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# A Model of Fixed Capital Without Substitution

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## I. INTRODUCTION

We analyze in this paper a completely aggregated model of production in which output is produced by inputs of homogeneous labor and heterogeneous capital goods, and allocated either to consumption or to use as capital goods. Allocations are irreversible; capital goods can never be directly consumed. Fixed coefficients rule: any concrete unit of capital has a given output capacity and requires a given complement of labor. Technological progress continuously differentiates new capital goods from old. But we assume that the "latest model" in capital goods has no smaller capacity and no higher labor requirement than any older-model capital goods with the same reproduction cost. Thus each instant's gross investment will take the form of the latest-model capital. There is no problem of the optimal "depth" of capital. The main effect of an <u>increase</u> in gross investment is to modernize the capital stock in use.

One normal consequence of technological progress will be a rising trend of the real wage rate. Since existing capital operates under fixed coefficients, there will eventually come a time in the life of every vintage of investment when the wage costs of using it to produce a unit of output will exceed one unit of output. At that instant the investment may be said to have become obsolete as a result of the competition of more modern capital; it will be retired from production -- permanently, unless the real wage should temporarily fall.

We have several motives for wishing to analyze so special a model.

1. Capital theory seems -- perhaps inevitably -- to consist of a catalog of special models, distinguished by the different ways time and durable commodities enter the process of production. Since this simple, but not trivial, model has not been studied as a growth model before, we think it a worthwhile addition to the catalog.[1/]

2. The model contributes something more than mere completeness to the catalog. It isolates the effects of what has been called "quickening" -- hastening the practical introduction of newly-discovered techniques into production -- from those of "deepening" of capital. "Widening" can also be analytically excluded by considering the special case of a constant labor force.

3. The literature sometimes suggests, or seems to suggest, that what are called "neo-classical" modes of analysis -- we emphasize that we do not refer to assumptions of Say's Law -- require for their validity or utility that capital and labor be directly and smoothly substitutable for one another. This paper provides a counterexample. Although there is no scope for substitution *ex post* or *ex ante*, we show that the basic neo-classical methods do function and give the expectable results. No use is made of any "generalized stock of capital."

4. What is true is that the basic neo-classical methods apply when and only when output is limited by the availability of resources, not by effective

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1/ The model was formulated and studied in detail by Salter [2], from a point of view which is somewhat different from ours.

demand. Most of our argument is conducted under the assumption that full employment of labor is the bottleneck to production. This assumption may be regarded as appropriate to a planned economy, or to a decentralized economy with an effective fiscal policy. An important task of economic theory is to find some way of unifying the theory of production and the theory of effective demand. The model of this paper is, we believe, particularly suited for this purpose, precisely because it gives effect to the common casual-empirical belief that in the short run the scope for changing factor proportions is small. On the other hand, the model no doubt limits excessively the scope for changing factor proportions over long periods of time. Like all aggregate models, it must ignore the effects of inter-commodity shifts.

5. Finally, it is sometimes asserted that in modern industrial economies *ex ante* choice of techniques is in fact unimportant; that at any instant of time one technique -- the latest one -- effectively dominates all others for all thinkable configurations of factor prices. We do not know how nearly true this assertion is (particularly in macroeconomic terms). But the model of production studied in this paper is presumably the appropriate vehicle for studying the implications of the assertion.

## II. PHYSICAL RELATIONS

### 1. Technological assumptions

The model assumes fixed-coefficient technology with embodied technical progress. Once capital has been put into place, there is no possibility of substituting capital for labor or vice versa; the output-capital and output-labor coefficients are fixed for the life of the capital. Neither are there any effective possibilities of *ex ante* substitution between labor and capital. For a business investing in new capital, only one pair of these coefficients, the pair which will characterize this capital so long as it is operating, is available. (This is not strictly true, since an investing business could always use older technology characterized by different coefficients. But this is an empty qualification, because in the model an investor will never prefer older technology to new technology no matter what wage rate and interest rate he faces.) Technical progress consists of improvement in one or both of the output-input coefficients. But the improved coefficients apply only to new vintage capital, not to investments made in the past. Since the model has only one commodity, serving indifferently as capital good and consumer good, investment can be measured unambiguously in physical units equal to the opportunity cost of one unit of consumption.

Formally, let:

$Y(t,v)dv$ be the rate of gross output (physical units per year) at time $t$, produced on capital of vintage $v$, i.e. capital installed during a period $(v, v + dv)$, where necessarily $v \leq t$.

$I(v)$ the rate of gross investment (physical units per year) at time $v$.

$I(v)dv$ the amount of capital (physical units) installed in the period $(v, v + dv)$.

$N(t,v)dv$ the rate of employment of labor (men) at time $t$ on capital of vintage $v$.

$\lambda(v)$ the technologically determined output per year per man producible on capital of vintage $v$.

| | |
|---|---|
| $\mu(v)$ | the technologically determined output per year producible with one unit of capital of vintage $v$ . |
| $Y(t)$ | total gross output per year, summed over all vintages of capital, at time $t$ . |
| $N(t)$ | total employment (men), summed over all vintages of capital, at time $t$ . |
| $L(t)$ | total labor supply at time $t$ . |
| $w(t)$ | the real wage rate (physical units per man-year). |
| $\rho(t,v)dv$ | the quasi-rent earned at time $t$ on one unit of capital of vintage $v$ (a pure number). |
| $m(t)$ | the age of the oldest capital in use at time $t$ (years). |

The assumptions about production outlined verbally above can be summarized in the following production function for output from capital of vintage $v$ ($\leqq t$)

$$(1) \qquad Y(t,v) = \text{Min} \left\{ \lambda(v)N(t,v),\ \mu(v)I(v) \right\}$$

This formulation ignores physical depreciation and assumes that capital is perfectly durable. This assumption has the advantage of simplicity, and it permits the model to bring out clearly the economics of obsolescence. Capital wears forever, but it is not in general used forever -- better, more modern, capital displaces it. As the same time, physical depreciation of simple types can be allowed without essentially altering the behavior of the economy described by the model. In Part VII below, two kinds of physical depreciation are mentioned: (1) exponential evaporation or decay; (2) "one-hoss-shay" collapse after a fixed lifetime at full strength. At that point we will also indicate how the model can be generalized to allow the productive capacity of a unit of capital to decline with age while the capital remains physically in existence with its original labor requirements.

In general, we shall be interested in situations where, for vintages $v$ in use:

(2) $$Y(t,v) = \lambda(v)N(t,v) = \mu(v)I(v) .$$

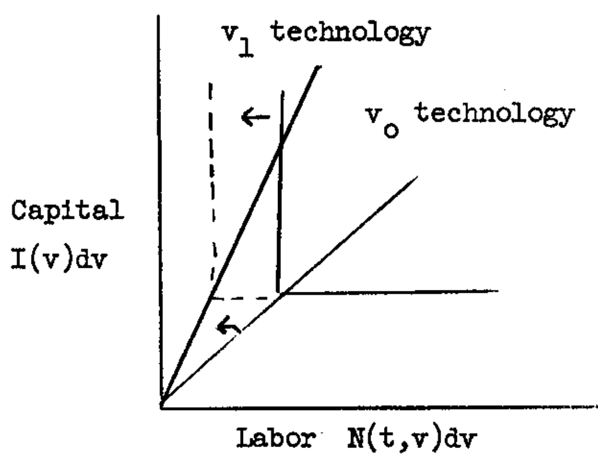
Unless this condition is met, capital of vintage $v$ is not being efficiently used. It makes no sense to overman capital, and in a continuous-time model it will not be under-manned either. In a discrete time model, it would be conceivable that some but not all of the capital invested during period $v$ might be in use at a later time $t$. This possibility does not arise here because there is not a finite mass of capital of any instantaneous vintage. If any vintage $v$ capital is in use, all of it is. Note that there is no specifically "vintage $v$" labor. Any labor available at time $t$ will do. One unit of vintage $v$ capital employs $\mu(v)/\lambda(v)$ workers when it is in use.

## 2. Kinds of technical progress.

The coefficients $\lambda(v)$ and $\mu(v)$ carry technical progress. We shall assume that each of these coefficients is a non-decreasing function of $v$. This guarantees that no earlier technology is ever preferred to the newest. The model does not explain the advance of technical knowledge; it is autonomous, requires no productive resources, and cannot be accelerated or retarded. A more complete model would relate progress not just to the passage of time but to production experience (as is done, for instance, by Arrow [1]) and to the use of resources in research and development (see, e.g., Uzawa [4]).

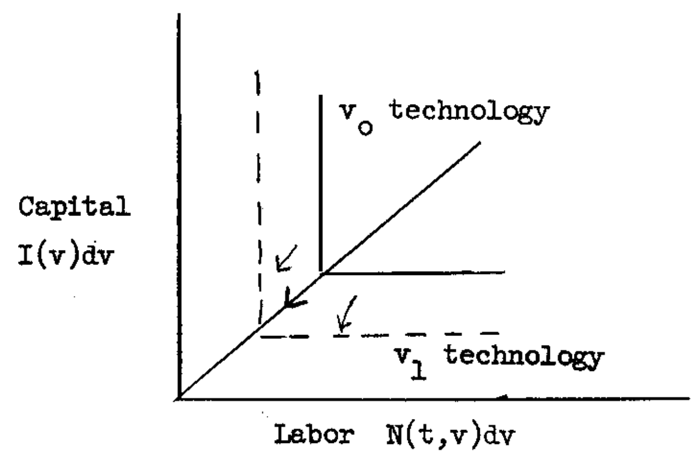
Three special kinds of technical progress are depicted in Figures 1a, 1b, and 1c. Capital-labor isoquants are shown for a fixed rate of output under vintage $v_o$ technology, and under technology of a later vintage $v_1$. The arrows show in each case the direction in which technical progress moves the isoquant.

Figure 1a



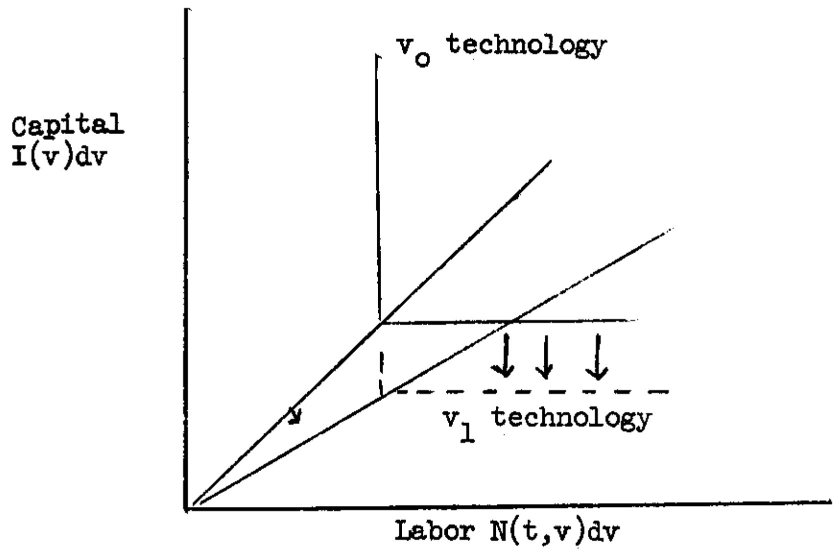

Purely labor-augmenting technical progress, "Harrod-neutral." Capital-labor ratio increases

Figure 1b




"Hicks-neutral." Capital-labor ratio constant

Figure 1c




Purely capital-augmenting
technical progress.
Capital-labor ratio falls

The three special cases are:

(a) $\lambda'(v) > 0$ , $\mu'(v) = 0$ . Purely labor-augmenting or "Harrod-neutral" progress.

(b) $\frac{\lambda'(v)}{\lambda(v)} = \frac{\mu'(v)}{\mu(v)} > 0$ . $\frac{\lambda(v)}{\mu(v)}$ constant. "Hicks-neutral" progress.

(c) $\lambda'(v) = 0$ , $\mu'(v) > 0$ . Purely capital-augmenting progress.

3. <u>Aggregative Implications.</u>

At any time $t$ , the total labor supply $L(t)$ is assumed to be given exogenously. This is not necessarily equal to aggregate employment $N(t)$ .

The past history of gross investment $I(v)$ determines the capital available for use at time $t$. The maximum possible employment which this investment history permits is:

$$N^*(t) = \int_{-\infty}^{t} \frac{\mu(v)}{\lambda(v)} I(v)dv$$

and this requires all (surviving) capital to be in use. The integral may diverge, in which case labor can never be in surplus. For simplicity we assume $N^*(t)$ finite. There are three important possible regimes:

(I) $L(t) \geq N^*(t) = N(t)$ Labor surplus.

All capital is in use. Labor is unemployed because of a shortage of capital. Or, when $L(t) = N^*(t)$, labor is just adequate to man all the capital.

(II) $N(t) = L(t) < N^*(t)$. Full employment.

Some capital is left unused because the labor supply is insufficient.

(III) $N(t) < L(t) \leq N^*(t)$. Keynesian unemployment.

Some labor, and an associated amount of capital, is unemployed because demand is insufficient.

## 4. Allocation of labor.

What is the optimal allocation of labor over the available capital of various vintages? Or, to put the same question somewhat differently, which vintages should be used and which left unused? Let $u$ be an unutilized vintage and $v$ a utilized vintage. If an allocation is optimal, it should not be possible to increase total output by shifting a unit of labor from vintage $v$ capital to vintage $u$ capital. Such a shift would increase output by $\lambda(u)$ and diminish it by $\lambda(v)$. Hence an optimal allocation requires that:

(3) $\lambda(u) \leqq \lambda(v)$ for any unutilized vintage $u$ and utilized vintage $v$.

Provided $\lambda'(v) > 0$, optimal allocation is very simple and obvious: $\lambda(u) < \lambda(v)$ if and only if $u < v$. No vintage should be left unutilized if an older vintage is in use. A rational planner allocating a given total employment $N(t)$ would first man the newest equipment, then the next newest, and so on until he runs out of labor (or out of equipment). This is also what the competitive market will do. As we shall see, except in the labor-surplus regime, the competitive real wage rate makes it unprofitable to operate the oldest equipment. Quasi-rents obtainable at time $t$ vary inversely with the age of capital -- highest for the most modern, zero for the "cut-off" age, and negative for economically obsolete vintages.

5. <u>The purely capital-augmenting case</u>.

