 - \frac{\text{var } \hat{e}_t}{\text{var } y_t}, \quad R_2^2 = 1 - \frac{\text{var } \hat{e}_t}{\text{var } \Delta y_t}$$

thus

$$(1 - R_1^2) \text{ var } y_t = (1 - R_2^2) \text{ var } \Delta y_t .$$

Now

$$\text{var } \Delta y_t = \text{var } (y_t - y_{t-1}) = 2(1 - \rho) \text{ var } y_t$$

where ρ is the first order autocorrelation coefficient for the y_t-series. Therefore

$$R_1^2 = 2\rho - 1 + 2(1-\rho)R_2^2$$

and $R_1^2 \gtrless R_2^2$ according as $\rho \gtrless 0.5$.

In many economic time series measured over short periods of time, ρ lies between 0.75 and 1, hence we would expect in such a case to have $R_2^2 \ll R_1^2$.

5.3. Estimates of the production change equation

The coefficients of the equation

$$(1) \quad \Delta P_t = \gamma_1(P_{t-1} + I_{t-1}) + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 U_{t-1} + e_t$$

were estimated by a procedure analogous to that described in Sections 4.2 and 4.3. The equation was first estimated by ordinary least squares

and the estimated residuals, $\hat{e}_t$, obtained. The spectrum of these residuals was then calculated, and the formula of (4.2.16) p. 56, was finally applied, to give the spectral analogue of generalized least squares. An example of the calculations is given in Appendix III.

The estimates which were obtained are presented in Table 3. As expected, the proportion of variance explained fell considerably, and the residual variance rose. We shall return to this subject in Section 5.4.

We now have a direct estimate of the production adjustment coefficient by changing the sign of the estimated coefficient of $(P_{t-1} + I_{t-1})$. Again, the greatest flexibility in production plans is shown by total nondurable goods manufacturing, while the greatest lags in production changes are found in transportation equipment manufacturing. The estimated coefficients can be compared to the coefficients of the lagged production term in Mills' regression equation, [26]. In three regressions fitted to monthly data, his estimates were as follows:

Southern pine lumber	0.343
Pneumatic tires	0.559
Department store shoes	0.420

These are broadly comparable with our results, the industries used by Mills falling towards the nondurable end of the production spectrum.

Mack [24] has used the reciprocal of such coefficients as an indication of the time taken to execute a significant change in planned levels; this would suggest that the transportation equipment industry requires five months to plan and carry out a revision in production levels,

TABLE 3

Estimates of the production change equation

	Total durables	Primary metals	Machinery	Transport equipment	Total nondurable
$(P_{t-1}+I_{t-1})$	-0.277 (0.044)	-0.257 (0.049)	-0.236 (0.038)	-0.197 (0.051)	-0.523 (0.038)
N_{t-12}	0.384 (0.071)	0.081 (0.057)	0.193 (0.044)	0.043 (0.041)	0.754 (0.059)
$(N_{t-1}-N_{t-13})$	0.194 (0.043)	0.111 (0.037)	0.054 (0.037)	-0.003 (0.031)	0.585 (0.066)
U_{t-1}	-0.028 (0.011)	0.0003 (0.013)	-0.021 (0.020)	0.007 (0.008)	-0.151 (0.110)
regression constant	2220	861	1216	660	2007
estimated q	0.51	1.36	0.28	-0.07	0.78
$\frac{\text{var } \hat{\Delta P}_t}{\text{var } \Delta P_t}$	0.16	0.15	0.13	0.12	0.43
$\frac{\text{var } \hat{e}_t}{\text{var } \Delta P_t}$	0.64	0.81	0.75	0.93	0.32
$\frac{2 \text{ cov } \hat{\Delta P}_t \hat{e}_t}{\text{var } \Delta P_t}$	0.21	0.04	0.13	-0.05	0.24

whereas this can be achieved in two months by the nondurable goods manufacturers. To some extent these effects reflect differences in the amount of time required to manufacture the products of different industries. For example, the transportation equipment industry includes manufacturers of aircraft, which take several months to construct, while total nondurable goods industries include a considerable amount of continuous-process manufacturing. Differences in the economic batch sizes of continuous-process manufacturers are also reflected in production flexibility variations. The costs of production for a car assembly plant, for example, are such that relatively long production runs are undertaken, while a manufacturer of clothing and apparel operates under fewer economic pressures for long production runs, and hence finds it easier to amend production levels.

With respect to unfilled orders, the coefficients are not, in general, significant. In only one case, total durables manufacturing, is the estimated coefficient greater than twice its standard error. Moreover, in this case the coefficient has the wrong sign, the estimate implying that the unfilled order backlog enters the equation for desired production with a negative coefficient, i.e. the greater the backlog the smaller the planned production should be. Clearly, this coefficient cannot be taken at its face value. We shall comment further upon the orders problem below; here we simply note that in this context the remaining estimated coefficients of unfilled orders support the conclusion observed in Lovell's work by Eisner and Strotz (p. 11 above). They noted that unfilled orders does not play a significant role in the determination of finished goods inventory investment, and a similar conclusion could

be drawn concerning the determination of production levels by using the identity $P_t \equiv \Delta I_t + S_t$ to change the dependent variable of Lovell's regression [20].

Again we obtain an estimate of q by dividing the coefficient of $(N_{t-1} - N_{t-13})$ by that of N_{t-12}, and this gives results which are very similar to those obtained in Section 5.1. In the transportation equipment category, neither coefficient is significant, hence not much can be said about the estimated q. For primary metals, the calculation of q again involves a nonsignificant coefficient, and again gives a value greater than 1. In the three remaining cases, the values of q lie between 0 and 1. Of these three cases, the expectations of nondurable goods manufacturers are least regressive, while the past represents the greatest drag on future expectations for manufacturers of machinery.

A further feature of the estimated coefficients of the new orders terms, which was also present in the unrestricted equation, Table 2, is now noted. If we examine the first four categories only, then the new orders coefficients for the total durables group are greater than the corresponding coefficients for the three categories which are part of the total durables group, and significantly so, paying attention to the estimated standard errors. We suggest that this is another manifestation of the aggregation problem. The estimated relationship for total durables may be regarded as being an aggregate of the relationships for the three industry subgroups together with an "other durables" category. Specifically, we suggest that it may be the case that new orders placed by a firm enter into its

decision function, in addition to new orders received. A firm placing orders for supplies to be received in the future is essentially committing itself to the future production which will utilize these supplies, while the firm receiving the order will either deliver the goods from stock or produce the goods on, or shortly after, receipt of the order. Thus a given new order enters the decision function of both firms. If firms in the four subgroups of total durables manufacturing trade between subgroups, then, for example, orders received by a firm in the primary metals industry will also feature in the production rule of the firm in the transportation equipment industry which placed the orders. When the aggregate total durable equation is estimated, the new orders term in the production equation will have to do the work of both these effects, and hence will have a larger coefficient.

If we were to regard equation (1) above as applying to the stock-production sector alone, then an estimate of the desired inventory coefficient, a, can be obtained as

$$\hat{a} = \frac{\hat{\gamma}_2}{\hat{\gamma}_1} - 1 .$$

In the three industry subgroups this estimate is negative, which suggests that this procedure is not really appropriate. However for total durables this gives 0.387, and for total nondurables 0.442. In his paper using quarterly data on actual sales anticipations, Lovell [23] obtained estimates of the desired inventory coefficient of 0.242 and 0.194 for durables and nondurables respectively. If we multiply these estimates by 3 in order to achieve comparability

with our estimates from monthly data, then the results, 0.726 and 0.582, are somewhat greater than the estimates presented above. Nevertheless, for strict comparability we should probably correct for the change in forecast horizon prior to multiplying by 3: the firm using sales forecasts over a quarter may require a higher buffer stock in order to compensate for the greater uncertainties involved in forecasting over a quarter rather than a month. If it were possible to remove this effect from Lovell's estimates before multiplying by 3, the resulting estimates would be closer to our figures of 0.387 and 0.442.

5.4. Goodness of fit

In the previous section, having estimated the coefficients of the relation

$$\Delta P_t = \gamma_1(P_{t-1} + I_{t-1}) + \gamma_2\hat{N}_t + \gamma_3 U_{t-1} + e_t ,$$

we calculated the variance of the regression estimate $\hat{\Delta P}_t$. This was presented in Table 3 as a proportion of the variance of ΔP_t. We can equally regard the estimated coefficients as providing an estimate of P_t, which we may call a restricted regression estimate P_t^*, as follows:

$$P_t^* = P_{t-1} + \hat{\gamma}_1(P_{t-1} + I_{t-1}) + \hat{\gamma}_2\hat{N}_t + \hat{\gamma}_3 U_{t-1} .$$

This may now be compared with $\hat{P}_t$, the unrestricted regression estimate of Section 5.1, to give an evaluation of the restriction imposed on the regression equation. This will permit us to abstract

from the effect of using a first-differenced variable as the dependent variable, described in Section 5.2 and observed in Section 5.3, and concentrate on the increase in residual variance caused by the imposition of the restriction.

In Table 4 are presented the variance of P_t^*, the variance of the regression residuals $\hat{e}_t$, and 2 x covariance $P_t^*\hat{e}_t$, expressed as proportions of the variance of P_t. (Note that the same variance of residuals was presented as a proportion of var ΔP_t in Table 3.) For comparison, the corresponding figures for the unrestricted equation are presented; these also are expressed as proportions of the variance of P_t, and were presented in Table 2.

In all five cases the variance of the residuals is increased by imposing the restriction on the equation, as one would expect, although the increases are relatively small. The increase in residual variance provides a measure of the severity of the restriction in practice. If the estimated coefficients of the original equation almost fit the restriction, then imposing the restriction will make little difference, whereas a large increase in variance means that the restricted equation is considerably different from the original equation, and it is therefore doubtful that the restriction is true. Had these equations been estimated by ordinary least squares, it would have been possible to test this increase in residual variance, a statistically insignificant increase implying that the restricted equation and the original equation both represent the same situation, and in that sense the restriction is correct. However, when estimation procedures analogous to generalized least squares are used, such tests

TABLE 4

Variances and covariances of estimates and residuals for restricted and unrestricted equations (as a proportion of var P_t)

	Total durables		Primary metals		Machinery		Transportation equipment		Total nondurables	
	Restrict equation	Unrestr. equation	Restrict equation	Unrestr. equation	Restrict equation	Unrestr. equation	Restrict equation	Unrestr. equation	Restrict equation	Unrestr. equation
var $\hat{P}_t$ (var P_t^*)	0.86	0.89	0.69	0.54	0.61	0.56	0.61	0.52	0.82	0.89
var $\hat{e}_t$	0.24	0.21	0.46	0.37	0.28	0.23	0.33	0.32	0.06	0.03
2 cov $\hat{P}_t\hat{e}_t$	-0.11	-0.10	-0.15	0.09	0.10	0.20	0.07	0.16	0.12	0.08

no longer apply, and we have to be content with a simple examination of the results.

As already noted, imposing the restriction increases the residual variances. These results are confounded, however, by the changes in the covariance term in moving from one situation to the other. In three cases the covariance term is reduced by such an amount that the "explained variance" is also increased by imposing the restriction. From the results concerning the proportion of variance explained, the overall fit of the regression equation based on 132 observations appears to be good.

In the conventional generalized least squares approach, while it is true that the covariance $\hat{P}_t \hat{e}_t$ term is still present, the testing problem is overcome by working with transformed variables, as follows. If we wish to estimate the equation

$$y = X\alpha + e$$

by generalized least squares, using the matrix $\underline{Eee}' = \Gamma$, then an equivalent approach is to transform the original equation by using the inverse square root of Γ, $\Gamma^{-\frac{1}{2}}$, thus:

$$\Gamma^{-\frac{1}{2}} y = \Gamma^{-\frac{1}{2}} X\alpha + \Gamma^{-\frac{1}{2}} e$$

$$= \Gamma^{-\frac{1}{2}} X\alpha + e_1 , \quad \text{and} \quad \underline{Ee_1 e_1'} = I .$$

Ordinary least squares may now be applied to the transformed equation; this will be identical with a direct application of generalized least squares to the original equation, and will give efficient coefficient

estimates. The usual tests can now be performed on the transformed variables and their ordinary least squares estimates; in particular the covariance of $\Gamma^{-\frac{1}{2}}\hat{y}$ and $\hat{e}_1$ will be zero.

However, by working in the frequency domain, the possibility of performing the usual tests on transformed variables is not available to us. The matrix which would be equivalent to Γ were we to transfer back into the time domain at an intermediate stage is not readily available, hence we must await further theoretical developments before being able to test our "explained variance" terms.

