# Classification of Distinct Fuzzy Subgroups of the Dihedral Group $D_{p^n q}$ for $p$ and $q$ distinct primes and $n \in \mathbb{N}$

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# Contents

# Notations

1 $\mathbb{N}$: The set of natural numbers

2 $\mathbb{Z}$: The set of integers

3 $f : X \to Y$ A mapping from set $X$ to set $Y$

4 $f(x)$: The image of $x$ under $f$

5 Im$\mu$: The image set of $\mu$

6 $\mathcal{F}_\mu$: The family $\{\mu_t : t \in Im\mu\}$



7 $\mathcal{I}^X$: The set of all fuzzy subsets $\mu$ of a non-empty set $X$

8 $\mathcal{I}^G$: The set of all fuzzy subsets $\mu$ of a group $G$

9 $\mathcal{F}(G)$: The set of all fuzzy subgroups $\mu$ of a group $G$

10 $G \approx H$: $G$ is isomorphic to $H$

11 $|G|$: The order of $G$

12 $\langle a \rangle$: The cyclic subgroup generated by $a$

13 $\langle a, b\rangle$: The subgroup generated by $a$ and $b$

14 $D_s^{a^ib} = \langle a^r, a^ib\rangle = \langle a^r, a^ib \mid (a^r)^s = e = b^2 = (ab)^2\rangle$: The dihedral subgroups generated by $a^r$ and $a^ib$, for $0 \leq i \leq r-1$

15 $x \sim y$: $x$ is related to $y$

16 $\cup_{i\in I} A_i$: Union of sets $A_i$, $i \in I$ (indexed set)

17 $(\mathcal{M}(D_{p^nq}))_c$: The number of cyclic maximal chains of $D_{p^nq}$

18 $(\mathcal{M}(D_{p^nq}))_d$: The number of $d$-cyclic maximal chains of $D_{p^nq}$



19 $(\mathcal{M}(D_{p^nq}))_{2d}$: The number of $2d$-cyclic maximal chains of $D_{p^nq}$

20 $(\mathcal{M}(D_{p^nq}))_b$: The number of $b$-cyclic maximal chains of $D_{p^nq}$

21 $\mathcal{F}(D_{p^nq})_d$: The number of distinct fuzzy subgroups from $d$-cyclic maximal chains of $D_{p^nq}$

22 $\mathcal{F}(D_{p^nq})_{2d}$: The number of distinct fuzzy subgroups from $2d$-cyclic maximal chains of $D_{p^nq}$

23 $\mathcal{F}(D_{p^nq})_{3d}$: The number of distinct fuzzy subgroups from $3d$-cyclic maximal chains of $D_{p^nq}$

# Abstract

In this dissertation, we classify distinct fuzzy subgroups of the dihedral group $D_{p^nq}$, for $p$ and $q$ distinct primes and $n \in \mathbb{N}$, under a natural equivalence relation of fuzzy subgroups and a fuzzy isomorphism. We aim to present formulae for the number of maximal chains and the number of distinct fuzzy subgroups of this group. Our study will include some theory on non-abelian groups since the classification of distinct fuzzy subgroups of this group relies on the crisp characterization of maximal chains. We give the definition of a natural equivalence relation introduced by Murali and Makamba in [67] which we will use in this study. Based on this definition, we introduce two counting techniques that we will use to compute the number of distinct fuzzy subgroups of $D_{p^nq}$. In this dissertation, we use the criss-cut counting technique as our primary method of enumeration, and the cross-cut method serves as a means of verifying results we obtain from our primary method.

To classify distinct fuzzy subgroups of this group, we begin by investigating the dihedral groups $D_{p^nq}$, for $p$ and $q$ distinct primes and specific values of $n = 2$ and 3 to observe a trend. We classify the flags of these groups using the characterization of flags introduced in [93]. From this characterization, we then present formulae for the number of distinct fuzzy subgroups attributed to the flags of $D_{p^2q}$ and $D_{p^3q}$.

To generalise results for $D_{p^nq}$, for $p$ and $q$ distinct primes and $n \in \mathbb{N}$, we characterize the flags of this group and classify them as either cyclic, $md$-cyclic for $1 \leq m \leq n$, or $b$-cyclic. Finally, we establish a general formula for the number of distinct fuzzy subgroups obtainable from these flags.

We conclude by comparing results obtained from using our general formula to those obtained by other researchers for the same group. Based on the results from this study, we give an outline of future research work.