If technical progress is purely capital-augmenting -- $\lambda'(v) = 0$, the third, (c), of the special cases listed above -- the allocation of employment among competing vintages of capital is indeterminate. Technical progress lowers the real cost of a unit of productive capacity. But once the capacity is in being, the marginal and average variable cost of output is the same on every vintage. Therefore, this case is not very interesting. It reduces to these possibilities:

(a) In regime I, there is always ample labor to man the whole capital stock. When $\lambda(v) = \lambda$, this implies:

(4) $$N(t) = N^*(t) = \frac{1}{\lambda} Y(t) = \frac{1}{\lambda} \int_{-\infty}^{t} \mu(v) I(v) dv .$$

Let $s(t)$ be the ratio of gross saving to gross output at time $t$. Correspondingly, $\frac{1}{\mu(t)}$ is the marginal or incremental capital requirement per unit of output. We have, therefore, the familiar Harrod-Domar equation for the rate of growth of output and employment:

(5) $$\frac{N'(t)}{N(t)} = \frac{Y'(t)}{Y(t)} = \mu(t)\ s(t) .$$

If labor is truly in excess supply, its marginal product is zero and so is its competitive real wage, or its shadow price in a planned economy. Correspondingly, the rent on capital of vintage $v$ is its average product: $\mu(v)$. If $L(t)$ is just equal to $N^*(t)$, then the price of labor $w(t)$ is indeterminate between zero and its average product $\lambda$. Correspondingly, the quasi-rent $\rho(t,v)$ on vintage $v$ capital is indeterminate between $\mu(v)$ and zero:

(6) $$\rho(t,v) = \mu(v)(1 - \frac{w(t)}{\lambda}) \geqq 0 .$$

(b) In the other two regimes, labor supply is not large enough to permit utilization of all vintages of capital. The marginal product of capital is zero, whatever its vintage. New capital has no advantage over old. If labor is fully employed, its real wage is $\lambda$, its average product. This situation may, of course, lead to Keynesian difficulties: full employment incomes might generate saving but, since profits are zero, not corresponding investment. Then the result would be under-utilization of both capital and labor, with the efficiency-prices of factors again indeterminate.

## 6. Obsolescence and income distribution.

So much for purely capital-augmenting technical progress. In all other cases new vintages will always be preferred to older vintages. We disregard the labor surplus regime as atypical for advanced economies. In cases of interest, then, the age of the oldest capital in use, $m(t)$, is related to total employment by the equation

$$(7) \qquad N(t) = \int_{t-m(t)}^{t} \frac{\mu(v)}{\lambda(v)} I(v)dv .$$

On the other hand, there is a relation between $m(t)$ and aggregate output:

$$(8) \qquad Y(t) = \int_{t-m(t)}^{t} \mu(v) I(v)dv .$$

Employment of a unit of additional labor at time $t$ would permit the use of capital just beyond the cutoff point $m(t)$, adding to total output the average product of labor on capital of this vintage. The marginal product of labor, therefore, is $\lambda(t-m(t))$. (This is the value of $\frac{\partial Y(t)}{\partial N(t)}$, as may be ascertained by differentiating (7) and (8) with respect to $N(t)$.) The marginal product of capital of any vintage may also be found. An additional unit of capital of an active vintage $v$ ($v$ greater than $t-m(t)$) would permit added output of $\mu(v)$. But it would require shifting $\frac{\mu(v)}{\lambda(v)}$ units of labor away from the oldest vintage capital, reducing output by $\frac{\mu(v)}{\lambda(v)} \lambda(t-m(t))$. An additional unit of capital of an idle vintage adds nothing to output.

Under competition, we can identify the marginal product of labor with the real wage and the marginal product of capital of any vintage with its quasi-rent:

(9) $$w(t) = \lambda(t-m(t))$$

(10) $$\rho(t,v) = \begin{cases} 0 & \text{if } v \leq t-m(t) \\ \mu(v)\left(1 - \dfrac{\lambda(t-m(t))}{\lambda(v)}\right) & \text{if } v \geq t-m(t)\ . \end{cases}$$

Together wages and quasi-rents exhaust the output of active capital.

The history of a particular investment is this: Its average product remains constant. At the beginning it earns a positive rent, because it is superior to earlier vintages. But as still better capital comes into existence, wages rise and the rents on the investment decline. Finally, wages are bid up so high by the owners of modern equipment that the rent on the investment vanishes. It is obsolete and ceases to operate.

7. The growth of income.

The growth of income may be decomposed into a part attributable to the growth of the labor force and another part associated with new investment. Differentiating (7) and (8), we obtain:

$$N'(t) = \frac{\mu(t)I(t)}{\lambda(t)} - \frac{\mu(t-m(t))}{\lambda(t-m(t))} I(t-m(t))(1-m'(t))$$

$$Y'(t) = \mu(t)I(t) - \mu(t-m(t))\, I(t-m(t))(1-m'(t))$$

$$\lambda(t-m(t))N'(t) = \mu(t)\, \frac{\lambda(t-m(t))}{\lambda(t)}\, I(t) - \mu(t-m(t))\, I(t-m(t))(1-m'(t))$$

(11) $$Y'(t) = N'(t)\ \lambda(t-m(t)) + I(t)(\mu(t))(1 - \frac{\lambda(t-m(t))}{\lambda(t)})$$

$$= N'(t)\ w(t) + I(t)\ \rho(t,t)$$

(12) $$\frac{Y'(t)}{Y(t)} = (\frac{w(t)N(t)}{Y(t)})\ \frac{N'(t)}{N(t)} + \rho(t,t)\ \frac{I(t)}{Y(t)}\ .$$

This decomposition is analogous to the more conventional one for models with substitution.

In regime II, full employment, causation may be interpreted in this manner: $L(t) = N(t) \rightarrow m(t) \rightarrow Y(t)$ and $w(t)$ . The first causal arrow stands for (7), the second for (8). In the Keynesian regime III, output is determined by effective demand. The causation then runs the other way: $Y(t) \rightarrow w(t)$ and $m(t) \rightarrow N(t) < L(t)$ . Now the first arrow stands for (8), the second for (7). In this interpretation, one can easily allow for feedback effects of income distribution on effective demand. Equations (11) and (12) apply under either interpretation.

If aggregate demand falls, the model says that plants shut down in order of their age. Aside from the usual complications of aggregation, this is realistic enough. Its corollary, however, is that the average and marginal products of labor rise as labor is laid off from the oldest and least efficient plants. Cyclical statistics indicate the opposite, apparently because in recessions believed to be temporary employers continue to man, at least partially, facilities which they are not using (and/or because the right-hand side of (7) contains an "overhead" component independent of current output).

## 8. Exponential Growth Under Full Employment: Labor-augmenting progress.

In what follows, both technical progress and labor force growth are assumed to be exponential:

$$(13) \qquad \begin{cases} L(t) = L_o e^{nt} \\ \mu(v) = \mu_o e^{\mu v} \\ \lambda(v) = \lambda_o e^{\lambda v} \end{cases}$$

The full employment regime is analyzed first: the labor supply is fully used but is insufficient to man all physically surviving capital. Moreover, the simplest kind of technical progress is assumed -- the purely labor-augmenting, "Harrod-neutral" variety, i.e., $\mu(v) = \mu_o$ for all $v$ .

## 9. Balanced growth paths.

Consider paths along which gross investment has been growing exponentially forever: $I(t) = I_o e^{gt}$ . From (7) and (13) we calculate:

$$L_o e^{nt} = \frac{\mu_o I_o}{\lambda_o (g-\lambda)} e^{(g-\lambda)t}(1 - e^{-(g-\lambda)m(t)}) \quad \text{for all } t .$$

If $g = n + \lambda$ , this equation can be satisfied with $m(t)$ constant. If $g \neq n + \lambda$ , the equation can not be satisfied even with variable $m(t)$ ; for $g < n + \lambda$ , the left-hand side must eventually outstrip the right while $g > n + \lambda$ implies that $m(t) \to 0$ which in turn implies that gross investment eventually exceeds gross output (see(18)). Therefore:

(i) $g = n + \lambda$ , the usual formula for the "natural rate of growth" under Harrod-neutral technical progress; and

(ii) $m(t)$ is a constant, say $m$, satisfying

(14)
$$L_o = \frac{\mu_o I_o}{\lambda_o n}(1 - e^{-nm})$$

$$m = -\frac{1}{n}\log\left(1 - \frac{\lambda_o n L_o}{\mu_o I_o}\right) .$$

For this formula to make sense, it is necessary that $\lambda_o n L_o < \mu_o I_o$. The meaning of this restriction is easily seen after it is rewritten:

$$n L_o e^{nt} < \frac{\mu_o I_o e^{gt}}{\lambda_o e^{\lambda t}} .$$

In this form it says that the increment to the labor force must be smaller than the labor required to man the brand new capital: the gap is to be filled with the labor that had been operating the capital (of age m) now being retired. If the inequality is not satisfied, the length of life of capital will have to be extended indefinitely and, if $N^*$ is finite, labor will eventually become surplus. This puts a lower limit on $I_o$(cf. (vi) below).

(iii) $\rho(t,t)$ is constant;

(15) $$\rho(t,t) = \mu_o(1 - e^{-\lambda m}) .$$

(iv) $w(t)$ grows exponentially at rate $\lambda$;

(16) $$w(t) = (\lambda_o e^{-\lambda m})e^{\lambda t}$$

(v) $Y(t)$ grows exponentially at rate $g$; from (8)

(17) $$Y(t) = \frac{\mu_o I_o}{g}(1 - e^{-gm})e^{gt} .$$

(vi) From (17) it follows that the gross saving ratio, $s(t)$, defined as $I(t)/Y(t)$, is a constant depending on $m$:

$$(18) \qquad s = \frac{g}{\mu_o(1 - e^{-gm})} = \frac{I_o}{Y_o}$$

If the saving rate thus calculated exceeds one, it means that even with consumption reduced to zero the economy is incapable of producing the minimal equipment required to employ the whole labor force, and eventually a labor surplus situation must emerge (cf. (ii) above).

10. <u>Other paths</u>.

We consider later the behavior of the model when it starts from some arbitrary collection of capitals of different vintages and then proceeds under its own power with a constant gross saving rate. Instead of characterizing balanced growth paths as those on which gross investment grows exponentially, we could have considered paths along which $m$ is constant. The constant-$m$ paths obviously include the exponential-$I$ paths. They include a few others as well. Rewriting equation (7) for the case at hand, we get

$$L_o e^{nt} = \int_{t-m}^{t} \frac{\mu_o}{\lambda_o} e^{-\lambda v} I(v)dv .$$

Differentiation leads to

$$nL_o e^{nt} = \frac{\mu_o}{\lambda_o} \left[ e^{-\lambda t} I(t) - e^{-\lambda(t-m)} I(t-m) \right] .$$

This difference equation tells us how big $I(t)$ must be in order to maintain full employment with unchanged $m$ , taking due account of new entrants into the labor force and those released from capital just reaching age $m$ . The general solution of this difference equation is given by

$$I(t) = I_0 e^{(\lambda+n)t} + e^{\lambda t} P(t)$$

Where $I_0$ is a constant and $P$ is an arbitrary periodic function of period $m$ which must be determined from the initial age distribution of capital. If that initial distribution is exponential (with rate $\lambda+n$) then $P$ must vanish and $I$ remains exponential forever, i.e., the solution follows the balanced growth path already discussed. But any other initial age distribution sets up a permanent "echo" in investment and output. The result is a "replacement cycle" which circles around a trend growing at the natural rate. During the cycles the real wage and employment grow steadily, but the saving rate is not constant. It will be shown below that the only paths whose saving rates have been constant forever are the balanced-growth paths with exponential investment history. (In some cases these "replacement cycles" superficially resemble accelerator-generated cycles, because $Y$ leads $I$ slightly. But the cycles are artificial; they arise simply from the requirement to keep $m$ constant with full employment despite an irregular investment history.)

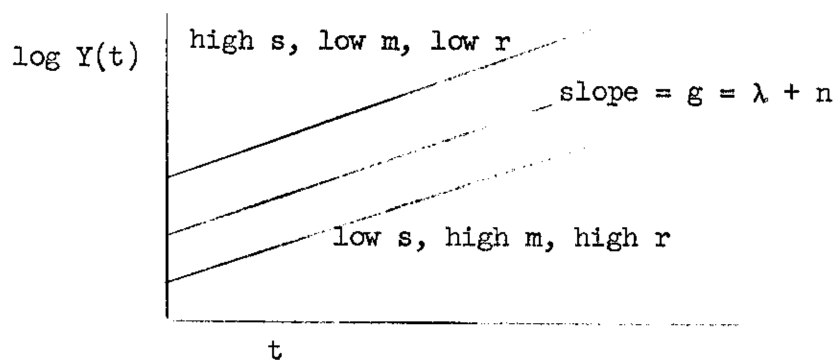
11. Alternative Saving Rates.

According to (18) the path corresponding to a high saving ratio is characterized by low $m$ , quick obsolescence, modern capital. In the same sense, a low saving ratio means a long economic life for capital. Eliminating $I_0$

between (14) and (17) shows that a path with low $m$ and high $s$ has a high $Y_o$, as in Figure 2.

Figure 2.

Balanced Growth Paths




Not all values of $s$ and $m$ are consistent with balanced growth of this kind, at the "natural" rate $g = \lambda + n$. At one extreme, the lower limit on the saving ratio $s$ is $g/\mu_o$. This is the value of $s$ for which $m$ must be infinity in (18). It corresponds, therefore, to a situation in which, according to (14), the rate of investment is just sufficient to employ increments to the labor force without transferring any workers from obsolescent capital. $L(t)$ and $N(t)$ are equal to $N^*(t)$ and all are growing at rate $n$. But because full employment requires that infinitely old capital be left in use, the competitive equilibrium real wage, according to (16), must be zero!

Suppose the saving ratio is still smaller, so that $s\mu_o$ is less than $g$. If no capital ever becomes obsolete, the stock of capital will grow at the rate $s\mu_o$. But with the number of workers growing at rate $n$ and the number of workers required per machine falling at rate $\lambda$, the stock of capital must grow at rate $g$ to provide enough places. If $s\mu_o < g$, therefore, new investment is insufficient to employ the natural increase in the labor force, much less to require release of any labor from older capital. So long as any capital is unused, previously submarginal vintages will be brought into use. As labor goes to work on older and older vintages, the real wage falls. The limit, of course, is the labor surplus regime.

The highest conceivable saving ratio (in a closed economy) is 1, and the correspondingly shortest capital lifetime $m$ is given by

$$1 = \frac{g}{\mu_o(1 - e^{-gm})} \quad .$$

(This has a positive solution for $m$ provided $g < \mu_o$; otherwise, as remarked above, the need for new capital surpasses total output.) But this path, which yields the highest output path in Figure 2, is obviously not the path of highest consumption.

12. <u>The Golden Rule Path</u>.

There is indeed a "golden rule" path -- the balanced growth path on which, given the development of the labor force $L(t)$, consumption is higher at every point in time than on any other balanced growth path. Along this path, (1) the saving ratio is equal to the share of capital in gross product; and (2) the rate of interest or marginal efficiency of capital is equal to the growth rate. These are familiar neo-classical or neo-neo-classical propositions, and it is of interest that they apply for the fixed-coefficient technology of the model under discussion here.

