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EQUAL PROFITS AS A FAIR DIVISION

James W. Friedman

April 18, 1966

EQUAL PROFITS AS A FAIR DIVISION*

James W. Friedman

1. INTRODUCTION

This paper reports the results of a duopoly experiment in which each firm was represented by one subject, and the two subjects comprising a duopoly could send messages to one another prior to making their decisions. The questions to which the experiment is addressed are: 1) To what extent are a pair of subjects able to agree on a joint course of action? 2) When subjects agree, is the profit point they agree upon Pareto optimal? and 3) Among Pareto optimal points chosen, does any particular point, such as the joint profit maximum, the Nash cooperative game solution, or the point of equal profits, predominate?^{1/} These questions will, of course, be formulated in a more precise manner below.

The present paper supplements results obtained from an earlier duopoly experiment.^{2/} The earlier experiment gave quite clear answers to the first two questions in the preceding paragraph: Subjects were able to agree a substantial majority of the time, and such agreements were usually Pareto optimal. On the third question, the answer was less clear and very likely to be sensitive to parameters of the market model which generated the payoff matrices. The central tendency among Pareto optimal choices was the Nash solution point; however, the joint maximum (in games with asymmetric payoffs) was an adjacent cell on the payoff matrices. Thus, the two points were so close together, it becomes

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^{1/}. The Nash solution is described in John F. Nash, "Two Person Cooperative Games," Econometrica, Vol. 21 (1953), pp. 128-140.

^{2/}. Friedman, James W., "An Experimental Study of Cooperative Duopoly," forthcoming, Econometrica.

an act of extreme optimism to believe one can distinguish between them on the basis of these results. As between the joint maximum-Nash solution (JM-NS) and the equal split, the case is more clear cut. At most, only 10% of the Pareto optimal choices could even loosely be described as attempts at reaching the equal split points while most of the remaining 90% are in the JM-NS neighborhood. The unpopularity of equal split may have been a consequence of the design of the market model which generated the payoff matrices, in that the equal split point was, in two senses, very far from the other two.

TABLE 1.

A Comparison Among Several Pareto Optimal Points

	Joint Max	Nash	Equal Split
Joint Profits divided by Joint Profits at Joint Max	1	.99	.75
Profits of first firm divided by profits of second	2.53	2.46	1

This is shown in Table 1. Note that the joint maximum and Nash solution points show nearly the same joint profits, and almost identical allocations of those profits between the two subjects. Contrasting these two points with the equal split point, the latter shows a joint profit $3/4$ the size of the former and, of course, an equal allocation rather than $5/7$ of the total to the first and $2/7$ to the second.

Two questions are raised by the preceding discussion: 1) If the joint maximum and Nash solution were physically separated by a greater distance on the payoff matrices, would experimental results distinguish clearly between them? and 2) If the equal split point were moved closer to the joint maximum and Nash solution, would it become more prominent? A desire to answer

these questions formed the main motivation for undertaking the experiment reported below. As long as the experiment was to be done, it was appropriate to alter the experimental design to eliminate problems encountered in the earlier experiment and to avoid complications of design which proved, ex post, of little interest. Also, it is of interest to conduct an analysis of agreement and of the relationship between agreement and Pareto optimality to see if the results of the earlier experiment are repeated.

Three sections follow. In section 2, the market model and experimental design are described, in section 3 the results are presented, analyzed and compared with the earlier experiment, and section 4 contains concluding remarks.

2. Market Model and Experimental Design.

The market model employs a linear demand function for each firm given by:

$$q_i = a_1 - a_2 p_i + a_3 p_j \quad \begin{array}{l} i, j = 1, 2 \\ i \neq j \end{array}$$
$$a_1, a_2, a_3 > 0$$
$$a_2 > a_3$$

where q_i is the quantity demanded of the i -th firm and p_i is the price charged by the i -th firm.

Total cost for a firm is a quadratic function of its own sales:

$$C_i = c_{i1} + c_{i2} q_i + c_{i3} q_i^2$$

Profit is given by $\pi_i = P_i q_i - C_i$

The experiment consisted of ten games. The parameter values for these games are given in Table 2.

TABLE 2.

