6-1-1960

# A New Technique in Equilibrium Analysis

Gerard Debreu

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 92



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June 10, 1960

* Research undertaken by the Cowles Commission for Research in Economics under Task NR 047-006 with the Office of Naval Research.

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The notation and the terminology are those of [3].

A quasi-equilibrium of the private ownership economy $\mathcal{E} = ((X_i, \precsim_i), (Y_j), (\omega_i), (\theta_{ij}))$ is an $(m + n + 1)$-tuple $((x_i^*), (y_j^*), p^*)$ of points of $R^\ell$ such that:

($\alpha$) $x_i \succ_i x_i^*$ implies $p^* \cdot x_i \geqq p^* \cdot \omega_i + \sum_{j=1}^{m} \theta_{ij} \, p^* \cdot y_j^*$, for every $i$,

($\beta$) $y_j^*$ maximizes profit relative to $p^*$ on $Y_j$, for every $j$,

($\gamma$) $x^* - y^* = \omega$,

($\delta$) $p^* \neq 0$.

Notice that this definition implies that $p^* \cdot x_i^* = p^* \cdot \omega_i + \sum_j \theta_{ij} \, p^* \cdot y_j^*$ for every $i$.

Theorem 1. $\mathcal{E}$ has a quasi-equilibrium if:

for every $i$ (a) $X_i$ is closed, convex, and has a lower bound for $\leqq$,

(b.1) there is no satiation consumption in $X_i$,

(b.2) for every $x_i'$ in $X_i$, the sets $\{x_i \in X_i \mid x_i \succsim_i x_i'\}$ and $\{x_i \in X_i \mid x_i \precsim_i x_i'\}$ are closed in $X_i$,

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* A technical report of research undertaken by the Cowles Foundation for Research in Economics under contract with the Office of Naval Research.

(b.3) if $x_i^1$ and $x_i^2$ are two points of $X_i$ and if $t$ is a real number in $]0, 1[$, then $x_i^2 \succ_i x_i^1$ implies $t\, x_i^2 + (1 - t)\, x_i^1 \succ_i x_i^1$,

(c) $(\{\omega_i\} + AY) \cap X_i \neq \emptyset$ ;

for every $j$ (d.1) $0 \in Y_j$ ;

(d.2) $Y$ is closed and convex,

(d.3) $Y \cap (-Y) \subset \{0\}$,

(d.4) $Y \cap \Omega \subset \{0\}$.

The proof differs from that of [3], section 5.7 in only two significant respects.

(1) In part 4, the correspondence $\hat{\xi}_i'$ is replaced by the correspondence $\psi_i$ defined as follows:

If $p.\omega_i + \sum_j \theta_{ij}\, \hat{\pi}_j\,(p) \neq \text{Min}\ p.\hat{X}_i$ , then $\psi_i(p) = \hat{\xi}_i'\,(p)$ .

If $p.\omega_i + \sum_j \theta_{ij}\, \hat{\pi}_j\,(p) = \text{Min}\ p.\hat{X}_i$ , then $\psi_i(p) = \{x_i \in \hat{X}_i \mid p.x_i = \text{Min}\ p.\hat{X}_i\}$ .

It is immediate that $\psi_i$ is upper semi-continuous on $P$ (which is now the intersection of the unit sphere with the polar of $AY$), and that for every $p$ in $P$ , $\psi_i(p)$ is non-empty, convex.

(2) In part 5, the strong market equilibrium theorem of [2] is used to obtain $p^*$ in $P$ and $z$ in $AY$ such that $z \in \sum_i \psi_i(p^*) - \sum_j \hat{\eta}_j(p^*) - \{\omega\}$. $AY$ cannot be a linear manifold on account of (d.3).

The proof is concluded along the same lines as before. For example, in part 7, if $w_i \neq \text{Min}\ p^*.\hat{X}_i$ , then (6) holds and, in particular, $p^*.x_i^* = w_i$ ; if $w_i = \text{Min}\ p^*.\hat{X}_i$ , then $p^*.x_i^* = w_i$ and, by the familiar argument on the interior of $K$ , $w_i = \text{Min}\ p^*.X_i$ . In both cases ($\alpha$) is true (in the first case because of (2) of 4.9 in [3]).

Theorem 2. Let $\mathcal{E}$ satisfy, in addition to the assumptions of Theorem 1,

(c') the relative interiors of $\{\omega\} + Y$ and $X$ intersect,

(e) if, in a quasi-equilibrium, $\overset{*}{p}\cdot\overset{*}{x}_i = \text{Min}\ \overset{*}{p}\cdot X_i$ occurs for some consumer, then it occurs for all.

Then every quasi-equilibrium of $\mathcal{E}$ is an equilibrium.

