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Conjugate Functions and Symmetric Duality

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1. INTRODUCTION

In a recent paper by Dantzig, Eisenberg, and Cottle [3] a very interesting and useful approach to duality in nonlinear programming has been presented. They consider the following dual programming problems:

	Primal	Dual
(1.1)	$\underset{x,\,y}{\text{Min.}}\ K(x, y) - y' \frac{\partial K}{\partial y}$	$\underset{x,\,y}{\text{Max.}}\ K(x, y) - x' \frac{\partial K}{\partial x}$
	s. t. $\frac{\partial K}{\partial y} \leq 0$	s. t. $\frac{\partial K}{\partial x} \geq 0$
	$x \in R^n_+,\ y \in R^m_+$	$x \in R^n_+,\ y \in R^m_+$

$$\frac{\partial K}{\partial x} = \left[\frac{\partial K(x, y)}{\partial x_1}, \frac{\partial K(x, y)}{\partial x_2}, \ldots, \frac{\partial K(x, y)}{\partial x_n} \right]'$$

$$\frac{\partial K}{\partial y} = \left[\frac{\partial K(x, y)}{\partial y_1}, \frac{\partial K(x, y)}{\partial y_2}, \ldots, \frac{\partial K(x, y)}{\partial y_m} \right]'$$

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Under the following conditions they prove certain duality theorems:

" (i) K is real valued on the cartesian product $U \times V$ where U and V are open subsets of R^n and R^m respectively, such that $R^n_+ \subset U$, $R^m_+ \subset V$.

(ii) K is twice continuously differentiable on $U \times V$

(iii) for each fixed $x \epsilon R^n_+$, K is strictly concave in y

(iv) for each fixed $y \epsilon R^m_+$, K is strictly convex in x ."

We concentrate on weakening the conditions (iii) and (iv) by requiring only that K be concave in y and convex in x instead of the strict concavity - convexity requirement. This weakening will allow the duality theorems to be applied to the general convex programming problem including such cases as linear programming and quadratic programming with a semi-definite quadratic form. Under the assumption of strict concavity - convexity these latter cases are, of course, excluded.

We shall apply the theory of conjugate functions as developed by W. Fenchel [4] to the function $K(x, y)$. In an earlier paper the present author [8] studied duality theory for nonlinear programming, using conjugate function theory. By also basing the present development on conjugate function theory we may be able to understand the differences and similarities between the various approaches to duality.

For use in the paper we record the following definitions and theorem concerning conjugate functions. Let $f(x)$ be a closed[2] convex function defined on a convex set C in E^n. Then we define the conjugate function $\varphi(\xi)$ by:

$$\varphi(\xi) = \sup_{x \in C} [(x' \xi) - f(x)] \tag{1.2}$$

$$\Gamma = \{\xi \in E^n \mid \sup_{x \in C} [(x' \xi) - f(x)] < +\infty\}$$

Let $g(x)$ be a concave closed function defined on a convex set D in E^n. Then we define the conjugate function $\psi(\xi)$ by:

$$\psi(\xi) = \inf_{x \in D} [(x' \xi) - g(x)] \tag{1.3}$$

$$\Delta = \{\xi \in E^n \mid \inf_{x \in D} [(x' \xi) - g(x)] > -\infty\} .$$

Theorem 1.1 If the sets $C \cap D$ and $\Gamma \cap \Delta$ are non void then

$$\sup_{x \in C \cap D} [g(x) - f(x)] = \inf_{\xi \in \Gamma \cap \Delta} [\varphi(\xi) - \psi(\xi)].$$

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2. General discussions of conjugate functions can be found in [4] and [6]. A convex function defined on a convex set S will be closed if for any x^0 where $\underline{\lim}_{x \to x^0} f(x)$ is finite, $f(x^0)$ is defined and is equal to $\underline{\lim}_{x \to x^0} f(x)$.

If the relative interior of $C \cap D \neq \emptyset$ then we may replace inf by min. Correspondingly if the relative interior of $\Gamma \cap \Delta \neq \emptyset$ we may replace sup by max.

Before proceeding with the formal discussion in the paper it may be useful to indicate the underlying ideas and the basic method of proof. The underlying idea behind the use of conjugate function theory and thus the duality theorems presented here is that we may represent a closed convex set either in terms of the locus of points of the boundary or as the intersection of its supporting hyperplanes. We may further characterize the supporting hyperplanes in terms of a space representing slope and intercept. Thus in problems concerning the minimization of a convex function we may consider either the problem in terms of the space of the locus of points or dually in terms of the space of slope and intercept.

