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Second Essay on the Golden Rule of Accumulation

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A few years ago we presented a theorem on maximal consumption in a golden age [6]. The same theorem was discovered and published by Allais [1], Desrousseaux [3], Mrs. Robinson [8], Swan [13] and von Weizsäcker [15].**

**Mention should also be made of an unpublished paper by Beckmann [2] in which the theorem is proved for the Cobb-Douglas case and the dissertation of Srinivasan [12] in which the existence of a state of maximum per capita consumption with a growing labor force is shown. All these authors made the finding independently, circa 1960.

The theorem established may be expressed as follows:

> If there exists a golden-age growth path*** on which the social net rate of return to saving equals the rate of growth (hence, in one class of models, the fraction of output saved equals the capital elasticity of output) -- or, in market terms, a golden-age path on which the competitive interest rate equals the growth rate and hence gross investment equals the gross competitive earnings of capital -- then this golden age produces a path of consumption which is uniformly higher than the consumption path associated with any other golden age.

***By a golden-age path we mean a growth path in which literally every variable changes (if at all) at a constant relative rate. It follows immediately that if investment is positive then output, investment and consumption must all grow at the same (constant) rate. Various other properties can be derived.

The consumption-maximizing golden age will be referred to in this paper,

*This paper owes a great debt to Tjalling C. Koopmans who contributed the basis for the theorems established in the second part of this paper. Robert M. Solow commented on an earlier draft. They are not responsible for any errors in the final product.

as in [6], as the Golden Rule or GR path.

The papers cited raise two sorts of questions. The first concerns the conditions for the existence of the GR path. Some of the papers (including my own) erroneously suggest that the GR path can exist only in "neoclassical" models, i.e., models in which capital and labor are continuously substitutable. Some of the papers leave the false impression that the GR path exists only if there is no technical progress while my own paper errs at the opposite extreme in suggesting that any type of technical progress, if it occurs at a constant relative rate, permits a GR path. The first part of this paper examines in two kinds of models the conditions for the existence of the GR path. We show that the GR path may exist in the un-neoclassical Harrod-Domar model as well as in the neoclassical model. And we prove that a positive-investment golden-age path can exist, and hence a GR path, only if technical progress can be described as purely labor-augmenting.

Question also arises as to the normative significance of the theorem. We called the saving rule prevailing on the consumption-maximizing golden-age path the Golden Rule of Accumulation because, on that path, each "generation" saves (on behalf of future generations as it were) that fraction of income which it would have past generations save, subject to the constraint that all generations past and present are to save the same fraction of income. But this is perhaps not much more than word-play: at any rate, no proof of the optimality of the GR path was given nor was any suggestion of its optimality seriously intended. Society need not confine itself to golden-age paths (should they exist) nor aim to achieve golden-age growth asymptotically. And even if some golden-age path should be utility-maximizing (at least for some initial conditions) the rate of time preference may make that path different from the GR path. It was evidently reflections such as these which led Pearce [5] and Samuelson [9] to doubt whether the GR path has any important normative significance at all.

In the second part of this paper it will be shown, however, that whether or not the GR path is "optimal" it has the following important normative property: Any growth path on which, at some point in time and forever after, the capital-output ratio always exceeds its GR level by at least some constant amount -- equivalently, any path which eventually keeps the social net rate of return to saving (or competitive rate of interest) permanently below its GR value by at least some finite amount -- is <u>dynamically inefficient</u>, for there always exists another path which, starting from the same initial capital stock, produces more consumption at least some of the time and never less consumption. This is the proposition conjectured by the author in reply to Pearce [7]. Its proof here is based on a proof provided by Tjalling C. Koopmans. The significance of the theorem is clear: no path which is dynamically inefficient can be optimal; hence no path which transgresses the GR path in the manner described can be optimal.

Since the conditions for a GR path are stringent this theorem is only of theoretical interest. But we are able to prove an analogous theorem even when no golden-age path, and hence no GR path, exists. Thus we show that the possibility of excessive capital deepening, despite a continuously positive net rate of return to saving, is quite general.

A fuller summary of the paper and some concluding remarks close the paper.

I. EXISTENCE OF THE GOLDEN RULE PATH

Section A will study the neoclassical and the Harrod-Domar models. In this section it is postulated that technical progress can be described as solely "labor-augmenting". Section B will introduce technical progress which must be described as (at least partially) "capital augmenting".

A. Labor-Augmenting Technical Progress

In both the neoclassical and Harrod-Domar cases, output, $Q(t)$, is a continuous function of capital, $K(t)$, labor, $L(t)$, and time:

$$Q(t) = F\,(K(t),\, e^{\lambda t}\, L(t)) , \qquad \lambda \geq 0 \tag{1}$$

It is assumed here that technical progress can be described as solely labor-augmenting -- time enters only in the second (labor) argument of the function -- and that it occurs at the constant rate λ. The function is supposed to be homogeneous of degree one (constant returns to scale).

We suppose that the labor force grows exponentially at rate γ:

$$L(t) = L_o\, e^{\gamma t} \qquad \gamma > 0 . \tag{2}$$

Capital is taken to be subject to exponential decay at rate δ

so that if $I(t)$ denotes the rate of gross investment:

(3) $$I(t) = \dot{K}(t) + \delta K(t) , \qquad \delta \geq 0 .$$

Finally, consumption, $C(t)$, is the difference between output and gross investment:

(4) $$C(t) = Q(t) - I(t) , \qquad C(t) \geq 0 .$$

1. The neoclassical case

We suppose now that the production function has the following "neoclassical" properties: it is twice differentiable (smooth marginal products), it is strictly concave (diminishing marginal products), and it has everywhere-positive first derivatives (marginal products). That is,

(1a) $$\frac{\partial F}{\partial K} > 0 , \quad \frac{\partial F}{\partial L} > 0 \ ;$$

(1b) $$\frac{\partial^2 F}{\partial K^2} < 0 , \quad \frac{\partial^2 F}{\partial L^2} < 0 .$$

By virtue of constant returns to scale and (2):

(5) $$Q(t) = L_o \, e^{(\gamma+\lambda)t} \, F\left(\frac{K(t)}{L_o \, e^{(\gamma+\lambda)t}} \ , 1\right)$$

Hence, if we let $k(t)$ denote capital per unit "effective labor",

(6) $$k(t) = \frac{K(t)}{L_o \, e^{(\gamma+\lambda)t}} \ ,$$

and if we define

$$f(k(t)) = F\,(k(t),\,1)\,, \tag{7}$$

we can express the production function as

$$Q(t) = L_o\, e^{(\gamma+\lambda)t}\, f(k(t))\,,\ f'(k(t)) > 0\,,\ f''(k(t)) < 0\,. \tag{8}$$

We show now that if $k(t)$ is equal to any positive constant $k > 0$, then the economy will grow in the manner of a golden age, provided of course that the constraint $I(t) \leq Q(t)$ is satisfied.

