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Non-Existence of Consistent Estimator Sequences and Unbiased Estimates: A Practical Example

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Correction sheet for CFDP No. 145

Page 1. First sentence of introduction: insert "for each sample size" after identifiability.

Last sentence: "distribution" should read "distributions"

Add the sentence: "The bracket indicates the parameters used to describe the family; μ ranges over the entire real line and the range of σ^2 and ρ is given in (2) below."

Page 3. (3) read ρ for p .

Page 10. To the sentence in the third line from the bottom append the footnote: "An essential role is played by the fact that τ^2 is identically zero if $\rho = 0$ but varies over the entire positive axis when $\rho \neq 0$."

Page 11. Middle of the page: refer to [6], not [2].

Page 12. First line: "we have, writing τ_{in}^2 for $\tau^2(i) + \mathcal{K}^2 n^{-1}$."
The title of section 6 is "A transformation..."

Page 13. Second line: omit "where $\underset{\sim}{x}_p = (x_{p1}, \ldots, x_{pn})$" .

Page 14. In the text following the first displayed formula insert V after $(m-k)^{-1}$.

After second displayed formula insert: "Note that these estimates are precisely the ones obtained for $\underset{\sim}{C}$ and $\mathcal{K}^2$ when $\rho = 0$."

NON-EXISTENCE OF CONSISTENT ESTIMATOR SEQUENCES AND UNBIASED ESTIMATES: A PRACTICAL EXAMPLE

Hendrik S. Konijn

0. Introduction and Summary

It is often thought that identifiability implies existence of consistent estimator sequences. A rather artificial counter example is given in [7]. We here consider a case which often arises in experimental and survey practice. The example concerns a model with intraclass correlation ρ . For ρ negative an indefinitely large sequence of observations cannot arise from such a model and so the discussion of consistency is restricted to $\rho \geq 0$. For any non-degenerate range of ρ we show that no unbiased estimate exists for the variance of the mean of the observations.

Certain other aspects of estimation in models of this sort are considered in [4].

1. Some families of distributions

Let $(y_1, \ldots, y_n)$ be multivariate nonsingular normal with $n > 1$, and with

$$\mu = \mathcal{E} y_j , \quad \sigma^2 = \mathcal{E}(y_j - \mu)^2 , \quad \rho = \mathcal{E}(y_j - \mu)(y_{j'} - \mu)/\sigma^2 \ (j \neq j')$$

unknown constants. We shall refer to the corresponding family of distribution as $\mathcal{N}_n[\mu, \sigma^2, \rho]$.

Such a family arises naturally in the study of experimental results [1] and in sample survey situations [2],[3]. It also seems natural in the study of interdependence among small clusters of individuals (such as arises in ecology, sociometry, and so forth) or of objects in a plane or space, being the simplest of a class of families in which the covariance of two elements is a function of their distance. Another extension is considered in section 6.

The joint density function is easily shown to have its logarithm proportional to

$$(1) \qquad c_1 - n \log \sigma^2 - m \log(1-\rho) - \log(1+m\rho)$$
$$- \sigma^{-2} \left[\Sigma(y_j - \mu)^2 (1-\rho)^{-1} - (\bar{y}-\mu)^2 n \left\{ (1-\rho)^{-1} - (1+m\rho)^{-1} \right\} \right],$$

where c_1 is a known constant and

$$m = n - 1 \quad .$$

From (1) it is seen that nonsingularity of the distribution implies

$$(2) \qquad \sigma^2 > 0 \, , \qquad -m^{-1} < \rho < 1 \, ;$$

-- an amusing interpretation of this arises when all members of a group of individuals make a conscious attempt to be nonconformists; if the model is appropriate the amount of possible nonconformity as measured by $-\rho$ appears limited by the size of the group.

