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New Concepts and Techniques for Equilibrium Analysis

Gerard Debreu

November 13, 1961

New Concepts and Techniques for Equilibrium Analysis[1]

by

Gerard Debreu

1. Introduction

In the study of the existence of an equilibrium for a private ownership economy, one meets with the basic mathematical difficulty that the demand correspondence of a consumer may not be upper semicontinuous when his wealth equals the minimum compatible with his consumption set.[2] One can prevent this minimum-wealth situation from ever arising by suitable assumptions on the economy; for example, in K. J. Arrow - G. Debreu [1], Theorem I, it is postulated that free disposal prevails and that every consumer can dispose of a positive quantity of every commodity from his resources and still have a possible consumption. However, assumptions of this type have not been readily accepted on account of their strength, and this in spite of th^ simplicity that they give to the analysis. Thus A. Wald [11], section II, K. J. Arrow - G. Debreu [1], Theorem II or II' ,

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[1]The research on which this paper reports was done partly at the Center for Advanced Study in the Behavioral Sciences and partly at the Cowles Foundation at Yale University under Task NR 047-006 with the Office of Naval Research.

I wish to acknowledge my debt to W. Isard for the stimulation I derived from the conversations I had with him on the possibility of weakening certain of the assumptions of W. Isard - D. J. Ostroff [5], to K. J. Arrow, D. Gale, L. Hurwicz, S. Kakutani, L. W. McKenzie and R. S. Phillips for their valuable comments and suggestions.

[2]Throughout this article I shall follow the notation and the terminology of [3].

L. W. McKenzie [7], [8], [9], D. Gale [4], H. Nikaido [10], and W. Isard - D. J. Ostroff [5] permit the minimum-wealth situation to arise but introduce features of the economy that nevertheless insure the existence of an equilibrium. The first purpose of the present article is to attempt to unify these various approaches. To this end, we use, for each consumer, a smoothed demand correspondence which coincides with the demand correspondence whenever the minimum-wealth situation does not arise and which is everywhere upper semicontinuous.[3] The existence proof is then carried out as before, but, because of the alteration of the demand correspondences, one obtains, instead of an equilibrium, a quasi-equilibrium, a formal definition of which follows.[4]

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[3]Similar smoothing operations have already been used in this area by H. W. Kuhn [6] and H. Nikaido [10].

[4]That definition is easily seen to imply $\overset{*}{p} \cdot \overset{*}{x}_i = \overset{*}{p} \cdot \omega_i + \sum_j \theta_{ij} \overset{*}{p} \cdot \overset{*}{y}_j$ for every i. From (α), $\overset{*}{p} \cdot \overset{*}{x}_i \leqq \overset{*}{p} \cdot \omega_i + \sum_j \theta_{ij} \overset{*}{p} \cdot \overset{*}{y}_j$ for every i. If the strict inequality occurred for some consumer, then, summing over i and using the fact that $\sum_i \theta_{ij} = 1$ for every j, one would obtain $\overset{*}{p} \cdot \sum_i \overset{*}{x}_i < \overset{*}{p} \cdot \omega + \overset{*}{p} \cdot \sum_j \overset{*}{y}_j$, a contradiction of ($\gamma$).

A quasi-equilibrium of the private ownership economy $\mathcal{E} = \left((X_i, \precsim_i), (Y_j), (\omega_i), (\theta_{ij}) \right)$ is an $(m + n + 1)$ - tuple $\left((x_i^*), (y_j^*), p^* \right)$ of points of $\left((X_i), (Y_j), R^\ell \right)$ respectively such that

(α) for every i, x_i^* is a greatest element of $\{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_j \theta_{ij} p^* \cdot y_j^*\}$

for $\precsim_i$ and/or $p^* \cdot x_i^* = p^* \cdot \omega_i + \sum_j \theta_{ij} p^* \cdot y_j^* = \operatorname{Min} p^* \cdot X_i$;

(β) for every j, $p^* \cdot y_j^* = \operatorname{Max} p^* \cdot Y_j$;

(γ) $\sum_i x_i^* - \sum_j y_j^* = \sum_i \omega_i$;

(δ) $p^* \neq 0$.

There remains only to establish that, in the private ownership economies for which an equilibrium has been proved to exist, there is a quasi-equilibrium which is an equilibrium. We will show in section 4 how this can be done.

The second purpose of this article is to deal with the fact, discovered by L.W. McKenzie [8], [9], that the irreversibility assumption on the total production set $(Y \cap (-Y) \subset \{0\})$ is superfluous by means of new techniques.

Instead of bounding the economy by a well-chosen cube, one uses an increasing sequence of cubes becoming indefinitely large. To each economy in this sequence one seeks to apply the general market equilibrium theorem of [2]. But the asymptotic cone AY of the total production set may be a linear manifold. This difficulty is resolved by adding to Y a certain cone $-\Delta$ with vertex 0 which has the properties that $AY - \Delta$ is not a linear manifold and that the solution of the problem is not altered by this addition.

Thirdly it will be proved that it suffices to assume, for every i, the insatiability of the i^{th} consumer in his attainable consumption set $\hat{X}_i$. This fact appeared as a simple remark in K. J. Arrow - G. Debreu [1]. But, in the presence of all the weakened assumptions that we are listing, its proof is no longer immediate. We shall further exploit the concept of attainability for consumption sets to strengthen the theorem in another way.[5]

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[5] According to the assumptions of the theorem, every $\hat{X}_i$ is compact (see the beginning of (b) in section 3, and the discussion in 5.4 of [3]). Thus if the economy has attainable states, i.e., if $\hat{X}_i$ is not empty, $\hat{X}_i$ has a greatest element $\bar{x}_i$ for $\precsim_i$. Assumption (b.1) then implies that $\{x_i \in X_i \mid x_i \succ_i \bar{x}_i\}$, which is equal to $\{x_i \in X_i \mid x_i \succ_i \hat{X}_i\}$, is not empty.

Moreover, if $x_i \succ_i \hat{X}_i$ for every i, then $\sum_i (x_i - \omega_i) \neq 0$. Equality to 0, which belongs to Y, would mean that every x_i is attainable. Therefore D is non-degenerate to $\{0\}$.

Finally, by (b.3), the set $\{x_i \in X_i \mid x_i \succ_i \hat{X}_i\} - \{\omega_i\}$ is convex for every i. Hence, the sum over i of these sets is convex. And D, which is the smallest cone with vertex 0 containing that sum, is also convex.

Let D be the smallest cone with vertex 0 owning all the points of the form $\sum_i (x_i - \omega_i)$, where $x_i \underset{i}{\succ} \hat{X}_i$ for every i. By adding $-D$ in (c.2) below we obtain a notably weaker assumption. Let us also note the connection between this problem and that discussed at the end of the last paragraph: one can choose for Δ any closed, convex cone with vertex 0, non-degenerate to $\{0\}$, contained in D and satisfying (c.2) when it is substituted for D.