Clearly, output will grow exponentially at rate $g = \gamma+\lambda$,

$$Q(t) = L_o\, e^{(\gamma+\lambda)t}\, f(k) = Q(0)\, e^{gt}\,, \tag{9}$$

as will the capital stock:

$$K(t) = L_o\, e^{(\gamma+\lambda)t}\, k = K(0)\, e^{gt}\,. \tag{10}$$

Hence, from (3) and the relation $\dot{K}(t) = g\,K(t)$, investment will also grow at the rate g:

$$I(t) = (g+\delta)\,K(0)\, e^{gt} = (g+\delta)\, L_o\, k_o e^{gt}\,. \tag{11}$$

Since investment and output will grow at the same rate, g, so will consumption, $C(t)$, (where $C(t) = Q(t) - I(t)$)

$$C(t) = (Q(0) - (g+\delta)\,K(0))\, e^{gt} = (f(k) - (g+\delta)\,k)\, L_o\, e^{gt}\,. \tag{12}$$

The gross investment-output ratio, s , will be constant:

$$(13) \qquad s = \frac{I(t)}{Q(t)} = \frac{(g+\delta)\ K(0)}{Q(0)} = \frac{(g+\delta)\ k}{f(k)}$$

So will the marginal productivity of capital:

$$(14) \qquad \frac{\partial Q(t)}{\partial K(t)} = \frac{\partial\ F\left(\frac{K(t)}{L_o\ e^{(\gamma+\lambda)t}}\ ,\ 1\right)}{\partial\left(\frac{K(t)}{L_o\ e^{(\gamma+\lambda)t}}\right)} = f'(k)$$

and so will the share of gross output going to capital, a , if capital receives its marginal product:

$$(15) \qquad a = \frac{\partial Q(t)}{\partial K(t)}\ \frac{K(t)}{Q(t)} = \frac{f'(k)k}{f(k)}$$

Conversely, it can be shown that every golden age path in which investment is positive implies a constant value of $k(t) > 0$ and a growth

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*In a golden age, if investment is positive, then investment, consumption and output must all grow at the same constant relative rate, denoted g . Hence $Q(t) = Q(0)\ e^{gt}$ and $I(t) = I(0)\ e^{gt}$. And capital must grow at some constant relative rate, denoted h . Hence $\dot{K}(t) = h\ K(t)$. Therefore, by (3), $I(t) = (h+\delta)\ K(t)$ which implies $h = g$. But if $K(t) = K(0)\ e^{gt}$ then, from (1) and the postulate that $\partial F/\partial L > 0$, it follows that $g = \gamma+\lambda$, hence that $k(t)$ is constant.

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rate equal to $\gamma+\lambda$.* Therefore, a golden age with positive investment occurs if and only if $k(t)$ is constant.

Thus, in every golden age with positive investment the growth rate of output, investment and consumption is $\gamma+\lambda$. These golden-age consumption paths are therefore logarithmically parallel. Associated with each golden age is a certain value of <u>s</u>, of $\frac{\partial Q}{\partial K}$, of $K(0)$ and of k. Let us assume for the moment (we drop this assumption later) that the golden age yielding the maximal consumption path, if such exists, is one in which k, and hence $K(0)$, is greater than zero. We assume, in other words, that if a maximum exists, it is an <u>interior</u> one rather than a corner maximum at $k = 0$. Then the following relationship derived from (12), must characterize that GR path:*

(16) $$\frac{\partial C(t)}{\partial K(0)} = \frac{\partial Q(0)}{\partial K(0)} - (g+\delta) = 0$$

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*Equivalently, one can differentiate (12) with respect to k to obtain the equivalent result:

(16a) $$f'(k) - (g+\delta) = 0 \ .$$

That is, on this assumption, the marginal product of capital will equal $g+\delta$ on the GR path (if it exists).** Since $\frac{\partial Q(t)}{\partial K(t)} = \frac{\partial Q(0)}{\partial K(0)}$ on any

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**A common sense explanation of (16) has been provided by Solow [11]. Imagine that capital is initially free but that we are to invest so as to maintain a golden age once the initial capital stock has been chosen. Consider a small increase of initial capital, $\Delta K(0)$. The rules of the game require that we then increase the rate of investment by $\Delta I(0) = (g+\delta)\,\Delta K(0)$ to make capital grow at rate g. The increase of initial capital will increase output by $\Delta Q(0) = \frac{\partial Q(0)}{\partial K(0)} \Delta K(0)$.

Hence consumption will increase by $\Delta C(0) = \Delta Q(0) - \Delta I(0) = \left[\frac{\partial Q(0)}{\partial K(0)} - (g+\delta)\right] \Delta K(0)$.

(Footnote continued on bottom of next page).

particular golden age path we may express (16) as

$$(17) \qquad \frac{\partial Q(t)}{\partial K(t)} - \delta = g \; .$$

The left-hand side of (17) is the social net rate of return to saving.*

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*By the (instantaneous) social net rate of return to saving at time t we mean

$$\lim_{h \to 0} \left\{ \left[\frac{\partial C(t+h)}{\partial C(t)} - 1 \right] / h \right\} .$$

For a discussion of the rate of return to saving see Solow [10].

Hence this result states that if an interior golden-age consumption maximum exists, it is where the social net rate of return to saving equals the golden-age growth rate. This is the first (and most general) way to characterize the GR path in purely technological terms.

The other technological characterization is obtained by multiplying both sides of (17) by $\frac{K(t)}{Q(t)}$ and rearranging terms:

Footnote continued from Page 8.

As long as $\frac{\partial Q(0)}{\partial K(0)} > g+\delta$ it pays to accept more capital. The consumption-maximizing golden age is reached when $K(0)$ has increased to the point where $\frac{\partial Q(0)}{\partial K(0)} - (g+\delta) = 0$.

$$\frac{\partial Q(t)}{\partial K(t)} \quad \frac{K(t)}{Q(t)} = (g+\delta) \frac{K(t)}{Q(t)} = \frac{I(t)}{Q(t)} \tag{18}$$

Hence

$$s = \frac{\partial Q(t)}{\partial K(t)} \quad \frac{K(t)}{Q(t)} \ . \tag{19}$$

This states that on the interior GR path the saving ratio is equal to the elasticity of output with respect to capital. (This was the characterization of the GR path employed by Swan and the present author; of course, such a capital elasticity exists only in one-commodity models in which output is a function of "capital".)

Conditions (17) and (19) can be translated into "market" terms if the economy is purely competitive and free of externalities in production. On these assumptions, $\frac{\partial Q(t)}{\partial K(t)}$ is the gross rental rate on capital and $\frac{\partial Q(t)}{\partial K(t)} - \delta$ is the (equilibrium) rate of interest. Then (17) implies that on the interior GR path the interest rate is equal to the golden-age growth rate. (19) implies that the saving ratio equals capital's gross relative share, or that net investment equals net profits.

