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Binary Choice Constraints on Random Utility Indicators

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BINARY-CHOICE CONSTRAINTS AND RANDOM UTILITY INDICATORS*

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1. The Problem.

This paper contains suggestions for combining various available types of information on economic choice. There are in the main three types:

1) Controlled experiments on choices from small sets (usually pairs) of alternatives;

2) Surveys of consumers' choices from large sets of alternatives, each set being determined by the market prices and by each consumer's monetary resources; similar surveys (*mutatis mutandis*) of producers' choices are also available, especially in the case of farmers;

3) Time series of total consumption (or production) of individual commodities, total incomes, and market prices, over a given geographical area.

Various researchers (in recent years notably Tobin) have worked on combining information on Types 2) and 3). The present paper deals with combining the materials 1) and 2), albeit only to the extent of outlining the general problem and giving a special example.

Suppose that, given a set of consumers' budgets at varying prices, we want to estimate a function on the commodity space that might be used as an utility indicator (a system of "indifference surfaces") describing the tastes of the sampled group of people. The desired statistical estimation will be usually preceded by the "specification"

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*) Dedicated to Ragnar Frisch, an old and esteemed friend, on the occasion of his 65th birthday. The paper was presented at the Stanford Symposium on Mathematical Methods in Social Sciences, June 1959, having been completed during a year's visit at the Carnegie Institute of Technology. The research was partly carried out under a contract of the Office of Naval Research with the Cowles Foundation for Research in Economics at Yale University. Previous collaboration with H. D. Block and discussions with Gerard Debreu, Morris DeGroot, and Duncan Luce were most useful.

of a class of eligible functions on the commodity space. The usual assumptions of economic theory—such as the convexity of indifference surfaces—permit such specification only within very broad limits.

Suppose, however, that, in addition to the consumers' survey data laboratory data are available which imply that the choices being made by people of a given socio-economic group obey certain constraints. The knowledge of these constraints might provide help in further restricting the set of eligible utility indicators.

It will be realistic to assume that a man's choice from a given set of alternatives is not unique but obeys some probability distribution. This agrees with the approach usual in psychometrics and psychophysics, and may also explain a part of the statistical variations in economic survey data. Accordingly, our approach will be "stochastic" throughout.

2. Binary-Choice Probabilities.

Unless specified to the contrary, all concepts in this paper are associated with a single given person, the "subject" of experiments. The only experiments considered in this paper—except for cursory remarks on other kinds of experiments, especially in Sections 4, 5, 11,—are binary choices: the subject is forced to choose one of the pair (x, y) of alternatives. On the other hand, the data of a consumer survey are of course multiple choices since a consumer's monetary resources permit him, given the prices of goods, to choose from among more than a pair of alternative budgets.

Definitions:

X : the set of alternatives, with generic elements x, y, ...

n : the number of elements of X when X is finite; in this case the elements will be identified by integers: $X = (1, \ldots, n)$.

$\underset{\sim}{p}$: the class of functions p on $\underset{\sim}{X} \times \underset{\sim}{X}$, such that

$$p_{xy} = 1 - p_{yx} \geq 0. \tag{2.1}$$

Hence $p_{xx} = 1/2$. When $x \neq y$, p_{xy} is called the probability of a binary choice, viz., of the choice of x out of the subset (x,y) of $\underset{\sim}{X}$. A region $\underset{\sim}{p}_C \subseteq \underset{\sim}{p}$ defines a condition (C) on all p_{xy} and is called a binary-choice constraint. It will be convenient to identify such constraints, and other conditions, by symbols in parentheses, thus: (C).

Binary-choice constraints involving no other quantities but the p_{xy} are called directly testable: it will be assumed that they can be accepted or rejected (in the sense of statistical inference) on the basis of binary-choice experiments.

Example: Let $\underset{\sim}{X} = (1,2,3)$. A function p is completely described by any six numbers p_{xy} obeying (2.1). The relation $1 \leq p_{12} + p_{23} + p_{31} \leq 2$ exemplifies a binary choice constraint (C); it is directly testable.

3. Random Utility Indicator.

Definition: $U^{(2)}$, a random function on $\underset{\sim}{X}$, is called a random utility indicator "in the binary sense" if the probability

$$Pr(U_x^{(2)} \geq U_y^{(2)}) = p_{xy}. \tag{3.1}$$

(The idea goes back to Thurstone's "law of comparative judgment": [ ]).

Theorem 1.1: Prob $(U_x^{(2)} = U_y^{(2)}) = 0$. Proof: Let Prob $(U_x^{(2)} = U_y^{(2)}) > 0$; then

$p_{xy} + p_{yx} =$ Prob $(U_x^{(2)} \geq U_y^{(2)})$ + Prob $(U_y^{(2)} \geq U_x^{(2)})$ = Prob $(U_x^{(2)} > U_y^{(2)})$ +

+ Prob $(U_y^{(2)} > U_x^{(2)})$ + 2 Prob $(U_x^{(2)} = U_y^{(2)})$ = $\{$Prob $(U_x^{(2)} > U_y^{(2)})$ +

+ Prob $(U_y^{(2)} > U_x^{(2)})$ + Prob $(U_x^{(2)} = U_y^{(2)})\}$ + Prob $(U_x^{(2)} = U_y^{(2)}) = 1 +$

+ Prob $(U_x^{(2)} = U_y^{(2)}) > 1$, contradicting (2.1). [The Theorem is due to DeGroot].

Examples: 1) Let $\underset{\sim}{X} = (1, 2, 3)$; then $U^{(2)}$ is a random utility indicator if the six numbers $\Pr(U_x^{(2)} > U_y^{(2)})$ are respectively equal to the six numbers p_{xy} which (or some constraints on which) are obtained from observation.