Fourthly after having exploited the concept of attainability for consumption sets, we exploit it for the total production set.[6] The basic concept is presented in the following definition:

An augmented total production set is a subset $\ddot{Y}$ of the commodity space containing Y and such that

$$(\{\omega\} + \ddot{Y}) \cap X = (\{\omega\} + Y) \cap X ,$$

i.e., such that $\ddot{Y}$ and Y give rise to the same attainable consumptions. The set $\ddot{Y}$ takes the place of the set Y in assumption (c.2) below. Here again there results a strengthening of the theorem, which is considerable for some economies.

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[6] That a strengthening of the theorem in this direction should be possible was suggested to me by K. J. Arrow and L. W. McKenzie.

Our fifth purpose will be to show that the weak-convexity assumption on preferences « for every x_i' in X_i , the set $\{x_i \in X_i \mid x_i \underset{i}{\succsim} x_i'\}$ is convex » suffices to establish the theorem. This can be done without great difficulty once the proper concept, namely the restricted demand correspondence φ_i of lemma 1 , has been introduced.

Finally two trivial improvements will be made. The lower boundedness of the consumption sets and the impossibility of free production will be replaced respectively by $A X \cap (- AX) = \{0\}$ and $A X \cap A Y = \{0\}$. This will have the advantage of yielding a coordinate-free theory.

In conclusion we shall prove the

Theorem:[7] The private ownership economy $\mathcal{E}$ has a quasi-equilibrium if

(a.1) $A X \cap (-A X) = \{0\}$;

for every i (a.2) X_i is closed and convex,

(b.1) for every consumption x_i in $\hat{X}_i$, there is a consumption in X_i preferred to x_i ,

(b.2) for every x_i' in X_i , the sets $\{x_i \in X_i \mid x_i \underset{i}{\succsim} x_i'\}$ and $\{x_i \in X_i \mid x_i \underset{i}{\precsim} x_i'\}$ are closed in X_i ,

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[7]In fact, as it will be proved, $\mathcal{E}$ has a quasi-equilibrium $\left((x_i^*), (y_j^*), p^* \right)$ such that $p^* \cdot \sum_j y_j^* = \operatorname{Max} p^* \cdot (\ddot{Y} - D)$.

(b.3) for every x_i' in X_i, the set

$$\{x_i \in X_i \mid x_i \underset{i}{\succsim} x_i'\} \text{ is convex ;}$$

(c.1) $(\{\omega\} + Y) \cap X \neq \emptyset$,

(c.2) there is a closed, convex augmented total production set $\ddot{Y}$ such that,

for every i, $(\{\omega_i\} + A\,\ddot{Y} - D) \cap X_i \neq \emptyset$;

for every j (d.1) $0 \in Y_j$;

(d.2) $A\,X \cap A\,Y = \{0\}$.

Assumption (c.2) is now too weak to insure that $\mathcal{E}$ has attainable states. It was, therefore, necessary to add (c.1).

With the exception of (c.2) every assumption is so simple as not to require comments. Let us stress, however, that the case of bounded consumption sets and/or bounded production sets (and in particular the pure exchange case where $Y = \{0\}$) is covered by the theorem. As for (c.2), its complexity has seemed justified by the gain in generality that it permits.

2. Lemmata

In this section all the assumptions of the theorem hold. Moreover X is assumed to be bounded.

Since X_i is compact, the demand correspondence ξ_i of the i^{th} consumer is defined for every pair of a price system p and a wealth w_i such that $w_i \geqq \text{Min } p \cdot X_i$. The elements of $\xi_i(p, w_i)$ are the consumptions in $\gamma_i(p, w_i) = \{x_i \in X_i \mid p \cdot x_i \leqq w_i\}$ to which no consumption in $\gamma_i(p, w_i)$ is preferred. However, instead of letting the i^{th} consumer choose any consumption in $\xi_i(p, w_i)$, we restrict his choice to the most expensive ones, i.e., to the set

$$\varphi_i(p, w_i) = \{x_i \in \xi_i(p, w_i) \mid p \cdot x_i = \text{Max } p \cdot \xi_i(p, w_i)\}$$

An essential property of φ_i will be its upper semicontinuity, therefore we state

Lemma 1: *If $w_i^o \geqq \text{Min } p^o \cdot X_i$, then $\varphi_i(p^o, w_i^o)$ is non-empty, convex. If $w_i^o > \text{Min } p^o \cdot X_i$, then φ_i is upper semicontinuous at (p^o, w_i^o).*

Proof: The first implication is immediate, let us therefore prove the second. That is, let us study two infinite sequences

$(p^q, w_i^q) \to (p^o, w_i^o)$ and $x_i^q \to x_i^o$ such that $x_i^q \in \varphi_i(p^q, w_i^q)$ for every q.

We must show that $x_i^o \in \varphi_i(p^o, w_i^o)$.

By (1) of 4.8 and (1) of 4.10 in [3], ξ_i is upper semicontinuous at (p^o,w_i^o), hence $x_i^o \in \xi_i\,(p^o,w_i^o)$. Therefore it suffices to show that

$$x_i \in \xi_i\,(p^o,w_i^o) \Longrightarrow p^o\cdot x_i \leqq p^o\cdot x_i^o \,,$$

i.e., $p^o\cdot x_i \leqq w_i^o$ and $x_i \underset{i}{\sim} x_i^o \Rightarrow p^o\cdot x_i \leqq p^o\cdot x_i^o$.

Since $p^q\cdot x_i^q \leqq w_i^q$ for every q , two cases will be distinguished:

(i) $p^o\cdot x_i^o = w_i^o$. Then, obviously, $p^o\cdot x_i \leqq p^o\cdot x_i^o$.

(ii) $p^o\cdot x_i^o < w_i^o$. Then, for q large enough, $p^o\cdot x_i^q < w_i^o$.

Hence $x_i^q \underset{i}{\precsim} x_i^o$, for $x_i^o \in \xi_i\,(p^o,w_i^o)$. Consider now a point x_i' different from x_i on the segment $[x_i^o,x_i]$. As $p^o\cdot x_i \leqq w_i^o$ and $p^o\cdot x_i^o < w_i^o$, one has $p^o\cdot x_i' < w_i^o$ and, for q large enough, $p^q\cdot x_i' < w_i^q$. Moreover $x_i' \underset{i}{\succ} x_i^o \underset{i}{\succsim} x_i^q$, the first relation following from $x_i \underset{i}{\sim} x_i^o$ and (b.3). But $x_i' \underset{i}{\succ} x_i^q$ with $p^q\cdot x_i' < w_i^q$ and $x_i^q \in \varphi_i\,(p^q,w_i^q)$ implies $p^q\cdot x_i' \leqq p^q\cdot x_i^q$. In the limit, $p^o\cdot x_i' \leqq p^o\cdot x_i^o$. And, since x_i' is arbitrarily close to x_i , also $p^o\cdot x_i \leqq p^o\cdot x_i^o$, Q. E. D.