Now we shall investigate the conditions for the existence of the GR path. For this purpose we adapt, in Figure 1, a diagram first presented by Pearce [5] and later employed by Koopmans [4]. It is a diagram of the relation between $K(0)$ and $C(0)$ in a golden age as given by (1) and (12):

$$C(0) = F(K(0), L_o) - (g+\delta) K(0) \tag{20}$$

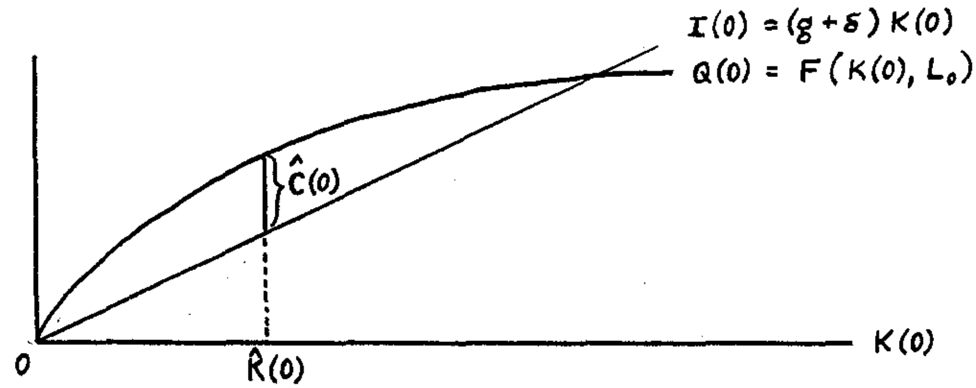

Figure 1

Figure 1 depicts a golden-age consumption maximum at $K(0) = \hat{K}(0)$ where $\frac{\partial Q(0)}{\partial K(0)} = g+\delta$. It is easy to see from the diagram, however, that there are two cases in which no such interior GR maximum exists.

In one case, neither an interior nor a corner maximum exists. This is the case in which $\lim_{K\to\infty} \frac{\partial Q(0)}{\partial K(0)} \geq g+\delta$; then the $Q(0)$ curve is everywhere steeper than the $I(0)$ line so that the distance between them always increases with $K(0)$. It is easy to see that this case implies $\lim_{K\to\infty} \frac{Q(0)}{K(0)} \geq g+\delta$. While our assumptions on the production function do not exclude this possibility, it can be shown however that this case can arise only if positive output can be produced without labor. Proof: $\frac{Q}{K} = F(1, \frac{L}{K})$. Hence $\lim_{K\to\infty} \frac{Q}{K} = F(1, 0)$. But $F(1, 0) = 0$ if $F(K, 0) = 0$. Hence $\lim_{K\to\infty} \frac{Q}{K} \geq g+\delta$ only if labor is not required for positive production.

The other case in which no interior maximum exists occurs when $\lim_{K\to 0} \frac{\partial Q(0)}{\partial K(0)} \leq g+\delta$; then the $Q(0)$ -curve is everywhere flatter than the

$I(0)$ line so that a corner maximum exists at $K(0) = 0$. We shall show that, in this case, $K(t) = 0$ can be considered the GR path.

There are two sub-cases to consider. Suppose first that $F(0, L_o) > 0$. Then $\lim_{K \to 0} \frac{\partial Q(t)}{\partial K(t)} \cdot \frac{K(t)}{Q(t)} = 0$ since $Q(t)$ does not go to zero in the limit. Hence, when $K(t) = 0$, $\frac{\dot{Q}(t)}{Q(t)} = \left[\lim \frac{\partial Q(t)}{\partial K(t)} \cdot \frac{K(t)}{Q(t)}\right] \frac{\dot{K}(t)}{K(t)} + \left[1 - \lim \frac{\partial Q(t)}{\partial K(t)} \frac{K(t)}{Q(t)}\right](\gamma+\lambda) = \gamma+\lambda$. That is, output grows at the usual golden-age rate, or "natural" rate $\gamma+\lambda$. So does consumption. This golden-age path, $C(t) = F(0, L_o) e^{gt}$, is clearly maximal and hence the GR path since $\lim_{K \to 0} \frac{\partial Q(0)}{\partial K(0)} \leq g+\delta$; investment would have to increase more than output to maintain a golden age with positive $k(t)$.*

*Note that on this GR path, where $K(t) = 0$, the saving ratio and capital's relative share are equal, since they are both equal to zero. But the interest rate may be less than the growth rate.

The other sub-case is $F(0, L_o) = 0$. In this case the $Q(0)$ curve lies uniformly below the $I(0)$ line (since they both start from the origin and $I(0)$ rises more steeply from the start). This implies that no golden age with $K(0) > 0$ is possible for it would require $I(t) > Q(t)$. But $K(t) = 0$ clearly implies a golden age for then $C(t) = Q(t) = I(t) = F(0, L_o\, e^{(\lambda+g)t}) = 0$. Since this is the only golden age that exists, it is the maximal golden age and hence the GR path.

Summarizing, if labor is required for positive output then a GR path always exists in the model under consideration. If there exists a golden-age capital path $K(t) = K(0)\ e^{gt}$ such that $\frac{\partial Q(t)}{\partial K(t)} = g+\delta$ then this is the GR path; if there does not exist such a path then $K(t) = 0$ is the GR path. In short, $K(t) = K(0)\ e^{gt}$ produces the GR path if $\frac{\partial Q(0)}{\partial K(0)} = g+\delta$ when $K(0) > 0$ or $\frac{\partial Q(0)}{\partial K(0)} \leq g+\delta$ when $K(0) = 0$.

2. The Harrod-Domar case

To illustrate the fact that no neoclassical assumptions are required for the existence of the GR path we now drop the assumptions of twice differentiability, strict concavity and everywhere positive marginal products and specialize (1) to the Harrod-Domar case:

$$\text{(1c)} \quad Q(t) = \min\ [\alpha\ K(t),\ \beta\ e^{\lambda t}\ L(t)]$$

We retain equations (2), (3) and (4).

By virtue of constant returns to scale and (2):

$$Q(t) = L_o\ e^{(\gamma+\lambda)t} \min \left[\alpha \frac{K(t)}{L_o\ e^{(\gamma+\lambda)t}}\ ,\ \beta\right] \tag{21}$$

or

$$Q(t) = L_o\ e^{(\gamma+\lambda)t} \min\ [\alpha\ k(t),\ \beta] \tag{21a}$$

It is easy to show again that if $k(t)$ is equal to any constant $k > 0$ then, provided the restraint $I(t) \leq Q(t)$ is satisfied, golden-age growth results. Clearly output, capital and investment will grow at the constant rate $g = \gamma+\lambda$; hence, so will consumption. As before, $s = (g+\delta) \frac{K(0)}{Q(0)}$; if $\alpha\ K(0) \leq \beta\ L_o$ (meaning that capital is not in surplus) then $\frac{K(0)}{Q(0)} = \frac{1}{\alpha}$ and if $\alpha K(0) > \beta L_o$ then $\frac{K(0)}{Q(0)} = \frac{K(0)}{\beta L_o}$. $\frac{\partial Q(t)}{\partial K(t)}$ will be constant, either equal to α (if labor is in surplus) or zero (if capital is in surplus).

