University of Nebraska at Omaha

DigitalCommons@UNO

Mathematics Faculty Publications

Department of Mathematics

2017

Nonminimal Cyclic Invariant Subspaces of Hyperbolic Composition Operators

Valentin Matache

Follow this and additional works at: https://digitalcommons.unomaha.edu/mathfacpub

Part of the Mathematics Commons

Nonminimal Cyclic Invariant Subspaces
of Hyperbolic Composition Operators

Valentin Matache

ABSTRACT. Operators on function spaces acting by composition to the right with a fixed self-map φ are called composition operators. We denote them C_φ. Given φ, a hyperbolic disc automorphism, the composition operator C_φ on the Hilbert Hardy space H^2 is considered. The bilateral cyclic invariant subspaces K_f, $f \in H^2$, of C_φ are studied, given their connection with the invariant subspace problem, which is still open for Hilbert space operators. We prove that nonconstant inner functions u induce non–minimal cyclic subspaces K_u if they have unimodular, orbital, cluster points. Other results about K_u when u is inner are obtained. If $f \in H^2 \setminus \{0\}$ has a bilateral orbit under C_φ, with Cesàro means satisfying certain boundedness conditions, we prove K_f is non–minimal invariant under C_φ. Other results proving the non–minimality of invariant subspaces of C_φ of type K_f when f is not an inner function are obtained as well.

1. Introduction

In this section, we set up the notation, introduce terminology, and report on the main results obtained in subsequent sections. We also give a brief survey of the results related to solving the invariant subspace problem by the study of the cyclic subspaces of a hyperbolic, automorphic, composition operator.

We call *operator* any bounded linear transformation of a Hilbert space into itself. By *invariant subspace* of an operator T, we mean a closed, linear manifold, left invariant by T. The collection $\operatorname{Lat} T$ of all invariant subspaces of T is a lattice (which is why we use notation $\operatorname{Lat} T$). If T is an operator, then the trivial elements of $\operatorname{Lat} T$ are the null subspace and the whole space.

The following is a famous unsolved problem called the *invariant subspace problem*.

PROBLEM 1. *Does any Hilbert space operator acting on a complex, separable, infinite dimensional space, always have nontrivial invariant subspaces?*

This problem has multiple equivalent reformulations. One of them is in terms of automorphic, hyperbolic, composition operators. To state it, denote by H^2 the

Hilbert Hardy space of all analytic functions on the unit disc $\mathbb{U}$ having square–summable Maclaurin coefficients. For the basics of Hardy space theory, the reader is referred to [**3**], [**8**], or [**19**]. It is known that any analytic self-map φ of $\mathbb{U}$ induces the necessarily bounded operator C_φ acting on H^2:

$$C_\varphi f = f \circ \varphi, \qquad f \in H^2,$$

called the *composition operator* with *symbol* φ. If the symbol is a *hyperbolic disc automorphism* (that is a conformal automorphism of $\mathbb{U}$ with 2 distinct fixed points situated on the unit circle $\mathbb{T} = \partial\mathbb{U}$), we call C_φ a *hyperbolic automorphic composition operator* or briefly, a *hyperbolic composition operator*.

Let C_φ be such a composition operator. In [**18**], the following theorem was proved.

THEOREM 1 ([**18**]). *The answer to* Problem 1 *is affirmative if and only if the only atoms contained by* $\mathrm{Lat}\, C_\varphi$ *are the 1–dimensional eigenspaces.*

By an *atom* of $\mathrm{Lat}\, T$ or *minimal invariant subspace* of T, one means any nonzero subspace $\mathcal{L} \in \mathrm{Lat}\, T$ so that the restriction $T|\mathcal{L}$ has trivial invariant subspace lattice, that is $\mathrm{Lat}(T|\mathcal{L}) = \{0, \mathcal{L}\}$.

Given T, a self-map of a set, and x an element of that set we call

$$O_T(x) := \{x, T(x), T(T(x)), T(T(T(x))), \dots\}$$

the *orbit* of x under T. If T is invertible, then

$$BO_T(x) := O_T(x) \cup O_{T^{-1}}(x)$$

is called the *bilateral orbit* of x under T, (or under T^{-1}). In [**10**], we made some obvious remarks on the atoms in $\mathrm{Lat}\, T$ when T is invertible (as is the operator in Theorem 1). More exactly, it is easy to prove that an atom of $\mathrm{Lat}\, T$ is doubly invariant, that is, it is invariant under both T and T^{-1}. Then, let us use the notation

$$K_x^+ = \bigvee_{n=0}^{\infty} T^n x \quad K_x^- = \bigvee_{n=0}^{\infty} T^{-n} x, \quad \text{and} \quad K_x = \bigvee_{n=-\infty}^{\infty} T^n x$$

for the closed subspace spanned by $O_T(x)$, $O_{T^{-1}}(x)$, respectively, $BO_T(x)$. If $\mathcal{L}$ is an atom of $\mathrm{Lat}\, T$, then

$$(1) \qquad\qquad \mathcal{L} = K_x^+ = K_x^- = K_x, \qquad x \in \mathcal{L} \setminus \{0\}.$$

In particular, if $\mathcal{L}$ is an atom of $\mathrm{Lat}\, T$, then

$$(2) \qquad \bigvee_{n \geq k} T^n x = \bigvee_{n \geq m} T^{-n} x = \mathcal{L}, \qquad x \in \mathcal{L} \setminus \{0\}, m, k \in \mathbb{Z}.$$

If $T = C_\varphi$ where φ is a hyperbolic automorphism, and $x = f \in H^2$, denote by $\mathcal{L}_f$ any of the spaces described in (1) or (2). This author raised in [**10**] the following :

PROBLEM 2. *Given* $f \in H^2 \setminus \{0\}$, *can one tell, by the properties of* f, *if* $\mathcal{L}_f$ *is an atom of* $\mathrm{Lat}\, C_\varphi$ *or not?*

Since the only known atoms are, so far, the 1–dimensional eigenspaces, this led to the characterization of eigenfunctions of C_φ as follows: the inner eigenfunctions were characterized in [**11**] (see also [**15**] for earlier partial characterizations) and

the outer eigenfunctions in [**6**], (see also an alternative point of view on inner eigenfunctions in [**6**]).

It should be recalled here that *inner functions* are bounded analytic functions whose radial limit functions are unimodular a.e. with respect to the arc–length measure. There are two basic kinds: *Blaschke products* and *singular inner functions*. Unimodular constants and products of finitely many disc automorphisms are called finite Blaschke products. The infinite Blaschke products are the functions of type

$$B(z) = \lambda z^p \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}, \qquad z \in \mathbb{U},$$

where $\{z_k\}$ is a sequence of nonzero numbers in $\mathbb{U}$ with property

$$(3) \qquad \sum_{k=1}^{\infty}(1 - |z_k|) < \infty,$$

$|\lambda| = 1$, and p is a nonnegative integer.