It is well known (see section 6) that

$$\bar{y} \quad \text{and} \quad V = \Sigma(y_j - \bar{y})^2$$

are independently distributed and that their density functions have logarithm proportional to

(3) $$c_2 - \log \sigma^2 - \log (1 + mp) - \sigma^{-2} (\bar{y} - \mu)^2 n(1 + m\rho)^{-1}$$

and

(4) $$c_3 - m \log \sigma^2 - m \log (1-\rho) + (m-2) \log V - \sigma^{-2} (\Sigma y_j^2 - \bar{y}^2 n) (1-\rho)^{-1} \quad ,$$

so that the conditional density of $(y_1, \ldots, y_n)$ given $(\bar{y}, V)$ has logarithm proportional to

(5) $$c_4 - (m - 2) \log V \ ,$$

which does not depend on μ , σ^2 or ρ

and so $(\bar{y}, V)$ is sufficient for the family $\mathcal{N}_n[\mu, \sigma^2, \rho]$.

Since $\sqrt{n}(\bar{y}-\mu)$ has a normal distribution with zero mean and variance

(6) $$\omega^2 = \sigma^2(1 + m\rho)$$

and V/κ^2 has a chi-square distribution with m degrees of freedom with

(7) $$\kappa^2 = \sigma^2(1 - \rho) ,$$

it is also convenient to consider a parametrization of $\mathcal{N}_n[\mu, \sigma^2, \rho]$ by μ , κ^2 and ω^2 with (2) replaced by the

(2') $$\kappa^2 > 0 , \quad \omega^2 > 0.$$

Frequently [1] we are really interested in estimating ω^2 and κ^2 rather than σ^2 and ρ .

In some problems it is possible to replace by

(2*) $$\sigma^2 > 0 , \quad \rho \geq 0 ;$$

we shall refer to that subfamily of $\mathcal{N}_n[\mu, \sigma^2, \rho]$ as $\mathcal{N}_{no}[\mu, \sigma^2, \rho]$.

2 Nonexistence of an unbiased estimate of ω^2

If f is a function of the observations and $\mathcal{E} f(y_1, \ldots, y_n)$ exists (as a Lebesgue integral), then the conditional expectation $\mathcal{E}\left\{f(y_1, \ldots, y_n) \mid \bar{y}, V\right\}$ exists, and, by the sufficiency of $(\bar{y}, V)$ does not depend on the parameters; call it $g(\bar{y}, V)$. Since the distribution of V does not depend on μ ,

$$g_o(\bar{y} \mid \kappa^2) = \mathcal{E}\left\{g(\bar{y}, V) \mid \bar{y}\right\}$$

exists and is not a function of μ .

So, if $f(y_1, \ldots, y_n)$ is an unbiased estimate of ω^2 , then, for each positive number κ_0^2, $\mathcal{E}\, g_o(\bar{y} \mid \kappa_0^2)$ equals ω^2 identically in μ and ω^2 . Consequently, writing $h(\bar{y}\, n^{\frac{1}{2}}) = g_o(\bar{y} \mid \kappa_0^2)$, z for $\bar{y}\, n^{\frac{1}{2}}$ and ν for $\mu\, n^{\frac{1}{2}}$,

$$\mathcal{E}\, h(z) = (2 \pi \omega^2)^{-\frac{1}{2}} \int h(z) \exp\left\{-\frac{1}{2}(z - \nu)^2 \omega^{-2}\right\} dz$$

identically in μ^2 and ω^2 . That would mean that there would exist an unbiased estimate of the variance ω^2 of a normal distribution with unknown mean ν based on a single observation. That this is not so is proved in [5] .

3. Sequences of families

In discussing asymptotic properties one also has to consider infinite sequences $\mathcal{N}[\mu, \sigma^2, \rho]$ of families $\mathcal{N}_n[\mu, \sigma^2, \rho]$ for $n = 2, 3, \ldots$. It should be noted that in such a sequence the second part of (2) is not tenable when ρ is taken to be constant throughout the sequence, i.e., that case (2) must be replaced by (2*). Therefore in this case the study of asymptotic properties is without sense, and we have to confine ourselves to the study of fixed sample size properties.

Alternatively, we can consider

(a) the sequence $\mathcal{N}'[\mu, \sigma^2, \lambda]$ of families $\mathcal{N}_n[\mu, \sigma^2, a_n(\lambda)]$ for $n = 2, 3, \ldots,$ where $a_2(\lambda)$, $a_3(\lambda), \ldots$ is a sequence of fully specified functions of a single unknown parameter λ .