The above observation forms one of the basic steps in the proof of the main result. Let $f(x)$ be a closed convex function defined on E^n. Then we may consider the problem

$$\min_{x \geqq 0} f(x) \tag{1.4}$$

Alternatively we may consider the problem in terms of the slope intercept space. In this space we have the problem: Among all hyperplanes in the non-negative slope, choose the one with a maximum intercept. We may write

$$(1.5) \quad \text{Max. } f(x) - x' \frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x_1} \cdots\cdots \frac{\partial f}{\partial x_n} \right)'$$

$$x \geqq 0, \ \frac{\partial f}{\partial x} \geqq 0$$

Problem (1.5) for the case of the function $f(x)$ differentiable is the dual problem of Fenchel and we may apply Theorem 1.1 to assert the equality, under certain conditions, of these two problems. However, because of the simple structure being dealt with here we may, instead, present the following:

<u>Theorem 1.2</u>

If problems (1.4) and (1.5) are solvable then we have

$$f(\hat{x}) = f(x^0) - x^0 \frac{\partial f}{\partial x^0}$$

where $\hat{x}$ solves (1.4) and x^0 solves (1.5).

<u>Proof</u>

By the convexity of the function $f(x)$ we have

$$f(x) \geq f(x^0) + (x - x^0)' \frac{\partial f}{\partial x^0}$$

$$\geq f(x^0) - x^{0'} \frac{\partial f}{\partial x^0}$$

where $x \geq 0$

$$f(\hat{x}) = \text{Min. } f(x) \geq f(x^0) - x^{0'} \frac{\partial f}{\partial x^0} = \text{Max. } f(x) - x' \frac{\partial f}{\partial x}$$

$$x \geq 0, \frac{\partial f}{\partial x} \geq 0$$

On the other hand we know that $\hat{x}$ must satisfy $\frac{\partial f}{\partial \hat{x}} \geq 0$ and $\hat{x}' \frac{\partial f}{\partial x} = 0$ and consequently $\hat{x}$ is a feasible solution to problem II.

Thus

$$\underset{x \geq 0,\ \frac{\partial f}{\partial x} \geq 0}{\text{Max.}}\ f(x) - x \frac{\partial f}{\partial x} \geq f(\hat{x}) = \text{Min.}\ f(x)$$

q. e. d.

2. SYMMETRIC DUALITY THEOREMS

Let $K(x, y)$ satisfy the conditions (i), (ii), (iii)' for each fixed $x \in R^n_+$, K is concave in y, (iv)' for each fixed $y \in R^m_+$, K is convex in x.

We first consider $K(x, y)$ as a convex function of x. Define

(2.1) $$\varphi(\xi) = \sup_{x \in X} [x'\xi - K(x, y)]$$

$$X = R^n_+$$

$$\Gamma_1 = \{\xi \in E^n \mid \sup_{x \in X} [x'\xi - K(x, y)] < +\infty\} .$$

For a maximum of (2.1) to exist[3] we must have for some $x \in X$, $\xi \in E^n$ and some $y \in E^m$

(2.2) $$\xi - \frac{\partial K}{\partial x} \leq 0$$

(2.3) $$x_i \left(\xi_i - \frac{\partial K}{\partial x_i}\right) = 0 \qquad i = 1, \ldots, n$$

(2.4) $$x \geq 0 .$$

Substituting (2.3) into (2.1) we have

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3. For the purposes of the present discussion we may, without loss of generality, only consider functions $K(x, y)$ for which if an extremal value of the function exists it is achieved on the domain of definition of the function. For a function not possessing this property the dual problems will have no feasible solution.

(2.5) $$\varphi(\xi) = [x' \frac{\partial K}{\partial x} - K(x, y)]$$

where x and ξ are related by (2.2), (2.3), and (2.4). We define

(2.6) $$\psi(\xi) = \inf_{x \in X} [x'\xi]$$

$$\Delta_1 = \{\xi \mid \inf_{x \in X} [x'\xi] > -\infty\} .$$

It is immediately seen that $\psi(\xi) = 0$ where $\Delta_1 = \{\xi \in E^n | \xi \geq 0\}$. We next wish to compute

(2.7) $$\inf_{\xi \in \Gamma_1 \cap \Delta_1} [x' \frac{\partial K}{\partial x} - K(x, y)]$$

where

$$\Delta_1 \cap \Gamma_1 = \{\xi | \xi - \frac{\partial K}{\partial x} \leq 0, x \geq 0, \xi \geqq 0\}$$

