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# Stationary Utility and Time Perspective

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Stationary Utility and Time Preference Perspective

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# Stationary Utility and Time Perspective

by*

Peter A. Diamond, Tjalling C. Koopmans and Richard E. Williamson

## 1. INTRODUCTION

In a previous article one of the authors ** studied some implications

** Tjalling C. Koopmans, "Stationary Ordinal Utility and Impatience," Econometrica, April, 1960, pp. 287-309.

of a set of postulates concerning a utility function that depends on a

consumption program for an infinite future. While the postulates themselves

appeared to be concerned only with properties more immediate and elementary

than any questions of timing preference, it was found that the postulates

implied, at least in certain parts of the program space, a preference for

advancing the timing of future satisfactions. This conclusion was expressed

by the concept of impatience. In its simplest form this concept was defined

to mean that, if in any given year the consumption of a bundle $x$ of commodities

is preferred over that of a bundle $x'$, then the consumption in two successive

years of $x$, $x'$, in that order, is preferred to the consumption of $x'$, $x$ .

Subsequently we have found a deeper property of the utility function in question, from which the previous result regarding impatience can be derived, and extended to a larger part of the program space. This property, to be called time perspective, conveys a quasi-cardinal character to a utility function originally conceived only as ordinal. It is found that, among all equivalent ordinal utility scales, there exists a subclass of quasi-cardinal scales permitting a comparison of utility differences in the following manner. Let there be two consumption programs, $(x,x',x'', \ldots)$ and $(y,y',y'', \ldots)$, of which the first is preferred to the second. Now postpone each entire program by one time unit, and insert a common consumption bundle z in the gap so created in both programs, to make $(z, x, x', \ldots)$ and $(z, y, y', \ldots)$, respectively. Then, while the postponed first program is still preferred to the postponed second, the difference in the utilities of the two programs has become smaller. We have called utility scales with this property quasi-cardinal because some pairs of utility differences can be ordered as larger, equal, or smaller, but no such differences can be compared numerically through ratios in a unique manner.

The term "time perspective" is derived from an analogy with perspective in space. As the timing of the differences between two programs is made to recede into a more distant future, the utility difference between the programs diminishes. To be precise, we call this property strong time perspective, as distinct from a property of weak time perspective, in which the utility difference may either remain the same or diminish. The proof of strong time

perspective found so far takes its point of departure from weak time
perspective, but requires lengthy reasoning beyond that, and also a slight
strengthening of Postulate 2. For these reasons, the present paper is limited
to weak time perspective only. However, the reasoning of the present paper
will suffice to show that, in any given quasi-cardinal scale, among all
program pairs subjected to a postponement as described, equality of utility
differences before and after postponement can only be in some sense an
exceptional case, whereas shrinkage of the utility difference occurs in some
average sense indicated in Section 5 below.

The notation and numbering of equations, postulates, and theorems
of the previous study will be continued here, and the reading of the present
paper will be facilitated by prior reading of the previous study.* Neverthe-
less, we will in Section 2 restate the postulates, so that our statements will be
complete in themselves, and also because we will introduce a strengthening
of the first postulate. This strengthening is needed to correct an error in the
previous study.**

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** This error was kindly brought to our attention by Richard Levitan
of the International Business Machines Corporation.

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In Section 3 we shall summarize enough of the results of the previous
study to be able to present, in Section 4, the main result of the present
paper. Sections 5 and 6 discuss further implications of this result.

Technical aspects of the reasoning are placed in starred sections

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* and by inspection of its diagrams.

generally bearing the same number as the section to which they refer.

## 2. RESTATEMENT OF THE POSTULATES

In the following, $_1x \equiv (x_1, x_2, \ldots) \equiv (x_1, {}_2x) \equiv ({}_1x_{t-1}, {}_tx)$ ,

$t = 2, 3, \ldots$ , denotes an infinite sequence of consumption vectors

$x_t \equiv (x_{t1}, \ldots, x_{tn})$ relating to successive periods $t = 1, 2, \ldots$ .

Interpretations of the following postulates have already been given in the

previous study. We shall enlarge somewhat on the interpretation of the

crucial Postulate 4. Also, in Section $2^*$, we shall comment on the

strengthening of Postulate 1.

POSTULATE 1 (Existence and continuity). There exists a utility function

$U({}_1x)$ , which is defined for all $_1x = (x_1, x_2, \ldots)$ such that, for all t, $x_t$

is a point of a bounded convex subset X of the n-dimensional commodity

space. The function $U({}_1x)$ has the continuity property that, if U is any

of the values assumed by that function, and if U' and U'' are numbers such

that $U' < U < U''$ , then there exists a positive number $\delta$ such that the

utility $U({}_1x')$ of every program $_1x'$ having a distance $d({}_1x', {}_1x) \leqq \delta$

from some program $_1x$ with utility $U({}_1x) = U$ satisfies $U' \leqq U({}_1x') \leqq U''$ .