Conversely, $k(t)$ is constant in every golden age with positive investment. If investment (hence output and consumption) is growing at some constant rate, $\underline{g}$, and capital is growing exponentially then capital must also be growing at rate g. Now if $\underline{g}$ were less then $\gamma+\lambda$ then labor would become redundant (if it was not initially) and the unemployment ratio would grow non-exponentially, which contradicts the notion of a golden age; if $\underline{g}$ were greater then $\gamma+\lambda$ then labor would eventually become scarce (if it were not initially) and growth of output at the rate $\underline{g}$ would then be impossible. Hence, in a golden age with positive investment, capital grows at the rate $\gamma+\lambda$ and $k(t)$ is therefore constant. Therefore, golden-age growth with positive investment occurs if and only if $k(t)$ is constant.

To investigate the GR path we use Figure 2 which differs from Figure 1 only in that, in (20), we have substituted the Harrod-Domar function $\min[\alpha K(0), \beta L_o]$ for $F(K(0), L_o)$:

$$C(0) = \min[\alpha K(0), \beta L_o] - (g+\delta) K(0) \tag{20'}$$

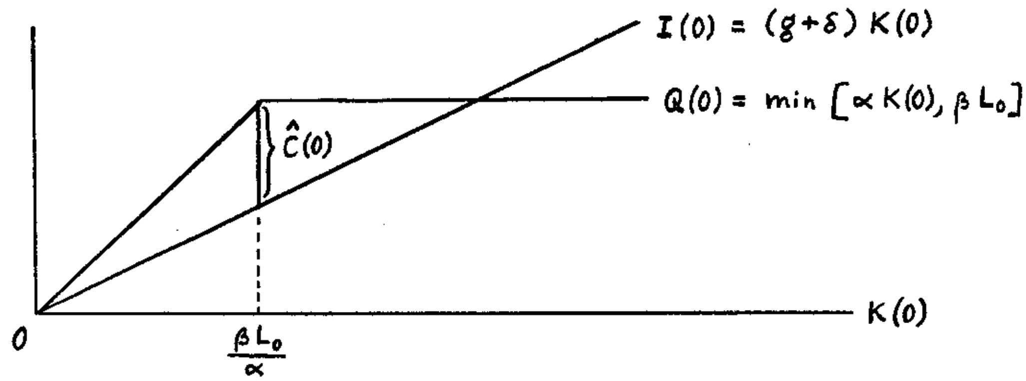

Figure 2

The diagram depicts an interior golden-age consumption maximum at $K(0) = \frac{\beta L_o}{\alpha}$. At this point the capital stock is just large enough to employ the entire labor force. A larger capital stock would put capital in surplus; a smaller stock would cause a surplus of labor. In the Harrod-Domar model, therefore, the interior GR path, if it exists, is the golden-age path in which there is full employment of both labor and capital.

What of the usual characterizations of the GR path in terms of the interest rate and capital's relative share? On the interior GR path the saving ratio is $\frac{(g+\delta)}{\alpha}$ and the growth rate $\gamma+\lambda$. But relative shares and the rate of interest are indeterminate: we can say only that capital's share is between zero and one and that the interest rate is between zero and $\alpha-\delta$. But it is true that this interior GR path is the only golden-age path with positive investment in which it is possible that the saving ratio equal's capital's share and the interest rate equals the growth rate; for in all other positive-investment golden ages capital's relative share and the interest rate are determinate and do not satisfy these equalities. Thus it remains valid that if there exists a golden age in which the interest rate equals the growth rate and the saving ratio equals capital's relative share then this golden-age path is the GR path. Hence the Golden Rule theorem applies to the Harrod-Domar model as well as to the neoclassical model.

As in the neoclassical case, however, an interior GR path may not exist. Figure 2 shows that if $\alpha < g+\delta$ then no golden age with positive investment exists, hence no interior GR path. In this case the golden age $K(t) = Q(t) = I(t) = C(t) = 0$ is the only possible golden age; hence it can be regarded as the GR path.

Note that, in the Harrod-Domar case, either an interior or a corner GR path must exist since positive labor input is required for positive output.

B. Capital-Augmenting Technical Progress

We have seen that golden-age paths with positive investment may exist if the labor force grows at a constant rate and if technical progress proceeds at a constant rate and can be described as purely labor-augmenting. Clearly no positive-investment golden age can exist if the labor force increases or the technology advances at a non-constant rate. We prove now that no positive-investment golden age can exist if technical progress cannot be described as solely labor-augmenting.

Suppose that technical progress must be described as at least partially capital-augmenting. To give the golden age every chance of existence, suppose that the rate of capital-augmenting technical progress is a constant, μ . Then the production function is of the form*

$$(22) \qquad Q(t) = F(e^{\mu t} K(t), e^{\lambda t} L(t)), \quad \mu > 0 , \quad \lambda \geq 0 .$$

*Note that the function $e^{\theta t} F(K, L)$ can be put into this form by setting $\mu = \lambda$.

We suppose that this function has the neoclassical properties.

Assume now the existence of at least one positive-investment golden age, hence a growth path in which output and capital both grow at some constant rate g . We shall show that (22) and this assumption imply that technical progress <u>cannot</u> be of the sort which must be described as (at least partially) capital-augmenting; that is, we shall show a contradiction.

Upon differentiating (22) totally with respect to time, dividing both sides of the resulting equation by $Q(t)$, we obtain the familiar growth rate formula

$$(23) \qquad \frac{\dot{Q}(t)}{Q(t)} = a_t\left(\mu + \frac{\dot{K}(t)}{K(t)}\right) + (1 - a_t)\left(\lambda + \frac{\dot{L}(t)}{L(t)}\right).$$

where $a_t = \frac{\partial Q(t)}{\partial K(t)} \cdot \frac{K(t)}{Q(t)}$. This formula states that the growth rate is a weighted average of the growth rates of "effective capital" and "effective labor".

But $\frac{\dot{L}(t)}{L(t)} = \gamma$ and, by our golden-age assumption, $\frac{\dot{Q}(t)}{Q(t)} = \frac{\dot{K}(t)}{K(t)} = g$; hence

$$g = a_t\ (\mu+g) + (1-a_t)\ (\lambda+\gamma)\ . \tag{24}$$

It follows that, since $\mu > 0$, a_t is a constant given by

$$a = \frac{g - \lambda - \gamma}{\mu + g - \lambda - \gamma}\ . \tag{25}$$

But if a is constant then, since $\frac{K(t)}{Q(t)}$ is a constant in a golden age, $\frac{\partial Q(t)}{\partial K(t)}$ is also a constant on the assumed golden-age path.