Parenthetically, we might comment that the only publicly available computer program for generalized least squares which is known to us in fact operates on the principle just described. The stepwise procedure begins with an ordinary least squares estimation of the equation, then calculates $\hat{\rho}_1$, the first order serial correlation coefficient of the residuals and transforms the regression variables:

$$y_t^{(1)} = y_t - \hat{\rho}_1 y_{t-1}, \qquad x_t^{(1)} = x_t - \rho_1 x_{t-1}.$$

The second step applies ordinary least squares to the transformed variables, estimates the residual first order serial correlation coefficient $\hat{\rho}_2$ and transforms the variables again:

$$y_t^{(2)} = y_t^{(1)} - \hat{\rho}_2 y_{t-1}^{(1)}, \qquad x_t^{(2)} = x_t^{(1)} - \hat{\rho}_2 x_{t-1}^{(1)}.$$

And so on. This is equivalent to filling in the parallel diagonals of the Γ - matrix one at a time. The procedure stops when no significant change is made between one step and its predecessor. This feature can be removed; when using monthly data we might expect significant changes

when we reach the twelfth step but not at the eighth step, say, yet we wish to ensure that the routine actually continues to the twelfth step. Another difficulty exists in the necessity of beginning with an ordinary least squares estimate, which will not be appropriate if the lagged dependent variable is among the regressors.

5.5. Spectra of regression estimates and residuals

In Figures 6-10 we present the estimated spectra of the restricted regression estimate P_t^*, calculated as described in the previous section, and the regression residual $\hat{e}_t$. Again, the spectra were estimated using the methods discussed in Appendix I.

We find the spectra of the regression estimates exhibiting behavior similar to that of the spectra of the original production series (Figures 1-5). In general, the spectra of P_t^* have high values at low frequencies, low values at high frequencies, and peaks at the seasonal frequencies $\theta_k = \frac{\pi k}{6}$, $k = 1, \ldots, 6$. These characteristics were also found in the spectra of the P_t series. The general close agreement underlines the conclusion of the previous section that the regression equation fits the data quite well; not only is the variance explained high, but the distribution of variance over the frequency range also corresponds quite closely to that of the original dependent variable. In comparing the spectra in detail, the regression estimates tend to have less pronounced seasonal peaks than the original production series, particularly at high frequencies.

By contrast, the spectra of $\hat{e}_t$, the regression residuals, show considerable peaks at all six seasonal frequencies superimposed

Figure 6

Spectra of regression estimates (solid line) and residuals (broken line), total durable goods

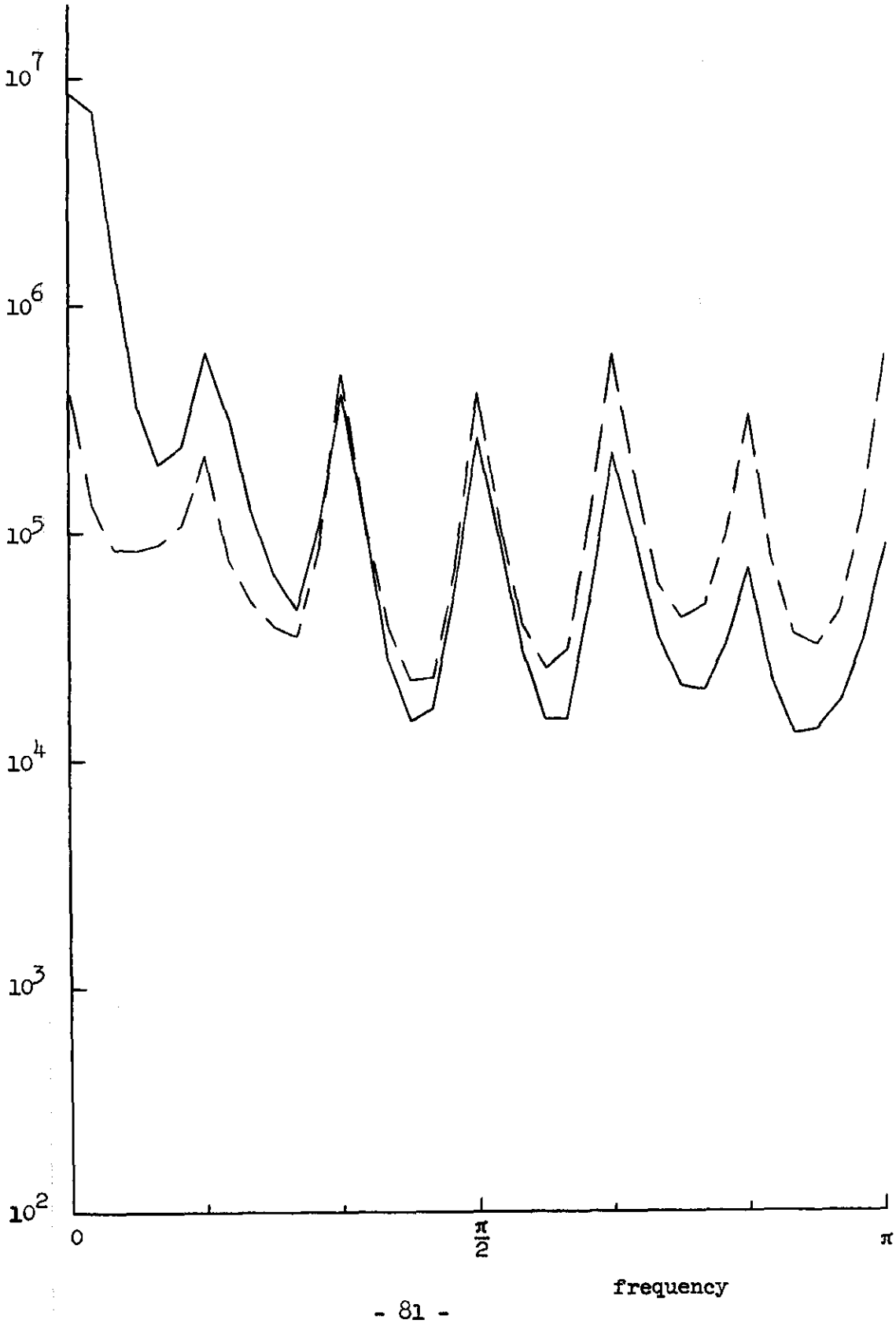


Figure 7

Spectra of regression estimates (solid line) and residuals (broken line), primary metals

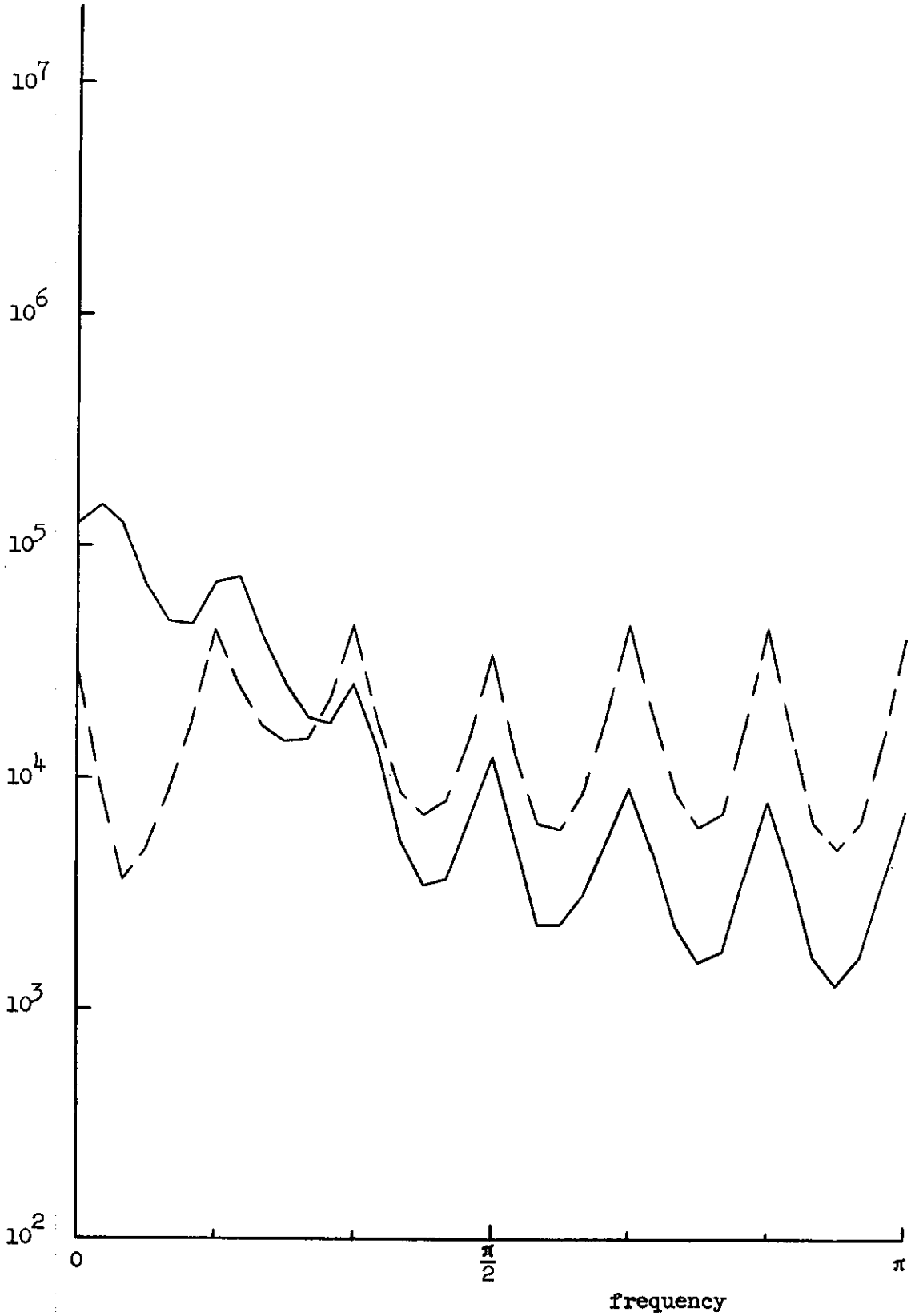


Figure 8

Spectra of regression estimates (solid line) and residuals (broken line), machinery

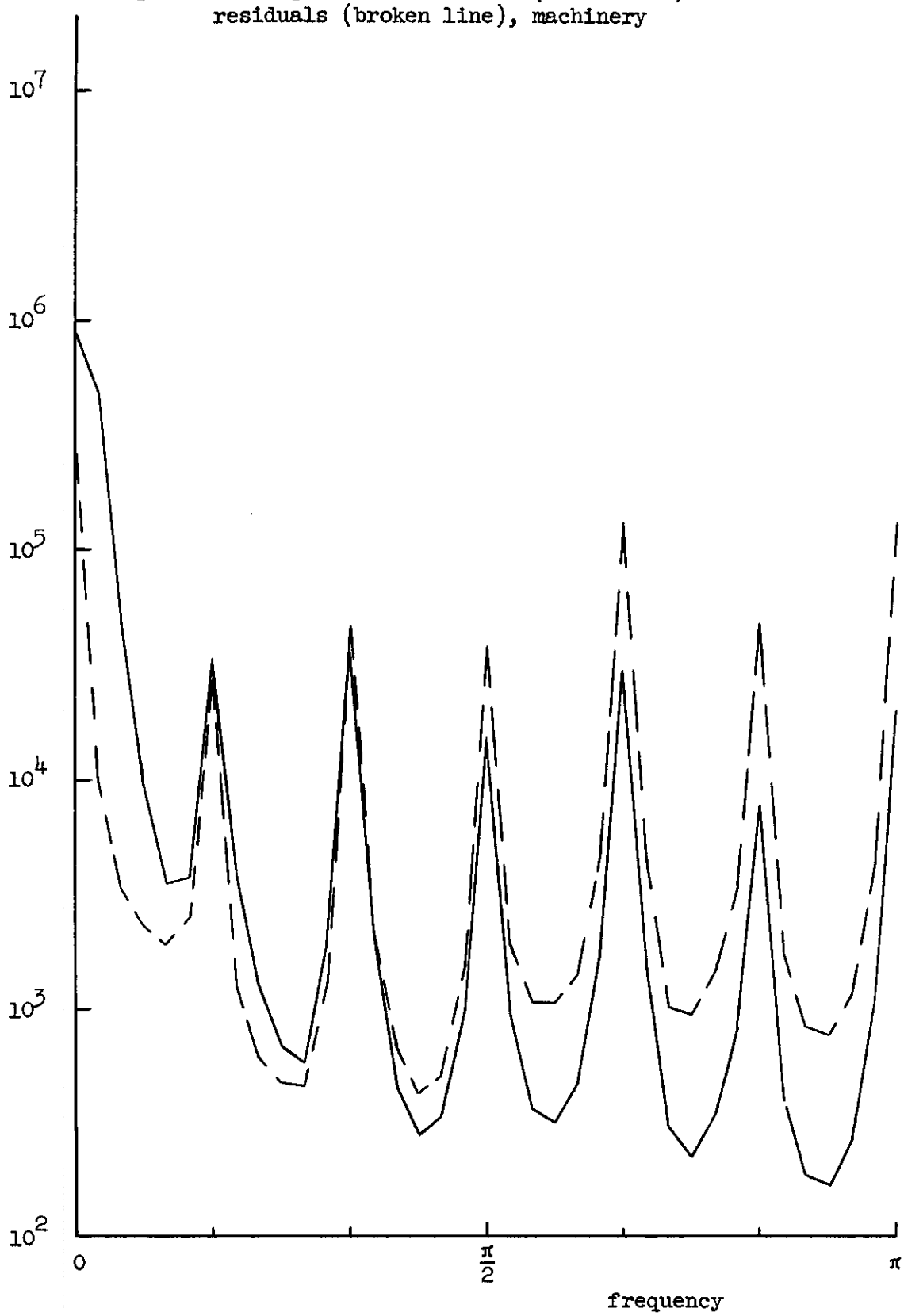


Figure 9

Spectra of regression estimates (solid line) and residuals (broken line), transportation equipment