Parameter Values Used for Deriving Payoff Matrices									
Game	<u>Demand for both</u>			<u>Cost for 1st firm</u>			<u>Cost for 2nd firm</u>		
	A ₁	A ₂	A ₃	C ₁₁	C ₁₂	C ₁₃	C ₂₁	C ₂₂	C ₂₃
1	3	.105	.070	0	10	3.3333	0	10	3.3333
2	15.9	.5565	.318	191	10	.62893	54.	0	6.2893
3	18.5	.6475	.370	213	10	.54054	78.9	0	5.4054
4	3	.105	.070	0	10	3.3333	0	10	3.3333
5	17.6	.616	.352	75.	0	5.68182	203	10	.568182
6	15.8	.553	.316	67.5	0	6.3291	183	10	.63291
7	13.6	.476	.272	46.2	0	7.353	164	10	.7353
8	14.4	.504	.288	48.8	0	6.9444	173	10	.69444
9	16.7	.5845	.334	193	10	.5988	71.1	0	5.988
10	15.1	.5285	.302	182	10	.662384	51.3	0	6.62384

From Table 2 it is clear that games 1 and 4 are each symmetric. The two firms have identical cost functions. The remaining eight games split naturally into two groups, consisting of 2, 7, 8 and 10, and 3, 5, 6 and 9.

Figures 1 and 2 illustrate the relevant features of the two sets of games. They differ in the placing of the equal split point relative to the joint maximum and Nash solution. This was done to facilitate testing whether the placement of the equal split point had any bearing on how frequently the joint maximum was chosen relative to the Nash solution. When this plan was made it was not expected that the equal split would be the most popular choice.