To smooth the correspondence φ_i we define the correspondence ψ_i by:

if $w_i > \text{Min } p \cdot X_i$, *then* $\psi_i(p, w_i) = \varphi_i(p, w_i)$;

if $w_i = \text{Min } p \cdot X_i$, *then* $\psi_i(p, w_i) = \{x_i \in X_i \mid p \cdot x_i = w_i\}$.

Lemma 2: *If* $w_i^o \geqq \text{Min } p^o \cdot X_i$, *then* $\psi_i(p^o, w_i^o)$ *is non-empty, convex and* ψ_i *is upper semicontinuous at* (p^o, w_i^o).

Proof: If $w_i^o > \text{Min } p^o \cdot X_i$, this is only a restatement of Lemma 1.

If $w_i^o = \text{Min } p^o \cdot X_i$, the proof is immediate.

The success of the technique that consists in bounding the economy by a sequence of cubes rests on the following simple remark.

Lemma 3: *Let* $\mathcal{Y}^q$ *be a non-decreasing infinite sequence of subsets of the commodity space having* $\mathcal{Y}$ *as their union. Let* p^q *be an infinite sequence of price systems tending to* p^*. *Then* $\lim (\text{Sup } p^q \cdot \mathcal{Y}^q) \geqq \text{Sup } p^* \cdot \mathcal{Y}$.

<u>Proof</u>: Let y be a point in $\mathcal{Y}$. For q large enough, $y \in \mathcal{Y}^q$, therefore $p^q \cdot y \leqq \operatorname{Sup} p^q \cdot \mathcal{Y}^q$. In the limit, $p^* \cdot y \leqq \underline{\lim} (\operatorname{Sup} p^q \cdot \mathcal{Y}^q)$. Hence the result.

The next four lemmata state fundamental properties of the sets $\dot{Y}$ and D. It will be convenient to agree that

<u>$E(Y)$ denotes the economy $\left((X_i, \precsim_i), Y, \omega\right)$</u>.

<u>Lemma 4</u>: <u>$A\ddot{Y} \cap D = \{0\}$</u>.

<u>Proof</u>: Let y be a point in the intersection. $y \in A\ddot{Y}$ and there are an m-tuple (x_i) such that $x_i \succ_i \hat{x}_i$ for every i, and a number $\lambda \geqq 0$ satisfying the equality $y = \lambda \sum_i (x_i - \omega_i)$. If $\lambda > 0$, we divide by λ and obtain $\frac{y}{\lambda} = \sum_i x_i - \omega$. The point $\frac{y}{\lambda}$ is also in $A\ddot{Y}$, which is contained in $\ddot{Y}$ by (14) of 1.9 in [3]. Thus (x_i) is attainable for $E(\ddot{Y})$, hence for $E(Y)$, a contradiction. Therefore $\lambda = 0$.

Lemma 6: $A\ddot{Y} - \Delta$ is not a linear manifold.

Proof: Assume the contrary. $A\ddot{Y} - \Delta$, which contains $-\Delta$, would also contain Δ. Thus, given δ_1 in Δ different from 0, there would be y in $A\ddot{Y}$ and δ_2 in Δ such that $y - \delta_2 = \delta_1$, i.e., $y = \delta_1 + \delta_2$. Since $\delta_1 + \delta_2 \in \Delta$, this implies, by Lemma 4, that $\delta_1 + \delta_2 = 0$. Hence, by Lemma 5, $\delta_1 = 0 = \delta_2$, a contradiction of $\delta_1 \neq 0$.

Lemma 7: If the consumption x_i is attainable for the economy $E(\ddot{Y} - D)$, then $x_i \succ_i \hat{X}_i$ does not hold.

Proof: Consider an attainable state of $E(\ddot{Y} - D)$. The sum of the consumptions in that state satisfies $\sum_i x_i - \omega = y - \delta$ where $y \in \ddot{Y}$ and $\delta \in D$. The last relation can also be written $\delta = \lambda \sum_i (x_i' - \omega_i)$ with $\lambda \geqq 0$ and $x_i' \succ_i \hat{X}_i$ for every i. Therefore

$$\sum_i x_i + \lambda \sum_i x_i' = \omega(1 + \lambda) + y .$$

Divide by $1+\lambda$, putting $\alpha = \frac{1}{1+\lambda}$ and $\beta = \frac{\lambda}{1+\lambda}$,

$$\sum_i (\alpha x_i + \beta x_i') = \omega + \alpha y .$$

Since $\alpha x_i + \beta x_i' \in X_i$ for every i and $\alpha y \in \ddot{Y}$, the consumption $\alpha x_i + \beta x_i'$ is attainable for $E(\ddot{Y})$, hence for $E(Y)$. If $x_i \succ_i \hat{x}_i$, then $\alpha x_i + \beta x_i' \succ_i \hat{x}_i$, a contradiction.

The last lemma concerns the approximation process by means of which the statement of footnote 7 will be proved.

Lemma 8: Let C be a convex cone with vertex 0 in the commodity space. There is a non-decreasing sequence (Γ^q) of closed, convex cones with vertex 0, contained in C and whose union contains the relative interior of C.

Proof: Since the problem can be treated in the smallest linear subspace containing C, there is no loss of generality in assuming that C has a non-empty interior. We shall also assume that C is non-degenerate to $\{0\}$;

in that case the theorem is trivially true. Denote by $|z|$ the norm of the vector z, by S the set of vectors with unit norm, $\{z \in R^{\ell} \mid |z| = 1\}$, and, given z in R^{ℓ} and a positive real number r, by $s(z,r)$ the set of points whose distance to z is less than r, $\{z' \in R^{\ell} \mid |z'-z| < r\}$. Consider the set $\{z \in S \mid s(z,\frac{1}{q}) \subset C\}$, which is not empty for q large enough. We will show that Γ^q, the smallest cone with vertex 0 containing that set, has all the required properties.

Γ^q is closed. To prove this, it suffices to study an infinite sequence (z^k) of points of $\Gamma^q \cap S$ tending to z^o. We wish to show that C contains $s(z^o, \frac{1}{q})$. Let z be a point of the latter set. One has $|z - z^o| < \frac{1}{q}$. Hence, for k large enough, $|z - z^k| < \frac{1}{q}$. Therefore z belongs to $s(z^k, \frac{1}{q})$, which is contained in C.