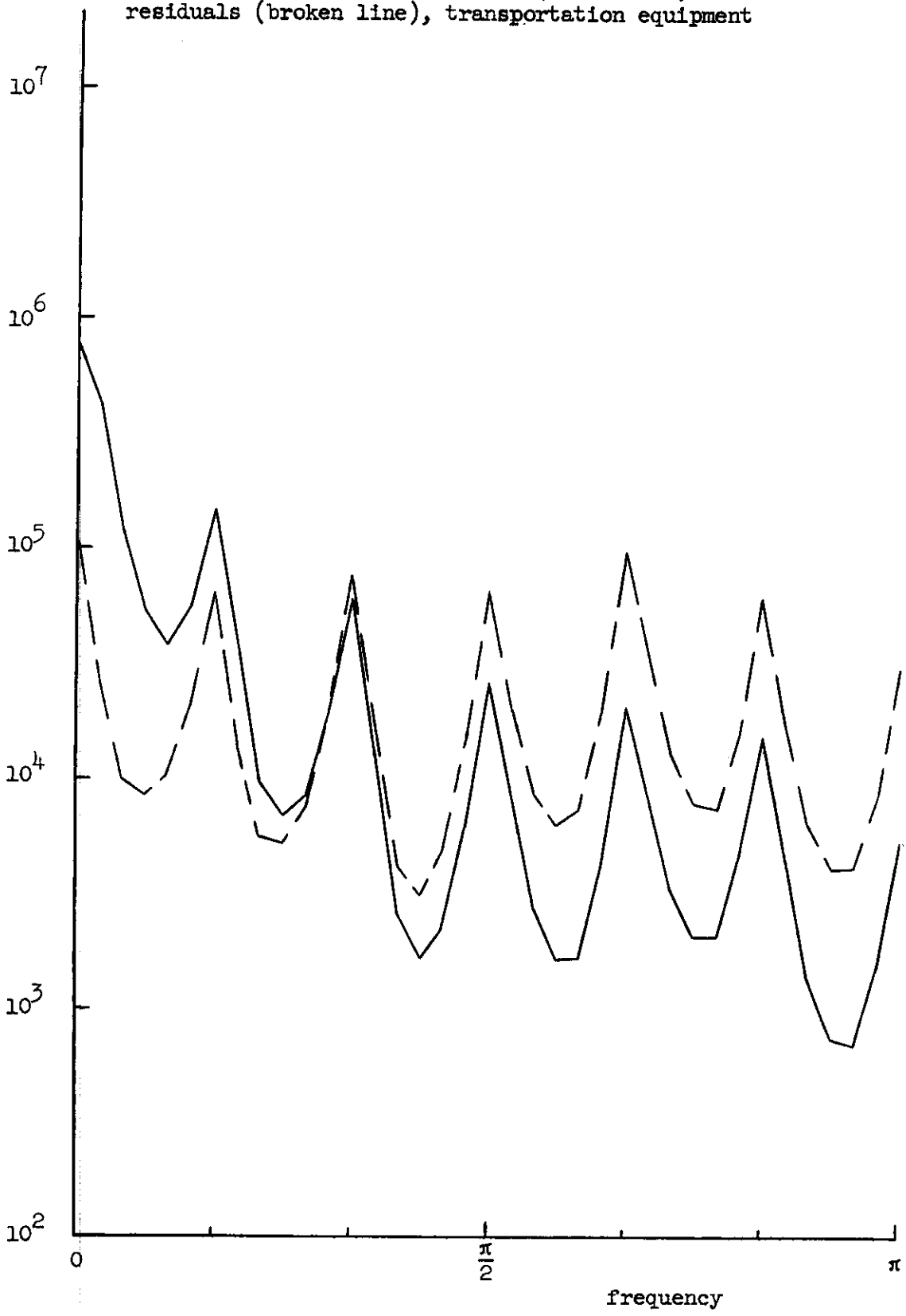
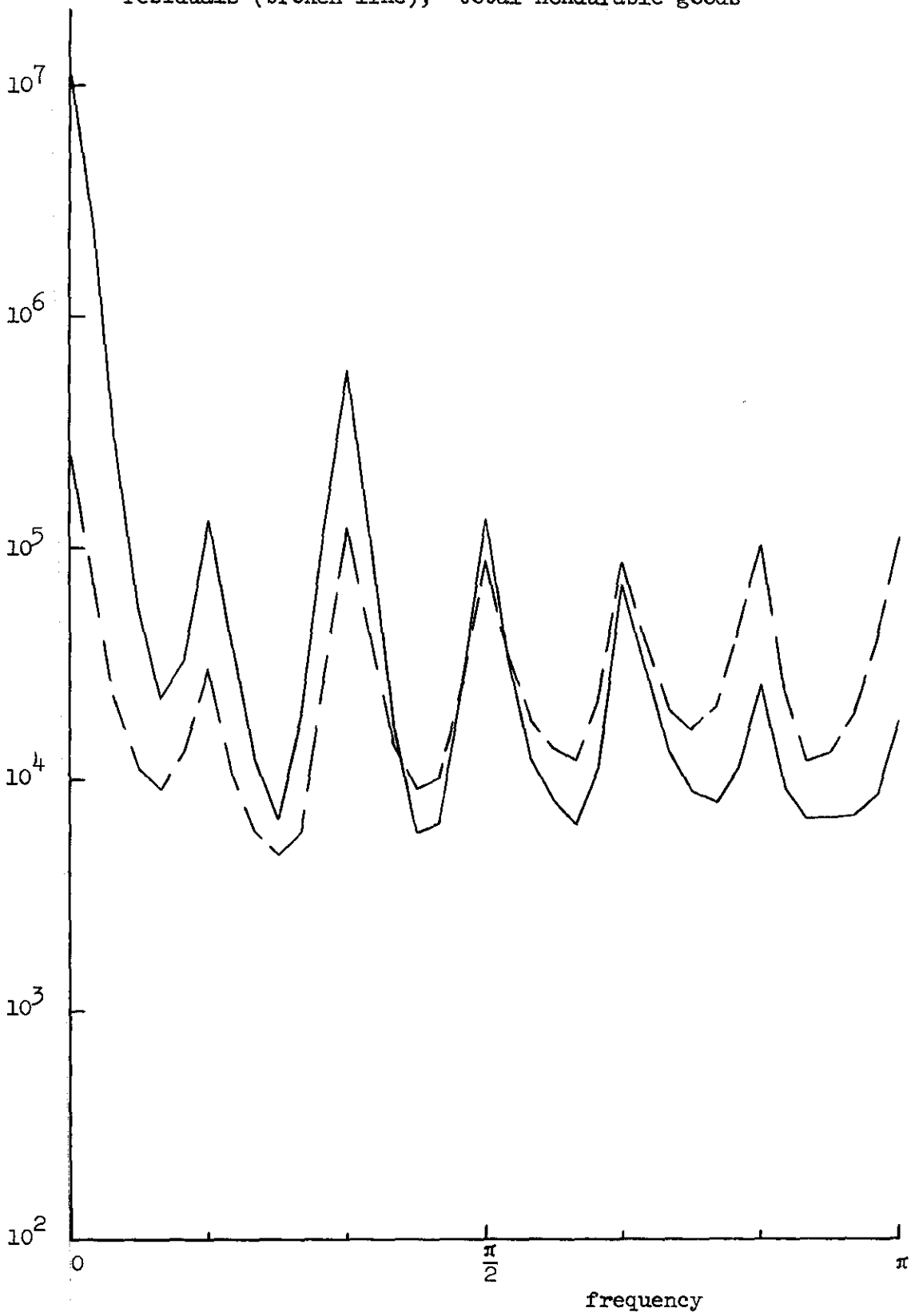


Figure 10

Spectra of regression estimates (solid line) and residuals (broken line), total nondurable goods



upon otherwise relatively flat spectra. The general height of these spectra reflects the total residual variance. As discussed in Section 4.1, we suggest that these spectra may be interpreted as follows: the regression variables account for the trend and other low-frequency components of the dependent variable, and a part of the seasonal; strong seasonality remains in the regression residuals. This considerable residual seasonality emphasizes the inapplicability of ordinary least squares procedures, for the residuals are by no means serially uncorrelated. It could probably be reduced by introducing dummy variables into the regression equation, and this would, moreover, increase the variance explained. Given that seasonal variations are not fixed but slowly changing, however, dummy variables cannot fully explain seasonality, and we prefer to leave it in the residuals. The residual seasonality could also be reduced by introducing a term in P_{t-12} into the regression equation. Our objective, however, was not to measure the seasonal pattern in the production series by itself, but to study the extent to which it could be regarded as being induced by seasonality in the regression variables, and the spectra indicate that this is the case to a considerable extent.

CHAPTER 6

CONCLUSIONS

In this study we consider a number of problems which arise in the analysis of inventory behavior. These problems are, on the one hand, methodological problems of wide applicability in econometrics and, on the other hand, questions surrounding the derivation and testing of hypotheses concerning the behavior of manufacturers' inventories of finished goods.

The methodological problems are, in the first place, those which arise from the use of seasonally unadjusted data in regression analysis. Such data are used in order to avoid the known inadequacies of seasonal adjustment procedures and their unknown effects on the relationships between time series. Seasonality in an economic time series is not an independent phenomenon, but influences, and is influenced by, seasonality in related series. Hence a model should attempt to explain seasonal variation in the dependent variables in much the same way that other types of variation are explained. Spectral analytic techniques have been used for the study of seasonal adjustment procedures, and it is proposed that one should use the spectra of the dependent variable, the regression estimate, and the regression residual in order to assess the extent to which the regression variables succeed in explaining the dependent variables at various frequencies, including seasonal frequencies. This is essentially a generalization of a well-known concept; one should study not only the amount of the variance of the dependent variable which is explained by the regression, but also

the way in which the explained variance is distributed over frequency components. In particular, the extent to which the regression variables succeed in explaining seasonality in the dependent variable can be ascertained by comparing the spectrum of the dependent variable with that of its regression estimate.

The second group of methodological problems concern serial correlation in regression estimation. Such correlation is increased as data for shorter time periods are employed, and ignoring its presence leads to inefficient estimates of the coefficients. The usual procedure for testing for the presence of serial correlation in regression residuals tests only for first order serial correlation, whereas the use of seasonally unadjusted data implies that serial correlation of higher orders may well be present and more important. Furthermore the usual test is inapplicable when the lagged dependent variable is among the regressors. As an alternative to the Durbin-Watson test, an examination of the spectrum of the residuals, can give general information on the nature of the stochastic process generating the regression residuals; such information should be incorporated into the estimation procedure in order to achieve efficient estimation. We await, however, the development of statistical distribution theory which would permit, for example, a significance test of whether or not the regression residuals possess serial correlation of any order. When the regression variables contain the lagged dependent variable, the consequences of ignoring serial correlation which is present are serious, hence in these circumstances we assume that the regression residuals are serially correlated,and proceed accordingly. We develop and apply an estimation procedure based on a spectral analytic analogue

of Aitken's generalized least squares, paying attention to the problem raised by the presence of the lagged dependent variable. For all practical purposes, this spectral procedure is free of assumptions about the time-structure of the regression residuals, and hence is able to handle more complex patterns than the simple first order Markov process.