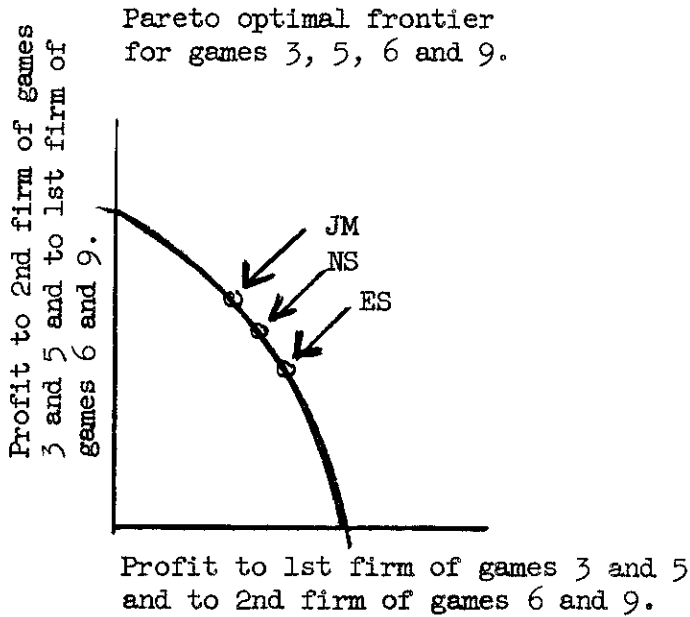


Figure 1

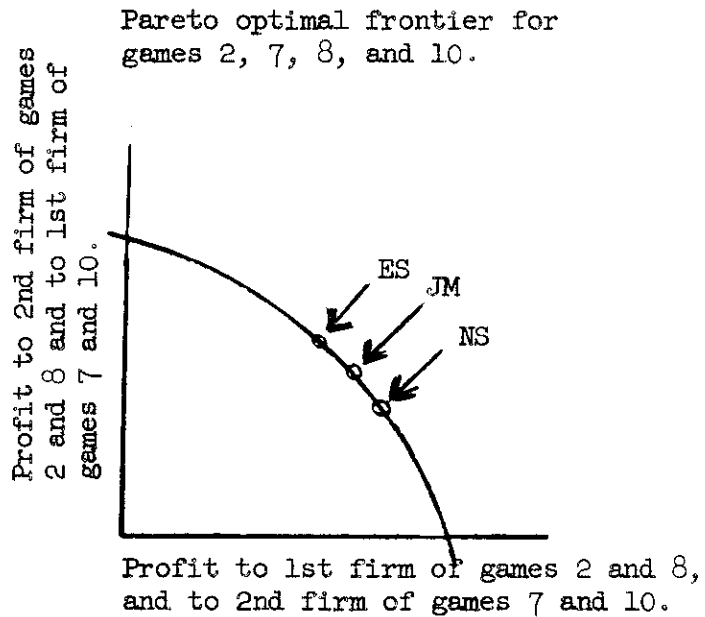


Figure 2

Table 3 shows the pairings of subjects used in the experiment. There were two identical replications, each involving 6 subjects. Within a replication, the Letter A represents the same subject. Similarly for B, C, D, E and F. Thus, each subject is paired twice with each of the other five subjects. Referring to Figures 1 and 2, a subject can be in one of four situa-

TABLE 3.

Pairings of Subjects

Game	1st firm	2nd firm
1	A C E	B D F
2	E D F	A B C
3	C E F	A B O
4	B D F	A C E
5	A B D	C E F
6	D F E	A B C
7	F C E	A B D
8	A B C	E D F
9	A B C	D F E
10	A B D	F C E

tions, corresponding to the horizontal and vertical axes of the two figures. Each subject was in each situation in two games. In addition, games 1 and 4 were symmetric.

The subjects were not allowed to see one another, nor were identities revealed. Each replication took place on two consecutive evenings of approximately five hours. With six subjects, three games were run simultaneously. They began and ended at the same time, and the three games of any triplet would run for the same number of periods. The games were generally 8 to 12 periods in length.

A period consisted of each player sending two messages to the other, then each choosing his price. One player was designated to send the first message. The second would read and reply. A second round of messages would ensue after which the first would choose a price, then the second would do so. The messages were carried back and forth by a project assistant. After collecting both prices, he would inform each of the price charged by the other; then the next period would begin. If someone wanted to send no message, he was instructed to write an X. On the average, a game took about an hour to play. This included about 8-10 minutes at the beginning when the subjects could examine the new payoff matrix, before the first period began.

3. Analysis.

There are three lines of analysis which are followed. The first is concerned with the probability that a pair of subjects will agree on a pair of prices to charge. The second line of analysis employs a limited

dependent variables model to estimate the probability that a decision will be Pareto optimal, and the expected value of its distance from the Pareto set as a function of the presence or absence of agreement and of other variables characterizing the game. The third line of analysis is concerned only with those points which are in the Pareto set and which come from asymmetric games. The object here is to determine whether the Nash solution, the joint maximum or the point of equal profits characterizes the Pareto optimal choices.

In studying agreement only two aspects of a period are taken into account: whether or not the subjects agreed upon a course of action, and, if there was agreement, whether it was honored. Each period is classified as:

non-A	nonagreement
A	agreement (honored by both)
AC	agreement (with one "cheating; i.e., not honoring it)
ACC	agreement (with both "cheating")

It is assumed that the probability of being in each of these four "states" in a given period depends only on the "state" which characterizes the preceding period. In other words a game is being viewed as a first order Markov process in which there are four possible states, non-A, A, AC and ACC. A matrix P in which each row corresponds to a "current state" and each column corresponds to a "next state," and in which an entry, p_{ij} , gives the probability that the next state j , given that the present state is state i is called a "transition matrix" or "matrix of transition probabilities."

In the earlier experiment, it was found that, over 18 games, the transition matrices were stable for the last 12 games and changing for the first 6. Testing for this was accomplished by estimating nine transition matrices, one for the first two games (across all replications and subjects) $P_{1,2}$ one for the third and fourth games $P_{3,4}$ etc.

A similar procedure here, estimating 5 transition matrices, requires testing the hypothesis:

$$H: P_{1,2} = P_{3,4} = P_{5,6} = P_{7,8} = P_{9,10}$$

by chi square. The χ^2 value is 51.6 and the number of degrees of freedom is 40. It is known that $\sqrt{2\chi^2} - \sqrt{79}$ is distributed approximately as a normal variate with zero mean and unit variance. $\sqrt{103.2} - \sqrt{79} = 1.27$ which is less than the critical value for the 1% level (one tailed test) of 2.33. The null hypothesis is accepted, and it is concluded that, in this experiment, changes in transition matrices from game to game are due to chance. Table 4 shows the transition matrix and associated equilibrium distribution. The equilibrium distribution is that vector x which satisfies the condition $Px = x$. The elements of x are non-negative and sum to unity.