2) Let $\underset{\sim}{X}$ = real m-space with generic point $x = (x_1, \ldots, x_m)$; let $T = (T_1, \ldots, T_m)$ be a random vector with $0 < T_i < 1$, $i = 1, \ldots, m$; $\Sigma T_i = 1$. Let

$$U_x^{(2)} = \prod_{i=1}^{m} x_i^{T_i}, \tag{3.2}$$

(the "Cobb-Douglas function" frequently used in economics, with $\underset{\sim}{X}$ = commodity space). If $\text{Prob}\,(U_x^{(2)} \geq U_y^{(2)}) = p_{xy}$ then $U^{(2)}$ is a random utility indicator "in the binary sense". This example will be used again in Section 12.

Condition (U): The function p is such that $U^{(2)}$ exists.

4. A Remark on Multiple-Choice Probabilities.

The concepts and the condition just defined can be easily generalized to multiple-choice probabilities. For every $x \in \underset{\sim}{X}^*$, $\underset{\sim}{X}^* \subseteq \underset{\sim}{X}$, one defines, with Duncan Luce [ ], $p_x(\underset{\sim}{X}^*)$, obeying

$$p_x(\underset{\sim}{X}^*) \geq 0, \quad \sum_{x \in \underset{\sim}{X}^*} p_x(\underset{\sim}{X}^*) = 1. \tag{4.1}$$

The number $p_x(\underset{\sim}{X}^*)$ is interpreted as the probability that the subject, forced to choose from the subset $\underset{\sim}{X}^*$, choose x. Clearly p_{xy} is a special case, with $\underset{\sim}{X}^* = (x, y)$. The following condition exemplifies an important and plausible constraint that would be directly testable if multiple choices from subsets of various sizes were observable:

$$\text{if } x \in \underset{\sim}{X}^{**} \subseteq \underset{\sim}{X}^* \subseteq \underset{\sim}{X} \text{ then } p_x(\underset{\sim}{X}^*) \geq p_x(\underset{\sim}{X}^{**}). \tag{4.2}$$

In [1], Block and Marschak dealt with multiple-choice probabilities, and had accordingly used the following definition of the random utility indicator, stronger than the one given in Section 3:

Definition: U, a random function on X, is a random utility indicator if, for all $X^* \subseteq X$, the probability

$$\Pr(U_x \geq U_y, \text{ all } y \text{ in } X^*) = p_x(X^*). \tag{4.3}$$

Correspondingly, condition $(U^{(2)})$ is replaced by the stronger

Condition (U): The set of probabilities $p_x(X^*)$, all x in X^*, all $X^* \subseteq X$, is such that U, in the sense of (4.3), exists. Clearly condition (U) implies $(U^{(2)})$ and also implies (4.2).

Some other properties of U will be used later, in Section 11. However, because of experimental difficulties with large subsets of alternatives, it seems worthwhile to devote a special and intensive study to binary choice probabilities. They cover much of the existing materials on human responses, and many practical situations ("pairwise comparisons").

Accordingly, it is not assumed in this paper that constraints on non-binary choice probabilities--such as 4.2--can be inferred from observations.

5. Probability of Ranking.

Similarly, and possibly on still stronger grounds, no reference will be made in this paper to experiments in which the subject is requested, not to make an actual choice, but to rank three or more alternatives according to his preferences (with two alternatives, choosing and ranking may be regarded as identical). It will not be assumed that responses obtained in such experiments are in any sense consistent with choices of single alternatives from subsets. To be sure we shall use the term "probability of ranking"; but not to denote a number which is estimated (or, more generally, whose properties are inferred) from experiments on ranking. Rather, we give the following

Definition: If $U^{(2)}$ is a random utility indicator then, for any $x_1, x_2, \ldots, x_m$ in $\underset{\sim}{X}$, the number

(5.1) $$P(x_1 x_2 \ldots x_m) = \Pr(U^{(2)}_{x_1} > U^{(2)}_{x_2} > \ldots > U^{(2)}_{x_m})$$

is called the probability of the ranking (permutation) $x_1\, x_2 \ldots x_m$.
It follows that $P(xy) = p_{xy}$.

Theorem 5.1: If $\left(U^{(2)}\right)$ is satisfied and $\underset{\sim}{X}^* = \underset{\sim}{M} = (1, \ldots, m)$ is a finite subset of $\underset{\sim}{X}$, then for any $i, j \in \underset{\sim}{M}$,

(5.2) $$p_{ij} = \sum_{r \in \underset{\sim}{R}_{ij}} P(r),$$

where R_{ij} is the set of all rankings r on $\underset{\sim}{M}$ in which i precedes j.
Proof: by (3.1), (5.1)

Theorem 5.2: $\left(U^{(2)}\right)$ is satisfied if $\underset{\sim}{X}$ is finite, $\underset{\sim}{X} = \underset{\sim}{N} = (1, \ldots, n)$, and there is a non-negative function p on the set of rankings r on $\underset{\sim}{X}$ such that equation (5.2) holds; provided $\underset{\sim}{R}_{ij}$ is the set of all rankings r on $\underset{\sim}{X}$ in which i precedes j. Proof. Denote by $r = 1_r\, 2_r \ldots n_r$ the ranking in which the i-th place is occupied by i_r. Assign the value $-i$ to the random variable $U^{(2)}_{i_r}$. Then $P(1_r\, 2_r \ldots n_r) = \Pr(U^{(2)}_{1_r} > U^{(2)}_{2_r} > \ldots > U^{(2)}_{n_r})$, and by (5.2), condition (3.1) is satisfied.

