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### Investment Criteria and Choice of Techniques of Production

T. N. Srinivasan

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Investment Criteria and Choice of Techniques of Production\*

Thirukodikaval N. Srinivasan

May 31, 1961

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\*Research undertaken by the Cowles Commission for Research in Economics under Task NR-047-006 with the Office of Naval Research.

INVESTMENT CRITERIA AND CHOICE OF  
TECHNIQUES OF PRODUCTION

By

Thirukodikaval N. Srinivasan

1961

A Dissertation  
Presented to the Faculty  
of the Graduate School of Yale University  
in Candidacy for the Degree  
of Doctor of Philosophy

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## CHAPTER 1

### INTRODUCTION

Recent literature on economic development has devoted considerable attention to the problem of deriving the criteria which should govern the choice among alternative techniques of production. Widely differing solutions have been proposed leading to widely differing choices, from the most capital intensive to the most labour intensive techniques of production. These differences result in part from differing assumptions about the economic structure of underdeveloped areas and in part from errors in logic.

The problem of investment criteria is a sub-problem of the more general problem of choice among alternative time paths attainable by an economy from some given initial and anticipated future conditions. The choice is made by maximizing some social welfare function. In evaluating alternative time paths, some authors have restricted themselves to a finite time horizon and ignored the effects of their choice beyond this horizon. One has to keep in mind the different social welfare functions and the different time horizons of the various authors in comparing their proposals.

The literature on this topic is vast and an excellent summary can be found in a paper by Hollis Chenery.<sup>1</sup> For our purposes it is sufficient

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<sup>1</sup>Hollis B. Chenery, "Comparative Advantage and Development Policy," American Economic Review, LI (March 1961), pp. 18-51.



to take up what we consider as the most important of the various proposals; (a) the Social Marginal Productivity (SMP) criterion of A. E. Kahn and Hollis Chenery, (b) the Marginal Per Capita Reinvestment Quotient (MRQ) criterion of Walter Galenson and Harvey Leibenstein, (c) the Marginal Growth Contribution (MGC) criterion of Otto Eckstein, and (d) the Re-investible Surplus criterion of Maurice Dobb and A. K. Sen.<sup>1</sup>

The social welfare function implicit in Kahn's work and made explicit by Chenery is a function of several variables, of which Chenery considers national income and balance of payments to be the most important. The problem they solve is the allocation of a given rate of investment of one period between alternative projects in such a way as to maximize the average annual increment in social welfare. This annual average is to be computed through a discounting process; the way in which the discount factor is to be chosen is not explained. Chenery's procedure is to rank projects according to their SMP and choose projects in decreasing order of SMP until the given rate of investment is exhausted. The SMP of a project is defined by Chenery as the average annual increment in national income (plus balance of payments effect) resulting from the

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<sup>1</sup>The relevant articles and books are: A. E. Kahn, "Investment Criteria in Development Programs," Quarterly Journal of Economics, LXV (February 1951), pp. 38-61; Hollis B. Chenery, "Application of Investment Criteria," Quarterly Journal of Economics, LXVII (February 1953), pp. 76-96; Walter Galenson and Harvey Leibenstein, "Investment Criteria, Productivity, and Economic Development," Quarterly Journal of Economics, LXIX (August 1955), pp. 343-370; Otto Eckstein, "Investment Criteria for Economic Development and the Theory of Intertemporal Welfare," Quarterly Journal of Economics, LXXI (February 1957), pp. 56-85; Maurice Dobb, Economic Growth and Planning, 1st ed., (New York, Monthly Review Press, 1960), chs. 3 and 4, pp. 29-75; A. K. Sen, Choice of Techniques, An Aspect of the Theory of Planned Economic Development, 1st ed., (Oxford, England; Basil Blackwell, 1960), chs. 2 and 3, pp. 21-51.

marginal unit of investment in that project. The increment in national income is to be computed after applying some corrections to market prices which take into account the divergence between social and private benefits that occur when external economies, unused resources, tariffs, and subsidies are present.

In summary, the Kahn-Chenery approach is static where the choice variables belong to a single period. In computing the increment to the flow of national income resulting from the given investment, planners need to know the future product and factor prices. But future relative scarcities of various factors and products will themselves depend on the current rate and composition of investment. The same remark applies to the corrections with which Chenery wants to alter the market prices. By taking a static view Kahn and Chenery are implying either (a) that the projected investment is too "small" to affect the future course of prices or (b) that the future course of prices is independent of current choices. Either assumption is untenable in the context of economic development.

Galenson and Leibenstein also criticize Chenery for using tools of economic statics in analyzing the problem of economic growth. The social welfare index they propose instead is an increasing function of the per capita output potential at some future point in time. The per capita output potential at any point in time depends, in their view, on a number of factors, of which capital per worker and the quality of the labour force are important. Capital per worker at any future point in time  $T$  depends on the accumulation of capital from now until  $T$  and the size of the labour force at  $T$ . Galenson and Leibenstein conclude that "best allocation of investment resources is achieved by equating the MRQ of

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capital in its various alternative uses."<sup>1</sup> The MRQ of a project is determined by the annual surplus it generates over and above wage and depreciation costs, taking into account the contribution of the project toward improving the quality of the labour force and bringing about a decline in its rate of growth. Galenson and Leibenstein come to an empirical judgement that the MRQ is likely to be higher in capital intensive than in labour intensive projects.

Galenson and Leibenstein have been criticized by John Moes,<sup>2</sup> Francis M. Bator<sup>3</sup> and others for the extreme nature of the assumptions needed to validate their theoretical and empirical conclusions. Even if one grants their assumption that only profit earners save, it is by no means necessary that profit maximization should always imply maximization of the capital labour ratio. Their prescription that governments in underdeveloped countries should maintain artificially high wages to induce producers to choose high capital labour ratios is questionable. If one is interested in retarding the rate of growth of the labour force, investing in capital intensive projects in a labour surplus economy is surely an indirect and perhaps an inefficient method. Lastly, the objective of maximizing the per capita output potential at some future point in time implies that a one commodity economy is being considered. This criterion is not very meaningful in an economy with more than one commodity. We shall come back to this point in Chapter 4.

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<sup>1</sup>Galenson and Leibenstein, op. cit., p. 351.

<sup>2</sup>John Moes, "Investment Criteria, Productivity and Economic Development: Comment," Quarterly Journal of Economics, LXXI (February 1957), pp. 161-164.

<sup>3</sup>Francis M. Bator, "On Capital Productivity, Input Allocation and Growth," Quarterly Journal of Economics, LXXI (February 1957), pp. 86-106.

Eckstein tries to synthesize the Kahn-Chenery approach and the Galenson-Leibenstein approach. His social welfare function is the sum of the discounted value of the stream of consumption resulting from a given current investment  $\bar{K}$  and the future reinvestments occasioned by it. Current investment is to be divided among alternative projects which differ in their output stream and reinvestment potential. All reinvestments (at future points in time) are to be directed to a single project. Maximizing his social welfare function, Eckstein arrives at his MGC criterion. The optimal allocation of  $\bar{K}$  results when the amounts invested in each project are such that the MGC's of the different projects are equal. The MGC of a project is the sum of two terms (a) the present value of the project's direct contribution to the consumption stream and (b) the present value of the consumption stream resulting from reinvestments associated with the project. Eckstein also tries to determine the rate of investment itself by determining the interest rate through a marginal utility approach.

Eckstein's approach is subject to a number of criticisms. First, if one wants to consider an infinite time horizon as Eckstein, one cannot proceed as if capital were the only scarce factor at all points in time. Second, there is no reason to direct all reinvestments to one project. One can, at the very least, postulate that the same projects are available at all future periods of time as at the beginning.<sup>1</sup>

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<sup>1</sup>Eckstein is aware of this problem but he does not offer a complete solution. (Eckstein, op. cit., p. 72.)

The approaches of Dobb and Sen are very similar and we shall discuss only that outlined by Sen. Sen considers an economy consisting of two sectors, an advanced sector and a backward sector. The backward sector produces a single consumer good, corn, presumably using only labour. The advanced sector about to be initiated consists of two departments, one which produces corn, using labour and machines and one which produces machines using only labour.<sup>1</sup> The corn productivity of labour in department 1 depends on the capital intensity of the process used. The capital intensity of a process is measured by the number of man years required in department 2 to produce enough capital (machines) to equip one labourer in department 1. The wage rate to be paid to labourers in the advanced sector is assumed given. The total wage bill in the advanced sector is to be met from corn production within the sector except in the first period--when the advanced sector is being initiated. The wage bill of the first period is to be met by an amount of corn surplus extracted from the backward sector.

Sen considers three possible objectives for his economy: (1) maximizing the corn output of the second period in the advanced sector, (2) maximizing the rate of growth of corn output, and (3) maximizing the undiscounted sum of the stream of output of corn over  $T$  periods of time. Naturally the optimal capital intensities for attaining the three objectives are different.

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<sup>1</sup>Sen relaxes the assumption that capital goods can be produced only with labour in Chapter III of his book. But he does not go beyond stating that a solution to the problem of choice of capital intensity in the two departments is not impossible. (Sen, op. cit., Ch. III.)

One can agree with Sen in dismissing the first objective as clearly not the one to be pursued if one is interested in the problem of growth, for maximization of total corn output need not necessarily result in maximum surplus (over wages) for reinvestment unless the wage rate is zero.<sup>1</sup> One can even construct examples in which maximization of corn output of the second period yields a negative surplus. But the second objective, maximizing the rate of growth of corn output by maximizing the rate of surplus in each period, is based on the assumption that any rate of growth can be sustained indefinitely by withdrawing labour from the backward sector without reducing its output. Once we introduce a limit to the withdrawal of labour from the backward sector, the optimal capital intensity under the second objective may be different. The third objective ignores completely the time path of consumption by considering only the aggregate consumption over an interval of time. Furthermore, such an approach precludes consideration of an infinite time horizon because undiscounted aggregate consumption over an infinite time horizon may well be infinite.

Thus our brief review of the more important articles in the literature reveals that most of these authors analyze the problem of growth in a static framework. Even the analysis of Galenson and Leibenstein, con-

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<sup>1</sup>Let  $\bar{s}$  be the surplus corn extracted from the backward sector in the first period. Assume with Sen a lag of one year in the production of machines and no lag in the production of corn. Let  $a$  be the capital intensity chosen and  $w$  the wage rate. Then the total output of corn in the advanced sector in the second period is  $\frac{\bar{s}}{wa} f(a)$  where  $f(a)$  is the corn productivity of labour when capital intensity  $a$  is used. But surplus over wages is  $\frac{\bar{s}}{wa} [f(a) - w]$ . The maximization of one is not equivalent to that of the other.

sidered by them to be dynamic, is static in our view because they limit themselves to the objective of maximizing per capita output potential at some chosen point of time in the future. They therefore have a finite time horizon and hence static tools could be used to analyze their problems. Sen and Dobb do introduce dynamic elements explicitly in their model; but they fail to consider restraints, such as the availability of labour, that may become operative at future points in time. We shall attempt in the next chapter to develop a dynamic model which explicitly introduces constraints operative at each point in time and contains some essential elements of choice inherent in growth problems. We shall consider only two constraints (other than the nonnegativity restraints) at each point in time. The first one limits the employment at any point in time to the available labour force. The second limits the rate of gross investment in any period to the rate of output of machines in that period. This choice of constraints does not imply that these are the only important constraints relevant to a problem. In fact, constraints such as the availability of foreign exchange, skilled labour and entrepreneurial skills may be equally important. Our choice of constraints is motivated purely by considerations of analytical convenience and our desire to analyze the problem of growth in a simple setting which still retains some essential elements.

In Chapter 2, we shall be concerned also with the derivation of the growth path called the "terminal path" that yields the maximum sustainable rate of per capita consumption (this concept will be defined later) at each point in time, assuming that the initial conditions are subject to choice.

In Chapter 3, we shall relax the assumption that the initial conditions are subject to choice and study the approach to the terminal path from historically given initial conditions. In particular, the growth path that minimizes the time taken to reach the terminal path will be derived.

In Chapter 4, the consequences of following an objective of maximizing the per capita consumption attainable at some point of time in the future, starting from given initial conditions, will be examined.

In Chapter 5 the objective of maximizing the discounted sum of the future stream of per capita consumption obtainable from given initial conditions will be studied. Chapter 6 will be concerned with characterizing the class of all efficient time paths (to be defined later). Chapter 7 will summarize and qualify some of the conclusions of the previous chapters.

Each of the following four chapters will be divided into two sections. The first section will introduce the problem to be considered and present the solution. The second section will be devoted to detailed proofs. The reader interested mainly in the results can omit the proofs without losing continuity.



## CHAPTER 2

### THE MODEL AND THE TERMINAL PATH

#### 2.1. Introduction and Results

We shall discuss the choice of techniques of production in terms of the following model. Consider an economy consisting of two sectors or departments: Sector 1 produces machines of various types and Sector 2 produces a homogeneous consumer good. Within Sector 2 there are alternative techniques of production, each technique characterized by the particular combination of labour and machines it uses. A machine associated with one technique of production in Sector 2 can not be used with any other technique of production. In Sector 1 one type of machine is employed with a fixed labour input to produce any machine. Thus any machine used in either Sector 1 or Sector 2 may be produced from this combination of inputs. This assumption enables us to use a common unit of measurement for various types of machines prior to their construction while retaining the assumption that once constructed, machines of various types are not substitutes for each other. Constant returns to scale are assumed to prevail in both sectors. All types of machines once constructed are assumed to depreciate physically at the same geometric rate per unit of time. The labour force is assumed to grow at a constant geometric rate per unit of time.

The following list of variables and notations is used throughout this thesis:

Class of variable	Variable	Notation <sup>1</sup>	Dimension <sup>2</sup>
Time <sup>3</sup>	time	$t$	$t$
Given constants	geometric rate of growth of the labour force	$\theta$	$1/t$
Given functions	available rate of labour services at $t$	$L(t)$	$l/t$
Technological constants	rate of input of labour services needed to operate a machine of type $i$ $i = 0, 1, 2, \dots$	$\lambda_i$	$l/m/t$
	rate of output of machines (of any type) per machine of type $0$ in use	$\gamma_0$	$1/t$
	rate of output of consumer good per machine of type $i$ in use $i = 1, 2, \dots$	$\gamma_0$	$c/m/t$
	geometric rate of depreciation of any type of machine	$\delta$	$1/t$
Policy variables	stock of machines of type $i$ in existence at time $t$ $i = 0, 1, 2, \dots$	$K_i(t)$	$m$
	stock of machines of type $i$ in use at time $t$ $i = 0, 1, 2, \dots$	$K_i^i(t)$	$m$
	rate of gross addition to the stock of machines of type $i$ at time $t$ $i = 0, 1, 2, \dots$	$X_i(t)$	$m/t$
	rate of output of consumer goods at time $t$	$C_t$	$c/t$
	rate of consumption per worker at time $t$	$c_t$	$c/l/t$

<sup>1</sup>Subscript  $0$  is associated with the technique of producing machines. Subscript  $i$  ( $i = 1, 2, 3, \dots$ ) is associated with the  $i^{\text{th}}$  technique of producing consumer goods.

<sup>2</sup>In this column  $t$  = time,  $c$  = unit of the consumer good,  $m$  = unit of machines,  $l$  = unit of labour services.

<sup>3</sup>Both continuous and discrete time versions of the model are con-

It is clear that by definition  $L(t) = L(0)[1 + \Theta]^t$  (or  $L(0)e^{\Theta t}$  if  $t$  is treated as a continuous variable). Without loss of generality we can assume that  $L(0) = 1$  and that the parameters of the technology of the consumer goods industry satisfy the following:<sup>1</sup>

$$\frac{\lambda_1}{\gamma_1} > \frac{\lambda_2}{\gamma_2} > \frac{\lambda_3}{\gamma_3} > \dots > \frac{\lambda_i}{\gamma_i} > \frac{\lambda_{i+1}}{\gamma_{i+1}} > \dots \quad (2.1)$$

$$\gamma_1 > \gamma_2 > \gamma_3 > \dots > \gamma_i > \gamma_{i+1} > \dots \quad (2.2)$$

$$\frac{\gamma_1 - \gamma_2}{\lambda_1 \gamma_2 - \lambda_2 \gamma_1} < \frac{\gamma_2 - \gamma_3}{\lambda_2 \gamma_3 - \lambda_3 \gamma_2} < \frac{\gamma_i - \gamma_{i+1}}{\lambda_i \gamma_{i+1} - \lambda_{i+1} \gamma_i} < \dots \quad (2.3)$$

(2.1) and (2.2) state that the techniques have been numbered according to decreasing labour intensity.<sup>2</sup> (2.3) ensures that techniques which require more labour and machines per unit of output than a suitable combination of other techniques have been eliminated from the set of alternative techniques. Figure 1 shows an isoquant of consumer goods production. This isoquant shows the alternative combinations of machines and labour that can be used to produce a rate of output of one unit of consumer good. It is worth emphasizing that this isoquant is meaningful only prior to the construction of the machines of various

sidered. In the continuous version rates refer to instantaneous rates and in the discrete version to rates per period. In the discrete version it is convenient to assume that the inputs in any period make available outputs that can be used in the next period.

<sup>1</sup>We shall be assuming that  $\lim_{i \rightarrow \infty} \gamma_i = \gamma > 0$  .  $\lim_{i \rightarrow \infty} \frac{\lambda_i}{\gamma_i} = \frac{\lambda}{\gamma} > 0$  .

<sup>2</sup>Labour intensity is defined as labour input per unit of output.

types. Once a machine of a type is constructed, it cannot be substituted for one of another type.