Turning now to the hypotheses which are developed about the behavior of firms, we first draw a distinction between firms that produce in anticipation of demand ("production to stock") and those that produce in response to established demand ("production to order"). In these two "pure" cases, firms which produce to order carry no inventory of finished goods, while firms which produce to stock have no backlog of unfilled orders. These two sectors are considered separately, and an explicit treatment is given of the aggregation problem which arises when a single equation doing duty for both sectors is estimated using aggregated data. The estimation of an aggregate relation requires that a variable common to both sectors be used as the dependent variable, hence the discussion of the firm's behavior is oriented towards its production decision.

The model contains assumptions of production smoothing, in that firms find it costly to change production levels rapidly. The inventory and production decisions involve the firm's forecast of new orders to be received, and an explicit structure for the forecasting procedure is hypothesized. This bases forecasts on the recent observed past of the new orders series. The deviation of actual production from the planned level is assumed to be random, and a restriction is imposed upon the

equation in order to achieve identification of the structural parameters from the estimated coefficients. The resulting equation is estimated using monthly, seasonally unadjusted, deflated data, 1953-1964, for five industry groups.

The coefficient estimates give considerable support to the production smoothing hypothesis, with the greatest lags in production changes being found in transportation equipment manufacturing, and total nondurable goods manufacturing showing the greatest flexibility in production planning. Despite our specific description of the way in which terms in unfilled orders enter the model, their estimated coefficients are not, in general, significant. However, when the coefficients of the new orders terms obtained for the two aggregates, total durables and total nondurables manufacturing, are compared with those for three industry subgroups, it is clear that there are significant aggregation effects in this direction also. This suggests that in the estimation of inventory and production relations both at the industry level and for higher aggregates, a detailed account of trading by firms between industries should be given, and a distinction drawn between orders placed by firms and orders received by firms. Thus one direction in which our study could be extended would lead to the construction of a simultaneous-equations model, in which specific attention could be given to the economics of ordering and the timing of procurement.

Firms' expectations are again found to have a regressive tendency, in that while the direction of changes is correctly anticipated, the magnitude of changes is underestimated. After making a simplifying assumption (that the order-production sector can be

ignored), desired inventory coefficients are observed for two aggregated industry groups, total durables and total nondurables manufacturing, which conform to those observed by other investigators. The desired inventory coefficients obtained for the three industry subgroups are not reasonable, however, which suggests that the assumption is grossly oversimplifying and hence that little meaning can be attached to the desired inventory coefficients which do appear to be reasonable.

Overall, the equation fits the data relatively well, judging both on the conventional variance-explained grounds and also on the basis of an examination of the spectra of the production series and the regression estimates and residuals. This spectral comparison shows that seasonality in the dependent variable is to a considerable extent the consequence of the seasonal influence of the regression variables, although strong seasonality remains in the regression residuals. We conclude that the comparison of spectra of the dependent variable, its regression estimate, and the regression residual is a useful technique for studying the causes of seasonal fluctuations in the dependent variable in a regression model.

APPENDIX I

AN INTRODUCTORY NOTE ON SPECTRAL DENSITY FUNCTIONS AND THEIR ESTIMATION

The first few pages of this appendix present the basic ideas much in the order that they were developed. As in many other branches of statistics, the historical development of harmonic analysis of time series began with sample statistics, and only later move to theoretical consideration of population parameters.

Given $T = 2n$ points of a time series $\{x_t\}$, we may exactly represent these observations as a sum of sine and cosine terms, as follows

$$(1) \qquad x_t = \sum_{j=0}^{n} (a_j \cos \theta_j t + b_j \sin \theta_j t)$$

where $\theta_j = \frac{2\pi j}{T}$. We have $b_o = b_n = 0$, and the T nonzero coefficients are given by

$$(2) \qquad a_j = \frac{1}{n} \sum_{t=1}^{T} x_t \cos \theta_j t \,, \quad b_j = \frac{1}{n} \sum_{t=1}^{T} x_t \sin \theta_j t \,, \quad j = 1, 2, \ldots, n-1$$

$$a_o = \frac{1}{T} \sum_{t=1}^{T} x_t \,, \qquad a_n = \frac{1}{T} \sum_{t=1}^{T} x_t \cos t\pi$$

Imposing the requirement that x_t has a zero mean then implies $a_o = 0$. In essence, these coefficients are calculated by forming normal equations with the sine and cosine variables, and using the following orthogonality conditions:

$$\sum_{t=1}^{T} \sin \theta_j t \sin \theta_k t = \begin{cases} 0 & j \neq k \\ n & j = k \end{cases}$$

$$\sum_{t=1}^{T} \sin \theta_j t \cos \theta_k t = 0$$

$$\sum_{t=1}^{T} \cos \theta_j t \cos \theta_k t = \begin{cases} 0 & j \neq k \\ n & j = k \neq 0 \text{ or } n \\ T & j = k = 0 \text{ or } n \end{cases}$$

where $\theta_j = \frac{2\pi j}{T}$, $\theta_k = \frac{2\pi k}{T}$, $\qquad j, k = 0, 1, \ldots n$.

The representation (1) can be interpreted as a decomposition of the time series $\{x_t\}$ into orthogonal frequency components. This orthogonality implies that the sum of squares of the observed series may be partitioned into contributions from each frequency component, cross-product terms being zero. Hence

$$\sum_{t=1}^{T} x_t^2 = \sum_{t=1}^{T} \sum_{j=0}^{n} (a_j \cos \theta_j t + b_j \sin \theta_j t)^2$$

$$= \sum_{j=0}^{n} \left\{ a_j^2 \sum_{t=1}^{T} \cos^2 \theta_j t + b_j^2 \sum_{t=1}^{T} \sin^2 \theta_j t \right\}$$

$$= n \sum_{j=0}^{n} (a_j^2 + b_j^2) \text{ , by orthogonality conditions.}$$

The term $n(a_j^2 + b_j^2)$ may be interpreted as the contribution of the component of frequency θ_j to the total sum of squares. From (2), this contribution may be written

$$(3) \qquad I(\theta_j) = \frac{2}{T} \left[\left(\sum_{t=1}^{T} x_t \cos \theta_j t \right)^2 + \left(\sum_{t=1}^{T} x_t \sin \theta_j t \right)^2 \right] ,$$

which is known as the Schuster periodogram. When this is plotted as a function of θ , a relative peak at a particular frequency θ_j has the interpretation that the component with this frequency has a strong

influence in the observed series. Such a component has frequency $\frac{j}{T}$ cycles/unit time, and hence represents a cycle with period $\frac{T}{j}$. Note that cycles of frequency greater than 0.5 cycles/unit time, i.e. periods less than twice the interval between observations, cannot be directly discerned in the data. The first frequency point, $\theta_1 = \frac{2\pi}{T}$, corresponds to a cycle with period T, i.e. a cycle which is completed once during the period over which we have data.

We have seen that the T observations on the time series can be described in the frequency domain via their Fourier series representation. Regarding these observations as a realization of some underlying stochastic process, we next ask whether any stochastic process can be equally well described in the frequency and time domains. The answer is in the affirmative for stochastic processes which are covariance stationary, i.e. for which the covariance function $c_{xx}(\ell) = E\, x_t x_{t+\ell}$ is independent of time, t. For a discrete covariance stationary stochastic process $\{x_t\}$ we have the Cramer representation

$$x_t = \int_{-\pi}^{\pi} e^{it\theta}\, dZ(\theta)$$

$$c_{xx}(\ell) = \int_{-\pi}^{\pi} e^{i\ell\theta}\, dF(\theta) .$$

$dZ(\theta)$ is a complex random function with the properties

$$E\, dZ(\theta_1)\, \overline{dZ(\theta_2)} = 0 , \qquad \theta_1 \neq \theta_2$$

$$E\, |dZ(\theta)|^2 = dF(\theta) ,$$

and $F(\theta)$ is the <u>spectral distribution function.</u> $F(\theta)$ can be decomposed into $F_1(\theta)$, an absolutely continuous function, $F_2(\theta)$, a step

function which corresponds to perfectly periodic components of the time series, and $F_3(\theta)$, a singular function which may be neglected in economic applications. The component $F_1(\theta)$ is the most interesting from an economic point of view, and our discussion now proceeds in terms of its derivative, $f_{xx}(\theta)$, the spectral density function, or spectrum. For a fuller discussion of spectral theory see Hannan [14].

The spectrum and covariance function are a Fourier transform pair:

$$c_{xx}(\ell) = \int_{-\pi}^{\pi} e^{i\ell\theta} f_{xx}(\theta)\, d\theta$$

$$f_{xx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{xx}(\ell)\, e^{-i\ell\theta} . \tag{4}$$

In particular $c_{xx}(0) = \int_{-\pi}^{\pi} f_{xx}(\theta)\, d\theta$,

and so we have an interpretation of the spectrum analogous to that of the periodogram given above: $f_{xx}(\theta)$ represents the portion of the variance of $\{x_t\}$ which is contributed by the component of frequency θ. Noting that $c_{xx}(\ell)$ and $\cos \ell\theta$ are even functions of ℓ, whereas $\sin \ell\theta$ is odd, we have

$$f_{xx}(\theta) = \frac{1}{2\pi}\left\{c_{xx}(0) + 2 \sum_{\ell=1}^{\infty} c_{xx}(\ell) \cos \ell\theta\right\} \tag{5}$$

Returning to the periodogram, we may write

$$\begin{aligned} I(\theta) &= \frac{2}{T} \sum_{t,s} x_t x_s (\cos \theta t \cos \theta s + \sin \theta t \sin \theta s) \\ &= \frac{2}{T}\left\{ \sum_{t=1}^{T} x_t^2 + 2 \sum_{t<s} x_t x_s (\cos \theta t \cos \theta s + \sin \theta t \sin \theta s)\right\} \\ &= \frac{2}{T}\left\{ \sum_{t=1}^{T} x_t^2 + 2 \sum_{\ell=1}^{T-1} \sum_{t=1}^{T-\ell} x_t x_{t+\ell} [\cos \theta t \cos \theta(t+\ell) \right. \\ &\qquad \left. + \sin \theta t \sin \theta(t+\ell)]\right\} \end{aligned}$$

$$(6) \qquad \therefore\ I(\theta) = 2\left\{\hat{c}_{xx}(0) + 2\sum_{\ell=1}^{T-1}\hat{c}_{xx}(\ell)\cos\ell\theta\right\}$$

where $\hat{c}_{xx}(\ell) = \frac{1}{T}\sum_{t=1}^{T-\ell} x_t x_{t+\ell}$ is an estimate of the autocovariance function $c_{xx}(\ell) = E\, x_t x_{t+\ell}$. Hence from equations (5) and (6), $I(\theta) \to 4\pi f_{xx}(\theta)$ in expectation as $T \to \infty$, and we might consider using the periodogram to estimate the spectrum. However, as is shown by Hannan [14], the variance of the periodogram does not converge to zero as $T \to \infty$, thus the periodogram does not provide a consistent estimate of the spectrum. Furthermore, whereas one's intuition is that the spectrum should be fairly smooth, calculated periodograms are often very erratic and difficult to interpret (for some theoretical reasons for this, see Hannan [14, Ch. 3]). But if we consider the spectral distribution function instead of the spectral density function, and equivalently consider the summed periodogram, then

$$\int_{\theta_1}^{\theta_2} I(\theta)d\theta \quad \text{converges to} \quad 4\pi\int_{\theta_1}^{\theta_2} f_{xx}(\theta)d\theta\ ,$$

and the integrated periodogram does provide a consistent estimate of the integral of $f_{xx}(\theta)$. We may write

$$\int_{\theta_1}^{\theta_2} I(\theta)d\theta = \int_{-\pi}^{\pi} S(\theta)\, I(\theta)d\theta$$

where $S(\theta) = 1$ for $\theta_1 \le \theta \le \theta_2$ and 0 elsewhere. This suggests considering spectral estimates of the form

$$(7) \qquad \hat{f}_{xx}(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} S(\omega-\theta)\, I(\omega)d\omega$$

i.e. at a particular frequency, θ, we take a weighted average centered on that frequency. $S(\omega-\theta)$ is called a spectral window.

