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COWLES FOUNDATION DISCUSSION PAPER NO. 179

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A PARAMETRIC SIMPLICIAL FORMULATION OF HOUTHAKKER'S CAPACITY METHOD

C. van de Panne and Andrew Whinston

November 13, 1964

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A PARAMETRIC SIMPLICIAL FORMULATION OF HOUTHAKKER'S CAPACITY METHOD

by

C. van de Panne and Andrew Whinston *

ABSTRACT

The paper reformulates Houthakker's capacity method for quadratic programming in the framework of the Simplex and dual methods for quadratic programming, thereby greatly reducing the conceptual and computational complexities of the method. It is shown that the method is applicable for all convex quadratic programming problems, including the case of a semi-definite matrix of the quadratic form and that of constraints in equality form. The method reduces in the linear programming case to a parametric version of the dual method.

1. INTRODUCTION

After linear programming has been developed, a profusion of methods for solving convex quadratic programming problems has emerged. The methods are based on rather different ideas, but almost all methods and certainly the most important ones, have the property of requiring only a finite number of iterations to reach the optimum solution.

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(Paper to be presented at the December, 1964 Meeting of the Econometric Society in Chicago, Illinois.)

Little has been done to relate the various methods. This paper intends to offer a contribution in this respect by relating Houthakker's capacity method [4] to the Simplex and dual methods for quadratic programming (see Dantzig [2] and van de Panne and Whinston [6] and [7]). A first advantage is that the computations of the capacity method can be performed in the framework of quadratic Simplex tableaux, which reduces the computational complexity of the method to a great extent.¹ A second advantage is that the properties of quadratic Simplex tableaux can be used to prove that the method works for all convex quadratic programming problems, so that the case of a semi-definite matrix of the quadratic form in the objective function is also covered; furthermore, it can be shown that degeneracy² creates no substantial difficulties. Finally, it can easily be shown that the capacity method is applicable for any form of the constraints.

We shall start by formulating the quadratic programming problem and explaining the tableaux used in quadratic Simplicial methods. In Section 3 the Simplex and the dual method for quadratic programming are briefly stated and explained. In Section 4 a demonstration is given of an application of the capacity method in Simplicial form to an example used by Houthakker; a more formal statement of the rules can be found in Section 5. In Section 6 it is shown how the objective function can be given as a function of the variable parameter for each solution. Section 7 contains a detailed comparison of the Simplicial formulation

¹ Houthakker already surmised that something like this could be done Cf. [4], p. 83.

² Our use of the word degeneracy differs from that of Houthakker, see Section 8.

of the capacity method and Houthakker's formulation. Section 8 deals with degeneracy, the special case of linear programming and the generalization of the capacity method for any form of the constraints. In Section 9 some concluding remarks are made.

2. THE QUADRATIC PROGRAMMING PROBLEM AND QUADRATIC SIMPLEX TABLEAUX

Let us consider the convex quadratic programming problem in the following formulation. Maximize with respect to the x -variables the function

$$(2.1) \quad p'x - \frac{1}{2}x'Cx$$

subject to

$$(2.2) \quad Ax \leq b ,$$

$$(2.3) \quad x \geq 0 .$$

p and x are vectors of n elements, b is a vector of m elements; A and C are $m \times n$ and $n \times n$ matrices, respectively; C is symmetric and positive semi-definite. An equivalent formulation which is frequently used, is one in which there is an equality instead of an inequality in (2.2); the present formulation is just as general and slightly more convenient for our purposes. (2.2) and (2.3) may be rewritten as

$$(2.4) \quad Ax + y = b ,$$

$$(2.5) \quad x, y \geq 0 ,$$

where y is a vector of m slack variables.

Necessary and sufficient conditions for an optimal solution to this problem are given by the Kuhn-Tucker conditions:

$$(2.6) \quad p - Cx = A'v - u ,$$

$$(2.7) \quad Ax + y = b ,$$

$$(2.8) \quad u'x + v'y = 0 ,$$

$$(2.9) \quad x, y, u, v \geq 0 ;$$

u and v are vectors of n and m elements and contain the so called dual variables. The x - and y -variables are called the primal variables; for each primal variable there is a corresponding dual variable, that is, the corresponding dual variable of x_i is u_i and the corresponding dual variable of y_j is v_j ; for $i = 1, \dots, n$ and $j = 1, \dots, m$. Note that according to (2.8) and (2.9) any pair of corresponding primal and dual variables in an optimum solution should have at least one variable with a value 0.

Quadratic Simplex tableaux are based upon (2.6) and (2.7). The tableau which is obtained by putting (2.6) and (2.7) into a Simplex tableau is called a set-up tableau; it is given in Table 1; the last row will be explained below. From the set-up tableau and its solution other tableaux and solutions may be generated. This is done by transformations of the Simplex tableaux which amount to transformations of the equations (2.6) and (2.7). The transformations are performed in exactly the same manner as in the Simplex method for linear programming, by pivoting on an element of the tableau. The selection of the pivot element which amounts to the selection of a variable to enter and a variable to leave the basis differs for the various Simplicial methods.

There are two types of tableaux which can be generated from the set-up tableau, namely tableaux in standard form and tableaux in nonstandard form. Tableaux in standard form are defined as tableaux which have no pair of corresponding primal and dual variables in the basis. In tableaux in nonstandard form there are corresponding primal and dual variables in the basis, but in the methods we shall treat here there will be only one such pair, which is called the basic pair of corresponding variables. In this case there will also be a pair of corresponding primal and dual variables which are both not in the basis; this pair is called the nonbasic pair.

As in linear programming it is also possible to give for each tableau the value of the objective function for the current solution. This is done as follows. Let us write $f(x)$ for twice the objective function:

$$(2.10) \quad f(x) = 2p'x - x'Cx .$$

Substituting for Cx according to (2.6) and for Ax according to (2.7), we find

$$(2.11) \quad f(x) = p'x + b'v - u'x - v'y .$$

For any tableau in standard form, $u'x$ and $v'y$ are zero since no pair of corresponding variables are simultaneously in the basis. Hence we may write for any tableau in standard form

$$(2.12) \quad f(x) - p'x - b'v = 0 .$$

This equation may be added to the set-up tableau, see Table 1; the corresponding row, which has as its basic variable $f(x)$ is then transformed as the other rows, but $f(x)$ never leaves the basis. The value of $f(x)$ which is twice the value

of the objective function appears then for standard tableaux always in the column of basic variables. For nonstandard tableaux a slight correction has to be made.³

TABLE 1. SET-UP TABLEAU FOR A QUADRATIC PROGRAMMING PROBLEM

Basic Var.	Values Bas.Var.	u	v	x	y
u	-p	I	-A'	-C	0
y	b	0	0	A	I
f	0	0	-b'	-p'	0

TABLE 2. GENERAL REPRESENTATION OF A TABLEAU IN STANDARD FORM

Basic Var.	Values Bas.Var.	x^1	u^2	u^1	x^2
f	q_0	$q^{1'}$	$-q^{2'}$	0	0
u^1	q^1	-Q	-P'	I	0
x^2	q^2	P	-R	0	I

Tableaux in standard form can be shown to have (see [7]) pattern of symmetry and skew-symmetry. A general representation of a tableau in standard form is given by Table 2. In the tableau x^1 stands for a vector containing all basic primal variables, u^1 for a vector containing all basic dual variables and u^2

³ Cf. [6].

for a vector containing all nonbasic dual variables. The arrangement of variables is supposed to be such that the u -vectors contain the dual variables of the variables in the x -vectors in the same sequence; Q and R can be shown to be positive semi-definite matrices. Note that in the set-up tableau the matrix $-R$ is a zero matrix. Note also that the symmetry and skew symmetry properties extend to the values of basic variables and the elements in the row of the objective function.

Tableaux in nonstandard form do not have these symmetry properties, but the methods we shall consider generate nonstandard tableaux which have similar essential properties.

3. THE SIMPLEX AND THE DUAL METHOD FOR QUADRATIC PROGRAMMING

The Simplex method for quadratic programming⁴ has been first discovered by Dantzig [2]; a set of complete proofs for the convergence of the algorithm can be found in [7]. It can be seen as a straightforward generalization of the Simplex method for linear programming. The dual method for quadratic programming, which is closely related to the Simplex method and which can be considered as a generalization of the dual method for linear programming has been proposed by the authors in [6]. Since the Simplicial version of Houthakker's capacity method is

⁴ It is difficult to find the right nomenclature in this case; Wolfe uses the same name for his method; Dantzig "a variant of the Wolfe-Markowitz algorithms." Despite the fact that the name Simplex method may lead to confusion, it is used because of the close relationships of the methods treated here with Simplex and dual methods for linear programming.

based on both methods, we shall give a brief description of these methods.