# Chapter 1

# Introduction

In literature, most of the theorems proved about groups are a generalisation of these groups. Even though many of them seem to be completely different from one another, they sometimes possess certain properties which might not be immediately evident from studying a specific group alone, which is why groups are classified according to the structure of their subgroup lattices and maximal chains. Counting the number of maximal chains of subgroups of a finite group is considered as one of the most important problems of combinatorial group theory, and exists as a significant combinatorial method of finding the number of distinct fuzzy subgroups of a particular finite group. The fascinating and appealing nature of groups of regular polygons facilitated the characterisation of the dihedral groups in the crisp case by Rotman [106] as well as Malone and Lyons in [64], who used the theory of successive decomposition generated by idempotence to establish the form of the endomorphism near-ring of the dihedral group of order $2n$. Zazkis and Dubinsky [42] studied the dihedral group $D_n$ with $n$ an odd number, while Cavior [32] presented the formula for counting the total number of subgroups of the dihedral group $D_n$ as $\tau(n) + \alpha(n)$ where $\tau(n)$ is the number of divisors of $n$ and $\alpha(n)$ is the sum of the divisors of $n$. Calhoun [29] extended this notion to the class of groups formed as cyclic extensions of cyclic groups, which include the dihedral groups, the $Z$-meta-cyclic groups, and the abelian groups of the form $\mathbb{Z}_m \times \mathbb{Z}_n$.

In the real physical world, however, human beings encounter objects which may not precisely constitute "classes" or sets as described mathematically in the crisp case; yet human thinking relies on these imprecisely defined classes, particularly in the areas of pattern recognition, communication of information and abstraction. The need for a conceptual framework that could address the issue of imprecision and uncertainty led to the development of fuzzy logic which dates back to Plato, and in the 1920s Lukasiewicz proposed the concept of many-valued logic. The notion of fuzzy sets developed by Zadeh [131] in the 1960s, emphasised the gap existing between perceptual depictions and the usual mathematical representations of reality. Fuzzy logic is a broad term that encompasses areas such as fuzzy arithmetic, fuzzy topology, fuzzy mathematical programming, fuzzy data analysis, and fuzzy graph theory, to name a few, and all of these are referred to as fuzzy set theory. The theory of fuzzy sets gave rise to a number of applications, one of which is the notion of fuzzy groups developed by Rosenfeld [105] in 1971. This inspired the development of fuzzy abstract algebra. The classification of fuzzy subgroups stands on its own, and rests mainly on the combinatorial analysis of the lattice of subgroups. This implies that it cannot be determined from the classification of crisp subgroups alone, a fact which has facilitated the study of classification of fuzzy subgroups.



The development of other notions pertaining to the characterisation of fuzzy subgroups, was studied by, among others, Bhattacharya [25] and Das [39], who used the notion of level subgroups to characterise fuzzy subgroups of finite cyclic groups. Murali and Makamba [67] introduced the idea of an equivalence relation of fuzzy subgroups which we will use in this study. Other researchers who discussed the notion of equivalence of fuzzy subgroups include, Ajmal and Thomas [10], Bentea and Tarnauceanu [20], Degang et al.

[34], Murali [85], Seselja and Tepavcevic [113], Tarnauceanu [123] and Zhang and Zou [136]. The generalised definition of an equivalence relation closely relates to the notion of level subgroups as studied by Bhattacharya [25] and Das [39]. Volf [128] concludes that two fuzzy subgroups $\mu$ and $\nu$ of $G$ are equivalent if they have the same set of level subgroups. Thus the equality of level subgroups in relation to this equivalence is the necessary and sufficient condition for equivalence of two fuzzy subgroups.