Constancy of $\frac{\partial Q(t)}{\partial K(t)}$ and $\frac{K(t)}{Q(t)}$ implies that technical progress cannot be of the sort which must be described as (as least partially) capital-augmenting. For Uzawa [14] has shown that if technical progress is Harrod-neutral -- meaning that the marginal product of capital is constant if the capital-output ratio is constant -- then the production function $F(K, L; t)$ can be written in the form $F(K, A(t)\ L)$; that is, technical progress can be described as solely labor-augmenting.*

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*There are cases in which Harrod-neutral technical progress can be described as capital-augmenting. The Cobb-Douglas function is such a case (and the only case if there are constant returns to scale) for the function $K^{\alpha}\ (A(t)\ L)^{1-\alpha}$ can be written as $(B(t)\ K)^{\alpha}\ L^{1-\alpha}$.

Therefore, a golden age with positive investment is possible only if technical progress can be represented as solely labor-augmenting, as in (1). If it is necessary to write the production function, as in (22), with a positive rate of capital-augmenting technical progress, then no golden-age path, and hence no GR path, exists.

But the general notion of the Golden Rule path does not lose its heuristic value in the event that technical progress must be described as capital-augmenting. It will be shown in the next part of this paper that there still exists in this event a particular constant interest - rate path -- which we shall call the Quasi-Golden-Rule path -- having the same normative significance as the GR path.

II. INEFFICIENT GROWTH PATHS

The preceding analysis shows immediately that there are some golden-age paths that are inefficient. Any golden age in which the capital-effective labor ratio exceeds its GR value is dominated by a policy of immediately gobbling up the "excess" capital and subsequently maintaining the capital-effective labor ratio at its GR value, i.e., following the GR path; such a policy will clearly make consumption higher at every point in time. It follows that any investment policy which at some point permanently fixes the capital-effective labor ratio at a level exceeding the GR level is inefficient and therefore cannot be optimal (since a policy to be optimal must be optimal at every stage).

In the author's reply to Pearce an obvious generalization of this result was conjectured: "Any policy which causes the capital-output ratio [equivalently, the capital-effective labor ratio, since the one ratio is a monotonically increasing function of the other] permanently to exceed -- always by some minimum finite amount -- its GR level is inefficient and hence cannot be optimal" [7, p. 1099]. A proof of this conjecture was later communicated to the author by Tjalling Koopmans. In what follows we present Koopmans' proof and then employ the technique to prove an analogous theorem for the case in which technical progress must be described as (at least partially) capital augmenting, for the case of non-exponential labor growth and technical progress, and finally for the case in which technical progress cannot necessarily be described as labor and capital-augmenting.

We confine our analysis to the neoclassical production function, although the theorems proved clearly carry over to the Harrod-Domar production function.

Suppose first that technical progress can be described as solely labor-augmenting and occurs at the constant rate λ . Then, as was shown above, when $k(t)$ is fixed the consumption path is given by the equation

$$C(t) = [f(k) - (\gamma+\lambda+\delta)\, k]\, L_o\, e^{(\gamma+\lambda)t} \qquad (12)$$

where $f'(k) > 0$, $f''(k) < 0$.

We show now that if $k(t)$ is not fixed, then the consumption path is given by the equation

$$(26) \qquad C(t) = [f(k(t)) - (\gamma+\lambda+\delta)\, k(t) - \dot{k}(t)]\, L_o\, e^{(\gamma+\lambda)t}$$

Proof: From (3), (4) and (9) we have

$$(27) \qquad C(t) + \dot{K}(t) + \delta\, K(t) = L_o\, e^{(\gamma+\lambda)t} f(k(t))$$

or

$$(28) \qquad \frac{C(t)}{L_o\, e^{(\gamma+\lambda)t}} = f(k(t)) - \delta\, k(t) - \frac{\dot{K}(t)}{L_o\, e^{(\gamma+\lambda)t}}$$

Now, differentiating $k(t)$ with respect to time, we have

$$(29) \qquad \dot{k}(t) = \frac{\dot{K}(t)}{L_o\, e^{(\gamma+\lambda)t}} - (\gamma+\lambda) \frac{K(t)}{L_o\, e^{(\gamma+\lambda)t}}$$

or

$$(30) \qquad \frac{\dot{K}(t)}{L_o\, e^{(\gamma+\lambda)t}} = \dot{k}(t) + (\gamma+\lambda)\, k(t) .$$

Therefore, substituting (30) into (28) yields (26).

Assume now that there exists a GR path, hence a GR value of $k(t)$, say $\hat{k}$. For simplicity only, we assume that the GR maximum is an interior one so that $\hat{k}$ is determined by the equation, derived from (12) (see also (16a))

$$(31) \qquad f'(\hat{k}) = \gamma+\lambda+\delta .$$

As a consequence of (31), the expression $f(k) - (\gamma+\lambda+\delta)k$ is monotonically increasing in k up to $k = \hat{k}$ and monotonically decreasing in k for all $k > \hat{k}$.

Consider now any capital-path which "violates" the Golden Rule in that, at some point in time (perhaps initially) and thereafter, it keeps the capital-effective labor ratio in excess of its GR value by some positive (possibly varying) amount. That is, consider any path $k(t)$ such that, for all $t \geq t_o \geq 0$,

$$k(t) \geq \hat{k} + \epsilon, \quad \epsilon > 0. \tag{32}$$

Then the following theorem can be proved:

> __Koopmans' Theorem:__ Any path satisfying (32) is "dynamically inefficient" or (equivalently) "dominated", for there always exists another path which starting from the same initial capital stock, provides more consumption at least some of the time and never less consumption.
>
> Proof: Define another path, $k^*(t)$, such that

$$k^*(t) = \begin{cases} k(t), & 0 \leq t < t_o; \\ k(t) - \epsilon, & t \geq t_o. \end{cases} \tag{33}$$

In the first interval, $0 \leq t < t_o$, the two paths are identical so that $C^*(t) = C(t)$ in this interval (which will not exist if $t_o = 0$). At $t = t_o$, the starred path gives a discontinuous consumption bonus equal to ϵ, for an amount of capital equal to ϵ is instantly consumed so as to make $k^*(t) = k(t) - \epsilon$ at $t = t_o$. In the remaining interval, $t > t_o$, the difference between the consumption rate offered by the starred path and the path specified in (32) is implied by (26) to be

(34) $$C^*(t) - C(t) = \{[f(k^*(t)) - (\gamma+\lambda+\delta)\, k^*(t) - \dot{k}^*(t)] - [f(k(t)) - (\gamma+\lambda+\delta)\, k(t) - \dot{k}(t)]\}\, L_o\, e^{(\gamma+\lambda)t}$$

But observe that, for all $t > t_o$, $\dot{k}^*(t) = \dot{k}(t)$ since the two paths differ after t_o by only a constant, ϵ. Hence (33) and (34) imply

(35) $$C^*(t) - C(t) = \{[f(k^*(t)) - (\gamma+\lambda+\delta)\, k^*(t)] - [f(k(t)) - (\gamma+\lambda+\delta)\, k(t)]\}\, L_o\, e^{(\gamma+\lambda)t}$$

The righthand side of (35) is strictly positive for all $t > t_o$ since $k^*(t) \geq \hat{k}$, $k(t) > k^*(t)$ and $f(k) - (\gamma+\lambda+\delta)k$ is strictly decreasing in k for all $k > \hat{k}$. Hence, in the interval $t > t_o$, the starred path gives more consumption at every point in time. Therefore, the starred path dominates the other path for it is never worse and is better for all $t \geq t_o$.