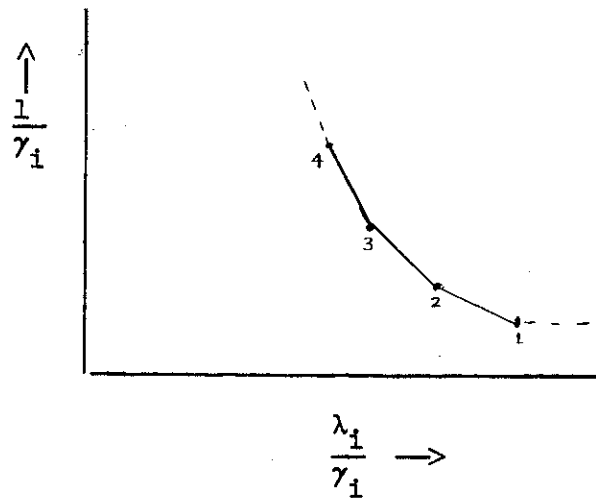


Figure 2a

Consider an economy described by the technology described by (2.1) - (2.3) and a labour force growing at constant proportionate rate  $\theta$ . Suppose now that this economy is free to choose the initial stocks of various types of machines. If the planners of this economy wish to maximize the sustainable rate of consumption per worker, what should be their choices regarding the initial values and growth paths of the stocks of various types of machines?<sup>1</sup> Before we can answer this question, we have to define our terms.

Definition 1

A time path is a vector  $c = (c_1, c_2, \dots, c_t, \dots)$  where  $c_t$  is the consumption per worker at time  $t$ .

---

<sup>1</sup>This particular maximand is not the only one that can be considered. This point is discussed subsequently.

Definition 2<sup>1</sup>

A time path  $c$  is said to be feasible if we can find  $K_i(0), K_i'(t), X_i(t)$  ( $i = 0, 1, 2, \dots$ ) such that the following holds for  $t = 0, 1, 2, \dots$

$$c_{t+1} = (1+\theta)^{-t-1} \sum_{i=1}^{\infty} \gamma_i K_i'(t) \quad (2.4)$$

$$K_i(t) = K_i(0)(1-\delta)^t + \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_i(t') \quad (2.5)$$

$i = 0, 1, \dots$

$$K_i(t) - K_i'(t) \geq 0 \quad i = 0, 1, 2, \dots \quad (2.6)$$

$$-\sum_{i=0}^{\infty} \lambda_i K_i'(t) \geq -(1+\theta)^t \quad (2.7)$$

$$\gamma_0 K_0'(t) - \sum_{i=0}^{\infty} X_i(t) \geq 0 \quad (2.8)$$

$$K_i(0), K_i'(t), X_i(t) \geq 0 \quad i = 0, 1, 2, \dots \quad (2.9)$$

Equation (2.4) states that consumption per worker at time  $(t+1)$  is the ratio of the output of consumer goods of the previous period to the labour force at that time.<sup>2</sup> Equation (2.5) states that the stock in existence at  $t$  of type  $i$  machines is equal to the depreciated level of the initial stock plus the depreciated level of the gross addition made at each point in time from  $0$  to  $t-1$ . Inequality (2.6) ensures that

<sup>1</sup>This definition has to be modified when  $K_i(0)$  ( $i = 0, 1, 2, \dots$ ) are treated as historically given. Then the definition will read "A time path is said to be feasible if we can find  $K_i'(t), X_i(t)$  ( $t = 0, 1, 2, \dots$ ) such that ...."

<sup>2</sup>This is because of our assumption of a lag of one period in production.

the stock in use of any type of machine does not exceed the stock in existence at any  $t$ . Inequality (2.7) ensures that the rate of employment at any  $t$  does not exceed the rate of labour services available at that  $t$ . Inequality (2.8) states that gross investment at any  $t$  does not exceed the rate of output of machines at that  $t$ . Inequality (2.9) ensures that stock variables and gross investment variables are nonnegative.

Definition 3

A feasible time path  $c$  is efficient if there is no other feasible time path  $\bar{c}$  such that (a)  $\bar{c}_t \geq c_t$  for all  $t$  and (b)  $\bar{c}_t > c_t$  for at least one  $t$ .

Definition 4

A rate of consumption per worker  $\underline{c}$  is said to be sustainable if there exists a feasible time path  $c$  such that  $c_t \geq \underline{c}$  for all  $t$ .

Clearly if  $\underline{c}$  is sustainable, so is any  $\underline{\underline{c}} < \underline{c}$ . Now from (2.7) it is clear that the stock of machines of any type in use cannot grow at a proportionate rate greater than  $\theta$  indefinitely, given the fact that  $\lambda_i > 0$  for all  $i$  and  $\lim_{i \rightarrow \infty} \lambda_i = \lambda > 0$ . This implies that there can be no feasible time path in which consumption per worker increases without any limit. Thus in the long run the economy under consideration can sustain only a constant rate of consumption per worker. It is therefore meaningful to derive that time path which maximizes the sustainable rate of consumption per worker (MSCP) assuming that the initial stocks of various types of capital are subject to choice.

It can be shown (see Section 2.2 for proof) that for the economy under consideration, the MSCP is associated with the time path

$c^j = (c_1^j, c_2^j, \dots)$  defined by the following:<sup>1</sup> for  $t = 0, 1, \dots$  :

$$c_{t+1}^j = \frac{\gamma_j(\gamma_0 - \theta - \delta)}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (2.10)$$

$$K_0(t) = K_0'(t) = \frac{(\theta + \delta)(1 + \theta)^t}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (2.11)$$

$$K_j(t) = K_j'(t) = \frac{(\gamma_0 - \theta - \delta)(1 + \theta)^t}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (2.12)$$

$$X_0(t) = (\theta + \delta)K_0(t) \quad (2.13)$$

$$X_j(t) = (\theta + \delta)K_j(t) \quad (2.14)$$

$$K_i(t) = K_i'(t) = X_i(t) = 0 \quad \text{for } i \neq 0, j \quad (2.15)$$

$j$  is an index which satisfied the following:<sup>2</sup>

$$\frac{\gamma_{j-1} - \gamma_j}{\lambda_{j-1}\gamma_j - \lambda_j\gamma_{j-1}} < \frac{(\gamma_0 - \theta - \delta)}{\lambda_0(\theta + \delta)} < \frac{\gamma_j - \gamma_{j+1}}{\lambda_j\gamma_{j+1} - \lambda_{j+1}\gamma_j} \quad (2.16)$$

The time path  $c^j$  described by (2.10) - (2.15) is a balanced growth path in which the stocks of machines of type 0 and  $j$  and the labour

<sup>1</sup>It will be noticed that  $\gamma_0 - \delta$  is assumed to be larger than  $\theta$ . For if  $\gamma_0 - \delta < \theta$  the economy cannot sustain even a constant rate of consumption per worker. This follows from the fact that  $\gamma_0 - \delta$  is the net reproduction rate of machines. If this is less than the rate of growth of labour force, consumption per worker cannot be sustained at a constant nonzero level. Since by assumption there is no technique which uses only labour to produce consumer goods.

<sup>2</sup>It is conceivable that for all  $i$ ,  $\frac{\gamma_{i-1} - \gamma_i}{\lambda_{i-1}\gamma_i - \lambda_i\gamma_{i-1}} < \frac{(\gamma_0 - \theta - \delta)}{\lambda_0(\theta + \delta)}$ . It can be shown that in this case the sustainable rates of consumption are bounded above; and this bound can either be attained or approached as closely as one wishes by choosing a sufficiently large  $j$ . One other possibility is that one of the inequalities in (2.16) is an equality. Only one can be an equality in view of (2.3). If this possibility occurs, the MSCP can be achieved by using either  $j$  or one of the techniques adjacent to it.

force grow at the same rate  $\theta$ . Consumption per worker is constant along this time path. Only one of the infinitely many alternative techniques is used for producing consumer goods. This result could be explained in two ways.

It can be seen from (2.4) - (2.8) that any feasible time path which yields a constant rate of consumption per worker is necessarily a balanced growth path. One can show that for each  $i \geq 1$ , a feasible balanced growth path  $c^i$  can be obtained by replacing  $j$  by  $i$  in (2.10) - (2.15). Such a path  $c^i$  uses only technique  $i$  for producing consumer goods. Any feasible balanced growth path which uses more than one technique for producing consumer goods can be expressed as a suitable combination of the paths  $c^i$  ( $i \geq 1$ ). The sustainable rate of consumption per worker (SCP) of the path  $c^i$  is

$$c^i = \frac{\gamma_i(\gamma_0 - \theta - \delta)}{\lambda_i(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (2.17)$$

Inequalities (2.16) together with (2.3) imply that  $\underline{c}^i$  attains its maximum value when  $i = j$ . In other words that among the class of time paths which yield constant rates of consumption per worker over time, the path  $c^j$  yields the maximum rate. Further (2.16) and (2.3) also imply that if we consider two economies which differ only in their rates of growth of labour force, the economy with the faster growing labour force will use a technique yielding its MSCP which is at least as labour intensive as the other economy. The MSCP of the former economy cannot exceed that of the latter. Figure 2b illustrates these results.<sup>1</sup>

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<sup>1</sup>This figure is for illustrative purposes only.



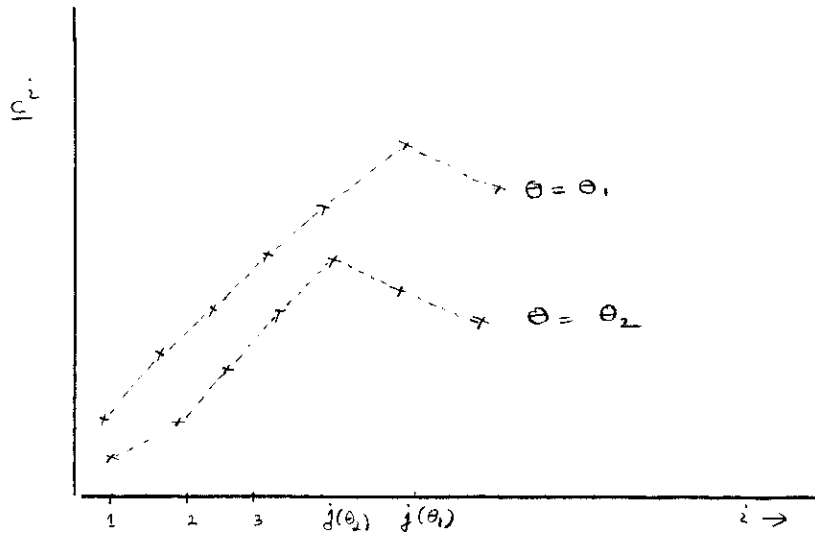


Figure 2b

Another way of interpreting the choice of  $j$  through (2.16) is the following. Suppose we wish to produce a SCP of one unit. Since the labour force at time  $t$  is  $(1+\theta)^t$ , the rate of output of consumer goods at time  $t$  should then be  $(1+\theta)^t$ . For producing this rate of output, one needs direct labour services of  $\lambda_1 (1+\theta)^t$  units if one uses technique 1. Further, it can be shown that one needs also in-

direct labour services of  $\left[ \frac{\lambda_0(\theta+\delta)}{\gamma_1(\gamma_0-\theta-\delta)} \right] (1+\theta)^t$  units for producing

enough machines of types 0 and 1 to maintain the desired rate of consumption per worker. Thus the total labour requirement<sup>1</sup> per unit of SCP

is  $\xi_1 = \left[ \lambda_1 + \frac{\lambda_0(\theta+\delta)}{\gamma_1(\gamma_0-\theta-\delta)} \right] (1+\theta)^t$ . Since in the final analysis labour

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<sup>1</sup>The ratio of direct and indirect labour services needed per unit of SCP seems to correspond to the concept of organic composition of capital of Marx.

is the only constraint, by minimizing  $\xi_1$  by a proper choice of  $i$  we can maximize the SCP. Inequalities (2.16) together with (2.3) establish that  $\xi_1$  attains its minimum value when  $i = j$ . In other words as  $i$  increases the direct labour requirement falls while the indirect labour requirement increases. Up to  $i = j$ , the fall in the direct labour requirement is greater than the rise in the direct labour requirement and beyond  $j$  the inequality is reversed and hence  $j$  is the technique which minimizes  $\xi_1$ .

In summary it has been shown that a particular two-sector economy is capable of sustaining a certain maximum rate of consumption per worker over time. This rate is attained at each point of time along a growth path in which only one of many alternative techniques of producing consumer goods is used. The stock of machines in both sectors grow at the same rate as the labour force. Thus such a path describes a stationary state in which every relevant variable is constant, not in absolute terms, but relative to the labour force.<sup>1</sup> The path can therefore be described by the stock per worker of various types of machines. We shall denote the growth path derived in this section as the "terminal path" for reasons to be explained in the next chapter.

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<sup>1</sup>The "Golden Age" of Mrs. Joan Robinson describes a similar path. See Joan Robinson, The Accumulation of Capital, London, Macmillan and Company Ltd., 1958. The model described by her incorporates neutral technical progress also.

## 2.2. Derivations

In this section we shall show that the time path represented by (2.10) - (2.16) yields the MSCP. First we need to establish the following lemma:

### Lemma:

For a sustainable rate of consumption per worker  $\underline{c}$  to be the MSCP, it is sufficient that there exist an efficient path  $c$  with  $c_t = \underline{c}$  for all  $t$ .

### Proof:

Suppose there exists an efficient  $c$  such that  $c_t = \underline{c}$  for all  $t$ . By the definition of efficiency there is no feasible  $c'$  such that

$$\begin{aligned} c'_t &\geq c_t = \underline{c} && \text{for all } t \text{ and} \\ c'_t &> c_t = \underline{c} && \text{for at least one } t. \end{aligned} \tag{2.18}$$

This implies that there exists no feasible  $c'$  such that  $c'_t > \underline{c} + \epsilon$  where  $\epsilon > 0$ . In other words no rate greater than  $\underline{c}$  can be sustained. Therefore  $\underline{c}$  is the MSCP.

In order to show that a feasible path is efficient we use a lemma due to Malinvaud.<sup>1</sup> This lemma states that a given feasible path is efficient if (a) there exists a set of nonnegative prices which sustains

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<sup>1</sup>Edmond Malinvaud, "Capital Accumulation and Efficient Allocation of Resources," Econometrica, XXI (April 1958), Lemma 5. Malinvaud's model assumes that there are only a finite number of commodities in each period. Our model has infinite number of commodities in the form of various types of machines. We are assuming with Malinvaud that his results could be extended to this case.

it and (b) the cost of inputs of period  $t$  evaluated with such a set of prices approaches zero as  $t \rightarrow \infty$ . A particular set of prices is said to sustain a given feasible path if it can be shown that the given path will result from profit maximizing behaviour under the particular set of prices. The profit maximization rule can be translated into the language of activity analysis as follows: under the chosen set of prices (a) no activity makes a positive profit and (b) any activity that is operated at a positive level in the given path makes zero profit.

The basic activities of our model are the following: (we use the same symbol for an activity and its level)

1.  $K_0'(t)$  : the activity which produces machines at  $t$ .
2.  $K_i'(t)$  : the activity which produces consumer goods with technique  $i$   $i = 1, 2, \dots$
3.  $X_i(t)$  : the activity which allocates a part of newly produced machines to increasing the stock of machines of type  $i$   $i = 0, 1, 2, \dots$

Let us use the following notation for prices: (all prices are present values, present meaning time zero)

$p(t)$  : price per unit of consumer good at  $t$

$q(t)$  : price per unit of a newly produced machine at  $t$

$r_i(t)$  : rate of rental per period of a unit of type  $i$  machine  
at time  $t$  .

$w(t)$  : rate of wages per unit of labour at time  $t$  .

The price conditions of efficiency can be stated as follows:

given a feasible solution  $X_i(t)$  ,  $K_i'(t)$  if a set of nonnegative prices  $p(t)$  ,  $q(t)$  ,  $r_i(t)$  ,  $w(t)$  could be found such that the following holds:<sup>1</sup>

$$\left. \begin{aligned} \gamma_i p(t+1) &\leq \lambda_i w(t) + r_i(t) & i = 1, 2, \dots \\ &= \lambda_i w(t) + r_i(t) & \text{if } K_i'(t) > 0 \end{aligned} \right\} (2.19)$$

$$\left. \begin{aligned} \gamma_0 q(t+1) &\leq \lambda_0 w(t) + r_0(t) \\ &= \lambda_0 w(t) + r_0(t) & \text{if } K_0'(t) > 0 \end{aligned} \right\} (2.20)$$

$$\left. \begin{aligned} q(t+1) &> \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') & i = 0, 1, 2 \dots \\ &= \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') & \text{if } X_i(t') > 0 \end{aligned} \right\} (2.21)$$

---

<sup>1</sup>A production lag of one period is assumed.

The right hand member of (2.19) represents the cost of producing  $\gamma_i$  units of the consumer good with technique  $i$ . If technique  $i$  is to be used then the cost of production of a unit of consumer good should equal its price. This is stated by the second part of (2.19). (2.20) represents a similar profit maximizing condition in the production of machines. (2.19) and (2.20) together represent one period efficiency conditions. (2.21) states intertemporal efficiency conditions. If a unit of newly produced machine is to be added to the stock of type  $i$  machines, then the present value of the returns from such an addition should equal the cost of production of the machine. This is precisely what the second part of (2.21) states.