TABLE 4.

The Matrix of Transition Probabilities and
Associated Equilibrium Distribution

	Transition Probabilities				Eq. Distribution
	Not A	A	AC	ACC	
Not A	.800	.100	.069	.031	.339
A	.029	.926	.045	0	.571
AC	.591	.114	.227	.068	.071
ACC	.5	0	.333	.167	.019

It is not obvious why the transition probabilities in the previous experiment should be different for the earlier games than they are in the later, and not be so in this experiment. In comparing between the earlier and later games, the changes in both experiments are mostly in the same direction: p_{11} and p_{22} ("not A to not A" and "A to A") are higher in the later games, for the most part, the other entries in the first two rows are smaller for the later games, and p_{31} (from AC to not A) is larger. Thus, in the later games, both non-agreement and honored agreement become more stable -- more likely to be repeated -- and when one person cheats, it becomes more likely that mutual trust will cease and non-agreement follow. Again, it is not obvious why these differences were sufficiently pronounced to be statistically significant in the earlier experiment and not so now. It is true there were approximately 2-1/2 times as many observations in the earlier experiments.

It is of interest to test whether the transition matrix in Table 4 is significantly different from that estimated for the previous experiment. The latter appears in Table 5.

TABLE 5.

Transition Matrix for Earlier Experiment,
and Eq. Dist.

	Transition Matrix				Eq. Dist.
	not A	A	AC	ACC	
not A	.720	.175	.088	.017	.163
A	.017	.934	.049	0	.759
AC	.410	.295	.279	.016	.073
ACC	.600	0	.200	.200	.005

The test of equality of these two matrices is χ^2 with 10 degrees of freedom. The χ^2 value is 12.8, which is far below the critical value for the 10% level of 23.29. Therefore, it is concluded that the transition matrices from the two experiments are the same, apart from chance variation.

It may be objected that a first order Markov process provides rather too simple minded a model in which to analyze the agreement process. This could be because the subject's whole past experience in the sort of environment provided by the experiment really determines his behavior. On a more modest level, one could ask "Why the last period? Why not the last two periods?" In fairness to critics with these views, a clear determination between rival hypotheses must be settled by an experiment carefully designed so as to distinguish among them. In the meantime the argument offered in defense of the hypothesis which is maintained here, is that the most recent period must have far more weight in determining what will happen next than all the preceding periods combined, because a subject will take as the best indication of his rival's intent his most recent action. If, for example, a decision is to be made for period t and the subject notes his rival honored an agreement in period $t-2$ but broke it in period $t-1$, he will place far more weight on the period $t-1$ action than the $t-2$ in determining whether to trust the rival again.

The remaining analysis utilizes distances measured in profit space. A pair of prices chosen by subjects determine a pair of profit levels

and, hence, a unique point in a space in which the profits of the first subject are measured on one axis and the profits of the second are measured on the other (as Figures 1 and 2). The Pareto set may also be drawn in this space, and the distance of a point from the Pareto set may be defined as the Euclidean distance from the point in question to that point of the Pareto set which is nearest the point in question. The distance concept requires some sort of normalization so that distances do not change when, for example, the unit of measurement of profit changes. Also, two games which are identical in every way, except that profit levels in one are double those of the other, would seem to present the same strategic possibilities. The unit of distance in the high profit game should be double the unit of the other in money terms so that the distance between a pair of corresponding points is the same in each game. It is taken as the length of the straight line segment from the Nash threat point to the Nash solution. This unit may not be an ideal choice; however, it has the virtue that in two games which are strategically identical, the distance between a pair of corresponding points will be the same.

The estimation of distance from the Pareto set is accomplished using the technique of limited dependent variables. ^{1/} This technique

^{1/}. On limited dependent variables see Tobin, James, "Estimation of Relationships for Limited Dependent Variables," Econometrica, Volume 26, (1958), pp. 24-36.

is appropriate for variables which: 1) cannot have values below (or above) a particular limit value, 2) have a positive probability, p , of being exactly equal to the limit value and 3) are distributed for values greater than the limit according to that part of the normal distribution for which

$$L < x < \infty, \text{ and } \int_L^{\infty} f(x) dx = 1-p, \text{ where}$$

L = the limit value
 x is the dependent variable in question
 $f(x)$ is the normal density function.

