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A Multi Centerpoint Theorem via Fourier analysis on the Torus

Yan Chen
Bard College, yc7007@bard.edu

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by
Jeffrey Yan Chen

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Abstract

The Centerpoint Theorem states that for any set S of points in $\mathbb{R}^d$, there exists a point c such that any hyperplane goes through that point divides the set. For any half-space containing the point c, the amount of points in that half-space is no bigger than $\frac{1}{d+1}$ of the whole set. This can be related to how close can any hyperplane containing the point c comes to equipartitioning for a given shape S. For a function from unit circle to real number, it has a Fourier interpretation. Using Fourier analysis on the Torus, I will try to find a multi centerpoint theorem for many points in the plane such that any hyperplanes go through those points are close to equipartitioning a given shape.

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Dedication

This senior project is dedicated to my beloved grandfather.

Acknowledgments

Without a doubt, the biggest thanks go to my advisor Steven Simon of whom I could not have asked for more. This project would not have been possible without Steve's valuable guidance and understanding.

Thanks to the following friends for supporting me and believing in me:
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You guys are the best part of my four year college experience.

1 Introduction

In this chapter, we are going to introduce the Centerpoint Theorem and the basic setup of the problem that I am working on. The Centerpoint Theorem states that for any set S of points in $\mathbb{R}^d$, there exists a point such that any hyperplane goes through that point divides the set into two half space, $|H^+ \cap S| \geq \frac{|S|}{d+1}$ for any half-space containing c. This can be related to how close can any line containing c comes to equipartitioning for a given shape S. My problem is a multi centerpoint Question. I am trying to find out how close can any two lines on a plane with a single mass come to equipartition the mass and try to find the upper bound for those centerpoints. Section 1.1 is an elaborations of two different perspectives of the Centerpoint Theorem, the discrete version and the continuous version. Section 1.2 gives an overview of what I am working on.

1.1 Centerpoint Theorem

In statistics and computational geometry, the notion of centerpoint is a generalization of the median to data in higher-dimensional Euclidean space. If a median is on a line, there is at least half of the points on one side of the median like the one showed in the picture below. In the problem that I am working on, the centerpoint is exactly the same as a median or a geometric median.

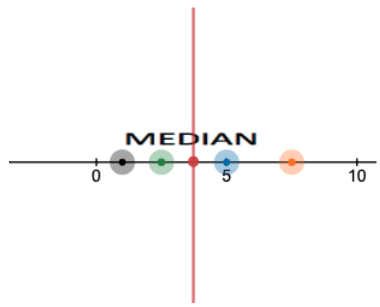


Figure 1.1.1. Median

Theorem 1.1.1. (Centerpoint Theorem). *Given a set of points in d-dimensional space, a centerpoint of the set is a point such that any hyperplane that goes through that point divides the set of points into two roughly equal subsets: each closed half-space contains at least a* $\frac{1}{d+1}$ *of the points. Conversely, if there is such a point exists in the space, then it's the centerpoint.*

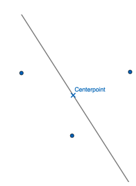
In the optimal case, each hyperplane that goes through the centerpoint is able to divide the whole space into two equal half spaces, which is called equipartition. In this situation, the centerpoint is working as a median. Under equipartition, the two half-spaces should have the same measure. For discrete cases, the same measure just means the two half spaces have the same amount of points. For continuous cases, instead of looking at points, we are going to view it as areas that are divided by hyperplane within the spaces. In this case, measure can be calculated by integrating the density function of the space. Now we are going to discuss the two different versions of Centerpoint Theorem.

1.1.1 Centerpoint Theorem Discrete Version

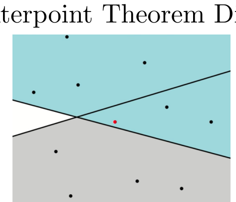
As it's the discrete version, we are going to consider all elements in the space as points and measure just means the numbers of points in the space. Let's call set A as a point set and set S as a subset of $\mathbb{R}^d$. Then, we define $\mu(S) = |S \cap A|$ for any subset S. H^+ and H^- are the closed half-spaces and the amount of points in half space can be expressed as $\mu(H^+) = |H^+ \cap S|$

Question 1. Does there exist some centerpoint c such that each half spaces created by the hyperplanes passing through it have exactly half amount of the points in the whole space S such that $\mu(H^+ \cap S) = \mu(H^- \cap S) = \frac{1}{2}\mu(S)$? Why $\frac{1}{d+1}$ is the optimal?

In other words, the first question means under what kind of situation can any hyperplane passes through point c comes to equipartition shape S. The answer is only when the shape is symmetrical and the hyperplane pass through the centerpoint is able to divide the whole shape into two equal half spaces. In general, for non-equipartition cases, we are able to have the following inequality $|H^+ \cap S| \geq \frac{|S|}{d+1}$ by the centerpoint theorem, and in Figure 1.1.2a it can be concluded the smaller part of the subset always has at least one point which is exactly $\frac{1}{3}$ of the points in this 2 dimensional case. This figure 1.1.2a also illustrates $\frac{1}{d+1}$ is the optimal case. Also if we rotate the line about the point, this property won't change. In Figure 1.1.2b, the red point is the centerpoint and there are two lines across the space. For each space contain the red point, it always has more than $\frac{|S|}{d+1}$ amount of points.



(a) Centerpoint Theorem Discrete 1



(b) Centerpoint Theorem Discrete 2

Figure 1.1.2. Centerpoint Theorem Discrete Version

1.1.2 Centerpoint Theorem Continuous Version

Now we are going to talk about the continuous version. In this version, instead of looking at points we are going to take it as the areas that are divided by lines within the space.

Let $h : \mathbb{R}^2 \to [0, +\infty)$ be a function such that $\iint_{\mathbb{R}^2} h(x, y)dA < +\infty$. x, y is the coordinates in the two dimensional space A. Such an h is called a density function. Given the density function, the measure is defined by the following definition.

Definition 1.1.1. (Measure). *For any measure for the line contain c such that $\mu(S) = \iint_S h(x, y)dA = \iint_{S\cap A} h(x, y)dA$, and this double integral equals to the common area of S and A, for some $S \subseteq \mathbb{R}^2$.*

As a result, for the measure of half spaces it can be expressed as $\mu(H^+) = \iint_{H^+} h(x, y)dA$, and for $\mu(\mathbb{R}^2)$, we have $\mu(\mathbb{R}^2) = \iint_{\mathbb{R}^2} h(x, y)dA = \iint_A 1dA$. It equals to the area of A and less than positive infinity.

Question 2. How close can you come to equipartition for non centrally symmetric shapes?

As we are looking at non centrally symmetric shapes, we know it's impossible for hyperplanes to perfectly divide the whole space into two equal half spaces. Now we are going to show how the centerpoint theorem can be rephrased into closeness of equipartition. Then, we have this inequality $|\mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2}| \leq \alpha\mu(\mathbb{R}^d)$ and we are trying to find this α.