First the Simplex method for quadratic programming will be described. The algorithm starts with a basic feasible solution to the constraints; this may be the solution given by the set-up tableau given in Table 1, or it may be another solution; in the latter case the corresponding tableau is easily generated from the set-up tableau, see [6]. The resulting tableau is in standard form. The tableau has then primal basic variables which are all nonnegative, but the dual basic variables can have either sign. The method consists of choosing variables to enter and to leave the basis, and transforming the tableau, using as a pivot the element in the column of the variable which is to be introduced into the basis and in the row of the variable which is to leave the basis.

In case the tableau is in standard form, the variable which is to be introduced into the basis is the corresponding variable of the dual basic variable which has the largest negative value. In case the tableau is in nonstandard form, the dual variable of the nonbasic pair is introduced into the basis. The variable which is to leave the basis is found by choosing the one connected with the smallest positive ratio of the elements in the column of values of basic variables and the column of the new basic variable and in the rows of the primal basic variables and in the row of the largest negative dual variable in case of a standard tableau and the row of the dual variable of the basic pair in case of a nonstandard. These rules guarantee that in any nonstandard tableau there will always be only one basic pair and one nonbasic pair of corresponding primal and dual variables. It can be shown that the objective function never decreases in successive solutions and that the optimum solution can be obtained in a finite

number of steps. Degeneracy in the sense of basic variables having zero values can be coped with by using the usual perturbation techniques.

The following properties of tableaux in nonstandard form play a rather crucial rôle in the method. They involve the elements of the column of the dual variable of the nonbasic pair, that is the variable which is to be introduced into the basis. The element in this column and in the row of the primal variable of the basic pair is nonpositive; if this element is zero, then all elements in the same column and in the rows of primal basic variables are zero. Furthermore, the element in the same column and in the row of the dual variable of the basic pair is negative. This means that there will always be at least one pivot in case of a nonstandard tableau, namely the element in the row of dual variable of the basic pair. For details and proofs, see [7].

In the dual method the rôles of the primal and dual variables are interchanged. We start with a basic solution of the tableau which is nonnegative in the dual variables, introduce into the basis in case the tableau is in standard form the dual variable with the largest negative primal variable and in the case of a nonstandard tableau the primal variable of the nonbasic pair, etcetera. The properties of a tableau in nonstandard form are the same after an interchange of the rôles of primal and dual variables.

If there is no basic feasible solution available in terms of either primal or dual variables various starting procedures can be used. In the context of Houthakker's approach one of the starting procedures for the dual method proposed in [6] is interesting. It consists of adding to the problem an "artificial constraint"

$$(3.1) \quad \sum_{i=1}^n x_i \leq \lambda,$$

where λ is taken so large that this constraint is not effective for the optimum solution. This device is the counterpart of introducing artificial variables with large negative coefficients in the objective function into the primal problem.

It turns out to be exactly the same additional constraint Houthakker is using.⁵ The difference between the dual method with starting procedure and Houthakker's capacity method is that Houthakker first obtains an optimal solution for $\lambda = 0$ and then traces through the changes in the optimal solution for increasing λ , while the dual method with starting procedure solves the problem by the dual method, taking a large value for λ . In Section 9 we shall indicate a variant of the dual method which can be shown to be equivalent to the capacity method. We shall apply Houthakker's idea using quadratic Simplex tableaux and the ideas of the Simplex and dual methods for quadratic programming.

4. HOUTHAKKER'S EXAMPLE SOLVED BY THE CAPACITY METHOD IN SIMPLICIAL FORM

In the capacity method for quadratic programming the so called capacity constraint

$$(4.1) \quad \sum_{i=1}^n x_i \leq \lambda$$

is added to the other constraints of the problem. In Houthakker's example this constraint forms already part of the problem and λ has an upper limit

⁵ Houthakker used β instead of λ .

of $1 \frac{2}{3}$. First we shall solve Houthakker's example by the Simplicial version of the capacity method; afterwards we shall attend to the details of the equivalence of both versions.

The problem is to maximize the function

$$(4.2) \quad [18 \quad 16 \quad 22 \quad 20] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \frac{1}{2} [x_1 \quad x_2 \quad x_3 \quad x_4] \begin{bmatrix} 6 & 1 & 8 & 0 \\ 1 & 10 & 1 & 4 \\ 8 & 1 & 17 & 3 \\ 0 & 4 & 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

subject to

$$(4.3) \quad \begin{bmatrix} 5 & 0 & 10 & 0 \\ 0 & 4 & 0 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \\ 1 \frac{2}{3} \end{bmatrix}$$

and

$$(4.4) \quad [x_1 \quad x_2 \quad x_3 \quad x_4] \geq 0.$$

The last equation of (4.3) serves as the capacity constraint and we replace therefore $1 \frac{2}{3}$ by the variable parameter λ . For this problem the set-up tableau can be given as in Table 1 for the general problem.

The result is Tableau 0 of Table 1. Since y_3 , the slack variable of the last equation of (4.3) is dependent on λ , the values of basic variables are in general dependent on λ . Hence the values of basic variables consist of two terms, a term independent of λ and a term dependent on λ , so that we can write for the value of the basic variable in the i^{th} row

$$(4.5) \quad r_i + s_i \lambda.$$

The r_i are given in the column of the independent term, the s_i in the column of the λ -term. In the set-up tableau the λ -term column has zeros everywhere except in the row of y_3 .

The solution given by the set-up tableau is not an optimal solution, since for an optimal solution we must have a standard tableau with nonnegative basic variables, whereas for any positive value of λ all u-variables are negative in Tableau 0. However, an optimum solution for $\lambda = 0$ can be easily achieved by the following two iterations, which are essentially an application of the Simplex method for quadratic programming. Note that the solution of the set-up tableau is a feasible solution because the primal variables are nonnegative for $\lambda = 0$.

According to the rules for the Simplex method, we introduce into the basis the corresponding primal variables of the largest negative dual variable; hence x_3 is introduced into the basis. In order to determine the variable to be replaced we compare the ratios

$$(4.6) \quad \frac{-22}{-17}, \quad \frac{2}{10}, \quad \frac{0 + \lambda}{1}.$$

Since the last ratio must be the smallest for $\lambda = 0$, y_3 has to leave the basis; Tableau 0 is then transformed into Tableau 1 with the underlined element -1 as a pivot.

The tableau is now in nonstandard form and we therefore introduce into the basis the dual variable of the nonbasic pair which is v_3 . In determining the variable to leave the basis, there is only one ratio to be considered, viz.,

$$(4.7) \quad -22/-1,$$

TABLE 3

SIMPLEX TABLEAUX FOR AN APPLICATION OF THE
SIMPLICIAL VERSION OF THE CAPACITY METHOD

Tabl.	Bas. Var.	Values Basic Variables				Nonbasic Variables							
		Ind. t.	λ -term	Cr. V. λ	$\lambda = 0$	v_1	v_2	v_3	x_1	x_2	x_3	x_4	
0	u_1	-18	0		-18	-5	0	-1	-6	-1	-8	0	
	u_2	-16	0		-16	0	-4	-1	-1	-10	-1	-4	
	u_3	-22	0		-22	-10	0	-1	-8	-1	-17	-3	
	u_4	-20	0		-20	0	-5	-1	0	-4	-3	-11	
	y_1	2	0		2	0	0	0	5	0	10	0	
	y_2	3	0		3	0	0	0	0	4	0	5	
	y_3	0	1		0	0	0	0	1	1	<u>1</u>	1	
					$\lambda = 0$	v_1	v_2	v_3	x_1	x_2	y_3	x_4	
1	u_1	-18	8		-18	-5	0	-1	2	7	8	8	
	u_2	-16	1		-16	0	-4	-1	0	-9	1	-3	
	u_3	-22	17		-22	-10	0	<u>-1</u>	9	16	17	14	
	u_4	-20	3		-20	0	-5	-1	3	-1	3	-8	
	y_1	2	-10		2	0	0	0	-5	-10	-10	-10	
	y_2	3	0		3	0	0	0	0	4	0	5	
	x_3	0	1		0	0	0	0	1	1	1	1	
					$\lambda = 1/7$	v_1	v_2	u_3	x_1	x_2	y_3	x_4	
2	u_1	4	-9	4/9	$2^5/7$	5	0	-1	-7	-9	-9	-6	
	u_2	6	-16	3/8	$3^5/7$	10	-4	-1	-9	-25	-16	-17	
	v_3	22	-17	$1^5/17$	$19^4/7$	10	0	-1	-9	-16	-17	-14	
	u_4	2	-14	1/7	0	10	-5	-1	-6	-17	-14	<u>-22</u>	
	y_1	2	-10	1/5	4/7	0	0	0	-5	-10	-10	-10	
	y_2	3	0		3	0	0	0	0	4	0	5	
	x_3	0	1		1/7	0	0	0	1	1	1	1	