POSTULATE 2 (Sensitivity). <u>There exist first-period consumption vectors</u> $x_1$, $x_1'$ <u>and a program</u> $_2x$ <u>from-the-second-period-on, such that</u>

$$U(x_1, {}_2x) > U(x_1', {}_2x) .$$

POSTULATE 3 (3a and 3b) (Limited Noncomplementarity). <u>For all</u> $x_1$, $x_1'$, $_2x$, $_2x'$

(3a) $\qquad U(x_1, {}_2x) \geqq U(x_1', {}_2x)$ <u>implies</u> $U(x_1, {}_2x') \geqq U(x_1', {}_2x')$ ,

(3b) $\qquad U(x_1, {}_2x) \geqq U(x_1, {}_2x')$ <u>implies</u> $U(x_1', {}_2x) \geqq U(x_1', {}_2x')$ .

POSTULATE 4 (Stationarity). <u>For some</u> $x_1$ <u>and all</u> $_2x, {}_2x'$ ,

$$U(x_1, {}_2x) \geqq U(x_1, {}_2x') \quad \underline{\text{if and only if}} \quad U({}_2x) \geqq U({}_2x') .$$

Postulate 4 says that the ordering of a subset of programs that differ only from the second period on is the same as that of corresponding programs obtained by advancing the timing of every future consumption vector by one period. This does <u>not</u> imply that, after one period has elapsed, the ordering then applicable to the "then" future will necessarily be the same as that now applicable to the "present" future. All postulates are concerned with only one ordering, that guiding decisions taken in the present. Any question of change or constancy of preferences as the time of choice changes

is therefore extraneous to the present study. Postulates 4 and 3b taken together express merely an invariance of the present ordering under postponement of entire programs, provided gaps created by such postponement are filled in the same way for all programs compared.

POSTULATE 5 (Extreme Programs). <u>There exist</u> $_1\underline{x}$ , $_1\bar{x}$ <u>such that</u>

$$U(_1\underline{x}) \leqq U(_1x) \leqq U(_1\bar{x}) \underline{\text{ for all }} _1x \quad .$$

2.[*] The <u>norm</u>, or concept of <u>distance between two programs</u>, used in Postulate 1, is defined by

$$(6) \qquad d(_1x', {}_1x) = \sup_t |x_t' - x_t|, \qquad |x_t' - x_t| = \max_k |x_{tk}' - x_{tk}| \quad .$$

The only difference between the previous and present versions of Postulate 1 is that the set $X$ of all possible one-period consumption vectors $x$ is now required to be convex and bounded. This means (convexity) that any weighted average $\theta x + (1-\theta)x'$ , $0 < \theta < 1$ of two feasible one-period consumption vectors $x, x'$ is again a feasible consumption vector, and (boundedness) that there are a lower[*] and an upper bound to the possible rates of consumption of any commodity.

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[*]While zero is a natural lower bound to all consumption proper, one may wish to treat labor of various kinds as negative consumption. In that case the lower bound for each type of labor expresses, in absolute value, the maximal amount of that labor that can be rendered.

### 3. SUMMARY OF PREVIOUS RESULTS

Postulates 1, 2, 3, 4 have been shown to imply that the aggregate utility function $U(_1x)$ satisfies a recurrent relation

$$(11) \qquad U(_1x) = V\left( u(x_1) , U(_2x) \right) .$$

Subject to supplementation on one open point discussed in Section $3^*$ below, the _aggregator function_ $V(u, U)$ has also been shown to be continuous and increasing in its two arguments $u, U$ . For the second argument $U$ , (11) specifies the aggregate utility $U(_2x)$ of that part $_2x$ of the given program $_1x$ that starts with the second period (evaluated as if it were to start immediately). For the first argument $u$ , (11) specifies the value assumed by an _immediate_, or _one-period_, utility function $u(x)$ for the consumption vector $x = x_1$ of the first period in the given program. The function $u(x)$ is defined and continuous on the set $X$ of all feasible consumption vectors.

By using also Postulate 5, it has been shown further that, by two independently chosen, continuous and increasing, transformations of the variables $U, u,$ respectively, one can make the range of variation of each of the functions $U(_1x)$ and $u(x)$ coincide with the closed unit interval $[0, 1],$

$$(12) \qquad 0 = U(_1\underline{x}) \leq U(_1x) \leq U(_1\bar{x}) = 1 \quad \text{for all programs } _1x ,$$

$$(13) \qquad 0 = u(\underline{x}) \leq u(x) \leq u(\bar{x}) = 1 \quad \text{for all vectors } x ,$$

and therefore have

(14)         $V(0, 0) = 0$ ,              $V(1, 1) = 1$ .