(b) the sequence $\mathcal{N}''[\mu, \sigma^2, \Lambda]$ of families $\mathcal{N}_n[\mu, \sigma^2, b_n(\Lambda)]$ for $n = 2, 3, \ldots,$ where $b_2(\Lambda), b_3(\Lambda), \ldots$ is a sequence of one-to-one functions of an ordered set Λ of at least two independent parameters, which cannot be represented as one-to-one functions of a single parameter.

Of course, $a_n(\lambda)$ and $b_n(\Lambda)$ must depend on n and must satisfy the second part of (2) for each n . The usual case of $\mathcal{N}''[\mu, \sigma^2, \Lambda]$ is the one in which for any n , the function b_n equals a quantity $\rho(n)$

of which we only know that it lies in the range specified in the second part of (2).

Similarly we can consider $\mathcal{N}'_o[\mu, \sigma^2, \lambda]$ or $\mathcal{N}''_o[\mu, \sigma^2, \Lambda]$.

4. Identifiability

The logarithm of the characteristic function of $(y_1, \ldots, y_n)$ is

$$(8) \qquad \psi(t_1, \ldots, t_n \mid \mu, \sigma^2, \rho) = \log \epsilon \exp (i \Sigma y_j t_j)$$

$$= i \mu n \bar{t} - \frac{1}{2} \sigma^2 (\Sigma t_j^2 + \rho \Sigma\Sigma t_j t_{j'}) \qquad (j \neq j') .$$

Consider a collection of specified functions $q, r, \ldots$ of the parameters. Necessary and sufficient for the identifiability of this collection in $\mathcal{N}_n[\mu,\sigma^2, \rho]$ is that for any two sets $(\mu_1, \sigma_1^2, \rho_1)$ and $(\mu_2, \sigma_2^2, \rho_2)$ of values of the parameters the identity over n space:

$$(9) \qquad \psi (t_1, \ldots, t_n \mid \mu_1, \sigma_1^2, \rho_1) \equiv \psi (t_1, \ldots, t_n \mid \mu_2, \sigma_2^2, \rho_n)$$

can hold if and only if all the functions $q, r, \ldots$ take on the same value for $(\mu_1, \sigma_1^2, \rho_1)$ and $(\mu_2, \sigma_2^2, \rho_2)$.

Suppose, for example, that $q(\mu, \sigma^2, \rho) = \mu$, $r(\mu, \sigma^2, \rho) = \sigma^2$ and $s(\mu, \sigma^2, \rho) = \rho$. For $t_2 = \ldots = t_n = 0$ and $t_1 \neq 0$, the real part of

(9) implies that $\sigma_1^2 = \sigma_2^2$, and the imaginary part that $\mu_1 = \mu_2$. This reduces (9) to the identity

$$(10) \qquad (\rho_1 - \rho_2) \; \Sigma \Sigma \, t_j \, t_{j'} \equiv 0$$

after division by the common, negative value of $-\frac{1}{2}\sigma^2$. By selecting any nonzero values for t_1 and t_2, and (if $n > 2$) setting $t_3 = \ldots = t_n = 0$, this yields $\rho_1 = \rho_2$. So $\{q, r, s\}$ is identifiable in $\mathcal{N}_n[\mu, \sigma^2, \rho]$.

It follows at once from the definition of identifiability that $\{q, r, s\}$ is also identifiable in $\mathcal{N}_{no}[\mu, \sigma^2, \rho]$ and that any collection of functions of (μ, σ^2, ρ) which depends on μ, σ^2, and ρ only through the value of (q, r, s) is identifiable in $\mathcal{N}_n[\mu, \sigma^2, \rho]$ and $\mathcal{N}_{no}[\mu, \sigma^2, \rho]$. Specifically if $t(\mu, \sigma^2, \rho) = \kappa^2$, defined in (7), and $u(\mu, \sigma^2, \rho) = \omega^2$, defined in (6), t and u are functions of r and s alone, and so $\{q, t, u\}$ is identifiable in $\mathcal{N}_n[\mu, \sigma^2, \rho]$. We can also show this directly: The right hand side of (5) can be written as

$$i \, \mu \, n \, \bar{t} - \frac{1}{2} \, \kappa^2 \, (\Sigma t_j^2 - n \, \bar{t}^2) - \frac{1}{2} \, \omega^2 \, n \, \bar{t}^2 \; .$$