TABLE 3 (Continued)

Tabl.	Bas. Var.	-Values Basic Variables				Nonbasic Variables						
		Ind. t.	λ -term	Cr. V. λ	$\lambda = 3/10$	v_1	v_2	u_3	x_1	x_2	y_3	u_4
3	u_1	$3^5/11$	$-5^2/11$	$2/3$	$1^9/10$	$2^3/11$	$1^4/11$	$-8/11$	$-5^4/11$	$-4^4/11$	$-5^2/11$	$-3/11$
	u_2	$4^5/11$	$-5^2/11$	$49/57$	$2^9/10$	$2^3/11$	$-3/22$	$-5/22$	$-4^4/11$	$-11^{19}/22$	$-5^2/11$	$-17/22$
	v_3	$20^8/11$	$-8^1/11$	$2^{50}/89$	$18^3/10$	$3^7/11$	$3^2/11$	$-4/11$	$-5^2/11$	$-5^2/11$	$-8^1/11$	$-7/11$
	x_4	$-1/11$	$7/11$		$1/10$	$-5/11$	$5/22$	$1/22$	$3/11$	$17/22$	$7/11$	$-1/22$
	y_1	$1^1/11$	$-3^7/11$	$3/10$	0	$-4^6/11$	$2^3/11$	$5/11$	$-2^3/11$	$-2^3/11$	$-3^7/11$	$-5/11$
	y_2	$3^5/11$	$-3^2/11$	$1^3/35$	$2^1/2$	$2^3/11$	$-1^3/22$	$-5/22$	$-1^4/11$	$3/22$	$-3^2/11$	$5/22$
	x_3	$1/11$	$4/11$		$1/5$	$5/11$	$-5/22$	$-1/22$	$8/11$	$5/22$	$4/11$	$1/22$
					$\lambda = 4/7$	y_1	v_2	u_3	x_1	x_2	y_3	u_4
4	u_1	4	-7	$4/7$	0	$1/2$	$2^1/2$	$-1/2$	$-6^1/2$	$-5^1/2$	-7	$-1/2$
	u_2	5	-7	$5/7$	1	$1/2$	1	0	$-5^1/2$	-13	-7	-1
	v_3	$21^3/5$	-11	$1^{53}/55$	$15^{11}/35$	$4/5$	5	0	-7	-7	-11	-1
	x_4	$-1/5$	1		$27/35$	$-1/10$	0	0	$1/2$	1	1	0
	y_1	$-6/25$	$4/5$		$38/175$	$-11/50$	$-1/2$	$-1/10$	$1/2$	$1/2$	$4/5$	$1/10$
	y_2	4	-5	$4/5$	$1^1/7$	$1/2$	0	0	$-2^1/2$	-1	-5	0
	x_3	$1/5$	0		$1/5$	$1/10$	0	0	$1/2$	0	0	0

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TABLE 3 (Continued)

Tabl.	Bas. Var.	Values Basic Variables				Nonbasic Variables						
		Ind. t.	λ -term	Cr. V. λ	$\lambda = 33/35$	y_1	v_2	u_3	u_1	x_2	y_3	u_4
5	x_1	$-8/13$	$1^1/13$		$2/5$	$-1/13$	$-5/13$	$1/13$	$-2/13$	$11/13$	$1^1/13$	$1/13$
	u_2	$1^8/13$	$-1^1/13$	$1^1/2$	$3/5$	$1/13$	$-1^3/26$	$11/26$	$-11/13$	$-8^9/26$	$-1^1/13$	$-15/26$
	v_3	$17^{19}/65$	$-3^6/13$	$4^{224}/225$	$14^1/35$	$17/65$	$2^4/13$	$7/13$	$-1^1/13$	$-1^1/13$	$-3^6/13$	$-6/13$
	x_4	$7/65$	$6/13$		$19/35$	$-4/65$	$5/26$	$-1/26$	$1/13$	$15/26$	$6/13$	$-1/26$
	v_1	$22/325$	$17/65$		$11/35$	$-59/325$	$-4/13$	$-9/65$	$1/13$	$1/13$	$17/65$	$4/65$
	y_2	$2^6/13$	$-2^4/13$	$1^1/15$	$2/7$	$4/13$	$-25/26$	$5/26$	$-5/13$	$1^3/26$	$-2^4/13$	$5/26$
	x_3	$33/65$	$-7/13$	$33/35$	0	$9/65$	$5/26$	$-1/26$	$1/13$	$-11/26$	$-7/13$	$-1/26$
					$\lambda = 1$	y_1	v_2	x_3	u_1	x_2	y_3	u_4
6	x_1	$2/5$	0		$2/5$	$1/5$	0	2	0	0	0	0
	u_2	$7^1/5$	-7	$1^1/35$	$1/5$	$1^3/5$	1	11	0	-13	-7	-1
	v_3	$24^2/5$	-11	$2^{12}/55$	$13^2/5$	$2^1/5$	5	14	0	-7	-11	-1
	x_4	$-2/5$	1		$3/5$	$-1/5$	0	-1	0	1	1	0
	v_1	$-1^{19}/25$	$2^1/5$		$11/25$	$-17/25$	-1	$-3^3/5$	$-1/5$	$1^3/5$	$2^1/5$	$1/5$
	y_2	5	-5	1	0	1	0	5	0	-1	-5	0
	u_3	$-13^1/5$	14		$4/5$	$-3^3/5$	-5	-26	-2	11	14	1

TABLE 3 (Continued)

Tabl.	Bas. Ver.	Values Basic Variables				Nonbasic Variables						
		Ind. t.	λ -term	Cr. V. λ	$\lambda = 1$	y_1	u_2	x_3	u_1	x_2	y_3	u_4
7	x_1	$2/5$	0		$2/5$	$1/5$	0	2	0	0	0	0
	v_2	$7^{1/5}$	-7		$1/5$	$1^3/5$	1	11	0	-13	-7	-1
	v_3	$-11^3/5$	24		$12^2/5$	$-5^4/5$	-5	-41	0	58	24	4
	x_4	$-2/5$	1		$3/5$	$-1/5$	0	-1	0	1	1	0
	v_1	$5^{11}/25$	$-4^4/5$		$16/25$	$23/25$	1	$7^2/5$	$-1/5$	$-11^2/5$	$-4^4/5$	$-4/5$
	y_2	5	-5		0	1	0	5	0	<u>-1</u>	-5	0
	u_3	$22^4/5$	-21		$-1^4/5$	$4^2/5$	5	29	-2	-54	-21	-4
					$\lambda = 1^{31}/665$	y_1	u_2	x_3	u_1	y_2	y_3	u_4
8	x_1	$2/5$	0		$2/5$	$1/5$	0	2	0	0	0	0
	v_2	$-57^4/5$	58		$2^{601}/665$	$-11^2/5$	1	-54	0	-13	58	-1
	v_3	$278^2/5$	-266	$1^{31}/665$	0	$52^1/5$	-5	249	0	58	<u>-266</u>	4
	x_4	$4^3/5$	-4	$1^3/20$	$55/133$	$4/5$	0	4	0	1	-4	0
	v_1	$-51^{14}/25$	$52^1/5$		$3^{244}/3325$	$-10^{12}/25$	1	$-49^3/5$	$-1/5$	$-11^2/5$	$52^1/5$	$-4/5$
	x_2	-5	5		$31/133$	-1	0	-5	0	-1	5	0
	u_3	$-247^1/5$	249		$13^{271}/665$	$-49^3/5$	5	-241	-2	-54	249	-4

so that u_3 leaves the basis. Hence Tableau 1 is transformed into Tableau 2 with the underlined element -1 as a pivot. Tableau 2 is in standard form; furthermore all basic variables are nonnegative for $\lambda = 0$, so that the solution is optimal for $\lambda = 0$. Note that the column λ -term and the y_3 -column are identical, as they have to be, since they were in Tableau 0 identical unit vectors.