As we are looking at half spaces, we know each half space has at least $\frac{\mu(\mathbb{R}^d)}{d+1}$ of area. Then, we have these two inequality $\mu(H^+) \geq \frac{\mu(\mathbb{R}^d)}{d+1}$ and $\mu(H^-) \geq \frac{\mu(\mathbb{R}^d)}{d+1}$. Then, we have $\mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2} \geq \frac{\mu(\mathbb{R}^d)}{d+1} - \frac{\mu(\mathbb{R}^d)}{2}$. It's lower bound for $\mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2}$. Now we are going to find the upper bound for it. As $\mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2} = \mu(\mathbb{R}^d) - \mu(H^-) - \frac{\mu(\mathbb{R}^d)}{2} = \frac{\mu(\mathbb{R}^d)}{2} - \mu(H^-)$. As we know $\mu(H^-) \geq \frac{\mu(\mathbb{R}^d)}{d+1}$, then $\frac{\mu(\mathbb{R}^d)}{2} - \mu(H^-) \leq \frac{\mu(\mathbb{R}^d)}{2} - \frac{\mu(\mathbb{R}^d)}{d+1}$. Thus, we have $-\frac{(d-1)}{2(d+1)}\mu(\mathbb{R}^d) \leq \mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2} \leq \frac{(d-1)}{2(d+1)}\mu(\mathbb{R}^d)$ As a result, $|\mu(H^+) - \frac{\mu(\mathbb{R}^d)}{2}| \leq \frac{d-1}{2(d+1)}\mu(S)$ and $\alpha = \frac{d-1}{2(d+1)}$ is the optimal constant. It represents the value as close as possible for each hyperplane to come to equipartition for non centrally symmetric shapes.

As my problem is a 2 dimensional case, so $d = 2$.

Lemma 1.1.1. (Centerpoint Theorem for 2 dimensional half spaces). *Give a measure μ, there exists some point c in $\mathbb{R}^2$ such that for every half space whose boundary line contain c, $|\mu(H^+) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{1}{6}\mu(\mathbb{R}^2)$. Conversely, if there exists some c such that $|\mu(H^+) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{1}{6}\mu(\mathbb{R}^2)$ for every half-space whose boundary line contains c, then c is a centerpoint.*

Definition 1.1.2. *Let f be a continuous map from S^1 to $\mathbb{R}$, then we define $f_c(\theta) = \mu(H^+(\theta))$ for all $c \in \mathbb{R}^2$. This function represents the amount of area in the half space $H^+(\theta)$ and the area depends on the amount of rotation of the lines.*

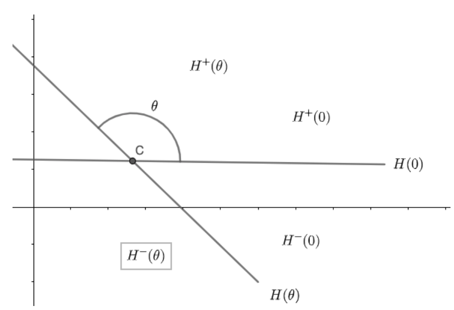


Figure 1.1.3. Half Spaces

In Figure 1.1.3, θ represents the angle the horizontal line passing through c rotates. Look at a horizontal line passing through c. We are going to call that line $H(0)$, $H^+(0)$ as the top half-space and $H^-(0)$ as the bottom half-space. $H(\theta)$ represents rotating $H(0)$ by θ counter clcok-wise about point c. Also it can be discovered that if we rotate the line counter clock-wise by π, we will get $H^-(\theta)$ such that $H^+(\theta + \pi) = H^-(\theta)$.

We have an equivalent interpretation for the Centerpoint Theorem in terms of angle θ when $d = 2$. For all c in $\mathbb{R}^2$, we have $f_c(\theta) = \mu(H^+(\theta))$. There exists some c in $\mathbb{R}^2$ such that $|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{\mu(\mathbb{R}^2)}{6}$ and it's true for all θ. Then, we can express the centerpoint theorem as follows: c is a centerpoint if and only if $|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{\mu(\mathbb{R}^2)}{6}$.

1.2 My problem: A Multi Centerpoint analogue of the centerpoint Theorem

For my problem, I'm looking for a multi-centerpoint theorem and there is only a single mass on the plane so it's a 2 dimensional case. Like the function for the single point case, there is a function for multiple points case. For all c_1, c_2 in $\mathbb{R}^2$, we have a continuous map $f_{(c_1,c_2)}(\theta_1, \theta_2) : S^1 \times S^1 \to \mathbb{R}$ or with another experssion $f_{(c_1,c_2)}(\theta_1, \theta_2) : T^2 \to \mathbb{R}$, $f_{(c_1,c_2)}(\theta_1, \theta_2) = \mu(H_1^+(\theta) \cap H_2^+(\theta_2))$. I am going to find out do there exist points c_1, c_2 in $\mathbb{R}^2$ such that for every line H_1 passing through c_1 and any line H_2 passing through c_2, is it true that

$$|f_{(c_1,c_2)}(\theta_1, \theta_2) - \frac{\mu(\mathbb{R}^2)}{4}| \leq \alpha\mu(\mathbb{R}^2) \tag{1.2.1}$$

for all θ_1, θ_2? What is the optimal α for this inequality? Or in other words, how close do any two lines passing through these respective points come to equipartitioning $\mu(\mathbb{R}^2)$?

As shown in Figure 1.2.1, there will be two lines H_1 and H_2 each contain c_1 and c_2 across the whole space and I will look at their common region and do Fourier Analysis on $f_{(c_1,c_2)}$. I will try to eliminate as many coefficients as possible in order to find the upper bound for α and the technique I am going to apply in finding α is Parseval's Identity. In the next chapter, I will fully discuss Fourier Analysis on the Torus.

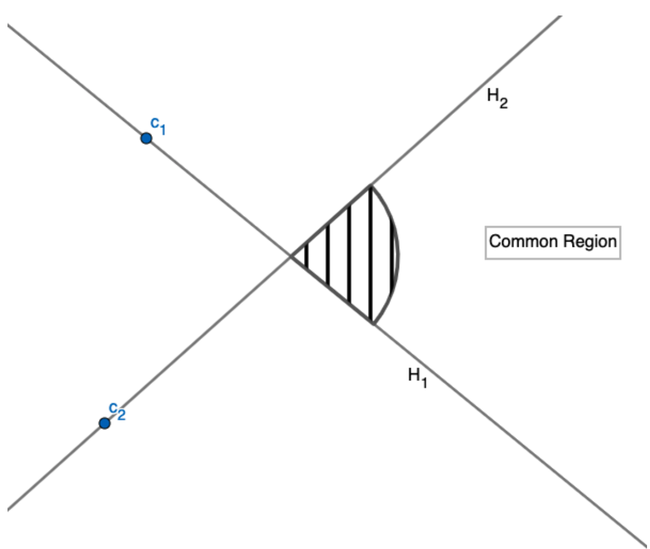


Figure 1.2.1. Setup

2

Fourier Analysis on the Torus

In this chapter, we are going to develop the the basic Fourier analysis on the Torus and the following propositions and theorems are the tools we are going to use.