The key property of the function $V(u, U)$ proved and used in the previous study concerns a result of its repeated application. We use again the notation

$$V_\tau \left( {}_1u_\tau; U \right) \equiv V(u_1, V(u_2, \ldots, V(u_\tau, U) \ldots)) ,$$

where the $u_t = u(x_t)$ are the immediate utility levels associated with the successive vectors $x_t$ of a program ${}_1x$ . The equation

(26)         $V_\tau \left( {}_1u_\tau; U \right) = U$

then expresses the condition that the postponement of a program of utility $U$ by $\tau$ periods is just compensated for by the insertion, in the $\tau$ periods so vacated, of consumption vectors $x_1, \ldots, x_\tau$ with a sequence of one-period utility levels

$${}_1u_\tau \equiv (u_1, u_2, \ldots, u_\tau) , \qquad u_t = u(x_t), \qquad t = 1, \ldots, \tau .$$

Obviously the utility

(27)         $U = U \left( {}_1x_\tau, {}_1x_\tau, {}_1x_\tau, \ldots \right)$

of the program indefinitely repeating the <u>consumption</u> pattern ${}_1x_\tau$ meets this

condition. It has been shown that, given the <u>utility pattern</u> $_1u_\tau$

associated with a consumption pattern $_1x_\tau$ , there exists one and only one value

(28) $\qquad U \equiv W_\tau \left(_1u_\tau\right)$

of $U$ that satisfies the condition (26). The <u>correspondence function</u> $W_\tau \left(_1u_\tau\right)$ is continuous and increasing in each of its arguments $u_1, \ldots, u_\tau$ . We are now able to state the key property (29) of $V_\tau \left(_1u_\tau; U\right)$ derived in the previous study (and illustrated in Figure 6 of that study for the case $\tau = 2$):

(29) $\quad$ If $U \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} W_\tau \left(_1u_\tau\right)$ then $U \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} V_\tau \left(_1u_\tau; U\right) \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} W_\tau \left(_1u_\tau\right)$ .

This indicates that repeated application of the function $V_\tau \left(_1u_\tau; U\right)$ to any initial value $U$ brings about a monotonic approach to $W_\tau \left(_1u_\tau\right)$ . It has been shown in (32) that $W_\tau \left(_1u_\tau\right)$ is also the limit for infinitely repeated application, regardless of the initial value $U$ used.

It will be useful to compare the already proved property (29) with the yet to be proved time perspective property described in Section 1. We can now state the latter as follows: There exists a continuous increasing transformation of the utility scale, as a result of which,

(49) $\left\{ \begin{array}{l} \text{if } U' > U, \ V_\tau \left(_1u_\tau, U\right) = U'', \ V_\tau \left(_1u_\tau, U'\right) = U''' \ , \ \tau \geqq 1, \text{ then} \\[2ex] \text{(49a)} \quad \text{weak time perspective, } \quad U''' - U'' \leqq U' - U \ , \\[2ex] \text{(49b)} \quad \text{strong time perspective, } \quad U''' - U'' < U' - U \ . \end{array} \right.$

Note that the strict inequalities in (29) represent special cases of (49b) obtained by those choices of $_1u_\tau$ that make $U'' = U$, or $U'''' = U'$, respectively. These are the only cases of (49) involving comparisons of utility levels rather than of utility differences. Thus (29) states the <u>ordinal</u> special cases contained in (49b), which are independent of continuous increasing transformations of the scale. In contrast, neither (49a) nor (49b) can be true for all equivalent ordinal scales. The main aim of the present study is to show that the ordinal comparisons in (49b) already known through (29) are sufficient, given the continuity and monotonicity of $V(u, U)$, to prove the existence of one or more scales for which the quasi-cardinal comparisons in (49a) are also valid.

$3^*_.$ The real-valued function $U(_1x)$ is defined on the cartesian product $_1X$ of an infinite sequence of identical sets $X$ assumed to be convex and bounded. In addition $U$ is continuous on $_1X$ in the topology defined by (6). We now show that $_1X$ is connected in that topology. Let $_1x$ and $_1x'$ be points of $_1X$. Because $X$ is convex, the segments defined by $x''_t(\theta) = \theta x_t + (1-\theta)x'_t$, $0 \le \theta \le 1$, lie in $X$ for each $t$. Because $X$ is bounded the functions $x''_t$ are equicontinuous. It follows that the function $_1x''(\theta)$ from $[0,1]$ to $_1X$ is continuous in the topology of (6), so $_1X$ is (are-wise) connected. It follows, by the continuity of $U(_1x)$, that the values assumed by $U(_1x)$ for all $_1x$ in $_1X$ fill an interval, which by Postulate 2 is nondegenerate. By Postulate 5, it is the closed interval $[U(_1\underline{x}), U(_1\overline{x})]$, which can by an appropriate continuous

increasing transformation be made to be the unit interval $\underline{\bar{I}} \equiv [0, 1]$ .
This proves (12) and (14). In particular, one can take $_1\underline{x} = (\underline{x}, \underline{x}, \underline{x}, \ldots)$
and $_1\bar{x} = (\bar{x}, \bar{x}, \bar{x}, \ldots)$ .

