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### On 'An Identity in Arithmetic'

Gerard Debreu

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On "An Identity in Arithmetic"\*

Gerard Debreu

April 29, 1959

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On "An Identity in Arithmetic"\*

H. D. Block and J. Marschak have presented in the Bulletin of the American Mathematical Society, 65, an identity which arose in a probability context. This note proves it by a probability theoretical argument.

Consider an experiment having the set of possible outcomes  $N = \{1, \dots, n\}$  with positive probabilities  $(u_1, \dots, u_n)$  and an infinite sequence of independent repetitions of that experiment. Let  $M$  be a subset of  $N$  and  $i$  be an element of  $M$ , and denote by  $A(i, M)$  the event that  $i$  is the first element of  $M$  which occurs in an infinite sequence of outcomes. If  $B_j(i, M)$  denotes the event that the first  $(j-1)$ st outcomes are not in  $M$  and the jth outcome is  $i$ , one has

$$A(i, M) = \bigcup_{j=1}^{\infty} B_j(i, M) .$$

Hence  $\Pr[A(i, M)] = \sum_{j=1}^{\infty} \Pr[B_j(i, M)] = \sum_{j=1}^{\infty} (1-u_M)^{j-1} u_i = \frac{u_i}{u_M}$ , writing  $u_M$

for  $\sum_{j \in M} u_j$ , and putting  $0^0 = 1$  for the degenerate case  $M = N$ .

Let now  $r$  be a permutation of  $N$  and  $k_r$  be the element of  $N$  ranked kth by  $r$ . Let also  $C(r)$  be the event that the first occurrences of the  $n$  elements of  $N$  in an infinite sequence of outcomes appear in the order  $r = (1_r, \dots, n_r)$ . If  $D_{j_1, \dots, j_n}(r)$  denotes the event that  $k_r$  occurs for

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the first time at the  $j_k$ th performance of the experiment (for every  $k = 1, \dots, n$ ), one has

$$C(r) = \bigcup_{j_1 < \dots < j_n} D_{j_1, \dots, j_n}^{(r)}.$$

$$\begin{aligned} \text{Hence } \Pr[C(r)] &= \sum_{j_1 < \dots < j_n} \Pr[D_{j_1, \dots, j_n}^{(r)}] = \\ &= \sum_{j_1 < \dots < j_n} u_{1_r}^{j_2 - j_1} u_{2_r}^{(u_{1_r} + u_{2_r})^{j_3 - j_2 - 1}} u_{3_r}^{(u_{1_r} + u_{2_r} + u_{3_r})^{j_4 - j_3 - 1}} \dots u_{n_r}. \end{aligned}$$

Putting  $j_{k+1} - j_k - 1 = h_k$  and  $u_{1_r} + \dots + u_{k_r} = v_{k,r}$ , one obtains

$$\Pr[C(r)] = \binom{n}{\prod_{j=1}^n u_j} h_1, \dots, h_{n-1} \geq 0 \quad v_{1,r}^{h_1} v_{2,r}^{h_2} \dots v_{n-1,r}^{h_{n-1}} = \binom{n}{\prod_{j=1}^n u_j} \prod_{k=1}^{n-1} (1 - v_{k,r})^{-1}.$$

Let finally  $R(i, M)$  be the set of permutations of  $N$  for which  $i$  is ranked first among the elements of  $M$ . Since the event that some element of  $N$  never occurs has probability zero, one has

$$\Pr[A(i, M)] = \sum_{r \in R(i, M)} \Pr[C(r)].$$

Hence the desired identity:

$$\frac{u_i}{\sum_{j \in M} u_j} = \binom{n}{\prod_{j=1}^n u_j} \sum_{r \in R(i, M)} \prod_{k=1}^n \binom{n}{\sum_{j=k}^n u_{j_r}}^{-1}.$$