Γ^q is convex. To prove this, it suffices to study two points z^1, z^2 in $\Gamma^q \cap S$ and one of their convex combinations $z^o = \alpha^1 z^1 + \alpha^2 z^2$ different from 0. We wish to show that C contains $s(\frac{z^o}{|z^o|}, \frac{1}{q})$. Let z be a point of the latter set. One has $|z - \frac{z^o}{|z^o|}| < \frac{1}{q}$. However $|z^o| \leq 1$ by convexity of the norm. Hence $|\,|z^o|z - z^o| < \frac{1}{q}$. Therefore, the points $z^1 + (|z^o|z - z^o)$ and $z^2 + (|z^o|\, z - z^o)$ both belong to C.

Thus, their convex combination with coefficients α^1, α^2, which is $|z^o|z$, also belongs to C. Hence z does.

It is clear that $q' > q$ implies $\Gamma^{q'} \supset \Gamma^q$, that the Γ^q are contained in C, and that their union contains the interior of C.

3. Proof of the Theorem.

The proof will be decomposed into two parts. Initially the total consumption set will be assumed to be bounded. Later the general case will be treated.

Let us remark at the outset that, according to (c.2), and because D is non-degenerate (see footnote 5), there is in D, for each i, a closed half-line L_i with origin 0 such that $\{\omega_i\} + A\ddot{Y} - L_i$ intersects X_i.

(a) Case of a bounded X.

The cone Δ, which will remain fixed until the end of (a), is chosen to be a closed, convex cone with vertex 0, containing the m half-lines L_i and contained in D. Such a choice is possible because D is convex (see footnote 5). Clearly, Δ is non-degenerate and satisfies (c.2) when one substitutes it for D.

Let now K^q be an increasing sequence of closed cubes with center 0, becoming indefinitely large. Remembering that n is the number of producers, we introduce the notation:

$$Y_j^q = Y_j \cap K^q , \quad Y^q = (\ddot{Y} - \Delta) \cap (n K^q) .$$

Given an arbitrary price system p , the supremum of profit on Y_j^q is finite (Y_j^q is bounded), and the maximum of profit on Y^q exists (Y^q is compact since $\ddot{Y} - \Delta$ is closed by lemma 4). We introduce the further notation:

$$\Pi_j^q(p) = \text{Sup } p.Y_j^q , \quad \Pi^q(p) = \text{Max } p.Y^q , \quad d^q(p) = \Pi^q(p) - \sum_j \Pi_j^q(p) .$$

As $\sum_j Y_j^q \subset Y^q$, we have

$$d^q(p) \geqq 0 \quad \text{for every } p .$$

Finally, we denote the set of y that maximize profit on Y^q by

$$\eta^q(p) = \left\{ y \in Y^q \mid p.y = \Pi^q(p) \right\} .$$

It follows immediately from (3) of 3.5 in [3] that the correspondence η^q is upper semicontinuous everywhere, and that the functions Π_j^q , Π^q , hence the functions d^q , are continuous everywhere.

We give to the i^{th} consumer the wealth

$$w_i^q(p) = p.\omega_i + \sum_j \Theta_{ij} \Pi_j^q(p) + \frac{1}{m} d^q(p) ,$$

m being the number of consumers. Notice that

(1) for every p , $w_i^q(p) \geqq p.\omega_i$ and $\sum_i w_i^q(p) = p.\omega + \Pi^q(p)$.

The first assertion follows from $\Pi_j^q(p) \geqq 0$ (since $0 \in Y_j^q$) and $d^q(p) \geqq 0$. The second follows from $\sum_i \Theta_{ij} = 1$ for every j and from the definition of d^q. Notice also that w_i^q is clearly continuous everywhere.

The price system p will now be restricted to the set

$$P = (A\ddot{Y} - \Delta)^o \cap S$$

where $(A\ddot{Y} - \Delta)^o$ is the polar of $A\ddot{Y} - \Delta$ and S is the set of vectors with unit norm. Every x_i in $(\{\omega_i\} + A\ddot{Y} - \Delta) \cap X_i$ satisfies $p.x_i \leqq p.\omega_i$ for every p in P. Hence $w_i^q(p) \geqq \text{Min}\, p.X_i$ for every p in P. Therefore the correspondence ζ^q such that

$$\zeta^q(p) = \sum_i \psi_i(p, w_i^q(p)) - \eta^q(p) - \{\omega\}$$

is defined everywhere on P. According to lemma 2, and on account of the continuity of w_i^q and of the upper semicontinuity of η^q, the correspondence ζ^q is upper semicontinuous on P; moreover, for every p in P, the set $\zeta^q(p)$ is easily seen to be non-empty, convex and to satisfy $p.\zeta^q(p) \leqq 0$ (since any x_i in $\psi_i(p, w_i^q(p))$ satisfies $p.x_i \leqq w_i^q(p)$, any y in $\eta^q(p)$ satisfies $p.y = \Pi^q(p)$, and $\sum_i w_i^q(p) = p.\omega + \Pi^q(p)$); finally, by lemmata 4 and 6, $A\ddot{Y} - \Delta$ is a closed, convex cone with vertex 0, which is not a linear manifold. Thus the theorem of [2] can be applied to the cone $(A\ddot{Y} - \Delta)^o$ and the correspondence ζ^q. There are

$p^q \in P$, $z^q \in A\ddot{Y} - \Delta$ such that $z^q \in \zeta^q(p^q)$.

In other words, there are $x_i^q \in \psi_i\ (p^q, w_i^q(p^q)\ )$ and $\bar{y}^q \in \eta^q\ (p^q)$ such that

$$\sum_i x_i^q - \bar{y}^q - \omega = z^q \ .$$

Introducing $y^q = \bar{y}^q + z^q$, one obtains

$$(2) \qquad \sum_i x_i^q - y^q - \omega = 0 \ .$$

However, $\bar{y}^q \in \ddot{Y} - \Delta$ and $z^q \in A\ddot{Y} - \Delta$ imply

$$(3) \qquad y^q \in \ddot{Y} - \Delta$$

(because $\ddot{Y} + A\ddot{Y} \subset \ddot{Y}$ by (14) of 1.9 in [3]). Therefore x_i^q is attainable for the economy $E(\ddot{Y} - \Delta)$. And, by lemma 7 , if $x_i \succsim_i \hat{X}_i$, then $x_i \succsim_i x_i^q$. This, jointly with $x_i^q \in \psi_i\ (p^q, w_i^q\ (p^q)\ )$, will be shown to imply

$$(4) \qquad p^q \cdot x_i^q = w_i^q\ (p^q) \ .$$

If $w_i^q(p^q) = \text{Min}\ p^q \cdot X_i$, then the equality is obvious.