Finally, by (11), for any given $x_1$, the values of the function
$U(x_1, {_2}x)$ for all $_2x \in {_2}X$ again fill the interval $\left[ V(u(x_1), 0), V(u(x_1), 1) \right]$ .
Since $V(u, U)$ has previously been proved to be increasing in $U$ for all $u$,
it follows that $V(u, U)$ is continuous in $U$ . This point, not covered by the
previous study, makes available all other conclusions of that study on the
basis of the strengthened Postulate 1.

## 4. PROOF OF THE WEAK TIME PERSPECTIVE PROPERTY

It will be useful to shift the discussion from points $U$ on the
utility scale to (closed nondegenerate) intervals, for which we shall use
the interchangeable notations

$$(50) \qquad \underline{\bar{U}} \equiv [\underline{U}, \bar{U}] \equiv \left\{ U \middle| \underline{U} \leqq U \leqq \bar{U} \right\} , \quad \text{where} \quad \underline{U} < \bar{U} .$$

In particular, the unit interval will be denoted

$$(51) \qquad \underline{\bar{I}} \equiv [0, 1]$$

The shift to intervals has the advantage that the set inclusion

symbol $\supset$ can be used to represent inequalities occurring frequently in the reasoning:

(52)  $\underline{\overline{U}} \supset \overline{\underline{U}}'$  stands for  $\underline{U} \lesssim \underline{U}' < \overline{U}' \lesssim \overline{U}$ .

Because  $V(u, U)$  is continuous and increasing in  $U$ ,  insertion in  $V$ of all the points  $U$  of an interval  $\overline{\underline{U}}$  gives another interval, which we denote by

(53)  $V(u, \overline{\underline{U}}) \equiv \left[ V(u, \underline{U}) , V(u, \overline{U}) \right].$

This operation can be iterated for a/sequence  $_1u_\tau$  of values of  $u$ , expressing
the effect of postponement of all programs with utilities in the interval  $\overline{\underline{U}}$
by  $\tau$  periods, with insertion of a common consumption sequence with one-period
utilities  $_1u_\tau$  in the gap created.  For further simplification of notation,
we shall use  $V$  as an operator symbol to denote any operation of this kind:

(54)  $\overline{\underline{U}}' = V \overline{\underline{U}}$  stands for  $\overline{\underline{U}}' = V_\tau(_1u_\tau; \overline{\underline{U}})$  for some  $\tau \gtrless 1$  and  $_1u_\tau$ .

We shall now list those properties of the class  $\mathcal{V}$ of all these
"postponement" operations that enter into the proof of weak time perspective.

(a)  Successive application of two operations  $V, V'$  of  $\mathcal{V}$ yields
another operation  $V'' = V' V$  of  $\mathcal{V}$ (i.e.,  $\mathcal{V}$ is a semi-group).

This property follows directly from the definition (54) of the generic
operation  $V$ .  For, if  $\overline{\underline{U}}' = V \overline{\underline{U}}$ ,  and  $\overline{\underline{U}}'' = V' \overline{\underline{U}}'$,  then obviously

(55) $\qquad \overline{U}'' = V_{\tau'} \left(_1 u'_{\tau'}; V_\tau \left(_1 u_\tau; \overline{U}\right)\right) = V_{\tau'+\tau} \left(_1 u'_{\tau'}, _1 u_\tau; \overline{U}\right)$ .

We have a semi-group rather than a group (in which each operation can be undone by an inverse operation) because the future has a beginning but no end. Hence the postponement of a program, while creating a gap to be filled, does not lead to any disappearance of consumption vectors. In contrast, a program cannot be advanced without suppressing one or more consumption vectors.

(b) As applied to points, each $V$ in $\mathcal{V}$ is a continuous increasing transformation from the unit interval $\overline{I}$ onto a subinterval $V\,\overline{I}$ thereof.

This property follows from the continuity and increasing character of $V(u, U)$ with respect to $U$.

(c) If $U$, $U'$ are any given points with $U' \neq 0$ or $1$ , then there exists an operation $V$ in $\mathcal{V}$ such that $V\,U = U'$ .