Definition 2.0.1. *S^1 is the unit circle. $S^1 = \{e^{i\alpha} : 0 \leq \alpha \leq 2\pi\} = \{z \in \mathbb{C}^* : |z| = 1\} = \{z \in \mathbb{C}^* : z\bar{z} = 1\}$.*

2.1 Hermitian Inner Product

Definition 2.1.1. *A Hermitian inner product on a complex vector space V is a complex-valued bilinear form on V which is antilinear in the second slot, and is positive definite.*

Remark 2.1.1. *Hermitian Inner Product has the following properties, where $\bar{z}$ denotes the conjugate of z*

1. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

   $$\langle u+v, w\rangle = \overline{\langle w, u+v\rangle} = \overline{\langle w, u\rangle + \langle w, v\rangle} = \overline{\langle w, u\rangle} + \overline{\langle w, v\rangle} = \langle u, w\rangle + \langle v, w\rangle$$

2. $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

   $$\langle u, v+w\rangle = \overline{\langle v+w, u\rangle} = \overline{\langle v, u\rangle + \langle w, u\rangle} = \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} = \langle u, v\rangle + \langle u, w\rangle$$

3. Linearity on the left: $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$

4. $\langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle$

   Conjugate linearity on the right: $\langle u, \alpha v\rangle = \overline{\langle \alpha v, u\rangle} = \overline{\alpha\langle v, u\rangle} = \bar{\alpha}\langle u, v\rangle$

5. $\langle u, v\rangle = \overline{\langle v, u\rangle}$

6. $\langle u, u\rangle \geqslant 0$

7. $\langle u, u\rangle = 0$, if and only if $u = 0$.

For the discussion of Fourier analysis: in all cases, the complex vector space is all continuous functions from G to $\mathbb{C}$, where G is either the unit circle, S^1, the torus, $T^2 = S^1 \times S^1$, or the real numbers $\mathbb{R}$ and they all satisfy the property stated above. For function on the unit circle, it's defined as $\langle f, \phi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$. We will further discuess this function later in this chapter.

Proposition 2.1.1. *Let f be a continuous function from S^1 to $\mathbb{C}$, then $\|f\|_2 < \infty$*

The L^2 norm of f is $\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \left(\int_{S^1} |f(z)|^2 dz\right)^{\frac{1}{2}}$ for some $z \in S^1$, with an interval $[-\pi, \pi]$. Hence, $\|f\|_2 < \infty$.

We will talk about the L^2 convergence of Fourier Series later in this chapter 2.4.

2.2 Character theory of S^1

Proposition 2.2.1. *S^1 is a group under complex multiplication.*

S^1 is set defined as $S^1 = \{e^{i\alpha} : 0 \leq \alpha \leq 2\pi\} = \{z \in \mathbb{C}^* : |z| = 1\} = \{z \in \mathbb{C}^* : z\bar{z} = 1\}$.

In order to show S^1 is a group under complex multiplication, we need to show these three properties:

1. S^1 is closed under multiplication

2. Associativity

3. The identity $1 \in S^1$

4. If $z \in S^1$, then its multiplicative inverse $z^{-1} \in S^1$

Proof. (1). Let $e^{i\theta}, e^{i\phi} \in S^1$. Then, by multiplication we have $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$. Let $z = e^{i\theta}$ and $w = e^{i\phi}$. As we know $|z| = |w| = 1$, then we have $|zw| = |z||w|$. As a result, we have $zw \in \phi$. Thus, S^1 closed under multiplication.

(2). Let $e^{i\theta}, e^{i\phi} \in S^1$. Then, we have $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)} = e^{i\phi} \cdot e^{i\theta}$. Hence, it's associative.

(3). Let $z \in \mathbb{C}$. Then, we have $|z| = 1$. Hence, the identity $1 \in S^1$.

(4). Let $z \in \mathbb{C}$. Then, we have $|z| = 1$. As z is the identity, we have $|\frac{1}{z}| = 1$. $|z| \cdot |\frac{1}{z}| = 1$. Thus, if $z \in S^1$, then its multiplicative inverse $z^{-1} \in S^1$.

From the four steps above, we know S^1 is a group under complex multiplication. □

2.2.1 The character theory of the circle group S^1 and Torus Group T^2

Definition 2.2.1. The Character group of an arbitrary group G. *The character group of group G is notified as $Ch(G)$ and it's a set of all continuous homomorphisms from G to S^1.*

Proposition 2.2.2. *$Ch(G)$ is a group under multiplication of functions.*

In order to prove $Ch(G)$ is a group under multiplication of functions, we need to show it in different three steps:

1. Associativity

2. identity

3. inverse

Proof. (1). Let $f, g, h \in Ch(G)$. Then, we have $((f \cdot g) \cdot h)(a) = ((f(a) \cdot g(a)) \cdot h(a) = f(a) \cdot (g(a) \cdot h(a)) = (f \cdot (g \cdot h))(a)$. Hence, we proved its associativity.

(2). Let $g \in Ch(G)$ such that $g(a) = 1$ and $1 \in Ch(G)$. Then, if we have $f \in Ch(G)$, then we have $(f \cdot g)(a) = f(a) \cdot 1 = f(a) = 1 \cdot f(a) = (g \cdot f)(a)$. As a result, g is the identity.

(3) Let $f \in Ch(G)$ and let $f^{-1}(a) = f(a)^{-1}$. We also need to show f^{-1} is homomorphism $f^{-1}(ab) = f(ab)^{-1} = (f(a)f(b))^{-1} = f(a)^{-1}f(b)^{-1} = f^{-1}(a)f^{-1}(b)$.

As a result, we prove $Ch(G)$ is a group under multiplication of functions. □

We will now characterize the character group when $G = \mathbb{R}$, S^1, and T^2.

Theorem 2.2.1. *Every group homomorphism of S^1 are the continuous homomorphisms of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$. In other words, $\phi : S^1 \to \mathbb{C}^*, \phi_n(z) = z^n$ and $\phi_n(e^{i\theta}) = e^{in\theta}$ for some $n \in \mathbb{Z}$. Vice versa: $\phi_n(z) = z^n$ is a continuous homoromorphism for every integer n. This thoerem can also be know as $Ch(\mathbb{S}^1) \cong \mathbb{Z}$.*

In order to prove this theorem, we need the result of $Ch(\mathbb{R}) \cong \mathbb{R}$ to prove it.

Proof. $Ch(\mathbb{R}) \cong \mathbb{R}$. $Ch(R)$ means all continuous homomorphism from $\mathbb{R}$ to S^1. In other words, consider $(\mathbb{R}, +)$, if $f : \mathbb{R} \to S^1$ is a continuous homomorphism, then f is given by the formula $f(x) = e^{i\alpha x}$, where α is any real number.

Suppose $r \in Ch(\mathbb{R})$. By definition, r is a continuous homomorphism from $\mathbb{R}$ to S^1. Claim: $r(x) = e^{2\pi i\alpha x}$, where $r(1) = e^{2\pi i\alpha}$. Then, we have $r(m) = e^{2\pi i\alpha m}$ for some $m \in \mathbb{R}$. Also $n \cdot r(\frac{1}{n}) = e^{2\pi i\alpha}$ and then $r(\frac{1}{n}) = e^{\frac{2\pi i\alpha}{n}}$ for some $n \in \mathbb{R}$. By multiplying them, we have $r(\frac{m}{n}) = e^{2\pi i\alpha\frac{m}{n}}, \forall \frac{m}{n} \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and r is continuous, we have $r(x) = e^{2\pi i\alpha x}$ for all $x \in \mathbb{R}$. As a result, $Ch(\mathbb{R}) \cong \mathbb{R}$.

$Ch(\mathbb{S}^1) \cong \mathbb{Z}$. $Ch(S^1)$ means all continuous homomorphism from S^1 to S^1. In other words, if $\psi : \mathbb{R} \to S^1 \to S^1$ is a continuous homomorphism and it's consist of two continuous homorphism, then there exists $\alpha \in \mathbb{R}$ such that $\psi(x) = e^{2\pi i\alpha x}$ for all $x \in \mathbb{R}$.