For any fixed unimodular constant λ and any nonnegative, finite, Borel measure μ, on the unit circle $\mathbb{T}$, which is singular with respect to the arc–length measure denote

$$(4) \qquad \lambda S_\mu(z) = \lambda e^{-\int_{\mathbb{T}}(u+z)/(u-z)\,d\mu(u)}, \qquad z \in \mathbb{U}.$$

Any function of type (4) is called a singular inner function. Any inner function is known to be equal to the product of a singular inner function and a Blaschke product, these factors being unique modulo unimodular coefficients. Given $f \in H^2$, the function

$$F(z) = e^{\int_{\mathbb{T}} \frac{u+z}{u-z} \log |f(u)|\,dm(u)},$$

where dm is the normalized arc–length measure on $\mathbb{T}$, is called an *outer function*, or more specifically, the outer factor of f. It is well known that any $f \in H^2$, $f \neq 0$, is representable as a product of an inner function (called the inner factor of f) and its outer factor, this factorization being unique. Again, we refer to [**3**], [**8**], or [**19**] for all these notions which will be used in the sequel.

Now, as observed in [**11**], if φ is a disc automorphism, then $f \in H^2$ is an eigenfunction of C_φ if and only if both the inner and the outer factor of f are eigenfunctions of C_φ. This property is not specific to automorphic composition operators; it holds for composition operators with inner symbols, as we prove in the second section of this paper (Proposition 1).

The first idea in [**10**] was "ruling out" classes of functions, in the quest for functions which might induce an infinite–dimensional atom (if any). More formally, we say $f \in H^2 \setminus \{0\}$ belongs to $\mathcal{N}_\varphi$ (where $\mathcal{N}$ stands for "nonminimal") if K_f is not an atom of $\text{Lat}\, C_\varphi$. Clearly, $\mathcal{N}_\varphi$ is *absorbent* in the following sense: if $K_f \cap \mathcal{N}_\varphi \neq \emptyset$ then K_f is nonminimal invariant under C_φ, and hence, $f \in \mathcal{N}_\varphi$. Problem 1 has an affirmative answer if and only if $H^2 \setminus \{0\} = \mathcal{N}_\varphi \cup \mathcal{E}_\varphi$, where $\mathcal{E}_\varphi$ denotes the set of all eigenfunctions of C_φ, a hyperbolic composition operator.

The notation $\varphi^{[n]}$ designates the n-fold iterate of φ, that is, $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, n times, if n is a positive integer, respectively, the $|n|$–fold iterate of φ^{-1}, when n is negative, and we denote by $\varphi^{[0]}$, the identity function. Then $O_\varphi(z) = \{\varphi^{[n]}(z) : n = 0, 1, 2, 3, \dots\}$ and $BO_\varphi(z) = \{\varphi^{[n]}(z) : n = 0, \pm 1, \pm 2, \pm 3, \dots\}$. We use the terminology $c \in \mathbb{C}$ is an *orbital cluster point of f under φ* if there is a sequence $\{n_k\}$ of positive integers and some $a \in \mathbb{U}$ so that $f \circ \varphi^{[n_k]}(a) \to c$. Below we

list a collection of types of functions known to belong to $\mathcal{N}_\varphi$, reviewing the main results known in that direction. We denote by α and β the attractive, respectively, the repulsive, fixed point of the hyperbolic automorphism φ. The following sets of functions, denoted $\mathcal{S}_k$, $k = 1, \ldots, 8$ are subsets of $\mathcal{N}_\varphi$:

(i) $\mathcal{S}_1$, the class of all nonconstant functions in H^2 having a nonzero nontangential limit at one of the fixed points of φ which are essentially bounded on some open arc containing that point; in particular, all nonconstant functions in H^∞, the space of bounded analytic functions, having nonzero nontangential limit at one of the fixed points of φ, [**15**, Proposition 1.1] and [**11**, Theorem 2.2] (hence all Blaschke products with zeros not clustering at one of the fixed points of φ, all singular inner functions whose support does not contain set $\{\alpha, \beta\}$).

(ii) $\mathcal{S}_2$, the class of all nonzero functions in $S_\mu H^2$ if $\mu(\{\alpha, \beta\}) > 0$, [**15**, Proposition 2.1].

(iii) $\mathcal{S}_3$, the class of all nonzero functions in $\sqrt{(z - \alpha)(z - \beta)}H^2$, [**21**, Theorem 3.5].

(iv) $\mathcal{S}_4$, the class of all functions $f \in H^2 \setminus \{0\}$ which belong to $(z - \alpha)^{\frac{2}{p}} H^p \setminus \{0\}$, for some $2 < p$ and are essentially bounded on some open arc about β([**11**, Theorem 5.5]).

(v) $\mathcal{S}_5$, the class of all functions $f \in H^2 \setminus \{0\}$ which are null and Lipschitz (or Hölder according to some) continuous of order $p > 0$ at a fixed point of φ (that is, $|f(z)| \leq c|\alpha - z|^p$ for all z in a neighborhood of α) and are bounded on a neigborhood of the other fixed point, [**2**].

(vi) $\mathcal{S}_6$, the class of all nonzero functions f in the unit ball of H^∞, having property

$$\lim_{n \to \infty} \left(|f(z_n)|^2 + |f'(z_n)|(1 - |z_n|^2) \right) = 1$$

where $\{z_n\}$ tends nontangentially to one of the fixed points of φ [**16**, Theorem 3.3].

In Section 2, Theorem 2, we add the following item to the list above:

(vii) $\mathcal{S}_7$, the class of all nonconstant inner functions with unimodular orbital cluster points, (hence all thin interpolating Blaschke products [**6**, Proposition 2.2]). Technically, we prove that $\mathcal{S}_7 \subseteq \mathcal{S}_1$

The proofs showing that $\mathcal{S}_k \subseteq \mathcal{N}_\varphi$, $k = 1, 4, 5$, consist of establishing the relation

$$(5) \qquad\qquad\qquad \mathcal{S}_k \subseteq \mathcal{S}_\mathbb{C} \quad k = 1, 4, 5,$$

where $\mathcal{S}_\mathbb{C}$ is the class of all nonconstant functions $f \in H^2$ with the property $\mathbb{C} \subseteq K_f$, where $\mathbb{C}$ is the subspace of constant functions. The obvious relation $C_\varphi 1 = 1$ implies that $\mathcal{S}_\mathbb{C} \subseteq \mathcal{N}_\varphi$.

In Section 3, we add to the list above the class

$(viii)$ $\mathcal{S}_8$ consisting of all nonzero H^2–functions whose bilateral orbit under a hyperbolic composition operator has associated Cesàro means which satisfy certain norm–boundedness conditions, (see Theorem 7 for an exact description of this result, and Corollary 3 for examples).

