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Buffer Stocks, Sales Expectations, and Stability:

A Multi-Sector Analysis of the Inventory Cycle

Michael C. Lovell

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A Multi-Sector Analysis of the Inventory Cycle[*]

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Implications for the stability of the economy of inventory
practices of individual firms are appraised in this study. Eric Lundberg
[28] and Lloyd Metzler [29] have both formulated macro-economic models of
the inventory cycle. My approach resembles Metzler's in that a simple servo-
mechanism type of behavior is attributed to the individual firm. Production
is conceded to be time consuming. Inventories of finished goods are held as
buffer-stocks in order that unanticipated demand may be satisfied. The
entrepreneur attempts to adjust inventories to the appropriate level in the
face of incomplete knowledge of future sales.

My approach departs from the macro-approach adopted by Metzler and
Lundberg in that I consider complications that arise in aggregating the
behavioral patterns assumed for the individual firm in deriving conclusions
concerning the dynamic properties of the economy. In contrast with the
procedure of traditional macro-analysis I consider the implications of a
multitude of interacting firms all attempting to adjust inventories to a
level deemed appropriate in the face of incomplete knowledge of future

market conditions.* An analysis of stability conditions for the multi-sector

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*The challenge raised is similar to that faced by Goodwin [16] and
Chipman [10] in their disaggregation of the multiplier process. My task
is more difficult in that a crucial role is explicitly assigned to errors
of expectations and inventory stocks. In the development of the matrix
and multi-sector multiplier theories, no explicit mention was made of the
role of inventories. Chipman acknowledges in a more recent paper that "in
the multiplier approach demand for outputs is regarded as preceding the
production of inputs, the initial production of outputs being made possible
by the temporary depletion of inventories of inputs" [11, p.5]. The specifica-
tion of stability conditions for the multi-sector multiplier has been made in
abstraction from this fact. It will be found that the multiplier models of
Goodwin and Chipman are only a special, restricted case of the model presented
here. Under more realistic assumptions, the conditions for stability are
found to differ fundamentally from those they specified.

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model reveals that dynamic properties depend upon a multitude of parameters,

some of which are suppressed in aggregative model construction. Conditions for

stability are found to differ fundamentally from those developed by Metzler

in his macro-analysis.

In a discussion of the implications for stability of alternative in-

ventory practices it is most appropriate to start, as in the theory of

competition, with a discussion of the behavior of the individual firm. The

second stage of the analysis is that of deriving the behavior of individual

industries from the assumed firm behavior. Then the interrelations between

the various industries must be considered in connecting the equations repre-

senting the behavior of the individual sectors in order to obtain a model

of the whole economy. Finally, dynamic properties of the model for

alternative values of its parameters have to be explored.

In order to make this difficult task tractable, numerous simplifying assumptions must be made. The inventory behavior pattern attributed to the individual firm is relatively simple. Capacity restraints on the level of output are ignored. Time is treated as a discrete unit. It is assumed that the production period is the same for all commodities and, in addition, that the production processes of all firms are synchronized so that output begins at the same point of time for all firms; essentially, the production process is of the "point-input point-output" type so frequently encountered in capital theory. In addition, price phenomena are neglected throughout the discussion. None of these assumptions is without precedent in either the literature of managerial economics on optimal inventory policy or in the multitude of aggregative economic models of the accelerator and related theories.

Granted that simplifying assumptions must be made in any analysis, it may still be asked whether an appropriate set has been chosen for the problem at hand. My own empirical investigation of the behavior of manufacturers' inventories in the United States suggests that the buffer-stock model does provide an appropriate framework for examining finished goods inventory behavior [27]. Manufacturers apparently attempt only a delayed adjustment type of inventory behavior. They tolerate a considerable departure of actual inventories from the optimum level rather than attempt an immediate adjustment of inventories on the basis of imprecise estimates of future market conditions. This complication was introduced into the theoretical analysis. Contrariwise, no empirical support was found for the hypothesis that

manufacturers speculate in inventories, adjusting their holdings of stocks

in response to anticipated price changes; consequently, it seems appropriate

to neglect the complications of speculation in the theoretical study.

Unfortunately, I have not succeeded in introducing unfilled orders into the

theoretical analysis, although the empirical evidence suggests they are a

factor of considerable importance in determining the desired level of stocks

of purchased materials and goods in process.

The assumption of price rigidity, motivated by convenience, serves to

suppress the roles of speculative inventory holdings and substitution. A statement

by J. R. Hicks suggests that a theoretical investigation of the properties of a

model involving price rigidity is worth while [22, p.145]:

> Both the manufacturer and the retailer are, for the most part,
> "price makers" rather than "price takers"; they fix their prices
> and let the quantities they sell be determined by demand. A
> model in which quantities bear the brunt of disequilibrium fits
> most of the facts distinctly better [than the model of Value and
> Capital].

In addition to facilitating the derivation of theorems concerning the

stability implications of buffer-stock inventory behavior, the assumption

of price rigidity makes it possible to muster within the confines of this

paper relevant data for the United States economy in an attempt to determine

whether the conditions for stability are satisfied. If the assumption of

price rigidity were abandoned, the difficulties involved in empirical investiga-

tion would be increased by whole orders of magnitude.

Empirical and theoretical research are complementary. A theoretical

investigation may assist the econometrician by providing limits on the range
of models that have to be tested. Although the assumption of profit maximiza-
tion may serve partially to restrict the hypotheses to be considered, major
simplification is almost inevitably involved in moving from theorems derived
from that assumption to the equations of a completely specified model.
Examination of the dynamic implications of conceivable modes of behavior
provides a second source of a priori knowledge. The theoretical investiga-
tion of stability conditions for the buffer-stock inventory model suggests
appropriate a priori restrictions for empirical research by revealing which
types of inventory behavior on the part of the individual firm imply reason-
able dynamic properties for the economy.

The model developed in this paper belongs to a general family of
multi-sector dynamic models which includes the input-output approach utilized
by the United States Government to investigate on an industry by industry
basis the impact of mobilization for the Korean war.*

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*A discussion of two air force models was presented by Holley [23, 24].
Chenery and Clark have discussed the United States Emergency Models [9].

In dynamic input-output analysis inventories usually are either relegated to
the final bill of goods or assumed to behave in accordance with an elementary
version of the accelerator principle uncomplicated by lags in the adjustment
process or errors made by firms in anticipating future sales volume. Although
it may be appropriate to neglect such complications in the analysis of
problems relating to long-run economic growth, my empirical investigations

of inventory behavior in the United States indicate that errors made by

firms in anticipating future sales and adjustment lags served to curtail sub-

stantially inventory accumulation during the Korean emergency [27]. The

theoretical framework presented in Part I of this paper provides an appro-

priate vehicle for considering these two complications of actual inventory

behavior in analyzing on an industry by industry basis such problems as the

economic impact of mobilization for limited war.

PART I A MULTI-SECTOR INVENTORY MODEL

A. The Behavior of the Firm

It seems appropriate to assume that under conditions of price

rigidity entrepreneurs will carry inventories for the purpose of avoiding

the unsatisfied market that would otherwise occur whenever demand exceeds

anticipated sales. When demand exceeds expectations, inventories are reduced

below the planned level; conversely, when sales fall short of anticipated

demand, unplanned inventories are accumulated. The production levels of future

periods is then set so as to either exceed or fall short of anticipated sales

in order to restore inventories to the desired level.* The excess over actual

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*Entrepreneurs in fact have the option of eliminating excess stocks by
price reductions and of raising prices in periods of shortage. This type
of behavior has been excluded by the assumption of price rigidity. Paul A.
Samuelson has derived stability conditions for a model in which price adjust-
ments occur when existing stocks exceed an equilibrium level as a result of a
divergence between current production and consumption [34,p. 275-6].

sales of goods available at the end of the production period is held as

inventory and must be considered in formulating production plans for the

next period. Although an essentially intuitive approach similar to that

of Metzler [30] will be followed in the subsequent paragraphs where the

details of this argument are spelled out, it must be mentioned that

Edwin Mills has demonstrated that under appropriate conditions this type

of firm behavior is at least consistent with the assumption of profit

maximization [31].