This theorem is based on $Ch(\mathbb{R}) \cong \mathbb{R}$. Suppose $\phi : S^1 \to S^1$. Let ψ be a continuous homomorphism from $\mathbb{R}$ to S^1 and ψ can be decomposed into two separate functions. $\psi = \phi \circ r$. The first one is what we used in the first theorem, r is a continuous homomorphism from $\mathbb{R}$ to S^1. The second one is what we just defined $\phi : S^1 \to S^1$. By $Ch(\mathbb{R}) \cong \mathbb{R}$, we know there exists $\alpha \in \mathbb{R}$ such that $\psi(x) = e^{2\pi i\alpha x}$. However, we know $\psi(x) = \psi(x+1)$ as $e^{2\pi i\alpha x} = e^{2\pi i\alpha(x+1)}$. Thus, $e^{2\pi i\alpha} = 1$ and we know $\alpha \in \mathbb{Z}$. Then, $\psi(x) = \phi(e^{2\pi ix}) = e^{2\pi i\alpha x} = (e^{2\pi ix})^\alpha$. Let $e^{2\pi i\alpha x} = y$, then we can conclude $\phi(y) = y^\alpha$.

Hence, we proved Theorem 2.2.1. □

Now we are going to prove the case for the Torus by using Theorem 2.2.1.

Theorem 2.2.2. *$Ch(T^2) \cong \mathbb{Z}^2$. $Ch(T^2) = \{\phi_{(n_1,n_2)} \mid (n_1, n_2)\} \in \mathbb{Z}^2\}$, where $\phi_{(n_1,n_2)}$ is a continuous homomorphism from T^2 to S^1.*

Proof. Suppose $\phi \in Ch(T^2)$. By definition, ϕ is a continuous homomorphism from T^2 to S^1 which also means ϕ is a continuous homomorphism from $S^1 \times S^1$ to S^1. As $S^1 \times S^1 = \{(z_1, z_1)|z_1, z_2 \in S^1\}$ and $(z_1, z_2) = (z_1, 1) \cdot (1, z_2)$, by homomorphism we have $\phi(z_1, z_2) = \phi(z_1, 1) \cdot \phi(1, z_2)$. Define ϕ_1 a continuous homomorphism from S^1 to S^1. Then, we have $\phi_1(z_1) = \phi(z_1, 1)$. $\phi_1(z_1 w_1) = \phi(z_1 w_1, 1) = \phi_1((z_1, 1) \cdot (w_1, 1)) = \phi_1(z_1, 1)\phi_1(w_1, 1) = \phi_1(z_1)\phi_1(w_1)$. Thus, we have $\phi(z_1, 1) = z_1^{n_1}$, for some $n_1 \in Z$ and $\phi(1, z_2) = z_2^{n_2}$, for some $n_2 \in Z$. Then, we have $\phi(z, 1)\phi(1, z_2) = z_1^{n_1} \cdot z_2^{n_2}$. Conversely, define $\phi : S^1 \times S^1 \to S^1$. Then, $\phi_{(n_1,n_2)}(z_1, z_2) = z_1^{n_1} \cdot z_2^{n_2}$. Then, this is a homomorphism. Hence, $Ch(T^2) \cong \mathbb{Z}^2$. □

We will now show that the collection of functions of $Ch(S^1)$ and $Ch(T^2)$ are essential in decomposing arbitrary functions $f : S^1 \to \mathbb{C}$ and $f : S^1 \times S^1 \to \mathbb{C}$ (respecively for Torus). Namely, they form an orthonomral basis for the space V above in the sense of series with L^2 convergence.

First, we show that they are orthonormal under the hermitian inner product as metioned in definition 2.1.1.

2.3 Fourier Series for Torus

Proposition 2.3.1. *Orthonormality of characters, $\langle \phi_n, \phi_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$*

Proof. Let V be the set of continuous function $\{f : S^1 \to \mathbb{C}\}$. We know basis for V is $Ch(S^1) \cong Z$ by theorem 2.2.1. As we know $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$ by definition 2.1. Hence, for $\langle \phi_n, \phi_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} e^{-im\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta = \frac{1}{2\pi} \cdot \frac{1}{(n-m)i} e^{i(n-m)\theta} \Big]_{-\pi}^{\pi}$. As a result, it has two different cases, when $n \neq m$, it equals to 0, by periodicity and when $n = m$, it equals to 1. □

Definition 2.3.1. Fourier Coefficients. *Fourier Coefficients are the weights (the scaling factor in front of each term of) the Fourier sinusoidal (functions). Suppose that f could be written as an infinite linear combination. Then, it's coefficients $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$. We will prove it in Proposition 2.3.2.*

Definition 2.3.2. Fourier Series. *Let f_c be a continuous function from S^1 to $\mathbb{R}$. Then, f_c has a Fourier decomposition such that $f_c = \sum_{n\in\mathbb{Z}} C_n e^{in\theta}$.*

Proposition 2.3.2. *If the Fourier Series $f_c = \sum_{n\in\mathbb{Z}} C_n\phi_n$, then its coefficients $C_n = \langle f, \phi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$*

Proof. As $f = \sum_{n\in\mathbb{Z}} a_n\phi_n$ and $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dz$, then by taking the inner product we have $C_n = \langle f, \phi_n\rangle = \langle f, e^{in\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ by $\phi_n = e^{in\theta}$. As a result, if $f = \sum_{n\in\mathbb{Z}} a_n\phi_n$, it's coefficient $C_n = \langle f, \phi_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$. □

Now we are trying to find the Fourier Coefficient for the Torus case.

Proposition 2.3.3. *If the Fourier Series $f_{(c_1,c_2)} = \sum_{m,n\in\mathbb{Z}} C_{(m,n)}\phi_m\phi_n$, then its coefficients $C_{(m,n)} = \langle f_{(c_1,c_2)}, \phi_m\phi_n\rangle = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-im\theta_1}e^{-in\theta_2}d\theta_1 d\theta_2$.*

Proof. It's the case for Torus. As $f_{(c_1,c_2)} = \sum_{m,n\in\mathbb{Z}} C_{(m,n)}\phi_m\phi_n$ and $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dz$, then by taking the inner product we have $C_{(m,n)} = \langle f_{(c_1,c_2)}, \phi_m\phi_n\rangle = \langle f_{(c_1,c_2)}, e^{im\theta}e^{in\theta}\rangle = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-im\theta_1}e^{-in\theta_2}d\theta_1 d\theta_2$. As a result, if $f_{(c_1,c_2)} = \sum_{m,n\in\mathbb{Z}} C_{(m,n)}\phi_m\phi_n$, its coefficient $C_{(m,n)} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-im\theta_1}e^{-in\theta_2}d\theta_1 d\theta_2$. □

If f is in V, then f is actually equal to its Fourier series in the L^2 sense in general. And likewise for the Torus $f_{(c_1,c_2)}$. This is the importance of the character group.

2.4 L^2 convergence of Fourier Series

This section is directly related to the next section 2.5.