The study of fuzzy groups extends to other related concepts such as, product fuzzy subgroups introduced by Foster [49] in 1979 using Rosenfeld's definition and in [98], Osman redefined the product fuzzy subgroups according to the definition by Sherwood [114]. Makamba [65] established the concept of internal and external direct products of fuzzy subgroups, showing that the internal direct product of two fuzzy subgroups of a group is isomorphic to their external direct product. In [26], Bhatacharya and Mukherjee introduced the notion of fuzzy normality. The following researchers also studied the concept of fuzzy normality; Ajmal [9], Ajmal and Thomas [10], Akgul [11], Anthony and Sherwood [14], Bhakat and Das [22], Dib and Hassan [41], Makamba and Murali [66] and Malik, Mordeson, and Nair [77], amongst others. In [105], Rosenfeld proved that while the homomorphic pre-image of a fuzzy subgroup is always fuzzy subgroup, only a homomorphic image of a fuzzy subgroup with the sup property is a fuzzy subgroup. Anthony and Sherwood [14], modified the definition of fuzzy subgroups and studied the effects of the sup property on their homomorphic images. Other researchers who extended the study of homomorphic images and pre-images are Ajmal [4] and [5], Bhatacharya and Mukherjee [26], Eroglu [46] and Mishref [81]. In [131], Zadeh introduced the notion of a fuzzy relation, which was further studied by Rosenfeld [105], Sanchez [109], and Bezdek and Harris [21]. In [132] Zadeh defined a generalisation of the notion of equivalence and its relationship with fuzzy relations. Further studies that combine fuzzy relations and equivalence were conducted by Murali [84], who defined properties of fuzzy equivalence relations on a set, showing the existence of a correspondence between fuzzy equivalence relations and certain classes of fuzzy subsets. Schmechel and Thiele [110] studied the concepts of fuzzy equivalence relations and fuzzy partitions, and Boixader, Jacas and Recasens [28] presented an overview of the different aspects of the concept of fuzzy equivalence relation. The natural equivalence we will use in this study was introduced by Murali and Makamba in [67], and further studied by the same researchers in [68] and [69]. In [90], Ndiweni and Makamba utilized the definition of an equivalence relation in [67], to obtain the number of distinct fuzzy subgroups of the dihedral groups $D_n$ for $n$ a product of distinct primes. In general, the number of fuzzy subgroups of a group is infinite if there is no equivalence relation, and this includes the trivial group $e$.

Equivalence relations are paramount in the characterisation of fuzzy subgroups of finite groups. Different researchers have developed and studied different concepts of equivalence relations, and these resulted in the development of various enumeration techniques for computing the number of distinct fuzzy subgroups of a group. In this dissertation, we use two enumeration methods derived from the definition of an equivalence relation in [67].



The classification of fuzzy subgroups of finite groups has largely been focused on finite abelian groups. In [48], Lazlo conducted a study on the construction of fuzzy subgroups of orders one to six. Zhang and Zou [136] determined a formula for the number of fuzzy subgroups of cyclic groups of order $p^n$, for $p$ a prime, while in [70] Murali and Makamba counted the number of fuzzy subgroups of the abelian $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ groups. Mordeson, Bhutani and Rosenfeld [27] found, up to a natural equivalence, the number of fuzzy subgroups of certain finite abelian groups. Ngcibi, Murali and Makamba [76] characterised and determined a formula for the number of fuzzy subgroups, up to a natural equivalence, of rank two abelian $p$-Groups. Ju-Mok Oh [97] extended this study to find an explicit formula $\forall\ n$ and $m$. Modifying the results of Murali and Makamba in [67], Tarnauceanu and Bentea [122] established a reccurence formula for the number of distinct fuzzy subgroups of finite cyclic groups. Using the definition of equivalence in [70], Ndiweni [88] conducted a study on the abelian groups $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m},\ \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m} + \mathbb{Z}_r\ and\ \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m} + \mathbb{Z}_{r^s}$ where $p,\ q$, and $s$ are distinct primes. Sulaiman and Ahmad [3] determined the number of fuzzy subgroups of the finite abelian group $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \times \mathbb{Z}_s$, where $p,\ q,\ r,\ s$ are distinct primes. Humera and Raza [56] classified the fuzzy subgroups of the finite abelian group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ where $p_1$, $p_2, \cdots, p_n$ are distinct primes, using the equivalence relation defined by Sulaiman and Ahmad in [3]. Tarnaceanu [127] introduced a new equivalence relation, extending the study on equivalence by Murali and Makamba in [67], [68] and [69]. He used this definition to classify fuzzy subgroups of cyclic groups, elementary abelian $p$-Groups, dihedral groups, and symmetric groups. Sehgal A, Sehgal B and Sharma [112] determined the number of fuzzy subgroups of the finite abelian $p$-Group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where $p$ is a prime, and Appiah and Makamba [15] determined the number of distinct fuzzy subgroups of some rank-3 abelian groups.