It is at this point that the parametric procedure is started. We are now looking for a value of λ which makes a basic variable⁶ zero. If basic variables have nonnegative entries in the λ -term column, their values do not decrease with increasing λ , so that their values cannot become zero or negative when λ is increased. Hence the value of λ at which one of the values of the basic variables becomes zero, the critical value of λ , is determined by

$$(4.8) \quad \text{Min}_i \left(\frac{r_i}{-s_i} \mid s_i < 0 \right).$$

Note that not both r_i and s_i can be negative, because the value of the basic variable would then be negative. The values $r_i/-s_i$ for negative s_i are written in the column Critical Value λ . We shall call the basic variable connected with the critical value of λ the critical variable.

In Tableau 2 we find that the critical value of λ is $1/7$ in the row of u_4 . Substituting $\lambda = 1/7$ into the values of basic variables, we find the numbers for these values as given in the column $\lambda = 1/7$ in Tableau 2. Note that this solution is a degenerate one because u_4 is zero. If λ is slightly larger than $1/7$ u_4 is negative. Hence, in order to keep the solution nonnegative (and therefore optimal) for $\lambda \geq 1/7$, we introduce according to the rule of the Simplex method for quadratic programming x_4 into the basis. In deciding

⁶ Or perhaps more variables in some cases. We discuss these cases in Section 8.

which variable has to leave the basis we evaluate the values of basic variables at $\lambda = 1/7$. Hence we must compare the ratios

$$(4.9) \quad \frac{0}{-22}, \frac{3}{5},$$

so that u_4 has to leave the basis.

Note that the variable which becomes zero will always leave the basis unless the element in the row of this variable and in the column of its corresponding variable is zero; this element is nonpositive (see Table 2). If this element is nonzero, the next tableau will be in standard form. Hence Tableau 2 is transformed into Tableau 3 with the underlined element -22 as a pivot. For $\lambda = 1/7$ the solution of Tableau 3 is the same as that of Tableau 2, but the solution of Tableau 3 is also valid for higher values of λ . Whereas the solution of Tableau 2 was nonnegative for the $0 \leq \lambda \leq 1/7$, we find that the solution of Tableau 3 is nonnegative for the range $1/7 \leq \lambda \leq 3/10$ where $3/10$ is the critical value of λ for Tableau 3. For $\lambda = 3/10$, y_1 becomes zero.

Here the rules of the dual method must be applied, which means that the dual variable of y_1 , v_1 , must be introduced into the basis. Since the element in the row of y_1 and the column of v_1 is $-4 \frac{6}{11}$ and therefore nonzero, v_1 replaces y_1 in the basis, and Tableau 3 is transformed into Tableau 4 with $-4 \frac{6}{11}$ as a pivot. The critical value of λ for Tableau 4 turns out to be $4/7$, so that its solution is nonnegative for $3/10 \leq \lambda \leq 4/7$. At $\lambda = 4/7$, u_1 becomes zero, so that x_1 must enter the basis according to the Simplex method rule. The element in the row of u_1 and the column of x_1 is nonzero, so that it can serve as a pivot and Tableau 5, another tableau in standard form is generated, which turns out to give a nonnegative solution for $4/7 \leq \lambda \leq 33/35$.

For $\lambda = 33/35$ x_3 becomes zero, so that u_3 is introduced into the basis. The element in the row of x_3 and the column of u_3 is nonzero, so that Tableau 5 is transformed into Tableau 6 with $-1/26$ as a pivot.

For Tableau 6 we find a critical value of λ of 1, at which value y_2 becomes zero. Hence we introduce (according to the rules of the dual method for quadratic programming) v_2 , but we find that the element in the row of y_2 and the column of v_2 is zero. We must therefore compare the ratios of the elements in the columns of values of basic variables for $\lambda = 1$, and the elements in the column of v_2 in the rows of dual basic variables.

$$(4.10) \quad \frac{1/5}{1}, \frac{13 \frac{2}{5}}{5}.$$

As a result it is decided that u_2 leaves the basis, so that the pivot element is the underlined element -1 . Tableau 6 is then in nonstandard form. Because we are in nonstandard form and are applying the dual method, we must introduce the primal variable of the nonbasic pair into the basis. This is x_2 . According to the properties of tableaux in nonstandard form, the element in the column of the primal variable of the nonbasic pair and in the row of the primal variable of the basic pair must be negative. This is the underlined element -1 . Since the value of the primal variable of the basic pair is zero for $\lambda = 1$, this variable leaves the basis and we pivot on -1 .

The next tableau is in standard form again. The critical value is now $1 \frac{31}{665}$ and is connected with v_3 . Hence y_3 is introduced into the basis and since the element in the row of v_3 and the column of y_3 is negative, it can serve as a pivot. In the next solution y_3 is basic, which means that any increase of λ does not affect the solution any more. This can also be concluded

from the λ -term of the values of basic variables; the λ -term column is in this case just a unit vector with the unit element in the y_3 row, so that any increase in λ only increases y_3 and leaves the other basic variables unaltered. It is not necessary to write down the next tableau since the solution of Tableau 8 for $\lambda = 1^{31}/665$ does not change because the value of v_3 in this tableau was 0 for $\lambda = 1^{31}/665$. Since $1^{31}/665$ is less than $1^2/3$, the upper limit of λ , the problem is solved.

5. RULES FOR THE CAPACITY METHOD IN SIMPLICIAL FORM.

It will now be clear that the rules for the Simplicial version of the capacity method are very simple ones indeed. We shall now state these rules explicitly, assuming that the constraints of the problem have the form (2.2), (2.3), and that the elements of b are nonnegative. In Section 8 the capacity method will be generalized for cases which do not satisfy these requirements.

First, the set-up tableau for the problem including the capacity constraint is constructed with the values of basic variables split into two columns, which give the independent term and the λ -term of these values; λ is initially taken as 0. The result is given in Table 4. Its last two rows are discussed in Section 6. The slack variable of the capacity constraint, which is basic in the set-up tableau, will be denoted by y_c , and its corresponding variable by v_c . The procedure is then given by the following three steps, of which the first one occurs at the start only.

STEP 0. Transform the tableau with the element in the row of y_c and the column of the corresponding variable of the largest negative dual variable as a pivot; transform then the tableau with the element in the row of the largest negative dual variable and the column of v_c as a pivot.

After this step we will have a standard tableau with a nonnegative solution for $\lambda = 0$; hence we have an optimal solution for $\lambda = 0$. We can then start the proper part of the capacity method which is connected with varying λ . Let us call the variable which is connected with a critical value of λ in the sense that it becomes zero for that critical value, the critical variable. The remainder of the procedure consists then of the following two steps.

TABLE 4. SET-UP TABLEAU FOR THE CAPACITY METHOD IN SIMPLICIAL FORM

Basic Var.	Values B.V.		u	v	v _c	x	y	y _c
	Ind.t.	λ -term						
u	-p	0	1	-A'	-i	-C	0	0
y	b	0	0	0	0	A	I	0
y _c	0	1	0	0	0	i'	0	1
f _i	0	0	0	-b	0	-p	0	0
f _{λ}	0	0	0	0	-1	0	0	0

STEP 1. Determine

$$(5.1) \quad \lambda_c = \text{Min}_i \left(\frac{r_i}{-s_i} \mid s_i < 0 \right);$$

if the element in the row of the critical variable and in the column of its corresponding variable is negative, transform the tableau with this element as a pivot and repeat Step 1; if this element is zero, apply Step 2.

STEP 2. Determine

$$(5.2) \quad \text{Min}_i \left(\frac{r_i + s_i \lambda_c}{t_i} \mid t_i < 0 \right),$$

where t_i stands for the element in the i^{th} row in the column of the corresponding variable of the critical variable. Transform the tableau with the element in the row connected with (5.2) and the column of the corresponding variable of the critical variable as a pivot. Transform the tableau with the element in the row of the critical variable and the column of the corresponding variable of the variable connected with (5.2). Return to Step 1.

TERMINATION. The procedure terminates when no critical value λ_c can be found any more. If y_c is then in the basis, the solution to the problem has been found, if y_c is not in the basis, the problem has an infinite solution.