For reasons of computational convenience it has become customary to estimate the spectrum by first computing an estimated autocovariance function and then proceeding as in equation (5), rather than by computing the periodogram as in equation (3). In this case it becomes necessary to truncate the estimated autocovariance function at some point m, since as ℓ approaches T, the function

$$\hat{c}_{xx}(\ell) = \frac{1}{T} \sum_{t=1}^{T-\ell} x_t x_{t+\ell}$$

is based on fewer time series observations and its coefficient of variation becomes large. Thus, rather than estimate the spectrum by taking a weighted average of the periodogram as in (7), we take a weighted average of the autocovariance function, using a lag window $W(\ell)$, which is the inverse Fourier transform of the spectral window. $W(\ell)$ is defined as an even function, with $W(0) = 1$ and $W(\ell) = 0$ for $\ell > m$, and the estimation formula is as follows:

$$\begin{aligned} \hat{f}_{xx}(\theta) &= \frac{1}{2\pi} \sum_{\ell=-m}^{m} W(\ell)\, \hat{c}_{xx}(\ell)\, e^{-i\ell\theta} \\ &= \frac{1}{2\pi} \left\{ \hat{c}_{xx}(0) + 2 \sum_{\ell=1}^{m} W(\ell)\, \hat{c}_{xx}(\ell) \cos \ell\theta \right\}, \end{aligned} \tag{8}$$

the spectral window being related to the lag window by

$$W(\ell) = \int_{-\pi}^{\pi} S(\theta) e^{-i\ell\theta} d\theta \,. \tag{9}$$

To show the equivalence of the two approaches, we rewrite (6) as

$$I(\theta) = 2\left\{\sum_{\ell=-T+1}^{T-1} \hat{c}_{xx}(\ell)\, e^{-i\ell\theta}\right\}$$

and substitute in (7):

$$\hat{f}_{xx}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega-\theta) \sum_{\ell=-T+1}^{T-1} \hat{c}_{xx}(\ell)\, e^{-i\ell\omega}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{\ell=-T+1}^{T-1} \hat{c}_{xx}(\ell)\, e^{-i\ell\theta} \int_{-\pi}^{\pi} S(\omega-\theta)\, e^{-i\ell(\omega-\theta)}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{\ell=-m}^{m} W(\ell)\, \hat{c}_{xx}(\ell)\, e^{-i\ell\theta} \quad .$$

Since $\hat{f}_{xx}(\theta)$ is an even function, we only consider the interval $(0, \pi)$. The spectrum is estimated at m discrete points on this interval, hence (8) may be written as

$$(10)\quad \hat{f}_{xx}(\theta_k) = \frac{1}{2\pi}\left\{\hat{c}_{xx}(0) + 2\sum_{\ell=1}^{m} W(\ell)\, \hat{c}_{xx}(\ell) \cos \ell\theta_k\right\}, \quad k = 0, 1, \ldots, m$$
$$\theta_k = \frac{\pi k}{m}$$

We could compute the estimated spectrum at additional points, but this would add no information, and would amount to a procedure of interpolation between the points θ_k, $k = 0, 1, \ldots, m$ by means of a trigonometric polynomial.

Ideally, we should like to choose a spectral window that would be rectangular -- weight all frequencies in the range $\theta_k \pm \frac{1}{2}\Delta\theta$ equally, and give zero weight to all other frequencies. Unfortunately, it is impossible to find a finite lag window corresponding to this spectral window; the approximations which can be derived have

oscillations in the neighborhood of the discontinuities. A considerable part of the literature of spectral analysis is devoted to the problem of the design of windows (see Hannan [14, Ch. 3]). We use the Parzen window given by

$$W(\ell) = \begin{cases} 1 - 6\frac{\ell^2}{m^2}(1-\frac{\ell}{m}) & 0 \leq \ell \leq \frac{m}{2} \\ 2(1-\frac{\ell}{m})^3 & \frac{m}{2} \leq \ell \leq m \\ 0 & \ell > m \end{cases}$$

or, equivalently,

$$S(\omega-\theta) = \frac{3}{4\pi m^3}\left[\frac{\sin m(\omega-\theta)/4}{\sin (\omega-\theta)/4}\right]^4$$

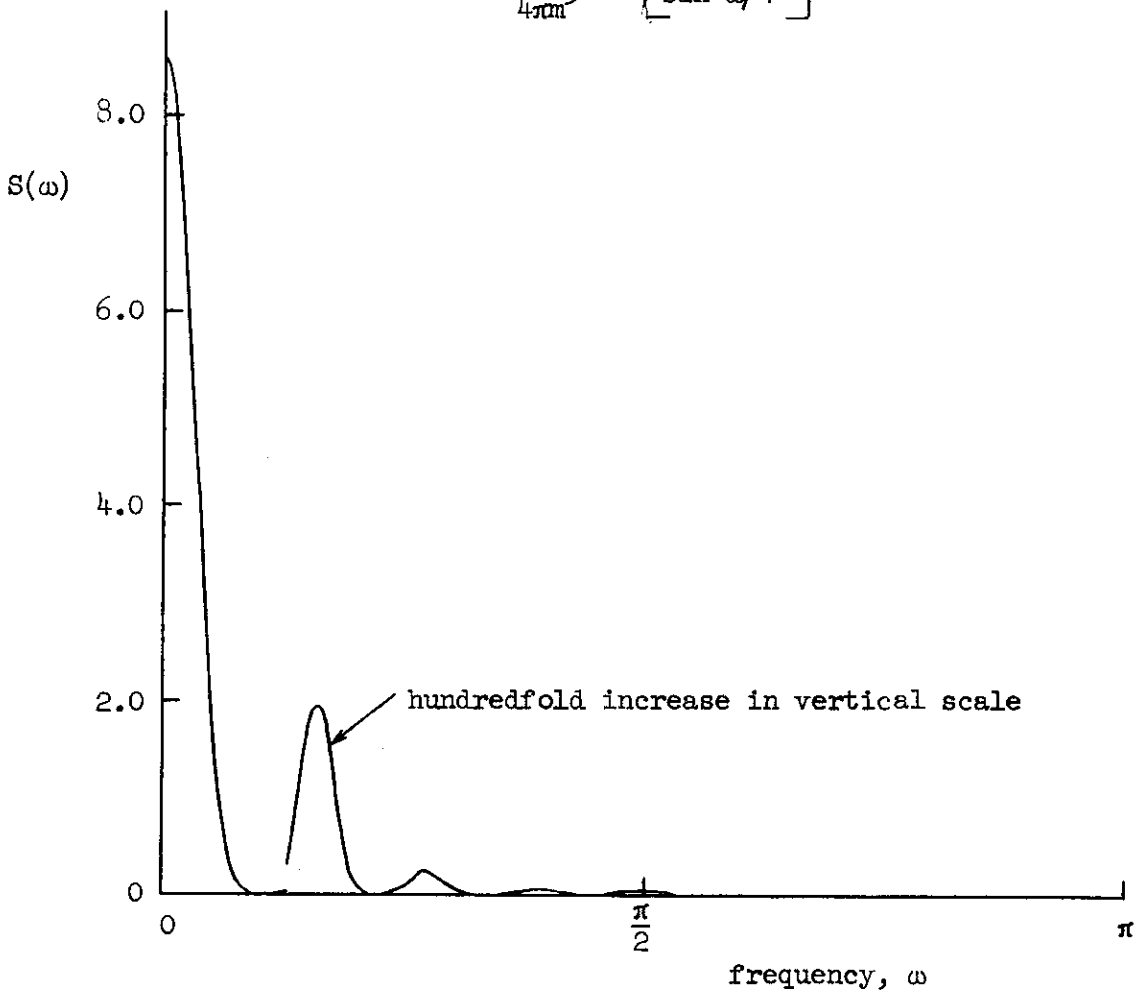
This spectral window, centered on $\theta = 0$, is plotted in Figure 11, using a value of $m = 36$, which was used in the estimation of the spectra described in Chapter 4. Since the spectral window is nowhere negative and since $\hat{c}_{xx}(\ell)$ is a positive definite function of ℓ, the estimated spectrum is always positive. (With other estimation procedures this is not always so; negative spectral estimates are, however, difficult to interpret, given that the spectrum represents a decomposition of the variance of the time series. Those who use such procedures argue, however, that such a negative estimate would provide useful evidence of a severe "leakage" problem in a particular case. See Granger and Hatanaka [10, p. 60 and p. 221]).

The covariance estimate which we use, namely $\hat{c}_{xx}(\ell)$, has a small-sample bias, although it is a minimum mean-square error estimate. For our purposes, its important property is positive

Figure 11

Parzen spectral window

$$S(\omega) = \frac{3}{4\pi m^3} \left[\frac{\sin m\omega/4}{\sin \omega/4}\right]^4, \quad m = 36$$



Given a time series $\{x_t\}$ we obtain the filtered series $\{\xi_t\}$ as follows

$$\xi_t = \sum_{r=o}^{p} a_r x_{t-r} \ .$$

The relationship between the covariance functions of the two series is given by

$$\begin{aligned} c_{\xi\xi}(\ell) &= E\ \xi_t \xi_{t+\ell} \\ &= E\left\{\sum_r a_r x_{t-r}\right\}\left\{\sum_s a_s x_{t+\ell-s}\right\} \\ &= \sum_r \sum_s a_r a_s\ c_{xx}(\ell+r-s)\ , \end{aligned}$$

thus the spectral density function is given by

$$\begin{aligned} f_{\xi\xi}(\theta) &= \frac{1}{2\pi} \sum_\ell c_{\xi\xi}(\ell)\ e^{-i\ell\theta} \\ &= \sum_r \sum_s a_r a_s\ e^{i(r-s)\theta}\ \frac{1}{2\pi} \sum_\ell c_{xx}(\ell+r-s)\ e^{-i(\ell+r-s)\theta} \\ &= \sum_r \sum_s a_r a_s\ e^{i(r-s)\theta}\ f_{xx}(\theta)\ . \end{aligned} \tag{11}$$

Writing $L(\theta) = \sum_r a_r e^{-ir\theta}$ as the frequency response function of the filter, we have

$$f_{\xi\xi}(\theta) = |L(\theta)|^2\ f_{xx}(\theta)\ .$$

Thus the estimated prewhitened spectrum $\hat{f}_{\xi\xi}(\theta_k)$ is first calculated, then an estimate of $f_{xx}(\theta)$ is obtained as follows:

$$\hat{f}_{xx}(\theta_k) = \frac{\hat{f}_{\xi\xi}(\theta_k)}{|L(\theta_k)|^2}$$

where the transfer function is calculated as

$$|L(\theta_k)|^2 = \sum_{r=o}^{p} a_r^2 + 2 \sum_{r=o}^{p-1} \sum_{s=r+1}^{p} a_r a_s \cos (r-s)\theta_k$$

for $\theta_k = \frac{\pi k}{m}$, $k = 0, 1, \ldots, m$.

The cross-spectrum

The lagged cross-covariance between two time series $\{y_t\}$ and $\{x_t\}$ is defined as $c_{yx}(\ell)$, where

$$c_{yx}(\ell) = E\, y_t x_{t+\ell} ,$$

and

$$c_{yx}(\ell) = c_{xy}(-\ell) .$$

Writing the cross-covariance function as the sum of an even part ${}_E c_{yx}(\ell)$ and an odd part ${}_O c_{yx}(\ell)$ then gives

$${}_E c_{yx}(\ell) = \frac{1}{2} [c_{yx}(\ell) + c_{yx}(-\ell)] = \frac{1}{2} [c_{yx}(\ell) + c_{xy}(\ell)]$$

$${}_O c_{yx}(\ell) = \frac{1}{2} [c_{yx}(\ell) - c_{yx}(-\ell)] = \frac{1}{2} [c_{yx}(\ell) - c_{xy}(\ell)] .$$

In similar fashion to equation (4), we define the cross-spectrum between series $\{y_t\}$ and $\{x_t\}$, $f_{yx}(\theta)$, as the Fourier transform of the cross-covariance function:

$$(12) \qquad f_{yx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{yx}(\ell)\, e^{-i\ell\theta}$$

This is a complex function, which we write as the sum of a real part, the co-spectrum $C_{yx}(\theta)$, and a negative imaginary part, the

quadrature spectrum $Q_{yx}(\theta)$ thus:

$$f_{yx}(\theta) = C_{yx}(\theta) - iQ_{yx}(\theta)$$

The co-spectrum and quadrature spectrum are respectively the cosine and sine transformations of the even and odd parts of the cross-covariance function:

$$C_{yx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{yx}(\ell) \cos \ell\theta$$

$$= \frac{1}{2\pi} \left\{ c_{yx}(0) + \sum_{\ell=1}^{\infty} [c_{yx}(\ell) + c_{xy}(\ell)] \cos \ell\theta \right\}$$

(13) $$Q_{yx}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{yx}(\ell) \sin \ell\theta$$

$$= \frac{1}{2\pi} \sum_{\ell=1}^{\infty} [c_{yx}(\ell) - c_{xy}(\ell)] \sin \ell\theta$$

Hence the co-spectrum is an even function of θ, while the quadrature spectrum is odd. Also

$$C_{yx}(\theta) = C_{xy}(\theta) , \qquad Q_{yx}(\theta) = - Q_{xy}(\theta) .$$

An interpretation of the cross-spectrum in the regression context is presented in Section 4.2; an alternative interpretation is given by writing the cross-spectrum in polar form:

$$f_{yx}(\theta) = A(\theta) \, e^{i\varphi(\theta)}$$

where the amplitude, $A(\theta)$, is equal to

$$A(\theta) = |f_{yx}(\theta)| = \sqrt{C_{yx}^2(\theta) + Q_{yx}^2(\theta)}$$

and the phase, $\varphi(\theta)$, is equal to

$$\varphi(\theta) = \arctan \left[\frac{- Q_{yx}(\theta)}{C_{yx}(\theta)} \right]$$

If the square of the amplitude is normalized by dividing by the spectra of the two series $\{y_t\}$ and $\{x_t\}$ at the given frequency θ, then we obtain a spectral analogue of the squared correlation coefficient, known as the coherence, $R^2(\theta)$.

$$R^2(\theta) = \frac{|f_{yx}(\theta)|^2}{f_{yy}(\theta)\, f_{xx}(\theta)}$$

The coherence lies between 0 and 1, and is essentially a measure of the association between the components of frequency θ of the two series. $\varphi(\theta)$ measures the phase relationship between these two components, i.e. the extent of the phase lag between the component of $\{y_t\}$ of frequency θ and the component of $\{x_t\}$ of frequency θ.