To prove the first proposition, it is necessary to show how the share of capital $\alpha$ depends on the obsolescence period $m$. The wage bill $N(t)w(t)$ is equal to $N(0)e^{nt}\, e^{\lambda(t-m)}$. Since $Y(t) = Y(0)e^{(n+\lambda)t}$, labor's share is constant over time along any path with exponential investment and, therefore, constant $m$ and constant $s$:

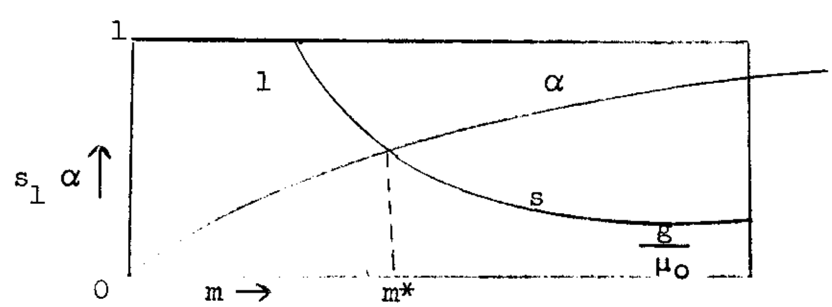
$$(19) \qquad 1 - \alpha = \frac{N(t)w(t)}{Y(t)} = \frac{N(0)\lambda_o e^{-\lambda m}}{Y(0)} .$$

From (14) and (17) this becomes:

$$(20) \qquad 1 - \alpha = \frac{g(e^{-\lambda m} - e^{-gm})}{n(1 - e^{-nm})} .$$

From (20) it follows that $\alpha$ is an increasing function of $m$ -- running from zero for $m = 0$, i.e., when all input is current labor input, to 1 for $m = \infty$, i.e., when labor is in surplus.

Figure 3
Balanced-Growth Paths
Relations of Capital Share $\alpha$ and
Saving Ratio $s$ to Obsolescence Period $m$



Similarly (18) shows that $s$, the saving ratio, is a decreasing function of $m$. Both these relationships are shown in Figure 3. At $m^*$, $s = \alpha$. That is:

$$(21) \qquad \frac{g}{\mu(1 - e^{-gm^*})} = \frac{n(1 - e^{-gm^*}) - g(e^{-\lambda m^*} - e^{-gm^*})}{n(1 - e^{-gm^*})}$$

We must show that this value of $m^*$ also maximizes $C(0)$.

$$C(0) = (1-s)Y(0) = \frac{1-s}{s} \cdot \frac{\lambda_o n\, N(0)}{\mu_o(1 - e^{-nm})}$$

For given $N(0)$, $C(0)$ will be maximized if

$$\frac{1 - s}{s(1 - e^{-nm})} \quad \text{is maximized,}$$

i.e., if $\dfrac{\mu_o(1 - e^{-gm})-g}{g(1 - e^{-nm})}$ is maximized with respect to $m$.

The condition for the maximum,

$$(22) \qquad g(1 - e^{-nm})\, \mu_o g e^{-gm} = (\mu_o(1 - e^{-gm}) - g)g\, ne^{-nm} ,$$

reduces to (21), the condition for $\alpha = s$. Since this equation determines a unique local extremum, which is a maximum, the first formulation of the golden rule theorem is proved. The second version of the theorem states that along the balanced growth path with maximum consumption the rate of interest is equal to the growth rate. That statement is also true in this model, but the proof is postponed until the interest rate or marginal efficiency of capital has been introduced more formally.

# III. ASYMPTOTIC BEHAVIOR UNDER PURELY LABOR-AUGMENTING-TECHNICAL PROGRESS WITH FULL EMPLOYMENT AT A CONSTANT SAVINGS RATIO

## 1. Preliminaries

Throughout the last few sections, we have been exploring the properties of full employment paths along which investment grows exponentially at the natural rate. We have observed in (14) and (18) that this restriction requires the economic lifetime of capital and the gross saving ratio to be constant, and fixes their values. Now we wish to postulate the saving behavior and then see if anything can be said about the path along which a full employment economy must travel.[2/] Our assumption will be the simplest one, i.e., that gross saving is a constant fraction of gross output. Other possible assumptions will be discussed later.

We adopt the exponential assumption (13) and for convenience we shall let $L_o = 1$, so that

$$L(t) = e^{nt} \quad \text{for all } t .$$

As for the technical progress functions, we assume that $\lambda_o = \mu_o = 1$, so that

$$\mu(v) = 1 \quad \text{and} \quad \lambda(v) = e^{\lambda v} \quad \text{for all } v .$$

Finally, let $s$ denote the (constant) savings ratio.

Before our economy can proceed to evolve, it must be endowed with an <u>initial capital profile</u>. Let $t = 0$ be the point in time at which the economy

2/ Uzawa [3] studied this problem in the framework of the no-obsolescence vintage model.

begins to evolve. Then the initial capital profile is given by an arbitrary nonnegative real function, which we denote $I$, on the interval $(-\infty, 0)$. In other words, $I(t)$ is predetermined and arbitrary for all $t < 0$. As a matter of convenience we shall assume that there exists a real number $h^* < 0$ such that

$$I(v) = 0 \quad \text{for} \quad v \leqq h^*$$

$$I(v) > 0 \quad \text{for} \quad v > h^*$$

where $h^* = -\infty$ is permissible. Vintages later than $h^*$ are all present in the initial capital profile in positive quantities. (We also assume that $I$ is a function which can be integrated.)

Instead of using the function $I$ for the initial capital profile, we shall use a transformed version. The reason for introducing this transformation will become apparent shortly. For every $t < 0$, define

$$f(t) = \frac{1}{s} I(t)e^{-(\lambda+n)t}$$

Apart from the multiplier $\frac{1}{s}$, $f$ is just the ratio of $I$ to an exponential trend, so specifying $f$ is equivalent to specifying $I$. Since $I(t)$ is intrinsically nonnegative, so is $f(t)$.

Starting at time $t = 0$, the economy proceeds under its own power. Its motion is determined by the following equations, which are obvious versions of (7) and (8)

(a) Full employment of labor

$$\int_{h(t)}^{t} e^{-\lambda v} I(v)dv = e^{nt} \quad \text{for all} \quad t \geqq 0 ,$$

where $h(t)$ is the vintage of the oldest capital in use at time $t$, so that $h(t) + m(t) = t$.

(b) Determination of output

$$\int_{h(t)}^{t} I(v)dv = Y(t) \quad \text{for all} \quad t \geqq 0 .$$

(c) Equality of gross saving and investment

$$I(t) = sY(t) \quad \text{for all} \quad t \geqq 0 .$$

These three equations may be collapsed into two. For every $t \geqq 0$, let $f(t)$ be defined by

$$f(t) = e^{-(\lambda+n)t} Y(t) ;$$

$f(t)$ is output per efficiency unit of labor. This definition is consistent with the one already made for $t < 0$, so we can proceed to write the basic equations which govern the motion of the economy as follows:

(7')
$$s \int_{h(t)}^{t} e^{-n(t-x)} f(x)dx = 1 \quad \text{for} \quad t \geqq 0$$

(8')
$$s \int_{h(t)}^{t} e^{-(\lambda+n)(t-x)} f(x)dx = f(t) \quad \text{for} \quad t \geqq 0 .$$

It is sometimes more convenient to write these equations somewhat differently:

$$(7'') \qquad s \int_0^{m(t)} e^{-nx} f(t - x)dx = 1 \quad \text{for} \quad t \geqq 0$$

$$(8'') \qquad s \int_0^{m(t)} e^{-(\lambda+n)x} f(t - x)dx = f(t) \quad \text{for} \quad t \geqq 0 ,$$

where $m(t)$ has its earlier meaning.

Remark 1: $f(t) > 0$ for all $t \geqq 0$

Proof: It follows from equation (8') that if $f(t_o) = 0$ for some $t_o \geqq 0$, then either $f(t) = 0$ for all $t \leqq t_o$ or $h(t_o) = t_o$. In either case, equation (7') cannot hold.

Remark 2: The functions $f$ and $h$ are both continuous on the interval $(0, \infty)$.

Proof: For $\lambda \geqq 0$, the integrand in (8') is no greater than that in (7'). Hence $f(t) \leqq 1$. Since $f$ is thus bounded, the continuity of $h$ follows from the fact that $f$ must be continuous on $(0, \infty)$.[3]

Remark 3: The functions $f$ and $h$ are, in fact, differentiable on $(0, \infty)$.

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[3] If we drop the assumption that the initial capital profile "has no holes" (i.e. the function $I$, once positive, remains positive) then $h$ may cease to be continuous (although it is not difficult to trace its discontinuities) while $f$ remains continuous throughout $(0, \infty)$.

Proof: Notice first that if $h$ is differentiable, then it follows from equation (8') that $f$ is also differentiable. To see that $h$ is differentiable, we write down equation (7') twice, once for time $t$ and once for time $t + \Delta t$, and then we subtract the latter from the former. This leads to

$$\int_{h(t)}^{h(t+\Delta t)} e^{nx} f(x)dx - \int_{t}^{t+\Delta t} e^{nx} f(x)dx = \frac{1}{s} e^{nt}(1 - e^{n\Delta t}) .$$

Since $f$ is continuous, we can use the mean value theorem and obtain, following division by $\Delta t$,

$$\frac{h(t+\Delta t) - h(t)}{\Delta t} e^{nx'} f(x') - e^{nx''} f(x'') = \frac{1}{s} e^{nt} \frac{1 - e^{n\Delta t}}{\Delta t}$$

where $x'$ is between $h(t)$ and $h(t + \Delta t)$ and $x''$ is between $t$ and $t + \Delta t$. Letting $\Delta t \to 0$, we see that

$$\lim_{\Delta t \to 0} \left\{ \frac{h(t+\Delta t) - h(t)}{\Delta t} e^{nx'} f(x') \right\}$$

must exist, since the other limits in the equation exist. Hence $h$ is differentiable at $t$ unless

$$\lim_{\Delta t \to 0} f(x') = 0 .$$

But $\lim f(x') = f(h(t))$, by continuity of $f$ and $h$. Now if $h(t) > h^*$, then $f(h(t)) > 0$. If $h(t) = h^*$ and $h'(t)$ does not exist, then it is easily seen that equation (7') cannot hold to the right of $t$, i.e., full employment ceases at $t$. This completes the proof.[4]

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[4] If the assumption that the initial capital profile "has no holes" were to be dropped, one would still have the differentiability of $f$ and $h$ in open intervals where $h$ is continuous.

## 2. Balanced growth paths

At every point of time $t$, the values of the function $f$ on the interval $(h(t),t)$ determine the immediate future of $f$. The values of $f$ on the interval $(h(0),0)$ are the initial conditions of the system. Our task in this section is to look for something analogous to an equilibrium point, namely for a set of self-sustaining initial conditions. In other words, we are looking for an initial capital profile which leads the function $m$ to be constant and the function $f$ to be periodic:

$$m(t) = m^* \text{ (a constant)} \qquad \text{for all } t \geqq 0$$

$$f(t) = f(t - m^*) \qquad \text{for all } t \geqq 0 .$$

A solution of equations (7') and (8') which satisfies these two requirements is called an equilibrium solution. An equilibrium solution for which $f$ is in fact constant is called a balanced growth solution. The discussion in 11.10 of "replacement echoes" shows that if $n \neq 0$ the only equilibrium solutions are actually balanced growth solutions, because the "echoes" in the function $e^{-(\lambda+n)t}I(t)$ cannot be strictly periodic. In any case, even if $n=0$ (which permits $f$ to be strictly periodic) the saving ratio cannot be constant unless $f$ is constant. In other words, the only equilibrium solutions are balanced growth solutions.

To find a balanced growth solution (if there is one) we must solve equations (7") and (8") under the assumption that $f$ and $m$ are both constant. Setting $f(t) = f^*$ and $m(t) = m^*$ for all $t$, where $f^*$ and $m^*$ are non-negative real numbers, causes equations (7") and (8") to reduce to

$$sf^* \int_0^{m^*} e^{-nx} dx = 1$$

and

$$s \int_0^{m^*} e^{-(\lambda+n)x} dx = 1$$

respectively. These equations have a unique solution, namely

(23) $$m^* = - \frac{1}{\lambda+n} \log \frac{s - \lambda - n}{s}$$

and

(24) $$f^* = \frac{n}{s(1 - e^{-nm^*})}$$

provided that $s \geqq \lambda+n$ . If $s = \lambda+n$ , we have $m^* = + \infty$ and $f^* = n/s$ , which we shall admit as a solution, provided $n > 0$ . If $s > \lambda+n$ , then $m^*$ is finite and $f^*$ exceeds $n/s$ . If $s < \lambda+n$ , full employment is in the long run impossible. This is another way of expressing the remarks made above in interpreting equation (14). Formally, (7") and (8") have no solution. To see this note that (8") implies for every $t \geqq 0$

$$f(t) = s \int_0^{m(t)} e^{-(\lambda+n)x} f(t - x)dx \leqq s \int_0^{\infty} e^{-(\lambda+n)x} f(t - x)dx \leqq \frac{s}{\lambda+n} \bar{f}$$

where $\bar{f}$ is the supremum of $f$ (finite by (7")). If $\frac{s}{\lambda+n} < 1$ , then either $\bar{f} = 0$ (where $f(t)$ is identically zero) or, if $\bar{f} > 0$ , a $t_o$ can be found for which $f(t_o) > \frac{s}{\lambda+n} \bar{f}$ . The first contingency contradicts (7"), the second contradicts the inequality just derived.

Note that in this section the capital-output ratio $\mu_o = 1$, so that $s$ is Harrod's warranted rate of growth. Comparisons between $s$ and $\lambda+n$ are comparisons between the warranted and natural rates.

## 3. A Basic Differential Equation

We return now to the general case where $f(t)$ and $m(t)$ need not be constant. Since we know that both $f$ and $m$ are differentiable, we may differentiate equations (7") and (8") and obtain, after some calculation, the differential equation:

$$f'(t) = (s - \lambda - n)f(t) - (sf(t) - n)e^{-\lambda m(t)}$$

for all $t > 0$. This is actually the differential equation which we have already seen in different guise, (11), expressing the rate of change of output in terms of marginal productivities and factor rewards. It can be usefully transformed with the help of (23) and (24). Note that (23) may be rewritten

$$s - \lambda - n = se^{-(\lambda+n)m^*}$$

and that equation (24) may be rewritten

$$e^{-nm^*} = 1 - \frac{n}{sf^*} \quad .$$

Thus,

$$s - \lambda - n = s(1 - \frac{n}{sf^*})e^{-\lambda m^*}$$

We now have

$$f'(t) = s(1 - \frac{n}{sf^*})e^{-\lambda m^*}f(t) - (sf(t) - n)e^{-\lambda m(t)} \quad .$$

But we have observed that $f(t) > 0$ for all $t$, so it is permissible to divide the second term by $f(t)$ and thus obtain

$$(25) \qquad f'(t) = sf(t)\left\{e^{-\lambda m^*}(1 - \frac{n}{sf^*}) - e^{-\lambda m(t)}(1 - \frac{n}{sf(t)})\right\}$$

for all $t > 0$.