6. A Necessary Condition for The Existence of a Random Utility Indicator.

Condition (Δ) ("triangular"):

(6.1) $$p_{xy} + p_{yz} \geq p_{xz}.$$

Because of (2.1), this can be rewritten more symmetrically:

(6.1a) $$1 \leq p_{xy} + p_{yz} + p_{zx} \leq 2.$$

Define the quantities Δ and Δ' by

(6.2) $$\Delta \equiv p_{xy} + p_{yz} + p_{zx} - 1; \quad \Delta' \equiv p_{yx} + p_{zy} + p_{xz} - 1;$$

then by (2.1), $\Delta + \Delta' = 1$; and (6.1a) can be rewritten as:

(6.1b) $$0 \leq \Delta \leq 1, \quad 0 \leq \Delta' \leq 1.$$

Condition (Δ) was probably first noticed by Guilbaud [ ].

Theorem 6.1: $\left(U^{(2)}\right)$ implies (Δ). [And therefore, clearly, also (U) implies (Δ)]. Proof: Denote by x, y, z the generic elements of the subset (1,2,3) of X. Then, by the definition (5.1), the six $P(xyz)$ add up to 1, and each is ≥ 0. If $\left(U^{(2)}\right)$ is true then by Theorem 5.1,

(6.3) $$\begin{aligned} p_{12} &= P(132) + P(123) + P(312) \\ p_{23} &= P(123) + P(213) + P(231) \\ p_{31} &= P(312) + P(231) + P(321). \end{aligned}$$

Assign the values of x, y, z so as to make (6.2) into

(6.4) $$\Delta = p_{12} + p_{23} + p_{31} - 1 - \Delta'.$$

In (6.3) the sum of the left sides is $1+\Delta$ by (6.2); on the right side, the three off-diagonal terms of the symmetric matrix add up to s, say, where $0 \leq s \leq 1$; and all 9 terms add up to $1+s$. Hence $s = \Delta$, and (6.1b) is satisfied.

Theorem 6.2. If $X = (1,2,3)$ then (Δ) implies $\left(U^{(2)}\right)$.

Proof. Denote by x, y, z the generic elements of X. We shall show that, for any six p_{xy} satisfying (2.1) and such that

$$0 \leq \Delta \equiv p_{12} + p_{23} + p_{31} - 1 \equiv 1 - \Delta' \leq 1,$$

one can find six numbers $P(xyz)$, non-negative, adding up to 1 and satisfying (6.3). We have seen in the proof of the preceding Theorem that (6.3) implies

(6.5) $$\Delta = P(123) + P(231) + P(312).$$

Subtracting (6.5) from each of the equations (6.3)

$$(6.6)\qquad \begin{aligned} p_{12} - \Delta &= P(132) - P(231) \\ p_{23} - \Delta &= P(213) - P(312) \\ p_{31} - \Delta &= P(321) - P(123). \end{aligned}$$

Put $0 = P(231)$ or $P(132)$ according as $p_{12} \geq$ or $\leq \Delta$; put $0 = P(312)$ or $P(213)$ according as $p_{23} \geq$ or $\leq \Delta$. Substituting into (6.6), (6.5), one obtains the remaining four $P(xyz)$, non-negative and adding up to 1. Hence, by Theorem 5.2, $\left(U^{(2)}\right)$ is satisfied.

Remark. For $n \geq 4$ no necessary and sufficient condition for $\left(U^{(2)}\right)$ is known to the author. Nor has it been proved that (Δ) is not sufficient for $U^{(2)}$. But it is clear that (Δ) is not sufficient for (U) for (Δ) does not put any constraints on non-binary choices while (U) does: we have seen that (U) implies (4.2). Some sufficient conditions for $\left(U^{(2)}\right)$ and (U), respectively, will be given in Sections 10 and 11.

7. Transitivity of Stochastic Preferences.

Definition: x is stochastically preferred (stochastically indifferent) to y if $p_{xy} > 1/2$ ($p_{xy} = 1/2$). In what follows, the words "transitivity of stochastic preferences" will be abbreviated to "transitivity".

Three Conditions: (t_w)(= weak transitivity); (t_m)(= mild transitivity); (t_s)(= strong transitivity)*):

$$\text{If } \min(p_{xy}, p_{yz}) \geq 1/2 \text{ then } p_{xz} \begin{cases} \geq 1/2 & : \text{Condition } (t_w); \\ \geq \min(p_{xy}, p_{yz}) & : \text{Condition } (t_m); \\ \geq \max(p_{xy}, p_{yz}) & : \text{Condition } (t_s). \end{cases}$$

Equivalently, by (2.1):

$$\text{If } \max(p_{xy}, p_{yz}) \leq 1/2 \text{ then } p_{xz} \begin{cases} \leq 1/2 & (t_w) \\ \leq \max(p_{xy}, p_{yz}) & (t_m) \\ \leq \min(p_{xy}, p_{yz}) & (t_s) \end{cases}$$

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*) (t_w) and (t_s) were formulated by S. Valavanis-Vail [ ] and originated in his work with C. Coombs. (t_m) was formulated by N. Georgescu-Roegen [ ] and by John Chipman [ ]. Experimental tests of stochastic transitivity conditions were undertaken by Papandreon et al [ ] and by Davidson and Marschak [ ].

A more symmetric equivalent form for (t_w) is, by (2.1):

$$\min(p_{xy}, p_{yz}, p_{zx}) \leq 1/2 \leq \max(p_{xy}, p_{yz}, p_{zx}).$$

Theorem 7.1: (t_s) is equivalent to the following condition:

If $p_{xy} \geq 1/2$ then $p_{xz} \geq p_{yz}$, all $z \in X$;

and also to the condition:

If $z \in X$ and $p_{xz} \geq p_{yz}$, then $p_{xy} \geq 1/2$.