Not many studies have been conducted on the classification of fuzzy subgroups of non-abelian groups. Although there are a number of researchers that have studied these groups, they have not used the same definition of an equivalence relation as the Murali and Makamba definition in [67] which we will use in this dissertation. Tarneauceanu [124], described the fuzzy subgroup structure of some finite $p$-groups with cyclic maximal subgroups. In [123], he further extended the characterization of fuzzy subgroups to finite non-abelian groups, where he established the formula for the number of fuzzy subgroups of the dihedral group $D_{p_1^{n_1} p_2^{n_2} \ldots p_s^{n_s}}$, where each $p_i$ is a prime number and $n_i \in \mathbb{Z}^+$. Ndiweni [88] studied the symmetric group $S_3$, the dihedral group $D_4$ and the quaternion group $Q_8$. In [2], Sulaiman and Ahmad computed the number of fuzzy subgroups of symmetric groups $S_2$, $S_3$ and the alternating group $A_4$, and Sulaiman [115] counted the number of fuzzy subgroups of the symmetric group $S_4$. Darabi, Saeedi and Farrokhi [37] computed the number of fuzzy subgroups of the dihedral group $D_{2n}$, the quasi-dihedral groups $QD_{2^n}$, the generalized quaternion groups $Q_{4n}$, and modular $p$-groups $M_{p^n}$, while Davvaz and Ardekani [16] counted the number of fuzzy subgroups of a special class of non-abelian groups of order $p$. Naraghi [86], determined the number of distinct fuzzy subgroups of the dihedral groups $D_2$, $D_4$, $D_8$ and the symmetric group $S_3$ and in [87] he computed the number of distinct fuzzy subgroups of the dihedral group $D_{p_1 \times p_2 \times \cdots \times p_n}$. In [90], [91] and [93], Ndiweni and Makamba classified distinct fuzzy subgroups of the dihedral groups $D_p$, $D_{pq}$, $D_{p^n}$, $D_{pqr}$, $D_{pqrs}$, where $p$, $q$, $r$ and $s$, are distinct primes, and $n \in \mathbb{N}$. Ndiweni [89], expanded the study in [93] to classify fuzzy subgroups of the dihedral group $D_{pqrst}$, for distinct primes $p$, $q$, $r$, $s$ and $t$. Tarnauceanu [123] developed a general formula to determine the number of distinct fuzzy subgroups of the dihedral groups $D_{p^n}$ and $D_{p^n q}$ using the equivalence relation introduced by Tarnauceanu and Bentea [121]. Darabi and Inamparast [38] used the equivalence relation in [121] to compute the number of fuzzy subgroups of the dihedral groups $D_{2p^n}$ for $p$ a prime number, and $D_{2p_1 p_2 \dots p_n}$ where the $p_i$s are distinct primes. Sehgal, A, Sehgal, S, and Sharma [111] use a recurrence relation to give a formula for the number of fuzzy subgroups of the dihedral group $D_{p^m q^n}$. In this dissertation, we include some of the work by [90] and [92], and we compare our results to the results obtained by Tarnauceanu in [123] and Sehgal, A, Sehgal, S, and Sharma [111]. The rest of the dissertation is organised as follows:

In Chapter 2, we give the definition of a fuzzy set and list down a few properties of fuzzy sets. We define a fuzzy point, listing down the characterisation of fuzzy sets using $\alpha$-cuts. We give the definition of the image of a fuzzy set and introduce the notion of fuzzy subgroups as defined by Rosenfeld in [105]. We define level subgroups according to Das in [39] and investigate some properties of fuzzy subgroups that include fuzzy normality, and the homomorphic images and pre-images of fuzzy subgroups. We then define maximal chains (flags) of subgroups of a group $G$ and the concept of a keychain and its components. We give a brief definition and characterisation of the index of a keychain. These concepts give rise to the notion of pinned flags, and are significant tools utilised in the enumeration of distinct fuzzy subgroups of a group.

In Chapter 3, we introduce the notion of fuzzy equivalence. We begin by defining equivalence relations and subsequently, fuzzy relations. We give the Murali and Makamba definition of an equivalence relation on all fuzzy subgroups of a group in [67]. This equivalence relation is stronger than other notions of equivalence, and is a special case of the concept of fuzzy isomorphism, which is an equivalence relation on all fuzzy subgroups of a group. It is also in this chapter that we introduce the criss-cut and cross-cut counting techniques that are used to determine the number of distinct equivalence classes of fuzzy subgroups of the groups we study. We give examples on how to determine the number of distinct fuzzy subgroups by using both procedures. In the criss-cut counting technique we utilise the flags of a group and the equivalence relation defined in [67]. To use this technique, it is important to correctly identify factors (components) that distinguish the flags from each other as these determine the number of distinct fuzzy subgroups contributed by each flag. In the cross-cut counting technique we use the concept of keychains and pinned-flags, where the levels and components of the flags play an important role in the enumeration process.