If $w_i^q(p^q) > \text{Min}\ p^q \cdot X_i$, then $x_i^q \in \varphi_i\ (p^q, w_i^q\ (p^q)\ )$. Hence

$p^q \cdot x_i > w_i^q(p^q)$. Therefore, if $p^q \cdot x_i^q < w_i^q(p^q)$, the points of the segment $[x_i^q, x_i]$ close enough to x_i^q would satisfy the wealth constraint defined by $(p^q, w_i^q(p^q))$, be at least as desired as x_i^q, <u>and be more expensive than x_i^q</u>. This would contradict the definition of φ_i.

Summing (4) over i, and using (1) one obtains $p^q \cdot \sum_i x_i^q = p^q \cdot \omega + \Pi^q(p^q)$. According to (2), this proves that

$$p^q \cdot y^q = \Pi^q(p^q) .$$

Now, the p^q belong to the bounded set S; the m-tuples (x_i^q) belong to $\prod_i X_i$, which is a product of bounded sets; therefore, by (2), the y^q are bounded; and the numbers $p^q \cdot y^q$ are also bounded. The $\Pi_j^q(p^q)$ are non-negative and their sum over j is at most equal to $\Pi^q(p^q)$, that is to $p^q \cdot y^q$. Hence the n-tuples $(\Pi_j^q(p^q))$ are bounded. Let us therefore extract a subsequence of the $(p^q, (x_i^q), (\Pi_j^q(p^q)))$ converging to $(p^*, (x_i^*), (\Pi_j^*))$, still using the index q for the convergent subsequence since no ambiguity can arise. According to (2), y^q tends to y^* which satisfies

$$\sum_i x_i^* - y^* - \omega = 0 . \tag{5}$$

And, by (3) and the closedness of $\ddot{Y} - \Delta$,

(6) $$y^* \in \ddot{Y} - \Delta .$$

Also $d^q(p^q)$ tends to $d^* = p^*.y^* - \sum_j \Pi_j^*$, and for every i,

(7) $$w_i^q(p^q) \text{ tends to } w_i^* = p^*. \omega_i + \sum_j \Theta_{ij} \Pi_j^* + \frac{1}{m} d^* .$$

While, by upper semicontinuity of ψ_i,

(8) $$x_i^* \in \psi_i (p^*, w_i^*), \text{ for every } i .$$

By a first application of lemma 3, $p^q. y^q = \text{Max } p^q. Y^q$ implies $p^*. y^* \geqq \text{Sup } p^*. (\ddot{Y} - \Delta)$. But $y^* \in \ddot{Y} - \Delta$, therefore

(9) $$p^*. y^* = \text{Max } p^*. (\ddot{Y} - \Delta).$$

By a second application of lemma 3, $\Pi_j^q (p^q) = \text{Sup } p^q. Y_j^q$ for every j implies

(10) $$\Pi_j^* \geqq \text{Sup } p^*. Y_j \quad \text{for every } j .$$

According to (6),

(11) $$y^* = y' - \delta ,$$

where $y' \in \ddot{Y}$ and $\delta \in \Delta$. Since, by (9), y^* maximizes profit relative to p^* on $\ddot{Y} - \Delta$, so do y' on $\ddot{Y}$ and $- \delta$ on $- \Delta$. The latter implies that

$$p^*. \delta = 0 .$$

As $\delta \in D$, it has the form $\delta = \lambda \sum_i (x_i - \omega_i)$ where $\lambda \geqq 0$ and $x_i \succ_i \hat{X}_i$ for every i. But (5) and (6) show that each x_i^* is attainable for the economy $E(\ddot{Y} - \Delta)$. Hence, by lemma 7, $x_i^* \prec_i x_i$. This establishes

(12) if $w_i^* > \text{Min } p^* \cdot X_i$, then $p^* \cdot x_i > w_i^*$,

for $w_i^* > \text{Min } p^* \cdot X_i$ implies, by (8), that $x_i^* \in \varphi_i(p^*, w_i^*)$. On the other hand, it is obvious that

(13) if $w_i^* = \text{Min } p^* \cdot X_i$, then $p^* \cdot x_i \geqq w_i^*$.

To conclude the first part of the proof we distinguish two cases:

(a.a) <u>$w_{i'}^* > \text{Min } p^* \cdot X_{i'}$ for some i'</u>.

Then, from (12) and (13), $p^* \cdot \sum_i x_i > \sum_i w_i^* = p^* \cdot \omega + p^* \cdot y^*$. Therefore $p^* \cdot \sum_i (x_i - \omega_i) > p^* \cdot y^* \geqq 0$, the last inequality resulting from the fact that y^* maximizes profit relative to p^* on a set owning 0. However, $p^* \cdot \sum_i (x_i - \omega_i) > 0$ and $p^* \cdot \delta = 0$ yield $\lambda = 0$, i.e., $\delta = 0$. Thus, by (11), $y^* \in \ddot{Y}$ and, on account of (5), $y^* \in Y$. As $Y \subset Y - \Delta$, (9) implies $p^* \cdot y^* = \text{Max } p^* \cdot Y$. But summing (10) over j, one obtains $\sum_j \Pi_j^* \geqq \text{Sup } p^* \cdot Y$. Consequently, $d^* = p^* \cdot y^* - \sum_j \Pi_j^*$, which is non-negative, is actually zero. And, for every j, $\Pi_j^* = \text{Sup } p^* \cdot Y_j$. It now suffices to take in each Y_j a y_j^* in such a way that $\sum_j y_j^* = y^*$ to obtain a quasi-equilibrium $\left((x_i^*), (y_j^*), p^* \right)$ of $\mathcal{E}$. Indeed (δ) of the definition of a quasi-equilibrium is satisfied because $p^* \in P$; (γ) is (5); (β) is fulfilled because $p^* . y^* = \text{Max } p^* . Y$ implies $p^* . y_j^* = \text{Max } p^* . Y_j$ for every j; (α) is satisfied because of (8) and because (7) has become $w_i^* = p^* . \omega_i + \sum_j \Theta_{ij} \, p^* . y_j^*$.

(a.b) $\underline{w_i^* = \text{Min } p^* . X_i}$ <u>for every i</u> .