Proof: see [1], Theorem IV.1.

Theorem 7.2: (t_s) is strictly stronger than (t_m), and (t_m) is strictly stronger than (t_w). Proof: obvious.

Theorem 7.3: (t_w) is neither sufficient nor necessary for (Δ). Proof: let $n = 3$, and $p_{13} = .6$. If $p_{12} = p_{23} = .9$, (t_w) is satisfied but (Δ) is not. If $p_{12} = p_{23} = .4$, (Δ) is satisfied but (t_w) is not.

Theorem 7.4. (t_m) is strictly stronger than (Δ). Proof: 1) Sufficiency: Consider $(1,2,3) \subseteq X$. Using the notation of (6.4), we have to prove that (t_m) implies $0 \leq \Delta \leq 1$, $0 \leq \Delta' \leq 1$. By (2.1), we can let $p_{12} \geq p_{23} \geq 1/2 \geq p_{32} \geq p_{21}$ without loss of generality; then $p_{12} + p_{23} \geq 1 \geq p_{32} + p_{21}$; $p_{12} + p_{23} + p_{31} - 1 = \Delta \geq 0$; $p_{21} + p_{32} + p_{13} - 1 = \Delta' \leq 1$. Now assume (t_m). Then $p_{31} \leq \max(p_{32}, p_{21}) = p_{32}$, $\Delta \leq p_{12} + p_{23} + p_{32} - 1 = p_{12} \leq 1$; and $p_{13} \geq \min(p_{12}, p_{23}) = p_{12}$, $\Delta' \geq p_{21} + p_{32} + p_{12} - 1 = p_{32} \geq 0$. 2) No necessity: by Theorems 7.2, 7.3.

Theorem 7.5. (t_m) is strictly stronger than the conjunction of (t_w) and (Δ). Proof: 1) Sufficiency: by Theorems 7.2, 7.4. 2) No necessity: let $X = (1,2,3)$, $p_{12} = .8$, $p_{23} = .7$, $p_{31} = .4$. Then (t_w) and (Δ) are satisfied. But since $p_{12} > p_{23} > .5$ yet $p_{13} = .6 < p_{23}$, (t_m) is violated.

Write "$\longrightarrow$" for "implies";

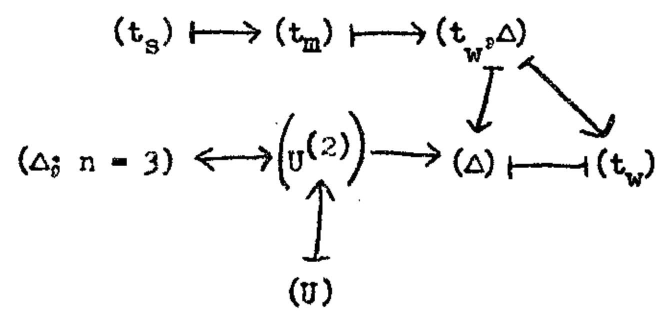
"$\longleftrightarrow$" for "implies and is implied by"

"$\longmapsto$" for "implies but is not implied by"

"$\vdash\!\!\!-\!\!\!\dashv$" for "does not imply nor is implied by".

Then our Theorems can be summarized in

Theorem 7.6;



8. Weak Utility Function.

Definition. w, a real valued function on X, is called a weak utility function if

$$w_x \geq w_y \text{ when and only when } p_{xy} \geq 1/2.$$

Condition (w): The function p is such that w exists.

It follows that w is unique up to an increasing monotone transformation.

Theorem 8.1 $(w) \longmapsto (t_w)$. Proof. 1) Sufficiency. Let $p_{ab} \geq 1/2$, $p_{bc} \geq 1/2$, $a, b, c \in X$. If (w) is true then $w_a \geq w_b \geq w_c$, $p_{ac} \geq 1/2$. 2) No necessity. Let X = real 2-space with generic points $x = (x_1, x_2)$ and assume that $p_{xy} > 1/2$ when either $x_1 > y_1$ or $x_1 = y_1$, $x_2 > y_1$ (lexicographical ordering). Compare Debreu [ ].

Condition (D): X is perfectly separable; and for every a in X, the sets $(x \;\; p_{xa} \geq 1/2)$ and $(x \;\; p_{xa} \leq 1/2)$ are closed. This is adapted from Debreu's paper just cited, which also contains, mutatis mutandis, the proof of

Theorem 8.2. If (D) is satisfied, (w) and (t_w) are equivalent.

Remark. Condition (D) is trivially satisfied on $\underset{\sim}{X}$ finite. Moreover, every subset of a finite-dimensional real space is separable; and it is reasonable to assume that Condition (D) is satisfied on a commodity space.

We summarize some of the previous Theorems in

Theorem 8.3.

$$(t_w;\ n \text{ finite}) \longmapsto (t_w, D) \longmapsto (w) \longmapsto (t_w) \leftrightarrow (t_w, \Delta) \longleftarrow\!\dashv (t_m) \longleftarrow\!\dashv (t_s)$$

$$(\Delta;\ n = 3) \longmapsto U^{(2)} \longrightarrow (\Delta) \longleftarrow (t_w, \Delta)$$

9. Strong Utility Function.

Definition. A real valued function v on $\underset{\sim}{X}$ is called a strong utility function if for any x, y, z, t in $\underset{\sim}{X}$ and a monotone function ϕ,

$$(9.1) \qquad \phi_x(v_x - v_y) = p_{xy}\ ;\ \phi_v(0) = 1/2\ .$$

Theorem 9.2. If $\underset{\sim}{X}$ is continuous and ϕ_v is strictly monotone then (1) v is unique up to an increasing linear transformation; (2) for any real number λ,

$$(9.3) \qquad \phi_v(\lambda) + \phi_v(-\lambda) = 1.$$

Proof: for (1) see [1], Section II; (2) is obvious.