(d) As applied to intervals, no $V$ transforms any interval $\overline{U}$ into an interval $\overline{U}'$ containing $\overline{U}$ :

(56) $\qquad$ If $\overline{U}' = V\,\overline{U}$ then $\overline{U}' \not\supset \overline{U}$

Properties (c) and (d) will be proved in Section 4[*] below.

It may be empahsized again that all the properties (a), (b), (c), (d) are ordinal. In particular, the translation (56) of the key property (29)

into "interval language" uses only the ordinal concept of one interval being contained in another.

It will be clear that, if (56) were violated by any operation $V$ and interval $\bar{\underline{U}}$ , then at least the strong time perspective condition (49b) could not be satisfied. For, if any $V\,\bar{\underline{U}}$ were to contain $\bar{\underline{U}}$ , then there could be no scale in which $V\,\bar{\underline{U}}$ is shorter than $\bar{\underline{U}}$ . It is somewhat less obvious that a converse statement is also true: that if (56) holds throughout, then at least a scale with the weak time perspective property (49a) can be constructed. According to a mathematical theorem, to be published elsewhere by two of the present authors,[*] the conditions (a), (b), (c), (d) above suffice for the

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[*] Tjalling C. Koopmans and Richard E. Williamson, Cowles Foundation Discussion Paper No. 143, "On the Existence of a Subinvariant Measure," July 31, 1962.

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existence of at least one, and possibly infinitely many, such scales. A few further remarks on the nature of the proof are given in Section 4[*] below.

So far, we have not been able to make sure that scales with the time perspective property exist that have a finite range. That is, if utility levels in the new scale are denoted by asterisks, the utility levels associated with the worst and best programs $_1\underline{x}$ and $_1\bar{x}$ , respectively, may have to be assigned the values

$$(57) \qquad U^*(_1\underline{x}) \ = -\infty \ , \qquad U^*(_1\bar{x}) \ = +\infty \ .$$

4[*]. Thus far we have allowed independent transformations of the

arguments  u, U  of  V(u, U) .  It will now be convenient, rather than neces-

sary, to apply the transformation (23) of the previous study to the one-period

utility scale, so as to make, in accordance with (14),

(58)    $W(u) \equiv W_1(u) = u$ ,   so  $V(U, U) = U$ ,   for all  $u, U \in \overline{I}$ .

These relations will be conserved if from here on we apply any required

transformations simultaneously to  u  and  U .

To prove property (c) we note that, if  $U < U' < 1$ ,   the sequence

defined by

$$U_1 = V(1, U) , \qquad U_{t+1} = V(1, U_t) , \quad t = 1, 2, \ldots$$

is,  by (29) and (58), an increasing sequence, of which the limit is  1  by (58),

(32).  Hence there is a  $\tau$  such that  $U_{\tau-1} \leqq U' < U_\tau$ ,   and a  u  such

that  $U' \leqq u < 1$  and

$$V_\tau (u, 1, \ldots, 1; U) = V(u, U_{\tau-1}) = U' ,$$

because  V(u, U)  is continuous and increasing in  u .  The proof is similar

for  $U > U' > 0$ .  In case  $U = U'$ ,   clearly  $U' = V(U, U)$  by (58).

To prove (d), we shall show that the assumption that  $\underline{U}' = V \overline{U} \supset \underline{U}$ ,

and hence

$$\underline{U}' \leqq \underline{U} < \overline{U} \leqq \overline{U}'$$

contradicts (29). If we should have $\underline{U}' = \underline{U}$ , a contradiction with (29) would already have occurred. But if

$$\underline{U}' = V \underline{U} \equiv V_\tau(_1u_\tau \; ; \; \underline{U}) \; < \; \underline{U} \; ,$$

then by (58) and the fact that $V(u, U)$ increases with $u$ , at least one of the elements $u_t$ , $t = 1, \dots, T$ in $_1u_\tau$ must satisfy

$$(59) \qquad u_t \; < \; \underline{U} \; .$$

We arrange the $u_t$ for which (59) holds in order of increasing $t$ , and increase each of these in succession continuously from the given value up to $\underline{U}$ until, by the continuity and increasing property of $V(u, U)$ with regard to both of its variables, we have reached a sequence $_1u_\tau'$ such that

$$\underline{U}'' \equiv V_\tau(_1u_\tau' \; ; \; \underline{U}) = \underline{U} \; , \quad \overline{U}'' \equiv V_\tau(_1u_\tau' \; ; \; \overline{U}) \; > \; V_\tau(_1u_\tau \; ; \; \overline{U}) = \overline{U}' \; \geqq \; \overline{U} \; ,$$

again contradicting (29). Such a sequence $_1u_\tau'$ is bound to be reached because, if we continue the increases in the $u_t$ satisfying (59) until all of them have been raised to $\underline{U}$ , we will obtain a sequence $_1u_\tau''$ such that, using (58),

$$u_t'' \; \geqq \; \underline{U} \; , \quad t = 1, \dots, \tau \; , \text{ hence } V_\tau(_1u_\tau'' \; ; \; \underline{U}) \; \geqq \; \underline{U} \; .$$