Step 0 must lead to a nonnegative solution for $\lambda = 0$. This can be proved as follows. The first transformation does not lead to any change in the independent-term column of values of basic variables, and in the column of v_c . In the second transformation the values of the y -variables remain equal to the elements of b , but the u -variables change as follows. Let the u -variable with the largest negative value be u_m ; this variable is replaced in the basis by v_c ; the value of v_c is then $-p_m / -1$. For the other variables we have

$$(5.3) \quad u_i = -p_i - (-1) \frac{-p_m}{-1} = -p_i - (-p_m)$$

which must be nonnegative since $-p_m \leq -p_i$.

After the optimum solution for $\lambda = 0$ is obtained, λ is increased and the optimum solutions corresponding to increasing ranges of λ are traced. Step 1 starts by taking the ratios of the elements of the independent term and the λ -term of the values of basic variables. The critical value of λ is then given by (5.1); for $\lambda = \lambda_c$ the critical variable becomes zero. According

to the Simplex or the dual method for quadratic programming we should introduce into the basis the corresponding variable of the critical variable, and take ratios for either the primal or the dual variables (according to whether the critical variable is a primal or a dual one), and for the row of the critical variable, taking $\lambda = \lambda_c$. Since for $\lambda = \lambda_c$, the critical variable is zero, the ratio in the critical variable is always the smallest, provided the denominator is negative (it is always nonpositive according to the properties of a general tableau in standard form). Hence the critical variable always leaves the basis if the element in its row and in the column of its corresponding variable is negative. The next tableau is then again in standard form and we can repeat Step 1.

It may occur that the minimum in (5.1) is not unique. In this case two or more basic variables become zero for the same critical value of λ . A simple rule is then to select any of the variables connected with the minimum as the critical variable, which, however, theoretically may result in cycling. For a rule which avoids cycling, see Section 8.

Since for $\lambda = \lambda_c$ the value of the basic variable in the pivot row is zero, the solution must be the same for that value of λ in the tableau before the transformation and the one after the transformation. For other values of λ these two solutions are of course different. In the old tableau the critical variable was nonnegative for $\lambda \leq \lambda_c$, whereas in the new tableau its corresponding variable is nonnegative for $\lambda \geq \lambda_c$, because its value is

$$(5.4) \quad \frac{r_c}{a_c} + \frac{s_c}{a_c} \lambda,$$

where a_c denotes the negative pivot.

It can now be shown that the next critical value of λ is higher than or equal to λ_c . For the value of basic variables other than in row c we have

$$(5.5) \quad \left(r_i - \frac{a_i}{a_c} r_c \right) + \left(s_i - \frac{a_i}{a_c} s_c \right) \lambda,$$

where a_i denotes the element in the old tableau in row i and in the column of the new basic variable. We have then to show that

$$(5.6) \quad \frac{r_i - \frac{a_i}{a_c} r_c}{s_i - \frac{a_i}{a_c} s_c} \geq \frac{r_c}{-s_c} \quad \text{for} \quad s_i - \frac{a_i}{a_c} s_c < 0.$$

r_c and $-s_c$ are nonnegative and positive, respectively. Suppose $r_c = 0$, then $\lambda_c = 0$; r_i must then be nonnegative, since it stands for the value of a basic variable in an optimal solution for $\lambda = 0$. Hence (5.6) is valid in this case. If r_c is positive, (5.6) can be written as

$$(5.7) \quad \frac{r_i}{r_c} \geq \frac{s_i}{s_c}.$$

For $s_i < 0$ this can be reduced to

$$(5.8) \quad \frac{r_i}{-s_i} \geq \frac{r_c}{-s_c} = \lambda_c,$$

which must hold since it involves the definition of the critical value of λ .

In the case $s_i \geq 0$ and $r_i \geq 0$, (5.7) must obviously be valid.

In the case $s_i \geq 0$ and $r_i < 0$, (5.7) follows from

$$(5.9) \quad r_i + s_i \lambda_c \geq 0.$$

The situation is more complicated when the element in the row of the critical variable and the column of its corresponding variable is zero. This is the nonstandard case of what Houthakker calls degeneracy, and what we shall call the case of a nonstandard iteration or a nonstandard case; if the element is nonzero, we speak of a standard case or standard iteration. In order to deal with this case, let us consider Table 5, which gives the relevant parts of a general tableau in standard form. The notation of the elements in the body of the tableau is in accordance with the one used in Table 2. The values of basic dual variables are denoted by q 's and those of basic primal variables by r 's, the λ -term elements being distinguished by a bar. Dual variables are denoted as u -variables and primal ones as x -variables. We assume that the critical variable is an x -variable; if it is a u -variable, the rôles of primal and dual variables are interchanged. Since x_c is the critical, we must, in accordance with the dual method, introduce u_c into the basis, but now the element $-r_{cc}$ is found to be zero. According to the properties of a general tableau in standard form,

TABLE 5. TABLEAU FOR A NONSTANDARD ITERATION

B.V.	Val.	B.V.	x_b	u_c
u_i	q_i	q_i	$-q_{ib}$	$-p_{ci}$
u_b	q_b	q_b	$-q_{bb}$	$-p_{cb}$
x_c	r_c	r_c	p_{cb}	$-r_{cc}$
x_i	r_i	r_i	p_{ib}	$-r_{ic}$

the elements $-r_{ic}$ for $i \neq c$ should also be zero. In order to find the variable to leave the basis, we have to compare the ratios of the values of basic variables, taken at $\lambda = \lambda_c$, and the corresponding elements in the column of u_c . Since now $-r_{cc}$ and hence all $-r_{ic}$ are assumed to be zero, a positive ratio, if it exists, must be found in the rows of the dual variables; let us assume that the smallest positive ratio occurs in the row of u_b . u_b leaves the basis and the tableau is transformed with the element $-p_{cb}$ as a pivot. Note that in this transformation the rows of the primal variables remain entirely unchanged. Note further that these must always be at least one positive element $-p_{ci}$ since otherwise there would only be nonnegative elements p_{ci} in the row of x_c , which contradicts the fact that x_c may become negative. For details, see [7], Section 4.

According to the rules of the dual method for quadratic programming, the corresponding variable of the variable which just left the basis is introduced into the basis. Since p_{cb} is known to be negative ($-p_{cb}$ was found to be positive) and $x_c = 0$ for $\lambda = \lambda_c$, x_c must leave the basis, so that p_{cb} is the pivot of the transformation. For x_b we have then

$$(5.10) \quad x_b = \frac{r_c}{p_{cb}} + \frac{\bar{r}_c}{p_{cb}} \lambda$$

which is nonnegative for $\lambda \geq \frac{r_c}{-r_c}$,

so that we have found a solution for a new range of λ . It can be proved that the critical value of λ for this tableau is larger than or equal to the previous initial value. Since we are in standard form again, we can repeat Step 1.

Summarizing the procedure, we see that finding optimal solutions for increasing values of λ can result in two cases. In the standard case we find immediately a pivot in the critical row. The next tableau is then again in standard form. In the nonstandard case we use first a pivot in another row than that of the critical variable; the transformation results then in a tableau in nonstandard form. In the next transformation a pivot is used in the row of the critical variable, and we are back in standard form. The standard case requires only one transformation, the nonstandard case requires two. Note that in the standard case the value of both primal and dual variables are the same for the two successive standard tableaux for the critical value of λ . In the nonstandard case this is generally not true. If the critical variable is a primal variable, the value of the primal variables in the three successive tableaux are the same for λ equal to its critical value, but the values of the dual variables will in general differ for the first standard tableau and the nonstandard tableau (compare Tableaux 6 and 7 of Table 3). The values of all basic variables are again equal for the critical value of λ in the nonstandard tableau and the second standard tableau. If the critical variable would have been a dual one, the dual variables would have remained the same for the critical value of λ , but the primal variables would differ in the first standard tableau and the nonstandard tableau.

The situation is the same as in linear programming, when one of the basic variables is zero; in this case there are two sets of basic variables giving the same optimal solutions with different shadow prices. The case when a dual variable is the critical variable in a nonstandard case is equivalent to the case in linear programming when one of the nonbasic variables has a zero element in the last row.

of the optimal tableau, so that there are multiple solutions. The difference is that in linear programming the dual variables are usually given in the last row, whereas in the Simplicial formulation of quadratic programming they are considered as regular basic variables. In Section 8 we shall see that in the case of a linear programming problem, which is essentially a special case of the quadratic programming with $C = 0$, only nonstandard cases occur.