Thus for $t_2 = \ldots = t_n = 0$ and $t_1 \neq 0$, the identity corresponding to (9) yields $\mu_1 = \mu_2$ and $\kappa_1^2 = \kappa_2^2$, and on substitution of these equalities becomes

$$-\frac{1}{2}(\omega_1^2 - \omega_2^2)\, n\bar{t}^2 \equiv 0$$

so that also $\omega_1^2 = \omega_2^2$.

Now consider the family $\mathcal{N}_n[\mu, \sigma^2, a_n(\lambda)]$ defined in section 3 under (a). We see at once that, if $v(\mu, \sigma^2, \lambda) = \lambda$, $\{q, r, v\}$ is identified in this family.

Let us proceed to $\mathcal{N}_n[\mu, \sigma^2, b_n(\Lambda)]$ defined in section 3 under (b). Let $w(\mu, \sigma^2, \Lambda)$ depend effectively on at least two components of the sequence Λ and not be definable in the form $\bar{w}(\mu, \sigma^2, b_n(\Lambda))$. Then evidently $\{w\}$ is not identifiable in the family, since (10) will lead to the identifiability of $b_n(\Lambda) = \rho(n)$ only for that value of n which coincides with the particular size of the sample that was taken. On the other hand, it was shown in the second paragraph of this section that $\{q, r, s\}$ is identifiable in $\mathcal{N}_n[\mu, \sigma^2, b_n(\Lambda)]$: $s\big(\mu, \sigma^2, b_n(\Lambda)\big) = b_n(\Lambda)$, $b_n = \rho(n)$.

5. Non-existence of a consistent estimator sequence for ρ in $\mathcal{N}_0[\mu, \sigma^2, \rho]$.

If σ^2 and ρ are constant then ω^2 depends on n; in particular it equals $\omega_n^2 = \kappa^2 + n\,\sigma^2\rho$. By a consistent sequence of estimators of ω_n^2 is meant a sequence of functions f'_n of the observations such that for all $\mathcal{E} > 0$

$$\text{(11)} \qquad \lim \Pr\left\{ |f'_n(y_1, \ldots, y_n) - \omega_n^2| > \mathcal{E} \right\} = 0 .$$

If such a sequence exists then there also exists a sequence of functions f''_n with

$$\lim \Pr\left\{ |f''_n(y_1, \ldots, y_n) - \tau_n^2| > \mathcal{E} \right\} = 0 ,$$

where

$$\tau_n^2 = n^{-1}\omega_n^2 .$$

Moreover, writing

$$\tau^2 = \lim \tau_n^2 = \sigma^2 \rho ,$$

we also have

$$\text{(12)} \qquad \lim \Pr\left\{ | f''_n(y_1, \ldots, y_n) - \tau^2 | > \mathcal{E} \right\} = 0 .$$

We shall show that no such sequence exists. Note that not only $\mathcal{K}^2$ and τ_n^2, but also $\mathcal{K}^2$ and τ^2, or $\mathcal{K}^2$ and ρ are independent parameters; it follows that no consistent estimator sequence exists for ρ.

To show that no sequence satisfying (12) exists, we first change variables from $(y_1, \ldots, y_n)$ to $(z_1, \ldots, z_n)$ with $z_1 = n^{\frac{1}{2}} \bar{y}$ and the other components $\underset{\sim}{z}^*$ having a distribution $\varphi^*_{n-1}(\underset{\sim}{z}^* \mid \mathcal{K}^2)$ independent of μ and τ_n^2 or τ^2. That this can be done follows from (1) and (2) of section 2 and is shown more explicitly in section 6. Since the z's depend on n, we show this more explicitly; in particular denote $n^{-\frac{1}{2}} z_1$ by $\bar{y}_n$ and $n^{-\frac{1}{2}} \underset{\sim}{z}^*$ by $\underset{\sim}{z}^*_{n-1}$. Then the joint density φ_n of $\bar{y}_n$ and $\underset{\sim}{z}^*_{n-1}$ is