Besides the aforementioned results, Sections 2 and 3 contain several stronger versions of results originally obtained by this author or others, focusing on the minimality or non–minimality of cyclic invariant subspaces of hyperbolic composition operators.

2. Inner functions

This section is dedicated to the study of the cyclic spaces generated by inner functions under hyperbolic composition operators and related results. We begin by proving the announced property of composition operators with inner symbols, namely, the one saying that $f \in H^2$ is an eigenfunction of such a composition operator C_φ, if and only if both the inner and the outer factor of f are eigenfunctions of C_φ.

We prove first the principle "outer $\circ$ inner=outer". More formally, in [**1**, Lemma 2.9] an older formula of Nordgren [**17**] is presented in the form

$$(6) \qquad \int_{\mathbb{T}} P(\varphi(z), u) f(u) \, dm(u) = \int_{\mathbb{T}} P(z, u) f \circ \varphi(u) \, dm(u) \qquad f \in L^1_{\mathbb{T}},$$

where, $P(z, u)$, $z \in \mathbb{U}$, $u \in \mathbb{T}$ is the usual *Poisson kernel*. Formula (6) is valid for any inner function φ.

LEMMA 1. *If F is an outer function in H^2 and φ is an inner function, then $F \circ \varphi$ is an outer function.*

PROOF. Let

$$F(z) = e^{\int_{\mathbb{T}} \frac{u+z}{u-z} \log |f(u)| \, dm(u)}.$$

Then

$$\log |F \circ \varphi(0)| = \int_{\mathbb{T}} \log |f(u)| P(\varphi(0), u) \, dm(u) = \int_{\mathbb{T}} \log |f \circ \varphi(u)| \, dm(u),$$

by (6). This, according to [**19**, Theorem 17.17], ends the proof. $\qquad \square$

The announced result can now be stated and proved.

PROPOSITION 1. *If C_φ is a composition operator with inner symbol, then $f \in H^2$ is an eigenfunction of C_φ if and only if both the inner and the outer factor of f are eigenfunctions of C_φ.*

PROOF. The statement in this proposition is an immediate consequence of the well-known fact that a composite of two inner functions is an inner function, Lemma 1, and the uniqueness of the inner-outer factorization of an H^2–function. $\qquad \square$

It should be observed that if F in Lemma 1 is bounded, then the requirement that φ be inner can be dropped:

REMARK 1 ([**20**, Chapter III, Corollary of Proposition 3.3]). *If the outer function F is bounded, then $F \circ \varphi$ is an outer function for all analytic self-maps φ of $\mathbb{U}$.*

Next, we will prove that $u \in \mathcal{N}_\varphi$ when u is a nonconstant inner function with unimodular orbital cluster points. First, we develop some necessary technical results. Recall that, as observed in [**12**, Lemmas 1 and 2]:

LEMMA 2. *In a Hilbert space, a sequence $\{v_n\}$ in the closed unit ball of that space, tends weakly to a norm–one vector v if and only if that sequence is norm convergent to v. Therefore, a sequence of inner functions $\{u_n\}$ is convergent weakly in H^2 to a norm–one function u if and only if $\|u_n - u\|_2 \to 0$ and, in that case, u is also an inner function.*

Above and throughout this paper, the notation $\| \quad \|_2$ designates the norm of H^2. A first application of Lemma 2 is the following:

PROPOSITION 2. *The set $\mathcal{I}$ of all inner functions is a norm–closed subset of H^2. If u is inner, φ is an inner self-map of $\mathbb{U}$, and $\{\lambda_n\}$ is a sequence in $\mathbb{T}$ such that the sequence $\{\lambda_n u \circ \varphi^{[n]}\}$ is $\| \quad \|_2$–convergent, then K_u is not a minimal invariant subspace of C_φ, unless u is an eigenfunction of C_φ.*

PROOF. Indeed, assume that $\{v_n\}$ is a sequence of inner functions and $\|v_n - v\|_2 \to 0$. Then $\|v\|_2 = 1$ and hence, $v \in \mathcal{I}$ by Lemma 2. Assume that the sequence $\{\lambda_n u \circ \varphi^{[n]}\}$ tends to some $f \in H^2$. Then f must be an inner function, hence $f \neq 0$. Also, there is some $c \in \mathbb{T}$, so that $C_\varphi f = cf$, that is, f is an eigenfunction of C_φ. Indeed,

$$(7) \qquad C_\varphi \lambda_n u \circ \varphi^{[n]} = \frac{\lambda_n}{\lambda_{n+1}} \left(\lambda_{n+1} u \circ \varphi^{[n+1]} \right) \quad n = 1, 2, 3, \dots$$

and one can find a sequence $\{n_k\}$ of distinct positive integers so that $\{\lambda_{n_k}/\lambda_{n_k+1}\}$ converges to some c, substitute n by n_k in (7), then let $k \to \infty$. One gets $f \circ \varphi = cf$. $\qquad \square$

Recall that the space H^2 is a RKHS (Reproducing Kernel Hilbert Space) where the kernel functions are uniformly bounded on compacts. In a RKHS, a sequence of functions is weakly convergent if and only if it is norm–bounded and pointwise convergent (a well-known fact). If that space consists of analytic functions and the kernel functions are uniformly bounded on compacts, then a normal family argument can be used to show that actually, any weakly convergent sequence is not just pointwise convergent, but even uniformly convergent on compacts to its limit (see [**14**]).

PROPOSITION 3. *If $\{u_n\}$ is a sequence of inner functions and $\lambda \in \mathbb{T}$, then the following statements are equivalent:*

$$(8) \qquad\qquad\qquad\qquad \|u_n - \lambda\|_2 \to 0.$$

$$(9) \qquad\qquad\qquad\qquad u_n \to \lambda \quad \textit{weakly in } H^2.$$

$$(10) \qquad\qquad\qquad\qquad u_n \to \lambda \quad \textit{uniformly on compacts.}$$

$$(11) \qquad\qquad\qquad\qquad u_n \to \lambda \quad \textit{pointwise.}$$

$$(12) \qquad\qquad\qquad\qquad u_n(a) \to \lambda \quad \textit{for some } a \in \mathbb{U}.$$

PROOF. It is well known that $(8) \Longrightarrow (9) \Longrightarrow (10) \Longrightarrow (11) \Longrightarrow (12)$. Assume first that $a = 0$ and (12) holds. Then

$$\|u_n - \lambda\|_2^2 = 2 \left(1 - \Re(\bar{\lambda} u_n(0)) \right) \to 0.$$

This ends the proof if $a = 0$. If $a \neq 0$, let $\alpha_a(z) = (a - z)/(1 - \bar{a}z)$, $z \in \mathbb{U}$. The condition $u_n(a) \to \lambda$ can be written as $u_n \circ \alpha_a(0) \to \lambda$, and so one deduces, by what we have already proved, that $\|C_{\alpha_a} u_n - \lambda\|_2 \to 0$. Therefore, $\|C_{\alpha_a} C_{\alpha_a} u_n - C_{\alpha_a} \lambda\|_2 \to 0$. Given that $C_{\alpha_a} C_{\alpha_a} = I$ and $C_{\alpha_a} \lambda = \lambda$, the conclusion follows. $\qquad \square$

Proposition 3 extends and completes a result in [**7**] (see also [**4**]). Based on it, we can write a very short proof for the following:

THEOREM 2. *Let u and φ be inner functions. If, for some $a \in \mathbb{U}$, one has that*

(13) $$\limsup_{n \to \infty} |u \circ \varphi^{[n]}(a)| = 1,$$

then $\mathbb{C} \subseteq K_u$ and so K_u is minimal invariant if and only if u is constant.