(2.7) may be written

(2.8) $$\inf_{x} [x' \frac{\partial K}{\partial x} - K(x, y)]$$

$$\frac{\partial K}{\partial x} \geqq 0$$

$$x \geq 0$$

(2.8) can be written

$$(2.9) \qquad - \sup_{x} \left[K(x, y) - x' \frac{\partial K}{\partial x}\right]$$

$$\frac{\partial K}{\partial x} \geq 0$$

$$x \geq 0 .$$

By Fenchel's Theorem 1.1

$$(2.10) \qquad \inf_{x \in X} K(x, y) = \sup_{x \in X} \left[K(x, y) - x' \frac{\partial K}{\partial x}\right]$$

$$\frac{\partial K}{\partial x} \geq 0$$

as long as there exists a feasible solution to the right hand side. Finally we have

$$(2.11) \qquad \sup_{y \in Y} \inf_{x \in X} K(x, y) = \sup_{y \in Y} \sup_{x \in X} \left[K(x, y) - x' \frac{\partial K}{\partial x}\right]$$

$$Y = R^n_+$$

$$\frac{\partial K}{\partial x} \geq 0$$

Considering $K(x, y)$ as a concaved function of y we derive, proceeding as above, the following:

$$(2.12) \qquad \inf_{x \in X} \sup_{y \in Y} K(x, y) = \inf_{x \in X} \inf_{y \in Y} \left[K(x, y) - y' \frac{\partial K}{\partial y}\right]$$

$$\frac{\partial K}{\partial x} \geq 0$$

We thus consider the two dual programming problems

	I	II
(2.13)	$\underset{x,\,y}{\text{Max.}}\ K(x, y) - x' \frac{\partial K}{\partial x}$	$\underset{x,\,y}{\text{Min.}}\ K(x, y) - y' \frac{\partial K}{\partial y}$
	$\frac{\partial K}{\partial x} \geq 0$	$\frac{\partial K}{\partial y} \leq 0$
	$x \geq 0, y \geq 0$	$x \geq 0, y \geq 0$.

When considering duality relations between problems I and II we shall assume that the constraint sets of each problem have non-empty interiors.

Theorem 2.1 [4]

$$\sup_{y \in Y} \sup_{x \in X} \left[K(x, y) - x' \frac{\partial K}{\partial x}\right] \leq \inf_{x \in X} \inf_{y \in Y} \left[K(x, y) - y' \frac{\partial K}{\partial y}\right] \tag{2.14}$$

$$\text{s. t. } \frac{\partial K}{\partial x} \geq 0 \qquad\qquad \text{s. t. } \frac{\partial K}{\partial y} \leq 0$$

Proof

We first note that

$$\sup_{y \in Y} \inf_{x \in X} K(x, y) \leq \inf_{x \in X} \sup_{y \in Y} K(x, y) \tag{2.15}$$

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4. We set $\sup_{y \in Y} \sup_{x \in X} K(x, y) - x' \frac{\partial K}{\partial x} = -\infty$ if the constraint set for this problem is empty. Corresponding $\inf_{x \in X} \inf_{y \in Y} \left[K(x, y) - y' \frac{\partial K}{\partial y}\right] = +\infty$ under the same conditions.

Then using (2.11) and (2.12) we have the result.

q. e. d.

<u>Lemma 2.1</u>

If $\underset{y \in Y}{\text{Max.}}\ \underset{x \in X}{\text{Min.}}\ K(x, y) = K\,(x^0, y^0)$ then $K\,(x^0, y^0)$ is a saddle point of $K(x, y)$. Correspondingly if $\underset{x \in X}{\text{Min.}}\ \underset{y \in Y}{\text{Max.}}\ K(x, y) = K(x^0, y^0)$ then $K(x^0, y^0)$ is a saddle point. Under either hypothesis we have

$$\underset{y \in Y}{\text{Max.}}\ \underset{x \in X}{\text{Min.}}\ K(x, y) = \underset{x \in X}{\text{Min.}}\ \underset{y \in Y}{\text{Max.}}\ K(x, y)$$

<u>Proof</u>

We give the proof for the first statement. Define the closed sets

$$X_z = \{x \mid 0 \leq x_i \leq x_{iz}, \qquad i = 1, \ldots, n\}$$

$$Y_z = \{y \mid 0 \leq y_i \leq y_{iz}, \qquad i = 1, \ldots, m\}$$

and the half-open sets

$$\tilde{X}_z = \{x \mid 0 \leq y_i < y_{iz}, \qquad i = 1, \ldots, n\}$$

$$\tilde{Y}_z = \{y \mid 0 \leq y_i < y_{iz}, \qquad i = 1, \ldots, m\}$$

where $(x^0, y^0) \in \tilde{X}_z \times \tilde{Y}_z$. Since X_z and Y_z are closed, bounded, convex sets, by the min-max theorem for general finite games [5] there exists (x^*, y^*), which is a saddle point for $K(x, y)$ defined on $X_z \times Y_z$. Thus on $X_z \times Y_z$ we have

$$\text{(2.23)} \qquad \underset{x}{\text{Min.}}\ \underset{y}{\text{Max.}}\ K(x, y) = K(x^*, y^*) = \underset{y}{\text{Max.}}\ \underset{x}{\text{Min.}}\ K(x, y)$$