As the Pareto set is the frontier of attainable profit points, no actual point can lie above. Thus distance from the set can be only distance from below and left. Distance is taken as positive; hence zero is the lower limit value. Maximum likelihood estimates are calculable for the parameters of a limited dependent variables model, and hypotheses may be tested by means of likelihood ratio tests. ^{2/}

The data were divided into six subsamples and a mean and variance were calculated for each. The grouping was done according to market model, and the presence or absence of agreement. The first subsample consists of the agreement periods from games 1 and 4, the symmetric games. The second is the agreement periods from the games illustrated by Figure 2 (games 2,7,8,10), etc.

^{2/}. For any periods of agreement which involve cheating, the profits used are the agreed upon profits, rather than the observed. The validity of this procedure hinges on the assumption that the distance from the Pareto set of agreements which are broken is the same, other things equal, as of agreements which are honored. It is also worth noting that cheating occurred in less than 10% of all periods.

TABLE 6

	<u>Agreement</u>			<u>Non-Agreement</u>		
	1,4	2,7,8,10	3,5,6,9	1,4	2,7,8,10	3,5,6,9
games	1,4	2,7,8,10	3,5,6,9	1,4	2,7,8,10	3,5,6,9
prob. of lim response	.865	.799	.585	.047	.0089	.128
max. like. est. expected value	.0055	.0524	.1164	.5333 .54	.8592 .862	.9444 .9974
no. of limit obs.	64	166	72	0	0	1
no. of non- limit obs.	10	55	107	31	67	78
$\hat{\sigma}$.081	.466	.387	.319	.362	.8317
no. of obs. within .008 of limit	64	213	146	0	0	1

Table 6 gives for each of the subsamples the probability that a period drawn randomly will be in the Pareto set, the maximum likelihood estimate of distance, the expected value of distance, the number of observations on the Pareto set, the number off, the standard error of distance and the number of observations within .008 of the limit. The latter number is the number of observations in the Pareto set plus the number corresponding to points where one subject received .1 cents less than at a Pareto optimal point, and the other received the same. Among agreement periods in the two subsamples of asymmetric games there were many of these nearly Pareto optimal points. Payoffs were given in cents with one place to the right of the decimal, and along the Pareto set, joint profits ran in the neighborhood of 35 to 50 cents. In all cases the points within .008 of the Pareto set were dominated by points within the Pareto set at which one subject received the same payoff and the other, .1 cent more. It is likely these points were chosen because the ones which dominated them chanced to go unnoticed.

The results in Table 6 are substantially like those found in the earlier experiment. The probability of limit response is substantially above .5 for agreement periods and substantially below for non-agreement periods.

Three hypotheses regarding these results are tested. They are:

$$H_1: \mu_1 = \mu_2 = \mu_3, \mu_4 = \mu_5 = \mu_6, \sigma_1 = \sigma_2 = \dots = \sigma_6$$

$$H_2: \sigma_1 = \sigma_2 = \dots = \sigma_6$$

$$H_3: \mu_2 = \mu_3, \mu_5 = \mu_6, \sigma_2 = \sigma_3, \sigma_5 = \sigma_6$$

Each is tested against the alternate hypothesis that the six subsamples have different means and variances. It is known that $-2 \ln \lambda$, where λ is the likelihood ratio, is distributed approximately as χ^2 with n degrees of freedom. n is the number of restrictions on parameters imposed by the null hypothesis as compared with the alternate. n is 9, 5 and 4 for the three hypotheses, $-2 \ln \lambda$ is 163.6, 111.4 and 70.8. The critical values for the 1 percent level are 21.67, 15.09 and 13.28, thus the three hypotheses are all rejected, and it is concluded that the six subsamples have different means and variances.

It is in regard to where the Pareto optimal points are located that the present results differ markedly from earlier results. The data are summarized in Figures 3, 4 and 5. In all three figures, zero is taken at the equal split point. The data comprise the points which are within .008 of the Pareto set. At a glance, the figures indicate an overwhelming preference for the equal split point. In the three sets of games some 202 of the 423 points are precisely at the equal split. Table 7 summarizes the distribution of points according to which of the three points are nearest. The figures in parentheses give the corresponding entry as a proportion of the row total.