The distinction between the equilibrium and planned level of in-

ventories proves to be crucial. It will be assumed that the firm's

equilibrium level of inventory is linearly related to sales. If $I^e(t)$

represents the equilibrium inventory for the firm at time t and $\overline{X}(t)$

anticipated sales, this assumption may be expressed by the equation

(1.1) $\qquad I^e(t) = c + b\,\overline{X}(t), \qquad t = 1,2,3, \ldots$

The parameter b , the marginal desired inventory coefficient, relates the

equilibrium level of inventories to sales volume. Metzler and Lundberg

assumed that firms attempt an immediate adjustment of inventories to this

equilibrium level. In actual practice, costs involved in changing produc-

tion levels and adjusting the size of stocks apparently lead firms to

attempt only a partial adjustment of actual inventories to the desired

level in any one period. If this flexible accelerator complication is

introduced, the level of inventories planned for period t , $I^p(t)$, is

determined by the equation

$$(1.2) \qquad I_t^p = d[I^e(t) - I^a(t\text{-}1)] + I^a(t\text{-}1) \ , \quad 0 < d \leq 1 \ ,$$

where $I^a(t\text{-}1)$ represents actual inventories in the preceding period.[*]

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[*]This is the distributed lag type of investment function suggested by Goodwin [17] and utilized by Chenery [8] and others in empirical work.

If the reaction coefficient d is precisely one, planned and equilibrium inventories are identical. But if d is less than unity, the planned adjustment in inventory is a proportion d of the discrepancy between the actual and the equilibrium level of inventories.

Entrepreneurs set the level of output at the beginning of the production period in an attempt to obtain the planned level of inventories on the basis of anticipations concerning future sales volume. Therefore, output $Q(t)$ is determined by the equation

$$(1.3) \qquad Q(t) = \overline{X}(t) + I^p(t) - I^a(t\text{-}1) \ .$$

If sales anticipations turn out to be correct, production will just suffice to meet the demand and to adjust inventories to the planned level. But when actual sales, $X(t)$, exceed anticipations, the extra demand can be met only by running inventory down below the planned level. On the other hand, surplus

inventory will be accumulated when sales fall short of anticipations. Actual inventory, therefore, is given by the equation

$$(1.4) \qquad I^a(t) = I^p(t) + \overline{X}(t) - X(t)$$

$$= cd + (1+bd)\, \overline{X}(t) - X(t) + (1-d)\, I^a(t-1) \ .$$

Returning to equation (1.3) the following substitutions are permitted by (1.1), (1.2) and (1.4)

$$Q(t) = \overline{X}(t) + d[c + b\overline{X}(t)] + (1-d)I^a(t-1) - cd - (1+bd)\overline{X}(t-1)$$

$$+ X(t-1) - (1-d)I^a(t-2) \ .$$

$$= (1+bd)\overline{X}(t) - (1+bd)\overline{X}(t-1) + (1-d)\, [I^a(t-1) - I^a(t-2)]$$

$$+ X(t-1) \ .$$

Further simplification is possible for the actual change in inventory is identical to the discrepancy between output and actual sales; that is,

$$I^a(t-1) - I^a(t-2) \equiv Q(t-1) - X(t-1) \ .$$

Simple substitution now serves to eliminate the inventory terms from the equation explaining the level of output

$$(1.5) \qquad Q(t) = (1+bd)\overline{X}(t) - (1+bd)\overline{X}(t-1) + (1-d)Q(t-1) + d\, X(t-1) \ .$$

The output of the firm may be determined from anticipated sales and last period's output and sales volume if the magnitude of the marginal desired inventory and reaction coefficients are known.

While it will be assumed that the output of each firm in the economy is determined in this fashion, it must be observed that wide fluctuations in sales might lead to conditions in which equation (1.5) could not represent the determination of a firm's output. First, a rapid fall in sales might imply negative outputs in order that the desired fraction of excess inventories could be eliminated within a single period; in reality, inventories cannot be liquidated at a rate higher than actual sales. Second, inventories cannot be negative. At least for relatively small fluctuations, however, these complications may be neglected: the output of each firm in the economy may be assumed to be determined by equation (1.5).

It is interesting to note that if the product of the marginal desired inventory coefficient times the delayed adjustment term equals minus one $(bd=-1)$, the output of the firm is completely independent of whatever sales anticipations the entrepreneur happens to hold and depends only upon output and sales in the preceding period. Again, if sales anticipations are at any fixed unchanging level, $\bar{X}_t = \bar{X}_{t-1}$, output is again found to depend only upon past output and sales. The complications created by the anticipations term may also be avoided if it is assumed that firms correctly anticipate next period's demand for their product. With perfect foresight, $\bar{X}_t = X_t$, equation (1.5) reduces to

(1.6) $\qquad Q(t) = (1+bd)X(t) + (d-1-bd)X(t-1) + (1-d)Q(t-1)$.

Such a procedure achieves simplification at the expense of suppressing the
complications of real interest. Sales anticipations and consequent errors
in planning are most easily introduced if naive expectations are assumed.
If $\bar{X}(t) = X(t-1)$, equation (1.5) becomes

(1.7) $Q(t) = (1+d+bd)X(t-1) - (1+bd)X(t-2) +(1-d)Q(t-1)$.

B.   The Industry and the Economy

A uniformity assumption facilitates the derivation of the economic
implications of the assumption that the multitude of firms in the economy
behave according to equation (1.7); it will be assumed that all firms producing a
given commodity have the same marginal desired inventory and delayed adjustment
coefficients. Since these are the only parameters entering into the linear
equations determining the output of each firm, the total output of a commodity may
be derived as a function of past industry sales and outputs by simply summing
the quantities and sales figures over all the individual firms in the industry.
If $X_{ij}(t)$ and $Q_{ij}(t)$ stand for the sales and output of the ith firm in
the jth industry at time t , the uniformity assumption implies the following
relationship between sales and outputs in each of the n industries con-
stituting the economy

(1.8) $\sum_{i} Q_{ij}(t) = (1+b_j d_j+d_j)\sum_{i} X_{ij}(t-1) - (1+b_j d_j)\sum_{i} X_{ij}(t-2)$

$+ (1-d_j)\sum_{i} Q_{ij}(t-1)$, $j = 1,2, \ldots, n$.

Thus the uniformity assumption permits a reduction in the complexity of

the system by replacing the multitude of equations in a given industry by

a single equation explaining industry output on the basis of past sales.*

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*It may be observed that the output of a whole industry, but not that
of any individual firm, is determined on the basis of past sales and output
of that industry. The output of the individual firm is indeterminate, a
problem also encountered in the theory of pure competition under the
assumption of constant costs. Samuelson suggests that "under the purest
conditions of competition the boundaries of the (firm) ... become vague
and ill-defined, and also unimportant." [34, p.79]. Here too we need
not be concerned with the scale of operation of the individual firm; only
the output of the industry is of interest.

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It proves convenient to rewrite equation (1.8) in more compact matrix

notation

$$(1.8') \qquad Q_t = (I+BD+D) X_{t-1} - (I+BD) X_{t-2} + (I-D) Q_{t-1} \cdot$$

Here Q_t and X_t are column vectors whose components represent industry

output and sales respectively, I is the identity matrix, and B and D

are diagonal matrices of marginal desired inventory and adjustment coeffi-

cients respectively.

The assumption of uniform desired inventory and adjustment coeffi-

cients for all firms in a given industry does not suffice to provide a

deterministic economic model. In addition to (1.8) other equations

connecting sales volume with the outputs of the various industries are

required. One possible procedure might be to assume that only one commodity

is produced in the economy and that all sales are made directly to consumers. The consumption function could then be employed to relate sales to the level of output. This procedure was utilized by Metzler in the development of his theory of the inventory cycle. Although this is common practice in the construction of business cycle models it is not necessary to neglect all aggregation problems in this way. There is an alternative, less restrictive, disaggregated approach. When this more realistic procedure is followed the conditions for stability are altered in a most fundamental fashion.

A system of equations relating the sales of each sector to the level of output of all sectors is required. Let production conditions be assumed to be the same for all the firms in a given industry. Let sales be regarded as the sum of purchases by all other firms for production purposes and a final bill of goods representing government demand, consumption, and sales that are not related to current and past levels of output.[*] Since we assume that no stocks of inputs are held, purchases

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[*]Later it is explained how consumption may be included endogenously within the basic framework of the model rather than assumed to be independent of current levels of activity.

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of commodities at time t by each sector must be related to the quantity that the sector plans to produce in the next period. The inputs required for production purposes may be considered as specified by a matrix of technological coefficients $A = (a_{ij})$, where $a_{ij} \geq 0$

represents the quantity of output of sector i required per unit of output

of the jth sector. The n equations relating sales of each sector to the

output of all other industries, because of the production lag, have the form

$$x_{it} = \sum_j a_{ij} Q_{j\ t+1} + y_{it} \ , \qquad (i = 1, \ldots, n)$$

or in matrix notation

$$(1.9) \qquad X_t = A Q_{t+1} + Y_t \ .$$

Sales are equal to the inputs required for output forthcoming in the next

period[*] plus Y_t , the final bill of goods vector whose components represent

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[*]The time lag in equation (1.9) is crucial. This type of
formulation is essential if production is regarded as a time consuming
process. The differential equation models discussed by Georgescu-Roegen
[15], David Hawkins [20], and Leontief [25] assumed that production is
instantaneous. The essence of the inventory problem may well lie in the
fact that production does require time. Although an alternative time lag
formulation may be appropriate for analyzing other problems relating to
growth as opposed to cyclical phenomena, it seems clear that a model of
inventory behavior has to recognize that the inputs used in the production
of a commodity must have been fabricated in an earlier period.