Since by our hypothesis we have

$$\text{(2.24)} \qquad \underset{y}{\text{Max.}}\ \underset{x}{\text{Min.}}\ K(x, y) = K(x^0, y^0)$$

(x^0, y^0) must be a saddle point in $X_z \times Y_z$. The following conditions are necessary and sufficient that (x^0, y^0) be a saddle point:[5]

$$\text{(2.25)} \qquad \frac{\partial K}{\partial x^0} \geq 0 \qquad\qquad \frac{\partial K}{\partial y^0} \leq 0$$

$$x_i^o \frac{\partial K}{\partial x_i^0} = 0 \quad i = 1, \ldots, n \qquad y_i \frac{\partial K}{\partial y_i^0} = 0 \quad i = 1, \ldots, m$$

$$x^0 \geq 0 \qquad\qquad y^0 \geq 0 \, .$$

---

5. These conditions are presented in [7]. Note that $(x^0, y^0) \in \tilde{X}_z \times \tilde{Y}_z$ and thus only lower bounds of the set are relevant.

These conditions are independent of the end points (x_z, y_z) chosen to define $X_z \times Y_z$ and are therefore satisfied for end points (∞, ∞). Therefore (x^0, y^0) is a saddle point on the space $X \times Y$.

Theorem 2.2[6]

If either dual problem possesses a solution then the other problem possesses a solution, and the optimal values of the two problems are equal.

Proof

We give the proof for the case where a solution (x^0, y^0) exists for

$$\text{(2.26)} \qquad \underset{x,\, y}{\text{Max.}} \left[K(x, y) - x' \frac{\partial K}{\partial x}\right]$$

$$\frac{\partial K}{\partial x} \geq 0$$

$$x \geq 0, \quad y \geq 0 .$$

By assumption the constraint set of (2.26) has a nonempty interior so that there exist points $(\hat{x}, \hat{y})$ such that $\underset{y \in Y}{\text{Max.}}\ \underset{x \in X}{\text{Min.}}\ K(x, y) = K(\hat{x}, \hat{y})$. By Lemma 2.2 we have $\underset{x \in X}{\text{Min.}}\ \underset{y \in Y}{\text{Max.}}\ K(x, y) = K(\hat{x}, \hat{y})$. Let Γ_2 and Δ_2 be the

6. This theorem was first proved in [3] under conditions (i), (ii), (iii), and (iv).

domains of definition of the conjugate functions obtained by considering $K(x, y)$ as a function of y. Since $K(\hat{x}, y)$ is bounded above by $K(\hat{x}, \hat{y})$, $\xi = 0$ is in the set $\Gamma_2 \cap \Delta_2$ and therefore $\Gamma_2 \cap \Delta_2 \neq \emptyset$. By Theorem 1.1 and noting that the interior of the set Y is nonempty we have

$$\underset{y \in Y}{\text{Max.}}\ K(\hat{x}, y) = \underset{y \in Y}{\text{Min.}}\ [K(\hat{x}, y) - y' \left(\frac{\partial K}{\partial y}\right)_{\hat{x}}]$$

$$\left(\frac{\partial K}{\partial y}\right)_{\hat{x}} \leq 0$$

and

$$\underset{x \in X}{\text{Min.}}\ \underset{y \in Y}{\text{Max.}}\ K(x, y) = \underset{x \in X,\ y \in Y}{\text{Min.}}\ [K(x, y) - y' \frac{\partial K}{\partial y}]$$

$$\frac{\partial K}{\partial y} \leq 0$$

q. e. d.