The construction of a scale with the weak time perspective property, given in the reference, is analogous to, but not identical with, the construction of Haar measure [Banach, 1937, or Halmos, 1950, Ch. XI]. It starts from a "counting

function" $\underline{\overline{U}} : \underline{\overline{T}}$ of two intervals, $\underline{\overline{U}}$, $\underline{\overline{T}}$, which, while using only ordinal

concepts, roughly measures $\underline{\overline{U}}$ as a multiple of $\underline{\overline{T}}$ . If we call any interval

$V \underline{\overline{T}}$ for $V \in \mathcal{V}$ a <u>descendant</u> of $\underline{\overline{T}}$ , the counting function is defined as

the minimum number of descendants of $\underline{\overline{T}}$ required to cover $\underline{\overline{U}}$ . To derive a

continuous measure from this function, one needs to form the ratio

$$\left( \underline{\overline{U}} : \underline{\overline{T}} \right) \Big/ \left( \underline{\overline{S}} : \underline{\overline{T}} \right)$$

of the count of $\underline{\overline{U}}$ to that of a standard interval $\underline{\overline{S}}$ , before one can shrink the

interval $\underline{\overline{T}}$ down to an arbitrarily chosen point. One wishes to make that

limit transition in such a way as to obtain an additive interval function,

that is, a function $\lambda (\underline{\overline{U}})$ satisfying

if $\overline{U} = \underline{U}'$ then $\lambda(\underline{\overline{U}} \cup \underline{\overline{U}}') = \lambda(\underline{\overline{U}}) + \lambda(\underline{\overline{U}}')$ .

This can be achieved by using a generalized limit[*] [Banach, 1932, II, § 3]

$$\lambda(\underline{\overline{U}}) \equiv \lim_{\substack{\overline{T}-\underline{T} \to o, \\ \overline{T} \geqq T_0 \geqq \underline{T}}} \left( \underline{\overline{U}} : \underline{\overline{T}} \right) \Big/ \left( \underline{\overline{S}} : \underline{\overline{T}} \right) \quad .$$

The resulting function is found to be positive and finite if $\underline{\overline{S}}$ and $\underline{\overline{U}}$

are nondegenerate intervals in the interior $\overset{o}{\underline{\overline{I}}}$ of $\underline{\overline{I}}$ . One also has

$$\lim_{\overline{U} \to \underline{U}} \lambda(\underline{\overline{U}}) = 0 \quad .$$

[*]In the reference, instead of a construction using Banach's generalized
limit, an existence proof along the lines of Halmos' discussion of Haar
measure is given.

Finally, due to the properties of the counting function used in the construction of $\lambda$ , one obtains

$$\lambda \left(\bigvee \overline{\underline{U}}\right) \leqq \lambda \left(\overline{\underline{U}}\right)$$

for all $\bigvee \in \mathcal{V}$ and all $\overline{\underline{U}} \subset \overset{o}{\underline{I}}$ . It follows that the continuous increasing transformation

$$U^* = \Phi \left(U\right) \equiv \begin{cases} C + \lambda([\tfrac{1}{2}, U]) & \text{if } U > \tfrac{1}{2} \\ C & \text{if } U = \tfrac{1}{2} \\ C - \lambda([U, \tfrac{1}{2}]) & \text{if } U < \tfrac{1}{2} \end{cases}$$

defines a utility scale satisfying (49a), provided the aggregator function $\bigvee(u, U)$ is likewise transformed by

$$\bigvee{}^* (u^*, U^*) \equiv \Phi\left(\bigvee\left(\Phi^{-1}(u^*), \Phi^{-1}(U^*)\right)\right) .$$

The construction is by no means unique (even apart from the choice of $C$ ) . From simple examples such as $\bigvee (u, U) = \tfrac{1}{2}(u + U)$ it is easily seen that in general no unique scale with either the weak or the strong time perspective property exists.