We are now in a position to prove the efficiency of the time path  $c^j$ . The quantity solutions of  $c^j$  and the associated price solutions are: (for  $t \geq 0$ )

$$\left. \begin{aligned} K'_0(t) &= \frac{(\theta+\delta)(1+\theta)^t}{\lambda_j(\gamma_0-\theta-\delta) + \lambda_0(\theta+\delta)} & X_0(t) &= (\theta+\delta) K'(t) \\ K'_j(t) &= \frac{(\gamma_0-\theta-\delta)(1+\theta)^t}{\lambda_j(\gamma_0-\theta-\delta) + \lambda_0(\theta+\delta)} & X_j(t) &= (\theta+\delta) K'_j(t) \end{aligned} \right\} (2.22)$$

$$K'_i(t) = 0 \quad i \neq 0, j \quad X_i(t) = 0 \quad i \neq 0, j$$

$$\left. \begin{aligned}
 q(t+1) &= q(0) [1+u]^{-t-1} \quad \text{where } q(0) = \frac{\lambda_0 \gamma_j}{\lambda_j(\gamma_0 - u - \delta) + \lambda_0(u + \delta)} \\
 w(t) &= \left[ \frac{\gamma_0 - u - \delta}{\lambda_0} \right] q(t+1) \quad r_0(t) = r_j(t) = (u + \delta)q(t+1) \\
 p(t+1) &= [1+u]^{-t-1} \quad r_i(t) = \alpha_1 r_j(t) \quad \text{for } i < j \\
 r_i(t) &= \alpha_2 r_j(t) \quad \text{for } i > j \quad \text{where } u, \alpha_1, \alpha_2 \text{ are to be} \\
 &\text{chosen such that } \Theta < u < (\gamma_0 - \delta) \quad 0 < \alpha_1, \alpha_2 < 1.
 \end{aligned} \right\} (2.23)$$

It is easy to verify that under the prices (2.23) the second parts of (2.19) - (2.21) hold. It remains to show that for  $i \neq 0, j$  the following holds to show that first parts of (2.19) - (2.21) are also satisfied.

$$\begin{aligned}
 \gamma_i p(t+1) &\leq \lambda_i w(t) + r_i(t) && i \neq 0, j && (2.24) \\
 q(t+1) &\geq \sum_{t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t')
 \end{aligned}$$

Now from (2.23) it can be shown that  $q(t+1) = \sum_{t+1}^{\infty} (1-\delta)^{t'-t-1} r_j(t')$

and that  $r_i(t) < r_j(t)$  for  $i \neq 0, j$ . Therefore the second part of (2.24) holds. For proving the first part (2.24) let us substitute

$\frac{\lambda_j w(t) + r_j(t)}{\gamma_j}$  for  $p(t+1)$ . We need to show that

$$\gamma_i \frac{\lambda_j w(t) + r_j(t)}{\gamma_j} \leq \lambda_i w(t) + r_i(t) \text{ for } i \neq 0, j \quad (2.25)$$

Substituting for  $r_i(t)$  from (2.23), we can reduce (2.25) to

$$\frac{\gamma_i - \alpha_1 \gamma_j}{\lambda_i \gamma_j - \lambda_j \gamma_i} \leq \frac{w(t)}{r_j(t)} = \frac{(\gamma_0^{-u-\delta})}{\lambda_0(u+\delta)} \text{ for } i < j \quad (2.26)$$

$$\frac{\alpha_2 \gamma_2 - \gamma_i}{\lambda_j \gamma_i - \lambda_i \gamma_j} \geq \frac{w(t)}{r_j(t)} = \frac{(\gamma_0^{-u-\delta})}{\delta_0(u+\delta)} \text{ for } i > j \quad (2.27)$$

Using (2.23) it is easy to show that

$$\frac{\gamma_i - \alpha_0 \gamma_j}{\lambda_i \gamma_j - \lambda_j \gamma_i} < \frac{\gamma_{j-1} - \alpha_1 \gamma_j}{\lambda_{j-1} \gamma_j - \lambda_j \gamma_{j-1}} \text{ for } i < j-1$$

$$\frac{\alpha_2 \gamma_j - \gamma_i}{\lambda_j \gamma_i - \lambda_i \gamma_j} > \frac{\alpha_2 \gamma_j - \gamma_{j+1}}{\lambda_j \gamma_{j-1} - \lambda_{j+1} \gamma_j} \text{ for } i > j+1$$

Therefore if (2.26) and (2.27) hold for  $i = j - 1$  and  $i = j + 1$  respectively, they hold for all other relevant values of  $i$ . By (2.16) we know that  $j$  is an index which satisfies



$$\frac{\gamma_{j-1} - \gamma_j}{\lambda_{j-1}\gamma_j - \lambda_j\gamma_{j-1}} < \frac{(\gamma_0 - \Theta - \delta)}{\lambda_0(\Theta + \delta)} < \frac{\gamma_j - \gamma_{j+1}}{\lambda_j\gamma_{j+1} - \lambda_{j+1}\gamma_j}$$

By choosing  $u$  sufficiently close to but greater than  $\Theta$  and  $\alpha_1, \alpha_2$  sufficiently close to unity we can make (2.26) and (2.27) hold for  $i = j - 1$  and  $i = j + 1$  respectively. Thus we have succeeded in showing that we can associate a set of nonnegative prices with the path  $c^j$  that sustains it. Now we have to show that the cost of inputs of period  $t$  evaluated at the prices (2.23) goes to zero as  $t \rightarrow \infty$ . This is easily proved. For along the path  $c^j$  all inputs of the form  $A[1 + \Theta]^t$  and prices given by (2.23) are of the form  $B[1 + u]^{-t}$ . Therefore the total cost of inputs is of the form  $D(1+\Theta)^t (1+u)^{-t}$  which goes to zero as  $t \rightarrow \infty$  since  $u$  has been chosen to be greater than  $\Theta$ .

It is interesting to note the effect of changes in  $\Theta$  on the index  $j$  of the optimal technique and  $\underline{c}^j$ , the MSCP associated with  $j$ . It will be recalled that  $j$  is chosen that<sup>1</sup>

$$\frac{\gamma_{j-1} - \gamma_j}{\lambda_{j-1}\gamma_j - \lambda_j\gamma_{j-1}} < \frac{(\gamma_0 - \Theta - \delta)}{\lambda_0(\Theta + \delta)} < \frac{\gamma_j - \gamma_{j+1}}{\lambda_j\gamma_{j+1} - \lambda_{j+1}\gamma_j} \quad (2.29)$$

$$\text{and } \underline{c}^j = \frac{\gamma_j(\gamma_0 - \Theta - \delta)}{\lambda_j(\gamma_0 - \Theta - \delta) + X_0(\Theta + \delta)} \quad (2.29)$$

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<sup>1</sup>It will be assumed that the  $j$  which satisfies (2.28) is such that  $\lambda_j \geq \lambda_0$ . If  $\lambda_0 < \lambda_j$  it can be shown that feasible paths exist which converge to the path  $c^j$  from above. Thus the path which yields the MSCP is not unique.

It is obvious that  $\frac{(\gamma_0 - \Theta - \delta)}{\lambda_0(\Theta + \delta)}$  is a decreasing function of  $\Theta$ . From

(2.23) we know that

$$\frac{\gamma_{i-1} - \gamma_i}{(\lambda_{i-1}\gamma_i - \lambda_i\gamma_{i-1})} < \frac{\gamma_i - \gamma_{i+1}}{(\lambda_i\gamma_{i+1} - \lambda_{i+1}\gamma_i)} \quad \text{for all } i .$$

Therefore as  $\Theta$  increases the optimal  $j$  can never increase. This also implies that  $\underline{c}^j$  decreases as  $\Theta$  increases. For, given any  $j$ ,  $\underline{c}^j$  decreases as  $\Theta$  increases.

## CHAPTER 3

### THE APPROACH TO THE TERMINAL PATH

#### 3.1. Results

In Chapter 2 the "terminal path" was defined to be a path which maximized the sustainable rate of consumption per worker. It was assumed that even the initial stocks of machines of various types were subject to choice, and this enabled us to initiate the economy on the terminal path itself. But in most development problems it is more realistic to assume that initial conditions, such as stocks of machines of various types, are historically given. These initial stocks need not be equal to their corresponding values on the terminal path. Thus we may not be able to start the economy on the terminal path.

The choices open to such an economy could be analyzed in a formal way as follows: define a feasible path  $c$ , as in Chapter 2, modifying it to take into account that we can choose only  $K_1(t)$  for all  $t$  but not  $K_1(0)$ .<sup>1</sup> Let us postulate that social preferences could be represented in terms of an ordinal utility function  $U(c)$ .<sup>2</sup> The objective of

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<sup>1</sup>See footnote 1, Chapter 2, p. 12.

<sup>2</sup>We are not starting from individual preference orderings to obtain a social preference ordering. By postulating a social preference ordering itself, we assume away the problem of relating social preference ordering to individual preference orderings.

the planners, then, is to choose a feasible path  $c$  which maximizes  $U(c)$ .

Such a formulation is much too general. In the present chapter we consider one particular utility indicator. We restrict ourselves to the subset  $S$  of the set of all feasible paths, which consists of time paths for which the following is true:  $c_t = c^E$  for all  $t \geq$  some finite  $T$  (depending on the particular path  $c$ ) where  $c^E$  is the rate of consumption per worker associated with the terminal path. The objective will then be to choose a  $c \in S$  such that the associated  $T$  is minimized. In other words, starting from given initial conditions, we wish to choose that feasible time path which reaches the terminal path in the minimum time and remains on it thereafter.

This objective completely disregards the time path of the rate of consumption per worker prior to the time of arrival at the terminal path. A somewhat more realistic approach will be to minimize the time of arrival  $T$  at the terminal path subject to the constraint that the rate of consumption per worker never falls below a certain floor at any time  $t < T$ . This approach is not followed in this chapter for reasons of analytical convenience.

Even with the simplifying assumption of disregarding consumption prior to  $T$ , we have not been able to solve the problem of choosing a time path  $c \in S$  which minimizes  $T$ . However we have been able to solve for a time path  $c$  which minimizes  $T$  among the class of time paths which are members of a subset  $S'$  of  $S$ . The character of the solution obtained leads us to the conjecture that the solution does, in minimize  $T$  among all  $c \in S$  as well.

The particular subset  $S'$  will consist of feasible paths  $c$  with the following characteristics: (1) each such path reaches the terminal path at a time  $T(\sigma)$ ;<sup>1</sup> (2) during an interval  $[0, t_1(\sigma)]$  of time ( $t_1(\sigma) < T(\sigma)$ ), the stock of machines of type 0 is made to grow at a rate  $\sigma > \Theta$  from an initial level below its value on the terminal path; (3) at any  $t$  in  $0 \leq t \leq t_1(\sigma)$ , machine production in excess of that needed to assure a growth rate  $\sigma$  of type 0 machines is invested in type  $j$  machines;<sup>2</sup> (4) at  $t_1(\sigma)$  the stock of type 0 machines is sufficiently above its corresponding value on the terminal path that we can attain the terminal path at  $T(\sigma)$  by letting it depreciate; and (5) the entire output of machines at any  $t$  in  $t_1(\sigma) < t \leq T(\sigma)$  is invested in type  $j$  machines. Each path  $c \in S'$  can therefore be defined by its  $\sigma$ . The range for  $\sigma$  is the semi-open interval  $[\Theta, \gamma_0 - \delta]$ .

Unless  $\sigma$  is greater than the rate of growth of labour force,  $\Theta$ , the stock of machines of type 0 cannot grow from an initial level below that corresponding to the terminal path to a level above the terminal path at  $t_1(\sigma)$ . The rate of growth of the stock of machines (and hence  $\sigma$ ) cannot exceed  $(\gamma_0 - \delta)$ , the maximum net reproduction rate of machines.

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<sup>1</sup>In this chapter time will be treated as a continuous variable.

<sup>2</sup>It is worth recalling that on the terminal path the stocks of machines 0 and  $j$  only are different from zero and grow at the rate  $\Theta$ .

The path which minimizes  $T(\sigma)$ , the time of arriving at the terminal path, is one for which  $\sigma = \gamma_0 - \delta$ . This means that during an interval  $[0, t_1(\gamma_0 - \delta)]$  of time, the stock of machines of type 0 grows at the maximum feasible rate  $(\gamma_0 - \delta)$  and necessarily the existing stocks of machines of all other types decline at the maximum possible rate  $\delta$ . Hence consumption per worker declines during the same interval of time. In algebraic terms the path which minimizes  $T(\sigma)$  is given by the following equations:<sup>1</sup>

$$\left. \begin{aligned} K_0(t) &= K_0(0) e^{(\gamma_0 - \delta)t} \\ K_j(t) &= K_j(0) e^{-\delta t} \end{aligned} \right\} 0 \leq t \leq t_1(\gamma_0 - \delta) \quad (3.1)$$

$$K_0(t) = K_0(0) e^{[(\gamma_0 - \delta)t_1 - \delta(t - t_1)]} \quad (3.3)$$

$$K_j(t) = \left[ \gamma_0(t - t_1) K_0(0) e^{(\gamma_0 - \delta)t_1} + K_j(0) e^{-\delta t_1} \right] e^{-\delta(t - t_1)} \quad (3.4)$$

where  $t_1(\gamma_0 - \delta)$  and  $T(\gamma_0 - \delta)$  are to be solved from

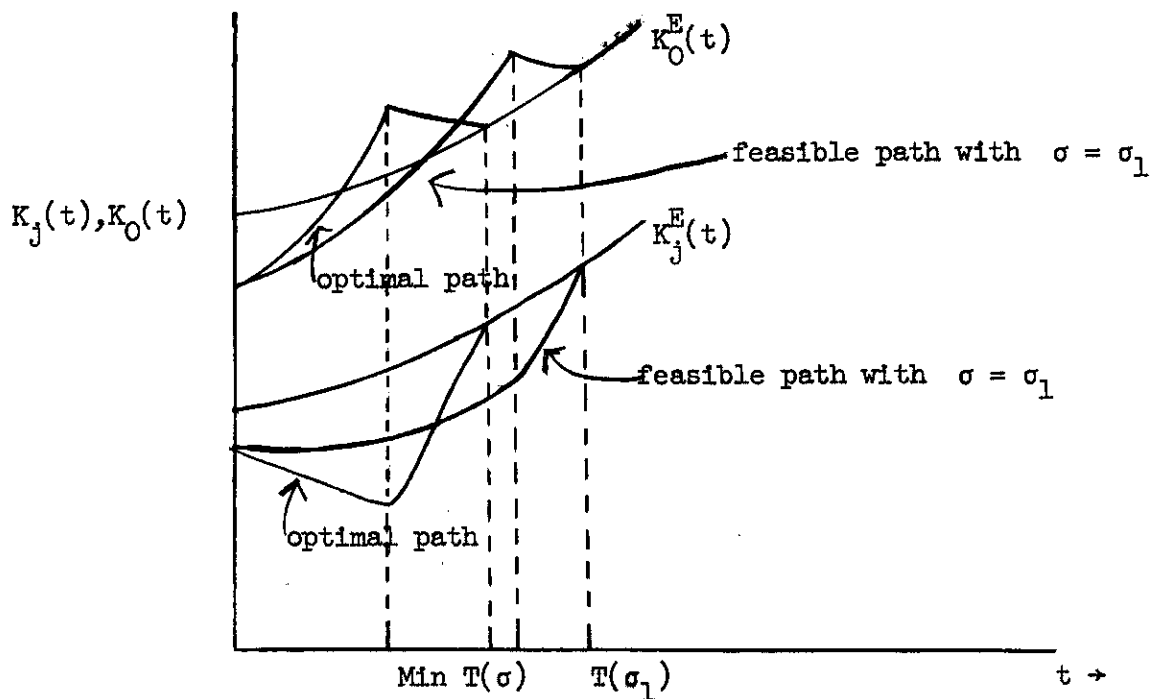
<sup>1</sup>For brevity, only the stocks (in existence) of machines of type 0 and j are given explicitly. The given initial stocks of all other types of machines are allowed to depreciate along this path. In (3.3) - (3.6) the argument  $(\gamma_0 - \delta)$  associated with  $t_1(\gamma_0 - \delta)$  and  $T(\gamma_0 - \delta)$  is omitted to avoid confusion. A superscript E is used to denote terminal path values.

$$K_0(T) = K_0^E(T) = \frac{(\gamma_0 - \theta - \delta)e^{\theta T}}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (3.5)$$

$$K_j(T) = K_j^E(T) = \frac{(\theta + \delta)e^{\theta T}}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)} \quad (3.6)$$

We can illustrate two members of  $S'$ , one of which minimizes  $T(\sigma)$ , in the following figure:<sup>1</sup>

Figure 3a



<sup>1</sup>This and all subsequent figures are for illustrative purposes only. No quantitative significance is intended.

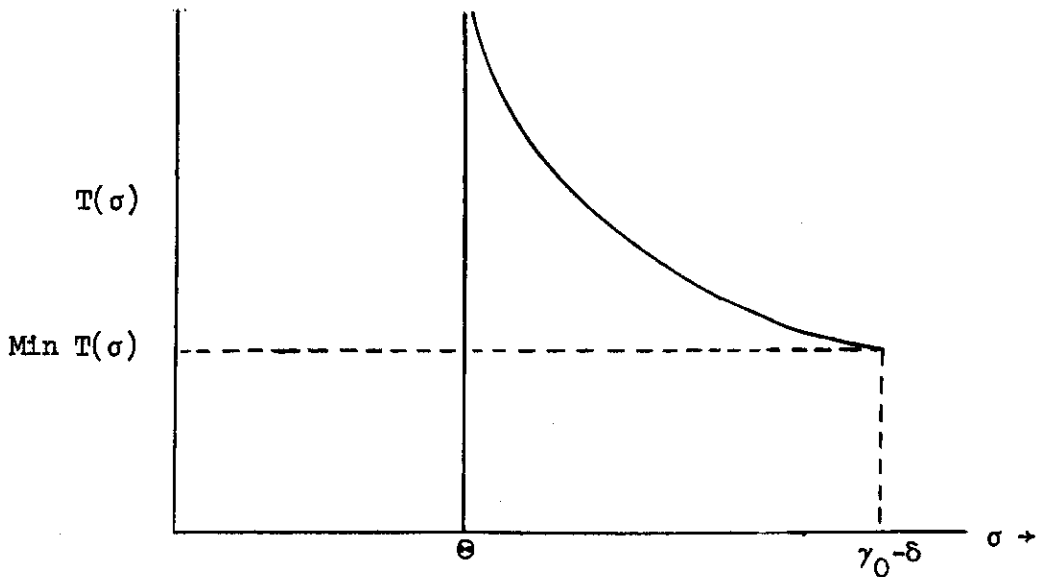


Figure 3b

This trade-off diagram shows that as  $\sigma$  increases from  $\Theta$  to its maximum possible value  $\gamma_0 - \delta$ ,  $T(\sigma)$  declines continuously. The larger the value of  $\sigma$  the greater is the cost (loosely speaking) in terms of consumption foregone prior to  $T(\sigma)$ , of reaching the terminal path. This can be seen as follows. Suppose the planners have made a decision to reach the terminal path through a member of the class of paths  $S'$ .<sup>1</sup> Then their choice is restricted to some  $\sigma$  in  $\Theta < \sigma \leq (\gamma_0 - \delta)$ .