This differential equation says something about the derivative of $f$ in terms of the deviation of the system from balanced growth. Specifically,

if $m(t) \geqq m^*$ and $f(t) \leqq f^*$ then $f'(t) \geqq 0$, and

if $m(t) \leqq m^*$ and $f(t) \geqq f^*$ then $f'(t) \leqq 0$.

## 4. Asymptotic Behavior in the Case $s \leqq \lambda + n$

Since the initial capital profile is, to a large extent, arbitrary, we cannot hope to characterize the solution of equations (7') and (8') fully. However, we can hope that as $t$ becomes very large, the influence of the initial capital profile wanes, so that assertions can be made about the behavior of the economy for large $t$. This hope turns out to be realized. This section and the next are devoted to asymptotic analysis.

It has already been shown that if $s < \lambda + n$, continued full employment is not possible. The economy does not save enough to provide for the growing effective labor force. So we turn to the other cases, and first to the case $s = \lambda + n$ . We define a function $F$ as follows:

$$F(t) = s \int_{-\infty}^{t} e^{-(\lambda+n)(t-x)} f(x)dx \qquad \text{for } t \geqq 0 .$$

$F(0)$ is finite if the stock of capital at time zero is finite. We assume this to be the case.

Differentiating $F$ , one obtains

$$F'(t) = sf(t) - (\lambda + n)F(t) = (\lambda + n)[f(t) - F(t)] \leqq 0 .$$

Thus $F$ is a non-increasing function. Since it is nonnegative, and therefore bounded below, it must converge. Furthermore, its derivative must approach $0$ . Therefore

$$\lim_{t\to\infty} [f(t) - F(t)]$$

exists and is equal to $0$ . Comparison of the definition of $F$ with (8') implies immediately that $h(t) \to -\infty$ and, therefore,

$$\lim_{t\to\infty} m(t) = \infty .$$

Now $f$ is known to converge, because $F$ does, and $\lim(f(t) - F(t)) = 0$ . Let $\delta$ be the limit of $f$ . For an arbitrary $\epsilon > 0$ , let $T$ be defined by

$$\epsilon = s \int_{T}^{\infty} e^{-nx}dx = \frac{s}{n} e^{-nT} .$$

It now follows from equation (7") and the fact that $f \leqq 1$ that

$$1 - \epsilon \leqq s \int_{0}^{T} e^{-nx} f(t - x)dx \leqq 1 ,$$

where the first inequality holds for all $t$, and the second inequality holds for large $t$ (since $m(t) \to \infty$). Letting $t \to \infty$, we obtain

$$1 - \epsilon \leqq \frac{s\delta}{n}\,[1 - e^{-nT}] \leqq 1$$

$$1 - \epsilon \leqq \delta\,[\frac{s}{n} - \epsilon] \leqq 1 \ .$$

But $\epsilon$ is arbitrary, so we must have $\delta = \frac{n}{s}$.

If $n = 0$, the proof must be modified slightly: in that case, for any arbitrary $T$,

$$s \int_0^T f(t - x)dx \leqq 1$$

provided $t$ is sufficiently large. (This is true because $m(t)$ tends to $\infty$.) Letting $t \to \infty$, we obtain

$$s\ \delta\ T \leqq 1 \ .$$

But $T$ is arbitrary, so it must be true that $\delta = 0$. Thus, when $s = \lambda + n$, the functions $f$ and $n$ tend to the balanced growth values derived in (23) and (24).

## 5. Convergence to Balanced Growth in the Case $s > \lambda + n$

This section is devoted in its entirety to a proof of the following theorem:

*If* $s > \lambda + n$, *then* $\lim_{t\to\infty} f(t) = f^*$ *and* $\lim_{t\to\infty} m(t) = m^*$.

The proof is rather elaborate, but the techniques used may be of some interest. We shall develop the theorem in a series of lemmas.

<u>Lemma 1</u>: <u>There exists a</u> $t_o$ <u>such that</u>

$$f(t) \geqq \frac{n}{s} \quad \underline{\text{for}} \quad t \geqq t_o$$

<u>Proof:</u> It follows from the differential equation (25) that if $f(t) \leqq \frac{n}{s}$ then $f'(t) > 0$ , so that if there exists a $t_o$ such that $f(t_o) \geqq \frac{n}{s}$ , then $f(t) \geqq \frac{n}{s}$ for all $t \geqq t_o$ . It remains to be shown, therefore, that $f(t) < \frac{n}{s}$ for <u>all</u> $t \geqq 0$ is an impossibility. Assume that in fact $f(t) < \frac{n}{s}$ for all $t \geqq 0$ . Then

$$f'(t) > 0 \qquad \text{for all } t$$

$$h'(t) < 0 \qquad \text{for all } t$$

where the second inequality follows from differentiation of equation (7'). Thus,

$$\lim_{t \to \infty} f(t)$$

exists. Call it $\delta$ . By assumption,

$$f(t) < \delta \leqq \frac{n}{s} \qquad \text{for all } t .$$

Substituting $\delta$ in equation (7") we obtain

$$1 < \frac{s\delta}{n} (1 - e^{-nm(t)}) \leqq 1 - e^{-nm(t)}$$

whence

$$e^{-nm(t)} < 0 ,$$

an impossibility. This completes the proof.

<u>Lemma 2</u>: <u>If</u> $s > \delta + n$ , <u>then</u> $\lim_{t\to\infty} h(t) = \infty$ .

<u>Proof</u>: Differentiating equation (7') with respect to t leads to

$$se^{-nm(t)}f(h(t))h'(t) = sf(t) - n .$$

If $f(t) \geqq \frac{n}{s}$ , then $h'(t) \geqq 0$ . By Lemma 1, there exists a $t_o$ such that $f(t) \geqq \frac{n}{s}$ for $t \geqq t_o$ , so we have $h'(t) \geqq 0$ for $t \geqq t_o$ . Therefore,

$$\lim_{t\to\infty} h(t) = \infty$$

or

$$\lim_{t\to\infty} h(t) = K < \infty .$$

Assume the latter. It implies (e.g. by looking at the equation in $h'(t)$ above) that $\lim f(t) = \frac{n}{s}$ . But now equation (25) tells us that as $t \to \infty$ ,

$$f'(t) \to n(1 - \frac{n}{sf^*})e^{-\lambda m^*}$$

$$= n(s - \lambda - n) > 0 \qquad\qquad (\text{for } n > 0)$$

since $s > \lambda + n$ . This is impossible since f has a limit. A similar argument holds when $n = 0$ .

What Lemma 2 tells us is that if $s > \lambda + n$ then every piece of capital is eventually discarded, never to be brought again into use.

Let $t_o$ be the point beyond which the function $h$ is increasing. Such a point exists by Lemma 1. Since, by Lemma 2, $h(t) \to \infty$ as $t \to \infty$, we can divide the interval $[t_o, \infty)$ as follows: Let a sequence $\{t_n\}$ be defined by

$$h(t_{n+1}) = t_n \qquad\qquad n = 0, 1, \ldots$$

Thus, $t_{n+1}$ is the time at which capital of vintage $t_n$ becomes obsolete.

We shall now study the asymptotic behavior of the functions $f$ and $m$ by looking at successive intervals of the form $[t_k, t_{k+1})$ .

<u>Lemma 3</u>: $\limsup_{t\to\infty} f(t) \leqq f^*$

<u>Proof</u>: Suppose that we have been able to find a real number $a_{n-1}$ such that for all $t \geqq t_{n-1}$ ,

$$f(t) \leqq a_{n-1} .$$

This means that if we take any $t \geqq t_n$ , we have

$$f(t - x) \leqq a_{n-1} \qquad\qquad \text{for } 0 \leqq x \leqq m(t) .$$

Consider an arbitrary $t \geqq t_n$ and fix it. We shall attempt to construct a real number $a_n$ such that $a_n < a_{n-1}$ and $f(t) \leqq a_n$ . For convenience, we shall refer to $f(t - x)e^{-nx}$ as $\varphi(x)$ .

<u>Problem:</u> Among all real functions $\varphi$, defined on $[0, m(t)]$, and satisfying the two conditions

$$0 \leqq \varphi(x) \leqq a_{n-1} e^{-nx}$$

$$s \int_0^{m(t)} \varphi(x)dx = 1 ,$$

find a function $\varphi^*$ which <u>maximizes</u> the quantity

$$s \int_0^{m(t)} e^{-\lambda x} \varphi(x)dx .$$

The solution of this maximization is given by the function $\varphi$ which is as concentrated as possible near zero. In other words,

$$\varphi^*(x) = a_{n-1}e^{-nx} \qquad \text{for } 0 \leqq x < T$$

$$= 0 \qquad \text{for } T \leqq x \leqq m(t)$$

where $T$ is determined from

$$s \int_0^T a_{n-1}e^{-nx}dx = 1$$

which reduces to

$$T = - \frac{1}{n} \log(1 - \frac{n}{sa_{n-1}}) .$$

Note that the inequality $T \leqq m(t)$ is assured because $T$ is the <u>smallest</u> value which $m(t)$ could take on in equation (7'), with $f(t - x)$ satisfying the constraints. We may now write

$$\text{maximum} = s \int_0^{m(t)} e^{-\lambda x} \varphi^*(x) dx$$

$$= sa_{n-1} \int_0^T e^{-(\lambda+n)x} dx$$

$$= \frac{sa_{n-1}}{\lambda+n} [1 - e^{-(\lambda+n)T}] \ .$$

Using the expression for $T$, we have

$$\text{maximum} = \frac{sa_{n-1}}{\lambda+n} \left\{ 1 - \left(1 - \frac{n}{sa_{n-1}}\right)^{\frac{\lambda+n}{n}} \right\}$$

Let us call this last quantity $a_n$

$$a_n = \frac{sa_{n-1}}{\lambda+n} \left\{ 1 - \left(1 - \frac{n}{sa_{n-1}}\right)^{\frac{\lambda+n}{n}} \right\}$$

Now $a_n$ is the <u>largest</u> value which $f(t)$ could take on, with $f(t - x)$ satisfying the constraints for all $x$. In other words,

$$f(t) \leqq a_n \ .$$

It remains to be shown that $a_n < a_{n-1}$, which is equivalent to

$$\frac{s}{\lambda+n} \left\{ 1 - \left(1 - \frac{n}{sa_{n-1}}\right)^{\frac{\lambda+n}{n}} \right\} < 1 \ .$$

But $a_{n-1} > f^*$, or else there is nothing to prove. The left-hand side of the last inequality is <u>precisely</u> equal to 1 when $a_{n-1} = f^*$, so it is clearly <u>less</u> than 1 when $a_{n-1} > f^*$. Thus, we have produced a new upper bound for $f$, namely $a_n$ , which is good for all $t \geqq t_n$ . In the same fashion, we can produce still newer upper bounds, with the result that we shall have a sequence $\{a_k\}$ such that

$$f(t) \leqq a_k \qquad\qquad \text{for all } t \geqq t_k .$$

To complete the argument, we must find an $a_o$ such that $f(t) \leqq a_o$ for $t \geqq t_o$ . But a brief look at equations (7') and (8') will reveal that $a_o = 1$ will do nicely. Finally, it remains to be verified that the sequence $\{a_k\}$ converges to $f^*$ . This follows immediately, from the definition of $f^*$ , upon solving the equation for $a_n$ under the condition $a_n = a_{n-1}$ . Thus, the proof is complete.

<u>Lemma 4</u>: $\liminf_{t\to\infty} f(t) \geqq f^*$

<u>Proof</u>: The proof is similar to that of Lemma 3, with one added complication. Suppose that we have found a real number $c_{n-1}$ such that

$$f(t) \geqq c_{n-1} \quad \text{for } t \geqq t_{n-1} .$$

Then, let $t$ be an arbitrary number satisfying $t \geqq t_n$ , and let $m(t)$ be denoted $m$ for short. Note that $m$ is restricted by

$$s \int_0^m c_{n-1} e^{-nx} dx \leqq 1$$

which reduces to

$$m \leqq -\frac{1}{n} \log \left(1 - \frac{n}{sc_{n-1}}\right) .$$

We know that $c_{n-1} \leqq f(t - x) \leqq 1$ for all $x$ in $[0, m]$ . Let us refer to $e^{-nx}f(t - x)$ as $\varphi(x)$ and consider the following problem: Among all functions $\varphi$ on $[0, m]$ , which satisfy

$$c_{n-1}e^{-nx} \leqq \varphi(x) \leqq e^{-nx} \qquad \text{for all } x$$

$$s \int_0^m \varphi(x)dx = 1 ,$$

find the function $\varphi^*$ which <u>minimizes</u> the quantity

$$s \int_0^m e^{-\lambda x}\varphi(x)dx .$$

The solution is given by

$$\begin{aligned} \varphi^*(x) &= c_{n-1}e^{-nx} && \text{for } 0 \leqq x < T \\ &= e^{-nx} && \text{for } T \leqq x \leqq m \end{aligned}$$

where $T$ is determined from

$$s \int_0^T c_{n-1}e^{-nx}dx + s \int_T^m e^{-nx}dx = 1 ,$$

which can be written

$$\frac{s}{n}\left\{ c_{n-1} + (1 - c_{n-1})e^{-nT} - e^{-nm}\right\} = 1 .$$

We note for later reference that

$$\frac{dT}{dm} = \frac{e^{-n(m-T)}}{1 - c_{n-1}} .$$

Now let the minimum of the problem be denoted $f_\ell(t)$ :

$$f_\ell(t) = s\int_0^m e^{-\lambda x}\varphi^*(x)dx$$

$$= \frac{s}{\lambda+n}\left\{ c_{n-1} + (1 - c_{n-1})e^{-(\lambda+n)T} - e^{-(\lambda+n)m} \right\} .$$

We know that $f(t) \geqq f_\ell(t)$ . But $f_\ell(t)$ depends on the unknown $m$ , which is awkward. In order to get rid of this dependence upon $m$ , let us evaluate

$$\frac{df_\ell(t)}{dm} = \frac{s}{\lambda+n}\left\{ -(\lambda+n)(1 - c_{n-1})e^{-(\lambda+n)T}\frac{dT}{dm} + (\lambda+n)e^{-(\lambda+n)m} \right\}$$

$$= se^{-\lambda m}\left\{ e^{-nm} - e^{-nT}\right\} \leqq 0$$

since $T \leqq m$ . Hence, if we let $m$ become as large as it can be, we shall still have a lower bound on $f(t)$ . But the largest $m$ can get is given by

$$m = -\frac{1}{n}\log(1 - \frac{n}{sc_{n-1}}) .$$

So, our new lower bound is obtained by setting $m$ equal to this quantity in the definition of $f_\ell(t)$ , whereupon $T$ becomes equal to $m$ . Doing this, one obtains a new lower bound, to be denoted $c_n$ , where

$$c_n = \frac{sc_{n-1}}{\lambda+n}\left\{ 1 - (1 - \frac{n}{sc_{n-1}})^{\frac{\lambda+n}{n}} \right\}$$

From here on, the proof proceeds as in Lemma 3. The sequence of lower bounds, $\{c_n\}$ , obtained in the manner described, is increasing and it converges to $f^*$ .