Theorem 2.4.1. *Let $f : S^1 \to \mathbb{C}$ is continuous, then Fourier series for f converges to f and also in L^2; i.e.,*

$$\lim_{N\to\infty} \int_0^{2\pi} \|f - F_N\|_2^2 = 0$$

Proposition 2.4.1. *$\|f - \sum_{n=-N}^{N} a_n\phi_n\|_2$ converges to 0, as $N \to \infty$.*

We know $f = \sum_{n\in Z} a_n\phi_n$. For $g_n \to g$, it uses the L^2 norm: $\lim_{n\to\infty} \|g_n - g\|_2 = 0$

2.5 Parseval's identity

Theorem 2.5.1. *Parseval's identity is about the sum of the squares of the Fourier coefficients of a function is equal to the integral of the square of the function. $||f||_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |C_n|^2$. The Fourier coefficients is given by $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$*

Proof.

$$\begin{aligned}
\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \int_0^{2\pi} \overline{f(re^{i\theta})} f(re^{i\theta}) d\theta && (2.5.1)\\
&= \int_0^{2\pi} \sum_m \overline{a_m} r^m e^{-im\theta} \sum_n a_n r^n e^{in\theta}\ d\theta && (2.5.2)\\
&= \sum_{m,n} \overline{a_m} a_n r^{m+n} \int_0^{2\pi} e^{i\theta(m-n)} d\theta && (2.5.3)\\
&= \sum_{m,n} \overline{a_m} a_n r^{m+n} 2\pi\delta_{mn} && (2.5.4)\\
&= 2\pi \sum_n \overline{a_n} a_n r^{n+n} && (2.5.5)\\
&= 2\pi \sum_n |a_n|^2 r^{2n} && (2.5.6)
\end{aligned}$$

There is also another way of proving just by using the he orthonormality of characters.

$||f||_2^2 = \langle f, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{f(x)}dx = \sum_{-\infty}^{\infty} |C_n|^2$ □

For the Parseval's identity of the Torus, we have $||f_{(c_1,c_2)}||_2^2 = \langle f_{(c_1,c_2)}, f_{(c_1,c_2)}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{(c_1,c_2)}(x)\overline{f_{(c_1,c_2)}(x)}dx = \sum_{-\infty}^{\infty} |C_{(m,n)}|^2$

3

Applying Fourier Analysis on the problem

As we have talked about A Multi Centerpoint analogue of the centerpoint Theorem in subsection 1.2, I am going to specifically talk about how Fourier Analysis is applied on my problem.

3.1 $f_{(c_1,c_2)}(\theta)$

Firstly, let's go over the single centerpoint case. For each c in $\mathbb{R}^2$, define $f_c : S^1 \to \mathbb{R}$ by $f_c(\theta) = \mu(H^+(\theta))$ and f_c a continuous map. As we have seen, the centerpoint theorem is equivalent to the existence of some c in $\mathbb{R}^2$ such that $|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{\mu(\mathbb{R}^2)}{6}$ for all θ. Then, we can express the centerpoint theorem for the single centerpoint as follows: c is a centerpoint if and only if

$$|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}| \leq \frac{\mu(\mathbb{R}^2)}{6}. \tag{3.1.1}$$

Now let's go over the multicenterpoint case. For each c_1, c_2 in $\mathbb{R}^2$, define $f_{c_1,c_2}(\theta_1, \theta_2) : S^1 \times S^1 \to \mathbb{R}$ by $f_{c_1,c_2}(\theta_1, \theta_2) = \mu(H_1^+(\theta) \cap H_2^+(\theta_2))$ and similar to the case of single centerpoint case f_{c_1,c_2} is also a continuous map. There are two different kinds of expression for this continuous map $f_{c_1,c_2}(\theta_1, \theta_2) : S^1 \times S^1 \to \mathbb{R}$ or $f_{c_1,c_2}(\theta_1, \theta_2) : T^2 \to \mathbb{R}$ as $S^1 \times S^1$ is a torus. As we have seen, the centerpoint theorem is equivalent to the existence of some c_1, c_2 in $\mathbb{R}^2$ such that $|f_{c_1,c_2}(\theta_1, \theta_2) - \frac{\mu(\mathbb{R}^2)}{4}| \leq \alpha\mu(\mathbb{R}^2)$ for all θ. Then, we can express the centerpoint theorem for the multicenterpoint as follows: c_1 and c_2 are centerpoints if and only if

$$|f_{c_1,c_2}(\theta_1,\theta_2) - \frac{\mu(\mathbb{R}^2)}{4}| \leq \alpha\mu(\mathbb{R}^2). \tag{3.1.2}$$

As any two lines passing through those two points in the whole plane, in the optimal case, there are two lines are going to cut the whole plane into four equal quarters. For the single centerpoint case, I am trying to show how close can the line passes through the centerpoint is able to divide the whole plane into two closed equal half spaces. For the multicenterpoint case, I am trying to find how close can the two lines pass through the two centerpoints are able to divide the whole place into four closed equal quarters spaces. As a result, we have this inequality 3.1.2 and I am trying to find the optimal α for this inequality.

3.2 Fourier Decomposition

As we know for a continuous map from S^1 to $\mathbb{R}$, it has a Fourier decompostion. Likewise the same for a continuous map from $S^1 \times S^1$ to $\mathbb{R}$, it also has a Fourier decomposition. I will try to use Fourier Analysis to find out how close can the multicenterpoint case come to equipartitioning. For the single centerpoint case, $f_c : S^1 \to \mathbb{R}$ is a continuous map so it has a Fourier decomposition. Then, we have $f_c(\theta) = \sum_{-\infty}^{\infty} C_n e^{in\theta}$. Since f_c is real-valued, then we know f_c and it's conjugate are equal. Thus, we have $f_c(\theta) = \overline{f_c(\theta)}$ which also means $C_m = \overline{C_{-m}}$.

Proposition 3.2.1. *If $f_c(\theta) = \overline{f_c(\theta)}$, then $C_m = \overline{C_{-m}}$.*

Proof. As we know if $f_c(\theta)$ is real-valued, then $f_c(\theta) = \overline{f_c(\theta)}$. Then, we have $\overline{C_m} = \overline{\frac{1}{2\pi}\int_0^{2\pi} f_c(\theta)e^{-im\theta}d\theta} = \frac{1}{2\pi}\int_0^{2\pi} \overline{f_c(\theta)e^{-im\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f_c(\theta)e^{im\theta} = C_{-m}$. Hence, if $f_c(\theta) = \overline{f_c(\theta)}$, then $C_m = \overline{C_{-m}}$. □

Similarly, it's the same for $f_{(c_1,c_2)}(\theta_1,\theta_2)$. If $f_{(c_1,c_2)}(\theta_1,\theta_2) = \overline{f_{(c_1,c_2)}(\theta_1,\theta_2)}$, then $C_{(m,n)} = \overline{C_{(-m,-n)}}$.

As a result, under absolute value both positive and negative coefficient need to be considered. Likewise $|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}|$ have a Fourier decomposition such that $|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}| = |\sum_{-\infty}^{\infty} C_n e^{in\theta}|$.

In order to compute the Fourier coefficient, we need to use the idea of Parseval's identity and we are computing the minimum α what we mention in chapter 1 by using the L^2 norm. Before talking about how to calculate the Fourier coefficient, there is a question that why calculating Fourier coefficient is significant in finding those points. In order to answer this question, I am going to prove this following lemma.

Lemma 3.2.1. Fourier character of full equipartition.*A point c for which any line passes through it equipartitions the whole plane if and only if all the Fourier coefficients of f_c, $C_n = 0$ for all $n \neq 0$.*

The reason for why $n \neq 0$ is $C_0 = \frac{\mu(\mathbb{R}^2)}{2}$ which means there are half amount of points are eliminated. The computation for why $C_0 = \frac{\mu(\mathbb{R}^2)}{2}$ is in chapter 4.