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sales of each commodity that are independent of output. The equation is

meaningful, of course, only if $X_t \geq 0$ and $Q_t \geq 0$. The components of the

matrix A of technological coefficients might be regarded as fixed if substi-

tution were not a technological possibility and if constant returns to scale

prevailed. The technological assumption of fixed proportions need not be adopted as the assumption of price rigidity, already introduced in the development of the inventory behavior equation for individual firms, together with constant returns to scale, serves to establish the same result. Under either set of assumptions, the empirical counterpart of the A matrix is provided by the Leontief input-output matrix of flow coefficients.

Alternative interpretations of the A matrix are possible. For example, we could restrict attention to three sectors only: manufacturing, wholesale, and retail trade. Another interpretation of the A matrix is provided by the Hayekian type of technology in which higher stages of production feed their output to the lower stages. Such special cases of the general problem are all subsumed within the analysis that follows.

With the aid of equation (1.9), a simple process of substitution suffices to eliminate the expression for sales from equation (1.8') so as to have a system of equations involving past and present levels of industry outputs and the final bill of goods alone. This yields

$$(1.10) \qquad Q_t = (I+BD+D)(AQ_t+Y_{t-1}) - (I+BD)(AQ_{t-1}+Y_{t-2}) + (I-D)Q_{t-1} .$$

A transformation of this expression for quantity reveals that current levels of output may be explained by past output levels and the magnitude of the final bill of goods

$$Q_t = [I - (I+BD+D)A]^{-1} \left\{ [I-D-(I+BD)A]Q_{t-1} +(I+BD+D)Y_{t-1} - (I+BD)Y_{t-2} \right\} ,$$

or more briefly

(1.11) $Q_t = TQ_{t-1} + K(I+BD+D)Y_{t-1} - K(I+BD)Y_{t-2}$,   where

$K \equiv [I-(I+BD+D)A]^{-1}$, and

$T \equiv K[I-D-(I+BD)A] = I - KD(I-A)$.

From this last equation it is apparent that a complete, deterministic model of the economy is obtained when the technological coefficients describing production conditions, as summarized by matrix equation (1.9), are combined with the set of equations giving the level of output for each industry when firms pursue a buffer-stock inventory policy.

Utilization of a matrix of technological coefficients to achieve closure abstracts from fixed investment in plant and equipment, buildings, and so forth. Baumol has implied, in a discussion of an aggregative model, that the buffer-stock type of behavior may be attributed to all investment, to explain a divergence between ex ante and ex post investment [5]. Only a rephrasing of the argument to follow rather than a substantive change would be required in order to include fixed investment in this way. Alternatively, non-inventory investment may be relegated to the final bill of goods. Another procedure would be to rewrite equation (1.9) in the form $X_t = AQ_t + E(Q_t - Q_{t-1})$, where E is a matrix of capital and purchased material accelerator coefficients. When such a procedure is followed in order to explain capital accumulation the inventory model proves difficult

to handle analytically, but it might be the most fruitful method to
apply in empirical applications of the model.

Consumption expenditure can be relegated to the final bill of
goods if it may be assumed to be independent of current income. Alterna-
tively, consumption expenditure may be made dependent upon the current level
of output by a slight reinterpretation of certain coefficients. No fundamen-
tal change in the structure of the model is required. The relation of con-
sumption of individual commodities to labor income may be expressed by
the equation

$$(1.12) \qquad C_t = C + A^c y_t \; ,$$

where A^c is a column vector whose components represent marginal propen-
sities to consume by commodity type and C is a vector of constant
components of consumption expenditure. Income, the scaler y_t , may in
turn be considered to depend upon the level of output, Q_{t+1} , according
to the extent to which labor is utilized to produce a unit of output, and
possibly the level of income itself, for labor may be consumed directly.
This gives the equation

$$(1.13) \qquad y_t = A^r Q_{t+1} + a_{oo} y_t \; ,$$

where $A^r = (a_{01}, a_{02}, \ldots, a_{on})$, a row vector, and a_{oo} are defined
in a fashion similar to the coefficients of A . Substituting, one obtains

$$(1.14) \qquad C_t = C + \left(\frac{1}{1 - a_{oo}}\right) A^c A^r Q_{t+1} \ .$$

Here $A^c A^r$ is an $n \times n$ matrix, of course. Equation (1.9) can be replaced by the relation

$$(1.15) \qquad X_t = [A + \left(\frac{1}{1 - a_{oo}}\right) A^c A^r] Q_{t+1} + Y_t + C \ .$$

By substituting $A^* = A + \left(\frac{1}{1 - a_{oo}}\right) A^c A^r$ into equation (1.15) it is made identical in form to (1.9); the equations derived with (1.9) for the case in which all consumption was relegated to the final bill of goods may clearly be made immediately applicable to the more general case. No additional equations for the final model are obtained when this procedure is utilized to open the buffer-stock inventory model with respect to consumption; there remains but one equation of (1.11) for each industry in the economy.[*]

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[*] An alternative procedure giving somewhat greater flexibility may be followed in order to introduce consumption complications. Let us regard the oth sector as the labor sector. Then a_{oj} must be the quantity of labor required per unit of the jth output and a_{io} the slope of the (linear) Engel's curve relating consumption of commodity i to income. Various lags in consumption behavior may now be considered by appropriate specification of the parameter b_o. Labor income at time t is $x_{ot} = \sum_j a_{oj} q_{j,t+1}$ by equation (1.9). Now if $b_o = -1$ and $d_o = 1$ we have by equation (1.8) that $q_{o,t+1} = x_{ot}$; since the goods required for "production" at time $t+1$ are purchased at time t , this is equivalent to unlagged consumption behavior! If, on the other hand, $b_o = -2$, we have $q_{ot} = x_{t-2}$ and consumption is lagged one period, as with the Robertsonian consumption function. If b_o lies between these two limits, consumption depends on both past and current income.

PART II STATICS AND DYNAMICS:

Implications for the behavior of the economy through time of alternative assumptions concerning the nature of expectations, of technological improvement, and of adjustments in marginal inventory and reaction coefficients are of fundamental interest. As a prelude to the analysis of these complex questions within the framework of the multi-sector inventory model developed in the preceding section, it is convenient to review certain basic concepts.

A. The Static Solutions:

The first step in the analysis of the properties of the buffer-stock inventory model is to observe its behavior under static conditions. If output and the final bill of goods remain at some fixed levels, call them Q and Y , we have from equation (1.10)

$$(2.1) \qquad Q = (I{+}BD{+}D)(AQ{+}Y) - (I{+}BD)(AQ{+}Y) + (I{-}D)Q .$$

This expression simplifies to

$$(2.2) \qquad Q = AQ + Y ,$$

an equation identical to a balance relation encountered in open input-output analysis. The static equilibrium of the inventory model for an unchanging final bill of goods, a vector of constant outputs, is

$$(2.3) \qquad Q = [I-A]^{-1} Y \ .$$

It is clear that the equilibrium output for each sector of the economy

depends only upon the matrix of flow coefficients and the final bill of

goods; the magnitudes of the marginal desired inventory and reaction co-

efficients do not enter into the static solution. The equilibrium solution

and certain complications of mathematical interest are familiar from static

input-output analysis.[*]

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[*] The model is called feasible in the static sense if $Y \geq 0$ implies
that the static solution Q is non-negative. A theorem established by
Hawkins and Simon [21] implies that the system is feasible unless it is in-
efficient in the sense that some productive process requires to produce a unit
of output (both directly as an input and indirectly in the production of other
commodities required as inputs) one or more units of its own output. Efficiency
is a most reasonable assumption for an economy in which production does not
require time, for an inefficient production process would be unprofitable
under any set of non-negative prices, not all zero. When production requires
time, however, it is conceivable that an inefficient process would not be un-
profitable if the rate of interest were negative. While this possibility has
been considered by von Neumann [35] and Irving Fisher [14, pp. 191-2], I
shall exclude from consideration technologies implying a negative rate of
interest and assume that the technology is that of an efficient economy so
that the model is feasible in the static sense.