Lemma 2.2

If a solution to the program

$$\underset{x,\ y}{\text{Max.}}\ [K(x, y) - x' \frac{\partial K}{\partial x}]$$

$$\frac{\partial K}{\partial x} \geq 0$$

$$x, \geq 0, \qquad y \geq 0$$

exists then there always exists a solution (x^0, y^0) such that

$$x_i^0 \frac{\partial K}{\partial x_i^0} = 0 \qquad i = 1, \ldots, n$$

Correspondingly if a solution to the program

$$\underset{x,\, y}{\text{Min.}} \ [K(x, y) - y' \frac{\partial K}{\partial y}]$$

$$\frac{\partial K}{\partial y} \leq 0$$

$$x \geq 0 \,, \quad y \geq 0$$

exists, then there always exists a solution (x^0, y^0) for which

$$y_i^0 \frac{\partial K}{\partial y_i^0} = 0 \qquad i = 1, \ldots, m$$

<u>Proof</u>

We give a proof for the first part of the lemma. By the assumptions of the lemma

$$\underset{y \epsilon Y}{\text{Max.}} \ \underset{x \epsilon X}{\text{Min.}} \ K(x, y) = K(x^0, y^0) \ .$$

The point x^0 satisfies $x_i^0 \frac{\partial K}{\partial x_i^0} = 0$

$$\frac{\partial K}{\partial x^0} \geq 0$$

and consequently (x^0, y^0) is the required solution to the dual problem.

q. e. d.

Theorem 2.3

If either dual program is solvable then there exists a common solution to both programs.

Proof

Assume that there exists an (x^0, y^0) such that

$$\text{Max. } [K(x, y) - x' \frac{\partial K}{\partial x}] = K(x^0, y^0)$$

$$\frac{\partial K}{\partial x} \geq 0$$

$$x \geq 0, \quad y \geq 0$$

Such a solution is guaranteed by the previous lemma. The point $K(x^0, y^0)$ is a saddle point and consequently (x^0, y^0) is a solution to the dual program.

$$\underset{x, y}{\text{Min. }} K(x, y) - y' \frac{\partial K}{\partial y}$$

$$\frac{\partial K}{\partial y} \leq 0$$

$$x \geq 0, \quad y \geq 0$$

q. e. d.

3. APPLICATION TO PROGRAMMING PROBLEMS

3.1 Linear Programming

Consider

$$px - y(Ax - b) = K(x, y) .$$

Note, of course, that $K(x, y)$ is convex in x and concave in y. Then we have two dual programming problems.

$$\min_{x,\,y}\; px - y(Ax - b) + y(Ax - b) \tag{3.1}$$

$$Ax \geq b$$

$$x \geq 0, \; y \geq 0$$

which gives

$$\min_{x}\; px \tag{3.2}$$

$$Ax \geq b$$

$$x \geq 0$$

and

$$\text{Max. } px - y(Ax - b) - x(p - yA) \tag{3.3}$$

$$p - yA \geq 0$$

$$x \geq 0, \; y \geq 0$$

which gives

$$\text{(3.4)} \qquad \begin{array}{l} \text{Max. } yb \\ yA \leq p \\ y \geq 0 \,. \end{array}$$

From Theorem 2.1 we have that

$$\sup_{y} yb \leq \inf_{x} px^{7}$$

$$\begin{array}{ll} yA \leq p & Ax \geq b \\ y \geq 0 & x \geq 0 \,. \end{array}$$

From Theorem 2.2 we know that if there exists a y^* which solves the maximum problem then there exists a feasible x^* which solves the minimum problem such that

$$y^* b = px^* .$$

The reverse implication is also true.

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7. We may, of course, replace sup with max and inf with min if the problems possess feasible solutions.

3.2 Nonlinear Programming

Let

$$K(x, y) = f(x) + y' g(x)$$

where

$$g(x) = \begin{vmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{vmatrix}$$

i.e., $g(x)$ is a vector of convex function, and $f(x)$ is convex. We obtain the following dual programming problems from 2.13.[8]

Min. $f(x)$	Max. $f(x) + y' g(x) - x' [\frac{\partial f}{\partial x} + y \frac{\partial g}{\partial x}]$
s. t. $g(x) \leq 0$	s. t. $\frac{\partial f}{\partial x} + y' \frac{\partial g}{\partial x} \geq 0$
$x \geq 0$	$x \geq 0, \qquad y \geq 0$.

Applying Theorem 2.2 we have that if either problem has an optimal solution, then the other problem has an optimal solution and the values of the criterion functions are equal.

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8. We may assume here that the constraint set has a non-empty interior. See [9] for this type of duality theorem.

4. CONSTRAINED GAMES[9]

We now consider more general constraints on the space of strategies. Let

$$X = \{x | x \geq 0,\ a(x) \geq 0\}$$
$$Y = \{y | y \geq 0,\ h(y) \leq 0\}$$
$$a(x) = [a_1(x) \ldots a_e(x)]'$$
$$h(y) = [h_1(y) \ldots h_k(y)]' .$$

$a(x)$ is a vector of concave continuously differentiable functions and $h(y)$ is a vector of convex continuously differentiable functions. The sets X and Y have non-empty interiors.

$$\nabla a(x) = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial a_e}{\partial x_1} & \cdots & \frac{\partial a_e}{\partial x_n} \end{bmatrix}$$

$$\nabla h(y) = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial h_k}{\partial y_m} & & \frac{\partial h_k}{\partial y_m} \end{bmatrix}$$

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9. The topic of constrained games was first introduced by A. Charnes in [1] and further developed in [2].