---

<sup>1</sup>A complete discussion of the cost in terms of consumption foregone (suitably defined) in reaching the terminal path in a finite time will involve comparing feasible paths  $c \in S$ . We are again restricting ourselves to paths  $c \in S' \subset S$  by assuming that the planners have made an arbitrary decision to choose a  $c \in S'$ . Since the discussion above is of a qualitative character, this assumption is not very serious.



(3.2) It can be shown (see Section 2) that along any time path  $c \in S'$ , the rate of labour services devoted to the production of machines at any  $t$  in  $0 \leq t \leq T(\sigma)$  is less than the available supply. Thus a surplus of labour is available at each  $t$  for employment in the production of consumer goods. Further, by definition of  $S'$ ,  $\sigma$  is the rate of growth of the stock of machines of type 0 (machine producing machines) at any  $t$  in  $0 \leq t \leq t_1(\sigma) < T(\sigma)$ . Hence the larger the value of  $\sigma$ , the smaller is the surplus of labour available for employment in the production of consumer goods and smaller is also the stock of various types of machines capable of producing consumer goods. Because of both these factors, the possible rate of production of consumer goods is smaller (at any  $t$  in  $0 \leq t \leq t_1(\sigma)$ ) the larger the value of  $\sigma$ . It can be shown that  $\sigma = \gamma_0 - \delta$  minimizes both  $t_1(\sigma)$  and  $T(\sigma)$ . Hence the path associated  $\sigma = (\gamma_0 - \delta)$  involves the maximum sacrifice in terms of consumption foregone during  $0 \leq t \leq \text{Min}_{\sigma} t_1(\sigma) = t_1(\gamma_0 - \delta)$ . We can illustrate this in terms of the following figure.

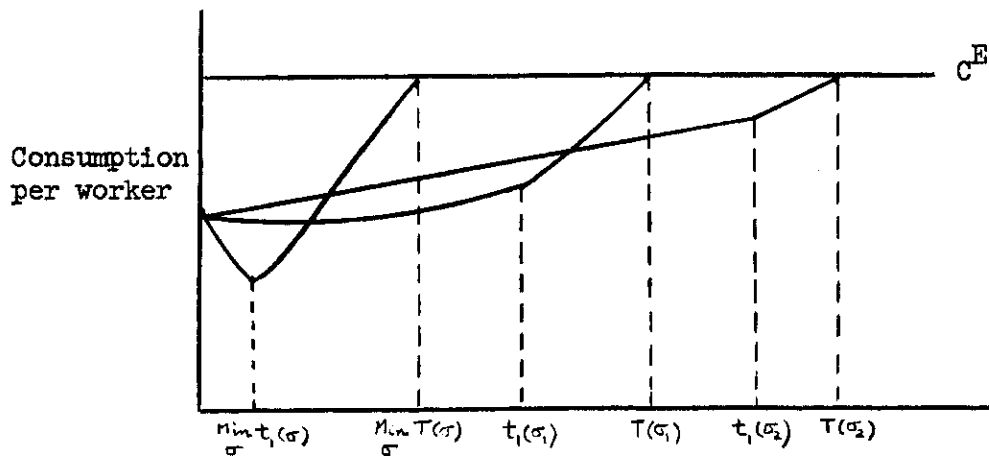


Figure 3c

In figure 3c three possible time paths of consumption per worker are shown corresponding to three different values of  $\sigma$  in the order  $\gamma_0 - \delta > \sigma_1 > \sigma_2$ . The figure as drawn shows a path which permits consumption per worker to increase continuously during  $0 \leq t \leq T(c)$ . In general such a path need not exist. For if the initial stock of machines of type 0 is not sufficiently large, any path whether it leads to the terminal path or not will involve a decline in the rate of consumption per worker during an initial interval of time.<sup>1</sup> In any case, it is clear that the path minimizes the time of reaching the terminal path and involves the largest decline in the rate of consumption per worker during an initial interval  $[0, t_1(\gamma - \delta)]$ . But this path may still be optimal if one considers an alternative objective such as minimizing the (undiscounted) sum of the deviations of actual rate of consumption per worker from the maximum sustainable rate. In other words, the path which minimizes  $T(\sigma)$  may also minimize the integral  $\int_0^{\infty} [c^E - c(t)] dt$ . However we have not investigated this approach.

---

<sup>1</sup>This assertion can be proved as follows: Suppose the initial rate of output of consumer goods be  $C(0)$ . If the initial consumption per worker is to be maintained, the rate of output of consumer goods has to increase at the rate  $\theta C(0)$ . The rate of gross increase in machines needed to achieve this is  $\frac{(\theta + \delta)C(0)}{\gamma_1}$  if we use technique 1, the least 'machine intensive' of the alternative techniques of producing consumer goods. On the other hand the maximum rate of output of machines is  $\gamma_0 K_0(0)$  where  $K_0(0)$  is the initial stock of machines of type 0. Thus if  $\gamma_0 K_0(0) < \frac{(\theta + \delta)C(0)}{\gamma_1}$  any time path will involve a decline in the rate of consumption per worker, at time 0.

### 3.2. Derivations

In this section we derive the path which reaches the terminal path from given initial conditions in minimum time and coincides with the terminal path thereafter. It can be easily shown that along any path which coincides with the terminal path beyond a certain point in time the stocks of various types of machines necessarily have to equal their corresponding values on the terminal path.<sup>1</sup> Therefore the problem of minimizing the time of arrival at the terminal can be stated as follows:

Minimize  $T$  subject to

$$K_i(T) = K_i^E(T) \quad i = 0, j \quad (3.7)$$

$$K_i(t) = \left[ K_i(0) + \int_0^t e^{\delta t'} X_i(t') dt' \right] e^{-\delta t} \quad i = 0, 1, \dots \quad (3.8)$$

$$K_i(t) - K_i'(t) \geq 0 \quad (3.9)$$

$$- \sum_{i=0}^{\infty} \lambda_i K_i'(t) \geq -e^{\theta t} \quad (3.10)$$

$$\gamma_0 K_0'(t) - \sum_{i=0}^{\infty} X_i(t) \geq 0 \quad (3.11)$$

$$K_i'(t), X_i(t) \geq 0 \quad (3.12)$$

$$0 \leq t \leq T$$

---

<sup>1</sup>This result is true under the assumption that  $\lambda_0 \leq \lambda_j$ .

Restraint (3.7) states that the terminal path is attained at  $T$ . Equation (3.8) gives the stock in existence of type  $j$  machine at any time  $t$ . Restraint (3.9) ensures that the stock in use of any type of machine does not exceed the stock in existence. (3.10) is the labour restraint. Restraint (3.11) ensures that gross investment does not exceed the rate of output of new machines. (3.12) is the nonnegativity restraint.

It is clear that if one wanted to minimize  $T$  subject to an additional constraint that the rate of consumption per worker at any  $t < T$  never fell below a certain floor  $\bar{c}$  then one must add the following to the set of restraints (3.7) - (3.12)

$$\sum_{i=1}^{\infty} \gamma_i K'_i(t) \geq \bar{c} e^{\theta t}.$$

However, we shall not follow this approach.

It was mentioned in Section (3.1) that  $T$  is to be minimized among paths belonging to a particular subset  $S'$  of the class of paths satisfying (3.7) - (3.12). We can express a member of  $S'$  in algebraic terms as follows:<sup>1</sup>

for  $t$  in  $0 \leq t \leq t(\sigma) = t_1$  define

$$K_0(t) = K'_0(t) = K_0(0)e^{\sigma t} \tag{3.13}$$

---

<sup>1</sup>It is convenient to write  $t_1$  for  $t(\sigma)$  and  $t_2$  for  $T(\sigma)$  to avoid confusion.

$$K_j(t) = \left( \frac{\gamma_0 - \sigma - \delta}{\sigma + \delta} \right) K_0(0) e^{\sigma t} + \left\{ K_j(0) - \left( \frac{\gamma_0 - \sigma - \delta}{\sigma + \delta} \right) K_0(0) \right\} e^{-\delta t} \quad (3.14)$$

$$K_i(t) = K_i(0) e^{-\delta t} \quad \text{for } i \neq 0, j \quad (3.15)$$

$$K_i'(t) = 0 \quad \text{for } i \neq 0 \quad (3.16)$$

$$X_0(t) = (\sigma + \delta) K_0(0) e^{\sigma t} \quad (3.17)$$

$$X_j(t) = (\gamma_0 - \sigma - \delta) K_j(0) e^{\sigma t} \quad (3.18)$$

$$X_i(t) = 0 \quad \text{for } i \neq 0, j \quad (3.19)$$

and for  $t$  in  $t(\sigma) < t \leq T(\sigma) = t_2$  define

$$K_0'(t) = K_0(t) = K_0(t_1) e^{-\delta(t-t_1)} \quad (3.20)$$

$$K_j(t) = [K_j(t_1) + \gamma_0(t-t_1) K_0(t_1)] e^{-\delta(t-t_1)} \quad (3.21)$$

$$K_i(t) = K_i(0) e^{-\delta t} \quad \text{for } i \neq 0, j \quad (3.22)$$

$$K_i'(t) = 0 \quad \text{for } i \neq 0 \quad (3.23)$$

$$X_j(t) = \gamma_0 K_0(t_1) e^{-\delta(t-t_1)} \quad (3.24)$$

$$X_i(t) = 0 \quad \text{for } i \neq j \quad (3.25)$$

$t_1$  and  $t_2$  are determined from

$$K'_0(t_2) = K_0^E(t_2) \quad (3.26)$$

$$K'_j(t_2) = K_j^E(t_2) \quad (3.27)$$

By varying  $\sigma$  in the semi-open interval  $0 < \sigma \leq (\gamma_0 - \delta)$  we can get all the members of  $S'$ . We have not yet shown that  $S'$  is a subset of the class of paths satisfying (3.7) - (3.12). Assuming that for a given  $\sigma$  we can find  $t_2 \geq t_1 > 0$  such that (3.26) and (3.27) hold, we can verify that each member of  $S'$  satisfies all the relevant restraints except (3.10). We shall prove that (3.10) is also satisfied after proving that  $t_1$  and  $t_2$  exist.

After substituting for  $K'_0(t_2)$ ,  $K'_j(t_2)$ ,  $K_0^E(t_2)$  and  $K_j^E(t_2)$ , (3.26) and (3.27) can be rewritten as

$$K_0(0)e^{\sigma t_1 - \delta(t_2 - t_1)} = K_0^E(0)e^{\Theta t_2} \quad (3.28)$$

$$\begin{aligned} & \left[ \left\{ \left( \frac{\gamma_0 - \sigma - \delta}{\sigma + \delta} \right) + \gamma_0(t_2 - t_1) \right\} K_0(0)e^{\sigma t_1} \right. \\ & \quad \left. + \left\{ K_j(0) - \left( \frac{\gamma_0 - \sigma - \delta}{\sigma + \delta} \right) K_0(0) \right\} e^{-\delta t_1} \right] e^{-\delta(t_2 - t_1)} \\ & = K_j^E(0)e^{\Theta t_2} \end{aligned} \quad (3.29)$$

$$\text{Define } \sigma_0 + \delta = \frac{\gamma_0 K_0(0)}{K_0(0) + K_j(0)} \quad (3.30)$$

Now  $\frac{K_0^E(0)}{K_j^E(0)} = \frac{(\Theta + \delta)}{(\gamma_0 - \Theta - \delta)}$  by definition of  $K_0^E(0)$  and  $K_j^E(0)$ .

Eliminating  $t_2$  between (3.28) and (3.29) and using (3.30) we get

$$\begin{aligned} & \left( \frac{\sigma - \Theta}{\Theta + \delta} \right) t_1 + \left[ \frac{1}{\sigma_0 + \delta} - \frac{1}{\sigma + \delta} \right] e^{-(\sigma + \delta)t_1} = \left( \frac{1}{\Theta + \delta} - \frac{1}{\sigma + \delta} \right) \\ & \quad + \frac{1}{(\Theta + \delta)} \log_e \left( \frac{K_0^E(0)}{K_0(0)} \right) \end{aligned} \quad (3.31)$$

Let us define

$$f(t) = \left( \frac{\sigma - \theta}{\theta + \delta} \right) t + \left[ \frac{1}{\sigma_0 + \delta} - \frac{1}{\sigma + \delta} \right] e^{-(\sigma + \delta)t} \quad (3.32)$$

and

$$k = \left( \frac{1}{\theta + \delta} - \frac{1}{\sigma + \delta} \right) + \frac{1}{(\theta + \delta)} \log \left( \frac{K_0^E(0)}{K_0(0)} \right) \quad (3.33)$$

We wish to examine whether a nonnegative solution to the equation  $f(t) = k$  exists. The discussion is facilitated by distinguishing a few cases:

Case 1.  $\sigma_0 \geq \theta$

It is obvious that<sup>1</sup>  $f(0) < k$  given that  $K_0^E(0) > K_0(0)$ .

Now

$$\dot{f}(t) = \frac{df}{dt} = \left( \frac{\sigma - \theta}{\theta + \delta} \right) - \left( \frac{\sigma - \sigma_0}{\sigma_0 + \delta} \right) e^{-(\sigma + \delta)t} \quad (3.34)$$

clearly  $\dot{f}(t) > 0$  for all  $t \geq 0$  since  $\sigma_0 \geq \theta$  and  $\sigma > \theta$ . It is also clear that  $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Therefore a unique  $t_1 > 0$  exists such that  $f(t_1) = k$ .

---

<sup>1</sup>

It is worth remembering that approaches to the terminal from below are being considered. In other words we are starting from  $K_0(0) < K_0^E(0)$  and  $K_j(0) < K_j^E(0)$ .



Case 2.  $\sigma_0 < \Theta$

It is clear from (3.34) that  $\dot{f}(t) < 0$  for  $0 \leq t < t'$  and  $\dot{f}(t) > 0$  for  $t > t' > 0$  where  $t'$  is given by

$$\dot{f}(t') = 0 \tag{3.35}$$

One can verify by examining the second derivative of  $f(t)$  at  $t = t'$  that  $f(t)$  attains its minimum value at  $t = t'$ . Now it can be shown that<sup>1</sup>  $f(t') < k$ . Therefore there exists a unique  $t_1 > t' > 0$  such that  $f(t_1) = k$ .

We now show that  $t_2 > t_1$ . From (3.28) we have

$$(t_2 - t_1) = \left( \frac{\sigma - \Theta}{\Theta + \delta} \right) t_1 - \left( \frac{1}{\Theta + \delta} \right) \log \left( \frac{K_0^E(0)}{K_0(0)} \right) \tag{3.36}$$

Using (3.31) with (3.36) we get

$$(t_2 - t_1) = \left( \frac{1}{\Theta + \delta} - \frac{1}{\sigma + \delta} \right) - \left( \frac{1}{\sigma_0 + \delta} - \frac{1}{\sigma + \delta} \right) e^{-(\sigma + \delta)t_1} \tag{3.37}$$

$$= \frac{1}{(\sigma + \delta)} \dot{f}(t_1) \tag{3.38}$$

---

<sup>1</sup>The proof of this assertion is rather long and tedious. Since it does not yield any results that are used subsequently, it has been omitted.

We know that if  $\sigma_0 \geq \Theta$   $\dot{f}(t) \geq 0$  for all  $t \geq t_1$  and hence  $(t_2 - t_1) > 0$ . If  $\sigma_0 < \Theta$ , we know that  $t_1 > t'$  and  $\dot{f}(t) > 0$  for  $t > t'$ . Therefore in either case  $(t_2 - t_1) > 0$ . Thus we have succeeded in showing the existence of  $t_2 > t_1 > 0$  satisfying (3.26) and (3.27).