The estimation procedure is the same as that used to estimate the spectrum. We first estimate the cross-covariance functions up to some maximal lag m:

$$\hat{c}_{yx}(\ell) = \frac{1}{T} \sum_{t=1}^{T-\ell} y_t x_{t+\ell}, \qquad \hat{c}_{xy}(\ell) = \frac{1}{T} \sum_{t=1}^{T-\ell} x_t y_{t+\ell},$$

and then substitute these estimates, weighted by the Parzen window $W(\ell)$, into equations (13):

$$\hat{C}_{yx}(\theta_k) = \frac{1}{2\pi} \left\{ \hat{c}_{yx}(0) + \sum_{\ell=1}^{m} [\hat{c}_{yx}(\ell) + \hat{c}_{xy}(\ell)]\, W(\ell) \cos \ell\theta_k \right\}$$

$$\hat{Q}_{yx}(\theta_k) = \frac{1}{2\pi} \sum_{\ell=1}^{m} [\hat{c}_{yx}(\ell) - \hat{c}_{xy}(\ell)]\, W(\ell) \sin \ell\theta_k$$

$$\theta_k = \frac{\pi k}{m}, \qquad k = 0, 1, \ldots, m.$$

For prewhitening and recoloring co-spectra and quadrature spectra by means of linear filters, the basic approach is the same as that for spectra described above. A theoretical development of the prewhitening and recoloring of cross-spectra is given by Nerlove [30]. Given series $\{y_t\}$, $\{x_t\}$, we apply linear filters to obtain series $\{\eta_t\}$, $\{\xi_t\}$:

$$\eta_t = \sum_{r=o}^{p_1} a_r y_{t-r} , \qquad \xi_t = \sum_{s=o}^{p_2} b_s x_{t-s} .$$

For ease of exposition, let $p = \max(p_1, p_2)$ and let either $a_r = 0$ for $r = p_1 + 1, \ldots, p$ or $b_s = 0$ for $s = p_2 + 1, \ldots, p$, which enables us to write

$$\eta_t = \sum_{r=o}^{p} a_r y_{t-r} , \qquad \xi_t = \sum_{s=o}^{p} b_s x_{t-s} .$$

Then
$$\begin{aligned} c_{\eta\xi}(\ell) &= E\, \eta_t\, \xi_{t+\ell} \\ &= E \left(\sum_r a_r y_{t-r} \right) \left(\sum_s b_s x_{t+\ell-s} \right) \\ &= \sum_r \sum_s a_r b_s c_{yx}(\ell+r-s) , \end{aligned}$$

and, as in (11) above

$$f_{\eta\xi}(\theta) = \sum_r \sum_s a_r b_s\, e^{i(r-s)}\, f_{yx}(\theta) .$$

Writing

$$L_1(\theta) = \sum_r a_r\, e^{-ir\theta} , \quad L_2(\theta) = \sum_s b_s\, e^{-is\theta}$$

recoloring is accomplished as follows:

$$f_{yx}(\theta) = \frac{f_{\eta\xi}(\theta)}{\overline{L_1(\theta)}\, L_2(\theta)} = f_{\eta\xi}(\theta) \cdot \frac{L_1(\theta)\, \overline{L_2(\theta)}}{|L_1(\theta)|^2\, |L_2(\theta)|^2} \quad .$$

If we consider real and imaginary parts,

$$f_{yx}(\theta) = C_{yx}(\theta) - iQ_{yx}(\theta)$$

$$f_{\eta\xi}(\theta) = C_{\eta\xi}(\theta) - iQ_{\eta\xi}(\theta)$$

$$L_1(\theta)\, \overline{L_2(\theta)} = u(\theta) + iv(\theta)$$

Then estimates of the recolored co-spectrum and quadrature spectrum are obtained from the prewhitened estimates as follows:

$$\hat{C}_{yx}(\theta_k) = \frac{\hat{C}_{\eta\xi}(\theta_k)\, u(\theta_k) + \hat{Q}_{\eta\xi}(\theta_k)\, v(\theta_k)}{u^2(\theta_k) + v^2(\theta_k)}$$

$$\hat{Q}_{yx}(\theta_k) = \frac{\hat{Q}_{\eta\xi}(\theta_k)\, u(\theta_k) - \hat{C}_{\eta\xi}(\theta_k)\, v(\theta_k)}{u^2(\theta_k) + v^2(\theta_k)}$$

where $u(\theta_k)$ and $v(\theta_k)$ are calculated as follows:

$$u(\theta_k) = \sum_r \sum_s a_r b_s \cos (s-r)\theta_k$$

$$\theta_k = \frac{\pi k}{m} \quad , \quad k = 0, 1, \ldots, m \, .$$

$$v(\theta_k) = \sum_r \sum_s a_r b_s \sin (s-r)\theta_k$$

For further discussion of the estimation and interpretation of spectra and cross-spectra of economic time series see Granger and Hatanaka [10].

APPENDIX II

ON THE ASYMPTOTIC EFFICIENCY OF GENERALIZED LEAST SQUARES ESTIMATES

In this appendix we give a proof of the statement made in Section 4.3 that when Aitken's generalized least squares is applied using an estimated variance-covariance matrix of residuals, the resulting estimates are not asymptotically efficient if the lagged dependent variable is among the regressors. In part, this proof follows Amemiya and Fuller [2, part V].

Consider the model

$$y_t = \alpha_1 y_{t-1} + \alpha_2 x_t + u_t$$

$$u_t = \rho u_{t-1} + \epsilon_t$$

under the usual assumptions of well-behaved moments, where ϵ_t is a white noise series, and $|\alpha_1|, |\rho| < 1$. Writing X for the Tx2 matrix of regression variables, $[\underline{y}_{-1}, \underline{x}]$, and $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ we may write $y = X\alpha + u$.

$$E\, uu' = \Gamma = \frac{\sigma_\epsilon^2}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \rho \\ \rho^{T-1} & \cdots & \cdots & \rho & 1 \end{bmatrix}$$

and

$$\Gamma^{-1} = \frac{1}{\sigma_\epsilon^2} \begin{bmatrix} 1 & -\rho & & & 0 \\ -\rho & 1+\rho^2 & & & \\ & & \ddots & & \\ & & & 1+\rho^2 & -\rho \\ 0 & & & -\rho & 1 \end{bmatrix} .$$

First, if ρ and hence Γ is known, then

$$\hat{\alpha} = (X'\Gamma^{-1}X)^{-1} X'\Gamma^{-1}y$$

is a consistent and asymptotically efficient estimate of α. (An estimator is said to be asymptotically efficient if it is asymptotically normal and it attains the Cramer-Rao lower bound asymptotically.)
The asymptotic variance-covariance matrix of the estimated coefficients is V_A,

where $$V_A = \text{A.E.}\{T(\hat{\alpha}-\alpha)(\hat{\alpha}-\alpha)'\} = \text{plim } T(X'\Gamma^{-1}X)^{-1}$$

Now, suppose ρ is not known, and we use $\hat{\Gamma}$ based on some estimate $\hat{\rho}$ in the generalized least squares calculation.
i.e.

$$\hat{\alpha} = (X'\hat{\Gamma}^{-1}X)^{-1} X'\hat{\Gamma}^{-1}y .$$

$$(1) \qquad \therefore \sqrt{T}(\hat{\alpha}-\alpha) = \left(\frac{X'\hat{\Gamma}^{-1}X}{T}\right)^{-1} \frac{X'\hat{\Gamma}^{-1}u}{\sqrt{T}} ,$$

and we now consider the second term on the R.H.S. of this expression.
Expanding $\hat{\Gamma}^{-1}$ in a Taylor series gives

$$\hat{\Gamma}^{-1} = \Gamma^{-1} + \frac{\partial\Gamma^{-1}}{\partial\rho}(\hat{\rho}-\rho) + O(\hat{\rho}-\rho)^2$$

$$(2) \qquad \therefore \quad \frac{X'\hat{\Gamma}^{-1}u}{\sqrt{T}} = \frac{X'\Gamma^{-1}u}{\sqrt{T}} + \sqrt{T}(\hat{\rho}-\rho)\,\frac{X'\left[\frac{\partial\Gamma^{-1}}{\partial\rho}\right]u}{T} + O\left(\frac{1}{\sqrt{T}}\right)$$

since $O(\hat{\rho}-\rho)^2 = O\left(\frac{1}{T}\right)$ if $\hat{\rho}$ is an asymptotically normal estimate of ρ.

$$X'\left[\frac{\partial\Gamma^{-1}}{\partial\rho}\right]u = \frac{1}{\sigma_\epsilon^2}\begin{bmatrix} y_0 \cdots y_{T-1} \\ x_1 \cdots x_T \end{bmatrix}\begin{bmatrix} 0 & -1 & & \\ -1 & 2\rho & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix}$$

The second element of this 2x1 matrix is zero in expectation since x and u are independent and $Eu = 0$.

Considering the first element, we have

$$\begin{aligned} &- y_o u_2 + y_1[- u_1 + 2\rho u_2 - u_3] \\ &\quad + y_2[- u_2 + 2\rho u_3 - u_4] \\ &\quad \vdots \\ &\quad + y_{T-2}[- u_{T-2} + 2\rho u_{T-1} - u_T] - y_{T-1} u_{T-1} \end{aligned}$$

$$= - \sum_{1}^{T-1} y_t u_t - \sum_{0}^{T-2} y_t u_{t+2} + 2\rho \sum_{1}^{T-2} y_t u_{t+1} .$$

Now

$$y_t = \alpha_2 \sum_{0}^{t-1} \alpha_1^j x_{t-j} + \sum_{0}^{t-1} \alpha_1^j u_{t-j} + \alpha_1^t y_o .$$

$$\therefore E\, y_t u_t = E\, u_t \sum_{0}^{t-1} \alpha_1^j u_{t-j} + E\, \alpha_1^t y_o u_t$$

$$= \sum_{0}^{t-1} \alpha_1^j \rho^j \sigma_u^2 + E\, \alpha_1^t y_o u_t$$

$$\to \frac{\sigma_u^2}{1 - \alpha_1 \rho} \quad \text{as} \quad t \to \infty .$$

Similarly $E\, y_t u_{t+1} \to \dfrac{\rho \sigma_u^2}{1 - \alpha_1 \rho}$, $E\, y_t u_{t+2} \to \dfrac{\rho^2 \sigma_u^2}{1 - \alpha_1 \rho}$

Thus, considering the expectation in the limit of the first element of $\frac{1}{T} X' \left[\frac{\partial \Gamma^{-1}}{\partial \rho}\right] u$ gives

$$\lim \frac{1}{T} \frac{\sigma_u^2}{\sigma_\epsilon^2} \left\{ - \frac{(T-1)}{1-\alpha_1\rho} - \frac{(T-1)\rho^2}{1-\alpha_1\rho} + \frac{2\rho(T-2)\rho}{1-\alpha_1\rho} \right\}$$

$$= \frac{\sigma_u^2}{\sigma_\epsilon^2} \frac{(-1+\rho^2)}{1-\alpha_1\rho} = \frac{-1}{1-\alpha_1\rho}$$

since $\sigma_\epsilon^2 = \sigma_u^2(1-\rho^2)$.

$$(3) \qquad \therefore \text{ plim } \frac{X'\left[\frac{\partial \Gamma^{-1}}{\partial \rho}\right]u}{T} = \begin{bmatrix} \frac{-1}{1-\alpha_1\rho} \\ 0 \end{bmatrix}$$

(plim = lim E in this case since a law of large numbers applies)

Thus, returning to equation (2), we see that $\frac{X'\hat{\Gamma}^{-1}u}{\sqrt{T}}$

has the same limiting distribution as

$$\frac{X'\Gamma^{-1}u}{\sqrt{T}} + \begin{bmatrix} \frac{-1}{1-\alpha_1\rho} \\ 0 \end{bmatrix} \sqrt{T}(\hat{\rho}-\rho) .$$

Applying the Taylor series expansion as above in $\frac{X'\hat{\Gamma}^{-1}X}{T}$ we see that

$$\text{plim}\left(\frac{X'\hat{\Gamma}^{-1}X}{T}\right)^{-1} = \text{plim}\left(\frac{X'\Gamma^{-1}X}{T}\right)^{-1} = V_A .$$

Hence, substituting in equation (1) we have that $\sqrt{T}(\hat{\alpha}-\alpha)$ is asymptotically distributed as

$$(4) \qquad \frac{1}{\sqrt{T}} V_A X'\Gamma^{-1}u + V_A \begin{bmatrix} \frac{-1}{1-\alpha_1\rho} \\ 0 \end{bmatrix} \sqrt{T}(\hat{\rho}-\rho)$$

Thus, while the first term in this expression will result in a term V_A in the expression for A.E. $\{T(\hat{\alpha}-\alpha)(\hat{\alpha}-\alpha)'\}$, the second term will result in some additional terms in the asymptotic covariance matrix, implying that $\hat{\alpha}$ is not asymptotically efficient. The actual magnitude of these terms will depend on the asymptotic distribution of $\sqrt{T}(\hat{\rho}-\rho)$.