Thus we consider the game $K(x, y)$ where player one must choose an $x \in X$ and player two a $y \in Y$. In order to motivate the choice of dual problems related to the constrained game, we apply Fenchel's conjugate function theory to $K(x, y)$.

We write the constraint $a(x_1) - x_2 = 0$ where $x_2 = (x_{21}, \ldots, x_{2n})'$, $x_1 = (x_{11}, \ldots, x_{1n})'$, $C = \{x_1, x_2 \mid a(x_1) - x_2 = 0, x_1 \geq 0\}$ Let $\xi_1 = (\xi_{11}, \ldots, \xi_{1n})'$, $\xi_2 = (\xi_{21}, \ldots, \xi_{2m})'$

(4.1)
$$\varphi(\xi) = \sup_{\substack{a(x_1)-x_2 = 0 \\ x_1 \geq 0, \quad x_2 \geq 0}} [x_1' \xi_1 + x_2' \xi_2 - K(x_1, y)]$$

$$\Gamma = \{\xi_1, \xi_2 \mid \sup_{\substack{a(x_1)-x_2 = 0 \\ x_1 \geq 0, \quad x_2 \geq 0}} x_1'\xi_1 + x_2'\xi_2 - K(x_1, y) < \infty\} .$$

Conditions for $\varphi(\xi)$ to have a maximum are that:

(4.2)
$$\xi_1 \leq \frac{\partial K}{\partial x} - u' \nabla a(x_1)$$

$$\xi_2 \leq u$$

$$x_{1i} \, \xi_{1i} = x_{1i} \, [(\frac{\partial K}{\partial x_i} - u' \nabla a \, (x_{1i})]$$

$$i = 1, \ldots, n$$

$$\xi_{2i} x_{2i} = u_i x_{2i} \qquad i = 1, \ldots, m$$

$$a(x_1) - x_2 = 0$$

$$x_1, x_2 \geq 0 .$$

Using these conditions we may write

(4.3) $$\varphi(\xi) = x_1' \left[\frac{\partial K}{\partial x} - u' \nabla a (x_1) \right] - K(x_1, y) + u' a(x_1)$$

where (x_1, x_2) and (ξ_1, ξ_2) are related by conditions of 4.2

Let

$$D = \{x_1, x_2 \mid x_1 \geqq 0, x_2 \geqq 0\}$$

Then

(4.4) $$\psi(\xi) = \inf_{x_1 \geqq 0, x_2 \geqq 0} [x_1' \xi_1 + x_2' \xi_2]$$

$$\Delta = \{\xi_1, \xi_2 \mid \inf_{x_1 \geqq 0, x_2 \geqq 0} x_1' \xi_1 + x_2' \xi_2 > -\infty\}$$

Thus

$$\Delta = \{\xi_1 ' \xi_2 \mid \xi_1 \geqq 0, \xi_2 \geqq 0\}$$

and

$$\psi(\xi) = 0 \tag{4.5}$$

where

$$x_1'\xi_1 = 0 \quad x_2'\xi_2 = 0 \ .$$

We may write

$$\Gamma \cap \Delta = \{x_1, x_2 \mid 0 \leq \frac{\partial K}{\partial x} - u'\nabla a(x_1), \ u \geq 0, \ a(x_1) \geqq 0, \ x_1 \geq 0, \ x_2 \geq 0\}.$$

We thus have computing

$$\varphi(\xi) - \psi(\xi) \tag{4.6}$$

the expression

$$x' \left[\frac{\partial K}{\partial x} - u'\nabla a(x)\right] - K(x, y) + u'a(x) \tag{4.7}$$

$$0 \leq \frac{\partial K}{\partial x} - u'\nabla a(x) \qquad a(x) \geqq 0$$

$$u \geq 0, \quad x \geq 0$$

where we may drop the variable x_2 and write $x_1 = x$. We may consider the following dual programming problems:

I

$$\operatorname*{Max.}_{x,\, y,\, u} \ K(x, y) - x' \left[\frac{\partial K}{\partial x} - u'\nabla a(x)\right] - u'a(x) \tag{4.8}$$

$$\text{s. t.} \quad \frac{\partial K}{\partial x} - u'\nabla a(x) \geq 0$$

$$h(y) \leq 0, \quad a(x) \geqq 0$$

$$x \geq 0,\ y \geq 0,\ u \geq 0 \ .$$

II

(4.9) $$\underset{x,\ y\ \lambda}{\text{Min.}}\ K(x, y) - y' \left[\frac{\partial K}{\partial x} - \lambda'\nabla h(y) \right] - \lambda' h(y)$$

$$\frac{\partial K}{\partial x} - \lambda'\nabla h(y) \leq 0$$

$$a(x) \geq 0, \quad h(y) \leqq 0$$

$$x \geq 0, \quad y \geq 0, \quad \lambda \geq 0 \ .$$

Theorem 4.1

If either dual problem has a solution then the other problem possesses a solution and the values of the criterion functions are equal.

Proof

The proof proceeds in a manner similar to section two and we omit the details.

Example: Let

$$K(x, y) = - \sum_{ij} p_i a_{ij} q_j$$

$$\text{s. t. } \sum_i p_i = 1, \quad \sum_j q_j = 1$$

(4.10) $$\sum_i p_i d_{is} \geq d_s, \; \sum_j b_{rj} q_j \leq b_r$$

$$p_i \geq 0, \; q_j \geq 0 \; .$$

Then we have the following dual programming problems:

I

(4.11) $$\underset{p_i,\, q_j,\, x_s}{\text{Max.}} \quad -\sum p_i a_{ij} q_j - \sum_i p_i \Big(-\sum_j a_{ij} q_j - \delta - \sum_s x_s d_{is}\Big)$$

$$-\Big(\sum_s x_s \sum_i p_i d_{is} - d_s\Big)$$

$$\text{s. t. } -\sum_j a_{ij} q_j - \delta - \sum_s x_s d_{is} \geq 0$$

$$\sum_j q_j = 1$$

$$\sum_j b_{rj} q_j \leq b_r$$

$$q_j \geq 0, \; p_i \geq 0, \; x_s \geq 0 \; .$$

II

(4.12) $$\underset{p_i,\, q_j,\, w_r}{\text{Min.}} \quad -\sum_{ij} p_i a_{ij} q_j - \sum_j q_j \Big(-\sum_i a_{ij} p_i - \rho - \sum_r w_r b_{rj}\Big)$$

$$-\sum_r w_r \Big(\sum_j b_{rj} q_j - b_r\Big)$$

$$\text{s. t.} -\sum_i a_{ij}p_i - \rho - \sum_r w_r b_{rj} \leq 0$$

$$\sum_i p_i d_{is} \geq d_s$$

$$p_i \geq 0,\ q_j \geq 0,\ w_r \geq 0\ .$$

Combining terms we have[10]

I

(4.13)
$$\text{Max. } \delta + \sum x_s d_s$$

$$\text{s. t. } \sum_j a_{ij}q_j + \delta + \sum_s x_s d_{is} \leq 0$$

$$\sum_j q_j = 1$$

$$\sum_j b_{rj}q_j \leq b_r$$

$$q_j \geq 0,\ x_s \geq 0$$

II

(4.14)
$$\text{Min. } \rho + \sum_r w_r b_r$$

$$\text{s. t. } \sum_i a_{ij}p_i + \rho + \sum_r w_r b_{rj} \geq 0$$

$$\sum_i p_i = 1$$

$$\sum_i p_i d_{is} \geq d_s$$

$$p_i \geq 0,\ w_r \geq 0\ .$$

10. See page 774 of [2] for the following two dual programs.

5. FURTHER EXTENSIONS

In this section we shall indicate how the duality conditions discussed earlier, may be generalized to cases where $K(x, y)$ is not necessarily differentiable everywhere.[11] This will indicate that the concept of symmetric duality is not limited to differentiable functions.

Considering $K(x, y)$ as a convex function of x and recalling the earlier definitions and Theorem 1.1 in section 1 we have

$$\text{(5.1)} \qquad \underset{x \geq 0}{\text{Min.}}\; K(x, y) = \underset{\xi \in \Gamma_1 \cap \Delta_1}{\text{Max.}} \quad \underset{x \geq 0}{\text{Min.}}\; [K(x, y) - x'\xi]$$

We assume that the relative interior of $\Gamma \cap \Delta \neq \phi$. Note that $\Delta_1 = E^n_+$ from 2.6, and thus we write

$$\xi \in \Gamma_1 \cap E^n_+$$

<u>Lemma 5.1</u>

If a solution exists to

$$\text{(5.2)} \qquad \underset{\xi \in \Gamma_1 \cap E^n_+}{\text{Max.}} \quad \underset{x \geq 0}{\text{Min.}}\; [K(x, y) - x'\xi]$$

---

11. It is well known that $K(x, y)$ is differentiable except at most on a denumerable set of points.

then

$(\hat{x}, \hat{\xi})$ where $\hat{x}'\hat{\xi} = 0$ is a maximal solution.