(9.3) implies that ϕ_v is anti-symmetrical about $(0, 1/2)$; the median and the mean (if it exists) are $= 0$.

The strong utility v_x of alternative x corresponds to the "sensation" produced by a stimulus x, as defined by Fechner [ ] in 1859. It also corresponds to the relative position of a gene in the chromosome when derived from the probability of a "crossover": this generation of a real line from a set of probabilities was noticed by D. Hilbert.*)

*) In a paper [ ] pointed out to me by Hans Rademacher.

Condition (v). The function p is such that a function v exists.

Condition (q) (quadruple condition): For any x, y, z, t in $\underset{\sim}{X}$,

(9.4) if $p_{xy} \geq p_{zt}$ then $p_{xz} \geq p_{yt}$.

Theorem 9.3

(v) implies (q) [Proof: by (9.1)];

(q) implies (t_s) [Proof: in (9.4) put $z = t$ and use Theorem 7.1].

(v) implies (w) [Proof: in (9.1) put $z = t$].

Theorem 9.4: If $\underset{\sim}{X}$ is finite, $\underset{\sim}{X} = (1, \ldots, n)$, then (q) implies (v) only for $n \leq 4$. Proof: see [1], Theorem IV.1.

For $\underset{\sim}{X}$ continuous it is useful to consider the following

Condition (s.c)("stochastic continuity"). If $p_{xy} < q < p_{xz}$ then there is a t in $\underset{\sim}{X}$ such that $p_{xt} = q$. In [ ] Debreu has proved

Theorem 9.5. If Condition (s.c) is satisfied then (q) implies (v).

If both $\left(U^{(2)}\right)$ and (v) are satisfied we can define a random function V_x by

$$U_x^{(2)} = v_x + V_x, \tag{9.5}$$

and set up the following three conditions, each stronger than the preceding one:

There exist two functions on $\underset{\sim}{X}$, a real-valued v and a random-valued V, and a distribution function ϕ_V such that

$$p_{xy} = \phi_V (v_x - v_y),$$

$$\underline{Pr}\,(V_y - V_x \leq \lambda) = \phi_V(\lambda)$$

where

- $\lambda = v_x - v_y$: Condition (V)
- λ = any real number: Condition (V')
- $\underline{Pr}\,(V_x \leq \alpha, V_y \leq \beta) = \underline{Pr}\,(V_x \leq \beta, V_y \leq \alpha)$; λ, α, β = any real numbers: Condition (V'').

In Condition (V) the distribution of the differences $V_x - V_y$ is defined only for each pair of alternatives (possibly finite in number); in (V') this distribution is defined for any real argument; in (V''), in addition, the distribution of V is a symmetrical function. From (9.5) and Theorems 9.1, 9.2 one obtains

Theorem 9.5. $(V'') \longmapsto (V') \longmapsto (V) \longleftrightarrow (U^{(2)}, v)$ if $\underset{\sim}{X}$ is finite.

$(V'') \longmapsto (V') \longleftrightarrow (V) \longleftrightarrow (U^{(2)}; v)$ if $\underset{\sim}{X}$ is continuous.

In the work of Thurstone and his school a condition even stronger than (V'') is used: the distribution function of V is assumed to be normal and symmetric (zero-means, equal variances, equal covariances). Actually, a normal distribution of differences ϕ_v (as assumed by Fechner), is consistent with certain non-symmetric normal distributions of V itself.

For possible application to the commodity space some of our results can be summarized in

Theorem 9.6. If $\underset{\sim}{X}$ is continuous and the conditions (D) and (s.c) are satisfied then

$$\begin{array}{ccccccccccccc}
(V'') & \longmapsto & (V') & \longleftrightarrow & (U^{(2)}, v) & \longmapsto & (v) & \longleftrightarrow & (q) & \longmapsto & (t_m) & \longmapsto & (t_w) \\
 & & & & \downarrow & & & & & & \downarrow & & \updownarrow \\
 & & & & (U^{(2)}) & & \longrightarrow & & & & (\Delta) & \longmapsto\!\!\dashv & (w)
\end{array}$$

10. Strict Utility Function (in the "binary" sense).

Definition. $u^{(2)}$, a positive-valued function on $\underset{\sim}{X}$, is called a strict utility function (in the "binary" sense) if

$$p_{xy} = u_x^{(2)}/(u_x^{(2)} + u_y^{(2)}). \tag{10.1}$$

Clearly, every positive multiple of a strict utility function is also a strict utility function.

Condition $\left(u^{(2)}\right)$. *The function* p *is such that* $\left(u^{(2)}\right)$ *exists.*

Applying (10.1) one obtains

Theorem 10.1. $\left(u^{(2)}\right)$ *is true if and only if, for any subset* $(1,2,\ldots,m)$ X,

$$(10.2) \qquad p_{12} \cdot p_{23} \cdots p_{m-1,m} \cdot p_{m1} = p_{21} \cdot p_{32} \cdots p_{m,m-1} \cdot p_{1m} \, .$$

Note that, when put in the equivalent form (10.2), condition $\left(u^{(2)}\right)$ is directly testable. Condition $\left(u^{(2)}\right)$ was formulated by Tornqvist [ ], Bradley and Terry [ ], and L. Ford Jun. [ ]; it is a weaker form of a postulate that has been proposed by Luce [ ] and involves multiple-choice probabilities: see Section 11 below.