It is clear from the derivation prior to equation (3) that this effect is a consequence of the joint occurrence of two factors, namely the presence of the lagged dependent variable and the use of an estimate $\hat{\Gamma}$. If Γ were known, then only the first term of expression (4) would be present; if the matrix X contained only exogenous variables, then the right-hand side of (3) would be zero.

In order to evaluate a particular case, we consider the situation where ρ is estimated by the following procedure:

(i) estimate α by instrumental variables, using lagged x_t as an instrumental variable for y_{t-1}

(ii) Using the resulting estimates $\tilde{\alpha}$, calculate the residuals $\hat{u} = y - X\tilde{\alpha}$

(iii) estimate $\hat{\rho}$ as the first order serial correlation coefficient of these residuals.

Writing $\bar{X} = [\underline{x}_{-1}, \underline{x}]$ as the instrumental variable matrix,

$$\tilde{\alpha} = (\bar{X}'X)^{-1}\bar{X}'y .$$

$$\hat{u} = y - X\tilde{\alpha} = X\alpha + u - X(\bar{X}'X)^{-1}\bar{X}'[X\alpha + u]$$

$$= u - X(\bar{X}'X)^{-1}\bar{X}'u$$

$$\hat{u}_{-1} = y_{-1} - X_{-1}\tilde{\alpha}$$

$$= u_{-1} - X_{-1}(\bar{X}'X)^{-1}\bar{X}'u$$

$$\therefore \sqrt{T}\,\hat{\rho} = \frac{\{u'_{-1} - u'\bar{X}(X'\bar{X})^{-1}X'_{-1}\}\{u - X(\bar{X}'X)^{-1}\bar{X}'u\}/\sqrt{T}}{\{u'_{-1} - u'\bar{X}(X'\bar{X})^{-1}X'_{-1}\}\{u_{-1} - X_{-1}(\bar{X}'X)^{-1}\bar{X}'u\}/T}$$

Let plim $\frac{\bar{X}'X}{T} = H$.

The denominator of $\sqrt{T}\,\hat{\rho}$ converges in probability to σ_u^2 , since plim $\frac{u'\bar{X}}{T} = 0$; and in the numerator $\frac{u'_{-1}u}{\sqrt{T}}$ converges to $\sqrt{T}\,\rho\sigma_u^2$.

$$\therefore \sqrt{T}(\hat{\rho}-\rho) \simeq \frac{1}{\sigma_u^2}\left\{ - \frac{u'_{-1}X}{T}\left[\frac{\bar{X}'X}{T}\right]^{-1} \frac{\bar{X}'u}{\sqrt{T}} - \frac{u'\bar{X}}{\sqrt{T}}\left[\frac{X'\bar{X}}{T}\right]^{-1}\frac{X'_{-1}u}{T}\right\}$$

$$\frac{u'_{-1}X}{T} = \frac{u'_{-1}}{T}[\underline{y}_{-1}, \underline{x}] = \frac{1}{T}\left[\Sigma\, u_{t-1}y_{t-1}, \Sigma\, u_{t-1}x_t\right]$$

$$\therefore \text{plim}\ \frac{u'_{-1}X}{T} = \left[\frac{\sigma_u^2}{1-\alpha_1\rho}\ ,\ 0\right] \text{ using results obtained above}$$

and, similarly, $\text{plim}\ \frac{X'_{-1}u}{T} = \begin{bmatrix} \frac{\sigma_u^2\rho^2}{1-\alpha_1\rho} \\ 0 \end{bmatrix}$.

$\therefore \sqrt{T}(\hat{\rho}-\rho)$ is asymptotically distributed as

$$-\left\{\left[\frac{1}{1-\alpha_1\rho}\ ,\ 0\right]H^{-1}\frac{\bar{X}'u}{\sqrt{T}} + \frac{u'\bar{X}}{\sqrt{T}}H^{-1\prime}\begin{bmatrix}\frac{\rho^2}{1-\alpha_1\rho}\\ 0\end{bmatrix}\right\}$$

These are scalar quantities, so we may transpose the second term, giving

$$-\left[\frac{1+\rho^2}{1-\alpha_1\rho}\ ,\ 0\right]H^{-1}\frac{\bar{X}'u}{\sqrt{T}}$$

Substituting this into equation (4), we have that $\sqrt{T}(\hat{\alpha}-\alpha)$ is

asymptotically distributed as

$$V_A \frac{X'\Gamma^{-1}u}{\sqrt{T}} + V_A \begin{bmatrix} -1 \\ 1 - \alpha_1\rho \\ 0 \end{bmatrix} (-1) \begin{bmatrix} \frac{1+\rho^2}{1-\alpha_1\rho} , & 0 \end{bmatrix} H^{-1} \frac{\bar{X}'u}{\sqrt{T}}$$

$$= V_A \frac{X'\Gamma^{-1}u}{\sqrt{T}} + V_A Q H^{-1} \frac{\bar{X}'u}{\sqrt{T}} ,$$

writing

$$Q = \begin{bmatrix} \frac{1+\rho^2}{(1-\alpha_1\rho)^2} & 0 \\ 0 & 0 \end{bmatrix} .$$

Now A.E. $\{T(\hat{\alpha}-\alpha)(\hat{\alpha}-\alpha)'\}$

$$= \text{A.E.} \{ \frac{1}{T} V_A X'\Gamma^{-1}uu'\Gamma^{-1}XV_A + \frac{1}{T} V_A X'\Gamma^{-1}uu'\bar{X} H^{-1'} Q V_A$$
$$+ \frac{1}{T} V_A Q H^{-1}\bar{X}'uu'\Gamma^{-1}XV_A + \frac{1}{T} V_A Q H^{-1}\bar{X}'uu'\bar{X} H^{-1'} Q V_A \}$$
$$= V_A + 2V_A Q V_A + V_A Q V_{IV} Q V_A$$

where $V_{IV} = H^{-1} \operatorname{plim} \frac{\bar{X}'\Gamma\bar{X}}{T} H^{-1'}$ is the asymptotic covariance matrix of the instrumental variable estimate $\tilde{\alpha}$.

Thus we see, in this one particular example, the extent to which the use of an estimate of ρ and the presence of the lagged dependent variable cause the covariance matrix of the estimated coefficients to deviate from V_A .

APPENDIX III

AN EXAMPLE OF THE COMPUTATIONS IN DETAIL

In this appendix we illustrate the estimation procedure described in Section 4.3 and summarized on pages 61-62 by presenting the intermediate results obtained at each stage of the procedure for one category, namely total durable goods manufacturing. The basic equation to be estimated is

$$(1) \qquad P_t = \gamma_1 P_{t-1} + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 U_{t-1} + \gamma_4 I_{t-1} + e_t \,.$$

(i) Regressing P_t on the exogenous variables, current and lagged, using ordinary least squares, gives the following estimates:

$$P_t = \underset{(0.067)}{0.542}\, N_{t-12} + \underset{(0.094)}{0.538}(N_{t-1} - N_{t-13}) - \underset{(0.107)}{0.419}\, U_{t-1}$$

$$+ \underset{(0.447)}{1.595}\, I_{t-1} + \underset{(0.085)}{0.059}\, N_{t-13} + \underset{(0.067)}{0.106}(N_{t-2} - N_{t-14})$$

$$+ \underset{(0.104)}{0.442}\, U_{t-2} - \underset{(0.403)}{1.105}\, I_{t-2} + \underset{(1181)}{1038}$$

$$n = 132 \qquad R^2 = 0.83$$

(ii) These estimated coefficients are then used to construct the lagged regression estimate, $\hat{P}_{t-1}$, which is used as an instrumental variable for P_{t-1} in an ordinary least squares estimation of the basic equation.

(iii) This gives the following results:

$$P_t = \underset{(0.100)}{0.123}\, P_{t-1} + \underset{(0.068)}{0.537}\, N_{t-12} + \underset{(0.048)}{0.456}(N_{t-1} - N_{t-13})$$

$$+ \underset{(0.018)}{0.017}\, U_{t-1} + \underset{(0.186)}{0.420}\, I_{t-1} + \underset{(1209)}{1108}$$

$$n = 132 \qquad R^2 = 0.80$$

(iv) The regression residuals from this equation and their estimated spectrum are then calculated, and the method of equation (4.2.16) is then applied.

(v) This gives the following results, as reported in Table 2:

$$P_t = \underset{(0.054)}{0.562}\, P_{t-1} + \underset{(0.067)}{0.455}\, N_{t-12} + \underset{(0.041)}{0.254}(N_{t-1} - N_{t-13})$$

$$- \underset{(0.010)}{0.030}\, U_{t-1} - \underset{(0.089)}{0.068}\, I_{t-1} + 1965$$

$$\frac{\text{var } \hat{P}_t}{\text{var } P_t} = 0.89 \,, \quad \frac{\text{var } \hat{e}_t}{\text{var } P_t} = 0.21 \,, \quad \frac{2 \text{ cov } \hat{P}_t \hat{e}_t}{\text{var } P_t} = -0.10$$

We now give some computational details concerning this last step. Equation (4.2.16), namely

$$\hat{\gamma} = \left\{ \sum_k \frac{\hat{F}_{zz}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\}^{-1} \left\{ \sum_k \frac{\hat{f}_{yz}(\theta_k)}{\hat{f}_{ee}(\theta_k)} \right\} \tag{2}$$

may be abbreviated to

$$\hat{\gamma} = \underset{\sim}{G}^{-1} \underline{H}$$

In our application, the row vector of regression variables, $\underline{z}_t'$, is

$$[P_{t-1}, N_{t-12}, (N_{t-1} - N_{t-13}), U_{t-1}, I_{t-1}],$$

and the elements of the 5x5 matrix $\underset{\sim}{G}$ consist of the cross-spectra

between these five variables divided by the spectrum of residuals and summed over k. For example, the element $G(2,4)$ is based on the cross-spectrum between N_{t-12} and U_{t-1}. Similarly, the five elements of the column vector $\underline{H}$ are based on the cross-spectra between P_t and the regression variables.

Considering the summation process, we need to determine the range of k such that θ_k ranges over the interval $(-\pi,\pi)$; in effect, we are summing the cross-spectrum around the unit circle. Writing the cross-spectrum as

$$f_{yx}(\theta) = C_{yx}(\theta) - iQ_{yx}(\theta)$$

we recall that the co-spectrum, $C_{yx}(\theta)$, is an even function of θ while the quadrature spectrum, $Q_{yx}(\theta)$, is odd. Thus, $Q_{yx}(\theta)$ will sum to zero over the interval $(-\pi,\pi)$, and in summing $C_{yx}(\theta)$ we need only consider the half-range $(0,\pi)$. Hence we sum $\hat{C}_{yx}(\theta_k)$ over $k = 0, 1, \ldots, m$, multiplying the values for $k = 1, 2, \ldots, m-1$ by a factor of 2. These considerations remain unaltered after division by $\hat{f}_{ee}(\theta_k)$, for this also is an even function. The fact that we need only consider the co-spectrum means that the matrix $\underset{\sim}{G}$ is symmetric, for $C_{yx}(\theta) = C_{xy}(\theta)$.