By (8), $p^* . x_i^* = \text{Min } p^* . X_i$ for every i , therefore $p^* . \Sigma x_i^* = \text{Min } p^* . X$ while, by (9), $p^* . y^* = \text{Max } p^* . (\ddot{Y} - \Delta) \geqq \text{Sup } p^* . Y$. Hence, the hyperplane H with normal p^* going through the point $\sum_i x_i^*$, which is also $\omega + y^*$ by (5), separates X and $\{\omega\} + Y$. But, by (c.1), the economy E(Y) has attainable states. We now show that any one of them $((x_i'), (y_j'))$ forms with p^* a quasi-equilibrium of $\mathcal{E}$. Indeed the point $\sum_i x_i' = \omega + \sum_j y_j'$ is necessarily in the hyperplane H . Therefore $p^* . \sum_i x_i' = \text{Min } p^* . X$, and $p^* . \sum_j y_j' = \text{Max } p^* . Y$. These equalities respectively imply $p^* . x_i' = \text{Min } p^* . X_i$ for every i , and $p^* . y_j' = \text{Max } p^* . Y_j$ for every j . Finally, we recall that, by (10), $\Pi_j^* \geqq \text{Max } p^* . Y_j$ for every j , that $d^* \geqq 0$, and that $\sum_j \Pi_j^* + d^* = p^* . y^*$. As $\omega + y^*$ is in the hyperplane H , one has

$p^* \cdot y^* = \operatorname{Max} p^* \cdot Y = \sum_j \operatorname{Max} p^* \cdot Y_j$, and all the inequalities above must be equalities. Therefore (7) becomes $w_i^* = p^* \cdot \omega_i + \sum_j \Theta_{ij}\, p^* \cdot y_j'$.

(b) <u>General case.</u>

An immediate transposition of the proof of (2) of 5.4 in [3] shows that the set of attainable states of the economy $E(Y)$ is bounded; it is also closed for it coincides with the set of attainable states of the economy $E(\ddot{Y})$, to which one applies (1) of 5.4 in [3]. Hence $\hat{X}_i$ is compact for every i . Let then K^q be an increasing sequence of closed cubes with center 0 , becoming indefinitely large, containing the $\hat{X}_i$ and owning, for each i , a consumption preferred to $\hat{X}_i$ and a consumption in the intersection of X_i and $\{\omega_i\} + A\,\ddot{Y} - L_i$, where the L_i are the half-lines described at the outset of this section. We introduce the notation

$$X_i^q = X_i \cap K^q .$$

Consider now a sequence (Γ^q) of cones with vertex 0 having all the properties listed in lemma 8, with D substituted for C . We define Δ^q as $\Gamma^q + \sum_i L_i$. This is a convex cone with vertex 0 , non-degenerate, contained in D and satisfying (c.2) for the private ownership economy $\mathcal{E}^q = \left((X_i^q, \precsim_i), (Y_j), (\omega_i), (\Theta_{ij})\right)$. The cone Δ^q is also closed as a sum of closed cones with vertex 0 , all contained in D which, by lemma 5, satisfies $D \cap (-D) = \{0\}$ (see (9) of 1.9 in [3]). Moreover the sequence of the Δ^q is non-decreasing and their union contains the relative interior of D since the union of the Γ^q does.

According to part (a) of the proof (see (9) in particular), for every q, the economy $\mathcal{E}^q$ has a quasi-equilibrium $\left((x_i^q), (y_j^q), p^q \right)$ such that $p^q \in S$ and $p^q \cdot \sum_j y_j^q = \operatorname{Max} p^q \cdot (\ddot{Y} - \Delta^q)$.

The m-tuples (x_i^q) are attainable for $E(Y)$, hence bounded; the total productions $\sum_j y_j^q$, which equal $\sum_i x_i^q - \omega$, are therefore bounded; and the p^q are bounded since they have a unit norm. Putting

$$(14) \qquad \Pi_j^q = p^q \cdot y_j^q = \operatorname{Max} p^q \cdot Y_j ,$$

and noting that $\Pi_j^q \geqq 0$ for every j and that $\sum_j \Pi_j^q = p^q \cdot \sum_j y_j^q$, which is bounded, we establish that the n-tuples (Π_j^q) are bounded.

Let us then extract a subsequence of the $\left((x_i^q), (\Pi_j^q), p^q \right)$ converging to $\left((x_i^*), (\Pi_j^*), p^* \right)$, still using the index q for the convergent-subsequence. $\sum_j y_j^q$ tends to $y^* = \sum_i x_i^* - \omega$.

Since the total production $\sum_j y_j^q$ is attainable for $E(Y)$, it belongs to $Y \cap (X - \{\omega\}) = \ddot{Y} \cap (X - \{\omega\})$. As the latter is closed, $y^* \in Y$.

Thus we can choose, for every j, a y_j^* in Y_j in such a way that $\sum_j y_j^* = y^*$. We shall prove that $\left((x_i^*), (y_j^*), p^*\right)$ is a quasi-equilibrium of $\mathcal{E}$, and that $p^* \cdot y^* = \operatorname{Max} p^* \cdot (\ddot{Y} - D)$.

We deal with the last fact, first. Let z be an arbitrary point in $\ddot{Y}$ - relative interior of D, i.e., $z = y - \delta$ where y is in $\ddot{Y}$ and δ is in the relative interior of D. For q large enough, $\delta \in \Delta^q$. Therefore $p^q \cdot (y - \delta) \leqq p^q \cdot \sum_j y_j^q$. In the limit, $p^* \cdot (y - \delta) \leqq p^* \cdot y^*$. Hence $p^* \cdot (\ddot{Y}$ - relative interior of $D) \leqq p^* \cdot y^*$. Hence also $p^* \cdot (\ddot{Y} - D) \leqq p^* \cdot y^*$.

By lemma 3, (14) implies $\Pi_j^* \geqq \operatorname{Sup} p^* \cdot Y_j$. However $\sum_j \Pi_j^* = p^* \cdot y^*$, while $y^* \in Y$ implies $p^* \cdot y^* \leqq \operatorname{Sup} p^* \cdot Y$. Consequently, $p^* \cdot y^* = \operatorname{Sup} p^* \cdot Y$ and $\Pi_j^* = \operatorname{Sup} p^* \cdot Y_j$, for every j. This means that y^* maximizes profit relative to p^* on Y, hence so does every y_j^* on Y_j. Therefore

$$\Pi_j^* = p^* \cdot y_j^* = \operatorname{Max} p^* \cdot Y_j \quad \text{for every } j .$$

There remains to check that (α) of the definition of a **quasi**-equilibrium is satisfied. Denote $p^q \cdot \omega_i + \sum_j \Theta_{ij} \Pi_j^q$ by w_i^q, and its limit, $p^* \cdot \omega_i + \sum_j \Theta_{ij} \Pi_j^*$, by w_i^*. According to footnote 4, $p^q \cdot x_i^q = w_i^q$ for every (i, q), hence, in the limit, $p^* \cdot x_i^* = w_i^*$ for every i . <u>Let us, therefore, assume that $w_i^* = \text{Min } p^* \cdot X_i$ does not hold for the i^{th} consumer.</u>

Consider x_i' in X_i such that $p^* \cdot x_i' < w_i^*$. The existence of such points is insured by the assumption. For q large enough, $p^q \cdot x_i' < w_i^q$ and $x_i' \in X_i^q$, hence $w_i^q > \text{Min } p^q \cdot X_i^q$ and, by definition of a quasi-equilibrium for $\mathcal{E}^q$, we have $x_i^q \underset{i}{\succsim} \{x_i \in X_i^q \mid p.^q x_i \leqq w_i^q\}$. Therefore $x_i' \underset{i}{\precsim} x_i^q$. In the limit, $x_i' \underset{i}{\precsim} x_i^*$.