Lemmas 3 and 4 together constitute the following theorem:

<u>The function</u> $f$ <u>converges and</u> $\lim_{t\to\infty} f(t) = f^*$ .

As an immediate corollary one now obtains that

<u>The function</u> $m$ <u>converges and</u> $\lim_{t\to\infty} m(t) = m^*$ .

## IV. COMPETITIVE VALUE RELATIONS

### 1. <u>Wages, quasi-rents, and marginal products</u>

The impossibility of direct substitution between labor and capital goods in this model means that there is no "intensive margin." But there is an "extensive margin" at which, under competition, price relationships are determined. The elementary calculations have been made in section II.6-7 and we recapitulate them here.

Capital goods of age $m(t)$ are on the verge of obsolescence; they are "no-rent" capital. If they earned a positive rent their owners would not be about to withdraw them from production under tranquil competitive conditions. Since wages are the only prime cost in this model, the real wage must equal the average product of labor on no-rent capital. This yields, as before,

$$w(t) = \lambda(t - m(t)) . \tag{9}$$

Younger goods are intra-marginal, and earn a differential quasi-rent equal to the difference between output and labor costs; older ones could not cover prime costs if they were operated. Thus, with $\rho(t,v)$ representing the real quasi-rent earned at time $t$ by capital goods of vintage $v$,

$$(10) \qquad \rho(t,v) = 0 \qquad\qquad v \leqq t - m(t)$$
$$= \mu(v)\left(1 - \frac{\lambda(t-m(t))}{\lambda(v)}\right) \qquad\qquad v \geqq t - m(t) .$$

In II.6-7 it is shown that the competitive real wage and quasi-rent play the role of social marginal product of labor and of capital goods of vintage $v$ : $w(t)$ is the increase in aggregate output permitted by one extra unit of employment, and $\rho(t,v)$ is the increase in aggregate output permitted by the availability of one extra unit of vintage $v$ capital.

2. Capital values

Under conditions near to steady growth, the economic lifetime of capital will not change very much and, therefore, $\rho(t,v)$ will fall through time for each fixed $v$ . (In the short run a sharp increase in output and employment may require a sudden increase in $m(t)$ and bring about a temporary rise in the quasi-rents on existing capital. Even previously retired capital will be activated.) If $m(t)$ does not fluctuate much, it is reasonable to suppose that the market can foresee with fair accuracy the pattern of quasi-rents a unit of capital can be expected to earn. The market value of any existing unit of capital will be the present value of the expected quasi-rents, discounted at the market rate of interest. Let $P(t,v)$ be the price at time $t$ of a unit of capital of vintage $v$ , and let $r(t)$ be the force of interest at time $t$ . Then

$$(26) \qquad P(t,v) = \int_t^{\tau} \rho(u,v) e^{-\int_t^u r(z)dz} du = \mu(v) \int_t^{\tau} \left[1 - \frac{\lambda(u-m(u))}{\lambda(v)}\right] e^{-\int_t^u r(z)dz} du$$

In this expression $\tau = \tau(v)$ is the root of the equation $\rho(\tau,v) = 0$; that is, it is the instant at which capital of vintage $v$ will be retired.[5] If $m$ is constant, then of course $\tau = v + m$, and in any case $\tau = v + m(\tau)$.

For existing capital (26) is all there is to be said. When $v = t$, (26) gives $P(t,t)$, the market price of a new machine at the moment of its construction. In tranquil competitive equilibrium, $P(t,t)$ must also equal the cost of production of a new machine of vintage $t$. ($P(t,t)$ can fall short of the cost of production if gross investment is zero, but we shall ignore that possibility.) Since this is a one-sector model we can, as mentioned in II.1, measure capital goods in units identical with the unit of output. Thus $P(t,t) = 1$, and we have for every $t$

$$(27) \qquad 1 = \int_t^{\tau} \rho(u,t)e^{-\int_t^u r(z)dz} du \quad [6]$$

or

$$(27') \qquad 1 = \int_0^{m(\tau)} \rho(x+t,t)e^{-\int_t^{t+x} r(z)dz} dx .$$

---

[5] We assume for simplicity that it is correctly foreseen that capital, once retired, will never be called back into production by a "cyclical" increase in output and employment.

[6] This can be regarded as an integral equation for the unknown interest rate as a function of time. The substitution

$R(u) = \exp(-\int_0^u r(z)dz)$ throws (27) into the more familiar form

$R(t) = \int_t^{\tau(t)} \rho(u,t)\, R(u)du$. Similarly for (27').

From (26) we can extract the well-known equilibrium condition

(28) $$\rho(t,v) + \frac{\partial P(t,v)}{\partial t} = r(t)P(t,v) .$$

The value of the stock of capital is

(29) $$K(t) = \int_{t-m(t)}^{t} P(t,v)I(v)dv .$$

(Here we use again the assumption that the earnings of any particular investment fall eventually to zero and do not revive.) Now, by total differentiation with respect to $t$ and use of (28) we find

(30) $$I(t) - K'(t) = \int_{t-m(t)}^{t} \rho(t,v)I(v)dv - r(t)K(t) .$$

$K'(t)$ can be identified as net investment and $r(t)K(t)$ as net profits. Thus the difference between gross investment and net investment is the same as the difference between gross quasi-rents and net profits. Both can be identified as "true depreciation"; since we have ignored physical depreciation, only "obsolescence" remains. We can let $Z(t)$ stand for net output and $C(t)$ for consumption and write the accounting identities

$$Y(t) = C(t) + I(t) = w(t)N(t) + \int_{t-m(t)}^{t} \rho(t,v)I(v)dv$$

$$Z(t) = C(t) + K'(t) = w(t)N(t) + r(t)K(t) .$$

Equipped with these definitions and relations we can experiment with hypotheses that make net saving depend in one way or another on net income or net profits. But not much can be accomplished at this level of generality, so we turn to our standard special case.

## 3. Harrod-neutrality and balanced growth: the interest rate

Under the assumptions of section II.9 technical progress is purely labor-augmenting and gross investment grows exponentially. Along such a path, as we saw, $m(t)$ is constant. From (10) and (13)

$$\rho(t,v) = 0 \qquad \text{if } v \leqq t-m$$

(31) $$= \mu_o(1 - e^{\lambda(t-v-m)}) \; ; \qquad \text{if } v \geqq t-m \; .$$

(27) becomes

(32) $$1 = \mu_o \int_t^{t+m} (1 - e^{\lambda(u-t-m)}) e^{-\int_t^u r(z)dz} du \; ;$$

Solution of this integral equation gives the equilibrium interest-rate as a function of time on a balanced-growth path. Experience with Harrod-neutrality and balanced growth in other models suggests that the interest rate will be constant. Since the interest rate is required to discount to unity the stream of quasi-rents expected from any newly-built item of capital, and since (31) shows that the current quasi-rent depends only on the age $(t-v)$ of a unit of capital, it is indeed hard to see how any non-constant interest rate can do the trick. In fact, none can.

Substitution of $r(z) = r$ in (32) and integration yields

(33) $$1 = \frac{\mu_o}{r}(1 - e^{-rm}) - \frac{\mu_o}{r-\lambda}(e^{-\lambda m} - e^{-rm})$$

$$= F(r) \; .$$

It is easily seen that $F(-\infty) = \infty$ and $F(\infty) = 0$ ; since $F(r)$ is continuous, (33) has at least one root. Since $F'(r) < 0$ (best seen directly from (32)) there is exactly one root. (That root may be negative; but not if the undiscounted sum of quasi-rents exceeds unity.) Thus if technical progress is Harrod-neutral there is one and only one constant interest rate compatible with competitive equilibrium along a path of steady growth.

It is more complicated to prove that the interest rate <u>must</u> be constant. In the form (27') the basic integral equation can be written

$$1 = \mu_o \int_0^m (1 = e^{\lambda(x-m)}) e^{-\int_t^{t+x} r(z)dz} dx = \int_0^m g(x) e^{-\int_t^{t+x} r(z)dz} dx$$

where $g(x) > 0$ for $x < m$ and $g(m) = 0$ . The substitution $R(-t) = e^{-\int_0^t r(z)dz}$ transforms the equation into

$$(27'') \qquad R(t) = \int_0^m g(x)R(t-x)dx \ .$$

$R(t)$ is, from its definition, intrinsically positive. We will show that the only positive solution of (27") valid for all $t$ is $R(t) = e^{rt}$ , where $r$ satisfies (33). Constancy of the interest rate follows.

We observe first that it is only necessary to settle the case $\int_0^m g(x)dx = 1$ (i.e. the case in which the constant interest rate is zero). If $\int_0^m g(x)dx \neq 1$ , there is a unique constant $h$ such that $\int_0^m e^{-hx} g(x)dx = 1$ ; and it is easily checked that $R^*(t) = R(t)e^{-ht}$ satisfies $R^*(t) = \int_0^m R^*(t-x)g^*(x)dx$

with $g^*(x) = e^{-hx} g(x)$ . We will show that if $\int_0^m g(x)dx = 1$ , the only positive solution of (27") are constant, whence h is the constant rate of interest, as it should be.

Any solution of (27") tends to a constant as $t \to \infty$ . Let $M_n = \max_{(n-1)m \leqq t \leqq nm} R(t)$ and $m_n = \min_{(n-1)m \leqq t \leqq nm} R(t)$ . Since $R(t)$ is a true weighted average of its own past values over an interval of length $m$ , the $M_n$ form a non-increasing and the $m_n$ a non-decreasing sequence, with $M_n \geqq m_n$ . The $M_n$ are bounded below by any $m_n$ and the $m_n$ bounded above by any $M_n$ . Both sequences therefore have limits: $\lim_{n\to\infty} M_n = M^*$, $\lim_{n\to\infty} m_n = m^*$ , and $M^* \geqq m^*$ . It remains to prove that $M^* = m^*$ . Suppose $M^* - m^* = \Delta > 0$ . Consider an arbitrary $t$ and suppose $\operatorname{Max}_{0 \leqq x \leqq m} R(t-x) \leqq M^* + \epsilon$ and $\operatorname{Min}_{0 \leqq x \leqq m} R(t-x) \geqq m^* - \epsilon$ . Divide the interval from $M^* + \epsilon$ to $m^* - \epsilon$ into four subintervals from $M^* + \epsilon$ to $M^*$ , from $M^*$ to $M^* - \frac{\Delta}{2}$ , from $M^* - \frac{\Delta}{2}$ to $m^*$ and from $m^*$ to $m^* - \epsilon$ . Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the integrals of $g(x)$ over the x-values for which $R(t-x)$ lies in each of the subintervals respectively.[7] Obviously $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$ .

From these definitions it follows that

$$R(t) \geqq \delta_1 M^* + \delta_2\left(M^* - \frac{\Delta}{2}\right) + \delta_3 m^* + \delta_4(m^* - \epsilon) = (\delta_1 + \delta_2)M^* + (\delta_3 + \delta_4)m^*$$

$$- \delta_2 \frac{\Delta}{2} - \delta_4 \epsilon$$

---

7/ The smoothness of $R(t)$ as an integral of an integral ... of an integral guarantees all the required measurability.

whence

$$\text{(a)} \qquad R(t) - m^* \geqq \left(\delta_1 + \frac{\delta_2}{2}\right)\Delta - \delta_4 \epsilon ;$$

and

$$R(t) \leqq \delta_1(M^* + \epsilon) + \delta_2 M^* + \delta_3(M^* - \frac{\Delta}{2}) + \delta_4 m^* = (1 - \delta_4)M^* + \delta_4 m^* + \delta_1\epsilon - \delta_3 \frac{\Delta}{2}$$

whence

$$\text{(b)} \qquad M^* - R(t) \geqq \left(\delta_4 + \frac{\delta_3}{2}\right)\Delta - \delta_1 \epsilon .$$

As $t \to \infty$, the $\delta_i$ and $\epsilon$ vary, but $\epsilon \to 0$. If $R(t) - m^*$ were bounded above zero, it would contradict the definition of $m^*$. Therefore, from (a), $\delta_1 + \frac{\delta_2}{2} \to 0$ as $t \to \infty$. But then $\delta_4 + \frac{\delta_3}{2}$ is surely bounded away from zero; from (b) $M^* - R(t)$ is bounded away from zero, which contradicts the definition of $M^*$. Therefore $\Delta = 0$, and $\lim_{t\to\infty} R(t) = M^* = m^*$.

The same sort of argument, worked in reverse, shows that $M_n - m_n \to \infty$ as $n \to -\infty$. Thus any solution to (27") is either constant or unbounded.

The rest of the argument we owe to Professor Frank Stewart of Brown University. Define $S(x) = \int_0^x R(t)dt$. From (27")

$$S(x) = \int_0^x \int_0^1 R(t-s)g(s)dsdt = \int_0^1 \int_0^x R(t-s)g(s)dtds$$

$$= \int_0^1 S(x-s)g(s)ds - \int_0^1 S(-s)g(s)ds .$$

Let $c = \frac{\int_0^1 S(-s)g(s)ds}{\int_0^1 s\, g(s)ds}$ , so $c < 0$ if $R(t) > 0$ , and define

$T(x) = S(x) + cX$ . It is easily verified that $T(t)$ satisfies (27"). Moreover $T'(x) = S'(x) + c = R(x) + c \geqq c$ , so $T(t)$ is a solution of (27") whose derivative is bounded below. In turn this entails the boundedness of $T(t)$. If $T$ were unbounded then, as before, $M_n - m_n$ would become arbitrarily large as $n \to -\infty$ . One could then find an $n$ for which $M_n - m_{n+1} > -2mc$ . $T(t)$ thus falls by more than $-2mc$ in an interval no larger than $2m$ ; by the mean value theorem $T'(t) < c$ at some intermediate point, a contradiction. It follows that $T(t)$ is a bounded solution of (27"). It is therefore constant. Thus $T'(t) = R(t) + c = 0$ and $R(t) = -c$ for every $t$ .