Since the enumeration of fuzzy subgroups of a group is dependent on the crisp characterisation of groups, in Chapter 4 we study the dihedral groups $D_n$ for $n \in \mathbb{N}$ in general, and give the characterisation of maximal chains as studied in [90]. We list properties of the dihedral groups $D_{p^n}$ and $D_{pq}$ studied in [90] and [92] which we use in our classification problem. This chapter serves as a basis for Chapter 5.

In Chapter 5, we introduce the dihedral group $D_{p^nq}$, for $p$ and $q$ distinct primes, and $n \in \mathbb{N}$. We first characterize maximal chains of specified dihedral groups $D_{p^nq}$ where $n = 2$ and 3 according to their cyclic, $d$-cyclic, $2d$-cyclic, $3d$-cyclic and $b$-cyclic maximal chains. From this characterization and our identification of distinguishing factors of each maximal chain, we use the criss-cut counting technique described in Chapter 3, to establish a formula for the number of distinct fuzzy subgroups and non-isomorphic fuzzy subgroups obtainable from the maximal chains of these groups. The trends and patterns observed in our investigation of $D_{p^2q}$ and $D_{p^3q}$ enable us to generalize results for the dihedral group $D_{p^nq}$ for $p$ and $q$ distinct primes and $n \in \mathbb{N}$. Therefore, we characterize the maximal chains of this group and classify them in terms of cyclic, $md$-cyclic for $1 \leq m \leq n$ and $b$-cyclic maximal chains. We then use this characterization of maximal chains to obtain a general formula for the number of distinct fuzzy subgroups of $D_{p^nq}$.

In the concluding chapter we make comparisons between results obtained from our formula for the number of distinct fuzzy subgroups of $D_{p^nq}$, and formulae presented by other researchers for the same group. We also give a brief summary of our future study on the classification of a dihedral group that is an extension of this group, based on the outcomes of this study.

# Chapter 2

# Fuzzy Sets and Fuzzy Subgroups

## 2.1 Introduction



Let $X$ be a non-empty set, and $S \subseteq X$. The crisp characterization of subsets $S$ of the set $X$ can be defined by the following characteristic function:

$$\chi(x) = \begin{cases} 1 & if \quad x \in S \\ 0 & if \quad x \notin S \end{cases}$$

Now, suppose the set $X$ is a collection of sets in the "real world" and suppose that the elements of $S$ have no definitive conditions. Since the property of the relation $\in$ is such that for each $x \in X$, $x \in S$ or $x \notin S$, we cannot declare, with absolute certainty that the statement $x \in S$ is either true or false. It is because of this uncertainty that in 1965, Zadeh [131] developed and studied the idea of fuzzy subsets of a set, and in [105], Rosenfled pioneered the study of fuzzy subgroups which extended the notion of fuzzy subsets to the structural setting of groups. In this chapter, we give the basic definition and general properties of a fuzzy set, we define $\alpha$-cuts according to the definitions given by Mordeson [27], Ndiweni [88] and Zimmerman [137]. We then define the notion of fuzzy subgroups according to Rosenfeld [105], as well as providing properties that characterise fuzzy subgroups, including the notion of level subgroups that was introduced by Das in [39] in 1981. A number of researchers, namely, Murkherjee and Bhattacharya [26]; Akgul [11]; Dixit, Kumar, and Ajmal [7], amongst them, classified fuzzy subgroups using level subgroups. Since they play a critical role in the study of general fuzzy subgroup structures, we define fuzzy normal subgroups as given in [27].

### 2.1.1 Fuzzy Sets

In the crisp case, when we speak of elements belonging to a non-empty set $X$, we normally refer to every element $a \in X$ completely belonging to the set $X$. In other words, the element $a$ having total membership. Fuzzy sets allow for the depiction of elements as partially belonging to a set.