With respect to the computation of the cross-spectra required for $\underset{\sim}{G}$ and $\underline{H}$, we compute the six cross-spectra for the four series P_t, N_t, U_t and I_t taken pair-by-pair, and then use the following results. Given time series $\{y_t\}$ and $\{x_t\}$, consider the series formed by lagging the original series r and s periods respectively. The covariance function is given by

$c_{y_{-r}x_{-s}}(\ell) = c_{yx}(\ell+r-s)$, and the cross-spectrum between the lagged series is derived as follows:

$$f_{y_{-r}x_{-s}}(\theta) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{y_{-r}x_{-s}}(\ell) e^{-i\ell\theta}$$

$$= \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_{yx}(\ell+r-s) e^{-i(\ell+r-s)\theta} \; e^{i(r-s)\theta}$$

$$= f_{yx}(\theta) \; e^{i(r-s)\theta}$$

$$= [C_{yx}(\theta) - iQ_{yx}(\theta)][\cos (r-s)\theta + i \sin (r-s)\theta]$$

Thus the co-spectrum between the two lagged series is given by

$$C_{y_{-r}x_{-s}}(\theta) = C_{yx}(\theta) \cos (r-s)\theta + Q_{yx}(\theta) \sin (r-s)\theta \; .$$

Secondly, consider the series $\{z_t\}$ formed from two series $\{x_{1,t}\}$ and $\{x_{2,t}\}$ by addition,

i.e. $$z_t = x_{1,t} + x_{2,t}$$

The cross-spectrum between y_t and z_t is given by

$$f_{yz}(\theta) = f_{yx_1}(\theta) + f_{yx_2}(\theta)$$

while the spectrum of z_t is obtained as follows:

$$c_{zz}(\ell) = E(x_{1,t} + x_{2,t})(x_{1,t+\ell} + x_{2,t+\ell})$$

$$= c_{x_1x_1}(\ell) + c_{x_2x_2}(\ell) + c_{x_1x_2}(\ell) + c_{x_2x_1}(\ell)$$

thus

$$f_{zz}(\theta) = f_{x_1x_1}(\theta) + f_{x_2x_2}(\theta) + 2C_{x_1x_2}(\theta)$$

These relations enable us to compute the co-spectra needed to calculate the elements of $\underset{\sim}{G}$, given the cross-spectra between P_t, N_t, U_t and I_t. For example, $G(1,3) = G(3,1)$ requires computation of the co-spectrum between P_{t-1} and $(N_{t-1} - N_{t-13})$, which is given by

$$f_{P_{-1}(N_{-1}-N_{-13})}(\theta) = f_{P_{-1}N_{-1}}(\theta) - f_{P_{-1}N_{-13}}(\theta)$$

$$= f_{PN}(\theta) - f_{PN}(\theta)\, e^{-i12\theta} .$$

$$\therefore C_{P_{-1}(N_{-1}-N_{-13})}(\theta) = C_{PN}(\theta)[1 - \cos 12\theta] + Q_{PN}(\theta) \sin 12\theta$$

$$\therefore G(1,3) = 2 \sum_{k=1}^{m-1} \frac{1}{\hat{f}_{ee}(\theta_k)} \left\{ \hat{C}_{PN}(\theta_k)[1 - \cos 12\theta_k] + \hat{Q}_{PN}(\theta_k) \sin 12\theta_k \right\}$$

since $C_{P_{-1}(N_{-1}-N_{-13})}(\theta)$ is zero for $\theta = 0$ and $\theta = \pi$.

As a second example, $H(4)$ is based on the co-spectrum between P_t and U_{t-1}:

$$C_{PU_{-1}}(\theta) = C_{PU}(\theta) \cos\theta - Q_{PU}(\theta) \sin\theta$$

$$\therefore H(4) = \frac{\hat{C}_{PU}(0)}{\hat{f}_{ee}(0)} + 2 \sum_{k=1}^{m-1} \frac{1}{\hat{f}_{ee}(\theta_k)} \left\{ \hat{C}_{PU}(\theta_k) \cos\theta_k - \hat{Q}_{PU}(\theta_k) \sin\theta_k \right\} - \frac{\hat{C}_{PU}(\pi)}{\hat{f}_{ee}(\pi)}$$

Turning now to the restricted equation introduced in Section 5.2, namely

$$(3) \qquad \Delta P_t = \gamma_1(P_{t-1} + I_{t-1}) + \gamma_2 N_{t-12} + \gamma_2 q(N_{t-1} - N_{t-13}) + \gamma_3 U_{t-1} + e_t ,$$

the first stage consists of an ordinary least squares estimation of this equation. For the total durable goods category, this gives the following estimates:

$$\Delta P_t = \underset{(0.050)}{-\ 0.480}(P_{t-1} + I_{t-1}) + \underset{(0.067)}{0.615}\, N_{t-12} + \underset{(0.045)}{0.333}(N_{t-1} - N_{t-13})$$

$$\underset{(0.012)}{-\ 0.054}\, U_{t-1} + \underset{(1063)}{4802}$$

$$n = 132 \qquad R^2 = 0.44$$

The estimated residuals from this equation, and their spectrum, are then calculated, and the method of equation (4.2.16) (equation (2) above) applied, giving the following results, as reported in Table 3.

$$\Delta P_t = \underset{(0.044)}{-\ 0.277}(P_{t-1} + I_{t-1}) + \underset{(0.071)}{0.384}\, N_{t-12} + \underset{(0.043)}{0.194}(N_{t-1} - N_{t-13})$$

$$\underset{(0.011)}{-\ 0.028}\, U_{t-1} + 2220 .$$

$$\frac{\text{var } \hat{\Delta P}_t}{\text{var } \Delta P_t} = 0.16 , \qquad \frac{\text{var } \hat{e}_t}{\text{var } \Delta P_t} = 0.64 , \qquad \frac{2 \text{ cov } \hat{\Delta P}_t\, e_t}{\text{var } \Delta P_t} = 0.21$$

In applying (2) above, the 4x4 matrix $\underset{\sim}{G}$ is now based on co-spectra between the series $(P_{t-1} + I_{t-1})$, N_{t-12}, $(N_{t-1} - N_{t-13})$, and U_{t-1}, and the vector $\underline{H}$ is based on co-spectra between $(P_t - P_{t-1})$ and the four regression variables. These are calculated from the spectra and cross-spectra of the P_t, N_t, U_t and I_t series using the relations derived above.

REFERENCES

[1] Abramovitz, Moses, Inventories and Business Cycles, Studies in Business Cycles No. 4, National Bureau of Economic Research, 1950.

[2] Amemiya, Takeshi, and Wayne A. Fuller, "A Comparative Study of Alternative Estimators in a Distributed-Lag Model," Technical Report No. 12 under NSF Grant GS-142, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1965.

[3] Arrow, K. J., S. Karlin and H. Scarf, Studies in the Mathematical Theory of Inventory and Production, Stanford: Stanford University Press, 1958.

[4] Bureau of the Census, Manufacturers' Shipments, Inventories, and Orders: 1947-1963 (Revised), U.S. Government Printing Office, Washington, D. C., 1963.

[5] Darling, Paul G., "Manufacturers' Inventory Investment, 1947-58," American Economic Review, 49, (December 1959).

[6] _____, "Inventory Fluctuations and Economic Instability," Inventory Fluctuations and Economic Stabilization, III, Joint Economic Committee, 87th Congress 1st Session, (December 1961).

[7] Eisner, Robert, and Robert H. Strotz, "Determinants of Business Investment," in Impacts of Monetary Policy, A Series of Research Studies Prepared for the Commission on Money and Credit, Englewood Cliffs, N. J.: Prentice-Hall, 1963.

[8] Goodwin, Richard M., "Secular and Cyclical Aspects of the Multiplier and Accelerator," in Income, Employment and Public Policy, New York: Norton, 1948.

[9] Granger, C. W. J., "The Typical Spectral Shape of an Economic Variable," Econometrica, 34, (January 1966).

[10] _____, and M. Hatanaka, Spectral Analysis of Economic Time Series, Princeton, N. J.: Princeton University Press, 1964.

[11] Grenander, U., and M. Rosenblatt, Statistical Analysis of Stationary Time Series, New York: Wiley, 1957.

[12] Grunberg, Emile, and Franco Modigliani, "The Predictability of Social Events," Journal of Political Economy, 62, (December 1954).

[13] Hamon, B. V., and E. J. Hannan, "Estimating Relations Between Time Series," Journal of Geophysical Research, 68, (November 1963).

[14] Hannan, E. J., Time Series Analysis, London: Methuen, 1960.

[15] Hannan, E. J., "Regression for Time Series," in M. Rosenblatt (ed.) Time Series Analysis, New York: Wiley, 1963.

[16] Johnston, J., "An Econometric Study of the Production Decision," Quarterly Journal of Economics, 75, (May 1961).

[17] Keynes, John Maynard, The General Theory of Employment, Interest, and Money, New York: Harcourt, Brace, 1936.

[18] Klein, L. R., R. J. Ball, A. Hazlewood, and P. Vandome, An Econometric Model of the United Kingdom, Oxford: Blackwell, 1961.

[19] Klein, Lawrence R., and Joel Popkin, "An Econometric Analysis of the Postwar Relationship between Inventory Fluctuations and Changes in Aggregate Economic Activity," Inventory Fluctuations and Economic Stabilization, III, Joint Economic Committee, 87th Congress 1st Session, (December 1961).

[20] Lovell, Michael C., "Manufacturers' Inventories, Sales Expectations, and the Acceleration Principle," Econometrica, 29, (July 1961).

[21] _____, "Factors Determining Manufacturing Inventory Investment," Inventory Fluctuations and Economic Stabilization, II, Joint Economic Committee, 87th Congress 1st Session, (December 1961).

[22] _____, "Determinants of Inventory Investment," in Models of Income Determination, Studies in Income and Wealth, Vol. 28, National Bureau of Economic Research, 1964.

[23] _____, "Sales Anticipations, Planned Inventory Investment, and Realizations," paper presented to the National Bureau of Economic Research Conference on Investment Behavior, Madison, Wisconsin, June 1965.

[24] Mack, Ruth P., "Comment," (on [22] above), in Models of Income Determination, Studies in Income and Wealth, Vol. 28, National Bureau of Economic Research, 1964.

[25] Metzler, Lloyd A., "The Nature and Stability of Inventory Cycles," Review of Economics and Statistics, 23, (August 1941).

[26] Mills, Edwin S., Price, Output, and Inventory Policy, New York: Wiley, 1962.

[27] Modigliani, Franco, "Business Reasons for Holding Inventories and their Macro-Economic Implications," in Problems of Capital Formation, Studies in Income and Wealth, Vol. 19, National Bureau of Economic Research, 1957.

[28] _____, and Owen H. Sauerlender, "Economic Expectations and Plans of Firms in Relation to Short-Term Forecasting," in Short-Term Economic Forecasting, Studies in Income and Wealth, Vol. 17, National Bureau of Economic Research, 1955.

[29] Nerlove, Marc, "Spectral Analysis of Seasonal Adjustment Procedures," Econometrica, 32, (July 1964).

[30] Nerlove, Marc, "A Comparison of a Modified 'Hannan' and the BLS Seasonal Adjustment Filters," Journal of the American Statistical Association, 60, (June 1965).

[31] _____, and Kenneth F. Wallis, "Use of the Durbin-Watson Statistic in Inappropriate Situations," Econometrica, 34, (January 1966).

[32] Smyth, D. J., "The Inventory and Fixed Capital Accelerators," Economic Record, 36, (August 1960).

[33] Stanback, Thomas M., Postwar Cycles in Manufacturers' Inventories, Studies in Business Cycles No. 11, National Bureau of Economic Research, 1962.

[34] Terleckyj, Nestor E., Measures of Inventory Conditions, Technical Paper No. 8, National Industrial Conference Board, 1960.

[35] Theil, H., "Specification Errors and the Estimation of Economic Relationships," Review of the International Statistical Institute, 25, (1957).

[36] _____, Economic Forecasts and Policy, 2nd ed., Amsterdam: North-Holland, 1961.

[37] Wallis, Kenneth F., "Distributed Lag Relationships between Retail Sales and Inventories," Technical Report No. 14 under NSF Grant GS-142, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1965.

[38] Watson, G. S., "Serial Correlation in Regression Analysis, I," Biometrika, 42, (December 1955).

[39] Whittle, P., Prediction and Regulation by Linear Least-Square Methods, London: English Universities Press, 1963.

[40] Wold, Herman, A Study in the Analysis of Stationary Time Series, 2nd. ed., Stockholm: Almqvist and Wiksell, 1953.

[41] Zarnowitz, Victor, "Unfilled Orders, Price Changes, and Business Fluctuations," Review of Economics and Statistics, 44, (November 1962).