To elaborate a little on the last step of the proof, note that $k^*(t) \geq \hat{k}$ because $k^*(t)$ is only ϵ smaller than $k(t)$ and the latter is at least ϵ larger than $\hat{k}$ for all t. Figure 3 illustrates why $f(k^*(t)) - (\gamma+\lambda+\delta)\, k^*(t) > f(k(t)) - (\gamma+\lambda+\delta)\, k(t)$ for any $t > t_o$.

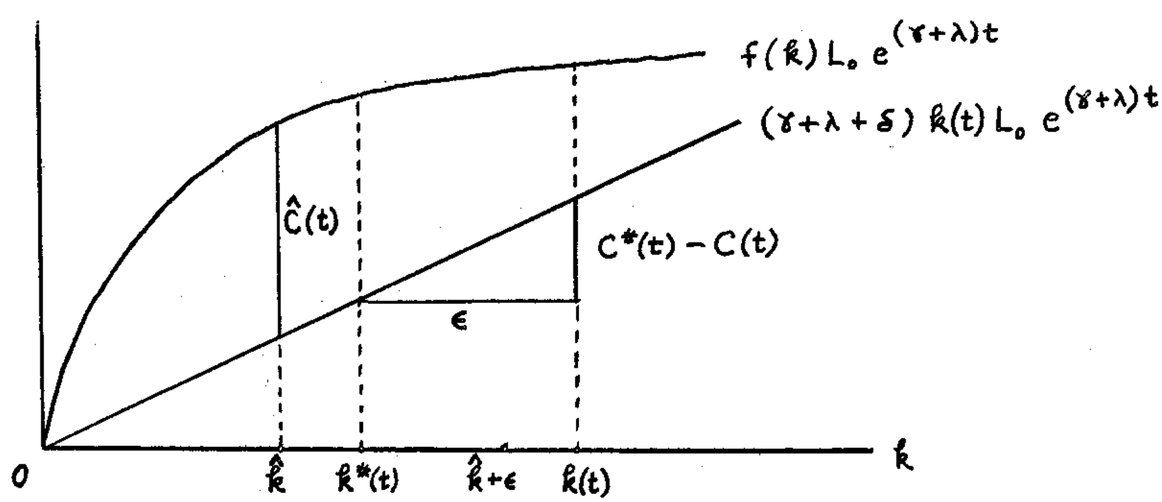

Figure 3

The theorem can be expressed in a variety of ways. For example, since the social net rate of return to saving (and the competitive rate of interest) $f'(k(t)) - \delta$, is a monotonically decreasing function of $k(t)$, an equivalent proposition is that any growth path which keeps the rate of return to saving permanently below, by some finite amount, its GR value (the golden-age growth rate on the assumption expressed by (31)) is dynamically inefficient. Or the proposition can be expressed in terms of the capital-output ratio, as we first conjectured it.

Another remark is that the neoclassical assumptions $f'(k) > 0$, $f''(k) < 0$ are far stronger than necessary for the theorem. If $f''(k) = 0$ for all $k > \hat{k}$, for example (where $\hat{k}$ is now defined as the smallest k for which $f'(k) = (\gamma+\lambda+\delta)$, then, while the two paths will yield the same consumption path after t_0 , the starred path still offers the consumption bonus at t_0 and hence dominates the other path. Secondly, the theorem is trivial in the Harrod-Domar case for it simply means that any path which keeps capital permanently in surplus is inefficient, and this hardly needs proving.

We turn now to the case in which technical progress can be described as input-augmenting but is (partially) capital-augmenting. Suppose that the rate of capital-augmenting technical progress is a constant, μ . And suppose once again that we have a neoclassical production function. Then

$$Q(t) = F(e^{\mu t} K(t) , \ e^{\lambda t} L(t)) , \qquad \mu > 0 , \ \lambda \geq 0 . \tag{36}$$

In the spirit of the first part of this paper we define

$$(37)\qquad k(t) = \frac{K(t)}{L_o\, e^{(\gamma+\lambda-\mu)t}}$$

which is the ratio of "effective capital" to "effective labor", and

$$(38)\qquad f(k(t)) = F(k(t), 1)$$

to obtain, by virtue of constant returns to scale,

$$(39)\qquad Q(t) = L_o\, e^{(\gamma+\lambda)t} f(k(t)) .$$

To obtain the consumption path as a function of $k(t)$ we follow the same procedure used to obtain (26). From

$$(40)\qquad C(t) + \dot{K}(t) + \delta K(t) = L_o\, e^{(\gamma+\lambda)t} f(k(t))$$

we have

$$(41)\qquad \frac{C(t)}{L_o\, e^{(\gamma+\lambda-\mu)t}} = e^{\mu t} f(k(t)) - \delta k(t) - \frac{\dot{K}(t)}{L_o\, e^{(\gamma+\lambda-\mu)t}}$$

From

$$(42)\qquad \dot{k}(t) = \frac{\dot{K}(t)}{L_o\, e^{(\gamma+\lambda-\mu)t}} - (\gamma+\lambda-\mu)\ \frac{K(t)}{L_o\, e^{(\gamma+\lambda-\mu)t}}$$

we have

$$(43)\qquad \frac{\dot{K}}{L_o\, e^{(\gamma+\lambda-\mu)t}} = \dot{k}(t) + (\gamma+\lambda-\mu)$$

Hence, from (41) and (43),

$$C(t) = \left\{ e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\, k(t) - \dot{k}(t) \right\} L_o\, e^{(\gamma+\lambda-\mu)t} \tag{44}$$

(If $\mu = 0$, we obtain (26) again.)

Now we define $\hat{k}(t)$ as the value of $k(t)$ which, for fixed $\dot{k}(t)$ and a particular t, maximizes $C(t)$. For simplicity only we assume an interior maximum is attained so that $\hat{k}(t)$ is defined by*

$$e^{\mu t} f'(\hat{k}(t)) = \gamma+\lambda+\delta-\mu \,. \tag{45}$$

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*If $f'(k) > 0$ for all k, as we assume, then $\gamma+\lambda+\delta-\mu > 0$ is required for the existence of such a value of $k(t)$.

Note that $\hat{k}(t)$ must be increasing over time if $\mu > 0$; and if $\lambda+\delta-\mu > 0$ then so must $\hat{K}(t)$, by (37).