**Definition 2.1.1.1.** *[27] A fuzzy subset $\mu$ of a group $X$ is the function*
$\mu : X \to I = [0, 1]$
*We denote the set of all fuzzy subsets $\mu$ of $X$ by $\mathcal{I}^X$.*

**Remark 2.1.1.0.1.** If we define the function $\mu : X \to [0, 1]$ by $\mu(x) = \lambda$, $\mu(y) = \beta$ and $\mu(z) = \alpha$ for $1 \geq \lambda \geq \beta \geq \alpha \geq 0$, we say that $\lambda$ is the degree that the element $x$ belongs to the fuzzy subset $\mu$, $\beta$ is the degree that $y$ belongs to $\mu$ and $\alpha$, the degree that $z$ belongs to $\mu$. Hence, when $\lambda = 0$ there is absolute non-membership of $x \in X$, when $\beta = 0$, $y \notin X$ and $\alpha = 0$ $\Rightarrow z \notin X$. When $\lambda = 1$, we have absolute membership of $x \in X$, similarly for $\beta = 1$ and $\alpha = 1$. When $1 \geq \mu(x) \geq \mu(y) \geq \mu(z) \geq 0$ then we say that $y$ belongs to $\mu$ more than $z$ belongs to $\mu$, and $x$ belongs to $\mu$ more than either $y$ or $z$. The image set $\{0, 1\}$ is clearly a crisp set.

**Definition 2.1.1.2.** *[27] Let $\mu \in \mathcal{I}^X$. Then the set $\{\mu(x) : x \in X\}$ is called the image set of $\mu$ and is denoted by $\mu(X)$ or $Im(\mu)$.*

**Definition 2.1.1.3.** *[108] The height of a fuzzy subset $\mu$ of $X$, denoted by $hgt(\mu)$ is the least upper bound of $\mu(x)$. Thus*
$hgt(\mu) = \sup_{x \in X}(\mu(x))$

**Fuzzy Points**

The concept of fuzzy points was developed from the notion of a fuzzy singleton that was introduced by Zadeh in [133]. Fuzzy singletons are instrumental in the study of convergence and local properties of fuzzy topology, as well as in the construction of fuzzy topological spaces. In [100] Pu and Liu gave a definition of a fuzzy point, and Murali [85] defined a fuzzy point of a fuzzy subset under a natural equivalence. We define a fuzzy singleton using the definitions given by Goguen [53], Kerre [60], Mordeson et al [27], and Amin et al. [13].

**Definition 2.1.1.0.1.** *[53] Let $X$ be a non-empty set, $x \in X$, and let $\mu_X$ be a fuzzy subset of $X$, with $\lambda \in [0, 1]$. The fuzzy subset $\mu_X$ is called a fuzzy singleton if $\forall y \in X$*



$$\mu_X(y) = \begin{cases} \lambda & if \quad y = x \\ 0 & if \quad y \neq x \end{cases}$$

In this case, $\mu_X$ is denoted by $x^\lambda$. To expound the definition of a fuzzy point $x^\lambda$ of a set $X$, Murali [85] made an implicit assumption referred to as the null set axiom (NULSAX), based on the vacuous "satisfaction" principle.

**Axiom:** (NULSAX) [85]
There is only one fuzzy subset $\mu$ of the empty set $\varnothing$ which takes the membership value 1 on the empty set, $\mu : \varnothing \to I$ so that $\mu(\varnothing) = 1$. We call such a $\mu$, the NULSAX fuzzy subset. The empty subset $\varnothing$ of $X$ is denoted by $X_0$.

**Definition 2.1.1.0.2.** *[85] Let $a \in X$ and $0 < \lambda < 1$. A fuzzy point, $a^\lambda$ of $X$, is a fuzzy set $a^\lambda : X \to [0,1]$ defined, $\forall x \in X$, by*

$$a^\lambda(x) = \begin{cases} 1 & if \quad x \in X_0 \\ \lambda & if \quad x = a \\ 0 & if \quad x \in X, \ x \neq a \end{cases}$$

If $\lambda = 1$ then $\mu(x) = 1$ when $x = a$ and $0$ when $x \neq a$, the fuzzy set is the crisp singleton $\{a\}$

**Definition 2.1.1.0.3.** *[85] By $a^\lambda \in \mu$ we mean $\mu(a) \geq \lambda$ and $a^\lambda \in_s \mu$ means $\mu(a) > \lambda$, where $\in_s$ implies strict "belonging".*

**Definition 2.1.1.0.4.** *[108] The support of a fuzzy subset $\mu$ of $X$, denoted by supp($\mu$) is the set of points in $X$, at which $\mu(x)$ is positive. Thus supp($\mu$) $= \{x \in X : \mu(x) > 0\}$*



If $\mu \in \mathcal{I}^X$, then it is said to have a **supp property** if every subset of $Im(\mu)$ has a maximal element.