PROOF. Condition (13) and the compactness of $\mathbb{T}$ imply the fact that there is some subsequence $\{u \circ \varphi^{[n_k]}(a)\}$ and a constant $\lambda \in \mathbb{T}$ so that $u \circ \varphi^{[n_k]}(a) \to \lambda$. By Proposition 3, one gets that $\|u \circ \varphi^{[n_k]} - \lambda\|_2 \to 0$, which implies $\mathbb{C} \subseteq K_u$. $\qquad\square$

Theorem 2 extends results in [**10**] and [**15**]. It is worth noting the following:

COROLLARY 1. *Let u and φ be inner functions. If, for some $a \in \mathbb{U}$, one has that*

$$\sum_{n=1}^{\infty} \left(1 - |u(\varphi^{[n]}(a))|\right) < \infty \quad \text{or, (if } C_\varphi \text{ is invertible),}$$

$$\sum_{n=1}^{\infty} \left(1 - |u(\varphi^{[-n]}(a))|\right) < \infty,$$

then K_u is not minimal invariant for C_φ. Hence, if u is an inner eigenfunction of C_φ, then

$$\sum_{n=1}^{\infty} \left(1 - |u(\varphi^{[n]}(a))|\right) = \infty \quad \text{and, (if } C_\varphi \text{ is invertible),}$$

$$\sum_{n=1}^{\infty} \left(1 - |u(\varphi^{[-n]}(a))|\right) = \infty.$$

It is known that several kinds of inner functions φ, (hyperbolic automorphisms included), have Blaschke summable orbits [**9**], that is, for all $a \in \mathbb{U}$, the sequence $\{z_n = \varphi^{[n]}(a)\}$ satisfies condition (3). Also, the orbit of any point in $\mathbb{U}$ under a hyperbolic disc automorphism is a sequence which tends to the attractive fixed point α of that automorphism through a Stolz angle with vertex at α. Therefore, it is worth raising the following:

PROBLEM 3. *Can one describe in easier terms, the set $\mathcal{S}_\alpha$ consisting of all inner functions which transform Blaschke summable sequences which converge to $\alpha \in \mathbb{T}$ through some Stolz angle with vertex at α, into Blaschke summable sequences?*

What we mean is, given $\alpha \in \mathbb{T}$, for what kind of inner functions u is it true that, if $\{z_n\}$ is a sequence in $\mathbb{U}$, which satisfies (3) and $z_n \to \alpha$ through some Stolz angle with vertex at α, it follows that sequence $\{u(z_n)\}$ satisfies

$$\sum_{n=1}^{\infty} (1 - |u(z_n)|) < \infty?$$

REMARK 2. *If φ is a hyperbolic disc automorphism with attractive fixed point α, then $\mathcal{S}_\alpha \subseteq \mathcal{S}_7 \subseteq \mathcal{S}_\mathbb{C} \subseteq \mathcal{N}_\varphi$.*

It is beyond the scope of this paper to solve Problem 3. However, let us give examples of functions in $\mathcal{S}_\alpha$.

EXAMPLE 1. *If u is an inner function so that there is a Stolz angle Γ with vertex at $\alpha \in \mathbb{T}$ and some $C > 0$ so that*

(14) $$1 - |u(z)| \le C(1 - |z|), \quad z \in \Gamma,$$

then $u \in \mathcal{S}_1 \cap \mathcal{S}_\alpha$.

It is evident that $u \in \mathcal{S}_\alpha$ if (14) holds. Also, by condition (14),

$$\liminf_{z \to \alpha} \frac{1 - |u(z)|}{1 - |z|} < \infty$$

which, by the well-known Julia–Carathéodory theorem, implies the fact that u has a finite angular derivative at α and hence a unimodular nontangential limit at α. Therefore, one has that $u \in \mathcal{S}_1$.

In the following, we will use the terminology "the greatest common inner divisor" of a family of inner functions, rather than "a greatest common inner divisor", recalling that any two greatest common inner divisors of that family of functions are unimodular scalar multiples of each other. In [**10**], the author proved the following theorem about the minimality of K_u when u is inner:

THEOREM 3 ([**10**, Theorem 3]). *If u is inner and K_u a minimal invariant subspace for C_φ, a hyperbolic composition operator, then for all n, the greatest common inner divisor of the functions in $\{u \circ \varphi^{[k]} : k \geq n\}$, respectively, in $\{u \circ \varphi^{[-k]} : k \geq n\}$ must be an eigenfunction of C_φ.*

Theorem 3 is not specific to cyclic subspaces induced by inner functions, as we prove in the following. First let us introduce some terminology and needed notation. If two inner functions u and v divide each other, that is, if $u|v$ and $v|u$, we denote this fact by $u \sim v$. As is well known, $u \sim v$ if and only if u is a unimodular multiple of v, that is, if and only if there is some $\lambda \in \mathbb{T}$, so that $u = \lambda v$. Given $f \in H^2 \setminus \{0\}$ and some fixed $n \in \mathbb{Z}$, we denote by v_n^+ the greatest common inner divisor of the inner factors of $f \circ \varphi^{[k]}$, $k \geq n$, and by v_n^- the greatest common inner divisor of the inner factors of $f \circ \varphi^{[k]}$, $k \leq n$. By v we denote the greatest common inner divisor of the inner factors of $f \circ \varphi^{[k]}$, $k \in \mathbb{Z}$. With this notation, Theorem 3 upgrades to:

THEOREM 4. *If $f \in H^2 \setminus \{0\}$ induces the minimal invariant subspace K_f, of C_φ an automorphic composition operator, then*

(15) $v \sim v_n^+ \sim v_k^-   n, k \in \mathbb{Z}.$

Whether, K_f is minimal or not, v is an eigenfunction of C_φ.

PROOF. Clearly, one has that $v|v_n^+$ and $v|v_k^-$, for all $n, k \in \mathbb{Z}$. On the other hand, if K_f is minimal, then $K_f = \bigvee_{j \geq n} C_\varphi^j f \subseteq v_n^+ H^2$, so $v_n^+|v$, $n \in \mathbb{Z}$. Thus $v \sim v_n^+$, $n \in \mathbb{Z}$. By a similar argument, one gets $v \sim v_k^-$, $k \in \mathbb{Z}$. The only thing left is to show $v \sim v \circ \varphi$, which means v is an eigenfunction of C_φ associated to a unimodular eigenvalue.