We still have to show that (3.10) is satisfied by each member of the set  $S'$  of paths described by (3.13) - (3.27). It can be seen that for any number  $S'$  if (3.10) holds at  $t = t_1$  then it holds for all  $t$  in  $0 \leq t < t_1$  and  $t_1 < t \leq t_2$ . Therefore we have to show that

$$\lambda_0 K_0(0) e^{\sigma t_1} \leq e^{\Theta t_1} \quad (3.39)$$

on

$$(\sigma - \Theta)t_1 + \log(\lambda_0 K_0(0)) \leq 0 \quad (3.40)$$

Substituting for  $(\sigma - \Theta)t_1$  from (3.31) we get

$$\begin{aligned} (\sigma - \Theta)t_1 + \log(\lambda_0 K_0(0)) &= \log \left\{ \lambda_0 K_0^E(0) \right\} + \left( \frac{\sigma - \Theta}{\sigma + \delta} \right) \\ &- \left[ \frac{(\sigma - \sigma_0)(\Theta + \delta)}{(\sigma + \delta)(\sigma_0 + \delta)} \right] e^{-(\sigma + \delta)t_1} \end{aligned} \quad (3.41)$$

Now

$$\left. \begin{aligned} & \left( \frac{\sigma - \Theta}{\sigma + \delta} \right) - \left[ \frac{(\sigma - \sigma_0)(\Theta + \delta)}{(\sigma + \delta)(\sigma_0 + \delta)} \right] e^{-(\sigma + \delta)t_1} < \left( \frac{\sigma - \Theta}{\sigma + \delta} \right) \\ & \text{if } \sigma > \text{Max}(\sigma_0, \Theta) \\ & \leq \left( \frac{\sigma_0 - \Theta}{\sigma_0 + \delta} \right) \text{ if } \sigma_0 \geq \sigma > \Theta \end{aligned} \right\} \quad (3.42)$$

By definition  $\sigma_0 < (\gamma_0 - \delta)$  and  $\sigma \leq (\gamma_0 - \delta)$  therefore we can assert that

$$(\sigma - \Theta)t_1 + \log(\delta_0 K_0^E(0)) \leq \frac{(\gamma_0 - \Theta - \delta)}{\gamma_0} + \log \left\{ \lambda_0 K_0^E(0) \right\}$$

Given that<sup>1</sup>  $\lambda_j > \lambda_0$  it can be shown that

$$\left( \frac{\gamma_0 - \Theta - \delta}{\gamma_0} \right) + \log(\lambda_0 K_0^E(0)) \leq 0. \quad (3.43)$$

This completes the proof that each member of the set  $S'$  of paths described by (3.13) - (3.27) is feasible in the sense of satisfying (3.7) - (3.12). We can now show that  $t_2(\sigma)$  decreases as  $\sigma$  increases and therefore the member of  $S'$  which minimizes  $t_2(\sigma)$  for  $\sigma$  in  $\Theta < \sigma \leq (\gamma_0 - \delta)$  is obtained by setting  $\sigma = (\gamma_0 - \delta)$ .

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<sup>1</sup>See footnote on page 26.

$$\text{By (3.36) } t_2 = \left( \frac{\sigma+\delta}{\theta+\delta} \right) t_1 - \frac{1}{(\theta+\delta)} \log \left( \frac{K_0^E(\sigma)}{K_0(\sigma)} \right)$$

Therefore

$$\begin{aligned} \frac{dt_2}{d\sigma} &= \left( \frac{1}{\theta+\delta} \right) \left\{ (\sigma+\delta) \frac{dt_1}{d\sigma} + t_1 \right\} \\ &= \frac{1}{(\theta+\delta)} \frac{d\eta}{d\sigma} \quad \text{where } \eta = (\sigma+\delta)t_1 \end{aligned}$$

We can rewrite (3.31) as

$$\left( \frac{1}{\theta+\delta} - \frac{1}{\sigma+\delta} \right) \eta + \left( \frac{1}{\sigma_0+\delta} - \frac{1}{\sigma+\delta} \right) e^{-\eta} = k \quad (3.44)$$

Differentiating both sides of (3.44) with respect to  $\sigma$  we have,

$$\begin{aligned} \frac{1}{(\sigma+\delta)^2} \cdot \eta + \left( \frac{1}{\theta+\delta} - \frac{1}{\sigma+\delta} \right) \frac{d\eta}{d\sigma} - \left( \frac{1}{\sigma_0+\delta} - \frac{1}{\sigma+\delta} \right) e^{-\eta} \cdot \frac{d\eta}{d\sigma} \\ + \frac{1}{(\sigma+\delta)^2} e^{-\eta} = \frac{dk}{d\sigma} \end{aligned} \quad (3.45)$$

Now  $\frac{dk}{d\sigma} = \frac{1}{(\sigma+\delta)^2}$ . Therefore (3.45) reduces to

$$\begin{aligned} \frac{d\eta}{d\sigma} & \left\{ \left( \frac{1}{\Theta+\delta} - \frac{1}{\sigma+\delta} \right) - \frac{1}{\sigma_0+\delta} - \frac{1}{\sigma+\delta} e^{-\eta} \right\} \\ & = \frac{1}{(\sigma+\delta)^2} (1 - e^{-\eta} - \eta) \end{aligned} \quad (3.46)$$

Using (3.37) we can rewrite (3.46) as

$$\frac{d\eta}{d\sigma} = \frac{(1 - e^{-\eta} - \eta)}{(\sigma+\delta)^2 (t_2 - t_1)} \quad (3.47)$$

Now  $(1 - e^{-\eta} - \eta) < 0$  for  $\eta > 0$ . We know that  $(t_2 - t_1) > 0$ .

Therefore  $\frac{d\eta}{d\sigma} < 0$  implying that  $\frac{dt_2}{d\sigma}$  as well as  $\frac{dt_1}{d\sigma}$  are negative.

Hence our assertion that  $\sigma = (\gamma_0 - \delta)$  yields the path which minimizes

$t_2(\sigma)$  for  $\sigma$  in  $\Theta < \sigma \leq (\gamma_0 - \delta)$  is proved.

## CHAPTER 4

### SOCIAL WELFARE AS A FUNCTION OF CONSUMPTION AT A POINT IN TIME

One of the many social welfare functions that have been proposed in the literature makes social welfare an increasing function of the rate of consumption per worker at some chosen future point in time, say  $T$ .<sup>1, 2</sup> Maximizing social welfare is then equivalent to maximizing the rate of consumption per worker at  $T$ . In the present chapter some consequences of following this objective are discussed.

The objective of maximizing the rate of consumption at any single point of time  $T$  in the future is rather difficult to justify. It implies that no consideration is to be given to the rate of consumption at any time before or after  $T$ . However under the following somewhat restrictive assumptions one can justify such an objective. Suppose we consider a one commodity economy, i.e., an economy in which a single commodity, say "Shmoo" serves both as a consumer good and as a capital good.<sup>3</sup> In other words, "Shmoos" can be either consumed or accumulated as capital stock.

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<sup>1</sup>In this chapter time is treated as a discrete variable.

<sup>2</sup>Galenson and Leibenstein, op. cit.

<sup>3</sup>Shmoo is an imaginary animal introduced to the world by Alfred Caplin. Shmoos "lays eggs at th' slightest excuse!! They also gives milk!! And as fo' meat, broiled they make the finest steaks -- fried they come out the yummiest chicken --- And as for upkeep they don't eat anything -- there is no waste!! The hide makes the finest leather -- or cloth, depending on how thick you slice it!" See Alfred Caplin (Al Capp), "The Life and Times of the Shmoo", (New York, Simon and Shuster Inc., 1948), pp. 14-15.

Suppose, further, that each unit of the stock of Shmoos at any period of time makes available a flow of (say)  $\rho$  Shmoos during that period of time. If the total human population and its rate of growth are exogenous factors, then maximizing the rate of consumption per head during any period  $T$  of time is equivalent to maximizing the stock of Shmoos at  $T$ . A necessary condition for the latter is that the stock of Shmoos be maximized at all periods before  $T$  as well. This implies that if the target date  $T$  is moved to  $T' > T$ , the optimal path of the stock of Shmoos (i.e., that path which maximizes the stock of Shmoos at  $T'$ ) up to the period  $T$  remains unaltered. This insensitivity of the optimal path to the choice of  $T$  makes the objective of maximizing the rate of consumption per head at some period  $T$  in the future not unreasonable. This is especially true if one imposes a floor on the rate of consumption per head at all periods before  $T$ .

Once the single commodity assumption of the previous paragraph is relaxed the objective of maximizing the rate of consumption per worker at some  $T$  loses its appeal. In an economy with two sectors, one of which produces consumer goods and the other, machines, maximizing the capacity (in terms of the stocks of various types of machines) of the sector producing consumer goods at all periods before  $T$  is not a necessary condition for maximizing its capacity at  $T$ . The optimal path of the capacity of the consumer goods sector is sensitive to the choice of  $T$ .

The reason for this sensitivity can be seen as follows. In a two sector economy there are two alternative investment opportunities in each period of time as opposed to only one in a Shmoo economy. One can either increase the stock of machines in sector I which produces machines or in sector II which produces consumer goods. These two investment opportunities make available different streams of output of consumer goods. In the notation of Chapter 2, an increment of one machine to the stock of type  $j$  machines in sector II in period  $(T - 1)$  makes available an additional (maximal) flow of  $\gamma_j [1 - \delta]^{t-T}$  units of consumer goods in each  $t \geq T$ . But an increment of one unit to the stock of type 0 machines (i.e., the machines that produce machines) in period  $(T - 1)$  increases the flow of consumer goods by  $\gamma_j [1 + \gamma_0 - \delta] (1 - \delta)^{t-T-1}$  in each  $t \geq T + 1$  assuming that the increased capacity of sector II is used to produce machines of type  $j$  in each period.<sup>1</sup> Thus by waiting for one period one can get an augmented floor of consumer goods.

The decision as to which of the two investment alternatives described above is to be chosen at each period prior to  $T$  depends on the choice of  $T$ . In the rest of this section we examine the time paths prior to  $T$  of the variables consumption per worker, capacity of sector II that result from alternative values of  $T$ . We conclude from this examination that the objective considered, viz., maximizing the rate

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<sup>1</sup>A production lag of one period is assumed in the production of machines as well as consumer goods. This is not essential to the argument.



consumption per worker at time  $T$  is not desirable in a two sector economy.

Since the initial stocks of various types of machines are historically given, it is enough if we devote our attention to describing the time paths of the addition to the stock of machines of various types. It can be shown (see section 4.2) that along a path which maximizes the rate of consumption per worker at time  $T$ , the gross addition to the stock of machines in Sector II at any  $t < T$  will be divided among alternative types in constant proportions independent of  $t$ .<sup>1</sup> Hence it is enough to describe the time paths of the following two components of gross investment: (a)  $X_0(t)$  the gross addition to the stock of machines (of type 0) in Sector I and (b)  $X(t)$  the gross addition to the stock of machines in Sector II without specifying the manner in which  $X(t)$  is to be distributed among alternative types.

The solutions for  $X_0(t)$  and  $X(t)$  fall under three cases depending on the value of  $T$ . In order to distinguish the three cases we need to define two nonnegative integers  $m$  and  $M$  as follows:

$$\gamma_0^m \leq (1 - \delta) \quad \text{and} \quad \gamma_0^{(m+1)} > (1 - \delta) \quad (4.1)$$

$$\lambda_0 K_0(0) [1 + \gamma_0 - \delta]^M < (1 + \theta)^M \quad \text{and} \quad \lambda_0 K_0(0) [1 + \gamma_0 - \delta]^{M+1} \\ \geq (1 + \theta)^{M+1} \quad (4.2)$$

---

<sup>1</sup>This assertion will be true only when there is no floor imposed on the rate of consumption per worker at any  $t < T$ .

From (4.2) it is clear that  $(M+1)$  is the number of periods needed to accumulate enough machines of type 0 to employ the entire labour force in Sector I. We can now enumerate the solutions for  $X_0(t)$  and  $X(t)$ .

Case 1 :  $T \leq (m+1)$ . The solution is<sup>1</sup>

$$\left. \begin{aligned} X(t) &= \gamma_0 K_0(0) [1 - \delta]^t \\ X_0(t) &= 0 \end{aligned} \right\} t = 0, 1, 2, \dots, T-2 \quad (4.3)$$

The equations of (4.3) imply that during the time span  $0 - (T - 2)$  positive gross investment is made only in the consumer goods industry. In other words, if the time point  $T$  at which the rate of consumption per worker is to be maximized is not too far in the future, the optimal policy is to devote the entire output of Sector I to increasing the stock of machines in Sector II during the periods  $0 - (T-2)$ . Thus by letting the stock of machines in Sector I to decline, consumption at some point of time in the near future is maximized at the expense consumption in the far future.

Case 2a :  $T > (m+1)$  and  $T - m - 2 \leq M$ .

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<sup>1</sup>The range for the time variable is from period 0 to period  $T - 2$ . This follows from our assumption of a lag of one period in the production of both machines and consumer goods. The rate of consumption at  $T$  depends on the stock of machines in Sector II at  $T - 1$  and this in turn depends on the investment policy from period 0 to period  $T - 2$ .

The solution in this case is

$$\left. \begin{aligned} X_0(t) &= \gamma_0 K_0(0) [1+\gamma_0-\delta]^t \\ X(t) &= 0 \end{aligned} \right\} t = 0, \dots, T-m-3 \quad (4.4)$$

$$\left. \begin{aligned} X_0(t) &= 0 \\ X(t) &= \gamma_0 K_0(0) [1+\gamma_0-\delta]^{T-m-2} (1-\delta)^{t-(T-m-2)} \end{aligned} \right\} t = T-m-2, \dots, T-2 \quad (4.5)$$

The implication of this solution is that for values of  $T$  in the range considered, the optimal investment policy is to devote the entire output of machines to increasing the stock of machines in Sector I during the periods 0 to  $(T-m-3)$  and to increasing the stock of machines in Sector II during the subsequent periods  $(T-m-2)$  to  $(T-2)$ . Thus the rate of consumption per worker decreases during the periods 0 to  $(T-m-3)$ .

Case 2b :  $T > (m+1)$  and  $(T-m-2) > M$ . The solution in this case is:

$$\left. \begin{aligned} X_0(t) &= \gamma_0 K_0(0) [1+\gamma_0-\delta]^t \\ X(t) &= 0 \end{aligned} \right\} t = 0, 1, \dots, M-1 \quad (4.6)$$

$$\left. \begin{aligned} X_0(t) &= \frac{1}{\lambda_0} [1+\theta]^{M+1} - (1-\delta) K_0(0) [1+\gamma_0-\delta]^M \\ X(t) &= -\frac{1}{\lambda_0} [1+\theta]^{M+1} + K_0(0) [1+\gamma_0-\delta]^{M+1} \end{aligned} \right\} t = M \quad (4.7)$$

$$\left. \begin{aligned} X_0(t) &= \frac{1}{\lambda_0} (\theta+\delta) (1+\theta)^t \\ X(t) &= \frac{(\gamma_0-\theta-\delta)}{\lambda_0} (1+\theta)^t \end{aligned} \right\} t = M+1, \dots T-m-3 \quad (4.8)$$

$$\left. \begin{aligned} X_0(t) &= 0 \\ X(t) &= \frac{1}{\lambda_0} (1+\theta)^{T-m-2} (1-\delta)^{t-(T-m-2)} \end{aligned} \right\} t = T-m-2, \dots T-2 \quad (4.9)$$

This solution is similar to that in Case 2a except that the labour constraint becomes binding at  $t = (M+1)$ . From period  $(M+1)$  to period  $(T-m-2)$  the entire labour force is employed in Sector I, thus reducing the rate of output of consumer goods to zero! Thus when  $T$  is sufficiently far off in the future, the optimal investment policy is to devote the entire output of machines to building up the stock of machines in Sector I during periods 0 to  $(M-1)$ ; from period  $M$  to period  $(T-m-3)$  gross investment is made in both sectors; from period  $(T-m-2)$  to period  $(T-2)$  the entire output of machines is invested in Sector II.

Maximizing consumption per worker at selected future point in time thus leads to one of the following two consequences: Either consumption in the near future is emphasized at the expense of consumption in the far future as in Case 1 or 2 consumption at some far off point  $T$  in time is emphasized at the expense of consumption before  $T$  as in Cases 2a and 2b . In either case, the maximized rate of consumption at  $T$  may not even be sustainable beyond  $T$  .

It might appear that one can modify the social objective considered above by maximizing the rate of consumption per worker and some  $T$  subject to the constraints that the rate of consumption before  $T$  does not fall below a certain given magnitude and that the rate of consumption achieved at  $T$  be sustainable thereafter. But we have already shown in Chapter 2 that there is a unique maximum sustainable rate of consumption per worker. This is attained on the terminal path. This modified objective differs from the objective<sup>1</sup> in Chapter 3 in that a floor on the pre-terminal path rate of consumption has been imposed.

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<sup>1</sup>In Chapter 3 the objective was to minimize the time taken to reach the terminal path from given initial conditions.

4.2. Derivations

We can formulate the problem of maximizing the rate of consumption per worker at time  $T$  (starting from given initial conditions and ignoring the rate of consumption prior to  $T$ ) as the following linear programming problem using the notation of Chapter 2.

Problem I

$$\text{Maximize } (1+\theta)^{-T} \sum_{i=1}^{\infty} \gamma_i K'_i(T-1) \quad \text{subject to}$$

$$\sum_{i=1}^{\infty} \lambda_i K'_i(T-1) \leq (1+\theta)^{T-1} \quad (4.10)$$

$$- \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X_i(t) + K'_i(T-1) \leq K_i(0)(1-\delta)^{T-1} \quad i = 1, 2, \dots \quad (4.11)$$

$$\sum_{i=0}^{\infty} X_i(t) - \gamma_0 K'_0(t) \leq 0 \quad (4.12)$$

$$- \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_0(t') + K'_0(t) \leq K_0(0)(1-\delta)^t \quad t = 0, 1, \dots, T-2 \quad (4.13)$$

$$\lambda_0 K'_0(t) \leq (1+\theta)^t \quad (4.14)$$

$$K'_0(t), X_i(t), K'_i(T-1) \geq 0 \quad (4.15)$$

The maximand is the rate of consumption per worker at time  $T$ . Restraint (4.10) is the labour restraint at time  $T$ . (4.11) states that the stock of machines of any type in use at time  $(T-1)$  does not exceed that in existence. Restraints (4.12) - (4.14) refer to periods prior to  $(T-1)$ . (4.12) states that the total gross investment in machines of various types at  $t$  does not exceed the rate of output of machines at  $t$ . (4.13) ensures that the stock of type 0 machines in use at  $t$  does not exceed that in existence. (4.14) is the labour restraint at  $t$ . It will be noticed that the variables  $K'_i(t)$   $i = 1, 2, \dots$ , viz., the stock of machines in use of types other than 0 do not enter (4.14). The reason is that we have chosen to ignore the rate of consumption prior to  $T$ . Restraint (4.15) is the nonnegativity restraint on our variables.