Consider a quasi-equilibrium of $\mathcal{E}$ and assume that $\overset{*}{p}\cdot\overset{*}{x}_i = \text{Min}\ \overset{*}{p}\cdot X_i$ occurs for some consumer. By (e) it occurs for all. Hence $\overset{*}{p}\cdot\overset{*}{x} = \text{Min}\ \overset{*}{p}\cdot X$. On the other hand $\overset{*}{p}\cdot\overset{*}{y} = \text{Max}\ \overset{*}{p}\cdot Y$. Therefore, the two sets $\{\omega\} + Y$ and $X$ can be separated by the hyperplane orthogonal to $\overset{*}{p}$ through $\overset{*}{x}$. But, following L.W. McKenzie [6], we can treat the problem in $L$, the smallest linear subspace of $R^\ell$ containing

$$Z = X - Y - \{\omega\} .$$

In $L$, the above separation cannot be achieved on account of (c') and the assumption introduced at the beginning of this proof thus leads to a contradiction. Consequently, $\overset{*}{p}\cdot\overset{*}{x}_i = \text{Min}\ \overset{*}{p}\cdot X_i$ cannot occur and ($\alpha$) implies that $\overset{*}{x}_i$ is a greatest element of $\{x_i \in X_i | \overset{*}{p}\cdot x_i \leq \overset{*}{p}\cdot\omega_i + \sum_j \theta_{ij}\ \overset{*}{p}\cdot\overset{*}{y}_j\}$ for $\underset{i}{\precsim}$, for every $i$ (by (1) of 4.9 in [3]).

## Applications

(1) The second equilibrium theorem given in note 3 of Chapter 5 of [3] asserts that a private ownership economy $\mathcal{E}$ satisfying the assumptions of Theorem 1, except for the fact that (c) is replaced by (c'') $(\{\omega_i\} + \text{Int}_L AY) \cap X_i \neq \emptyset$ , has an equilibrium. This Theorem is an immediate consequence of Theorem 1. According to the latter, $\mathcal{E}$ has a quasi-equilibrium. Because of (c''), in the subspace L , $p^*.\omega_i + \sum_j \theta_{ij} p^*.y^*_j = \text{Min } p^*.X_i$ cannot occur. Hence that quasi-equilibrium is an equilibrium.

(2) Theorem II (or II') of Arrow-Debreu [1].

The private ownership economy $\mathcal{E}$ covered by this Theorem satisfies all the assumptions of Theorem 1; therefore, it has a quasi-equilibrium $((x^*_i), (y^*_j), p^*)$ . We will show that the assumptions of Theorem 2 are also satisfied. Thus the above quasi-equilibrium will be an equilibrium.

That (c') is satisfied is clear by V of [1]. As for (e), if $p^*.x^*_{i'} = \text{Min } p^*.X_{i'}$ , it means (assumption IV'a or IV''a of [1] that some desired commodity has a zero price, or that some productive type of labor has a zero price, in which case some desired commodity has a zero price (if this were not so, the total profit of producers would not be at a maximum). In any case $p^*_h = 0$ for some $h \in \mathcal{D}$, the set of desired commodities. Consider then an arbitrary consumer, say the $i^{th}$. By definition of a desired commodity, there is a vector d parallel to the $h^{th}$ axis such that $x^*_i + d \succ_i x^*_i$ . Since $p^*_h = 0$ , one has $p^*.(x^*_i + d) = p^*.x^*_i$ . Consequently $x^*_i$ does not satisfy the preferences of that consumer under the constraint $p^*.x_i \leq p^*. x^*_i$ although it minimizes expenditure on the set $\{x_i \in X_i | x_i \succsim_i x^*_i\}$ (see (α) of the definition of a quasi-equilibrium). This can happen only if $p^*.x^*_i = \text{Min } p^*.X_i$ (by (1) of 4.9 in [3]), Q.E.D.

(3) Theorem of W. Isard and D.J. Ostroff [5].*

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* Walter Isard and I hope to be able to refine this Theorem by means of Theorems 1 and 2. I wish to acknowledge the stimulation I derived from the conversations I had with him on this point.

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Here, again, the assumptions of Theorem 1 are satisfied and the private ownership economy $\mathcal{E}$ has a quasi-equilibrium. If the price system in any region were 0, the price system in every region would be 0 (otherwise the total profit of producers would not be at a maximum). But this would contradict (δ) of the definition of a quasi-equilibrium. Hence, the price system in every region is different from 0 and assumption IVa of [5] ensures that $p^*.\omega_i + \sum_j \theta_{ij}\, p^*.y_j^*$ $= \text{Min}\ p^*.X_i$ cannot occur, Q.E.D.

(4) Irreducible economies of D. Gale [4] and L.W. McKenzie [6].