B. The Homogeneous Solution:

The task of analyzing the dynamic properties of the model represented

by the simultaneous system of equations (1.11) is facilitated, just as with

the single difference equations encountered in aggregative business cycle

analysis, by working in terms of deviations from the static solution. This

procedure will be most fully appreciated if a problem of prediction is considered. It may be observed that (1.11) is the "reduced form" of the system of equations (1.10); the vectors Y_t and Y_{t-1} are exogenous and the vector Q_{t-1} constitutes the predetermined variables of the system; the vector of endogenous variables Q_t is to be determined. If one desires to predict future levels of production on the assumption that equation (1.11) portrays the behavior of the economy, the prediction has to be conditional upon knowledge of the path of the unexplained exogenous variables, the final bill of goods vector Y_t . Suppose, in order to facilitate the argument, that the final bill of goods is a known constant vector $Y.^*$ If Q_0 is the vector of current output, the

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*The assumption that the final bill of goods is fixed is introduced only to simplify the discussion; it is not essential to the argument. The procedure for a fluctuating final bill of goods is analogous to that utilized in analyzing the elementary case where only a single difference equation is involved. The conditions for stability are independent of the time path of the final bill of goods. An economist's account of the procedure for the special, single difference equation problem is given by Allen [3, Ch. 6] and for the more general case by Leontief [25, pp. 63-5].

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level of output of each sector for the next period is given by the matrix expression $Q_1 = TQ_0 + KDY$; substituting this result into equation (1.11) yields $Q_2 = T^2Q_0 + (T+I)KDY$. Clearly, the assumptions permit prediction any number of periods into the future.

· This clumsy, iterative procedure may be circumvented by working in terms of deviations from equilibrium. Observe that by definition the static solution obtained by application of (2.2) must satisfy (2.1).

Consequently, subtraction of the equilibrium solution Q from the vector of actual outputs Q_t as given by (1.10) yields the homogeneous equation

$$(2.4) \qquad Q_t - Q = (I+BD+D)A(Q_t - Q) + [I-D-(I+BD)A](Q_{t-1} - Q) = T(Q_{t-1} - Q) \; ,$$

the last equality following from the definition of T presented in the derivation of (1.11). By induction on this last equation the homogeneous solution is obtained

$$(2.5) \qquad Q_t - Q = T^t(Q_0 - Q) \; .$$

All that is necessary to obtain Q_t from this homogeneous solution is to add Q to both sides

$$(2.6) \qquad Q_t = T^t(Q_0 - Q) + Q \; ,$$

where Q is provided by equation (2.2).

C. The Stability of Equilibrium:

The stability of a difference equation system is most conveniently discussed with reference to the homogeneous equation (2.5). Will the system converge to the static solution Q for all initial deviations from equilibrium? Equation (2.5) reveals that this concept of stability requires

$$\underset{t \to \infty}{\text{limit}} \quad T^t(Q_0 - Q) = 0 \; , \text{ for all vectors } (Q_0 - Q) \; .$$

Only the transition matrix T has to be considered in resolving the issue

of stability. Stability requires that all of the characteristic roots

of T lie within the unit circle on the complex plane. Is the multi-

sector buffer-stock inventory model stable? This issue is to be explored

in Part III of this paper.[*]

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[*]Other dynamic properties in addition to stability are also important.
Of particular interest is the question of dynamic feasibility. It is clear
that if the marginal desired inventory coefficients are non-negative,
sufficiently small disturbances could not cause a stable system to degenerate
into a state of negative stocks or outputs. The complication of stability
for larger disturbances might be dealt with by embedding the model discussed
here within a larger, possibly piecewise linear system; it would then be but
one of several possible regimes; the nature of the alternative regimes and
the rules for switching from one regime to the next would have to be
specified unambiguously. Leontief modified the Hawkins dynamic model in this
way; see [25, pp. 68-76]. For the buffer-stock model these complications
cannot be explored within the compass of this paper.

D. Stochastic Complications:

The buffer-stock inventory model has been presented in deterministic

form; the existence of stochastic disturbances was not considered in

developing the analysis. Stochastic disturbances are frequently introduced

into dynamic models in order to represent variables omitted from the

analysis. For the buffer-stock model they may be assigned a particularly

important role. Stochastic disturbances can result if the assumption of static

expectations is introduced only as an approximation. The assumption that

actual expectations are yesterday's sales plus a disturbance with zero

expected value is much more palatable than the rigid condition that

$$\overline{X}_t = X_{t-1} \; .$$

Fortunately, the dynamic properties of the type of linear system under consideration is not complicated in any essential way when stochastic shocks are introduced. Suppose that the true equation describing the generation of outputs is of the form

$$(2.7) \qquad Q_t = TQ_{t-1} + KDY + \epsilon_t \; ,$$

where ϵ_t is a vector of random variables with zero expected value. More precisely, if $\mathcal{E}$ is the expected value operated assume $\mathcal{E}(\epsilon_t) = 0$ and $\mathcal{E}(\epsilon_t \epsilon_t') = \mathcal{E}(\epsilon\epsilon')$ for all positive integers t. Then it can be shown, although not within the space available here, that the deterministic scheme already developed describes the path of <u>expected</u> if not actual output

$$(2.8) \qquad \mathcal{E}(Q_t) = T\mathcal{E}(Q_{t-1}) + KDY \; .$$

Of course, how concerned one is with $\mathcal{E}(Q_t)$ and in particular the limit $\underset{t \to \infty}{}$ $\mathcal{E}(Q_t)$ depends upon the magnitude of the discrepancies between the actual level of output determined by the stochastic scheme and its expected value. Let $E_t = Q_t - \mathcal{E}(Q_t)$ represent this error. Interest centers on the variance covariance matrix of errors $\mathcal{E}(E_t E_t')$. Although the matrix depends in an essential fashion upon the distance t into the

future which we are attempting to predict the linear nature of the system means that if the non-stochastic system is stable, if in other words $T^t \rightarrow 0$, then $\lim_{t \to \infty} \mathcal{E}(E_t E_t')$ exists. It follows immediately that the variance of the prediction errors is bounded. This means that the conditions of stability for the non-stochastic system are pertinent for the more realistic case in which random shocks are introduced. The theorems developed in the next section are of interest in the appraisal of a stochastic version of the buffer-stock inventory model.

PART III SOME DYNAMIC IMPLICATIONS OF BUFFER-STOCK INVENTORY BEHAVIOR

In comparative statics the effects of changes in a model's parameters upon its equilibrium solution are examined. For the multi-sector inventory model, we have seen, only the matrix A of technological coefficients together with the final bill of goods Y are involved in the determination of the static solution. Comparative dynamics is concerned with contrasting possible paths that an economy might follow in adjusting through time under alternative assumptions concerning the magnitude of the system's parameters. We shall see that the marginal desired inventory coefficients and the reaction coefficients as well as the A matrix influence the dynamic behavior of the buffer-stock inventory economy.

Two fundamental questions are to be considered: Does the delayed adjustment, flexible accelerator complication observed in actual buffer-stock inventory behavior help to stabilize the economy? Do errors made by firms in forecasting future sales volume contribute to economic instability;

would more accurate expectations serve to stabilize the economy? It proves convenient to follow a twofold line of attack in demonstrating that only the first of these questions can be answered in the affirmative. The behavior of the model with all reaction coefficients equal to unity will be contrasted with the delayed reaction case in which the flexible accelerator coefficients are less than one. In addition, the case of static expectations will be compared with the hypothetical situation in which entrepreneurs anticipate correctly next period's demand for their product.

In the presentation of the argument it proves essential to place certain restrictions upon the types of technologies that will be considered. The theorems are developed under the restriction that the economy be efficient in the sense of Hawkins and Simon [21]. This is equivalent to the condition that the characteristic roots of A are all within the unit circle on the complex plane. As has already been mentioned, this guarantees the existence of a feasible static solution for any positive final bill of goods. In addition, the matrix A of flow coefficients will be assumed to be equivalent under a similarity transformation to a diagonal matrix; while this is not truly restrictive,[*] it does facilitate the proofs of certain

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[*]No empirical investigation could lead to the rejection of this assumption for Bellman [6, p. 25] has shown that for any square matrix $A = (a_{ij})$ and $\epsilon > 0$ there exists a matrix $A^* = (a_{ij}{}^*)$ with the desired property and such that $|a_{ij} - a_{ij}{}^*| \leq \epsilon$.

theorems. It will also be helpful to agree as a matter of notation

that if A is any square matrix and λ_i is a characteristic root of

A , then $|\lambda_i|$ is the modulus of λ_i and $r(A) \equiv \max_i |\lambda_i|$. In the

development of the conditions for stability, frequent use is made of

properties of the characteristic roots of square non-negative (Frobenius)

matrices presented by Debreu and Herstein [12].

A. <u>Static Expectations with Immediate Adjustment Behavior</u>:

The contrast between the stability properties of the multi-sector

buffer-stock inventory model and those of the macro-inventory cycle theory

of Lloyd Metzler is most easily demonstrated by considering the restricted

case of immediate adjustment behavior under the assumption of naive

expectations. Metzler found that under the assumption of naive expectations

his single commodity economy was stable if and only if the marginal

propensity to consume is less than unity [29, pp. 117-8]. The following

theorem, proved in the Appendix, establishes a necessary condition for

stability of the multi-sector model.

