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On a Theorem of Halmos Concerning Unbiased Estimation of Moments

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Correction sheet for CFDP No. 144

Page 2. To "(a)" add: "In [3a] this seemingly uninteresting fact has been applied to a practical case."

Page 4. Title of section: leave off s in first word and append the following footnote: "It has been remarked that it is obvious that from a sample of 1 it is not possible to obtain an unbiased estimate of two independent parameters, i.e., two functions F_1 and F_2 on a class of distributions such that there exists no function φ in the plane with $\varphi[F_1(D), F_2(D)] = 0$ for all distributions D in the class. That this is not so is easily shown by an example. Let $\alpha^2 = \nu^2 + \omega^2$, then ν and α^2 are independent parameters of $\mathcal{D}$, but $\nu = \mathcal{E} z$, $\alpha^2 = \mathcal{E} z^2$.

Page 8. Add [3a] KONIJN, H. S. (196). Nonexistence of Consistent Estimator Sequences and Unbiased Estimates: A Practical Example, (Forthcoming).

ON A THEOREM OF HALMOS CONCERNING UNBIASED ESTIMATION OF MOMENTS

Hendrik S. Konijn

1. Introduction

In [3] Halmos considers the following situation. Let $\mathscr{D}$ be a class of one dimensional distribution functions and F a function over $\mathscr{D}$. He investigates which functions F admit estimates that are unbiased over $\mathscr{D}$ and what are all possible such estimates of any given F. In particular he shows that on the basis of a sample of size $n(\geq 1)$ one can always obtain an estimate of the first moment which is unbiased in $\mathscr{D}$ and that the central moments $\bar{F}_m$ of order $m \geq 2$ have estimates which are unbiased in $\mathscr{D}$ if and only if $n \geq m$, provided $\mathscr{D}$ satisfies the following properties: $\bar{F}_m$ exists and is finite for all distributions in $\mathscr{D}$ and $\mathscr{D}$ includes all distributions which assign probability one to a finite number of points.* Halmos also finds

* Let $E_{a,b,p}$ be the distribution which assigns probability p to the point a and $1 - p$ to the point b. One can easily modify Halmos' argument to show that the second condition on $\mathscr{D}$ can be replaced by: there exist two different points a and b and a set T of at least $m + 1$ different numbers in the closed interval from 0 to 1 such that $\mathscr{D}$ contains $\{E_{a,b,p} (p \in T)\}$.

that under additional conditions symmetric unbiased estimates are unique** and have smaller variances than the unsymmetric ones.

** It will be convenient to call a function on a k-dimensional Euclidean space the unique function satisfying a certain property if any other function on this space satisfying the property may differ from it only on a set of k-dimensional Lebesgue measure zero.

He recognizes that his assumptions are too restrictive for most applications and mentions in particular the case where $\mathcal{D}$ is the class of all normal distributions. The present paper addresses itself to that case.

2. Statement of results

If $\mathcal{D}$ is the class of all nondegenerate univariate normal distributions, then, on the basis of a sample of size $n (\geq 1)$ an estimate of the first moment which is unbiased over $\mathcal{D}$ exists (and is unique when $n = 1$); and a central moment of order $2r \geq 2$ has estimates which are unbiased over $\mathcal{D}$ if and only if $n \geq 2$, and is a unique symmetric unbiased estimate when $n = 2$ but not if $n > 2$.

Specifically, this means the following:

Let $z_1, \ldots, z_n$ be a sample for a normal distribution with mean ν and variance $\omega^2 > 0$. Let $\bar{z} = n^{-1} \Sigma z_i$, $S^2 = \Sigma (z_i - \bar{z})^2$. Recall that the even central moments $\bar{F}_{2r}$ equal $\omega^{2r} 2^{-r} (2r)!/r!$ and the odd ones vanish.

(a) If $n = 1$, $\bar{z} = z_1$ is the unique unbiased estimate of ν, and no unbiased estimate of $\bar{F}_{2r}$ exists for $r = 1, 2, \ldots$.

(b) If $n \geq 2$,

$$\bar{f}_{2r} = \frac{\left\{\frac{1}{2}(n-1) - 1\right\}! \, (2r)!}{\left\{\frac{1}{2}(n-1) + r - 1\right\}! \, r!} \left(\frac{1}{2} S\right)^{2r}$$

is an unbiased estimate of $\bar{F}_{2r}$ $(r = 1, 2, \ldots)$, and is the unique symmetric unbiased estimate if $n = 2$, but not if $n > 2$.

In the next two sections we prove the parts of (a) and (b) which are not immediately obvious.

It now follows from [4] that, if $n \geq 2$, $\bar{z}$ and $\bar{f}_{2r}$ are the unique unbiased estimates of ν and $\bar{F}_{2r}$ depending only on the sufficient statistic $(\bar{z}, S^2)$ and have the smallest variance among all unbiased estimates. Note that $\bar{z}$ and S^2 are symmetric functions of the observations. The usual symmetric estimate $\bar{f}'_{2r}$ for $\bar{F}_{2r}$, which is unbiased for all distribution functions for which $\bar{F}_{2r}$ exists, is defined only when $n \geq 2r$. It cannot be specified in a convenient general formula. When $r = 1$ it coincides with $\bar{f}_2$, when $r = 2$ it equals [1 , 27.6]:

$$\bar{f}'_4 = \frac{(n-4)!}{n!} \left\{ n(n^2-2n+3)\, \Sigma(z_i - \bar{z})^4 - 3(2n-3)\, S^4 \right\} \quad (n \geq 4) .$$

Being symmetric, $\bar{f}'_{2r}$ is unique among symmetric estimates which are unbiased in the class of all distribution with finite $\bar{F}_{2r}$ by Halmos' results. But in the normal case our results imply that $\bar{f}_{2r}$ has a smaller variance than $\bar{f}'_{2r}$ for $r > 1$.