We stated earlier that along the optimal path, the gross addition to the stock of machines in Sector II will be divided among alternative types in constant proportions independent of  $t$ . In other words, along an optimal path  $\frac{X_i(t)}{X(t)}$  will be a constant independent of  $t$ . For showing this, we first have to reduce problem I into an equivalent problem. We need to define a set of new variables. Let

$$Y_i(T-1) = \text{Addition (gross of depreciation of the initial stock)} \\ \text{to the stock of machines of type } i \text{ over the} \\ \text{periods } 0 - (T-2) .$$

$$I(T-1) = \sum_{i=1}^{\infty} Y_i(T-1) \quad \text{clearly}$$

$$Y_i(T-1) = \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X_i(t) \quad \text{It will be recalled that}$$

by definition  $X(t) = \sum_{i=1}^{\infty} X_i(t)$ . Now problem I is equivalent to the

following linear problem in a sense to be made more precise in the next paragraph.

Problem II

$$\text{Maximize } (1+\theta)^{-T} \sum_{i=1}^{\infty} \gamma_i K_i'(T-1) \quad \text{subject to}$$

$$\sum_{i=1}^{\infty} \lambda_i K_i'(T-1) \leq (1+\theta)^{T-1} \quad (4.16)$$

$$- Y_i(T-1) K_i'(T-1) \leq K_i(0) (1-\delta)^{T-1} \quad i = 1, 2, \dots \quad (4.17)$$

$$\sum_{i=1}^{\infty} Y_i(T-1) - I(T-1) = 0 \quad (4.18)$$

$$- \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X(t) + I(T-1) = 0 \quad (4.19)$$



$$X_0(t) + X(t) - \gamma_0 K'_0(t) \leq 0 \quad (4.20)$$

$$- \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_0(t') + K'_0(t) \leq K_0(0) (1-\delta)^t \quad t = 0, 1, \dots, T-2 \quad (4.21)$$

$$\frac{\lambda_0}{\beta_0} K'_0(t) \leq (1+\theta)^t \quad (4.22)$$

$$K'_0(t), X_0(t), X(t), I(T-1), Y_1(T-1), K'_1(T-1) \geq 0 \quad (4.23)$$

It is easy to see that any feasible solution of Problem I, i.e., any set of values of the variables  $X_1(t)$ ,  $K'_0(t)$  and  $K'_1(T-1)$  which satisfy (4.10) - (4.15), provides a feasible solution of Problem II. From any feasible solution of Problem II we can get a feasible solution of Problem I by setting<sup>1</sup>

$$X_1(t) = \begin{cases} \left[ \frac{Y_1(T-1)}{I(T-1)} \right] X(t) & \text{if } I(T-1) > 0 \\ 0 & \text{if } I(T-1) = 0 \end{cases}$$

The maximand is the same for both problems and the correspondence between the set of feasible solutions of the two problems leaves the values of

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<sup>1</sup>It is seen that our earlier assertion that  $X(t)$  will be divided among alternative types in constant proportions independent of  $t$  is true.

the variables that enter the maximand unaltered. Hence Problems I and II are equivalent and it is sufficient to solve Problem II.

We now assert that Problem II can be decomposed into two linear programming problems. First, from (4.17) and (4.18) it is clear that increasing the value of  $I(T-1)$  can never decrease the optimal value of the maximand.<sup>1</sup> Second, the set of variables that enter the maximand, and the restraints (4.16) - (4.18) and the set of variables that enter the restraints (4.19) - (4.22) have only  $I(T-1)$  in common.<sup>2</sup> Hence we can decompose Problem II into the following two linear programming problems:

Problem IIa

$$\text{Maximize } I(T-1) = \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X(t)$$

subject to

$$X_0(t) + X(t) - \gamma_0 K'_0(t) \leq 0 \quad (4.24)$$

$$-\sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_0(t') + K'_0(t) \leq K_0(0) (1-\delta)^t \quad (4.25)$$

$$\lambda_0 K'_0(t) \leq (1+\theta)^t \quad (4.26)$$

$$X_0(t), X(t) \geq 0 \quad (4.27)$$

t = 0, 1, ..T-2

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<sup>1</sup>By assuming that there exists an  $i$  with  $\lambda_i$  sufficiently small, we can assert that the optional value of the maximand is in fact increased by increasing  $I(T-1)$ .

<sup>2</sup>This assertion is possible only because we did not impose a floor on consumption at any  $t < T$ .

Problem IIb

$$\text{Maximize } (1+\Theta)^{-T} \sum_{i=1}^{\infty} \gamma_i K'_i(T-1)$$

subject to

$$\sum_{i=1}^{\infty} \lambda_i K'_i(T-1) \leq (1+\Theta)^{T-1} \quad (4.28)$$

$$- Y_i(T-1) + K'_i(T-1) \leq K_i(0) (1-\delta)^{T-1} \quad (4.29)$$

$$\sum_{i=1}^{\infty} Y_i(T-1) - I_{\max}(T-1) = 0 \quad (4.30)$$

$$K'_i(T-1), Y_i(T-1) \geq 0 \quad (4.31)$$

It is seen that problem IIb involves the period (T-1) only and its solution simply determines the distribution of the gross investment X(t) in Sector II between alternative types of machines. For as we showed earlier  $X_i(t)$  are to be determined by  $X_i(t) = \left[ \frac{Y_i(T-1)}{I_{\max}(T-1)} \right] X(t)$ .

The time path of the variables X(t) and  $X_0(t)$  can be obtained by solving Problem IIa. We present the solutions of Problem IIa only.

The optimal solutions to Problem IIa fall under three cases, as was mentioned earlier. We now present the solutions and establish their optimality by associating with each solution a set of nonnegative shadow prices and showing that under the chosen price system the solution is

sustained. In the language of activity analysis, we need to show that under the chosen price system no activity makes a positive profit and all activities operated at a positive level break even. We use the following notation to translate these price conditions into algebraic terms:

$q(t)$  : Shadow price of a unit of newly produced machine becoming available at time  $t$  .

$r(t)$  : Rent per period of a unit of type 0 machine at  $t$  .

$w(t)$  : Wages per period per unit of labour at  $t$  .

By  $X_0(t)$  ,  $X(t)$  ,  $K'_0(t)$  let us denote respectively the activities (at  $t$  ) of adding one machine to the stock of machines in Sector I, Sector II and the activity of producing machines. Let the same symbols represent the levels at which the corresponding activities are operated. Then the 'price conditions' associated with an optimal solution can be written as follows:

$$\left. \begin{aligned} q(t) &\geq \sum_{t'=t+1}^{T-2} (1-\delta)^{t'-t} r(t') \quad \text{for all } t \\ &= \sum_{t'=t+1}^{T-2} (1-\delta)^{t'-t} r(t') \quad \text{if } X_0(t) > 0 \end{aligned} \right\} \quad (4.32)$$

$$\left. \begin{aligned} q(t) &\geq (1-\delta)^{T-1-t} \quad \text{for all } t \\ &= (1-\delta)^{T-1-t} \quad \text{if } X(t) > 0 \end{aligned} \right\} \quad (4.33)$$

$$\left. \begin{aligned} \gamma_0 q(t+1) &\leq \lambda_0 w(t) + r(t) && \text{for all } t \\ &= \lambda_0 w(t) + r(t) && \text{if } K'_0(t) > 0 \end{aligned} \right\} \quad (4.34)$$

The first part of (4.32) states that the activity  $X_0(t)$  never makes a positive profit and the second part states that  $X_0(t)$  breaks even whenever it is operated at a positive level. Another way of stating the economic meaning of (4.32) is to say that the cost of producing a new machine should be equal to the sum of the stream of returns (rents) from it in Sector I over the relevant period if it is to be added to the stock of machines in that sector. (4.33) and (4.34) express similar profit maximizing conditions with respect to activities  $X(t)$  and  $K'_0(t)$ .<sup>1</sup>

Now the optimal quantity solutions and the corresponding price solutions can be given.<sup>2</sup>

Case 1 :  $T \leq (m+1)$  . The solutions are

$$\left. \begin{aligned} X_0(t) &= 0 \\ X(t) &= \gamma_0 K_0(0) (1-\delta)^t \\ K'_0(t) &= K_0(0) (1-\delta)^t \end{aligned} \right\} t = 0, 1, \dots, T-2 \quad (4.35)$$

<sup>1</sup>The assumed production lag of one period is built into the price inequalities. In other words, outputs are evaluated at the price of the period in which they become available and inputs are evaluated at the prices of the period in which they are used.

<sup>2</sup>The price of a machine in Sector II (irrespective of its type) available for use at time  $(T-1)$  is set equal to unity.

$$q(t) = (1-\delta)^{T-1-t} \quad t = 1, 2, \dots, T-1 \quad (4.36)$$

$$\left. \begin{aligned} \gamma(t) &= \gamma_0 [1-\delta]^{T-2-t} \\ w(t) &= 0 \end{aligned} \right\} \quad t = 0, 1, \dots, T-2 \quad (4.37)$$

$$\left. \begin{aligned} \gamma(t) &= \gamma_0 [1-\delta]^{T-2-t} \\ w(t) &= 0 \end{aligned} \right\} \quad t = 0, 1, \dots, T-2 \quad (4.38)$$

By direct substitution in (4.24) - (4.27) one can verify that the quantity solution is feasible. The price solution can be seen to sustain the quantity solution by substitution in (4.32) - (4.34). Therefore the above solution is optimal. As a check on our price computations we can verify that optimal value of the maximand is imputed to the scarce factors of production, i.e., the binding restraints. In other words, the following is seen to hold:

$$\begin{aligned} \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X(t) &= \sum_{t=0}^{T-2} \left\{ K_0(0) (1-\delta)^t \right\} r(t) \\ &= \gamma_0 (T-1) K_0(0) (1-\delta)^{T-2} . \end{aligned}$$

Case 2a :  $T > (m+1)$  and  $(T-m-2) \leq M$ . The solutions are

$$\left. \begin{aligned} X_0(t) &= \gamma_0 K_0(0) (1+\gamma_0-\delta)^t \\ X(t) &= 0 \\ K_0^*(t) &= K_0(0) (1+\gamma_0-\delta)^t \end{aligned} \right\} \quad t = 0, 1, \dots, T-m-3$$

$$\begin{array}{l}
 X_0(t) = 0 \\
 X(t) = \gamma_0 K_0(0) (1+\gamma_0-\delta)^{T-m-2} (1-\delta)^{t-(T-m-2)} \\
 K'_0(t) = K_0(0) (1+\gamma_0-\delta)^{T-m-2} (1-\delta)^{t-(T-m-2)}
 \end{array}
 \left. \vphantom{\begin{array}{l} X_0(t) \\ X(t) \\ K'_0(t) \end{array}} \right\} t = T-m-2, \dots, T-2$$
  

$$\begin{array}{l}
 q(t) = \gamma_0^{(m+1)} (1-\delta)^m (1+\gamma_0-\delta)^{(T-m-1)-t} \\
 = (1-\delta)^{T-1-t}
 \end{array}
 \left. \vphantom{\begin{array}{l} q(t) \\ = \end{array}} \right\} \begin{array}{l} t = 1, \dots, T-m-2 \\ t = T-m-1, \dots, T-1 \end{array}$$
  

$$\begin{array}{l}
 r(t) = \gamma_0^2 (m+1) (1-\delta)^m (1+\gamma_0-\delta)^{(T-m-2)-t} \\
 = \gamma_0 (1-\delta)^{T-2-t}
 \end{array}
 \left. \vphantom{\begin{array}{l} r(t) \\ = \end{array}} \right\} \begin{array}{l} t = 0, 1, \dots, T-m-3 \\ t = T-m-2, \dots, T-2 \end{array}$$
  

$$w(t) = 0 \quad t = 0, 1, \dots, T-2$$

It can be seen again by substitution in the relevant inequalities that the above quantity solution is feasible and it is sustained by the set of prices given above. Therefore the solution is optimal. One can verify also that the optimal value of the maximand is equal to the imputed value of the scarce factors. Stated in algebraic terms, we can show that

$$\begin{aligned}
 \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X(t) &= \sum_{t=0}^{T-2} \left\{ K_0(0) (1-\delta)^t \right\} r(t) \\
 &= \gamma_0^{(m+1)} K_0(0) (1-\delta)^m (1+\gamma_0-\delta)^{T-m-2}
 \end{aligned}$$

Case 2b :  $T > (m+1)$  and  $T-m-2 > M$  . The solutions are

$$\left. \begin{aligned} X_0(t) &= \gamma_0 K_0(0) (1+\gamma_0-\delta)^t \\ X(t) &= 0 \\ K_0'(t) &= K_0(0) (1+\gamma_0-\delta)^t \end{aligned} \right\} t = 0, 1, \dots, M-1$$

$$\left. \begin{aligned} X_0(t) &= \frac{1}{\lambda_0} [1+\theta]^{M+1} - (1-\delta) K_0(0) (1+\gamma_0-\delta)^M \\ X(t) &= -\frac{1}{\lambda_0} (1+\theta)^{M+1} + K_0(0) (1+\gamma_0-\delta)^{M+1} \\ K_0'(t) &= K_0(0) (1+\gamma_0-\delta)^M \end{aligned} \right\} t = M$$

$$\left. \begin{aligned} X_0(t) &= \frac{1}{\lambda_0} (\theta+\delta) (1+\theta)^t \\ X(t) &= \frac{(\gamma_0-\theta-\delta)}{\lambda_0} (1+\theta)^t \\ K_0'(t) &= \frac{(1+\theta)^t}{\lambda_0} \end{aligned} \right\} t = M+1, \dots, T-m-3$$

$$\left. \begin{aligned} q(t) &= (1-\delta)^{T-M-2} [1+\gamma_0-\delta]^{M+1-t} \\ &= (1-\delta)^{T-1-t} \end{aligned} \right\} \begin{aligned} t &= 1, 2, \dots, M \\ t &= M+1, \dots, T-1 \end{aligned}$$



$$\begin{aligned}
 r(t) &= \gamma_0(1-\delta)^{T-M-2} [1+\gamma_0-\delta]^{M-t} & t = 0, 1, \dots, M \\
 &= 0 & t = M+1, \dots, T-m-3 \\
 &= \gamma_0(1-\delta)^m [(1-\delta) - \gamma_0 m] & t = T-m-2 \\
 &= \gamma_0(1-\delta)^{T-m-2} & t = T-m-1, \dots, T-2
 \end{aligned}$$

$$\begin{aligned}
 w(t) &= 0 & t = 0, 1, \dots, M \\
 &= \frac{\gamma_0(1-\delta)^{T-t-2}}{\lambda_0} & t = M+1, \dots, T-m-3 \\
 &= \frac{1}{\lambda_0} (1-\delta)^m [\gamma_0(m+1) - (1-\delta)] & t = T-m-2 \\
 &= 0 & t = T-m-1, \dots, T-1
 \end{aligned}$$

One can again verify by direct substitution in the relevant inequalities that the quantity solution given above is feasible and it is sustained by the above set of prices. As a computational check, one can verify that the optimal value of the maximand is imputed to the scarce resources, i.e., binding constraints. In other words, one can show that

$$\begin{aligned}
 \sum_{t=0}^{T-2} (1-\delta)^{T-2-t} X(t) &= \sum_{t=0}^{T-2} \left\{ K_0(0) (1-\delta)^t \right\} r(t) + \sum_{t=0}^{T-2} (1+\theta)^t w(t) \\
 &= K_0(0) (1-\delta)^{T-M-2} (1+\gamma_0-\delta)^{M+1} \\
 &\quad + \frac{(1+\theta)^{T-m-2} (1-\delta)^m}{\lambda_0(\theta+\delta)} [(\gamma_0-\theta-\delta)(1-\delta)+\gamma_0(m+1)(\theta+\delta)] \\
 &\quad - \frac{\gamma_0(1+\theta)^{M+1}(1-\delta)^{T-M-3}}{\lambda_0(\theta+\delta)}
 \end{aligned}$$

## CHAPTER 5

### SOCIAL WELFARE AS A DISCOUNTED SUM OF THE FUTURE STREAM OF CONSUMPTION

#### 5.1. Results

In this chapter we examine another social welfare function. Some authors view social welfare as an increasing function of the sum of the discounted stream of future consumption per worker.<sup>1</sup> Stated in the notation of Chapter 2, social welfare  $U(c)$  associated with a time path  $c$  is given by the following equation:

$$U(c) = \sum_{t=1}^{\infty} (1 + \alpha)^{-t} c_t \quad (5.1)$$

$\alpha$  is called the social rate of time preference. One important criticism against the use of  $U(c)$  for development planning is that  $U(c)$  implies a marginal rate of substitution between consumption at any two periods  $t_1$  and  $t_2$  which depends on the difference between  $t_1$  and  $t_2$  and is

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<sup>1</sup>See for instance Eckstein, op. cit., and Chenery, loc cit.

independent of the rates of consumption at  $t_1$  and  $t_2$ . In this chapter it is shown that this property of  $U(c)$  leads to the choice of time paths of consumption of extreme character.<sup>1</sup>

A society with  $U(c)$  as its welfare function discounts future consumption simply because it is available only in the future. Such a behaviour is often justified in the case of an individual because his life span is finite and he is subject to the process of aging which might conceivably impair his capacity for consumption. But this argument is not valid for a society or nation because it does not necessarily have a finite lifespan.

This is not to say that a society ought to treat a unit of consumption at two different points of time as perfect substitutes. We are arguing only against discounting the future simply because it is future.<sup>2</sup>

Since our purpose in this chapter is only to demonstrate the extreme character of the time paths that result from maximizing  $U(c)$  as given by (5.1), we simplify the technology of production drastically. Assume that no labour is needed either for producing machines or for producing consumer goods. Under this assumption, only one technique in Sector II for producing

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<sup>1</sup>The results of this chapter are not new. Similar findings have been reported by Jan Tinbergen. Cf. Jan Tinbergen, "Optimum Savings and Utility Maximization Over Time," Econometrica 28 (April 1960), pp. 481-489.