### $\alpha$-Cuts or $\alpha$-Levels

**Definition 2.1.1.0.5.** *[137] The crisp set of all elements that belong to the fuzzy set $\mu$ of $X$, at least to the degree $\alpha$, is called the $\alpha$-level set, denoted by:*

$$\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$$

**Definition 2.1.1.0.6.** *[137] A strong $\alpha$-level set or "strong $\alpha$-cut" of $\mu$, for each real number $\alpha \in [0,1]$ is the crisp set of elements denoted by:*

$$\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$$

The characterization of fuzzy subgroups using $\alpha$-cuts is shown in the following propositions by [88]

**Proposition 2.1.1.0.0.1.** *For any fuzzy subset $\mu$, we have that*

$$\mu = \sup_{0<\alpha<1} \alpha\chi_{\mu_\alpha} = \vee_{\alpha\in(0,1)}\alpha\chi_{\mu_\alpha}$$

*Proof.* [88] □

**Proposition 2.1.1.0.0.2.** *Let $\mu$ be any fuzzy subset, then*

$$\mu(x) = \int_0^1 \alpha\chi_{\mu_\alpha}(x)dx$$

*Proof.* [88] □

**Definition 2.1.1.0.7.** *[137] Let $A$ be a fuzzy subset of a set $X$. The cardinality of $A$, denoted $|A|$ is defined as*

$$|A| = \sum_{x\in X} \mu_A(x)$$

*and $\|A\| = \frac{|A|}{|x|}$ is called the relative cardinality of $A$.*



### Images and Pre-images of Fuzzy Sets

One of the most important tools in fuzzy set theory is Zadeh's **Exention Principle**, described in [134]. This notion extends the operations of classical set theory to fuzzy set theory.

**Definition 2.1.1.0.8.** *[27] Let $f : X \to Y$ be a function from $X$ to $Y$, and let $\mu : X \to [0,1]$ be a fuzzy subset of $X$. Define fuzzy subsets $f(\mu)$ of $Y$, $\forall y \in Y$, and $f^{-1}(\nu)$ of $X$, $\forall x \in X$, by*

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x)|x \in X, f(x) = y\} \quad if \quad f^{-1}(y) \neq \varnothing \\ 0 \quad otherwise \end{cases}$$

*and $f^{-1}(\nu)(x) = \nu(f(x))$*
*The fuzzy subset $f(\mu)$ is called the image of $\mu$ under $f$, and the fuzzy subset $f^{-1}(\nu)$ is called the pre-image of $\nu$ under $f$.*

Thus the degree to which $y$ belongs to $f(\mu)$ is at least as much as the degree to which $x$ belongs to $\mu$ $\forall x$, for which $f(x) = y$.

**Theorem 2.1.1.0.1.** *[27] Let $f : X \to Y$ be a function from $X$ to $Y$, and let $g : Y \to Z$ be a function from $Y$ to $Z$. Then the following assertions hold:*

*(i)* $(g \circ f)(\mu) = g(f(\mu)) \quad \forall \mu \in \mathcal{I}^X$

*(ii)* $(g \circ f)^{-1}(\rho) = f^{-1}(g^{-1}(\rho)) \quad \forall \rho \in \mathcal{I}^Z$

*(iii) If $i : X \to X$ is the identity function, then $\mu(i) = \mu$*

*(iv)* $f^{-1}(f(\mu)) \supseteq \mu \quad \forall \mu \in \mathcal{I}^X$

*(v)* $f(f^{-1}(\nu)) \subseteq \nu \quad \forall \nu \in \mathcal{I}^Y$

*(vi)* $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2) \quad \forall \mu_1, \mu_2 \in \mathcal{I}^X$

*(vii)* $\nu_1 \subseteq \nu_2 \Rightarrow f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2) \quad \forall \nu_1, \nu_2 \in \mathcal{I}^Y$



*Proof.* [27] □

### 2.1.2 Fuzzy Subgroups

In [105], Rosenfeld developed the theory of fuzzy subgroups, showing that many concepts of group theory can be extended to the fuzzy group setting. In this dissertation, we use the following definition of fuzzy subgroups.