To that aim, for u inner and f in $H^2 \setminus \{0\}$, we will write $u|f$ and mean that u divides the inner factor of f. Note that, in that case, it follows that $u \circ \varphi^{[k]}|f \circ \varphi^{[k]}$, $k \in \mathbb{Z}$. Our proof will be over as soon as we prove $v \circ \varphi|v$ and $v \circ \varphi^{-1}|v$, since one gets immediately $v \circ \varphi^{-1} \circ \varphi|v \circ \varphi$, that is, $v \sim v \circ \varphi$. Whether K_f is minimal invariant or not, one has that

$$v|f \circ \varphi^{[n-k]}   n, k \in \mathbb{Z};$$

hence

$$v \circ \varphi^{[k]}|f \circ \varphi^{[n]}   n, k \in \mathbb{Z}$$

which implies that

$$v \circ \varphi^{[k]}|v \circ \varphi^{[n]}   n, k \in \mathbb{Z}.$$

$\square$

It should be added here that the fact that the function v in Theorem 4 is an eigenfunction was initially obtained for the case when f is inner in [**15**, Proposition 2.2]. It is true whether f is inner or not. Relative to Theorem 4, we prove:

PROPOSITION 4. *If v is an inner eigenfunction of C_φ, a composition operator, and $f \in H^2$, then K_f^+ is a minimal invariant subspace of C_φ if K_{vf}^+ is minimal invariant.*

PROOF. Let $V = M_v$ be the multiplication operator with symbol v. Clearly V is isometric, so $V^*V = I$. Since v is an eigenfunction of C_φ, $K_{vf}^+ = VK_f^+$. Therefore, if S is a closed subspace of K_f^+ left invariant by C_φ, then VS is a closed subspace of K_{vf}^+, invariant under C_φ. If K_{vf}^+ is a minimal invariant and $S \neq 0$, then $VS = K_{vf}^+ = VK_f^+$, hence $S = K_f^+$, so K_f^+ is also minimal invariant. $\square$

By Theorem 4, one gets the following:

COROLLARY 2. *If $f \in H^2 \setminus \{0\}$, K_f^+ is a minimal invariant subspace of C_φ, a hyperbolic composition operator, and v is the greatest common inner divisor of the inner factors of the functions in $\{f \circ \varphi^{[n]} : n \geq 0\}$, then K_g^+ is also minimal invariant, where $g = f/v$. Thus, if one studies if K_f^+ can be an infinite–dimensional atom in $\operatorname{Lat} C_\varphi$ (which of course happens if and only if K_f is such an atom), one can assume without loss of generality that the greatest common inner divisor of the inner factors of the functions in $\{f \circ \varphi^{[n]} : n \geq 0\}$ is 1.*

By Proposition 2, if for some inner u the sequence $\{u \circ \varphi^{[n]}\}$ is norm convergent to the greatest common inner divisor of the functions $\{u \circ \varphi^{[n]}\}$, then K_u is not a minimal invariant subspace of C_φ, unless u is an eigenfunction of C_φ. Here is a characterization of when a sequence of inner functions tends to the greatest common inner divisor of those functions.

PROPOSITION 5. *If v denotes a common inner divisor of the sequence $\{u_n\}$ of inner functions, and we denote $v_n = u_n/v$, $n = 1, 2, 3, \ldots$, then*

$$(16) \qquad\qquad\qquad \|u_n - v\|_2 \to 0$$

if and only if

$$(17) \qquad\qquad\qquad v_n(0) \to 1.$$

If any of the conditions (16) and (17) holds, then v is necessarily the greatest common inner divisor of the sequence $\{u_n\}$.

PROOF. Indeed $\|u_n - v\|_2 = \|v(v_n - 1)\|_2 = \|v_n - 1\|_2$, $n = 1, 2, 3, \ldots$. This equality combines with Proposition 3 into establishing the equivalence of conditions (16) and (17). In order to prove now that if any of those conditions holds, then v must be the greatest common inner divisor of the sequence $\{u_n\}$, begin by assuming that $v = 1$. If, arguing by contradiction, one assumes that the greatest common inner divisor of the sequence $\{u_n\}$ is a nonconstant inner function u, then $|u(0)| < 1$, and, since for all $n = 1, 2, 3 \ldots$, $u_n = uw_n$, where w_n is inner, $n = 1, 2, 3 \ldots$, one gets that

$$|u_n(0)| \leq |u(0)| \quad n = 1, 2, 3 \ldots .$$

By letting $n \to \infty$, one gets the contradictory relation $1 \leq |u(0)|$. In general, if v is a common inner divisor of the sequence $\{u_n\}$ of inner functions, then it is

elementary to prove that v is the greatest common inner divisor of the sequence $\{u_n\}$ if and only if the greatest common inner divisor of the sequence $\{v_n\}$ is 1.   □

The problem we want to pose here is:

PROBLEM 4. *If φ is a hyperbolic automorphism and u is inner, then K_u is minimal invariant if and only if u is an eigenfunction of C_φ. True or false?*

It is very likely that the answer is: TRUE.

3. Non–inner functions

Relative to functions with nontangential limit at each of the fixed points of a hyperbolic automorphism, the author proved:

PROPOSITION 6 ([**11**, Proposition 2.3]). *If φ is a hyperbolic automorphism and f is an eigenfunction of C_φ having finite nontangential limits at the fixed points of φ, then f is constant.*

Thus, if f is an eigenfunction in the disc algebra $\mathcal{A}$ then f is constant. Given that 1–dimensional subspaces spanned by eigenvectors are the only minimal invariant subspaces known so far, it makes sense to conjecture that the following problem has a POSITIVE answer:

PROBLEM 5. *If φ is a hyperbolic automorphism and $f \in \mathcal{A}$, then K_f is a minimal invariant subspace of C_φ if and only if f is a nonzero constant function. True or false?*

Relative to the above problem, it is worth noting that:

REMARK 3. *The only kind of nonconstant functions $f \in \mathcal{A}$, which might disprove the conjecture in Problem 5 are those with property $f(\alpha) = f(\beta) = 0$, where α and β are the fixed points of φ.*

Indeed, this is an immediate consequence of [**10**, Theorem 2], a theorem where it is shown that $\|f \circ \varphi^{[n]} - f(\alpha)\|_2 \to 0$ if α is the attractive fixed point of φ, provided that f is continuously extensible at α (see also [**5**, Lemma 1.1]). Based on the previous statements, one can prove:

LEMMA 3. *If $f \in H^2 \setminus \{0\}$ has a norm–bounded bilateral orbit under C_φ, a hyperbolic composition operator, (in particular, if f is continuously extendable at the fixed points of φ), and*

$$\text{(18)} \qquad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^n f\|_2} < 1$$

or

$$\text{(19)} \qquad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^{-n} f\|_2} < 1,$$

then the restriction $C_\varphi|K_f$ has a point spectrum containing a nonempty open annulus centered at the origin.