<sup>2</sup>Professor Maurice Dobb discusses the role of pure time preference in national planning quite extensively (Dobb, op. cit., Ch. 2).

consumer goods exists, namely the technique which produces the maximum rate of consumer goods per machine.<sup>1</sup> We no longer need to distinguish between the stock of machines in existence and the stock in use since it will be inefficient to leave any machine idle. In the notation of Chapter 2, the problem can be stated as follows:

$$\text{Maximize } U(c) = \sum_{t=1}^{\infty} (1+\alpha)^{-t} c_t \quad \text{subject to}$$

$$(1+\theta)^{t+1} c_{t+1} = \gamma_1 K_1(t) \quad (5.2)$$

$$K_i(t) = K_i(0)(1-\delta)^t + \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_i(t') \quad i = 0, 1 \quad (5.3)$$

$$X_0(t) + X_1(t) \leq \gamma_0 K_0(t) \quad (5.4)$$

$$X_0(t), X_1(t) \geq 0 \quad (5.5)$$

Equation (5.2) relates the rate of output of consumer goods during period  $t$  (becoming available for consumption in period  $(t+1)$ ) to the stock of machines in Sector II. Equations (5.3) give the stock of machines of machines of type 0 and 1 respectively. Inequality (5.4) states that the rate of gross investment in machines of type 0 and 1 together cannot exceed the rate of output of machines.

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<sup>1</sup>Technique 1 yields the maximum rate of output of consumer goods per machine.

The solution to the above maximization depends on the value of the discount factor  $(1+\alpha)$ . When  $(1+\beta) = (1+\alpha)(1+\theta) \geq 1 + \gamma_0 - \delta$ , the maximizing solution<sup>1</sup> is for  $t \geq 0$

$$\begin{aligned}
 X_0(t) &= 0 & X_1(t) &= \gamma_0 K_0(0) (1-\delta)^t \\
 K_0(t) &= K_0(0)(1-\delta)^t & K_1(t) &= \left[ (1-\delta)K_1(0) + \gamma_0 t K_0(0) \right] (1-\delta)^{t-1} \\
 c_{t+1} &= \gamma_1 [1+\theta]^{-t-1} \left[ (1-\delta) K_1(0) + \gamma_0 t K_0(0) \right] (1-\delta)^{t-1}
 \end{aligned}
 \tag{5.6}$$

This solution means that if the rate at which future consumption is discounted is greater than the maximum feasible rate of growth of the stock of machines per capita in the Sector producing machines, the optimal policy is to invest the entire output of new machines in the Sector producing consumer goods. In other words, the desire for current consumption is so great relative to future consumption that the society adopts a policy of permitting consumption per worker to decline steadily to zero from a certain point in time.

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<sup>1</sup>If  $\beta = (\gamma_0 - \delta)$  it is a matter of indifference how gross investment is divided between the two sectors. All feasible divisions yield the same value for  $U(c)$ . Therefore the maximization of  $U(c)$  by itself does not provide any basis for choice between alternative feasible divisions.

If  $(1+\beta) < (1+\gamma_0-\delta)$  there is no maximizing solution because the maximand is unbounded. In other words, given any  $M$ , however large, we can find a time path  $c$  such that  $U(c) > M$ . Thus it is clear that as a social objective the attempts to maximize  $U(c)$  leads to one of two extreme results: 1) either per capita consumption  $c_t$  steadily declines to zero beyond a certain time point, or 2) the process of maximization itself ceases to have any meaning with the maximand unbounded. It may appear that the removal of some of the simplifying assumptions on the technology of production modifies the extreme nature of this conclusion. But it is shown in Section 5.2 that reintroduction of labour as a factor of production in the technology does not make any difference in the case  $(1+\alpha)(1+\theta) > 1 + \gamma_0 - \delta$ . Consumption per worker along a time path which maximizes  $U(c)$  still declines to zero. The only difference is that  $U(c)$  is always bounded above as long as every technique of producing consumer goods uses nonzero amounts of labour services.

## 5.2. Derivations

Let us now solve the problem of maximizing  $U(c) = \sum_{t=1}^{\infty} (1+\alpha)^{-t} c_t$

subject to

$$(1+\theta)^{t+1} c_{t+1} = \gamma_1 K_1(t) \quad (5.7)$$

$$K_i(t) = K_i(0) (1-\delta)^t + \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_i(t') \quad i = 0, 1 \quad (5.8)$$

$$X_0(t) + X_1(t) \leq \gamma_0 K_0(t) \quad (5.9)$$

$$X_0(t), X_1(t) \geq 0 \quad (5.10)$$

Let us define a new parameter  $\beta$  by

$$(1+\alpha)(1+\theta) = 1 + \beta \quad (5.11)$$

Using (5.7), (5.8) and (5.11) and omitting terms that are unessential to the choice of  $X_0(t)$  and  $X_1(t)$  that maximize  $U(c)$ , we can rewrite the problem as:

$$\text{Maximize} \quad \sum_{t=0}^{\infty} (1+\beta)^{-t} X_1(t) \quad \text{subject to}$$

$$-\gamma_0 \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_0(t') + X_0(t) + X_1(t) \leq \gamma_0 K_0(0) (1-\delta)^t \quad (5.12)$$

$$X_0(t), X_1(t) \geq 0 \quad (5.13)$$

The problem is in a standard linear programming form and the optimality of a feasible solution to a linear programming problem can be tested by associating a set of shadow prices to the restraints and checking whether the given feasible solution is sustained by the price system.<sup>1</sup> Let  $u(t)$  be the shadow price associated with restraint (5.12). Then a feasible solution is optimal if the associated values of  $u(t)$  satisfy:

$$\left. \begin{aligned} u(t) &\leq \gamma_0 \sum_{t'=t+1}^{\alpha} (1-\delta)^{t'-t-1} u(t') && \text{for all } t \geq 0 \\ &= \gamma_0 \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} u(t') && \text{if } X_0(t) > 0 \end{aligned} \right\} (5.14)$$

$$\left. \begin{aligned} u(t) &\leq (1+\beta)^{-t} && \text{for all } t \geq 0 \\ &= (1+\beta)^{-t} && \text{if } X_1(t) > 0 \end{aligned} \right\} (5.15)$$

We mentioned in Section (5.1) that a maximizing solution exists only when  $\beta \geq (\gamma_0 - \delta)$ . The optimal solution and the associated prices in this case are:

$$\left. \begin{aligned} X_0(t) &= 0 && u(t) = (1+\beta)^{-t} \\ X_1(t) &= \gamma_0 K_0(0) (1-\delta)^t \end{aligned} \right\} (5.16)$$

---

<sup>1</sup>Standard linear programming problems involve a finite number of variables. We have here a denumerably infinite number of variables  $X_0(t)$ ,  $X_1(t)$ . It is shown in Appendix A that our procedure of testing optimality by prices works in this case also.



By substituting these values in (5.12) - (5.15) one can verify that the solutions (5.16) are optimal.<sup>1</sup> Consumption per worker at time  $t$  along the optimal path is given by  $c_t = (1+\theta)^{-t} \gamma_1 K_1(t-1)$ . Substituting for  $K_1(t-1)$  from (5.8) and using (5.16) we have

$$c_t = (1+\theta)^{-t} (1-\delta)^{t-2} [(1-\delta)K_1(0) + \gamma_0(t-1) K_0(0)]$$

It is seen that  $c_t$  steadily declines to zero from a certain point in time.

In order to show that the maximand is unbounded when  $\beta < (\gamma_0 - \delta)$  consider the following class of feasible solutions:

$$\left. \begin{aligned} X_0(t) &= \gamma_0 K_0(0) [1+\gamma_0-\delta]^t \\ X_1(t) &= 0 \end{aligned} \right\} t = 0, 1, 2, \dots, T-1 \quad (5.17)$$

$$\left. \begin{aligned} X_0(t) &= 0 \\ X_1(t) &= \gamma_0 K_0(0) [1+\gamma_0-\delta]^T (1-\delta)^{t-T} \end{aligned} \right\} t = T, T+1, \dots \quad (5.18)$$

It is easy to verify that for each  $T \geq 1$ , (5.17) and (5.18) provide a feasible solution. The value of the maximand for such a solution is given by

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<sup>1</sup>In the case  $(\gamma_0 - \delta) = \beta$  it is immaterial how investment is divided between  $X_0(t)$  and  $X_1(t)$ . See footnote on page 70.

$$\begin{aligned} \sum_{t=0}^{\infty} (1+\beta)^{-t} X_1(t) &= \gamma_0 K_0(0) (1+\gamma_0-\delta)^T \sum_{t=T}^{\infty} (1+\beta)^{-t} (1-\delta)^{t-T} \\ &= \frac{\gamma_0 K_0(0)}{(\beta + \delta)} \left[ \frac{1+\gamma_0-\delta}{1+\beta} \right]^T \end{aligned} \quad (5.19)$$

Given that  $\beta < (\gamma_0 - \delta)$  it is clear that  $\left[ \frac{1+\gamma_0-\delta}{1+\beta} \right]^T$  increases indefinitely as  $T$  increases. Therefore the maximand can be made to exceed any value  $M$  however large by choosing a feasible path given by (5.17) and (5.18) for some sufficiently large value of  $T$ .

Thus far we have been assuming that no labour was needed to produce either machines or consumer goods. Let us now relax this assumption, thereby reintroducing the whole range of techniques of production of consumer goods. The problem can now be stated as:

$$\text{Maximize } U(c) = \sum_{t=1}^{\infty} (1+\alpha)^{-t} c_t \quad \text{subject to (for } t \geq 0)$$

$$(1+\theta)^{t+1} c_{t+1} = \sum_{i=1}^{\infty} \gamma_i K_i'(t) \quad (5.20)$$

$$K_i(t) = K_i(0)(1-\delta)^t + \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_i(t') \quad i=0,1,2, \dots, \quad (5.21)$$

$$K_i'(t) - K_i(t) \leq 0 \quad (5.22)$$

$$\sum_0^{\infty} \lambda_i K'_i(t) \leq (1+\theta)^t \quad (5.23)$$

$$\sum_0^{\infty} X_i(t) - \gamma_0 K'_0(t) \leq 0 \quad (5.24)$$

$$X_i(t), K'_i(t), \geq 0 \quad i = 0, 1, \dots \quad (5.25)$$

The maximand is as before the sum of the discounted stream of future consumption. Equation (5.20) states that the rate of output of consumer goods becoming available at (t+1) is the sum of the output produced with various techniques. Equations (5.21) give the stocks of various types of machines in existence.

The exact form of the solution to the above problems depends on the initial conditions, discount factor and the parameters of the technology. We present here only two cases. Let us first define a time function  $g(t)$  as follows:

$$g(t) = \left[ (1-\delta) \left( \sum_{i=0}^{\infty} \lambda_i K_i(0) \right) + \left( \lambda_1 \gamma_0 K_0(0) t \right) \right] (1-\delta)^{t-1} \quad (5.26)$$

It can be seen that  $g(t)$  represents the potential level of employment in period  $t$  if (a) the existing stocks of each type of machine is used and (b) in each period  $t$  the entire output of new machines is added to the stock of type 1 machines. The behaviour of  $g(t)$  is relevant for distinguishing the two cases to be discussed.

As in earlier sections we prove the optimality of a feasible solution by associating with it a set of nonnegative prices which sustains it. In the notation of Section 2.2, we have to show that<sup>1</sup>

$$\left. \begin{aligned} \gamma_0 q(t+1) &\leq \lambda_0 w(t) + r(t) && \text{for all } t \\ &= \lambda_0 w(t) + r(t) && \text{if } K'_0(t) > 0 \end{aligned} \right\} (5.27)$$

$$\left. \begin{aligned} \gamma_i p(t+1) &\leq \lambda_i w(t) + r_i(t) && \text{for all } t \text{ and } i \neq 0 \\ &= \lambda_i w(t) + r_i(t) && \text{if } K'_i(t) > 0 \end{aligned} \right\} (5.28)$$

$$\left. \begin{aligned} q(t+1) &\geq \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') && \text{for all } t \\ &= \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') && \text{if } X_i(t) > 0 \end{aligned} \right\} (5.29)$$

Case I :  $g(t) \leq (1+\theta)^t$  for all  $t$  and  $\beta \geq (\gamma_0 - \delta)$ .

The quantity and price solutions are:

$$\left. \begin{aligned} X_1(t) &= \gamma_0 K_0(0) (1-\delta)^t \\ K_1(t) &= K'_1(t) = \left[ (1-\delta) K_1(0) + t \gamma_0 K_0(0) \right] (1-\delta)^{t-1} \end{aligned} \right\} \text{for all } t \quad (5.30)$$

---

<sup>1</sup>It can be seen that all prices are present values and the present value of unit of consumer good available at  $t$  is determined by the maximand as  $[1+\beta]^{-t}$  where  $(1+\beta) = (1+\alpha)(1+\theta)$ .

$$\left. \begin{aligned} X_i(t) &= 0 \\ K_i'(t) &= K_i(t) = K_i(0) (1-\delta)^t \end{aligned} \right\} \begin{array}{l} \text{for } i \neq 1 \text{ and} \\ \text{for all } t \end{array} \quad (5.31)$$

$$\left. \begin{aligned} p(t) &= [1+\beta]^{-t} \\ w(t) &= 0 \\ r_0(t) &= \frac{\gamma_0 \gamma_1 [1+\beta]^{-t-1}}{(\beta + \delta)} \\ r_i(t) &= \gamma_i [1+\beta]^{-t-1} \\ q(t) &= \frac{\gamma_1 [1+\beta]^{-t}}{(\beta + \delta)} \end{aligned} \right\} \text{for all } t \geq 0 \quad (5.32)$$

Given that  $g(t) \leq (1+\theta)^t$  it can be verified that the quantity solutions (5.30) and (5.31) satisfy (5.21) - (5.25). The condition that  $\beta \geq (\gamma_0 - \delta)$  ensures that the prices (5.32) satisfy (5.27) - (5.29).

Therefore the feasible solution given by (5.21) - (5.25) is optimal.

The rate of consumption per worker at  $t$  as per this solution is

$$c_t = (1+\theta)^{-t} (1-\delta)^{t-2} \left[ (1-\delta) \left( \sum_{i=1}^{\infty} \gamma_i K_i(0) \right) + \gamma_1 \gamma_0 (t-1) K_0 \right]$$

It is obvious that  $c_t \rightarrow 0$  as  $t \rightarrow \infty$ .

Case 2

(a) There exists a  $t_1$  such that

$$g(t_1) \leq (1+\theta)^{t_1} \quad \text{and} \quad g(t_1+1) > (1+\theta)^{t_1+1}$$

$$(b) \quad g(t_1+1) - \gamma_0(\lambda_1 - \lambda_2) K_0(0) (1-\delta)^{t_1} \leq (1+\theta)^{t_1+1}$$

$$(c) \quad \frac{\gamma_0(\lambda_1\gamma_2 - \lambda_2\gamma_1)}{\lambda_0(\gamma_1 - \gamma_2)} > (\beta+\delta) > \frac{\lambda_1\gamma_0}{\lambda_0} > (1-\delta)$$

Notice that (c) implies that  $(\beta+\delta) \geq \gamma_0$  since  $\lambda_1 \leq \lambda_0$  by assumption. The quantity solutions in this case are:<sup>1</sup>

$$K_i'(t) = K_i(0) (1-\delta)^t + \sum_{t'=0}^{t-1} (1-\delta)^{t-1-t'} X_i(t') \quad \text{for all } i$$

$$\text{and for all } t \tag{5.33}$$

$$X_i(t) = 0 \quad \text{for all } i \neq 1, 2 \quad \text{and for all } t \tag{5.34}$$

$$\left. \begin{aligned} X_1(t) &= \gamma_0 K_0(0) (1-\delta)^t \\ X_2(t) &= 0 \end{aligned} \right\} t = 0, 1, 2, \dots, t_1-1 \tag{5.35}$$

<sup>1</sup>The prices that sustain this quantity solution can be derived quite easily.

$$\left. \begin{aligned} x_1(t_1) &= \frac{(1+\theta)^{t_1+1} + \gamma_0(\lambda_1 - \lambda_2) K_0(0) (1-\delta)^{t_1} - g(t_1+1)}{(\lambda_1 - \lambda_2)} \\ x_2(t_1) &= \frac{g(t_1+1) - (1+\theta)^{t_1+1}}{(\lambda_1 - \lambda_2)} \end{aligned} \right\} (5.36)$$

$$\left. \begin{aligned} x_1(t) &= \frac{(\theta + \delta) (1+\theta)^t - \lambda_2 \gamma_0 K_0(0) (1-\delta)^t}{(\lambda_1 - \lambda_2)} \\ x_2(t) &= \frac{\lambda_1 K_0(0) (1-\delta)^t - (\theta + \delta) (1+\theta)^t}{(\lambda_1 - \lambda_2)} \end{aligned} \right\} t = t_1, \dots, t_2 \quad (5.37)$$

where  $t_2$  is defined by

$$\left. \begin{aligned} \lambda_1 K_0(0) (1-\delta)^{t_2} &\geq (\theta + \delta) (1+\theta)^{t_2} \\ \lambda_1 K_1(0) (1-\delta)^{t_2+1} &< (\theta + \delta) (1+\theta)^{t_2+1} \end{aligned} \right\} (5.38)$$

$$\left. \begin{aligned} x_1(t) &= \gamma_0 K_0(0) (1-\delta)^t \\ x_2(t) &= 0 \end{aligned} \right\} t \geq t_2+1 \quad (5.39)$$

It is easy to verify that this solution is feasible, given that (a) and (b) hold. Given (c) it can be shown that it is optimal as well. In this case also it can be seen that consumption per worker goes to zero as  $t \rightarrow \infty$ . For

$$c_{t+1} = (1-\delta) \left\{ \sum_1^{\infty} \gamma_1 K_1'(t_2) \right\} + \gamma_1 \gamma_0 K_0'(t_2) (t-t_2) (1-\delta)^{t-t_2-1} \quad t \geq t_2 \quad (5.40)$$

where  $K_1'(t_2)$  are given by (5.33).