Theorem 10.2 $\left(u^{(2)}\right) \longmapsto (v)$. **Proof** (due to Luce): put $u_y^{(2)} = \exp v_y$; then by (10.1) $p_{xy} = \phi_v(v_x - v_y)$ where

$$(10.3) \qquad \phi_v(\lambda) = 1/(1 - \exp \lambda):$$

the "logistic curve". Since ϕ_v must have this particular form (to transform ratios v_x/v_y into differences), the condition $\left(u^{(2)}\right)$ is strictly stronger than (v).

Theorem 10.3: *If* X *is finite,* $\left(u^{(2)}\right)$ *implies* $\left(U^{(2)}\right)$. **Proof.** We shall use an arithematical identity proved in [1] (Theorem III.6) and [ ]: **Lemma.** *Let* $N = (1,\ldots,n)$ *and denote by* $r = (1_r 2_r \ldots n_r)$ *the permutation on* N *in which the* k-*th place is occupied by the element* k_r; *denote by* R_{iM} *the set of permutations on* N *in which* i *precedes all other elements of* $M \subseteq N$. *Then, for any positive* $u_1,\ldots,u_n$,

$$(10.4) \qquad \frac{u_i}{\sum_{h \in M} u_h} = \sum_{r \in R_{iM}} \prod_{j=1}^{n-1} u_{j_r} / \sum_{k=j}^{n} u_{k_r} .$$

To prove Theorem 10.3 from this Lemma, let $X = N$, $M = (i,j)$ so that by (10.1), the left side of (10.4) become p_{ij}, and the set $R_{iM} = R_{ij}$ as defined in Theorem 5.1. Since each of the products on the right side is positive and they add up to p_{ij}, we can equate each of them to a probability of ranking so that, for any ranking r on $\underset{\sim}{N}$,

$$(10.5) \qquad P(r) = P(1_r 2_r \ldots n_r) = \frac{u_{1_r}^{(2)}}{u_{1_r} + \ldots + u_{n_r}} \cdot \frac{u_{2_r}^{(2)}}{u_{2_r} + \ldots + u_{n_r}} \cdots \frac{u_{(n-1)_r}}{u_{(n-1)_r} + u_{n_r}} .$$

If we now interpret each u_i as $u_i^{(2)}$ then, by Theorem 5.2, $\left(U^{(2)}\right)$ is satisfied.

If $\underset{\sim}{X}$ is continuous, the reasoning just used can be applied to any finite subset of $\underset{\sim}{X}$. Hence the following

Conjecture: *Theorem* 10.3 *applies to any* $\underset{\sim}{X}$.

Theorem 10.4: $\left(U^{(2)}\right)$ does not imply $\left(u^{(2)}\right)$. *Proof*: by Theorems 9.6 and 10.2.

We can summarize the results of this and the preceding section:

Theorem 10.5:

$$\begin{array}{ccccccccccc} \left(u^{(2)}\right) & \longmapsto & (v) & \longmapsto & (q) & \longmapsto & (t_s) & \longmapsto & (t_w) & \longleftrightarrow & (w) \\ \downarrow & & & & \downarrow & & & & & & \\ \left(U^{(2)}\right) & & \longrightarrow & & (\Delta) & . & & & & & \end{array}$$

Remark on the commodity space. If Condition (D) is satisfied on the continuous commodity space and if the Conjecture above is correct, Theorem (10.5) applies also to the commodity space; if, in addition, (s.c) is satisfied, then $(v) \longleftrightarrow (q)$.

11. *Some Results on Multiple-Choice Probabilities.*

In Section 4, the condition (U) was stated. Clearly (U) implies $\left(U^{(2)}\right)$. Similarly, $\left(u^{(2)}\right)$ can be replaced by the following stronger

Condition (u): *There exists a positive value function* u *on* $\underset{\sim}{X}$ *such that for any finite subset* $\underset{\sim}{M} = (x_1, \ldots, x_m) \subseteq \underset{\sim}{X}$, *the multiple-choice probability*

(11.1) $$p_{x_i}(\underset{\sim}{M}) = u_i / \sum_{j \in \underset{\sim}{M}} u_j \,.$$

The function u may be called strict utility function in the multiple-choice sense.

As was proved in [1], (u) implies that the probability of each ranking, $P(r)$ is precisely the expression on the right-hand side of (10.5). E.g., if $\underset{\sim}{X} = (1,2,3)$, we have (letting $u_1 + u_2 + u_3 = 1$ without loss of generality),

(11.2) $$P(xyz) = \frac{u_x \cdot u_y}{u_y + u_z} \,.$$

Applying the Lemma of the previous Section one obtains, using (11.1):

Theorem 11.1 (u) $\longrightarrow$ (U).

This result is discussed in more detail in [1]. Luce [ ] proposed a condition still stronger than (u), being a conjunction of (u) and the following

Condition (u'). Denote by $p'_x(\underset{\sim}{X}^*)$ the probability that x is the last choice out of the subset $\underset{\sim}{X}^* \subseteq \underset{\sim}{X}$. There exists a real valued function u' or $\underset{\sim}{X}$ such that for any finite subset $\underset{\sim}{M} = (x_1, \ldots, x_m) \subseteq \underset{\sim}{X}$, the probability

$$p'_{x_i}(\underset{\sim}{M}) = u'_i / \sum_{j \in \underset{\sim}{M}} u'_j \,.$$

u' may be called strict disutility function.

Theorem 11.2. If both (u) and (u') are true then (U) is in general not true. Proof: (for any finite $\underset{\sim}{X}$) was given in [1].