Of course, $e^{\mu t} f'(k(t))$ is just the marginal productivity of capital at time t.** Hence the path $\hat{k}(t)$ defined by (45) is a constant interest

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**

$$\frac{\partial F\,(e^{\mu t} K(t),\, e^{\lambda t} L(t))}{\partial K(t)} = e^{\mu t} \frac{\partial F\left(\frac{e^{\mu t} K(t)}{e^{(\gamma+\lambda)t} L_o},\, 1\right)}{\partial\left(\frac{e^{\mu t} K(t)}{e^{(\gamma+\lambda)t} L_o}\right)} = e^{\mu t} f'(k(t)) \,.$$

---

rate path in which the (competitive) interest rate is $e^{\mu t} f'(k(t)) - \delta = \gamma+\lambda-\mu$.

We know that $\hat{k}(t)$ is not the GR path; no positive-investment golden age exists when $\mu > 0$ and hence no GR path exists. Nevertheless we shall dub this path the Quasi-Golden-Rule path. For we shall demonstrate that it is like the GR path in the following respect: Any path which, at some point in time and thereafter, keep the interest rate below $\gamma+\lambda-\mu$ by a positive (non-vanishing) amount is dynamically inefficient, just as, when $\mu = 0$, any path which maintains the interest rate below its GR value, $\gamma+\lambda$, is dynamically inefficient.*

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*Does the Quasi-GR path dominate all other constant-interest-rate paths? We have been unable to show this and we suspect the answer is no. In that respect, the Quasi-GR is not like the GR path.

Such a path is one which causes $k(t)$ to satisfy, for all $t \geq t_o \geq 0$,

(46) $$k(t) \geq \hat{k}(t) + \epsilon$$

We show now that the following path dominates any such path:

(47) $$k^*(t) = \begin{cases} k(t), & 0 \leq t < t_o ; \\ k(t) - \epsilon , & t \geq t_o \end{cases}$$

Comparing the associated consumption paths, we observe first that the two paths yield identical consumption paths until t_o. At this point the starred path yields a dividend equal to ϵ, unlike the other path. Subsequently, $\dot{k}^*(t) = \dot{k}(t)$, since, for $t \geq t_o$, $k^*(t)$ and $k(t)$ differ only by the constant, ϵ. Hence, for all $t > t_o$,

$$(48)\qquad C^*(t) - C(t) = \left\{ \left[e^{\mu t} f(k^*(t)) - (\gamma+\lambda+\delta-\mu)\, k^*(t)\right] - \left[e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu\, k(t)\right]\right\} L_o\, e^{(\gamma+\lambda-\mu)t}$$

The right-hand side of (48) must be positive for every $\underline{t}$ since $k(t) > k^*(t) \geq \hat{k}(t)$ and $e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\, k(t)$ is, for every $\underline{t}$, monotonically decreasing in $k(t)$ in the range $k(t) > \hat{k}(t)$ (since $\hat{k}(t)$ is maximal and $f''(k(t)) < 0$). Hence, the starred path dominates the path which transgresses the Quasi-Golden-Rule path. Therefore, any path which violates the Quasi-Golden-Rule path in the manner described in (46) is dynamically inefficient.*

---

*We have just shown that (46) is a <u>sufficient</u> condition that a $k(t)$ path be dominated by another path on which $\overline{k(t)}$ is smaller by a constant amount. The following argues that (46) is also a necessary condition that a $k(t)$ path <u>be dominated in this way</u>.

First we show that every $k(t)$ path so dominated is a path along which $e^{\mu t} f'(k(t)) < \gamma+\lambda+\delta-\mu$ for all $t \geq t_o$. Proof: Choose any path $k(t) \geq 0$ and suppose that it is dominated by another path $k^*(t) = k(t) - \epsilon$, $\epsilon > 0$ for $t \geq t_o$. Then, for every $t \geq t_o$ we have

$$C^*(t) - C(t) = \left\{ \left[e^{\mu t} f(k(t) - \epsilon)) - (\gamma+\lambda+\delta-\mu)\, (k(t) - \epsilon)\right] - \left[e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\, k(t)\right]\right\} L_o\, e^{(\gamma+\lambda)t} \geq 0 .$$

Then it is immediately clear that, for every $t \geq t_o$, $k(t)$ must exceed $\hat{k}(t)$; that is, $k(t)$ must lie on the right side of the hill whose peak occurs at $k(t) = \hat{k}(t)$, i.e., where $e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\, k(t)$ is at a maximum; for if $k(t)$ were at the peak or the left side of the hill, $k^*(t)$ which lies left of $k(t)$ could not equal or exceed it.

This proves only that $k(t) > \hat{k}(t)$ is a necessary condition that a path be dominated in the manner described; it does not prove that (46) is a necessary condition. But it is easy to see that $k(t) > \hat{k}(t)$ is not a sufficient

(Footnote continued on bottom of next page)

We can relax without difficulty the assumptions that the labor force and the technology increase at constant rates. Further, we may a allow the depreciation rate at time t , $\delta(t)$, (the same for capital goods of every age) to vary with time. Write

$$(49) \qquad Q(t) = F\,(B(t)\,K(t),\, A(t)\,L(t))$$

where $A(t)$, $B(t)$ and $L(t)$ are continuously differentiable functions of time. Then, defining $k(t) = \frac{B(t)\,K(t)}{A(t)\,L(t)}$, one can easily derive

$$(50) \qquad C(t) = \left\{ B(t)\,f(k(t)) - \left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} + \delta(t) - \frac{\dot{B}(t)}{B(t)} \right) k(t) - \dot{k}(t) \right\} \frac{A(t)\,L(t)}{B(t)}$$

where $f(k(t)) = F(k(t), 1)$.

Footnote continued from page 28.

condition. For consider a path $k(t) > \hat{k}(t)$ with $\lim_{t\to\infty} k(t) = \lim_{t\to\infty} \hat{k}(t)$. Then, for any $\epsilon > 0$,

$$\lim_{t\to\infty} [C^*(t) - C(t)] = \lim_{t\to\infty} \Big\{ [e^{\mu t} f(k(t) - \epsilon)) - (\gamma+\lambda+\delta-\mu)(k(t) - \epsilon] - [e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu) k(t)] \Big\} L_o\, e^{(\gamma+\lambda)t} < 0$$

since

$$\lim_{t\to\infty} [e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\,k(t)] = \lim_{t\to\infty} [e^{\mu t} f(\hat{k}(t)) - (\gamma+\lambda+\delta-\mu)\,\hat{k}(t)]$$

and $\hat{k}(t)$ maximizes $[e^{\mu t} f(k(t)) - (\gamma+\lambda+\delta-\mu)\,k(t)]$ for every t .

Thus $k(t) > \hat{k}(t)$ is not a sufficient condition that the path $k(t)$ be dominated. We conclude that (46) is necessary and sufficient that a path be dominated by a path described in (47).

It should be emphasized however that (46) has not been proved a necessary condition for a $k(t)$ path to be dominated in <u>any</u> way. In other words, it has not been shown that (46) is a necessary condition for dynamical inefficiency; it has only been argued that (46) is a necessary condition for a path to be dominated by a path which relates to it in the particular way specified in (47).