Consider now $\{x_i \in X_i \mid p^* \cdot x_i \leqq w_i^*\}$. Any point x_i of that set can be approximated by points x_i' of X_i for which $p^* \cdot x_i' < w_i^*$. Since every such x_i' satisfies $x_i' \underset{i}{\precsim} x_i^*$, one also has $x_i \underset{i}{\precsim} x_i^*$. And x_i^* is indeed a greatest element of $\{x_i \in X_i \mid p^* \cdot x_i \leqq w_i^*\}$ for $\underset{i}{\precsim}$.

4. <u>Equilibrium and Quasi-equilibrium</u>.

To prove that a certain private ownership economy $\mathcal{E}$ has an equilibrium, it suffices to prove that $\mathcal{E}$ has a quasi-equilibrium in which

$$(\alpha.2) \qquad p^* \cdot x_i^* = p^* \cdot \omega_i + \sum_j \Theta_{ij} \, p^* \cdot y_j^* = \text{Min } p^* \cdot X_i$$

occurs for no consumer.

A simple way of obtaining such a quasi-equilibrium is to replace $\ll A\ddot{Y} - D \gg$ by $\ll$ Interior of $A\ddot{Y} - D \gg$ in assumption (c.2). According to footnote 7, $\mathcal{E}$ has a quasi-equilibrium whose price system p^* belongs to the polar of $A\ddot{Y} - D$. Therefore $p^* \cdot \omega_i > \text{Inf } p^* \cdot X_i$ for every i, and (α.2) cannot occur. Theorem I of K. J. Arrow - G. Debreu [1] is of this type, since it assumes implicitly that Y contains $-\Omega$, the non-positive orthant, and explicitly that

$$(\{\omega_i\} - \text{Interior of } \Omega) \cap X_i \neq \emptyset \quad \text{for every } i .$$

In W. Isard - D. J. Ostroff [5], the emphasis is on the location aspect of equilibrium. Let us suppose that their hypotheses on the technology are altered along the lines of the theorem of this article so as to insure that a quasi-equilibrium exists.[8] If free disposal prevails, the price system in this quasi-equilibrium is non-negative. According to [5], in each region, each consumer can obtain a possible consumption by disposing of a positive amount of every commodity located in his region. Therefore, (α.2) occurs for him only if the prices of all the commodities in his region are zero. Assume that such is the case. If there were, in some other region, a commodity with a positive price, the economy of [5] is such that an exporter from the first region to the second could increase his profit indefinitely. This contradicts (β) of the definition of a quasi-equilibrium. Hence, all prices would be zero, a contradiction of (δ). Consequently, (α.2) occurs for no consumer.

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[8] One can construct an economy with two regions, one good and one transportation service, and a constant returns to scale, free disposal technology satisfying all their assumptions and such that a total production, every coordinate of which is positive, is possible. That economy cannot have an equilibrium, since, for any price system different from 0, the total profit of producers can be indefinitely increased.

We now strengthen assumption (c.2) of the theorem, adding to it

<u>and that the relative interiors of $\{\omega\} + \ddot{Y}$ and of X have a non-empty intersection.</u>

And we call (c) the result of this addition, which makes (c.1) redundant. This strengthening is a generalization of the second part of assumption 5 of L. W. McKenzie [8]. We also assume[9]

(e) <u>if, in a quasi-equilibrium, $p^* \cdot x_i^* = \text{Min}\ p^* \cdot X_i$ occurs for some consumer, then it occurs for every consumer.</u>

We then prove

<u>Proposition:</u> <u>The private ownership economy $\mathcal{E}$ has an equilibrium if it satisfies (e) and the assumptions of the theorem where (c.1) and (c.2) are replaced by (c).</u>

<u>Proof:</u> Let $\mathcal{L}$ be the smallest linear manifold containing $X - \{\omega\} - \ddot{Y}$. According to (c), the origin belongs to $\mathcal{L}$, which is therefore a linear subspace of the commodity space. Since $0 \in \ddot{Y}$, the set $X - \{\omega\}$ is contained in $X - \{\omega\} - \ddot{Y}$, hence in $\mathcal{L}$. Moreover, $\ddot{Y} \subset (X - \{\omega\}) - (X - \{\omega\} - \ddot{Y})$. As both sets in this difference are

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[9] Notice, from the proof of the proposition, that it suffices to make this assumption for quasi-equilibria such that $p^* \cdot \sum_j y_j^* = \text{Max}\ p^* \cdot (\ddot{Y} - D)$.

contained in $\mathcal{L}$, so is $\ddot{Y}$. Consider now the set $\sum_i \left\{ x_i \in X_i \mid x_i \underset{i}{\succ} \hat{x}_i \right\} - \{\omega\}$ at the end of footnote 5. It is contained in $X - \{\omega\}$, hence in $\mathcal{L}$, and so is the cone D. According to (c), the set $X_i - \{\omega_i\}$ intersects $A\ddot{Y} - D$. But both $A\ddot{Y}$ and D are contained in $\mathcal{L}$. Therefore, every set $X_i - \{\omega_i\}$ intersects $\mathcal{L}$, while their sum $X - \{\omega\}$ is contained in $\mathcal{L}$. To see that this implies « $X_i - \{\omega_i\} \subset \mathcal{L}$ for every i », take x_i in $(X_i - \{\omega_i\}) \cap \mathcal{L}$ for each i. The sets $X_i - \{\omega_i\} - \{x_i\}$ own 0, hence their sum $X - \{\omega\} - \left\{ \sum_i x_i \right\}$ contains them all. However, this sum is contained in $\mathcal{L}$, since $\sum_i x_i$ belongs to $\mathcal{L}$. Finally, observe that $Y_j \subset \ddot{Y}$ for every j. In conclusion, $\mathcal{L}$ contains every $X_i - \{\omega_i\}$ and every Y_j, and, following L. W. McKenzie [8], we can treat the equilibrium problem <u>in $\mathcal{L}$</u>.