It is obvious from (5.40) that  $\lim_{t \rightarrow \infty} c_{t+1} = 0$ .



## CHAPTER 6

### SOME CHARACTERISTICS OF EFFICIENT TIME PATHS

#### 6.1. Results

In this chapter some common characteristics of all efficient time paths are derived. Previously,<sup>1</sup> an efficient time path was defined to be a feasible time path  $c$  such that there is no other feasible time path  $c'$  which satisfies

$$\left. \begin{array}{l} c'_t \geq c_t \quad \text{for all } t \\ c'_t > c_t \quad \text{for at least one } t \end{array} \right\} \quad (6.1)$$

The following theorem states a necessary and sufficient condition for a feasible path to be efficient.

#### Theorem 1

A feasible time path  $c$  is efficient if and only if, for all  $h$ , there is no feasible time path  $c'$  which satisfies

$$\left. \begin{array}{l} c'_t \geq c_t \quad t \leq h \\ c'_t = c_t \quad t > h \end{array} \right\} \quad (6.2)$$

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<sup>1</sup>See Definition 3, Ch. 2.

The meaning of this theorem is obvious: A feasible path  $c$  is efficient if and only if there is no feasible time path  $c'$  which provides at least as much consumption per worker as  $c$  at any  $t$  within a finite time horizon  $h$  and exactly as much consumption per worker as  $c$  at any  $t$  beyond  $h$ .

The following result relates the utilization of the existing stocks of various types of machines with efficiency.<sup>1</sup>

Theorem 2

Along any efficient path  $c$ , if at any  $t$ , the existing stock of type  $j$  ( $j > 1$ ) machines is not fully utilized, then the stock in use of type  $i$  machine will be zero for  $1 \leq i < j$ .

It is obvious why Theorem 2 should be true. For labour is more productive with type  $j$  machines than with type  $i$  for  $j > i \geq 1$ . If along any feasible path  $c$ , at any  $t$  (a) the existing stock of type  $j$  machines is not fully utilized and (b) the stock in use of type  $i$  machine is positive for some  $i$  in  $1 \leq i < j$ , then by transferring labour from employment with type  $i$  machines to employment with type  $j$  machines one can increase the rate of output of consumer goods at time  $t$  without reducing that at any other  $t' \neq t$ . Therefore  $c$  cannot be efficient.

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<sup>1</sup>It is worth recalling machines of type  $i$  ( $i \geq 1$ ) are used to produce consumer goods with technique  $i$ . The techniques are numbered in such a way that the rate of output of consumer goods per unit of labour increases with  $i$ .

Corollary 2.1

Along any efficient path  $c$  the following can be true for at the most one type of machine: The stock in use is greater than zero but less than the stock in existence.

Corollary 2.2

Along any efficient path  $c$  at least one of the following must hold at any  $t$

- (a) The existing stock of all types of machines which produce consumer goods are fully utilized.
- (b) The available labour supply is fully utilized.

This corollary expresses the simple idea that it is inefficient to keep both men and machines idle.

The following theorem provides a necessary condition for positive gross investment to be made in a type of machine which produces consumer goods.

Theorem 3

In an efficient path  $c$  with  $c_t > 0$  for all  $t$ , positive gross investment in type  $i$  machines ( $i \geq 1$ ) will be made at any  $t$  only if the existing stock of machines of type  $i$  will be fully utilized at at least one  $t' > t$ .

The economic meaning of this theorem is trivial. It is inefficient to add to the existing stock of a type of machine which is never going to be fully utilized.

## 6.2. Derivations

Let us now prove the results of the previous section.<sup>1</sup>

### Theorem 1

A feasible time path  $c$  is efficient if and only if, for all  $h$ , there is no other feasible time path  $c'$  which satisfies

$$\left. \begin{array}{ll} c'_t > c_t & t < h \\ c'_t = c_t & t = h \\ c'_t = c_t & t > h \end{array} \right\} \quad (6.3)$$

The necessity part of the theorem is a direct consequence of the definition of efficiency. To prove the sufficiency part we first state and prove a lemma.

### Lemma

If a time path  $c$  is feasible then any time path  $c'$  such that  $c'_t \leq c_t$  for all  $t$  is also feasible.

Given  $c$  is feasible it is clear that  $c'_t$  can be obtained from  $c$  by suitably reducing the stock in use of machines of various types. Therefore  $c'$  is also feasible.

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<sup>1</sup>Theorem 1 corresponds to Lemma 1 of Malinvaud's article (Malinvaud, op. cit., p. 244).

We can now prove the sufficiency part of Theorem 1 . Suppose contrary to the assertion of the theorem  $c$  is not efficient. Then there exists a feasible  $c'$  such that

$$c'_t \geq c_t \text{ for all } t \quad \text{and}$$

$$c'_t > c_t \text{ for at least one } t \quad \text{say } (t = h).$$

Now define a time path  $c''$  as follows

$$c''_t = c'_t \quad t \leq h$$

$$c''_t = c_t \quad t > h$$

(6.4)

Clearly  $c''_t \leq c'_t$  for all  $t$  . Therefore given that  $c'$  is feasible,  $c''$  is also feasible. Now  $c''$  violates the hypothesis of the theorem, thus bringing about a contradiction. Therefore  $c$  is efficient.

Theorem 2 and its corollaries and Theorem 3 are easily proved<sup>1</sup> by using the properties of the price system that can be associated with an efficient path. The existence of a sustaining price system associated with any efficient path has been proved by Malinvaud.<sup>2</sup> The existence

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<sup>1</sup>The same results can be also proved more directly without the use of efficiency prices. But the direct proofs are lengthy.

<sup>2</sup>Malinvaud, op. cit., Theorem 1, p. 245. This theorem is the converse of the Lemma we used in proving the efficiency of a given feasible path in Section 2.2. It must be remembered that we are extending Malinvaud's results to the case where there are infinite numbers of commodities in each period.

theorem can be stated as follows, using the notation of Chapter 2.

Given an efficient path  $c$ , there exist a set of nonnegative prices  $p(t)$ ,  $q(t)$ ,  $r_i(t)$ , and  $w(t)$  ( $t \geq 0$ ) such that:

$$\begin{aligned} \text{(a)} \quad \gamma_i p(t+1) &\leq \lambda_i w(t) + r_i(t) \quad \text{for all } i \geq 1 \\ &= \lambda_i w(t) + r_i(t) \quad \text{if } K'_i(t) > 0 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \gamma_0 q(t+1) &\leq \gamma_0 w(t) + r_0(t) \\ &= \lambda_0 w(t) + r_0(t) \quad \text{if } K'_0(t) > 0 \end{aligned}$$

$$\text{(c)} \quad r_i(t) = 0 \quad \text{if for any } i \geq 0 \quad 0 < K'_i(t) < K_i(t)$$

$$\text{(d)} \quad w(t) = 0 \quad \text{if the available supply of labour is not fully utilized at any } t$$

$$\begin{aligned} \text{(e)} \quad q(t+1) &\geq \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') \quad \text{for all } i \geq 0 \\ &= \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_i(t') \quad \text{if } X_i(t) > 0 \end{aligned}$$

(f) The cost of inputs evaluated in terms of the above prices for each period  $t$  is minimum for the input combination associated with  $c$ , among all input combinations for that period which permit the same consumption per worker as in  $c$  for periods  $t' \geq t$ .

It is easily seen that properties (a) and (b) of the set of prices imply that the quantity solutions  $K'_i(t)$  of  $c$  are profit maximizing levels, given the set of prices, of the activities that produce consumer goods and machines. Properties (c) and (d) state that unused resources earn zero rentals. Property (e) covers profit maximization in investment activities. Property (f) is needed only when we wish to prove the efficiency of a given feasible solution through prices.<sup>1</sup> It is possible that in certain cases one can associate with an inefficient path a set of prices with properties (a) - (e) .

Theorem 2

In an efficient path  $c$  , if  $0 < K'_j(t) < K_j(t)$  for some  $j > 1$  and some  $t \geq 0$  then  $K'_i(t) = 0$  for all  $i \leq j$  .

Suppose, on the contrary,  $K'_i(t) > 0$  for some  $i < j$  , then by (a) we have

$$\begin{aligned} \lambda_i w(t) + r_i(t) &= \gamma_i p(t+1) \\ \lambda_j w(t) + r_j(t) &= \gamma_j p(t+1) \end{aligned} \tag{6.5}$$

By (c) we know that  $r_j(t) = 0$  . . Now  $\frac{\lambda_i}{\gamma_i} > \frac{\lambda_j}{\gamma_j}$  and  $p(t+1) > 0$  for all  $t$  since the consumer good is the only desired commodity in our model

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<sup>1</sup>For a discussion of the property (f) see Tjalling C. Koopmans, Three Essays in the State of Economic Science, I ed. (New York, McGraw-Hill Book Company, Inc.), pp. 110-112.

and saturation in consumption is ruled out. If (6.5) were to hold, then  $r_i(t) < 0$ , violating the nonnegative character of the efficiency prices. Thus  $K'_i(t) = 0$  for all  $i$  in  $1 \leq i < j$  given that  $0 < K'_j(t) < K_j(t)$ . It is also clear that  $0 < K'_j(t) < K_j(t)$  can hold for at the most one  $j \geq 1$  proving corollary (2.1).

Corollary 2.2

Along any efficient path  $c$ , at least one of the following holds: for each  $t$ :

$$\sum_{i=0}^{\infty} \lambda_i K'_i(t) = (1+\theta)^t \quad (6.6)$$

$$K'_i(t) = K_i(t) \quad \text{for all } i \geq 1 \quad (6.7)$$

For if strict inequalities ( $<$ ) held in (6.6) and (6.7), (a) and (c) would imply  $w(t) = 0$  and  $r_i(t) = 0$  for all  $i \geq 1$ . These, in turn, would imply  $\lambda_i w(t) + r_i(t) < \gamma_i p(t+1)$ , thus violating a part of (a).

Theorem 3

In an efficient path  $c$  with  $c_t > 0$  for all  $t$ ,  $X_i(t_1) > 0$  for some  $i \geq 1$  and some  $t_1$  only if  $K'_i(t) = K_i(t)$  for some  $t' > t_1$ .

Suppose on the contrary  $K'_i(t) < K_i(t)$  for all  $t > t_1$ , then by (c)  $r_i(t) = 0$  for all  $t > t_1$ . Now given that  $X_i(t_1) > 0$ , by (e)



we know that

$$q(t_1+1) = \sum_{t'=t_1+1}^{\infty} (1-\delta)^{t'-t-1} r_1(t') = 0. \quad (6.8)$$

Another implication (6.8) is that  $r_j(t') = 0$  for all  $j$  and for all  $t' > t_1$ . For if  $r_j(t'') > 0$  for some  $t'' < t_1$  then for such a  $j$

$$q(t_1+1) < \sum_{t'=t+1}^{\infty} (1-\delta)^{t'-t-1} r_j(t')$$

This violates (e). Therefore  $r_j(t') = 0$  for all  $j$  and for all  $t' > t_1$ .

Now given  $c_t > 0$  for all  $t$ , it is clear that for each  $t$ ,  $K'_j(t) > 0$  for some  $j \geq 1$ . This implies that for such a  $j$ ,

$$\lambda_j w(t) + r_j(t) = \gamma_j p(t+1) \quad (6.10)$$

Now consider a  $t' > t_1$ . We have already shown that  $r_j(t') = 0$  for all  $j$ . Hence (6.10) implies

$$\lambda_j w(t') = \gamma_j p(t'+1) \quad (6.11)$$

Consider an  $i > j$  . For any  $i$   $r_i(t') = 0$  . Now (6.11) implies that

$$\lambda_i w(t') < \gamma_i p(t'+1) \quad \text{since} \quad \frac{\lambda_i}{\gamma_i} < \frac{\lambda_j}{\gamma_j}$$

Therefore  $\lambda_i w(t') + r_i(t') < \gamma_i p(t'+1)$  contradicting (a), thus proving the theorem.

## CHAPTER 7

### SUMMARY AND QUALIFICATIONS

In this Chapter, the conclusions of the previous chapters are summarized and qualified. In Chapter 1 we reviewed some important contributions to the problem of choice of investment criteria.

In Chapter 2 we considered a simple dynamic model of an economy with two sectors, sector I producing machines and sector II producing a homogeneous consumer good. The former used a single fixed combination of machines and labour to produce various types of machines. In the latter sector, many alternative techniques of production of differing labour intensities were available. It was demonstrated that there exists a maximum sustainable rate of consumption per worker for such an economy. This rate is achieved along a "terminal path". The terminal path was shown to be a balanced growth path in which the stocks of machines of various types grew at the same geometric rate as the labour force. While on the terminal path, only one of the many alternative techniques was used in sector II at every point of time, and consumption per worker was constant at the maximum sustainable level.

In Chapter 3, the time path was derived which minimizes the time taken to reach the terminal path and remain on it thereafter, starting

from given initial stocks of various types of machines.<sup>1</sup> This derived path implied a continuous decline in consumption per worker for a finite interval of time.

In Chapters 4 and 5 we examined time paths that result from maximizing two specific social welfare functions: (a) consumption per worker at some chosen point of time  $T$  in the future and (b) the discounted sum of the future stream of consumption per worker. We concluded that these time paths led to two possible results: (1) consumption per worker declines after a certain point of time, or (2) consumption at some point in time is maximized at the expense of consumption before that point and without any sustainability requirement on consumption beyond that point.

In Chapter 6 some common characteristics of efficient time paths were established.

It should be clear from our discussion in Chapters 3 - 5, that it is inherently difficult to arrive at an analytically convenient social welfare function, the maximization of which yields a satisfactory time path of consumption per worker. A modified version of the objective considered in Chapter 3 such as minimizing the time taken to reach the terminal path subject to the constraint that consumption per worker never falls below a certain floor at any point in time is

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<sup>1</sup>We considered only the approaches to the terminal path from below.

worth exploring. The most serious analytical difficulty in this approach is the problem of the choice of techniques of production of consumer goods at each point in time prior to the attainment of the terminal path. This problem disappears if the constraint of a floor in consumption is removed.

Even if a satisfactory solution could be found for the choice of a social welfare function, our model has to be generalized in a number of directions before it can be used for a discussion of a realistic planning problem. We considered a closed economy which produced the machines it needed. The underdeveloped areas of today have no heavy industry to speak of and have to rely on imports from advanced countries for industrialization. More importantly the introduction of international trade could conceivably alter the choice of techniques of production. An additional constraint in the form of the availability of foreign exchange will have to be introduced.

We made the rather convenient but unrealistic assumption that labour was homogeneous and can be transferred from one technique to another without incurring any cost. It is well known that one of the most serious obstacles to economic growth in underdeveloped areas is the lack of skilled manpower. A realistic model will have to include training activities, the input of which unskilled labour and the output trained workers.

Investment in training and education is one form of investment

in social overhead capital. Other examples are investment in transportation networks, power supply etc. Investment in social overhead capital may yield increasing returns to scale. This analytically complex phenomenon has to be recognized in a development model.

The assumption that the labour force grows at a constant geometric rate per period needs revision. The rate of growth of population may depend upon the level and rate of growth of consumption, and as such it may be one of the choice variables.

We assumed a simple and unchanging technology for our economy. It is clear that in a model which covers an infinite time horizon, unchanging technology is a restrictive assumption. A thorough understanding of the process by which technical change is brought about is essential before we can include this aspect in our model.

We conclude by expressing a hope that in spite of the above mentioned qualifications, this study has contributed towards a clearer understanding of the problem of the choice of investment criteria.

APPENDIX A

Consider the following linear programming problem involving finite number of variables and restraints:

$$\text{Minimize } c \cdot x \quad (\text{A.1})$$

$$\text{Subject to } Ax \geq b \quad (\text{A.2})$$

$$x \geq 0 \quad (\text{A.3})$$

where  $c$  is a row vector  $(c_1, \dots, c_n)$ ,  $x$  a column vector  $(x_1, \dots, x_n)^T$ ,  $A$  is a  $m \times n$  matrix and  $b$  a column vector  $(b_1, b_2, \dots, b_m)^T$ . Then the following theorem holds:

Theorem  $\bar{x}$  is optimal if we can find a row vector  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_m)$  such that

$$\bar{x} \geq 0 \quad (\text{A.4})$$

$$\bar{u} \geq 0 \quad (\text{A.5})$$

$$A\bar{x} \geq b \quad (\text{A.6})$$

$$\bar{u} A \leq c \quad (\text{A.7})$$

$$c \cdot \bar{x} \leq \bar{u} \cdot b \quad (\text{A.8})$$

Proof: Consider any feasible  $x$ , i.e., any  $x$  which satisfies (A.2) and (A.3).

Since  $\bar{u} \geq 0$  and  $Ax \geq b$  it is clear

that  $\bar{u} Ax \geq \bar{u} b$

Since  $\bar{u} A \leq c$  and  $x \geq 0$  we have

$$\bar{u} Ax \leq c x$$

Therefore  $c x \geq \bar{u} b \geq c \bar{x}$  showing that  $\bar{x}$  is optimal.

Suppose now that we admit a countably infinite number of variables keeping the number of restraints finite. It is clearly seen that if each of infinite sums in  $Ax$ ,  $\bar{u} A$  and  $c x$  is convergent for all feasible  $x$ , the above proof that (A.4) -- (A.8) are sufficient for the optimality of  $\bar{x}$  holds.



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