Proof. Suppose $C_n = 0$, for all $n \neq 0$, then as the line pass through the point equipartitions we have $||f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}||_2^2 = 0$. Then, we have $\|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}\|_2^2 = \sum_{n\neq 0} |C_n|^2 = 0$. Since we know $C_0 = \frac{\mu(\mathbb{R}^2)}{2}$, then $\|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}\|_2^2 = \|f_c(\theta) - C_0\|_2^2 = 0$. Hence, we have $f_c(\theta) = \frac{\mu(\mathbb{R}^2)}{2}$ and it implies $\mu(H^+(\theta)) = \frac{\mu(\mathbb{R}^2)}{2}$ for all θ. As a result, any line passes through c equipartitions, if and only if $C_n = 0$ for all $n \neq 0$. □

As a result, by applying parseval's identity, we have $||f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}||_2^2 = ||\sum_{n\neq 0} C_n e^{in\theta}||_2^2 = \sum_{n\neq 0} |C_n|^2$.

For the multicenterpoint case, $f_{(c_1,c_2)} : S^1 \times S^1 \to \mathbb{R}$, $f_{(c_1,c_2)}(\theta) = \sum_{m,n\neq 0} C_{(m,n)} e^{in\theta} e^{im\theta}$. Similar to the single point case lemma 3.2.1 also works for the multicenterpoint case such that any two lines passing through respective points equipartitions if and only if $C_{(m,n)} = 0$ for all $(m,n) \neq (0,0)$. The reason why $C_{(0,0)}$ can't be included will be showed in the next chapter by calculation. By applying Parseval's identity, we are able to annihilate some cofficients, $||\sum_{m\neq 0,n\neq 0} C_{(m,n)} e^{in\theta_1} e^{im\theta_2}||_2^2 = \sum_{n\neq 0,m\neq 0} |C_{(m,n)}|^2$. Then, we are able to find a bound on $\|f_{(c_1,c_2)} - \frac{\mu(\mathbb{R}^2)}{4}\|_2$.

The full calculation of Fourier coefficients is in the next chapter.

4

Calculation of Fourier Coefficients

In the last chapter, I explained we are going to use the Parseval's Identity to annihilate some coefficients in order to get a bound on $\|f_{(c_1,c_2)}(\theta_1,\theta_2) - \frac{\mu(R^2)}{4}\|_2^2$. In this chapter, I am going to demonstrate my calculations by firstly showing the one Centerpoint cases, then the two Centerpoints cases. In the end, it will reach the conclusion for the optimal upper bound that I find for the two centerpoints cases.

Before talking about calculations, there are several things that need clarification for techniques I used in calculation.

Question 1. Why is $f_c(\theta) + f_c(\theta+\pi) = \mu(\mathbb{R}^2)$?

As we discussed in Chapter 1, that $\mu(H^+(\theta+\pi)) = \mu(H^-(\theta))$. If we rotate the line anti-clockwise by π, its original closed half space will match with the rest of the whole plane. As a result, $\mu(H^+(\theta)) + \mu(H^-(\theta)) = \mu(\mathbb{R}^2)$. Also as $f_c(\theta) = \mu(H^+(\theta))$ which we discusses in the first chapter, then we have $f_c(\theta) + f_c(\theta+\pi) = \mu(\mathbb{R}^2)$. It's exactly the same case for the two centerpoint cases $f_{(c_1,c_2)}(\theta_1,\theta_2)$,

$$f_{(c_1,c_2)}(\theta_1,\theta_2) + f_{(c_1,c_2)}(\theta_1,\theta_2+\pi) + f_{(c_1,c_2)}(\theta_1+\pi,\theta_2) + f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi) = \mu(\mathbb{R}^2).$$

As we know $f_{(c_1,c_2)}(\theta_1,\theta_2) = \mu(H_1^+(\theta) \cap H_2^+(\theta_2))$ and if we rotate H_1 anti-clockwise by π, we will get $\mu(H_1^+(\theta+\pi) \cap H_2^+(\theta_2))$ and it equals to $f_{(c_1,c_2)}(\theta_1+\pi,\theta_2)$. Then, if we combine

$f_{(c_1,c_2)}(\theta_1+\pi,\theta_2)$ with $f_{(c_1,c_2)}(\theta_1,\theta_2)$, we will get $f(\theta_1,\theta_2)+f(\theta_1+\pi,\theta_2)=\mu(H_2^+(\theta_2))$ which is the region in figure 4.0.2. Hence, figure 4.0.1 shows the situation before rotation and figure 4.0.2 shows the situation after rotation. Similarly for $f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi)+f_{(c_1,c_2)}(\theta_1,\theta_2+\pi)=\mu(H_1^+(\theta)\cap H_2^+(\theta_2+\pi))+\mu(H_1^+(\theta+\pi)\cap H_2^+(\theta_2+\pi))=\mu(H_2^-(\theta_2))$. As $\mu(H_2^+(\theta_2)+\mu(H_2^-(\theta_2)=\mu(\mathbb{R}^2)$. Thus we have $f_{(c_1,c_2)}(\theta_1,\theta_2)+f_{(c_1,c_2)}(\theta_1,\theta_2+\pi)+f_{(c_1,c_2)}(\theta_1+\pi,\theta_2)+f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi)=\mu(\mathbb{R}^2)$.

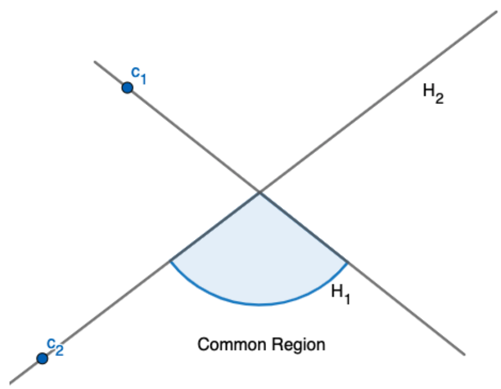


Figure 4.0.1. Rotation of line 1

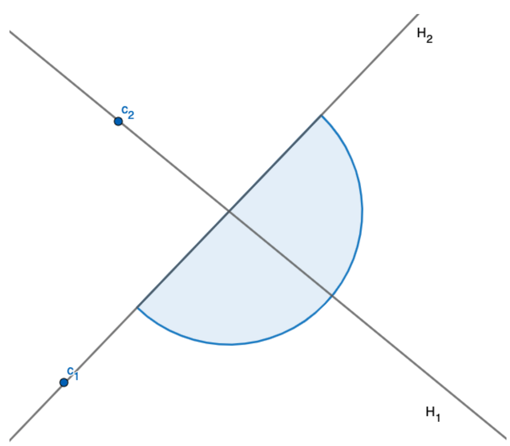


Figure 4.0.2. Rotation of line 2

4.1 One Centerpoint cases

We mentioned this one in Chapter 3.

Here is the calculation for C_0. Let $n = 0$.