The procedure terminates when no new critical value of λ can be found. If y_c is then in the basis, increasing λ will only result in increasing y_c ; all elements in the λ -column will be zero apart from a unit in the row of y_c , as it was in the set-up tableau. v_c will then be nonbasic, having a value zero, which indicates that an increase in λ no longer increases the objective function. It is clear that in this case the capacity constraint is redundant. If no critical value for λ is found and y_c is not in the basis, the capacity constraint remains effective for all value of λ . Note that in this case the element in the row of v_c in the λ -term column must be zero; it cannot be negative because then v_c would be the critical variable, and it cannot be positive because it is equal to the element in the same row in the column of y_c which is a diagonal element of a negative semi-definite matrix. Since in this case the values of the primal basic variables increase with λ , the problem must have an infinite solution.

6. THE VALUE OF THE OBJECTIVE FUNCTION

v_c can be interpreted as the partial derivative of the constrained objective function with respect to λ . This follows from the definition of v_c

as minus the partial derivative with respect to y_c ,⁷ and from the capacity constraint in which both y_c and λ appear. This can be demonstrated more clearly by introducing explicitly into the tableaux the value of the objective function, as was indicated in Section 2, see the last row of Table 1. The difficulty in this case is then that, since we added the capacity constraint to the problem, the row-vector b' depends on λ . The remedy consists in separating the parts of the last row independent of λ and dependent on λ .

The result is then as given in the last two rows of Table 4; the basic variable in the first of these two rows is indicated as f_1 , and in the second as f_λ ; the corresponding unit vectors are deleted. Since the values of basic variables consist already of an independent term and a λ -term, the value of $f(x)$ is given by 4 terms, which can be arranged in a 2×2 matrix as follows:

$$(6.1) \quad \begin{bmatrix} f_{11} & f_{1\lambda} \\ f_{\lambda 1} & f_{\lambda\lambda} \end{bmatrix}$$

in Table 4 all these elements are zero. Since each element in the λ -term column should be multiplied by λ and since also each element in the row of f_λ should be multiplied by λ , we find for twice the value of the objective function for a tableau in standard form⁸

$$(6.2) \quad f(\lambda) = f_{\lambda\lambda}\lambda^2 + (f_{1\lambda} + f_{\lambda 1})\lambda + f_{11};$$

⁷ See [6] or [7].

⁸ For another application of the same device, see [5].

for a standard tableau we have $f_{i\lambda} = f_{\lambda i}$

From (6.2) we can derive immediately

$$(6.3) \quad \frac{1}{2} \frac{df(\lambda)}{d\lambda} = f_{\lambda i} + f_{\lambda\lambda} \lambda,$$

which gives an explicit expression for the derivation of the objective function with respect to λ .

(6.3) should be equal to v_c , which is true for the set-up tableau since v_c is there nonbasic. It is also true for any other tableau generated in the procedure. This can be shown as follows. In the second transformation of Step 0 v_c is taken into the basis by pivoting on the element in its column and in the row of the largest negative u-variable. Since the element in the last row of the tableau and in the column of v_c is -1 , the last row becomes identical (apart from the elements in the column of v_c) to the row in which v_c is becoming a basic variable, and remains identical as long as v_c remains in the basis.

Table 6 gives the two rows of the objective function for the first four tableaux for Houthakker's example. Note that the symmetry properties of tableaux in standard form are also valid for these two rows and the two columns of the values of basic variables. Because of this, these rows contain no information which was not given in Table 3 except for the value of f_{ii} ; for the following tableau we give therefore only the values of f_{ii} .

Figure 1 and 2 give for the solutions of the capacity method for Houthakker's example the value of the objective function and v_c , its derivative with respect to λ , as functions of λ . Figure 1 consists of a

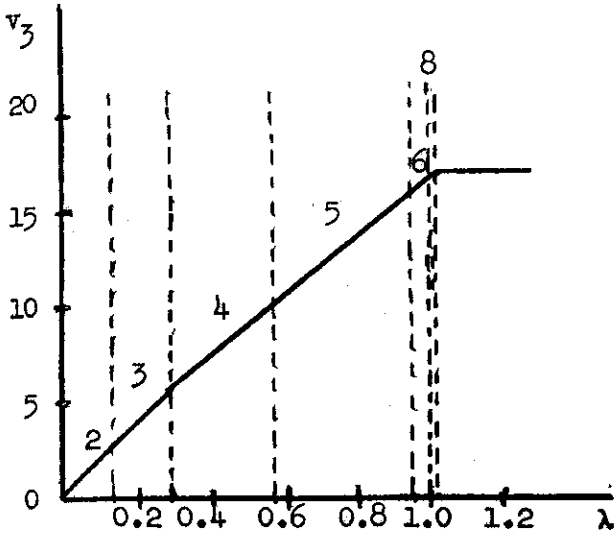


Figure 1.

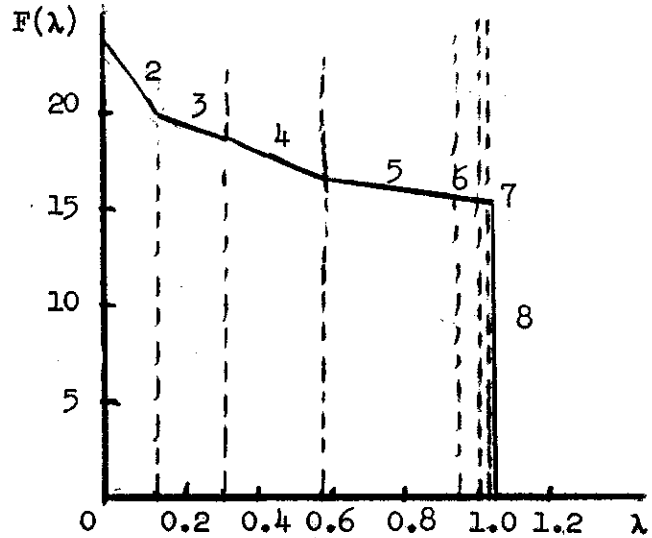


Figure 2.

series of concave parabola. The objective function increases, but at a decreasing rate until it does not increase any more at $\lambda = 1^{31/665}$. v_3 is a decreasing function of λ . Note that this function is discontinuous at $\lambda = 1$, where the nonstandard iteration occurred.

It can be proved that v_c decreases or does not increase in successive iterations. In standard iterations this is obvious, since in successive solutions the value of all basic variables remains the same for the critical value of λ , while it must decrease or stay the same for increasing λ in a particular solution, since the λ -term of the value of v_c is nonpositive. For a nonstandard iteration this is somewhat more difficult to prove. Let us look at the example treated in Table 3. v_3 can only increase in Tableau 6 if the element in the row of v_3 and the column of v_2 (which is 5) is negative. However, the element

must be positive, since according to the properties of symmetry and skew-symmetry of a standard tableau (see Table 2) it is equal to minus the element in the row of y_2 and the column of y_3 , which is in its turn equal to the λ -term of the value of y_2 ; this element is clearly negative because the critical value of λ is found in the row of y_2 .

7. COMPARISON WITH HOUTHAKKER'S CAPACITY METHOD

In this section it will be shown that the Simplicial version of the capacity method and Houthakker's own presentation are equivalent.

Key concepts in Houthakker's presentation are the effective set of variables and the effective set of constraints. The effective set of variables are the x -variables which are allowed to be nonzero; the other x -variables are kept at a zero level. The effective set of constraints contains the inequality constraints which are required to be satisfied as equalities. Houthakker obtains then a solution in terms of the variable parameter λ (in his notation β) which is optimal for a certain range of λ ; this solution is obtained by maximization of the objective function with respect to the set of effective variables, subject to the set of effective constraints.