We have established that, with exponential, purely labor-augmenting technical progress, the only competitive equilibrium interest rate compatible with a permanent path of balanced growth is a constant interest rate, namely the unique real root of (33). Since the instantaneous interest rate is constant, the yield curve or term structure of interest rates is flat.

According to (33) $r$ depends on $\mu_o$, $\lambda$ and $m$ ; through (18) $r$ depends also on the other parameters, $n$ and the gross saving ratio $s$ . By straightforward calculation, $\frac{dr}{dm} > 0$ ; if one compares two steady-growth paths with the same $\mu_o$ and $\lambda$ but with different $m$, the path with longer lifetime for capital will be the one with higher interest rate. This sounds "un-Austrian"; indeed the mechanism is very different from the economics of roundaboutness. From (18), a higher $m$ is associated with a lower $s$; with lower $s$, full employment requires the break-even margin to be pushed back to older machines. Thus a lower saving rate implies a higher $m$, which implies a higher rate of interest. This result is entirely conventional. Similarly (18) shows that, with given $s$, $\frac{\partial m}{\partial g} > 0$. Since $g = n + \lambda$, a steady-growth path with higher $n$ will have higher $m$ and higher $r$; other things equal, faster growth in the labor force favors a higher rate of profit. (Remember that full employment, or at least a constant unemployment rate, is simply assumed.)

The relation between $r$ and $\lambda$, for given $s$, is more complicated because $\lambda$ appears directly in (32) or (33). Nevertheless, it can be shown from (32) and (18) that $\frac{\partial r}{\partial \lambda} > 0$. In this model faster Harrod-neutral technical progress with unchanged saving ratio always implies a higher rate of interest. The key to this result is that, from (18), $\frac{\partial m}{\partial \lambda} = \frac{1}{g}\left(\frac{1}{s\mu_o - g} - m\right)$; and, again from (18), $s\mu_o - g = \frac{g}{1 - e^{-gm}} - g$ $= \frac{g}{e^{gn} - 1} < \frac{1}{m}$. Thus $\frac{\partial m}{\partial \lambda} > 0$; with given $s$, a faster rate of technical progress actually **lengthens** the economic lifetime of capital. The greater initial productivity advantage of new capital must outweigh the more rapid rate of improvement of capital still to come.

By letting $r \to 0$ in (33), we find the $m$ corresponding to a zero rate of interest. This $m_1$ satisfies

$$\lambda = \mu_o(\lambda m_1 - 1 + e^{-\lambda m_1}) .$$

Since the right-hand side increases monotonically from zero at $m = 0$ to $+\infty$ as $m \to \infty$, there is always a lifetime short enough to reduce the rate of interest to zero. From II.11, however, the shortest $m$, say $m_2$, attainable by a closed economy in balanced growth is associated with $s = 1$, and satisfies $\mu_o(1 - e^{-gm_2}) = g$. Depending on the other parameters, $m_1$ may exceed, equal, or fall short of $m_2$. In the first case, $r = 0$ for some saving rate less than unity; in the second case $r = 0$ for $s = 1$; in the third case, the rate of interest remains positive even if all of output is saved and invested.

At the other end of the spectrum, as $m \to \infty$, $r \to \mu_o$ and this is the highest profit rate the technology can generate. For then the real wage is zero and investment of one unit of output earns a perpetuity of $\mu_o$ units of output per unit time. The saving rate corresponding to infinite lifetime is $s = \frac{g}{\mu_o}$.

4. The Golden Rule path once more.

In II.12 it was shown that a steady-growth path on which gross investment is always equal to gross quasi-rent generates the highest consumption path among all steady-growth paths. We can now see that the other standard characterization of the "Golden Rule," that the rate of interest equals the rate of growth, also holds in this model. It is only necessary to put $r = g$ in (33) and observe that the resulting equation is the same as (21) or (22).

## 5. Harrod-neutrality and balanced growth: capital values.

Using (26) and (31), for Harrod-neutral balanced growth, it is easy to calculate that

$$P(t,v) = \frac{\mu_o}{r}\left(1 - e^{r(t-v-m)}\right) - \frac{\mu_o}{\lambda - r}\left(e^{r(t-v-m)} - e^{\lambda(t-v-m)}\right) .$$

(Putting $v = t$ and $P(t,t) = 1$ gives the equation for the rate of interest.) With this formula and (29), another straightforward calculation gives

$$(34)\quad K(t) = I_o e^{gt} \mu_o \left\{ \frac{1}{rg} - \frac{\lambda e^{-rm}}{r(\lambda-r)(g-r)} + \frac{e^{-\lambda m}}{(\lambda-r)(g-\lambda)} - \frac{\lambda e^{-gm}}{g(g-r)(g-\lambda)} \right\} .$$

$K(t)$ is a value; to be exact it is the competitive market value (in units of the single commodity) of the stock of diverse capital goods in existence at time $t$. Since we are limited, in any case, to paths of steady growth, the foresight involved in this valuation is no extra strain on the imagination. The ratio $K(t)/I(t)$ will be constant along a steady-growth path. Its value depends on all the main parameters of the model $\lambda$, $g$, and $\mu_o$, as well as on $m$ and $r$, and therefore on $s$.

Knowledge of $K(t)$ permits the calculation of various <u>net</u> magnitudes. To begin with, since $K' = gK$, (34) gives the ratio of net to gross investment as

$$(35)\quad \frac{K'}{I} = \frac{gK}{I} = \mu_o \left\{ \frac{1}{r} - \frac{\lambda g e^{-rm}}{r(r-\lambda)(r-g)} - \frac{g e^{-\lambda m}}{(r-\lambda)(g-\lambda)} + \frac{\lambda e^{-gm}}{(r-g)(g-\lambda)} \right\} .$$

$$= 1 - \lambda\mu_o \left\{ \frac{e^{-rm}}{(r-\lambda)(r-g)} + \frac{e^{-\lambda m}}{(r-\lambda)(g-\lambda)} - \frac{e^{-gm}}{(r-g)(g-\lambda)} \right\} .$$

(The last equality is obtained with the aid of (33).) Now if we define $\omega = 1 - \frac{K'}{I}$ = depreciation as a fraction of gross investment, we have

$$(36)\quad \begin{aligned} \omega &= \lambda\mu_o \left\{ \frac{e^{-rm}}{(r-\lambda)(r-g)} + \frac{e^{-\lambda m}}{(r-\lambda)(g-\lambda)} - \frac{e^{-gm}}{(r-g)(g-\lambda)} \right\} \\ &= \frac{\lambda\mu_o}{r-\lambda} \left\{ \frac{e^{-\lambda m}(1 - e^{-nm})}{n} - \frac{e^{-gm}(1 - e^{-(r-g)m})}{r-g} \right\} . \end{aligned}$$

Net output is gross output minus depreciation: $Z = Y - \omega I$; but $I/Y = s$, so $Z = (\frac{1}{s} - \omega)I$. Thus the ratio of net investment to net output is

$$\sigma = \frac{\dot{K}}{Z} = \frac{(1-\omega)I}{(\frac{1}{s} - \omega)I} = \frac{s - s\omega}{1 - s\omega} .$$

Since $\omega$ is nonnegative, $\sigma \leq s$. It would be interesting to know whether $\sigma$ is a monotone function of $s$ on steady-growth paths, i.e.

whether, as between steady-growth paths alike in all parameters except $s$, the one with higher $s$ always has higher $\sigma$. We have not settled this question; so far as we know, it may be that the higher gross saving ratio, associated with a shorter lifetime, may so accelerate depreciation as to result in a smaller net saving rate. It is clear, however, that "on the average" a higher $s$ is associated with a higher $\sigma$. From the discussion in V.3, the lowest gross saving rate compatible with full employment is $s = \frac{g}{\mu_o}$ (we must assume $\mu_o > g$ else continued full employment is not possible at all). Along such a path $\omega = 0$, since $r \to \mu_o > g$; hence $\sigma = s$. (Intuitively, as $m \to \infty$, the real wage tends to zero, and there is no obsolescence. Since we have ruled out physical depreciation, $\omega = 0$. If there were physical depreciation-by-evaporation, $\omega$ would tend to the rate of depreciation and $\sigma$ would be at its minimum when $s$ is at its minimum.) At the other extreme, when $s = 1$, $\sigma = 1$, so the net and gross saving rates reach their maxima together. But we do not know whether their overall positive association is broken for some values of $s$.

The symbol $\alpha$ has already been introduced for the share of gross quasi-rents in gross output. Let $\Pi$ be the share of net profits in net output. Then $\Pi = \frac{\alpha Y - \omega I}{Y - \omega I} = \frac{\alpha - \omega s}{1 - \omega s} = \alpha - \frac{\omega\sigma(1 - \alpha)}{1 - \omega}$.

Thus $\Pi < \alpha$. Also, when $\alpha = s$, $\Pi = \sigma$; the maximal-consumption or golden-rule path can be characterized in still a third way: net savings equal net profits.

## 6. Harrod-neutrality and balanced growth: alternative savings functions.

So far we have parametrized saving-investment behavior by the ratio of gross investment to gross output along full employment balanced-growth paths. The equations describing any such path may be collected:

$$(14)\quad L_o = \frac{\mu_o I_o}{\lambda_o n}\,(1 - e^{-nm})$$

$$(16)\quad w(t) = \lambda_o e^{-\lambda m}\, e^{\lambda t}$$

$$(18)\quad s = \frac{g}{\mu_o(1 - e^{-gm})} = \frac{I_o}{Y_o}$$

$$(33)\quad 1 = \frac{\mu_o}{r}\,(1 - e^{-rm}) - \frac{\mu_o}{r-\lambda}\,(e^{-\lambda m} - e^{-rm})\ .$$

If we treat $s$, the gross savings ratio, as a parameter, then the given constants in these equations are $L_o$, $\mu_o$, $\lambda_o$, $n$, $\lambda$, $g$ and $s$. The unknowns are $I_o$, $Y_o$, $w$, $m$, and $r$, and they are uniquely determined (subject to the restriction $\frac{g}{\mu_o} \leq s \leq 1$). These equations "decompose" in a particular way. We can say that (18) alone determines $m$, (14) determines $I_o$, (16) determines $w$, and the "no-pure-rent" equation (33) determines the rate of interest or rate of profit $r$.

The gross savings ratio is not the only possible parametrization of saving behavior. It is convenient because, in this one-commodity model at least, it is a purely "physical" description independent of all value considerations. But for that very reason it may also be inappropriate in an economy in which the capitalist motivations play a role Alternative descriptions have been proposed; the commonest are to make net saving proportional to net income, or to net profit, or to make gross or net saving a linear function of the wage bill and gross or net profits. These alternative saving functions do not introduce any new growth paths. It is clear from (33) and (36) that along a steady-growth path a constant $s$ is accompanied by constant $\omega$, constant $\sigma$, and constant $\Pi$. But different ways of characterizing saving behavior lead to different "decompositions" of the equivalent equilibrium conditions and therefore, from a superficial point of view, to different "theories" of interest and profit. Since this is sometimes misunderstood, we make some remarks here.

The most interesting alternative to consider from this point of view is the assumption that net saving is proportional to net profit:

$$K' = \sigma_r \Pi Z \ .$$

It can be verified that (18), (20), (33), and (34) imply that $\Pi Z = rK$, as they should, since both sides define net profits. It is obvious that along any balanced-growth path $K$ is a constant times $I$, so that $K' = gK$. It now follows that with the present savings function

$$(37) \qquad r\sigma_r = g .$$

This equation replaces the first equality in (18) among the equilibrium conditions for steady growth.

Full employment and competition in the labor market continue to imply (14) and (16). These two equations, plus the second equality in (18) and the newly-derived (37), are one equation short of determining all the unknowns along a steady-growth path. There seem to be two and only two consistent ways of completing the system. One is to adopt (33) as a market equilibrium condition: the rate of interest must equalize the present value of future quasi-rents from a new capital good to its cost of production. The other requires that the uses of gross output exhaust gross output: $Y = I + wN + (1-\sigma)\Pi Z$. But this requirement together with the other four equations just stipulated entails (33). So there is only one way to complete the system and we might as well let (33) stand.

The path just defined, with a particular value of $\sigma_r$ given, is of course the same as one of the paths defined earlier, namely the one with $s = \dfrac{g}{\mu_o(1 - e^{-gm})}$. But the new equilibrium equations decompose differently and so lend themselves to another interpretation. Now (37) involves only one unknown, $r$. Thus we must say that it determines the rate of interest/profit, the "no-pure-rent" condition (33) determines $m$, and the rest goes as before. Thriftiness conditions and the rate-of-return conditions have exchanged roles. However (33) still holds, though the "causal" interpretation has changed

The inessential character of the change is revealed by considering yet another assumption about saving, that net saving is proportional to net income: $K' = \sigma Z$. The determinate system of equilibrium conditions now consists of (14), (16), the second equality of (18), (33), and the new equation, which can be written

$$(38) \qquad 1 - \omega = \sigma \left(\frac{\mu_o(1 - e^{-gm})}{g} - \omega\right)$$

where $\omega$ is the ratio of depreciation to gross investment given in (36). Once again, the paths thus described are the same as those described earlier; they are merely characterized via a different parameter. But now the equations do not "decompose" at all. There is no one-at-a-time solution possible. Instead (33) and (38) must be solved <u>simultaneously</u> for $r$ and $m$, after which the other unknowns follow as before. (Something similar is the case if the propensities to save wages and profits are positive but different.)

The only safe statement, therefore, is that the rate of interest is determined, in general, both by thriftiness conditions and by "marginal" conditions. This result is not only safe, but satisfying.

## V. The Interest Rate and the Social Rate of Return on Investment

### 1. Definitions and preliminaries.

In this part of the paper we revert to the more general assumptions of Part II. To be precise, we assume $\mu(v)$ and $\lambda(v)$ to be continuous positive functions of $v$, with $\lambda(v)$ strictly increasing. The object of this part is to relate the competitive equilibrium interest rate defined in (27) to what we shall call the social rate of return on saving or investment. The point of the analogy is suggested by the fact that in a perfect capital market the ruling rate of interest functions as the private rate of return on savings.