According to the theorem, there is a quasi-equilibrium $\left((x_i^*), (y_j^*), p^*\right)$ such that $p^* \cdot \sum_j y_j^* = \text{Max } p^* \cdot (\ddot{Y} - D)$. We will show that $(\alpha.2)$ occurs for no consumer. Assume that it occurs for one of them; by (e), it occurs for all. Thus $p^* \cdot x_i^* = \text{Min } p^* \cdot X_i$ for every i, hence $p^* \cdot \sum_i x_i^* = \text{Min } p^* \cdot X$. Therefore, the hyperplane H with normal p^*, through $\sum_i x_i^*$, separates X and $\{\omega\} + \ddot{Y} - D$. <u>A fortiori</u> it separates

X and $\{\omega\} + \ddot{Y}$. H cannot contain both sets, for $\mathcal{L}$ would not be the smallest linear manifold containing $X - \{\omega\} - \ddot{Y}$. Thus, one of them has points strictly on one side of H. Consequently, its relative interior is strictly on that side of H and cannot intersect the relative interior of the other set, a contradiction of (c).

The proposition that we have just established generalizes the results of A. Wald [11] section II, K. J. Arrow - G. Debreu [1], Theorem II or II' , D. Gale [4], H. Nikaido [10], and L. W. McKenzie [8], [9] . The only assumption for which it is not obvious that it holds in these various cases is (e) . We will give two illustrations of the reasoning involved in checking this point.

In the economy of Theorem II' of K. J. Arrow - G. Debreu [1] , there is a (non-empty) set $\mathcal{D}'$ of always desired commodities such that, for every i, for every consumption x_i in $\hat{X}_i$, and for every h in $\mathcal{D}'$, the i^{th} consumer can obtain a consumption in X_i preferred to x_i by increasing the h^{th} coordinate of x_i. There is also a set $\mathcal{P}'$ of always productive commodities such that for every attainable total production y and for every h in $\mathcal{P}'$, one can obtain a production in Y whose output of every commodity different from h is at least as large as in y, and whose output of at least one commodity in $\mathcal{D}'$ is larger than in y. It is assumed that each consumer can dispose of a positive quantity of at least one commodity in $\mathcal{D}' \cup \mathcal{P}'$ from his resources and still have a possible consumption. The economy has a quasi-equilibrium $\left((x_i^*), (y_j^*), p^*\right)$

and p^* is non-negative since free disposal prevails. Let us suppose that (α.2) occurs for the i'^{th} consumer. Thus, at least one commodity in $\mathcal{D}' \cup \mathcal{P}'$ has a zero price. If this commodity is in $\mathcal{P}'$, some commodity in $\mathcal{D}'$ has a zero price (otherwise there would be a total production in Y yielding a total profit larger than $p^* \cdot y^*$). Hence, there is a commodity h in $\mathcal{D}'$ with a zero price. Consider now an arbitrary consumer, say the i^{th} one. By consuming more of the h^{th} commodity, he can obtain a consumption preferred to x_i^* without spending more. Consequently, x_i^* does not satisfy the preferences of the i^{th} consumer under the constraint $p^* \cdot x_i \leq p^* \cdot x_i^*$, and, by ($\alpha$) of the definition of a quasi-equilibrium, $p^* \cdot x_i^* = \text{Min } p^* \cdot X_i$. Therefore (e) is satisfied.

If I_k is a set of consumers, and if a_i is a real number, or a vector of the commodity space, or a subset of the commodity space associated with the i^{th} consumer, we now denote by a_{I_k} the sum $\sum_{i \in I_k} a_i$.

Generalizing a concept of D. Gale [4], L. W. McKenzie [8], [9] considers an economy that is irreducible in the following sense:[10]

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[10] The economy of K. J. Arrow - G. Debreu [1], Theorem II' is irreducible. But in this case, as well as in the case of D. Gale [4], it seems easier to establish (e) directly than to establish irreducibility.

Let (I_1, I_2) be a partition of the set of consumers into two non-empty subsets. If $\left((x_i), y\right)$ is an attainable state of the economy, then there is z in $Y + \{\omega_{I_2}\} - X_{I_2}$ such that $x_{I_1} - y + z$ can be allocated to the consumers in I_1 so as to make all of them at least as well off, and at least one of them better off, than in the given state.

Let then $\left((x_i^*), (y_j^*), p^*\right)$ be a quasi-equilibrium of the economy, and let I_2 be the set of consumers for whom $(\alpha.2)$ occurs. To show that irreducibility implies (e), we have to show that if $I_2 \neq \emptyset$, then its complement $I_1 = \emptyset$. For this, we assume $I_1 \neq \emptyset \neq I_2$ and derive a contradiction. One has $p^* \cdot x_{I_2}^* = \text{Min } p^* \cdot X_{I_2}$ and $p^* \cdot y^* = \text{Max } p^* \cdot Y$.

Hence $Y + \{\omega_{I_2}\} - X_{I_2}$ is below the hyperplane with normal p^*, through $y^* + \omega_{I_2} - x_{I_2}$, which is equal to $x_{I_1}^* - \omega_{I_1}$. By the definition of irreducibility, there is z in $Y + \{\omega_{I_2}\} - X_{I_2}$ $\left(\text{hence } p^* \cdot z \leqq p^* \cdot (x_{I_1}^* - \omega_{I_1})\right)$ such that $x_{I_1}^* - y^* + z$ is collectively preferred to $x_{I_1}^*$ by the consumers in I_1. Summing the wealth equations of these consumers, one obtains $p^* \cdot x_{I_1}^* = p^* \cdot \omega_{I_1} + \sum_j \Theta_{I_1 j} \, p^* \cdot y_j^*$, hence

$p^* \cdot (x_{I_1}^* - \omega_{I_1}) = \sum_j \Theta_{I_1 j} \, p^* \cdot y_j^* \leqq p^* \cdot y^*$. Therefore $p^* \cdot z \leqq p^* \cdot y^*$ and

$$(15) \qquad p^* \cdot (x^*_{I_1} - y^* + z) \leq p^* \cdot x^*_{I_1}$$

Since, for every i in I_1, the consumption x^*_i satisfies the preferences of the i^{th} consumer under the constraint $p^* \cdot x_i \leq p^* \cdot x^*_i$, inequality (15) means that $x^*_{I_1} - y^* + z$ cannot be collectively preferred to $x^*_{I_1}$ by the consumers in I_1 (if all the preferences satisfy the assumption « $x'_i \underset{i}{\succ} x_i$ implies $t\, x'_i + (1 - t)\, x_i \underset{i}{\succ} x_i$ if $0 < t < 1$ ») by the usual argument on Pareto optima.

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