Consider a private ownership economy $\mathcal{E}$ satisfying the assumptions of Theorem 1, assumption (c') of Theorem 2 and the irreducibility assumption 6 of [6] and for which Y is a cone with vertex 0. It has a quasi-equilibrium $((x_i^*), (y_j^*), p^*)$ . Let $I_2$ be the set of i for which $p^*.x_i^* = \text{Min}\ p^*.X_i$ and assume that $I_2$ is not empty. We will show that its complement, $I_1$ , is empty. This will establish assumption (e) of Theorem (2) and the above quasi-equilibrium will be an equilibrium.

Let us assume therefore that $I_1$ is non-empty. One has $p^*.x^*_{I_2} = \text{Min}\ p^*.X_{I_2}$ while $p^*.y^* = \text{Max}\ p^*.Y$ . Hence $Y - X_{I_2} + \{\omega_{I_2}\}$ is below the hyperplane orthogonal to $p^*$ through $y^* - x^*_{I_2} + \omega_{I_2}$ , which is equal to $x^*_{I_1} - \omega_{I_1}$ . According to assumption 6 of [6], there is $w$ in $Y - X_{I_2} + \{\omega_{I_2}\}$ such that $x'_{I_1} = \overset{*}{x}_{I_1} + w$ is preferred to $\overset{*}{x}_{I_1}$ in Pareto's sense, by the consumers in $I_1$. But $\overset{*}{p}.w \leqq \overset{*}{p}.(\overset{*}{x}_{I_1} - \omega_{I_1})$ which is equal to zero. Hence $\overset{*}{p}.w \leqq 0$ and $\overset{*}{p}.x'_{I_1} \leqq \overset{*}{p}.\overset{*}{x}_{I_1}$, a contradiction.

We have thus established Theorem 2 of L.W. McKenzie [6], except for the fact that this Theorem does not use the irreversibility assumption d.3 of our Theorem 1. We will come back to this point in the second part of the appendix.

## Appendix

The $i^{th}$ consumer has a closed, convex consumption set $X_i$ and insatiable, continuous, convex preferences $\precsim_i$. Take the asymptotic cone of the set $\{x_i \in X_i | x_i \succsim_i x_i^0\}$ for a given $x_i^0$. It is a closed, convex cone, non-degenerate to $\{0\}$. The intersection of all these cones when $x_i^0$ varies in $X_i$ is also a closed, convex cone $\Delta_i$, non-degenerate to $\{0\}$, which will be called the insatiability cone of the $i^{th}$ consumer.

If $x_i \in X_i$ and $\delta_i \in \Delta_i$, then $x_i + \delta_i \succsim_i x_i$. From this follows that if $x_i^*$ satisfies the preferences of the consumer under the constraint $p.x_i \leqq w_i$, then the hyperplane $p.x_i = w_i$ is supporting from below for the cone $\{x_i^*\} + \Delta_i$. Therefore if $p^*$ is an equilibrium price system for the private ownership economy $\mathcal{E}$, $-\Delta_i$ is below the hyperplane $p^*.x_i = 0$ for every $i$, and so is $-\Delta$, where $\Delta$ denotes, as usual, $\sum_i \Delta_i$.

Then if one wishes to study an economy $\mathcal{E}$ where the total production set $Y$ is a cone with vertex 0, not necessarily satisfying the irreversibility assumption d.3, one replaces $\mathcal{E}$ by the economy $\mathcal{E}'$ obtained by substituting $Y' = Y - \Delta$ for $Y$. It follows easily from the assumptions of Theorem 1 that $Y - \Delta$ cannot be a linear manifold. Thus the reasoning of (4) above yields, without difficulty, an equilibrium for $\mathcal{E}'$ (in 5.7 of [3] the irreversibility assumption is used to bound the individual attainable production sets $\hat{Y}_j$, which is irrelevant when $Y$ is a cone). Let $((x_i'), y', p^*)$ be that equilibrium. Since $y' \in Y - \Delta$, one has $y' = y^* - \sum_i \delta_i$ where $y^* \in Y$ and $\delta_i \in \Delta_i$ for every $i$. Define $x_i^*$ as equal to $x_i' + \delta_i$. It is readily checked that $((x_i^*), y^*, p^*)$ is an equilibrium of $\mathcal{E}$: $y'$ maximizes profit on $Y - \Delta$ , hence $y^*$ maximizes profit on $Y$ and $p^*.\delta_i = 0$ ; therefore $p^*.x_i^* = p^*.x_i' = p^*.\omega_i$ , and $x_i^* \underset{i}{\succsim} x_i'$ according to the basic property of $\Delta_i$ ; since $x_i'$ satisfies the preferences of the $i^{th}$ consumer under $p^*.x_i \leqq p^*.\omega_i$ , so does $x_i^*$ . This completes the proof of Theorem 2 of L.W. McKenzie [6].

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