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} f_c(\theta)d\theta \tag{4.1.1}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} f_c(\theta)d\theta + \int_{\pi}^{2\pi} f_c(\theta)d\theta] \tag{4.1.2}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} f_c(\theta)d\theta + \int_0^{\pi} f_c(\theta+\pi)d\theta] \tag{4.1.3}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} [f_c(\theta) + f_c(\theta+\pi)]d\theta] \tag{4.1.4}$$
$$= \frac{1}{2\pi}\mu(\mathbb{R}^2)\cdot\pi \tag{4.1.5}$$
$$= \frac{1}{2}\mu(\mathbb{R}^2) \tag{4.1.6}$$

As a result, $C_0 = \frac{1}{2}\mu(\mathbb{R}^2)$. For this case, we eliminate half amount of points in the whole plane.

Now let's look at the even coefficient. C_{2n}, for $n \neq 0$.

$$C_{2n} = \frac{1}{2\pi}\int_0^{2\pi} f_c(\theta)e^{-2in\theta}d\theta \tag{4.1.7}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} f_c(\theta)e^{-2in\theta}d\theta + \int_{\pi}^{2\pi} f_c(\theta)e^{-2in\theta}d\theta] \tag{4.1.8}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} f_c(\theta)e^{-2in\theta}d\theta + \int_o^{\pi} f_c(\theta+\pi)e^{-2in(\theta+\pi)}d\theta] \tag{4.1.9}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} f_c(\theta)e^{-2in\theta}d\theta + \int_0^{\pi} f_c(\theta+\pi)e^{-2in\theta}d\theta] \tag{4.1.10}$$
$$= \frac{1}{2\pi}[\int_0^{\pi} [f_c(\theta) + f(\theta+\pi)]e^{-2in\theta}d\theta] \tag{4.1.11}$$
$$= \frac{\mu(\mathbb{R}^2)}{2\pi}\int_0^{\pi} e^{-2in\theta}d\theta = 0 \tag{4.1.12}$$

As $e^{-2in\theta}$ is a periodic function and we are evaluating at π and 0, we have $2n\pi$ and $2n0 = 0$ in the antiderivative. Also we have proved $C_{-n} = \overline{C_n}$ in proposition 3.2.1 which we proved in last chapter, then we have $C_{2n} = 0$, for all $n \neq 0$.

Now let's discuss the case for C_1 and we need another theorem in order to calculate it.

Theorem 4.1.1. *For any mass μ on $\mathbb{R}^2$, there exists some c in $\mathbb{R}^2$ such that $C_1 = 0$ in the Fourier expansion of f_c.* [1]

For my case, the theorem states above can also be applied on my problem. Hence, we are able to know that there exist some points such that $C_1 = 0$.

Now we reach the question that what does the existence of some c with $C_1 = 0$ and $C_{2n} = 0$ imply ?

Based on this following lemma, we are able to find an upper bound on $\|f_c(\theta) - \frac{\mu(R^2)}{2}\|_2$.

Lemma 4.1.1. *For any mass μ on $\mathbb{R}^2$, there exists some c in $\mathbb{R}^2$ such that* [1]

$$\|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}\|_2 \leq \sqrt{\frac{1}{3} - \frac{2}{\pi^2} - \frac{1}{3 \cdot 2^2}}\mu(\mathbb{R}^2).$$

Base on the inequality we have:

$$\|f_c(\theta) - \frac{\mu(\mathbb{R}^2)}{2}\|_2 \leq \sqrt{\frac{1}{3} - \frac{2}{\pi^2} - \frac{1}{3 \cdot 2^2}}\mu(\mathbb{R}^2) = \sqrt{\frac{1}{4} - \frac{2}{\pi^2}}\mu(\mathbb{R}^2) = \sqrt{\frac{\pi^2 - 8}{4\pi^2}}\mu(\mathbb{R}^2) \tag{4.1.13}$$

Proof. As we know $\|f_c - \mu(\mathbb{R}^2)\|_2^2 = \sum_{n\in\mathbb{Z}} |C_n|^2$, then we can use the fact which we have already calculated and proved to find the upper bound. In last chapter, we proved $C_{-n} = \overline{C_n}$ which also implies $|C_{-n}| = |C_n|$. Then, we have $\sum_{n\in\mathbb{Z}} |C_n|^2 = 2 \cdot \sum_{n=1}^{\infty} |C_n|^2$. Also as we have shown that $C_{2n} = 0$ which means all even coefficient equal to 0, then we have $\sum_{n\in\mathbb{Z}} |C_n|^2 = 2 \cdot \sum_{n=1}^{\infty} |C_n|^2 = 2 \cdot \sum_{n>1(odd)}^{\infty} |C_n|^2$. From Prof. Simon's paper, I get this inequality $|C_n| \leq \frac{\mu(\mathbb{R}^2)}{\pi \cdot |n|}$ [1]. Then, we have $\sum_{n\in\mathbb{Z}} |C_n|^2 \leq 2 \cdot \sum_{n>1(odd)}^{\infty} \frac{(\mu(\mathbb{R}^2))^2}{\pi^2 \cdot n^2} \leq \frac{2(\mu(\mathbb{R}^2))^2}{\pi^2} \sum_{n>1(odd)}^{\infty} \frac{1}{n^2}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and from the calculation of $\sum_{n(even)}^{\infty} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{4k^2} = \frac{\pi^2}{24}$, we have $\sum_{n(odd)}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$. As a result, we have $\sum_{n\in\mathbb{Z}} |C_n|^2 \leq \frac{2(\mu(\mathbb{R}^2))^2}{\pi^2} \sum_{n>1(odd)}^{\infty} \frac{1}{n^2} = \frac{2(\mu(\mathbb{R}^2))^2}{\pi^2} \cdot (\frac{\pi^2}{8} - 1)$. The reason for why we need to minus 1 from $\frac{\pi^2}{8}$ is $C_1 = 0$. As a result, we have $\sum_{n\in\mathbb{Z}} |C_n|^2 \leq (\frac{1}{4} - \frac{2}{\pi^2}) \cdot (\mu(\mathbb{R}^2))^2$. Thus, we proved the lemma that $\|f_c - \mu(\mathbb{R}^2)\|_2 \leq \sqrt{\frac{1}{4} - \frac{2}{\pi^2}}\mu(\mathbb{R}^2)$.

□

4.2 Two Centerpoints cases

We mentioned this one in Chapter 3.

Let $m = 0$ and $n = 0$. $C_{(0,0)}$, for all $(\theta_1, \theta_2) \in [0, 2\pi] \times [0, 2\pi]$.

$$c_{(0,0)} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_{(c_1,c_2)}(\theta_1, \theta_2) d\theta_1 d\theta_2 \tag{4.2.1}$$

$$= \frac{1}{4\pi^2} \int_0^{\pi} \int_o^{\pi} f_{(c_1,c_2)}(\theta_1, \theta_2) + f_{(c_1,c_2)}(\theta_1 + \pi, \theta_2) \tag{4.2.2}$$

$$+ f_{(c_1,c_2)}(\theta_1, \theta_2 + \pi) + f_{(c_1,c_2)}(\theta_1 + \pi, \theta_2 + \pi) d\theta_1 d\theta_2 \tag{4.2.3}$$

$$= \frac{1}{4\pi^2} \mu(\mathbb{R}^2) \cdot \pi \tag{4.2.4}$$

$$= \frac{\mu(\mathbb{R}^2)}{4} \tag{4.2.5}$$

As a result, $C_{(0,0)} = \frac{\mu(\mathbb{R}^2)}{4}$. In this case, we are able to eliminate $\frac{1}{4}$ of the points in the whole Space.