THEOREM I; If T is defined by (1.11) with D = I and B $\geq$ 0, then

$$\lambda^+ < \frac{1}{3 + 2 \max(b_i)} \implies r(T) < 1 \text{ , and}$$

$$r(T) < 1 \implies \lambda^+ < \frac{1}{3 + 2 \min(b_i)} \quad .$$

Recognition of the multi-commodity nature of the economy reveals that

even if the marginal propensity to consume is zero, stability under the

assumption of immediate adjustment requires that the largest characteristic

root of the matrix of technological coefficients be less than one-third.[*]

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[*]Because Metzler neglects production conditions entirely, the
special single commodity case of the disaggregated model is not identical
to Metzler's. Metzler implicitly assumed that only labor enters into the
production process, excluding the possibility that today's output is begot
by the marriage of labor service with commodity inputs produced in the past.

This condition does not suffice if any of the marginal desired inventory

coefficients are greater than zero.[**] A priori considerations suggest

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[**]While negative marginal desired inventory coefficients seem unrea-
sonable, it is interesting to note that if $B = -2I$ and $D = I$ the buffer-
stock inventory model (1.10) simplifies to $Q_t = AQ_{t-1} + Y_t$, the matrix or
multi-sector multiplier of Goodwin [16] and Chipman [10]. It is well known
that this system is stable if the economy is feasible in the static sense that
$Y \geq 0$ implies that the static solution Q is necessarily non-negative.
The Hawkins-Simon concept of economic efficiency indicates that it is
reasonable to assume that this condition is met. While it can be shown that
the model is necessarily feasible in the dynamic sense that $Y_t \geq 0$ and $X_o \geq 0$
imply $X_t \geq 0$ for $t = 1,2, \ldots,$ the initial size of the inventory endowment
places an absolute ceiling on the level of output that can be obtained without
the contradiction of negative inventories. It seems appropriate to restrict
our discussion to the case in which $B \geq 0$.

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that Metzler's restriction upon the marginal propensity to consume is

satisfied. While a priori considerations also suggest that the characteris-

tic roots of A are all less than unity in absolute value, nothing implies

that they are less than one-third.

The theoretical results raise an empirical question concerning the
actual values of the characteristic roots of A.   Max A. Woodbury
reports a figure of 0.5414 for the largest characteristic root of an 18x18
input-output matrix based on data for the year 1939 [36]. My own calculations
performed on a 10x10 input-output matrix for the year 1947 yielded a dominant
root of 0.55, a figure remarkably similar to that reported by Woodbury.
Inspection of matrices of six, eleven, and twenty-one sectors published
by the Harvard Economic Research Project [32] revealed that in every case
the largest characteristic root was larger than one half.[*] This establishes

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[*]Although the precise determination of the characteristic roots of a
matrix is a difficult computational task, a lower bound for the largest character-
istic root can be easily determined for A is non-negative. Specifically,
$r(A) \geq \min_{j} \Sigma_{i} a_{ij}$ and $r(A) \leq r(A^*)$ if $A \leq A^*$.

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that stability is not compatible with immediate adjustment behavior for
reasonable values of λ^+ , the dominant root of A.

A further difficulty with the assumption that firms attempt an
immediate adjustment of inventories to their equilibrium level must be
mentioned. Even if we chose to reject the empirical evidence and assumed
that the immediate adjustment buffer-stock model were stable, the system
would still have a most undesirable property. If the reaction coefficient
is unity, stability implies that the transition matrix T is non-positive.[*]

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[*]This statement is established in the Appendix in proving Theorem I.

This means that if the system were stable, it would be prone to generate a cycle with most peculiar characteristics. A simple example will serve to illustrate this strange cycle. Suppose that the economy were initially in equilibrium with some given final bill of goods Y . If the final bill of goods changes to Y^*, $0 \leq Y^* \leq Y$, the output of each sector must fall in the next period below the new equilibrium level. Then in the subsequent period they must all rise above their equilibrium value, and so forth. For this type of disturbance, the length of the inventory cycle is two time periods. Such a saw-tooth cycle necessarily develops once every sector is producing below the equilibrium level, as in a depression. All this follows from the fact that a necessary condition for stability, $T \leq 0$, together with $(Q_t - Q) \leq 0$, implies $T (Q_t - Q) = Q_{t+1} - Q \geq 0$. The conclusion that the inventory cycle will be of two production time periods in length holds, of course, only for a particular if common type of disturbance. Nevertheless, the possibility of an inventory cycle of such curious form demonstrates that immediate adjustment behavior can give rise to cycles of an entirely different type from those that plague the American economy.

B. Static Expectations with Delayed Adjustment:

A first step in demonstrating that delayed adjustment behavior may contribute to stability is provided by the following theorem, proved in the Appendix, concerning necessary conditions for stability.

THEOREM II:

If $T = I - [I-'I+BD+D)A]^{-1}D(I-A)$,

where $I \geq) = \text{diag}(d_i) \geq 0$, $B = \text{diag}(b_i) \geq 0$, and $A \geq 0$, $r(A) < 1$,

then $r(T) < 1$ implies:

(i) $D > 0$

(ii) $r(K) < 1$, where $K \equiv (B+I)DA(I-A)^{-1}$

(iii) $\lambda^+ < \dfrac{2 - \min(d_i)}{2 - \min(d_i) + 2\min(b_i d_i + d_i)}$.

Of course, even if these conditions are satisfied, the economy may still be unstable.

If empirical evidence were to reveal that any of these three conditions were not satisfied. we could conclude that the system is unstable. Estimates derived from time series data must be utilized in calculating the upper bound on λ^+ compatible with stability as appropriate cross section data on inventory holdings is not currently available. In my empirical study of manufacturing inventory investment in the United States covering the period 1948-55 [27], estimates were derived of the marginal desired inventory and reaction coefficients for finished goods inventory held by both durable and by non-durable firms. These point estimates yield an upper bound on λ^+ of 0.93, a figure safely above the estimated λ^+ of 0.55. A second set of coefficients is provided by data including purchased material

and goods in process as well as finished goods inventory but broken

down into five durable goods industries; examination of these estimates

yielded an upper bound of 0.90 on λ^+ .[*]   These calculations neglect

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[*]The data appear in Tables II and III of [27]. For finished goods
inventory, the non-durable sector yields $\min(d_i)$ = .0649 and $\min(b_i d_i + d_i)$
= .0707. For the total inventory analysis values of .0554 and .1092
respectively are provided by primary metals and non-durables.

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wholesale and retail trade inventories. Since the point estimates

cannot be regarded as precise, we cannot be certain that the conditions

of Theorem II are satisfied.[**]   The evidence thus proves inconclusive;

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[**]The finished goods non-durable estimate of d is not significantly
different from zero at the .05 level, suggesting that there is reason to
suspect that condition (i) of Theorem II may not be satisfied.

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all that can be said is that no basis is provided for concluding that

the economy is unstable by considering the available data in conjuncture

with Theorem II.

The next theorem specifies conditions assuring that T has no

real roots greater than unity in absolute value.

THEOREM III:

If $t_i = \alpha_i + \beta_i \sqrt{-1}$ denotes a characteristic root of T, then

$$\lambda^+ < \frac{2 - \max(d_i)}{2 - \max(d_i) + 2\max(b_i d_i + d_i)} \quad \text{implies}$$

$$-1 < \alpha_i < 1 \; ,$$

provided $I \geq D > 0$ and $B \geq 0$.

If the conditions of Theorem III are satisfied, T might still be unstable, but it would have to be an instability dominated by an explosive cycle.

An inspection of empirical estimates of the parameters of the model obtained with the finished goods inventory data yields a bound on λ^+ of 0.80; for the inventory data including purchased materials and goods in process the bound is 0.68.[*] Since both these figures are well above

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[*]Durables provided $\max(d_i) = .1829$ and $\max(b_i d_i + d_i) = .2379$ for finished goods inventory. Transportation equipment data yielded maximum coefficients of 0.3160 and .3987 respectively.

$\lambda^+ \doteq 0.55$, the calculations suggest that the transitions matrix of the buffer-stock inventory model cannot have a real root greater than unity in absolute value. While the system might be unstable, it would have to be dominated by an explosive cycle.