Proof

Suppose the theorem is not true. Since (5.2) is assumed solvable then a solution to $\underset{x \geq 0}{\text{Min.}}\ K(x, y)$ exists. It is clear that $\hat{x}$ is such a solution.

By the Fenchel duality theorem we have for some value of y

$$(5.3) \qquad K(\hat{x}, y) = K(x^*, y) - x^{*\prime}\xi^*$$

where (x^*, ξ^*) is a solution to (5.2) and $x^{*\prime}\xi^* > 0$. Since $(\hat{x}, \hat{\xi})$ is assumed nonoptimal we have

$$K(x^*, y) - x^{*\prime}\xi^* > K(\hat{x}, y)$$

which is a contradiction.

q. e. d.

We may state the following dual programming problems:

$$(5.4) \qquad \underset{y \geq 0}{\text{Max.}}\ \underset{\xi \in \Gamma_1 \cap E_+^n}{\text{Max.}}\ \underset{x \geq 0}{\text{Min.}}\ K(x, y) - x'\xi$$

$$(5.5) \qquad \underset{x \geq 0}{\text{Min.}}\ \underset{\eta \in \Delta_2 \cap E_+^m}{\text{Min.}}\ \underset{y \geq 0}{\text{Max.}}\ K(x, y) - y'\eta$$

The set Δ_2 is determined when considering $K(x, y)$ as a concave function of y. If either dual problem possesses a solution we assume that the relative interior of the respective sets

$$\Gamma_1 \cap E_+^n \text{ or } \Delta_2 \cap E_+^m \text{ is nonvoid.}$$

Theorem

If either of the dual problems possesses a solution then the other problem possesses a solution.

Proof

Utilizing Lemma 5.1 we proceed exactly as in section two.

q. e. d.

REFERENCES

[1] Charnes, A., "Constrained Games and Linear Programming." Proceedings of the National Academy of Sciences, July 1953, pp. 639-641.

[2] Charnes, A. and W. W. Cooper, Management Models and Industrial Applications of Linear Programming. John Wiley and Sons, 1961. Chapter 19 and 20.

[3] Dantzig, G., E. Eisenberg and R. W. Cottle, "Symmetric Dual Nonlinear Programs." Research Report 30, Operations Research Center, University of California, Berkeley.

[4] Fenchel, W., Convex Cones, Sets and Functions. Princeton University Lecture Notes, Spring 1963.

[5] Kakutani, S. "A Generalization of Brouwer's Fixed Point Theorem." Duke Mathematical Journal, 1941, pp. 457-458.

[6] Karlin, S. Mathematical Methods and Theory in Games Programming and Economics. Addison-Wesley, 1959

[7] Kuhn, H. and A. W. Tucker, "Nonlinear Programming." Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, University of California Press, 1951, pp. 481-482.

[8] Whinston, A., "Conjugate Functions and Duality Theorems in Nonlinear Programming." Report 6305, Econometric Institute, Rotterdam.

[9] Wolfe, P., "A Duality Theorem for Nonlinear Programming." Quarterly of Applied Mathematics, Vol. 19 (1961), pp. 239-244.

BIBLIOGRAPHY

Charnes, A., W. W. Cooper and K. Kortanek, "Duality, Haar Programs and Finite Sequence Spaces," Proc. Nat. Acad. Science, Vol. 48, No. 5, (May 1962). pp. 783-786.

Dorn, W. S., "Duality in Quadratic Programming," Quarterly Applied Mathematics, Vol. 18, (1960). pp. 155-162.

Dorn, W. S., "A Symmetric Dual Theorem for Quadratic Programs," Journal Operations Research Society of Japan, Vol. 2, (1960). pp. 93-97.

Dorn, W. S., "A Duality Theorem for Convex Programs," IBM Journal of Research and Development, Vol. 4 (1960). pp. 407-413.

Hanson, M. A., "A Duality Theorem for Nonlinear Programming with Nonlinear Constraints," Australian Journal of Statistics, Vol. 3 (1961). pp. 64-72.

Huard, P., "Dual Programs," IBM Journal of Research and Development, Vol. 6, (1962). pp. 137-139.

Kretschmer, K. "Programmes in Paired Spaces" Can. Jour. Math, (1961) pp. 221-238.