PROOF. If the bilateral orbit of f under C_φ is norm–bounded, then

$$\text{(20)} \qquad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^n f\|_2} \le 1 \quad \text{and} \quad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^{-n} f\|_2} \le 1.$$

By (20), it follows that, if $\limsup\limits_{n\to\infty} \sqrt[n]{\|C_\varphi^n f\|_2} < 1$ or $\limsup\limits_{n\to\infty} \sqrt[n]{\|C_\varphi^{-n} f\|_2} < 1$, then

$$(21) \qquad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^n f\|_2} \cdot \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^{-n} f\|_2} < 1$$

and so, the restriction $C_\varphi|K_f$ has a point spectrum containing a nonempty open annulus centered at the origin, by [**11**, Theorem 5.1] □

For each arbitrary analytic self-map φ of $\mathbb{U}$, other than the identity or an elliptic automorphism, there is a remarkable point $\alpha \in \overline{\mathbb{U}}$ (called the Denjoy–Wolff point of φ), with property $\varphi^{[n]} \to \alpha$ uniformly on compacts. For a hyperbolic automorphism, the Denjoy–Wolff point is, of course, the attractive fixed point of that map. Whether φ is automorphic or not, it is known that $0 < \varphi'(\alpha) \le 1$, if $\alpha \in \mathbb{T}$, where $\varphi'(\alpha)$ denotes the angular derivative of φ at α. Maps φ with the property $\varphi'(\alpha) < 1$ are called maps of hyperbolic type, whereas those with property $\varphi'(\alpha) = 1$ are called maps of parabolic type. With this terminology, we note that:

LEMMA 4. *If φ is a map of parabolic or hyperbolic type with Denjoy–Wolff point α and $f \in H^2 \setminus \{0\}$ satisfies condition*

$$(22) \qquad \limsup_{n\to\infty} \frac{\|f \circ \varphi^{[n]}\|_2}{\|\alpha - \varphi^{[n]}\|_2^p} < \infty$$

for some $p > 0$ then

$$(23) \qquad \limsup_{n\to\infty} \sqrt[n]{\|C_\varphi^n f\|_2} \le (\varphi'(\alpha))^{p/2} .$$

PROOF. Observe that condition (22) is equivalent to the existence of some $c > 0$ so that

$$(24) \qquad \|f \circ \varphi^{[n]}\|_2 \le c \|\alpha - \varphi^{[n]}\|_2^p \qquad n = 1, 2, 3, \dots .$$

On the other hand, according to [**13**, Proposition 1],

$$(25) \qquad \limsup_{n\to\infty} \sqrt[n]{\|\alpha - \varphi^{[n]}\|_2} \le \left(\sqrt{\varphi'(\alpha)}\right) .$$

□

The immediate consequence of the above two lemmas is the following.

THEOREM 5. *Assume φ is a hyperbolic automorphism with attractive fixed point α and repulsive fixed point β. If $f \in H^2 \setminus \{0\}$ has a norm–bounded bilateral orbit under C_φ, and*

$$(26) \qquad \limsup_{n\to\infty} \frac{\|f \circ \varphi^{[n]}\|_2}{\|\alpha - \varphi^{[n]}\|_2^p} < \infty$$

or

$$(27) \qquad \limsup_{n\to\infty} \frac{\|f \circ \varphi^{[-n]}\|_2}{\|\beta - \varphi^{[-n]}\|_2^p} < \infty,$$

for some $p > 0$, then the point spectrum of the restriction of C_φ to K_f contains a nonempty open annulus about the origin and hence, K_f is not minimal invariant.

PROOF. If condition (26) holds, then the desired result follows by Lemmas 3 and 4, since $\left(\sqrt{\varphi'(\alpha)}\right)^p < 1$. If (27) holds, just recall that β is the attractive fixed point of φ^{-1}. □

One method to produce eigenfunctions of a hyperbolic composition operator was first given in [2] by V. Chkliar, who considered a formal series of form

$$(28) \qquad F_\lambda := \sum_{n=-\infty}^{\infty} \lambda^n f \circ \varphi^{[n]}$$

where λ is a scalar. If series (28) is weakly convergent, then $C_\varphi F_\lambda = \lambda^{-1} F_\lambda$ and so, if F_λ is a nonzero function and f is not an eigenfunction of C_φ, then K_f is not a minimal invariant subspace of C_φ. V. Chkliar in [2] used that idea and a complex analytic argument to prove that $\mathcal{S}_5 \subseteq \mathcal{N}_\varphi$. Others [11], [21] followed in his steps. An interesting idea in [21, Theorem 3.5] is to try a Fourier analysis argument and so, the main message of [21, Theorem 3.5] is to use Hilbert space Fourier series in order to prove:

THEOREM 6. *If φ is a hyperbolic disc automorphism and the bilateral orbit of $f \in H^2 \setminus \{0\}$ is square summable, that is, if*

$$(29) \qquad \sum_{n=-\infty}^{\infty} \|f \circ \varphi^{[n]}\|_2^2 < \infty,$$

then f is not an eigenfunction of C_φ and $\sigma_p(C_\varphi|K_f) \cap \mathbb{T} \neq \emptyset$, where $\sigma_p(C_\varphi|K_f)$ is the point spectrum the restriction of C_φ to K_f. Hence, K_f is not minimal invariant under C_φ.

The proof of the above principle is implicit in [21, Theorem 3.5], a theorem where more than the relation $\sigma_p(C_\varphi|K_f) \cap \mathbb{T} \neq \emptyset$ is proved, namely, it is proven that $\sigma_p(C_\varphi|K_f) \cap \mathbb{T}$ contains a set of positive Lebesgue measure. Also, J.H. Shapiro in [21] preferred to state his theorem by exhibiting a class of functions with property (29), namely, the following:

EXAMPLE 2. *If f is in $\sqrt{(z-\alpha)(z-\beta)}H^2 \setminus \{0\}$, then the bilateral orbit of f under C_φ, a hyperbolic composition operator with fixed points α and β, is square summable and hence K_f is not minimal invariant under C_φ.*

It should be observed that:

REMARK 4. *A function $f \in H^2 \setminus \{0\}$ has square summable bilateral orbits under C_φ, a hyperbolic composition operator, if and only if the outer factor of f has that property.*

The fact that such a function cannot be an eigenfunction of C_φ is rather obvious since, if arguing by contradiction, one assumes that $f \neq 0$ has property (29) and is an eigenfunction of C_φ associated to an eigenvalue λ, then the orbit of f under C_φ is not square summable if $|\lambda| \geq 1$, which is a contradiction. If $|\lambda| < 1$, then note that $\lambda \neq 0$, since C_φ is invertible, and f being an eigenfunction of C_φ^{-1} associated to eigenvalue λ^{-1}, one gets that the orbit of f under C_φ^{-1} is not square summable.