It is difficult to specify conditions assuring that explosive
oscillations will not take place. One procedure, not too pleasing
empirically, is to invoke a stronger uniformity assumption. Earlier
in the analysis when the problem of aggregating from the firm to the
industry level was encountered, it was assumed that all firms in a given
industry have the same marginal desired inventory coefficient. In order
to specify conditions assuring stability of the multi-sector model the
strong uniformity assumption that there are no interindustry differences
in the reaction and marginal desired inventory coefficients proves
convenient.[*]

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[*]While this is not too happy an assumption, it is a mistake to
conclude that it completely circumvents the problem of aggregation. The
strong uniformity assumption does not serve to collapse the multi-sector
model into the single equation form derived under the traditional
procedure or macro analysis. Roy Harrod is mistaken in arguing [25, p. 282]:
"... it is necessary to frame the concept of an entrepreneur, who is
representative in two respects, namely: (i) demand for his output must expand
at the same rate that the economy as a whole is expanding; and (ii) he must
be psychologically representative, in the sense that his reaction to recent ex-
perience is an average one. He must be average in his make-up of courage
and prudence, of optimism and pessimism...The formula that correctly describes
the behavior of this representative entrepreneur may be applied to the
macro-economy." While this uniformity assumption might be invoked to
justify the use of aggregative data to estimate the parameters of the buffer
stock inventory equation, it must be observed that under no circumstances
does it suffice to justify the application of the iterative predictive
procedure mentioned in Part II, Section B to aggregative empirical models
of the economy.

The strong uniformity assumption means that there exist scalars b
and d such that bI = B and dI=D. Under this assumption and with

static expectations, the transitions matrix reduces to

$$(3.1) \qquad T = [I-(1+bd+d)A]^{-1}[(1-d)I-(1+bd)A] \ .$$

With strong uniformity the following function maps the characteristic

roots λ_i of A into the roots t_i of the transitions matrix T.[*]

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[*]Since $PAP^{-1} = \Lambda = \text{diag}(\lambda_i)$, $PTP^{-1} = [I-(1+bd+d)\Lambda]^{-1}[1-d)I-(1+bd)\Lambda]$,
a diagonal matrix. This also implies that the characteristic vectors of A
and T are identical; if A is indecomposable, the strong uniformity
assumption implies that T can have but one characteristic vector with all
components of the same sign.

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$$(3.2) \qquad t(b,d,\lambda) = \frac{1-d-(1+bd)\lambda}{1-(1+bd+d)\lambda}$$

The strong uniformity assumption yields a necessary and sufficient

condition for stability.

THEOREM IV: If T is defined as in (1.11) with $B = bI \geq 0$

and $D = dI > 0$, then

$$r(T) < 1 \Longleftrightarrow \lambda^+ < \frac{2-d}{2+d+2bd} \ .$$

If the point estimates of $d = 0.152$ and $b = 0.276$ obtained

when finished goods inventory data aggregated over all manufacturing

industries are regarded as precise [27], Theorem IV implies that the

inventory cycle will be stable if and only if $\lambda^+ < .84$. While this

gives a considerable margin of safety over the estimated value of

λ^+ of 0.55, it neglects movements in purchased material and goods

in process stocks. If all inventories are assumed to behave in accordance

with the buffer stock model, the appropriate levels of d and b to

utilize in testing for stability are 0.430 and 0.389 respectively; if

these estimates are regarded as precise, stability is possible if and

only if $\lambda^+ < .57$. This bound is uncomfortably close to o.54, the

dominant root Max A. Woodbury calculated for a matrix of technological

flow coefficients, sans consumption. While the point estimates suggest

that the conditions for stability are satisfied, it is clear that a

slightly different configuration of measurement errors or the inclusion

of consumption might well have led to the opposite conclusion.[*]

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[*]Slight changes in λ^+ lead to large changes in t^+ ; for
$\lambda^+ = .54$, $d = .43$ and $b = .389$, yield $\dfrac{\partial t(d,b,\lambda)}{\partial \lambda} = -13$.

Under the strong uniformity assumption equation 3.2] may be utilized

in conjunction with estimates of d and b to transform the characteristic

roots of A into corresponding roots of T . The calculations of

Max A. Woodbury yielded the two complex roots $\lambda_2 = -.1969 \pm .0871$ i as well as

the dominant root of $\lambda_1^+ = .541$ already mentioned [36, p.381]. The correspond-

ing roots of T obtained by substituting these roots into equation (3.2)

together with the finished goods estimates of d and b are $t_1 = .803$

and $t_2 = .853 \pm .0016$i. When the estimates of d and b provided by

the regressions involving data on purchased materials and goods in process as well as finished goods are utilized in the calculations, the corresponding roots of T are $t_1 = -.459$ and $t_2 = .610 \overset{+}{-} 0.0128i$. As expected, both sets of roots imply a stable inventory cycle. It is interesting to observe that in both cases Woodbury's second root yields a root of T implying a cycle of long period, the transformation having reduced the imaginary portion of the root considerably. Notice too that although $|\lambda_1| > |\lambda_2|$, this ranking is reversed in the process of transforming the roots of A into roots of T ; in both cases $|t_1| < |t_2|$. It must be remembered that the dominating root of T is not necessarily obtained by transforming the larger roots of A . Since not all the roots of A are available, it is impossible to determine whether the roots of T reported above are the dominating ones.

These calculations have been presented only in order to illustrate the application of the theorems. Although the estimates of the characteristic roots of T may be suggestive, they can hardly be regarded as precise. More is involved than the errors of measurement inevitably encountered in estimating the parameters of a system. Although fluctuations in purchased material and goods in process as well as finished goods inventory were taken into account in deriving the estimates of the characteristic roots of T , the calculations were based on the assumption that both fixed investment and consumption spending were relegated to the final bill of goods. Another set of calculations involving a different level of

closure yielded a highly explosive characteristic root of 3.15![*] While

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[*]These calculations included consumption but excluded fixed invest-
ment and stocks of purchased materials and goods in process. The strong
uniformity assumption was not utilized. Immediate adjustment was assumed
for certain inventory holding sectors. Since performing the calculations
I have been advised that the unpublished stage-of-fabrication breakdown
of inventory data for individual industries utilized in estimating the
b_i and d_i are extremely unreliable and inappropriate for my purposes.
Details of the calculations are provided in [26, Ch. V].

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it can be claimed that for reasonable values of the parameters of the

system delayed adjustment buffer-stock inventory behavior may be stable,

the possibility of instability cannot be excluded. Even this weak con-

clusion is encouraging, however, for with the Metzler assumption of

immediate adjustment, instability was inevitable for reasonable values

of λ^+ .

The effects upon the stability of the economy of changes in the

parameters of the system may be evaluated, at least under the strong

uniformity assumption. The lower the marginal desired inventory coef-

ficient, other things being equal, the larger the dominant root of A

that is compatible with stability. The slower the speed of adjustment,

provided it remains positive, the more likely the economy is to be stable.

The effects of changes in technology upon the stability of the system can

also be evaluated. Suppose that the economy becomes more efficient as

a result of technological change that dominates in the sense that the

new input-output matrix $\tilde{A}$ has the property $0 \leq \tilde{A} \leq A$, where A

is the old input-output matrix; then $r(\tilde{A}) \leq r(A)$.[*] Clearly, an

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[*]If $\tilde{A}$ is indecomposable and $\tilde{A} \neq A$, then $r(\tilde{A}) < r(A)$.

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improvement in technology that dominates the old methods of production

does not contribute to instability.

C. Alternative Assumption Concerning Expectations:

Consideration of the stability properties of the buffer-stock

inventory model under the assumption that the volume of sales anticipated

by entrepreneurs are simply a naive projection of current levels revealed

that stability is incompatible with Metzler's assumption of immediate

adjustment behavior for reasonable values of the dominant root of the

input-output matrix. When the possibility of delayed adjustment was

admitted, stability was implied by estimates of the system's parameters

for the United States economy. Is stability a possibility even with

immediate adjustment behavior under alternative assumptions about

expectations? This question will now be considered.

Intuition might well suggest that instability is the consequence

of errors of judgment resulting from imperfect foresight. Furthermore,

my own empirical investigations of observed manufacturers' sales and

inventory behavior [27] suggest that manufacturers' expectations are

considerably more accurate than is implied by the assumption of naive

expectations. Analytical difficulties are presented by the most interesting

case in which expectations are assumed to lie between the value obtained by

a naive projection and actual development. Let us content ourselves with

exploring the consequences of the assumption that expectations are perfect

in the myopic sense that anticipations of sales volume for the next period

are precisely fulfilled.[*] An analysis of the implications of perfect

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[*]It might be objected that the assumption of perfect foresight
provides a meaningless context in which to explore the consequences of
buffer-stock behavior. In a similar vein, D. H. Robertson complained
that in the Keynesian analysis of the interest rate "the organ which
secretes it has been amputated, and yet it somehow still exists -- a
grin without a cat." [33, p. 36]. By assuming perfect foresight,
the very element of errors of judgment required to justify the existence
of buffer stock inventories has been eliminated. A sufficient defense
of the procedure is provided by observing that it is necessary to analyze
the consequences of perfect foresight in order to demonstrate that
instability is not simply the consequence of errors of expectations. The
problem is of further interest in that the implications of stochastic
disturbances may be analyzed within essentially the same framework, as
was pointed out above, Part II, Section D.