As an illustration let $\tilde{X} = (1,2,3)$. Then (u) implies (11.2); and (u') implies, by the same reasoning,

$$P(zyx) = \frac{u'_x \cdot u'_y}{u'_y + u'_z}$$

where $\sum_{x=1}^{3} u'_x = 1$. Interchanging x and z and using (11.2), one obtains a set of equations of the form

$$\frac{u'_x + u'_y}{u'_y \cdot u'_x} = \frac{u_x \cdot u_y}{u_y + u_z} ;$$

these, together with the normalizing condition $\sum_{x=1}^{3} u'_x = \sum_{x=1}^{3} u_x = 1$, yield

$$u_x = u'_x = 1/3 , \quad x = 1,2,3 .$$

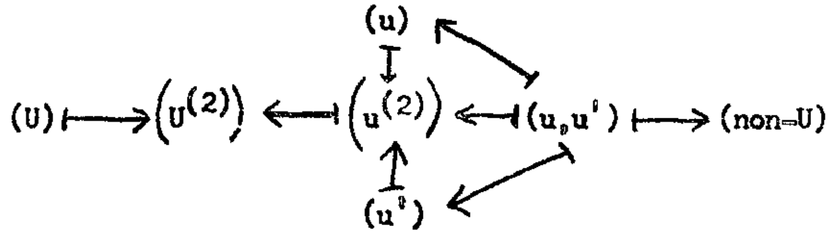
Thus, unless all three first-choice probabilities and all three last-choice probability are equal, the conjunction (u, u') is incompatible with the existence of a random utility indicator.

It has been pointed out by Luce (in private correspondence) that, while in experiments on choices the probabilities of last choices may or may not be accessible to observations, they definitely are in experiments or perceptions: the subject's decision as to which is the loudest of three sounds is as observable as his decision as to which sound is the softest. If one accepts (u), symmetry requires to accept also (u'). But then a random utility (or sensation) indicator U in the multiple-choice sense cannot exist.

Alternatively, one can limit oneself to the weaker condition $\left(u^{(2)}\right)$. It involves binary choices only; so that, when identifying the loudest sound, the subject identifies also the softest. That is, if the subset $\underset{\sim}{X}^*$ consists of two elements, then Conditions $\left(u^{(2)}\right)$, (u), and (u') are identical.

A strict utility function in this limited sense — $\left(u^{(2)}\right)$ rather than (u) — is compatible with the existence of a random utility indicator $U^{(2)}$ involving binary choices only.

As a summary of this Section, we have, for $\underset{\sim}{X}$ finite, Theorem 11.3.



12. Binary-Choice Constraints and the Form of the Random Utility Indicator.

Consider Example 2) of Section 3, and let $m=2$, $0 < T_1 = A < 1$, $T_2 = 1 - A$. Then

$$U_x^{(2)} = x_1^A x_2^{1-A} ; \tag{12.1}$$

the function $U^{(2)}$ is monotone increasing in x in the sense that if $x=(x_1,x_2)$, $y=(y_1,y_2)$, $x_1 \geq y_1$, $x_2 > y_2$ then $U_x^{(2)} > U_y^{(2)}$. $U^{(2)}$ is a random function depending on the random parameter A.*) If $U^{(2)}$ is a utility indicator in the "binary" sense of (3.1) then $p_{xy} = \Pr(U_x^{(2)} \geq U_y^{(2)}) = \Pr(y_1^A y_2^{1-A} \leq x_1^A x_2^{1-A})$. Let $y_1 > x_1$, $y_2 < x_2$ (so that x does not dominate y, nor conversely). Then $\log(y_1/x_1) > 0$, $\log(x_2/y_2) > 0$, and

$$p_{xy} = \Pr\left(\frac{A}{1-A} \leq \frac{\log(x_2/y_2)}{\log(y_1/x_1)}\right) = F(\mu_{xy}), \text{ where}$$

$$\mu_{xy} = \log(x_2/y_2)/\log(y_1/x_1),$$

and F is a monotone non-decreasing function depending on the distribution of A. We shall now show that the form (12.1) of the random utility indicator is not consistent with some binary-choice constraints:

Theorem 12.1. *If the random utility indicator has the form (12.1) then the strong transitivity condition (t_s) is not satisfied.* **Proof.** Consider three points x,y,z with $z_1 > y_1 > x_1$, $z_2 < y_2 < x_2$. Then

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*) Maximizing $U_x^{(2)}$ with respect to x subject to the constancy of income $x_1 + x_2$, we see that $A/(1-A)$ is the (random) proportion spent on the 1-st commodity. Hence the observed frequency distribution of this proportion should help to estimate the distribution of A--assuming that members of the sampled population were characterized by the same random function $U^{(2)}$. (But we shall not purseu this matter here).

$$p_{xy} = F(\mu_{xy}) , \qquad \mu_{xy} = (\log x_2 - \log y_2)/(\log y_1 - \log x_1)$$

$$p_{yz} = F(\mu_{yz}) , \qquad \mu_{yz} = (\log y_2 - \log z_2)/(\log z_1 - \log y_1)$$

$$p_{xz} = F(\mu_{xz}) , \qquad \mu_{xz} = (\log x_2 - \log z_2)/(\log z_1 - \log x_1).$$

$$\mu_{xz} = \frac{\mu_{xy}(\log y_1 - \log x_1) + \mu_{yz}(\log z_1 - \log y_1)}{(\log y_1 - \log x_1) + (\log z_1 - \log y_1)} ,$$

a convex combination of μ_{xy} and μ_{yz}. Hence

$$\min (\mu_{xy}, \mu_{yz}) \leq \mu_{xz} \leq \max (\mu_{xy}, \mu_{yz}) ,$$

and since F is non-decreasing,

$$\min (p_{xy}, p_{yz}) \leq p_{xz} \leq \max (p_{xy}, p_{yz}).$$

The strong transitivity condition (t_s), Section 7, contradicts the second inequality if p_{xy} and p_{yz} are both $> 1/2$; (t_s) contradicts the first inequality if p_{xy} and p_{yz} are both $< 1/2$.