$$\varphi_n (\bar{y}_n, \underset{\sim}{z}^*_{n-1} \mid \mu, \mathcal{K}^2, \tau_n^2)$$

$$= (2\pi\lambda_n^2)^{-\frac{1}{2}} \exp\left\{- \tfrac{1}{2}(\bar{y}_n-\mu)^2 \tau_n^{-2}\right\} \varphi^*_{n-1} \left(n^{\frac{1}{2}} \underset{\sim}{z}^*_{n-1} \mid \mathcal{K}^2\right) n^{\frac{1}{2}} .$$

Now it is shown in [2] that when τ can take two different values $\tau(1)$ and $\tau(2)$, then for a sequence of functions f_n over $(\bar{y}_n, \underset{\sim}{z}^*_{n-1})$ to be consistent estimator sequence of $\tau(1)$ and $\tau(2)$ it is necessary that, for $n \to \infty$,

$$\Delta^2(\tau^2(1) + \mathcal{K}^2 n^{-1}, \tau^2(2) + \mathcal{K}^2 n^{-1})$$

$$= \iint \left\{ \varphi_n(\bar{y}_n, \underset{\sim}{z}^*_{n-1} \mid \mu, \mathcal{K}^2, \tau^2_{1n}) \, \varphi_n(\bar{y}_n, \underset{\sim}{z}^*_{n-1} \mid \mu, \mathcal{K}^2, \tau^2_{2n})\right\}^{\frac{1}{2}} d\bar{y}_n \, d\underset{\sim}{z}^*_{n-1}$$

converges to zero. Since the integral with respect to $\underset{\sim}{z}^*_{n-1}$ gives unity, we have, writing τ_{in}^2 for $\tau^2(1)+K^2 n^{-1}$

$$\Delta^2(\tau_{1n}^2,\tau_{2n}^2) = (2\pi\,\tau_{1n}\tau_{2n})^{-\frac{1}{2}} \int \exp\left\{ -\frac{1}{2}(\bar{y}_n-\mu)^2\,\bar{\tau}_n^{-2}\right\} d\bar{y}_n$$

$$= \bar{\tau}_n\,(\tau_{1n}\,\tau_{2n})^{-\frac{1}{2}}$$

with

$$\bar{\tau}_n^{-2} = \frac{1}{2}(\tau_{1n}^{-2}+\tau_{2n}^{-2}) = \frac{1}{2}(\tau_{1n}^2+\tau_{2n}^2)\,\tau_{1n}^{-2}\,\tau_{2n}^{-2} \quad ,$$

so that

$$\Delta^2(\tau_{1n}^2,\ \tau_{2n}^2) = 2\,\tau_{1n}\,\tau_{2n}\,(\tau_{1n}^2+\tau_{2n}^2)\ .$$

So for $\tau(1)$ and $\tau(2)$ both positive, $\Delta^2(\tau_{1n}^2,\ \tau_{2n}^2)$ does not converge to zero. So there exist no consistent estimator sequences for τ when the τ can take on any two positive values and consequently none for τ when τ can be any nonnegative number or for ρ when ρ is nonnegative.

6. A transfunction and an extension.

We have used here the fact that $\bar{y}$ and V are independently distributed according to (3) and (4). This was shown by Walsh [9], but an examination of his proof shows that his argument is valid for ρ independent of n only if $\rho \geq 0$. Another argument, valid for the range (2), was given by Stuart [8]. It may be desirable, however, to give a more direct proof, and at the same time consider a more general form of problem.