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foresight demonstrates that the unstable elements of the buffer-stock

model are not the simple consequence of errors of expectations.

In order to explore the implications of perfect foresight it is

necessary to substitute $\tilde{X}_t = X_t$ in the matrix version of equation (1.5),

the expression for output in terms of current anticipations and past

experience. In addition, the input-output relationship $X_t = AQ_{t+1} + Y$

may be utilized in order to reduce the system to

$$(3.3) \qquad Q_t = (I+BD)(AQ_{t+1}+Y) - (I+BD)(AQ_t+Y) + (I-D)Q_{t-1} + D(AQ_t+Y) .$$

The following theorem is proved in the Appendix.

THEOREM V:

The system of simultaneous difference equations (3.3) is unstable if either

(i) $D = I$ and $B \geq 0$, or

(ii) there exist scalars b and d such that

$I > D = dI > 0$ and $B = bI \geq 0$.

Perfect foresight implies instability with immediate adjustment behavior; even when the assumption of immediate adjustment is relaxed, the system remains unstable, at least under the strong uniformity assumption.

The case of perfect foresight is not the only alternative to the assumption of naive expectations. As a third case, suppose that the expectations of entrepreneurs are completely independent of current and past sales; for definiteness, let us assume that $\tilde{X}_t = X$ for all t , including $t-1$. Then it follows from the matrix version of equation (1.5) and (1.9)

$$(3.4) \qquad Q_t = (I-D)Q_{t-1} + DX_{t-1} = (I-D)Q_{t-1} + DAQ_t + Y .$$

The dynamic behavior of this system is completely independent of the marginal desired inventory coefficients. In the appendix we establish

THEOREM VI:

Difference equation system (3.4) is stable provided $I \geq D > 0$.

Consideration of two alternatives to the assumption of naive

expectations suggests that instability is not simply the consequence

of errors of foresight. With perfect anticipations, stability is

incompatible with immediate adjustment behavior. Even with delayed

adjustment, perfect foresight implies instability under the strong

uniformity assumption. The second alternative considered, the case

in which expectations are completely independent of actual experience,

provided an example of a system that is necessarily stable; while this

alternative is unrealistic,it does serve to indicate that the issue of

stability hinges in part upon the particular assumptions one chooses to

make about the nature of expectations.

PART IV: SUMMARY AND CONCLUSIONS:

The stability conditions derived in this paper for the buffer-

stock inventory model stand in marked contrast to the dynamic properties

of the Walrasian model of multiple competitive markets. Lloyd Metzler

has shown in a brilliant article that the stability of competition does

not depend upon the speed of adjustment if all commodities are gross

substitutes [30]. Theoretical investigations of the role of expectations

within the framework of a purely competitive environment by Enthoven,

Arrow, and Nerlove suggest that the stability of a competitive economy

in which all commodities are gross substitutes may be independent of errors

of expectations [4, 13]. In contrast, both errors in anticipating future

sales volume and speeds of adjustment have been shown in this paper to play

an important role in the determination of the dynamic properties of an inven-

tory holding economy involving price rigidity. Both of these complications

have to be conridered in evaluating policy measures advanced as means of

stabilizing the economy.

The magnitude of fluctuations in the inventory component of GNP is

widely recognized.[*] As a consequence, tax measures designed to stabilize

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[*]In his analysis of inventory behavior in the inter-war period,
Moses Abramovitz revealed that the inventory component of GNP was subject
to major fluctuations from peaks to troughs of the business cycle [1, ch. 21].

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inventory investment have been proposed. Moses Abramovitz suggested that

a tax on the average value of inventories, by inducing firms to operate with

lower stocks, would contribute to economic stability 2, pp. 293-4].

Albert G. Hart argues that "a tax at a substantial rate (25 per cent, say)

to be applied each quarter to the value of any increase or decrease in each

firm's inventory, compared with the same date a year previously," might

better contribute to the same objective, provided the scheme were feasible

administratively [19, pp. 452-3].

The multi-sector buffer-stock inventory model provides a theoretical

framework helpful in evaluating such policy issues. A tax on the size of

inventory holdings designed to reduce the average value of inventories might

well miss its objective of dampening cycles in economic activity

engendered by fluctuations in inventories. A reduction in the size of

inventories that entrepreneurs desire to hold at relevant levels of output

does not insure increased stability, for the crucial marginal desired

inventory coefficients might still be larger than before. While it might

be argued that the adoption of the alternative proposal, a tax levied each

quarter on the change in the value of inventories from the corresponding

quarter of the preceding year, would necessarily lower the marginal desired

inventory coefficient, this would by no means establish that such a tax

would contribute to the stability of the economy. Ceteris paribus, the

faster entrepreneurs attempt to adjust inventories to the desired level,

the less likely the economy is to be stable. A tax on inventory investment

might well reduce the size of the marginal desired inventory coefficients;

the possibility that it would raise the reaction coefficient by inducing

firms to attempt a tighter inventory policy must also be admitted.

Restraint is also called for in evaluating the implications of a

possible trend on the part of manufacturers toward a policy of closer

control on inventories during the post-war period in the United States.

Arthur F. Burns has suggested [7, p. 14]:

> There is...strong evidence that the businessman of our
> generation manage inventories better than did their pre-
> decessors... Success in economizing on inventories has
> tended to reduce the fluctuations of inventory investment
> relative to the scale of business operations and this in
> turn has helped to moderate the cyclical swings in
> production.

The argument of this paper demonstrates that the adoption of inventory practices that are more efficient from the point of view of the individual firm and lead to a reduction in average inventory levels does not necessarily contribute to stability.

The multi-sector model incorporates the economy's technological coefficients, a set of additional parameters suppressed in macro-business cycle analysis. A government policy of encouraging innovation and technological advance might be expected to contribute to stability under fairly general conditions for a change in technology that dominates serves to reduce the largest characteristic root of A . While it is conceivable that a policy of encouraging technological advance might be more effective than taxing inventory investment as a means of stabilizing the economy as well as encouraging growth, it is clear that much involved analysis will be required before it will be possible to determine precisely what policy measures will indeed contribute to a more stable economy.

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APPENDIX

Before presenting proofs of the theorems stated in the text it is convenient to establish the following Lemma. $\varphi(M)$ denotes the largest of the real parts of the characteristic roots of M, any square matrix.

LEMMA:

Let $Q \equiv - (I-A)[I-A-(B+I)DA]^{-1}D$, so that $(I-A)T(I-A)^{-1} = I+Q$, and let $t_i = \alpha_i + \beta_i \sqrt{-1}$ be any characteristic root of T.

Then

$$\left. \begin{array}{c} r(Q) < 2 \\ \\ \varphi(Q) < 0 \end{array} \right\} \Longleftrightarrow |\alpha_i| < 1 .$$

Furthermore,

$$\left. \begin{array}{c} r(Q) < 2 \\ \\ \varphi(Q) < 0 \\ \\ Q + I \le 0 \end{array} \right\} \Longrightarrow r(T) < 1$$

PROOF: The roots q_i of Q and t_i of T are related by the equation $q_i + 1 = t_i$ for Q+I is equivalent under a similarity transformation to T ; the first statement follows immediately from this equality. These two conditions in themselves do not suffice to assure stability as T might have a large imaginary root. The third condition, Q+I $\le$ 0 , assures that the dominant root of T is real [12], hence less than unity an absolute value.

The proofs of Theorems II and III will be considered before that
of Theorem I.

PROOF OF THEOREM II

Suppose that $r(t) < 1$.

Let $Q \equiv - (I-A)[I-A-(B+I)DA]^{-1}D$, so that $(I-A)T(I-A)^{-1} = I+Q$.

Now if some $d_i = 0$, $|D| = 0$. Hence $|Q| = 0$ and therefore $\varphi(Q) \geq 0$.

But this leads to a contradiction, for $r(T) = r(I+Q) < 1$ implies
$\varphi(Q) < 0$, proving (i).

To prove (ii) observe that $Q^{-1} = D^{-1}(K-I)$, where $K \geq 0$ as it is
the product of non-negative matrices. Since $D > 0$, all the
off-diagonal elements of Q^{-1} as well as $(K-I)$ are non-negative;
therefore Q^{-1} and $K-I$ both belong to a generalization of a class
of matrices considered by Metzler. $\varphi(Q) < 0 \Leftrightarrow \varphi(Q^{-1}) < 0 \Leftrightarrow \varphi(K-I) < 0$,
the last implication following from a theorem of Metzler [30] as
generalized by Enthoven and Arrow [13]; see also [4]. Now $\varphi(K-I) < 0$
is equivalent to $r(K) < 1$, which is (ii), for $K \geq 0$ implies that
its dominant root is real and positive.