Thus (12.1) is inconsistent with the existence of strict or strong utility functions, since those imply strong transitivity: Theorems 9.3 and 10.5.

Consider now another random form of the "Cobb Douglas function":

(12.2) $$U_x^{(2)} = x_1^{\alpha} x_2^{1-\alpha} + V_x ,$$

where α is a constant, $0 < \alpha < 1$, and V is a random function on $\underset{\sim}{X}$. If $U^{(2)}$ is a random utility indicator then

$$p_{xy} = Pr(U_x^{(2)} \geq U_y^{(2)}) = Pr(V_y - V_x \leq y_1^{\alpha} y_2^{1-\alpha} - x_1^{\alpha} x_2^{1-\alpha}).$$

Suppose we have found V such that, for any real λ, $Pr(V_y - V_x \leq \lambda) = \phi(\lambda)$, ϕ being a distribution function independent of x, y. Then condition (V') of Section 9 is satisfied, and

(12.3) $$U_x^{(2)} - V_x = v_x = x_1^{\alpha} x_2^{1-\alpha}$$

is a strong utility function consistent with the assumed random utility indicator.*)

If we could accept the still stronger condition (V), i.e., let the (infinite-dimensional) distribution of V be symmetrical in its arguments $(x,y,\ldots)$, one would have to search for such distributions. However, the symmetry assumption is hardly realistic: for example, in the case of normality it implies that the deviations from the average budget composition have not only the same variances everywhere, but also have the same correlations for any two points of the $\underset{\sim}{X}$ - plane, however close or remote in relation to each other.

Given a distribution ϕ of differences $V_x - V_y$ (possibly observed from experiments), a random function V may not exist. For example, consider $\underset{\sim}{X}$ finite, $\underset{\sim}{X} = (1,2,3)$, and let

(12.4) $\text{Prob}\ (V_i - V_j = 1) = 1/2 = (V_i - V_j = -1),\quad i,j = 1,2,3.$

Then it is impossible to find a random vector (V_1, V_2, V_3): for if $V_1 = c,\ V_2 = c - 1,\ V_3 = 1 - c,$ then $V_1 - V_2 = 1 = V_3 - V_1$; $V_1 - V_3 = -1 = V_2 - V_1$; but $V_1 - V_3 = c$, a value not foreseen in (12.4).

The examples used in this Section have merely the purpose of illustrating the problem: find a class of random utility indicators consistent with a given set of binary-choice constraints.

*) One may want to use indifference lines to picture some relevant aspect of a given random utility indicator, and in particular, its mean. In the case just discussed, described by (12.2), (12.3), we may choose V to have zero expectation; then $EU_x^{(2)} = v_x$. Accordingly, an indifference line through x consists of all points y with $v_y = v_x$, $p_{yx} = \phi_v(0) = 1/2$ as in (9.2); it is the common boundary of the two sets used in condition (D), Section 8. On the other hand, if the random utility indicator is described by (12.1), equating $EU_y^{(2)} = EU_x^{(2)}$, or possibly $E \log U_y^{(2)} = E \log U_x^{(2)}$, gives quite a different result.

REFERENCES.

1 Block, H. D. and J. Marschak. "Random orderings and stochastic theories of responses." Cowles Foundation Discussion Paper No. 66 (1959) (Mimeogr.) to be published in the volume of statistical articles in honor of Harold Hotelling.

2 Block, H. D. and J. Marschak. "An identity in arithmetic". Bull. Am. Math. Soc., 65 (1959), pp.

3 Bradley, R. A. and M. E. Terry. "Rank analysis of incomplete block designs. I. The method of paired comparisons". Biometrika 39 (1952), 324-344

4 Chipman, J. S. "Stochastic choice and subjective probability". (Abstract). Econometrica 26 (1958), 613

5 Davidson, D. and J. Marschak. "Experimental tests of stochastic decision theories". Measurement: Definition and Theories, C. W. Churchman, Ed., New York, Wiley (in press)

6 Debreu, G. "Representation of a preference ordering by a numerical function". Decision Processes, Thrall, Coombs and Davis, eds., pp. 159-166, New York, Wiley, 1954

7 Debreu, G. "Stochastic choice and cardinal utility". Econometrica 26 (1958)

8 Debreu, G. On "An identity in arithmetic". Cowles Foundation Discussion, Paper No. 72 (1959).

9 Fechner, G. Th. Elemente der Psychophysik, 1859 (see especially pp. 70-103 of the 1889 edition).

10 Ford, L. R. "Solution of a ranking problem from binary comparisons." Amer. Mathem. Monthly Vol. 64 (1957), No. 8 (supplement), 28-33

11 Georgescu-Roegen, N. "Threshold in choice and the theory of demand". Econometrica 26, (1958), 157-168

12 Guilbaud, G. "Sur une difficulté de la théorie du risque." Colloques Internationaux du Centre National de la Recherche Scientifique 40 Econometrie (1953), pp. 19-25

13 Hilbert, D. Gesammelte Abhandlungen, 3, 378-387

14 Luce, R. D. Individual choice behavior (in press)

15 Papandreou, A. G. (with the collaboration of O. H. Sauerlender, O. H. Brownlee, L. Hurwicz, W. Franklin) "A test of a stochastic theory of choice", University of California Publications in Economics 16, 1-18, Berkeley, 1957

16 Thurstone, L. L. "A law of comparative judgment," Psychological Review 34 (1927), 273-286

17 Tornqvist, L. "A model for stochastic decision making", Cowles Commission Discussion Paper, Economics 2100, 1954 (dittoed)

18 Valavanis-Vail, S. "A stochastic model for utilities" Seminar on the application of mathematics to social sciences. University of Michigan, 1957 (Mimeogr.)