Consider a person who disposes of a certain amount of wealth, $W(0)$, at time zero and who is obliged for some reason to pursue a saving program such that his wealth at some given later time $T$ is equal to a given amount $W(T)$. His wealth at time zero may consist of a current stock plus the sum of his discounted wage income in the time interval [0,T]. This person is free to choose any stream of consumption $c(t)$, which has a present value equal to $W(0)$ minus the present value of $W(T)$ or, in a formula,

$$(39) \qquad \int_0^T e^{-\int_0^t r(u)du} c(t)dt = W(0) - e^{-\int_0^T r(t)dt} W(T) .$$

Hence, as before, $r(t)$ is the instantaneous interest rate ruling in the market at time $t$. If we compare any two such programs $c_1(t)$ and

$c_2(t)$ and define $\Delta c(t) = c_1(t) - c_2(t)$ we get from (39)

$$(40) \quad \int_0^T e^{-\int_0^t r(u)du} \Delta c(t)dt = \int_0^T e^{-\int_0^t r(u)du} [c_1(t) - c_2(t)]dt = 0 .$$

Because of (40) it is natural to refer to the expression $e^{\int_0^t r(u)du}$ as the rate of transformation between consumption at time $t$ and consumption at time zero. Giving up one unit of present consumption permits $e^{\int_0^t r(u)du}$ additional units of consumption at time $t$. Thus the instantaneous rate of return on savings is the geometric rate of change of the transformation rate as $t$ varies. Especially, as $t$ approaches zero we get

$$(41) \qquad e^{\int_0^t r(u)du} \simeq 1 + r(o)t$$

If we choose the time unit small enough, we can say that $r(o)$ expresses the net gain in total consumption, if consumption is reduced by one unit at time zero and correspondingly increased after one period. The same interpretation will be given to the social rate of return on savings.

2. The social rate of return on saving.

Under competitive conditions, the private rate of return on savings is independent of the individual's decisions. Like any price, however, the social rate of return depends on the aggregate of investment decisions. It can only be determined after the whole investment path (for the past as well as for the future) has been decided. Let us call this predetermined path $I^*(t)$ . The development of the labor force, $L(t)$ , is also given. We can now compute the corresponding values for $Y^*(t)$ and $C^*(t)$ .

$$(42) \qquad Y^*(t) = \int_{t-m^*(t)}^{t} \mu(v)I^*(v)dv$$

$$(43) \qquad C^*(t) = Y^*(t) - I^*(t)$$

where the age of the oldest capital in operation at time $t$ , expressed by $m^*(t)$ , is given by the now familiar equation

$$(44) \qquad L(t) = \int_{t-m^*(t)}^{t} \frac{\mu(v)}{\lambda(v)} I^*(v)dv \ .$$

As in II.3, our assumptions about the technology entail that the economy is efficient in the sense that $Y^*(t)$ is the capacity output of the economy at each instant of time. Without a change in the investment path $I^*(t)$ , no higher gross output than $Y^*(t)$ is possible for any $t$ . In other words, in order to achieve a higher volume of gross output in the future the economy has to increase the current rate of accumulation, which means that it has to reduce current consumption.

The concept of a social rate of return on savings makes sense only in the case of efficient paths. For, if $Y(t)$ were limited not by the capacity to produce but by effective demand, then a rise in consumption today could be effected without changing the future capacity to produce consumption goods. But for efficient paths $Y^*(t)$ the social rate of return on investment links small changes in $I(t)$ to small changes in $C(t)$ for given $L(t)$ , just as the marginal productivity of labor relates small changes in $L(t)$ to the resulting small changes in $C(t)$ for given $I(t)$ .

There are of course infinitely many ways in which marginal changes of the function $I^*(t)$ can be introduced. $\Delta I(t)$ , the difference between the old and the new investment path, may be almost any function of time, as long as $t$ lies in the interval $[0,T]$, which we are considering. Since the past is history, $\Delta I(t) = 0$ for $t < 0$ . We will confine ourselves to the effects of variations in the finite period from zero to some arbitrary $T$ $(0 < T < \infty)$ , and assume that $\Delta I(t) = 0$ for $t > T$ . Let us write $\Delta I(t) = \epsilon\psi(t)$ where $\epsilon$ is a constant and $\psi(t)$ is any bounded function of $t$ for $0 \leq t \leq T$ and equal to zero otherwise. Note that $I(t)$ itself is not entirely arbitrary. It must be nonnegative and no bigger than $Y(t)$ . To be sure of room to introduce small changes of $I^*(t)$ in either direction, we assume that

$$(45) \qquad \inf_{0 \le t \le T} I^*(t) > 0 \quad \text{and} \quad \inf_{0 \le t \le T} [Y^*(t) - I^*(t)] > 0 \ .$$

Also, as will be seen, we have to assume that $I^*(t)$ is such that

$$(46) \qquad \lim_{t \to \infty} \left( t - m^*(t) \right) = \infty \ .$$

(See III.5 Lemma 2 for the case of constant $s$ .)

This innocuous assumption means that along the original path the volume of investment is adequate to ensure that the economic lifetime of capital remains finite (or becomes infinite slowly). We now have

$$(47) \qquad I(t) = I^*(t) + \epsilon\psi(t) \ ;$$

for $\epsilon$ sufficiently close to zero $I(t)$ is a feasible investment program, differentiable with respect to $\epsilon$ at the point $\epsilon = 0$ . From (47) we infer

$$(48) \qquad L(t) = \int_{t-m(t)}^{t} [\mu(v)/\lambda(v)] \, [I^*(v) + \epsilon\psi(v)] dv$$

which is an equation for $m(t)$ , and

$$(49) \qquad Y(t) = \int_{t-m(t)}^{t} \mu(v) \, [I^*(v) + \epsilon\psi(v)] dv$$

which determines $Y$ after $m(t)$ has been computed from (48). Now we can differentiate both sides of (48) and (49) with respect to $\epsilon$ for $\epsilon = 0$. Then we get from (48), since $L(t)$ does not depend on $\epsilon$,

$$0 = \frac{\partial L(t)}{\partial \epsilon}\Big|_{\epsilon = 0} = \int_{t-m^*(t)}^{t} [\mu(v)/\lambda(v)]\psi(v)dv + \frac{\mu(t-m^*(t))}{\lambda(t-m^*(t))} I^*(t-m^*(t)) \frac{\partial m(t)}{\partial \epsilon}\Big|_{\epsilon = 0}$$

or, since $I^*(t) > 0$ and $\frac{\mu(t-m^*(t))}{\lambda(t-m^*(t))} > 0$

$$(50) \quad \frac{\partial m(t)}{\partial \epsilon}\Big|_{\epsilon = 0} = - \frac{\int_{t-m^*(t)}^{t} [\mu(v)/\lambda(v)]\psi(v)dv}{\frac{\mu(t-m^*(t))}{\lambda(t-m^*(t)} I^*(t-m^*(t))} .$$

Differentiation of (49) yields

$$\frac{\partial Y(t)}{\partial \epsilon}\Big|_{\epsilon = 0} = \int_{t-m^*(t)}^{t} \mu(v)\psi(v)dv + \mu(t-m^*(t))I^*(t-m^*(t)) \frac{\partial m(t)}{\partial \epsilon}\Big|_{\epsilon = 0}$$

But $\frac{\partial m(t)}{\partial \epsilon}\Big|_{\epsilon = 0}$ is given by (50). So we have

$$(51) \quad \frac{\partial Y}{\partial \epsilon}\Big|_{\epsilon = 0} = \int_{t-m^*(t)}^{t} \mu(v)\psi(v)dv - \lambda(t-m^*(t)) \int_{t-m^*(t)}^{t} \frac{\mu(v)}{\lambda(v)} \psi(v)dv$$

If we subtract $\left.\frac{\partial I(t)}{\partial \epsilon}\right|_{\epsilon = 0} = \psi(t)$ from (51) we get an expression for the marginal change in $C(t)$, which we may call $\delta C^*(t)$

$$(52)\quad \delta C^*(t) = \left.\frac{\partial C(t)}{\partial \epsilon}\right|_{\epsilon = 0} = \int_{t-m^*(t)}^{t} \psi(v)\, \mu(v)\, [1 - \frac{\lambda(t-m^*(t))}{\lambda(v)}]dv - \psi(t) .$$

Comparison with (10) in II.6 shows that the marginal change in consumption at time $t$ is equal to the competitive quasi-rents earned at time $t$ by the incremental investment less the current cost of incremental investment. $\delta C^*(t)$ is the infinitesimal expression that corresponds to $\Delta C(t)$ in equation (40). It is the marginal rate of change from one consumption program to another. If we consider very small $\epsilon$, we can write

$$C(t) \approx C^*(t) + \epsilon\delta C^*(t) .$$

Hence in the neighbourhood of $C^*(t)$, $\epsilon\delta C^*(t)$ plays the same role as $\Delta C(t)$ in the case of the private individual. In that case there existed a discount factor

$$e^{\int_0^t r(u)du} ,$$

such that any admissible $\Delta C(t)$ satisfied

$$\int_0^T e^{-\int_0^t r(u)du} \Delta C(t)dt = 0 .$$

Can we find a corresponding function $r^*(t)$ such that $\epsilon\delta C^*(t)$ must satisfy a similar equation? Observe that $\psi(t) = 0$ for $t > T$ and that because of (46) there exists a $T_1$ such that $t - m^*(t) > T$ for $t > T_1$. Therefore after $T_1$ the economy is no longer affected by the perturbation function $\psi(t)$. This means that we have to look for a function $r^*(t)$ (depending on $I^*(t)$ but not on $\psi(t)$) with

$$(53) \qquad \epsilon \int_0^{T_1} e^{-\int_0^t r^*(u)du} \delta C^*(t)dt = 0$$

for any $\delta C^*(t)$, which can be generated by some admissible $\psi(t)$.

If such a function exists, we may call $e^{\int_0^t r^*(u)du}$ the marginal <u>social</u> rate of transformation between consumption at time zero and consumption at time $t$. For any fixed $t$ society could increase $\delta C^*(t)$ by $e^{\int_0^t r^*(u)du}$ units, if it were to reduce $\delta C^*(0)$ by one unit (this change would be achieved by changing the function $\psi(t)$). It is then natural to call $r^*(t)$ the social rate of return on savings. For a small unit period we could express the marginal rate of transformation between consumption now and consumption $t$ periods later by

$$e^{\int_0^t r^*(u)du} \approx 1 + r^*(0)t$$

By giving up one unit of consumption today society could gain $1 + r^*(0)$ additional units of consumption after one period.

## 3. Equality of private and social rates of return.

We proceed now to prove that in the model of this paper for any given $I^*(t)$ (fulfilling the requirements (45) and (46)) there exists a function $r^*(t)$ such that (53) holds. Moreover we shall see that this unique social rate of return $r^*(t)$ is equal to the instantaneous rate of interest $r(t)$, which in turn, of course, depends on the particular reference path $I^*(t)$.

Substituting (52) into (53) and cancelling the $\epsilon$ on the left hand side of (53) produces the following double integral

$$(54) \quad 0 = \int_0^{T_1} \left[ e^{-\int_0^t r^*(u)du} \left\{ \int_{t-m^*(t)}^{t} \psi(v)\mu(v) \left[1 - \frac{\lambda(t-m^*(t))}{\lambda(v)}\right] dv - \psi(t) \right\} \right] dt$$

Now we introduce a set function $X^*(v)$ defined by

$$X^*(v) = \left\{ t : t-m^*(t) \le v \le t \right\} .$$

$X^*(v)$ has a simple economic interpretation: it is the set of all instants $t$ at which machines of vintage $v$ are operating. From this interpretation of $X^*(v)$ we can infer the equation

$$(55)\qquad \int_{X^*(v)} \mu(v)\left[1 - \frac{\lambda(t-m^*(t))}{\lambda(v)}\right] e^{-\int_v^t r(u)du}\, dt = 1 .$$

This is merely (27) of IV.2 in a slightly altered notation. We write $\rho^*(t,v)$ for the quasi-rent at time $t$ of a capital good of vintage $v$ along the reference path $I^*(t)$.

We define the set $X_S^*(v)$ as

$$X_S^*(v) = \left\{t : t \in X^*(v),\ 0 \le t \le S\right\}$$

$X_S^*(v)$ is the set $X^*(v)$ restricted to the time interval [0,S]. Now, by simple arithmetic, we get equation (56) for any $S \geq 0$.

$$(56)\qquad \int_0^S \left\{ e^{-\int_0^t r^*(u)du} \int_{t-m^*(t)}^{t} \psi(v)\rho^*(t,v)dv \right\} dt =$$

$$\int_{-m^*(o)}^{S} \left\{ e^{-\int_0^v r^*(u)du} \psi(v) \int_{X_S^*(v)} e^{-\int_v^t r^*(u)du} \rho^*(t,v)dt \right\} dv$$

(56) can be proved as follows. First observe that for $S = 0$ both sides are equal to zero. Differentiation of the left hand side with respect to $S$ yields the expression

$$(57) \qquad e^{-\int_0^S r^*(u)du} \int_{S-m^*(S)}^{S} \psi(v)\rho^*(S,v)dv$$

Now observe that the set $V_S^*$ defined by

$$V_S^* = \left\{v : S \epsilon X^*(v)\right\}$$

is just equal to the interval $[S-m^*(S), S]$ . For $V_S^*$ consists of all vintages which are in use at time $S$ . Therefore we can write the expression (57) as

$$(58) \qquad e^{-\int_0^S r^*(u)du} \int_{V_S^*} \psi(v)\rho^*(S,v)dv$$

Now differentiate the right hand side of (56) with respect to $S$ to get

$$(59) \qquad e^{-\int_0^S r^*(u)} \psi(S) \int_{X_S^*(S)} e^{-\int_S^t r^*(u)du} \rho^*(t,v)dt$$

$$+\int_{V_S^*} e^{-\int_0^v r^*(u)du} \psi(v)\, e^{-\int_v^S r^*(u)du} \rho^*(S,v)dv$$

It is easily seen that $X_S^*(S)$ consists only of the point $S$ . Hence the first term in expression (59) is zero. The second term is equal to (58), which proves that the derivatives with respect to $S$ are equal on both sides of (56) for every $S$ . This, together with the fact that both sides have the same value for $S = 0$ , proves (56).

If we take into account that $\psi(v) = 0$ for $v < 0$ and $v > T$ (56) turns into

$$\int_0^{T_1} \left\{ e^{-\int_0^t r^*(u)du} \int_{t-m^*(t)}^{t} \psi(v)\ \rho^*(t,v)dv \right\} dt =$$

(60)

$$\int_0^{T_1} \left\{ e^{-\int_0^v r^*(u)du} \psi(v) \int_{X^*(v)} e^{-\int_v^t r^*(u)du} \rho^*(t,v)dt \right\} dv$$

This is so, because for the relevant $v$'s $(0 \leq v \leq T)$ we have $X_{T_1}^*(v) = X^*(v)$ . Now we can substitute (60) into (54):

$$(61)\quad 0 = \int_0^{T_1} e^{-\int_0^v r^*(u)du} \psi(v) \left[ \int_{X^*(v)} e^{-\int_v^t r^*(u)du} \rho^*(t,v)dt - 1 \right] dv$$

For (61) to be true for all admissible functions $\psi(v)$ it is a necessary and sufficient condition that

$$(62) \qquad \int_{X^*(v)} e^{-\int_v^t r^*(u)du} \rho^*(t,v)dt = 1$$

for all $v$ in the interval $[0,T]$. (The left hand side must be equal to unity at least for a dense subset of $[0,T]$. But it can easily be shown to be a continuous function of $v$, whence it must be equal to unity everywhere).