To prove (iii) observe that $r(T) = r(Q+I) < 1$ implies $r(Q) < 2$.
By conditions (i) and (ii) we also have $D > 0$ and $(K-I)^{-1} \leq 0$;
hence $Q = (K-I)^{-1}D \leq 0$.

Now $Q \frac{1}{\min(d_i)} \leq (K-I)^{-1} \leq Q \frac{1}{\max(d_i)} \leq 0$. Application of a well
known theorem concerning non-negative matrices [12] together with

$r(Q) < 2$ yields:

$$\frac{2}{\min(d_i)} > \frac{r(Q)}{\min(d_i)} \geq r[(K-I)^{-1}] = \frac{1}{1-r(K)} \geq \frac{r(Q)}{\max(d_i)} \quad ,$$

or $\min(d_i) < 2-2r(K)$.

In addition, the inequality $\min(b_i d_i + d_i)A(I-A)^{-1} \leq (B+I)DA(I-A)^{-1} = K$

implies $\min(b_i d_i + d_i) \, r[A(I-A)^{-1}] = \min(b_i d_i + d_i) \left(\frac{\lambda^+}{1-\lambda^+} \right) \leq r(K)$.

Consequently, $\min(d_i) < 2 - 2r(K) \leq 2 - 2\min(b_i d_i + d_i) \left(\frac{\lambda^+}{1-\lambda^+} , \right)$ from which

condition (iii) follows immediately.

PROOF OF THEOREM III:

The restriction on λ^+ implies $\max(d_i) < 2 - 2\max(b_i d_i + d_i) \left(\frac{\lambda^+}{1-\lambda^+} \right)$.

Now $0 \leq K = (B+I)DA(I-A)^{-1} \leq \max(b_i d_i + d_i)A(I-A)^{-1}$; therefore the

restriction on λ^+ implies

$$r(K) \leq \max(b_i d_i + d_i) \left(\frac{\lambda^+}{1-\lambda^+} \right) < 1 \quad \text{and also} \quad 2[1-r(K)] > \max(d_i) \ .$$

Since it was demonstrated in establishing condition (ii) of Theorem II

that $r(K) < 1 \Longrightarrow \varphi(T-I) = \varphi(Q) < 0$, it only remains to establish

that $r(T-I) < 2$.

Observe that $r(K) < 1$ implies $(K-I)^{-1} \leq Q \frac{1}{\max(d_i)} \leq 0$, yielding

$$r[(K-I)^{-1}] = \frac{1}{1-r(K)} \geq \frac{r(Q)}{\max(d_i)} \quad , \text{ or } \max(d_i) \geq r(Q)[1-r(K)] \; .$$

This, together with the fact that $2[1-r(K)] > \max(d_i)$ yields

$r(Q) = r(T-I) < 2$, as required.

PROOF OF THEOREM I:

Since $D = I$, the first inequality of Theorem I is equivalent

to that of Theorem III. Consequently, the inequality implies via

the Lemma, that T has no root with that real part greater than

unity. In order to establish the first statement of Theorem I it

is only necessary to observe that $D = I$ implies that

$Q+I = (K-I)^{-1}D + I = I \; - D - KD - K^2D - \; \ldots \; \leq 0$, for $K \geq 0$ and

the convergence of the series following from the condition that

$r(K) < 1$. The second statement of Theorem I follows immediately

from Theorem II with $D = I$.

PROOF OF THEOREM IV:

Let $t(b,d,\lambda) = \frac{1 - d - \lambda - \frac{bd\lambda}{}}{1 - (1+bd+d)\lambda}$ be the function mapping the

roots of A into the corresponding roots of T under the assumption

that $D = I$.

Theorems II and III establish that $|t(b,d,\lambda)| < 1 \quad \lambda < \frac{2-d}{2+d+2bd}$

for _real_ λ . It is only necessary to demonstrate that $|t(b,d,\lambda^+)| < 1$

implies $|t(b,d,\lambda)| < 1$ where $\lambda = \gamma + \delta\sqrt{-1}$ is any other, possibly

complex root of A . From the definition of $t(b,d,\gamma)$ it follows that

$$|t(b,d,\lambda)| = \frac{|\ 1-d-\lambda-bd\ \lambda\ |}{|\ 1-(1+bd+d)\ \lambda\ |} = +\sqrt{\frac{[1-d-(1+bd)\gamma]^2 + (1+bd)^2\delta^2}{[1-(1+bd+d)\gamma]^2 + (1+bd+d)^2\delta^2}}$$

By Theorem III

$-1 < t(b,d,\gamma) < 1$, or $[1-d-(1+bd)\gamma]^2 < [1-(1+bd+d)\gamma]^2$. In addition, $d > 0$, so $(1+bd)^2 > (1+bd+d)^2$. This establishes that the ratio of the terms under the radical in the equation is less than unity, for both terms in the denominator of the ratio are larger than the corresponding terms of the numerator. Consequently, $t(b,d,\lambda^+) < 1$ implies $t(b,d,\lambda) < 1$.

PROOF OF THEOREM V

$D = I$ implies that the homogeneous form of (3.3) reduces to

$$Q_t = (I+B)AQ_{t+1} - (I+B)AQ_t + AQ_t \text{ or } (I-A)Q_t = (I+B)AQ_{t+1} - (I+B)AQ_t .$$

Premultiplying by $(I-A)^{-1}$ yields $Q_t = RQ_{t+1} - RQ_t$, where

$R \equiv (I-A)^{-1}(I+B)A \geq 0$.

Although stability of $Q_{t+1} = R^{-1}(I+R)Q_t$ requires $r[R^{-1}(I+R)] < 1$,

$R > 0$ implies that $r[R^{-1}(I+R)] = \dfrac{1 + r(R)}{r(R)} > 1$, establishing

instability for the case of immediate adjustment.

The strong uniformity assumption facilitates the analysis of the delayed adjustment case by permitting us to write the homogeneous form of (3.3) as

$$Q_{t+1} = \left\{ [(1+bd)A]^{-1}(1-da) + I \right\} Q_t + [(1+bd)A]^{-1} (d-1) Q_{t-1}$$

If we adopt a new definition of commodities in terms of the composite bundles of goods $Z_t = PQ_t$, this becomes

$$Z_{t+1} = \left\{ I + [(1+bd)\varLambda]^{-1}(1-d-\varLambda) \right\} Z_t + [(1+bd)\varLambda]^{-1}(d-1)Z_{t-1} .$$

But each of the matrices in this last equation is diagonal. Consequently, the new definition of commodities in terms of composite bundles of goods Z_t effectively separates variables so that there are now n independent second order difference equations of the form

$$(2.15) \quad z_i(t) + \gamma_{i1} z_i(t-1) + \gamma_{i2} z_i(t-2) = 0 \qquad \begin{aligned} &i = 1,2,\ldots, n \\ &t = 1,2,\ldots \end{aligned}$$

where

$$\gamma_{i1} = -1 - \frac{(-d\lambda_i)}{(1+bd)\lambda_i}$$

$$\gamma_{i2} = \frac{1-d}{(1+bd)\lambda_i}$$

Clearly, all n of these difference equations must be stable if (2.13) is to be stable. It will be shown that if the matrix of technological coefficients $A \geq 0$ is indecomposable and $r(A) < 1$, then at least one of these n difference equations is unstable. $\lambda_i = r(A) > 0$ is a root of A . Consequently, the corresponding difference equation of the set (2.15) has real coefficients. Samuelson has specified that a necessary condition for stability of a second order difference equation with real coefficients is that

$1 + \gamma_1^+ + \gamma_2^+ > 0$. [34, p. 456]. Applying this test to the equation corresponding to the largest root of A , however, reveals that $1 + \gamma_1^+ + \gamma_2^+ = d(\lambda^+ - 1)/[(1+bd)\lambda^+] < 0$, the numerator being negative and the denominator positive for b and d are positive and λ^+ is positive but less than unity. Perfect expectations, at least under the strong uniformity assumption, imply instability regardless of the speed of adjustment.

PROOF OF THEOREM VI:

The homogeneous form of equation (3.4) implies

$(I-DA)Q_t = (I-D)Q_{t-1}$ or $D(I-A)Q_t = -(I-D)Q_t + (I-D)Q_{t-1}$.

Consequently, $Q_t = (I+R)^{-1}R \, Q_{t-1}$ where $R \equiv (I-A)^{-1}(D^{-1}-I) > 0$.

Since $R > 0$, $r[(I+R)^{-1}R] = r(R)[1+r(R)]^{-1} < 1$, as was to be shown.