In Houthakker's presentation the solution of this classical maximization problem is found in the usual manner. Suppose that the set of effective variables are the first n_1 ones and the set of effective constraints are the first m_1 ones. The vectors and matrices of the quadratic programming problem can then be partitioned as

$$(7.1) \quad x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad p = \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}, \quad b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

TABLE 6

ROWS OF THE SIMPLEX TABLEAU FOR THE OBJECTIVE FUNCTION

Tabl.	B. V.	Ind. t.	$\lambda-t$	v_1	v_2	v_3	x_1	x_2	x_3	x_4
0	f_i	0	0	-2	-3	0	-18	-16	-22	-20
	f_λ	0	0	0	0	-1	0	0	0	0
				v_1	v_2	v_3	x_1	x_2	y_3	x_4
1	f_i	0	22	-2	-3	0	4	6	22	2
	f_λ	0	0	0	0	-1	0	0	0	0
				v_1	v_2	u_3	x_1	x_2	y_3	x_4
2	f_i	0	22	-2	-3	0	4	6	22	2
	f_λ	22	-17	10	0	-1	-9	-16	-17	-14
				v_1	v_2	u_3	x_1	x_2	y_3	u_4
3	f_i	$2/11$	$20^8/11$	$-1^1/11$	$-3^5/11$	$-1/11$	$3^5/11$	$4^5/11$	$20^8/11$	$1/11$
	f_λ	$20^8/11$	$-8^1/11$	$3^7/11$	$3^2/11$	$-4/11$	$-5^2/11$	$-5^2/11$	$-8^1/11$	$-7/11$
Tableau		4	5	6	8					
f_{11}		$-2/25$	$2^{124}/325$	$-4^8/25$	$-257^8/25$					

The problem is then to maximize with respect to x^1

$$(7.2) \quad \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}' \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}' \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

subject to

$$(7.3) \quad \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = b^1 .$$

From this we form the Lagrangean expression

$$(7.4) \quad \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}' \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}' \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} - v^1' \left(\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} - b^1 \right) .$$

Taking the partial derivative of this expression with respect to the variables in x^1 and putting these equal to zero, we find, substituting $x^2 = 0$,

$$(7.5) \quad p^1 - c_{11}x^1 - A_{11}'v^1 = 0 .$$

From (7.3) we have, substituting $x^2 = 0$,

$$(7.6) \quad A_{11}x^1 = b^1 .$$

(7.5) and (7.6) can be solved for x^1 and v^1 . Since the capacity constraint is always effective,⁹ b^1 is a linear function of λ and x^1 and v^1 will therefore also be linear functions of λ . They must be nonnegative for the range of λ under consideration if the solution is to be optimal. From x^1 and v^1 we can derive upper limits for which the solution is still optimal.

⁹ As a matter of definition, Houthakker does not include the capacity constraint in the set of effective constraints, though it is always effective.

Another requirement for an optimal solution of the quadratic programming problem is that the variables in x^2 do not increase the objective function when given small positive values. Hence we should have that the vector of partial derivatives of (7.4) with respect to the variables in x^2 is non-positive:

$$(7.7) \quad p^2 - c_{21}x^1 - A'_{12}v^1 \leq 0.$$

Since x^1 and v^1 are both linear functions of λ , the left-hand side is a linear function of λ . For the range of λ under consideration (7.7) should be valid and hence we can determine upper limits for λ for which (7.7) is still valid.¹⁰

Finally it is required that the solution obtained satisfies the constraints not in the set of effective constraints, so that we must have

$$(7.8) \quad A_{21}x^1 \leq b^2.$$

Since x^1 is a linear function of λ , upper bounds for λ can be found for which (7.8) is still satisfied.

Houthakker solves for each iteration separately the four sets of relations (7.5) - (7.8) and derives from these the upper limits for λ . The Simplicial version of the capacity method does the same, but in a much more efficient way. The solutions to (7.5) and (7.8) can be obtained by partitioning the vectors and matrices in the set-up tableau as indicated in (7.1); in this case u^2

¹⁰ Houthakker presented this slightly different. He determined values of λ by solving each equation of (7.7) as an equation for λ . In this case upper as well as lower limits for λ are obtained after which the lower limits are discarded. In the case when lower and upper limits coincide, it is then not possible to distinguish between lower and upper limits, which gave Boot [1] occasion to claim that the capacity method breaks down in this case.

stands for minus the left-hand side in (7.7) and y^2 for the vector of slack variables in (7.8). The solution for x^1 , v^1 , u^2 and y^2 is then obtained by having all these variables in the basis; since they are linear functions of λ , they can be divided into an independent term and a λ -term, and the upper bounds of λ can be obtained as indicated in the rules. Hence each maximization problem that Houthakker considers is the solution of a standard tableau.

We shall now compare step by step Houthakker's capacity method with the Simplicial formulation of this method; for Houthakker's method we will refer to the steps given in the Appendix of [4].

Step I is the initial step in which the x -variable connected with the largest positive element of p is taken into the set of effective variables. The capacity constraint is the only effective constraint. It is obvious that the Simplicial version is equivalent in this respect. In Step II and partly in Step III, (7.5) - (7.8) are solved. In Steps III a, b, c, d and e Houthakker computes upper bounds for λ for v_c , the dual variable of the capacity constraint, for the x -variables, for the v -variables, for the u -variables, and for the y -variables, respectively. In Step IV the smallest upper limit for λ , that is the critical value of λ , for these five cases are determined. The equivalent operation in the Simplicial version is given by (5.1).

If the critical value of λ is connected with v_c , the problem is solved in both versions; the solution to the quadratic programming problem is found by substitution of the critical value of λ in the solution. If the critical value of λ is connected with an x -variable, v -variable, u -variable or y -variable, and there is no degeneracy in Houthakker's terminology, then Houthakker respectively deletes the connected x -variable from the set of

effective variables, deletes the connected constraint from the set of effective constraints, adds the connected variable to the set of effective variables or adds the connected constraint to the set of effective constraints. It is obvious that the same occurs in the Simplicial version if the element in the row of the critical variable and the column of its corresponding variable is nonzero.

The situation when this element is zero corresponds with Houthakker's degeneracy case. Since Houthakker deals only with the case of a strictly concave objective function, his maximization problem must always have solutions unless the number of effective constraints (including the capacity constraint) is greater than the number of effective variables. A situation of no solution can occur when an x -variable has to be deleted or a constraint added, that is in the case an x -variable or a y -variable is connected with the critical value of λ . Houthakker detects the no-solution case by counting constraints and variables; in the Simplicial version the element in the critical row and the column of the corresponding variable is found to be zero.

Houthakker then proposes the following. Take the same effective variables and constraints as in the no solution case, but add one x -variable or delete a constraint. Consider all resulting solutions, selecting from these all which are nonnegative for λ larger than the critical value and choosing among those the solution which has the highest value of v_c , the dual variable of the capacity constraint for λ equal to its critical value. We shall show that the Simplicial version leads to the same result, except in one particular case which Houthakker partly overlooked.

Let us consider Houthakker's example and look at Tableau 6 of Table 3, where y_2 is the critical variable; the element in the row of y_2 and the

column of v_2 is found to be zero; in Houthakker's terms we have a case of degeneracy. The solutions we have to consider according to Houthakker have now two basic variables different from those in Tableau 6; y_2 must be replaced in the basis by another x - or y -variable and v_2 must come into the basis replacing the corresponding variable of the x - or y -variable. This can be done by first introducing v_2 into the basis in the row of the dual variable of the new primal variable and after this introducing the primal variable into the basis replacing y_2 . This last iteration will not change the values of basic variables for λ equal to its critical value, since y_2 is then zero. Hence the first iteration, when v_2 is introduced into the basis, determines the values of the basic variables and hence of v_3 . Now according to Houthakker we must select a dual variable to leave the basis in such a way that v_3 is decreased least. Since the element in the row of v_3 and the column of v_2 has to be positive, this is accomplished by selecting the dual variable (except v_3) which has the minimum positive ratio of its value for $\lambda = 1$ and the corresponding element in v_2 . This is entirely the same as in the Simplicial version except for the fact that the Simplicial version also allows v_3 to leave the basis. Houthakker did not treat this case, except when no positive ratios can be found. This case could occur easily if the ratio in the row of v_2 , $\frac{13^2/5}{5}$ was smaller than in the row of u_2 , $\frac{1/5}{1}$. This would be so for $p_2 > -13 \frac{13}{25}$.

We have seen that Houthakker's capacity method and the Simplicial formulation of this method are equivalent apart from some very minor points. The Simplicial formulation has clearly the advantages of a much simpler organization and a far greater computational efficiency. Furthermore, the Simplicial version can easily deal with the case when the matrix of the quadratic form is semi-definite.¹¹

¹¹ The same extension could be made for Houthakker's presentation, but this would increase the complexity of an already rather complex method.

8. DEGENERACY, SPECIAL CASES AND GENERALIZATIONS

The word degeneracy can in the capacity method refer to various situations. Houthakker uses degeneracy for the situation which we denote as a nonstandard case. It can also refer to the fact that one of the basic variables is zero for λ equal to the critical value; the situation is very similar to that in linear programming, but it is quite typical for a parametric programming method and it raises no difficulties.