For that we change the assumption $\mathcal{E} y_j = \mu$ to $\mathcal{E} y_j = \mu + \Sigma_{p=1}^{k} C_p x_{pj}$ where $x_p = (x_{p1}, \ldots, x_{pn})$ with $p \leq k$ and $0 \leq k < n-1$. Here the x_p are fixed and known, linearly independent vectors; without loss of generality we assume that for each p the components of x_p add to zero. Let

$$x_p = (x_{p1}, \ldots, x_{pn}), \quad x' = [x_1', \ldots, x_k'], \quad C = (C_1, \ldots, C_k) .$$

Like in the case in which ρ is known to vanish, our objectives are attained by using a Helmert matrix H, viz., an orthogonal matrix with each element in the first column equal to $n^{-\frac{1}{2}}$ and with the other columns having sum of elements equal to 0. For $z = y H$, $z_1 = n^{\frac{1}{2}} \bar{y}$ and the covariance matrix of z is

$$H' u' u H = \sigma^2 H' \left\{ \rho(1\ldots1)' (1\ldots1) + (1-\rho) I \right\} H$$

$$= \sigma^2 \left[\begin{array}{c|c} 1 + m\rho & 0 \\ \hline 0' & (1-\rho) I \end{array} \right]$$

Since the rows of x add to 0, the first column of $x H$ is a zero column; call the remaining columns x^*. If we denote $(z_2, \ldots, z_n)$ by z^*, we have: z_1 and z^* are independently and normally distributed, the former with mean $\nu = n^{\frac{1}{2}} \mu$ and variance $\omega^2 = \sigma^2(1 + m\rho)$, the latter with a vector mean $C x^*$ and covariance matrix $\kappa^2 I$ with $\kappa^2 = \sigma^2(1-\rho)$, and ν and the components of C range over the entire real line while ω^2 and κ^2 range over the entire positive line.

The analysis of $\underset{\sim}{z}^*$ is an ordinary regression problem (through the origin); for example, the minimum variance linear unbiased estimate of $\underset{\sim}{C}$ is

$$\underset{\sim}{C}^o = \underset{\sim}{z}^* \underset{\sim}{x}^{*\prime} (\underset{\sim}{x}^* \underset{\sim}{x}^{*\prime})^{-1} = [z_1 \vdots \underset{\sim}{z}^*] [\underset{\sim}{Q}' \vdots \underset{\sim}{x}^*]' \left\{ [\underset{\sim}{Q}' \vdots \underset{\sim}{x}^*] [\underset{\sim}{Q}' \vdots \underset{\sim}{x}^*]' \right\}^{-1}$$

$$= \underset{\sim}{z}(\underset{\sim}{x} \underset{\sim}{H})' (\underset{\sim}{x} H H' \underset{\sim}{x}')^{-1} = \underset{\sim}{y} \underset{\sim}{x}' (\underset{\sim}{x} \underset{\sim}{x}')^{-1} ,$$

and the usual estimate of $\mathcal{H}^2$ is $(m - k)^{-1}$ with

$$V = \| \underset{\sim}{z}^* - \underset{\sim}{C}^o \underset{\sim}{x}^* \|^2 = \underset{\sim}{z} \left\{ \underset{\sim}{I} - \underset{\sim}{x}^{*\prime} (\underset{\sim}{x}^* \underset{\sim}{x}^{*\prime})^{-1} \underset{\sim}{x}^* \right\} \underset{\sim}{z}' - z_1^2$$

$$= \underset{\sim}{y} \left\{ \underset{\sim}{I} - \underset{\sim}{x}' (\underset{\sim}{x} \underset{\sim}{x}')^{-1} \underset{\sim}{x} \right\} \underset{\sim}{y}' - n \bar{y}^2 = \| \underset{\sim}{y} - \bar{y}(1 \ldots 1) - \underset{\sim}{C}^o \underset{\sim}{x} \|^2 .$$

It is now easily seen that x_1, $\underset{\sim}{C}^o$ and V are sufficient for the family of distributions of $\underset{\sim}{y}$; families (1), (3), (4) and (5) are still valid, except that in (4) and (5) m is replaced by $m - k$ and that in (1) and (3) μ is replaced by $\mu + \Sigma C_p x_{pj}$, and when $k > 0$ the logarithms of the joint density of the components of $\underset{\sim}{C}^o$ is proportional to

$$c_5 - k \log \sigma^2(1-\rho) - (\underset{\sim}{C}^o - \underset{\sim}{C}) \underset{\sim}{x} \underset{\sim}{x}' (\underset{\sim}{C}^o - \underset{\sim}{C})'/\sigma^2(1 - \rho) .$$

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