Comparison of (62) with (17) or (27) of V.2 shows that $r^*(t)$ satisfies the same integral equation as $r(t)$, whose solution is known to be unique. It follows that $r^*(t) = r(t)$, as was to be proved.

## VI. The Keynesian Case: Output Limited by Effective Demand

### 1. Output and Employment

Up to now we have dealt only with the case of full employment. Without inquiring into the causal mechanism, we have assumed that employment could be identified with the exogenous supply of labor. This is a double assumption (a) that at each moment of time the stock of surviving capital is adequate to employ the whole labor force, and (b) that effective demand is always adequate to buy the output producible at full employment from the existing stock of capital. Thus we have placed ourselves in the second of the three regimes mentioned in II.3: output is limited by the supply of labor.

Regime I, in which output is limited by a shortage of capital while labor is redundant, has no application to an advanced industrial economy, though it may be relevant for the advanced sector of a developing economy. We turn in this part of the paper to the third, or Keynesian, regime, when both capital and labor are unemployed, and output is limited not by scarce resources but by effective demand.

The basic equations (7) and (8) continue to hold, but their interpretation is different. In the full-employment regime, $N(t)$ is replaced by $L(t)$, $m(t)$ is determined by (7) and $Y(t)$ by (8). That is: the margin separating active from idle capital is fixed by the requirement that the entire labor force find employment; and output is whatever they are capable of producing. This would presumably be true in a planned economy, or in one where a flexible fiscal policy regulated aggregate demand accurately. Pre-Keynesian neo-classical economics relied on a market mechanism: so long as there was unemployment the real wage would fall; older and older vintages of capital would be able to earn positive quasi-rents; as they entered production, employment would rise. Modern short-run income analysis rests on the presumption that this cannot or does not happen, or does not happen quickly enough to matter. The causal structure in (7) and (8) is reversed. If we take aggregate demand $Y(t)$ as given (in the simplest case, from exogenous investment via the multiplier), (8) determines $m(t)$ and (7) determines $N(t)$. That is: the margin separating active from idle capital is fixed by the requirement that output just match real effective demand; and employment is whatever is necessary to man that capital. If the division of output between consumption and investment is determined, a model like this is clearly able to generate its own future time-path.

## 2. Aggregate Supply and Demand

There must, of course, be a market mechanism underlying a Keynesian economy, though it can not be the same as the neoclassical mechanism. For one thing, a Keynesian economy must have at least one more asset, money, and therefore one more price, the money wage, than the neoclassical aggregative model we have been discussing. Otherwise there is no explanation for over-saving. Without the attraction of some other store of value, investors would simply increase consumption whenever capital accumulation became unattractive. Nor is there, without money, any opening for the trouble that may arise from stickiness of money wages and prices. A mechanism close to that described in The General Theory itself is the following. Suppose $W(t)$ , the money wage, is given in the short-run; $W(t)$ or its rate of change may depend on past unemployment, but for the current instant it is given. Now equation (9) can be rewritten

$$(9') \qquad P(t) = \frac{W(t)}{\lambda(t - m(t))}$$

where $P(t)$ is the money price level. Together (8) and (9') define an aggregate supply curve, giving $Y(t)$ as a function of $P(t)$. Any $P(t)$ determines, via (9'), an $m(t)$. That $m(t)$, inserted in (8), yields the corresponding $Y(t)$. More descriptively, with a sticky money wage, any arbitrary price level fixes the margin between those vintages of capital which can operate at a profit and those which can not; the corresponding supply of output is the capacity of the profitable vintages. Obviously $Y(t)$ is an increasing function of $P(t)$.

A detailed treatment of aggregate demand would be out of place in this essay. One limiting possibility is that real aggregate demand is independent of the price level for a given money wage. More generally, real demand might depend on the price level through the distribution of income, through the real volume of cash balances, or in other ways. In any case, the intersection of the aggregate demand and supply curves determines the price level and real output.

This is a perfectly-competitive Keynesian model, with (9') doing the work of a marginal-product-of labor equation. Here -- as in the full employment model -- it is possible to allow for imperfectly-competitive pricing. Then (9') can be altered to

$$P(t) = (1+\eta) \frac{W(t)}{\lambda(t - m(t))} \; ; \qquad (9'')$$

$\eta$ is the percentage by which price is marked up over prime costs on <u>no-rent capital</u>. So long as $\eta$ is roughly constant the theory can be worked out as before, though the results differ in more or less predictable ways. (Of course the equality of private and social rates of return is broken.) It should be realized that in short-run equilibrium the real wage may be at its competitive level for the current level of employment. Real-wage rigidity means only that unemployment does not make the real wage fall.

## VII. EXTENSIONS AND OPEN QUESTIONS

### 1. Depreciation and loss of productivity.

It is quite straightforward to make allowance for age-dependent physical depreciation within our simple technology. Let $\delta(x)$ be the proportion of an instant's gross investment that survives to age $x$. Then the fundamental employment and output equations become

$$N(t) = \int_{t-m(t)}^{t} \frac{\mu(v)}{\lambda(v)} \delta(t-v)I(v)dv$$

$$Y(t) = \int_{t-m(t)}^{t} \mu(v)\, \delta(t-v)I(v)dv$$

The easy special case is, of course, depreciation-by-evaporation: $\delta(x) = e^{-\delta x}$. The results are generally predictable. The case of "one-hoss-shay" depreciation is mixed. If $\theta$ is the physical lifetime of capital, $\delta(x) = \begin{cases} 1 & 0 \leqq x < \theta \\ 0 & \theta \leqq x \end{cases}$. Whenever the economic facts require

$m(t) \leqq \theta$ , the physical lifetime is irrelevant, and the analysis is exactly as in the body of the paper. But when the economic facts would make $m(t) > \theta$ , the physical lifetime has primacy and shortage of capital supervenes. It is laborious to piece the two regimes together but it can be done.

Related to, but not identical with, the idea of physical depreciation is the notion that capital goods lose productivity (or require increasing maintenance) over their lifetime. Suppose, for concreteness, that plant constructed at time $v$ has a capacity at time $t \geqq v$ of $\mu(t,v)$ units of output. It is now complicated to say what "technological progress" means, since it may well be desirable to have capital goods which are less productive when new but lose productivity more slowly with age. It is unambiguously progress if $v' > v$ implies $\mu(v' + x, v') > \mu(v + x, v)$ for all $x \geqq 0$ (and labor requirements are not higher on vintage $v'$ capital). But this is unnecessarily strong. The simple special case is $\mu(t,v) = \psi(t-v)\,\mu(v)$ . Then one has

$$N(t) = \int_{t-m(t)}^{t} \frac{\mu(v)}{\lambda(v)}\, \delta(t-v) I(v) dv$$

$$Y(t) = \int_{t-m(t)}^{t} \psi(t-v)\, \mu(v)\, \delta(t-v) I(v) dv$$

The distinction between the phenomenon and depreciation is that halfdepreciated capital is assumed to require only half the labor it did when new; but capital that has lost half its productivity is assumed to retain its original labor requirement. (There is a symmetric hypothesis; output capacity remains but labor requirement increases with age.) It is not hard to see that -- ignoring depreciation again -- the real wage must be $\psi(m)\,\lambda(t-m)$ where, as usual, $m$ is the economic lifetime of capital. It follows that

$$\rho(t,v) = \mu(v)[\psi(t-v) - \frac{\psi(m)\lambda(t-m)}{\lambda(v)}] .$$

If we revert to the exponential Harrod-neutral case and put $\psi(x) = e^{-\psi x}$, it turns out that equilibrium paths have constant $m$ and a constant interest rate satisfying

$$1 = \mu_o \frac{1 - e^{-(r+\psi)m}}{r+\psi} - \frac{\mu_o}{r-\lambda} (e^{-(\psi+\lambda)m} - e^{-(\psi+r)m}) .$$

Comparing this equation with (33) one sees that if the triple $(r^*, \lambda^*, m^*)$ satisfies (33), then the triple $(r^* - \psi, \lambda^* - \psi, m^*)$ satisfies the equation above.

## 2. Partially capital-augmenting technical progress

Although the basic model is quite general -- within its fixed coefficient limitations -- we have concentrated very heavily on the case of Harrod-neutral or purely-labor-augmenting technical progress. There are two reasons for this. First, only in this case can we get clear and simple analytical results. Second, it may be that the broad outlines of economic history -- in particular, the apparent long-run trendlessness of the marginal efficiency of capital -- suggest Harrod-neutrality more than they suggest any other simple hypothesis about technical progress.

Under Harrod-neutrality, constant m and constant s go together. When there is any capital-augmenting technical progress -- including the other standard case of Hicks-neutrality -- we must choose between them. In general a constant gross saving ratio requires $\lim_{t\to\infty} m(t) = 0$ [8] and therefore, a rate of interest falling toward zero. On the other hand, if $m(t)$ is constant, then output will grow at the usual natural rate while the gross saving ratio will fall exponentially and the rate of interest will rise. We illustrate these remarks by sketching the Hicks-neutral case.

Let $\lambda(v) = \lambda_o e^{\lambda v}$ and $\mu(v) = \mu_o e^{\mu v}$, and let employment be $N_o e^{nt}$. Then the fundamental equations for employment and output are

$$N(t) = N_o e^{nt} = \frac{\mu_o}{\lambda_o} \int_{t-m(t)}^{t} e^{(\mu-\lambda)v} I(v)dv$$

$$Y(t) = \mu_o \int_{t-m(t)}^{t} e^{\mu v} I(v)dv ;$$

8/ We owe Mr. Geroge Akerlof of M.I.T. the observation that this is so if there is any capital-augmenting progress.

$\mu = 0$ returns us to Harrod-neutrality, while $\mu = \lambda$ gives Hicks-neutrality. On a path with gross investment growing exponentially at the rate $g$,

$$N_o e^{nt} = \frac{\mu_o}{\lambda_o(\mu-\lambda+g)} I_o e^{(\mu-\lambda+g)t} (1 - e^{-(\mu-\lambda+g)m(t)})$$

$$Y(t) = \frac{\mu_o}{g+\mu} I_o e^{(\mu+g)t} (1 - e^{-(\mu+g)m(t)}) .$$

If $m(t)$ is to be constant along this path, it is necessary that the rate of growth of investment $g = n + \lambda - \mu$.

In that case

$$N_o = \frac{\mu_o I_o}{n\lambda_o} (1 - e^{-nm})$$

$$Y(t) = Y_o e^{(n+\lambda)t} = \frac{\mu_o I_o}{n+\lambda} e^{(n+\lambda)t} (1 - e^{-(n+\lambda)m}) ;$$

thus output must grow at the rate $n + \lambda$. Constant economic lifetime for capital is not compatible with a constant gross savings ratio; indeed

$$\frac{I(t)}{Y(t)} = \frac{n + \lambda}{\mu_o(1 - e^{-(n+\lambda)m})} e^{-\mu t} ;$$

the gross savings rate must fall at the rate of capital-augmenting technical progress. Output grows more rapidly than investment.

Even if $m(t)$ is permitted to vary in time, a constant gross savings rate is incompatible with exponentially-growing gross investment and exponentially growing employment. The fundamental equation for output implies

$$m(t) = -\frac{1}{n+\lambda} \log \left(1 - \frac{n+\lambda}{s\mu_o} e^{-\mu t}\right).$$

Thus $m(t) \to \infty$ as $t$ decreases to $\frac{1}{\mu} \log \left( \frac{s\mu_o}{n+\lambda} \right)$ . Even for larger values of $t$ , substitution of this equation for $m(t)$ into the fundamental equation for employment leads to

$$N_o = \frac{\mu_o I_o}{\lambda_o(\mu-\lambda+g)} e^{(\mu-\lambda+g-n)t} \left[1 - \left(1 - \frac{n+\lambda}{s\mu_o} e^{-\mu t}\right)^{\frac{n}{n+\lambda}}\right]$$

which is an impossibility if $\mu > 0$ .

Akerlof has pointed out the following line of argument which proves that $m(t) \to 0$ whenever $\mu > 0$ and $s$ is constant. Although we know that the investment and output paths are not exponential, we can be sure that the output path corresponding to $\mu > 0$ is no lower than the output path corresponding to $\mu = 0$ and the same values for all the other parameters. Therefore, it follows from the theorem of III.5 that for sufficiently large $t$ $Y(t) \geqq c\, e^{(\lambda+n-\epsilon)t}$ where $c$ is a constant and $\epsilon$ is any positive constant, however small. Gross investment, then, eventually exceeds $sce^{(\lambda+n-\epsilon)t}$. Each unit of new gross investment provides employment equal to $\frac{\mu_o}{\lambda_o} e^{(\mu-\lambda)t}$. Thus current gross investment provides employment in excess of $\frac{s\mu_o}{\lambda_o} c\ e^{(n+\mu-\epsilon)t}$. If $\epsilon$ is chosen to be less than $\mu$, employment on currently-produced capital will eventually grow faster than an exponential itself growing faster than the labor force. From what has been said about constant-m paths, it is clear that $m(t)$ can have no positive lower limit. So $\lim_{t\to\infty} m(t) = 0$.

If we turn to the price relationships, (10) says that

$$\rho(t,v) = \mu_o\, e^{\mu v}\, (1 - e^{\lambda(t-m-v)}) ,$$

no longer a function only of the age of capital.

If the saving rate behaves so as to keep $m$ constant, it is clear that the quasi-rent on capital $t-v$ years old grows exponentially with $v$. This is true for the whole stream of quasi-rents from $t = v$ to $t = v + m$. Hence the rate of interest must be rising with calendar time in order to keep the present value of quasi-rents from new investment always equal to 1. On the other hand, if the saving ratio is constant, or falls slowly enough so that $m(t) \to 0$, the shortening of the length of life offsets the rising trend of quasi-rents. The result may be a constant or falling rate of interest.

For the equilibrium interest rate one can apply (27') along a path with constant $m$ to get

$$1 = \mu_o \, e^{\mu t} \int_o^m (1 - e^{\lambda(x-m)}) \, e^{-\int_t^{t+x} r(z)dz} \, dx \quad .$$

Trial shows that the interest rate cannot be constant, i.e., the maturity structure of interest rates cannot be flat. The equation for $r(t)$ can be transformed by differentiating it with respect to $t$ . The result, after some rearrangement is

$$r(t) + \lambda \, \mu_o \, e^{\mu t} \int_t^{t+m} e^{-\int_t^{\mu} r(z)dz} \, du = \mu_o \, e^{\mu t} \, (1 - e^{-\lambda m}) + \lambda - \mu \quad .$$

We have not been able to solve this equation.

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