A potentially more dangerous situation occurs when the same critical value for λ is found in more than one row. If we then choose any of the rows as the critical row, we may in the next iteration find the same critical value of λ . Hence the range of λ for which the solution is valid has not increased; it is then in principle possible that after a number of iterations with the same critical value of λ the same solution reappears.

This is a case which is very similar to cycling in linear programming, and it can be treated in the same way. We can either choose perturbation methods, or concept of lexicographic ordering.¹² Both devices amount to replacing for first rows the elements in the independent term column by elements of other columns until a unique minimum ratio is found. We shall not go into details here.

Another kind of degeneracy can be found when ties occur in a nonstandard iteration in (5.2). This kind of degeneracy is harmless, however, since the increasing ranges of λ are determined by the choice of the critical variable.

Since the capacity method works for a positive definite matrix C as well,

¹² See e.g. [3].

as for a positive semi-definite one, it is interesting to see how the algorithm works when $C = 0$, that is in the linear programming case. A first observation which can be made is that the element in the row of the critical variable and in the column of its corresponding variable will always be zero, as will all elements in rows of dual variables and in columns of primal variables and in rows of primal variables and in columns of dual variables; see for example the set-up tableau in Table 1. From this it follows that the algorithm will only have nonstandard iterations. Furthermore it is not necessary to use the whole quadratic Simplex tableau; only the part containing elements in rows of primal basic variables and in columns of primal nonbasic variables are necessary. In this case only one transformation is necessary for each nonstandard iteration.

The algorithm turns then out to be very similar to the dual method for linear programming. The variable to leave the basis is determined by the critical variable, and the variable to enter into the basis is determined in the manner of the dual method for linear programming, that is by comparing ratios of elements in the last row of the tableau and those in the row of the leaving basic variable. For reasons of space we shall not go into details, but give a small example of an application instead.

Let us consider the following problem. Maximize

$$(8.1) \quad 3x_1 + 4x_2$$

subject to

$$(8.2) \quad -x_1 + 2x_2 \leq 2,$$

$$(8.3) \quad x_1 - x_2 \leq 1,$$

$$(8.4) \quad x_1, x_2 \geq 0.$$

We add to this the capacity constraint

$$(8.5) \quad x_1 + x_2 \leq \lambda .$$

The initial tableau is given by Tableau 0 of Table 7. y_1 , y_2 and y_3 are the slack variables of (8.2), (8.3) and (8.5); f stands for the value of the objective function. The algorithm is started by making the variable having the largest negative entry in the last row basic; in this case this is x_2 . y_3 leaves the basis and the underlined element -1 is the pivot of the iteration. In the resulting tableau all elements in the last row in columns of basic variables have to be nonnegative. Now the proper part of the procedure can start. For the critical value of λ we find 1 and y_1 is the critical variable which has to leave the basis. The variable to leave the basis is determined by comparing ratios

$$-\frac{1}{-3}, \quad -\frac{4}{-2},$$

of which the smallest is the first so that x_1 enters the basis and -3 is the pivot of the transformation. In Tableau 2 we find for the critical value of λ

$$-\frac{2^{1/3}}{-1/3} = 7$$

so that y_2 leaves the basis. There is now only one variable which can enter the basis, which is y_3 , so that the problem is solved.

It is interesting to observe that the capacity method applied to the linear programming problem can be regarded as a special case of a procedure described by Dantzig, which he called "a self dual parametric algorithm." ¹³

¹³ Cf. Dantzig [2], p. 245, seq.

TABLE 7

APPLICATION OF THE CAPACITY METHOD TO A LINEAR PROGRAMMING PROBLEM

Tableau	Bas. V.	Ind.t.	λ -t.	x_1	x_2	Tableau	Bas. V.	Ind.t.	λ -t.	y_1	y_3
0	y_1	2	0	-1	2	2	x_1	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
	y_2	1	0	1	-1		y_2	$2\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
	y_3	0	1	1	<u>1</u>		x_2	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	f	0	0	-3	-4		f	$\frac{2}{3}$	$3\frac{1}{3}$	$\frac{1}{3}$	$3\frac{1}{3}$
				x_1	y_3					y_1	y_2
1	y_1	2	-2	<u>-3</u>	-2	3	x_1	4	0	1	2
	y_2	1	1	2	1		y_3	-7	1	-2	-3
	x_2	0	1	1	1		x_2	3	0	1	1
	f	0	4	1	4		f	24	0	7	10

It is a special case of this algorithm because Dantzig also allows the elements in the last row of the linear programming tableau to depend on a variable parameter. This throws an interesting light on the capacity method for quadratic programming. This method can be regarded as a kind of generalization of Dantzig's self-dual parametric algorithm for quadratic programming.

The capacity method as stated by Houthakker required a special form of the problem, namely that the constraints have the form

$$(8.6) \quad Ax \leq b ,$$

with $b \geq 0$, so that $x = 0$ is a feasible solution. However, the method can be generalized for any form of the constraints, provided that a basic feasible solution to the constraint is known.

Let the constraints have the general form

$$(8.7) \quad Ax = b$$

and let the first m x -variables be basic in the feasible solution. The vectors and matrices of the problem can then be partitioned accordingly

$$(8.8) \quad x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} , \quad p = \begin{bmatrix} p^1 \\ p^2 \end{bmatrix} , \quad A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} , \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} .$$

For (8.7) we have then

$$(8.9) \quad A_1 x^1 + A_2 x^2 = b ,$$

from which we obtain

$$(8.10) \quad x^1 = A_1^{-1} b - A_1^{-1} A_2 x^2 = b^* ;$$

the last equality involves definition of b^* . This equation may be used to substitute for x^1 in the objective function, which becomes, apart from constant terms,

$$(8.11) \quad p^* x^* - \frac{1}{2} x^* C^* x^* ;$$

the constraints can be written according to (8.10) as

$$(8.12) \quad A^* x^* \leq b^*$$

$$(8.13) \quad x^* \geq 0 .$$

where of course $x^* = x^2$; the other starred vectors and matrices can be obtained in a straightforward manner. Now the problem (8.11) - (8.13) is equivalent to the original problem, so that if we solve it we have immediately the solution to the original problem. This reformulation of the problem can be done in the framework of the quadratic Simplex tableaux. For details, see [6] and [7].

9. CONCLUDING REMARKS

In the previous sections the capacity method was formulated as a parametric variation of optimal tableaux. Another equivalent formulation of the capacity method can be made which is very similar to the dual method with starting procedure. The set-up tableau is as before, but in the first tableau we introduce v_c into the basis, replacing the largest negative dual variable. The solution obtained will have nonnegative values for both primal and dual variables, but it occurs in a nonstandard tableau. The rules to be applied in the successive iterations are

then as follows. Introduce into the basis the corresponding variable of the variable which left the basis in the last iteration; delete from the basis that variable (primal or dual) which corresponds to the minimum of the positive ratios of values of basic variables and the corresponding elements in the column of the new basic variables. In this formulation, apart from the initial and the final tableaux, only nonstandard tableaux will occur.

It is obvious that this formulation combines the Simplex and the dual method for quadratic programming. The successive solutions will be the same as in the capacity method, and the critical values of λ can be obtained from equating the minimum ratio to that in the row of y_c ; y_c will depend linearly upon λ . The procedure terminates when v_c or y_c leaves the basis; in the first case the solution of the problem has been found and in the second case the problem has an infinite solution. In some cases it is not possible to find a ratio in the row of y_c because the denominator is zero; this corresponds to a nonstandard iteration. The successive tableaux obtained by the two formulations differ because they have one different basic variable, but they can easily be derived from each other.

If the capacity method in Simplicial formulation is compared with any of the other methods for quadratic programming, there is no reason why it should be found to be less effective. Comparing it with, for example, the Simplex method for quadratic programming, we find that the organization of the methods is very similar. A slight disadvantage is that the capacity method needs one more row and one more column. On the other hand, it may well be that for problems of a certain structure the number of iterations in the capacity method will be smaller. Furthermore, in cases

when the capacity constraint forms part of the problem, the successive solutions generated by the capacity method may be of interest because they show the rôle played by the capacity constraint for the various values of λ .

Though the organization of the capacity method in its original formulation seemed very complex indeed, it turns out to be very simple and straightforward in its Simplicial formulation. As to the generality of the method, we have shown that it can be used for any convex quadratic programming problem, provided that a feasible solution to the constraints exists. Hence the capacity method should be considered among the more practicable algorithms for quadratic programming.

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