Next we define the Generalized Quasi-GR path, $\hat{k}(t)$, by

$$B(t)\, f'(\hat{k}(t)) = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} + \delta(t) - \frac{\dot{B}(t)}{B(t)} \; . \tag{51}$$

It can then be shown, in precisely the same manner as before, that any path which makes $k(t) \geq \hat{k}(t) + \epsilon$, $\epsilon > 0$, is dynamically inefficient. That is, any path which (immediately or eventually) keeps the interest rate path finitely below the path of $\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} - \frac{\dot{B}(t)}{B(t)}$ is dominated by another path.

Note that if technical progress is Hicks-neutral, so that $Q(t) = A(t)\, F(K(t), L(t))$ then, since by constant returns to scale, $A(t)\, F(K(t), L(t)) = F(A(t)\, K(t), A(t)\, L(t))$, we have $B(t) = A(t)$ and $\frac{\dot{B}(t)}{B(t)} = \frac{\dot{A}(t)}{A(t)}$ in (51). In this case the interest rate path corresponding to the Generalized Quasi-GR path is the same as for the case of no technical progress; it is just the path of $\frac{\dot{L}(t)}{L(t)}$.

This remark suggests another proposition: If input-augmenting technical progress rates are not defined then the Generalized Quasi-GR interest rate path is just the path of $\frac{\dot{L}(t)}{L(t)}$. We now demonstrate this.

Let us, in this last case, make no assumption as to whether technical progress is input-augmenting and write the production function in the form

$$Q(t) = F(K(t), L(t); t) \; . \tag{52}$$

Then, by constant returns to scale,

$$Q(t) = L(t)\, f(k(t); t) \tag{53}$$

where

$$k(t) = \frac{K(t)}{L(t)} \tag{54}$$

and

$$f(k(t);\ t) = F\left(\frac{K(t)}{L(t)}\ ,\ 1\ ;\ t\right) . \tag{55}$$

From (53), (3) and (4) we have

$$\frac{C(t)}{L(t)} = f(k(t);\ t) - \delta\ k(t) - \frac{\dot{K}(t)}{L(t)} \quad . \tag{56}$$

From (54) we have

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)}\ k(t) \tag{57}$$

Equations (56) and (57) yield

$$C(t) = \left\{ f(k(t);\ t) - \left(\frac{\dot{L}(t)}{L(t)} + \delta \right) k(t) - \dot{k}(t) \right\} L(t) \tag{58}$$

It is clear now that the Generalized Quasi-GR path, $\hat{k}(t)$, is defined by

$$f_k\ (\hat{k}(t);\ t) = \frac{\dot{L}(t)}{L(t)} + \delta\ . \tag{59}$$

It can be shown, by the same method that we have been using, that any path which, at t_o and thereafter, keeps $k(t) \geq \hat{k}(t) + \epsilon$ is dominated by a path $k^*(t) = k(t),\ t < t_o,\ k^*(t) = k(t) - \epsilon,\ t \geq t_o$ so that such a path is dynamically inefficient.

Note that the interest rate, $f_k - \delta$, associated with the

Generalized Quasi-GR path is the path of $\frac{\dot{L}(t)}{L(t)}$ which is independent of $\underline{t}$. Hence, as we conjectured, if technical progress cannot be described in purely input-augmenting terms then the critical interest rate path is just the path of $\frac{\dot{L}(t)}{L(t)}$. This proof also shows that, quite generally, any path which maintains the interest rate path below $\frac{\dot{L}(t)}{L(t)}$ is dynamically inefficient.

We have now to relate our various results. First, clearly, the last result, in which $\frac{\dot{L}(t)}{L(t)}$ is the critical interest rate path, is weaker than can be obtained if technical progress is known to be purely labor augmenting; for, in that case, any path which keeps the interest rate below $\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}$, not just below $\frac{\dot{L}(t)}{L(t)}$, can be shown to be dynamically inefficient. That is, if we know that technical progress is purely labor-augmenting, we can show the inefficiency of a greater number of paths than if we do not have this information. Second, by obvious extension, the last result is weaker than can be obtained if technical progress is purely input-augmenting and if $\frac{\dot{A}(t)}{A(t)} > \frac{\dot{B}(t)}{B(t)}$; for, in that case, the interest rate path corresponding to the Generalized Quasi-GR path, $\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} - \frac{\dot{B}(t)}{B(t)}$ is higher -- and hence rules out as dynamically inefficient more paths -- than the interest rate path given by $\frac{\dot{L}(t)}{L(t)}$. But, third, if $\frac{\dot{A}(t)}{A(t)} < \frac{\dot{B}(t)}{B(t)}$, then our last result is stronger than any obtained up to that point; for, if $\frac{\dot{A}(t)}{A(t)} < \frac{\dot{B}(t)}{B(t)}$, the critical path $\frac{\dot{L}(t)}{L(t)}$ below which the interest rate must not be maintained is higher than the path $\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} - \frac{\dot{B}(t)}{B(t)}$. Thus, the last case provides us with additional information even if technical progress is known to be purely input-augmenting.

III. CONCLUDING REMARKS

We demonstrated that a Golden Rule path, that is, a consumption-maximizing golden age path, always exists in the neoclassical and Harrod-Domar models if the labor force increases at a constant rate, the depreciation rate is constant, technical progress, if any, is purely labor-augmenting and occurs at a constant rate and positive labor is required for positive output. It was also demonstrated that a positive-investment GR path never exists if technical progress must be described as (at least partially) capital-augmenting.

It was then shown that any path which permanently deepens capital in excess of the GR path is dynamically inefficient -- it is dominated with respect to consumption by another path. Further, if technical progress or labor force growth is non-exponential or if technical progress cannot be described as purely labor-augmenting, then, while no GR path will exist, there may exist a Generalized Quasi-GR path having the same property, namely, that any path which permanently deepens capital in excess of that path is dynamically inefficient. (Note that such paths may not exhaust the class of dynamically inefficient paths. For example, even if no Quasi-GR path exists, the growth path produced by a permanently unitary saving ratio is clearly dynamically inefficient.)

Concerning the significance of these findings, we believe that it may sometime and somewhere be of real practical importance to know that certain growth paths, even growth paths with positive interest rate and less-than-unitary saving ratio, are dynamically inefficient.* It is

*It should be admitted that, as Pearce [5] has observed, an economy may rationally deepen capital "excessively" in order to possess a "war chest" of capital for consumption in the event of earthquakes, wars or a hankering for a consumption brings in the future. If these events actually occur and the excess capital is completely consumed, the war chest will clearly have been a good strategy; if they do not, the strategy, while rational, will always be regretted _ex post_.

not only in Soviet-type economies that the extent of capital-deepening deserves watching; capitalist countries too may, by the exercise of their fiscal and monetary controls, excessive capital deepening. For while it may be conjectured that no competitive laissez-faire economy free from externalities and having perfect futures markets would ever choose a dynamically inefficient growth path -- unless wealth _per se_ possessed utility -- there seems to be no mechanism which will insure against an excessively austere fiscal policy. Thus even the economist without a social utility function and without a set of fiscal and monetary principles may have a role to play in growth policy.

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