TABLE 7.

	Eq. Split	Joint Maximum	Nash	Total
games 2, 7, 8, 10	157 (.737)	55 (.258)	1 (.005)	213
games 3, 5, 6, 9	95 (.651)	13 (.089)	38 (.260)	146

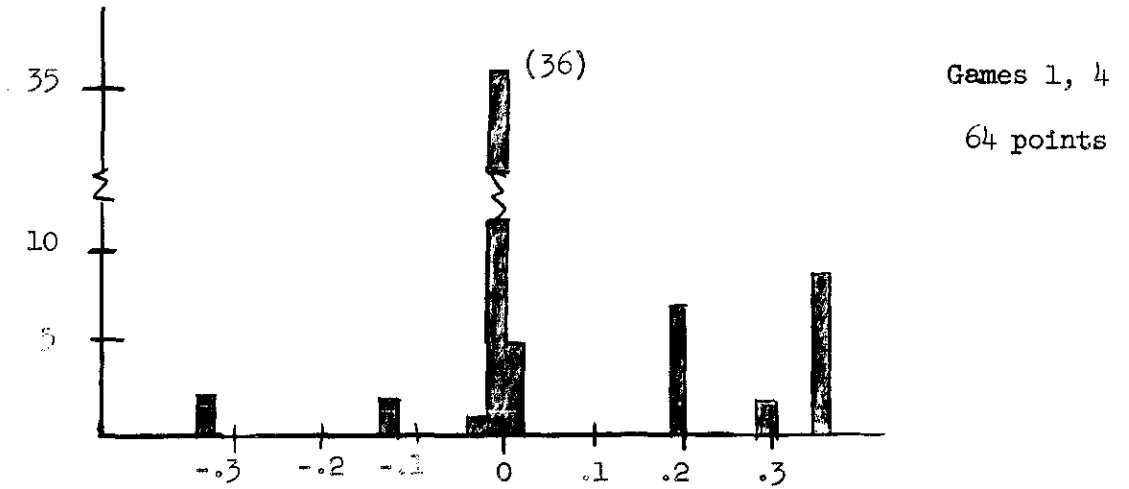


Figure 3

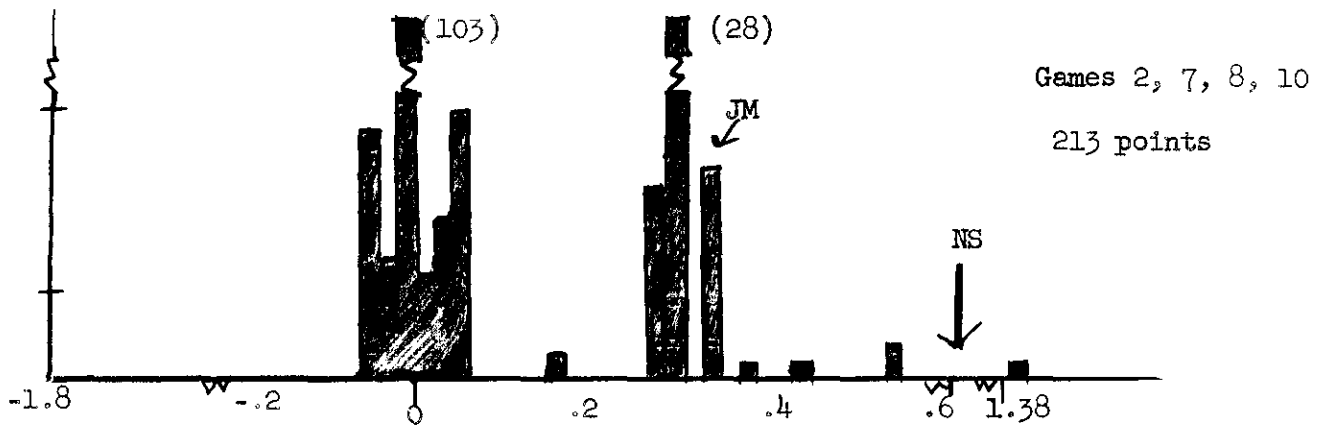


Figure 4

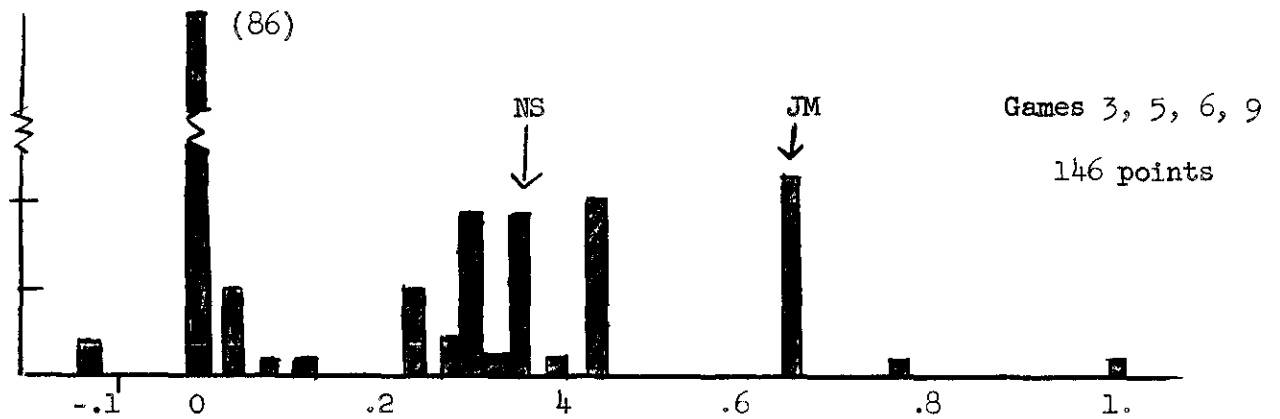


Figure 5

In games 3, 5, 6, and 9 it is unclear whether the Nash solution is sought the 55 times it is the nearest point, or whether those 55 points represent compromises between the equal split point and joint maximum.

Comparing the previous experiment to this, in terms of the unit of measure employed in Figures 4 and 5, the equal split point was about 6.0 to the right of the Nash solution and the joint maximum .3 to the left. About 60 percent of the points fell within .25 of either the Nash solution or the joint maximum, and some 90 percent of all points were within 2. units of the pair. It was noted above that joint profits at the equal split point, as a percentage of joint profits at the joint maximum were considerably less in the previous experiment than here. It appears, in comparing the present results to the earlier, that the popularity of the equal split point depends on the sacrifice of joint profits made in moving from the joint maximum to the equal split. As to comparison between the joint maximum and Nash solution, it appears, tentatively, as if the Nash solution is of little importance.

4. Conclusion.

Between the present experiment and its predecessor, the evidence is quite strong that subjects in cooperative duopoly situations are able to agree a majority of the time, and their agreements are, in the main, Pareto optimal. These results must not be taken as a hint of what would occur in a three person game. In such a game, the rules regarding messages would have to be expanded. Would a subject be allowed to send a message which

just one of the others would see? Or would each message have to be seen by all? The results of such games might be greatly affected by the way this condition is specified. In a triopoly, there are three pairs who could make bilateral agreements, in addition to a coalition of the whole. Probably agreement of all three would be less frequent than agreement of a pair of duopolists, merely because more people are involved.

Returning to duopoly, the present experiment leaves somewhat open the question of how the structure of the game helps determine which Pareto optimal point is chosen. In the earlier experiment, equal split was far from the joint maximum and Nash solution and it was seldom chosen. In the present experiment, it was close to them and was chosen most often. It is possible that within any one experiment, one particular point will predominate because early in the experiment it happens to be frequently chosen and a precedent is established. If this is so, it may be that the proportion of identical replications in which, say, equal split is the predominant choice depends on its distance from, say, the joint maximum.

Or, it may be that precedents do not get established in this manner. The proportion of equal split choices among Pareto optimal points may vary, within experiment, with the distance of equal split from the joint maximum.

Finally, both experiments leave the Nash solution in limbo. It remains a mystery whether the subjects have any feeling for it, intuitive or otherwise. A set of interesting and useful experiments could be devised to examine in detail the choices subjects make in cooperative duopoly games among Pareto optimal points.