All the above leads to the question: Exactly when is the bilateral orbit of a function under a hyperbolic composition operator square summable? Here is a quick answer. Let us consider hyperbolic disc automorphisms with fixed points ± 1, the attractive fixed point being 1: $\varphi(z) = (r+z)/(1+rz)$ for some fixed $0 < r < 1$. Denote $\mu = (1+r)/(1-r)$. Then, by using formula (6) and the Lebsegue monotone convergence theorem, one gets:

PROPOSITION 7. *If $f \in H^2$ and φ is as above, then*

$$(30) \qquad \sum_{n=0}^{\infty} \|f \circ \varphi^{[n]}\|_2^2 = \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \psi_+(e^{i\theta}) \, d\theta/2\pi$$

where

$$\psi_+(e^{i\theta}) := \sum_{n=0}^{\infty} \frac{4\mu^n}{4 + (\mu^{2n} - 1)|1 - e^{i\theta}|^2} = \sum_{n=0}^{\infty} \frac{\mu^n}{1 + (\mu^{2n} - 1)\sin^2(\theta/2)}.$$

and

$$(31) \qquad \sum_{n=1}^{\infty} \|f \circ \varphi^{[-n]}\|_2^2 = \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \psi_-(e^{i\theta}) \, d\theta/2\pi$$

where

$$\psi_-(e^{i\theta}) := \sum_{n=1}^{\infty} \frac{4\mu^n}{4 + (\mu^{2n} - 1)|1 + e^{i\theta}|^2} = \sum_{n=0}^{\infty} \frac{\mu^n}{1 + (\mu^{2n} - 1)\cos^2(\theta/2)}.$$

Above we denoted the normalized arc–length measure $d\theta/2\pi$ instead of dm.

PROOF. By formula (6) one has:

$$\|f \circ \varphi^{[n]}\|_2^2 = \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 P(\varphi^{[n]}(0), e^{i\theta}) \, d\theta/2\pi, \qquad n \in \mathbb{Z},$$

and it is easy to check the identity

$$\varphi^{[n]}(z) = \frac{z + r_n}{1 + r_n z} \qquad r_n = \frac{\mu^n - 1}{\mu^n + 1}, \quad n = 1, 2, 3, \ldots .$$

Combining the above equality with Lebesgue's monotone convergence theorem, one gets equalities (30) and (31). $\qquad\square$

Our last result is in the spirit of Theorem 6. Given the formal series (28), we denote by $S_n(\lambda, z)$ the symmetric partial sums of that series, that is,

$$S_n(\lambda, z) = \sum_{k=-n}^{n} \lambda^k f \circ \varphi^{[k]}(z), \qquad z \in \mathbb{U}, \ \lambda \in \mathbb{T}, \ n = 1, 2, 3, \ldots$$

Also we introduce

$$\sigma_n(\lambda, z) = (1/n) \sum_{k=0}^{n-1} S_k(\lambda, z), \quad \lambda \in \mathbb{T}, \ z \in \mathbb{U}, \ n = 1, 2, 3, \ldots .$$

With this notation, we can state and prove the following:

THEOREM 7. *Assume φ is a hyperbolic automorphism, $f \in H^2 \backslash \{0\}$, for almost all fixed $\lambda \in \mathbb{T}$, sequence $\{\sigma_n(\lambda, z)\}$ is pointwise convergent on $\mathbb{U}$ to some $\sigma(\lambda, z)$, and is $\| \ \|_2$–bounded, that is, there is some $M_\lambda > 0$ such that*

$$(32) \qquad \|\sigma_n(\lambda, z)\|_2 \le M_\lambda, \qquad n = 1, 2, 3, \ldots .$$

Assume further that, for all fixed $z \in \mathbb{U}$, there is some $M_z > 0$ so that

$$(33) \qquad \int_{\mathbb{T}} |\sigma_n(\lambda, z)| \, dm(\lambda) \le M_z, \quad n = 1, 2, 3 \ldots$$

Then the space K_f is not a minimal invariant subspace of C_φ because the point spectrum of $C_\varphi|K_f$ contains a measurable subset $E \subseteq \mathbb{T}$ with property $m(E) > 0$.

PROOF. For all considerations on Cesàro means contained in this proof, the reader is referred to [**8**, Ch. 2]. Observe that $\sigma(\lambda, z)$, the weak limit of $\sigma_n(\lambda, z)$ is an element of K_f for almost all λ in $\mathbb{T}$, according to our hypothesis. We claim that for almost all $\lambda \in \mathbb{T}$, one has that $C_\varphi \sigma(\lambda, z) = \lambda^{-1} \sigma(\lambda, z)$ and so, if for some $\lambda \in \mathbb{T}$, the function $\sigma(\lambda, z)$ exists and is not the null function, then it is an eigenfunction of $C_\varphi | K_f$ associated to the eigenvalue λ^{-1}.

We turn now to property (33). It says that for all $z \in \mathbb{U}$ fixed, the sequence $\{\sigma_n(\lambda, z)|\}$ is the sequence of Cesàro means of an $L^1_{\mathbb{T}}$–function G_z, namely, of that function whose sequence of Fourier coefficients is $\{f \circ \varphi^{[n]}(z)\}_{n=-\infty}^{\infty}$ and so, by Lebesgue's theorem on the Fourier coefficients of such a function, one has that $f \circ \varphi^{[n]}(z) \to 0$ as $n \to \infty$. Thus, one has

$$(34) \qquad \frac{1}{n} \sum_{j=1}^{n-1} \left(\lambda^{j-1} f \circ \varphi^{[j]}(z) + \lambda^j f \circ \varphi^{[j+1]}(z) \right) \to 0 \quad \text{when} \quad n \to \infty.$$