3. Nonexistences of an unbiased estimate of $\bar{F}_{2r}$ in a sample of 1 .

In this section we denote z_1 by z . If $h(z)$ is an unbiased estimate of $\bar{F}_{2r}$ then

$$\int_{-\infty}^{\infty} \left\{h(z+\nu) - z^{2r}\right\} \exp\left(- \frac{1}{2} z^2 \omega^{-2}\right) dz$$

should vanish for all ν and all $\omega > 0$. This integral can be written as an integral over the positive axis and then we can make the substitution $u = z^{\frac{1}{2}}$ and obtain, setting $\omega' = (2\omega^2)^{-1}$, that

$$\int_0^{\infty} \left\{h(- u^{\frac{1}{2}} + \nu) + h(u^{\frac{1}{2}} + \nu) - 2 u^r\right\} u^{-\frac{1}{2}} \exp(-u\,\omega')\, du$$

is zero for all ν and all $\omega' > 0$. This being a Laplace transform of $u^{-\frac{1}{2}}$ times the expression in brackets, it follows that

$$h(- z + \nu) + h(z + \nu) - 2 z^{2r} = 0$$

for all ν and almost all positive z . That is, there is a set T on the positive z axis such that the Lebesgue measure ℓ of the positive points z not in T is zero and such that the above equality holds on T for all real ν .

As we shall show below, there exist two points a and $\frac{1}{2}a$ in T . Choosing $\nu = a$ and $2a$ respectively gives for $z = a$

$$h(0) + h(2a) = 2a^{2r} , \qquad h(a) + h(3a) = 2a^{2r} ,$$

so that,

$$h(0) + h(a) + h(2a) + h(3a) = 4a^{2r} .$$

Choosing $\nu = \frac{1}{2} a$ and $2 \frac{1}{2} a$ respectively gives for $z = \frac{1}{2} a$

$$h(0) + h(a) = \frac{1}{2} a^{2r}, \qquad h(2a) + h(3a) = \frac{1}{2} a^{2r} ,$$

so that

$$h(0) + h(a) + h(2a) + h(3a) = a^{2r} .$$

Since $a \neq 0$, this is a contradiction.

To show that one can choose two points a and $\frac{1}{2} a$ in T, let a' be in T and let $0 < b < a'$. Define the disjoint intervals I_i from ia' to $i(a'+ b)$ for $i = 1, 2,$ which have $\ell(I_i T) = ib$. Denote by $p_j (I_i T)$ the set of points x in $I_j T$ such that $i\, x j^{-1}$ is in $I_i T$. Since given any $\eta > 0$ there is a denumerable collection of open intervals whose union contains the set $I_i - I_i T$ of points in I_i but not in $I_i T$ and whose total length is less than η (see e.g., [5, 19.15]), there is a sequence of intervals whose union contains $I_j - p_j(I_i T)$ and whose total length is less than $j \eta i^{-1}$, (see same reference) so/ $\ell\left\{ p_j(I_i T)\right\} = jb$. Now let

$$T_2 = I_2 T p_2(I_1 T), \quad T_1 = p_1(T_2) ,$$

then, since the T_i are subsets of T with $\ell(T_i) = ib$, there exist $a > 0$ such that $\frac{1}{2} ia$ is in T_i for $i = 1$ and 2, so that $\frac{1}{2} a$ and a are in T.

4. <u>Uniqueness of the unbiased symmetric estimate of</u> $\bar{F}_{2r}$ <u>in a sample of 2 and nonuniqueness in a larger sample.</u>

For $n \geq 2$ (so that S^2 is not identically zero) the sufficiency of the statistic $(\bar{z}, S^2)$ and the completeness of its distribution imply that $\bar{f}_{2r}$ is the unique unbiased estimate of its expectation $\bar{F}_{2r}$ among unbiased estimates depending on $(\bar{z}, S^2)$ only [4]. Now if $n = 2$, $(\bar{z}, S^2)$ determines the set $\{z_1, z_2\}$ of observations, but not their order. Therefore $\bar{f}_{2r}$ is also the unique unbiased estimate of $\bar{F}_{2r}$ among unbiased estimates which are symmetric in the observations.

That this is not so for $n \geq 2r > 2$ is shown by the unbiasedness of the symmetric estimates $\bar{f}'_{2r}$ which differ from $\bar{f}_{2r}$ for all $r > 1$, since $\bar{f}_{2r}$ when defined contains the factor $\Sigma(z_i - \bar{z})^{2r}\ (n-2r+1)^{-1}$ [3].

For $2 < n < 2r$ one can always combine expressions involving $\Sigma(z_i - \bar{z})^4$ and S^4 to get unbiased symmetric estimates of $\bar{F}_{2r}$. For example, if $n = 3$, $1\frac{1}{2}\,\mathcal{E}\,\Sigma(z_i - \bar{z})^4 = F_4 + \bar{F}_2^2$ and, in the normal case, S^4 has mean $8\,\bar{F}_2^2$, so that $1\frac{1}{2}\,\Sigma(z_i - \bar{z})^4 - S^4/8$ is a symmetric unbiased estimate of $\bar{F}_4$ different from $3\,S^4/8$.

5. Remarks

One could similarly discuss unbiased estimation of other functions over the class of normal distributions.

Fraser [2] adapts Halmos' argument to the case where $\mathcal{P}$ contains all distribution uniform over intervals,*** such as is the case when $\mathcal{P}$ is the class of absolutely continuous distributions.

The writer is much indebted to T. C. Koopmans and T. N. Srinivasan for helpful suggestions.

*** This requirement can be weakened; in fact, it is already a weakened version of Fraser's requirement.

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