## 5. WEAK VERSUS STRONG TIME PERSPECTIVE

An elementary consideration suffices to show that any scale with the weak time perspective property must in some average sense exhibit strong time perspective. Consider the effect of postponement of the best and worst programs by one period. In terms of the original scale where (12), (13), (14) hold, this effect is subject to the inequalities

$$(60) \quad \text{if} \quad 0 \left\{ \begin{matrix} = \\ < \end{matrix} \right\} u \left\{ \begin{matrix} < \\ = \end{matrix} \right\} 1 \quad \text{then} \quad 0 \left\{ \begin{matrix} = \\ < \\ < \end{matrix} \right\} V(u,0) < V(u,1) \left\{ \begin{matrix} < \\ < \\ = \end{matrix} \right\} 1 \ ,$$

because of (58) and the monotonicity of $V(u,U)$ . Being ordinal, (60) goes over into any new scale

$$u^* = \Phi(u) \ , \qquad U^* = \Phi(U) \ ,$$

constructed to have the weak time perspective property, provided 0 and 1 are replaced by $0^* = \Phi(0)$ , $1^* = \Phi(1)$ , respectively. If, contrary to (57), $0^*$ and $1^*$ are finite, then obviously for all $u^*$

$$(61) \quad \frac{V^*(u^*,1^*) - V(u^*,0^*)}{1^* - 0^*} < 1$$

Since for any partition $0^* = U_0^* < U_1^* < \ldots < U_N^* = 1^*$ of $[0^*, 1^*]$ the left hand member in (61) is a suitably weighted average of the corresponding ratios

$$\frac{\mathcal{V}^*(u^*, U^*_{n+1}) - \mathcal{V}^*(u^*, U^*_n)}{U^*_{n+1} - U^*_n}$$

for all intervals of the partition, the latter ratios average out at less than 1, whereas none exceeds 1 . If, on the other hand, one or both of $0^*$ , $1^*$ are infinite, one can for any finite $u^*$ construct a similar argument in which $0^*$ , $1^*$ are replaced by any $\underline{u}^*$ , $\overline{u}^*$ such that $\underline{u}^* \leqq u^* \leqq \overline{u}^*$ and $\underline{u}^* < \overline{u}^*$ .

We intend to return in a later paper to the problem of constructing a scale exhibiting strong time perspective throughout.

## 6. TIME PERSPECTIVE AND IMPATIENCE

The time perspective property (49a) or (49b), whichever applicable, directly implies two extensions of the results of the previous study with regard to impatience. Omitting asterisks, assume that the aggregator function $\mathcal{V}(u, U)$ satisfies (49a), and that the scales of $u$ and $U$ have been made comparable by the transformations (23) leading to (58). Let there be two consumption vectors $x'$ , $x''$ with immediate utilities $u'$ , $u''$ such that

(62) $\qquad u'' = u(x'') < u(x') = u' \cdot .$

Consider two programs $_1x' = (x', x'', {}_3x)$ and $_1x'' = (x'', x', {}_3x)$ of which the common continuation $_3x$ from period 3 on is such that,

$$(63) \qquad u'' < U \equiv U({}_3x) < u'$$

Then, by (58) and the monotonicity of $V(u,U)$,

$$(64) \qquad U'' \equiv V(u'',U) < U < V(u',U) \equiv U' ,$$

and by (49a), with $\tau = 1$,

$$(65) \qquad \begin{cases} V(u',U) - V(u',U'') \leqq U - U'' \\[2ex] V(u'',U') - V(u'',U) \leqq U' - U \end{cases}$$

By adding the inequalities (65) and using the definitional equalities in (64), we obtain

$$(66) \qquad V\left(u', V(u'',U)\right) \geqq V(u'', V(u',U) ) ,$$

the inequality defining <u>weak impatience</u> for the program $_1x'$ . If (49b) had been available, we would have concluded to <u>strong impatience</u>, with the > sign in (66), defined simply as "impatience" in Definition 1 of the previous study.

In the previous study strong impatience was established for $U$ in the "central" interval

$$(67) \qquad "U' \equiv W_2(u'',u') \underset{=}{\leq} U \underset{=}{\leq} W_2(u',u'') \equiv 'U''$$

or in either of the "lateral" intervals

$$(68) \qquad \underset{\approx}{U} \underset{=}{\leq} U \underset{=}{\leq} u'', \qquad u' \underset{=}{\leq} U \underset{=}{\leq} \overset{=}{U},$$