Now let's look at the even coefficients $C_{(2m,2n)}$, for all $(\theta_1, \theta_2) \in [0, 2\pi] \times [0, 2\pi]$ and $n, m \neq 0$.

$$C_{(2m,2n)} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_{(c_1,c_2)}(\theta_1, \theta_2) e^{-2im\theta} e^{-2in\theta} d\theta_1 d\theta_2 \tag{4.2.6}$$

$$= \frac{1}{4\pi^2} \int_0^{\pi} \int_0^{\pi} f_{(c_1,c_2)}(\theta_1, \theta_2) e^{-i2m\theta_1} e^{-i2n\theta_2} \tag{4.2.7}$$

$$+ f_{(c_1,c_2)}(\theta_1 + \pi, \theta_2) e^{-i2m(\theta_1+\pi)} e^{-i2n\theta_2} \tag{4.2.8}$$

$$+ f_{(c_1,c_2)}(\theta_1, \theta_2 + \pi) e^{-i2m(\theta_1)} e^{-i2n(\theta_2+\pi)} \tag{4.2.9}$$

$$+ f_{(c_1,c_2)}(\theta_1 + \pi, \theta_2 + \pi) e^{-i2m(\theta_1+\pi)} e^{-i2n(\theta_2+\pi)} d\theta_1 d\theta_2 \tag{4.2.10}$$

$$= \frac{\mu(\mathbb{R}^2)}{4\pi^2} \int_0^{\pi} e^{-i2m\theta_1} d\theta_1 \cdot \int_0^{\pi} e^{-i2n\theta_2} d\theta_2 = 0 \tag{4.2.11}$$

As $e^{-2in\theta}$ and $e^{-2im\theta}$ are periodic functions and we are evaluating at π and 0, we have $2n\pi$, $2n0 = 0$, $2m\pi$ and $2m0 = 0$ in the antiderivative. Also we have proved $C_{(m,n)} = \overline{C_{(-m,-n)}}$ in proposition 3.2.1 which we proved in last chapter, then we have $C_{(2m,2n)} = 0$, for $m, n \neq 0$.

Now let's look at $C_{(m,2n)}$ for all $(\theta_1,\theta_2) \in [0,2\pi] \times [0,2\pi]$.

$$C_{(m,2n)} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-im\theta}e^{-2in\theta}d\theta_1 d\theta_2 \quad (4.2.12)$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} f(\theta_1,\theta_2)e^{-im\theta_1}e^{-i2n\theta_2} \quad (4.2.13)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2)e^{-im(\theta_1+\pi)}e^{-i2n\theta_2} \quad (4.2.14)$$

$$+ f_{(c_1,c_2)}(\theta_1,\theta_2+\pi)e^{-im(\theta_1)}e^{-i2n(\theta_2+\pi)} \quad (4.2.15)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi)e^{-im(\theta_1+\pi)}e^{-i2n(\theta_2+\pi)}d\theta_1 d\theta_2 \quad (4.2.16)$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}(f(\theta_1,\theta_2) \quad (4.2.17)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2) + f_{(c_1,c_2)}(\theta_1,\theta_2+\pi) \quad (4.2.18)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi))e^{-im\theta_1}e^{-i2n\theta_2}d\theta_1 d\theta_2 \quad (4.2.19)$$

Now let's look at $C_{(2m,n)}$ For all $(\theta_1,\theta_2) \in [0,2\pi] \times [0,2\pi]$.

$$C_{(2m,n)} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-i2m\theta}e^{-in\theta}d\theta_1 d\theta_2 \quad (4.2.20)$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} f_{(c_1,c_2)}(\theta_1,\theta_2)e^{-i2m\theta_1}e^{-in\theta_2} \quad (4.2.21)$$

$$+ f_{(c_1,c_2)}(\theta_1+\pi,\theta_2)e^{-i2m(\theta_1+\pi)}e^{-in\theta_2} \quad (4.2.22)$$

$$- f_{(c_1,c_2)}(\theta_1,\theta_2+\pi)e^{-i2m(\theta_1)}e^{-in(\theta_2+\pi)} \quad (4.2.23)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi)e^{-i2m(\theta_1+\pi)}e^{-in(\theta_2+\pi)}d\theta_1 d\theta_2 \quad (4.2.24)$$

$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}(f_{(c_1,c_2)}(\theta_1,\theta_2) \quad (4.2.25)$$

$$+ f_{(c_1,c_2)}(\theta_1+\pi,\theta_2) - f_{(c_1,c_2)}(\theta_1,\theta_2+\pi) \quad (4.2.26)$$

$$- f_{(c_1,c_2)}(\theta_1+\pi,\theta_2+\pi))e^{-i2m\theta_1}e^{-in\theta_2}d\theta_1 d\theta_2 \quad (4.2.27)$$

Based on the calculation of $C_{(m,2n)}$ and $C_{(2m,n)}$, we are unable to eliminate any points from these two coefficients.

For the case of $C_{(0,1)} = C_{(1,0)}$, based on theorem 4.1.1 we should have a similar result to the one centerpoint case. As a result, we know there should be some $c_1, c_2 \in \mathbb{R}^2$ such that $C_{(0,1)} = C_{(1,0)} = 0$ in the Fourier expansion of $f_{(c_1,c_2)}$.

4.3 Conclusion

Based on the calculation we have for the two centerpoints Case, we know for some c_1, c_2 in $\mathbb{R}^2$ such that $C_{(1,0)} = C_{(0,1)} = 0, C_{(2m,2n)} = 0$, for $m, n \neq 0$ and $C_{(0,0)} = \frac{\mu(\mathbb{R}^2)}{4}$.

Similar to the procedure for the one centerpoint case, we have $\|f_{(c_1,c_2)}(\theta_1,\theta_2) - \frac{\mu(R^2)}{4}\|_2^2 = 2 \cdot \sum_{m\neq 0}^{\infty} |C_{(m,0)}|^2 + 2 \cdot \sum_{n\neq 0}^{\infty} |C_{(0,n)}|^2 + 2 \cdot \sum_{m,n\neq 0}^{\infty} |C_{(m,n)}|^2$ and it should has an upper bound.

These are the conjecture on Fourier coefficients for the two centerpoints case based on one centerpoint case:

$|c_{(m,0)}| \leq \frac{\mu(\mathbb{R}^2)}{|m|\cdot\pi}$ for $m \neq 0$

$|c_{(0,n)}| \leq \frac{\mu(\mathbb{R}^2)}{|n|\cdot\pi}$ for $n \neq 0$

$|c_{(m,n)}| \leq \frac{\mu(\mathbb{R}^2)}{|nm|\cdot\pi}$ for $m, n \neq 0$

Then, we have this inequality:

$\|_{(c_1,c_2)}(\theta_1,\theta_2) - \frac{\mu(R^2)}{4}\|_2^2 \leq \frac{2(\mu(\mathbb{R}^2))^2}{\pi^2} \cdot (\sum_{m\neq 0}^{\infty} \frac{1}{m^2} + \sum_{n\neq 0}^{\infty} \frac{1}{n^2} + \sum_{m,n\neq 0}^{\infty} \frac{1}{m^2 n^2})$.

I tried to find an upper bound by doing the similar procedure given in the one centerpoint case. However, there is not enough time to reach the final result.

Bibliography

[1] Steven Simon, *Measure Partitions via Fourier Analysis II: Center Transversality in the L2-norm for Complex Hyperplanes.* arXiv:1506.06610v2.

[2] Serge Lang, *Undergraduate Analysis*, Springer, 2005.