Relation (34) combines with the equality

$$\sigma_n(\lambda, z) \circ \varphi(z)$$

$$= \lambda^{-1} \sigma_{n-1}(\lambda, z) \circ \varphi(z) + \frac{1}{n} \sum_{j=1}^{n-1} \left(\lambda^{j-1} f \circ \varphi^{[j]}(z) + \lambda^j f \circ \varphi^{[j+1]}(z) \right) + \frac{f \circ \varphi(z)}{n}$$

to prove that $C_\varphi \sigma(\lambda, z) = \lambda^{-1} \sigma(\lambda, z)$, for almost all $\lambda \in \mathbb{T}$. On the other hand, the Cesàro means of an $L^1_{\mathbb{T}}$–function, are $\| \ \|_1$–convergent to that function, that is, $\|\sigma_n(\lambda, z) - G_z\|_1 \to 0$, so, as is well known, a subsequence of $\{\sigma_n(\lambda, z)\}$ is convergent a.e. to G_z. Thus $G_z(\lambda) = \sigma(\lambda, z))$ a.e. That is, for all fixed z, the function $\sigma(\lambda, z)$ is in $L^1_{\mathbb{T}}$, and therefore that function is not null a.e. when $f(z) \neq 0$, since $f(z)$ is a Fourier coefficient of $\sigma(\lambda, z)$. This means that if $f(z) \neq 0$, then $\sigma(\lambda, z) \neq 0$ for all $\lambda \in E$ where $E \subseteq \mathbb{T}$ is measurable with property $m(E) > 0$. Now choose any fixed $\lambda \in E$ and observe that the H^2–function $\sigma(\lambda, z)$, $z \in H^2$ is not the null function, which proves that E is a subset of the point spectrum of $C_\varphi | K_f$, by our previous considerations. $\qquad \square$

COROLLARY 3. *If φ is a hyperbolic automorphism, $f \in H^2 \setminus \{0\}$, condition* (32) *holds and*

$$(35) \qquad \sum_{n=-\infty}^{\infty} |f \circ \varphi^{[n]}(z)| < \infty, \quad z \in \mathbb{U},$$

then the hypothesis of Theorem 7 *is satisfied.*

Indeed, by condition (35), series (28) is convergent for all $\lambda \in \mathbb{T}$ and $z \in \mathbb{U}$. Denote by $F(\lambda, z)$ its sum and observe that $\sigma_n(\lambda, z) \to F(\lambda, z)$, $\lambda \in \mathbb{T}$ and $z \in \mathbb{U}$. Also, for all fixed $z \in \mathbb{U}$, the function $F(\lambda, z)$, $\lambda \in \mathbb{T}$, is continuous on $\mathbb{T}$, for which reason, condition (33) must hold.

As a final remark, nonzero functions with norm summable bilateral orbits satisfy the assumptions in both Theorem 6 and 7. More formally:

REMARK 5. *If $f \in H^2 \setminus \{0\}$ has the property*

$$(36) \qquad \sum_{n=-\infty}^{\infty} \|f \circ \varphi^{[n]}\|_2 < \infty,$$

then f satisfies the assumptions in Theorems 6 and 7.

The fact that the assumptions in Theorem 6 are satisfied is pretty obvious. Also observe that the assumptions in Corollary 3 hold as well, since in H^2 a norm convergent sequence is necessarily weakly convergent hence pointwise convergent, so (35) holds if (36) holds. Finally, condition (32) holds as well since for all $n = 1, 2, 3, \ldots$, one has the estimate

$$\|S_n(\lambda, z)\|_2 \leq \sum_{k=-n}^{n} \|f \circ \varphi^{[k]}\|_2 \leq \sum_{k=-\infty}^{\infty} \|f \circ \varphi^{[k]}\|_2 < \infty, \qquad \lambda \in \mathbb{T},$$

which implies the fact that

$$\|\sigma_n(\lambda, z)\|_2 \leq \sum_{k=-\infty}^{\infty} \|f \circ \varphi^{[k]}\|_2 < \infty, \qquad \lambda \in \mathbb{T}, \, n = 1, 2, 3, \ldots.$$

References

[1] P. S. Bourdon, V. Matache, J. H. Shapiro, *On convergence to the Denjoy-Wolff point*, Illinois J. Math. **49** (2005), no. 2, 405–430.

[2] V. Chkliar, *Eigenfunctions of the hyperbolic composition operator*, Integral Equations Operator Theory **29** (1997), no. 3, 364–367.

[3] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970; reprinted Dover, 2000.

[4] E. A. Gallardo-Gutierrez and P. Gorkin, *Cyclic Blaschke products for composition operators*, Rev. Mat. Iberoamericana **25** (2009), no. 2, 447–470.

[5] E. A. Gallardo-Gutierrez and P. Gorkin, *Minimal invariant subspaces for composition operators*, J. Math. Pures Appl. **95** (2011), 245-259.

[6] E. A. Gallardo–Gutiérrez, P. Gorkin, D. Suárez, *Orbits of non–elliptic disc automorphisms on H^p*, J. Math. Anal. Appl. **388** (2012), 1013–1026.

[7] P. Gauthier and J. Xiao, *The existence of universal inner functions on the unit ball of $\mathbb{C}^n$*, Canad. Math. Bull. **48** (2005), no. 3, 409–413.

[8] K. Hoffman, *Banach Spaces of Analytic Functions*, Dover Publ. Inc., Mineola, New York, 2007 (originally published by Prentice–Hall, Englewood-Cliffs, New Jersey 1962).

[9] V. Matache, *Invariant subspaces of composition operators*, J. Operator Theory **73** (2015), no. 1, 243–264.

[10] V. Matache, *On the minimal invariant subspaces of the hyperbolic composition operator*, Proc. of the American Mathematical Society **119** (1993), no 3, 837–841.

[11] V. Matache, *The eigenfunctions of a certain composition operator*, Contemporary Mathematics **213** (1998), 121–136.

[12] V. Matache, *Numerical ranges of composition operators with inner symbols*, Rocky Mountain J. Math. **42** (2012), 235–249.

[13] V. Matache, *On spectra of composition operators*, Oper. Matrices **9** (2015), no. 2, 277–303.

[14] V. Matache and W. Smith, *Composition operators on a class of analytic function spaces related to Brennan's conjecture*, Complex Anal. Oper. Theory **6** (2012), no. 1, 139–162.

[15] R. Mortini, *Cyclic Subspaces and Eigenvectors of the Hyperbolic Composition Operator*, Travaux mathématiques, Fasc. VII, 69–79, Sém. Math. Luxembourg, Centre Univ. Luxembourg, Luxembourg, 1995.

[16] R. Mortini, *Superposition operators, $\mathcal{B}$–universal functions, and the hyperbolic composition operator*, Acta Math. Hungar. **138** (2013), no. 3, 267–280.

[17] E. A. Nordgren, *Composition operators*, Canad. J. Math. **20** (1968), 442–449.

[18] E. A. Nordgren, P. Rosenthal, F. S. Wintrobe, *Invertible composition operators on H^p*, J. Funct. Anal. **73** (1987), no. 2, 324–344.

[19] W. Rudin, *Real and Complex Analysis*, 3-d edition, McGraw–Hill, Boston, 1987.

[20] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, Translated from the French and revised North-Holland Publishing Co., Amsterdam–London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.

[21] J. H. Shapiro, *The invariant subspace problem via composition operators–redux*, Topics in Operator Theory, Volume 1. Operators, Matrices and Analytic Functions, 519-534, Oper. Theory Adv. Appl., 202, Birkhuser Verlag, Basel, 2010.

Department of Mathematics, University of Nebraska, Omaha, NE 68182, USA
E-mail address: vmatache@unomaha.edu