where $\underset{\approx}{U}$ and $\overset{=}{U}$ were defined in (42) [where they were denoted $\underline{U}$ and $\overline{U}$, respectively]. The presently established interval of weak impatience contains the "central" interval and is adjacent to both "lateral" intervals, thus closing the gaps as indicated in Figure 10.
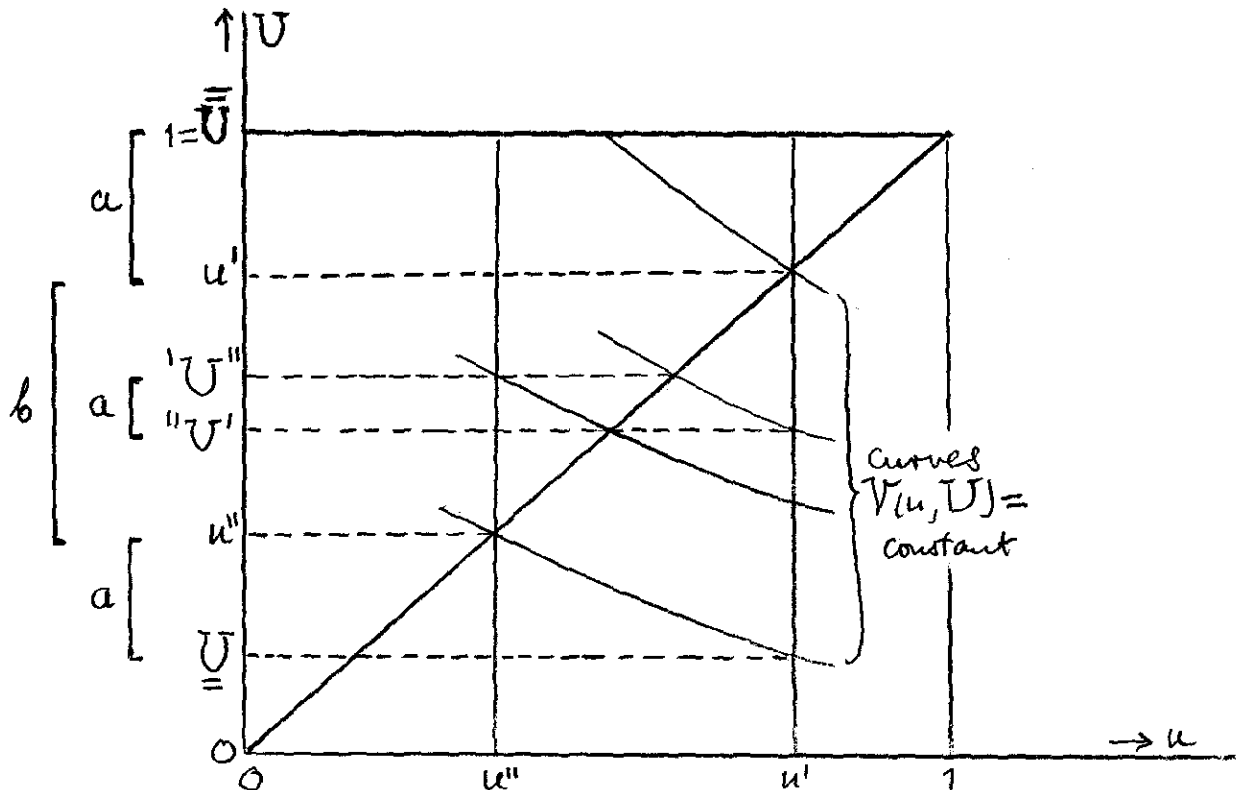


Figure 10. Zones (a) of strong impatience previously found and zone (b) of weak impatience added in the present study.

The second extension of previously announced results arises from the observation that, in all previous and present proofs of impatience relations, the symbols $u''$, $u'$ can without any change in the proof be re-interpreted as finite sequences, $_1u''_{\tau''}$, $_1u'_{\tau'}$ of one-period utility levels. In that case the symbols $u''$, $u'$ where occurring as scalars rather than as arguments of $V$ must be replaced by $W_{\tau''}(_1u''_{\tau''})$ and $W_{\tau'}(_1u'_{\tau'})$, respectively, and expressions such as $V(u', U)$ must be read as iterated functions $V_{\tau'}(_1u'_{\tau'}; U)$. The proof of (66) thus comes to rest on (49a) for arbitrary values of $\tau$. Careful reading of Section 13[*] of the previous study will show that its results are subject to the same generalizing reinterpretations. The redefinition of the end points $\underline{U}$, $\overline{U}$, of the entire interval of proved weak or strong impatience when $u'$ and $u''$ are sequences is given below.

It follows that impatience, weak or strong as the case may be, is also found, in corresponding zones, for the interchange of finite sequences of consumption vectors not necessarily equal in length. The question which is the "better" sequence is settled by comparison of programs in which the finite sequence in question is repeated infinitely often.

6[*]   The definitions of $\underline{U}$, $\overline{U}$, referred to are

$$(69) \begin{cases} \underline{U} = \max \left\{ 0, \text{ solution of } V_{\tau'}(_1u'_{\tau'}, \underline{U}) = W_{\tau''}(_1u''_{\tau''}) \right\} \\ \overline{U} = \min \left\{ 1, \text{ solution of } V_{\tau''}(_1u''_{\tau''}, \overline{U}) = W_{\tau'}(_1u'_{\tau'}) \right\} \end{cases}$$

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