**Definition 2.1.2.1.** *[14] A fuzzy subset $\mu : G \to I = [0, 1]$ of a group $G$ is a fuzzy subgroup of $G$ if:*

*i* $\mu(xy) \geq min\{\mu(x), \mu(y)\}$ $\forall x, y \in G$

*ii* $\mu(x^{-1}) = \mu(x)$ $\forall x \in G$

**Definition 2.1.2.2.** *[6] The range set of $\mu$ is defined as:*

$Im\mu = \{\mu(x) : x \in G\}$

**Definition 2.1.2.3.** *[39] Let $G$ be a group and $\mu$ be a fuzzy subgroup of $G$. The subgroups $\mu_\alpha$, $\alpha \in [0,1]$ and $\alpha \leq \mu(e)$ are called level subgroups of $G$*

In [4], Ajmal modified this definition of level subgroups by restricting $\alpha \in Im\mu$, giving the following definition:

**Definition 2.1.2.4.** *[4] The level subgroup of $G$ is* $\mu_\alpha^> = \{\mu(x) > \alpha : x \in G,\ \alpha \in Im\mu\}$

**Properties of Fuzzy Subgroups**

The following properties of fuzzy subgroups result from the above definitions.

**Proposition 2.1.2.0.0.1.** *[88] If $\mu$ is a fuzzy subset of a group $G$, then $\mu$ is a fuzzy subgroup of $G$ if and only if each $\mu_\alpha$ is a subgroup of $G$, $0 \leq \alpha \leq 1$*

*Proof.* [88] □

**Proposition 2.1.2.0.0.2.** *[88] Let $\mu$ be a fuzzy subset of $G$. Then $\mu$ is a fuzzy subgroup of $G \Leftrightarrow \forall a_\lambda, b_\beta \in \mu \Rightarrow a_\lambda(b_\beta^{-1}) \in \mu$*



*Proof.* [88] □

**Theorem 2.1.2.0.1.** *[27] Let $f : G \to H$ be a homomorphism from a group $G$ into a group $H$. If $\mu$ is a fuzzy subgroup of $G$, then $f(\mu)$ is a fuzzy subgroup of $H$.*

*Proof.* [27] □

**Theorem 2.1.2.0.2.** *[27] Let $f : G \to H$ be a homomorphism and let $\nu$ a fuzzy subgroup of $H$, then $f^{-1}(\nu)$ is a fuzzy subgroup of $G$.*

*Proof.* [27] □

**Fuzzy Normal Subgroups**

Just as normal subgroups play a critical role in the study of the general structure of classical groups, fuzzy normal subgroups play a similar role in the theory of fuzzy subgroups. In [26], Murkhejee and Bhattacharya gave a detailed study on the concept of fuzzy normal subgroups introduced by Liu in [63]. This concept has since been studied by various researchers under different contexts.

**Theorem 2.1.2.0.3.** *[27] Let $\mu \in \mathcal{F}(G)$. Then the following assertions are equivalent $\forall x, y \in G$:*

*(i)* $\mu(yx) = \mu(xy)$ *In this case, $\mu$ is called an Abelian fuzzy subgroup of $G$*

*(ii)* $\mu(y) \leq \mu(x^{-1}yx)$

*(iii)* $\mu(y) \geq \mu(x^{-1}yx)$

*(iv)* $\mu \circ \nu = \nu \circ \mu \ \forall \nu \in \mathcal{F}(G)$



*Proof.* [27] □

**Definition 2.1.2.0.1.** *[27] Let $\mu \in \mathcal{F}(G)$, then $\mu$ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$, $\forall x, y \in G$. i.e If $\mu$ is an Ableian fuzzy subgroup of $G$.*

**Proposition 2.1.2.0.3.1.** *[88] If $\mu,\ \nu \in \mathcal{F}(G)$, and $\mu$ is fuzzy normal, then $\mu\nu \in \mathcal{F}(G)$.*

*Proof.* [88] □

**Proposition 2.1.2.0.3.2.** *[88] If $\mu$ and $\nu$ are both normal fuzzy subgroups of a group $G$, then $\mu\nu \in \mathcal{F}(G)$*

*Proof.* [88] □

**Proposition 2.1.2.0.3.3.** *[88] If $\mu,\ \nu \in \mathcal{F}(G)$, and $\mu$ is fuzzy normal, then $\mu\nu = \nu\mu \in \mathcal{F}(G)$*

*Proof.* [88] □

**Proposition 2.1.2.0.3.4.** *[88] Let $\mu \in \mathcal{F}(G)$. Then $\mu$ is fuzzy normal $\iff$ each $\mu_\alpha$ is a normal fuzzy subgroup of $G$ $\forall \alpha \in [0, 1]$*

*Proof.* [88] □

### 2.1.3 Maximal Chains, Pins, Keychains and Pinned Flags

Due to the innate connection between subgroup lattices and fuzzy subgroups of a group $G$, counting the number of maximal chains, or flags of a group provides a combinatorial method of enumerating the number of fuzzy subgroups of a group $G$. The concept of keychains, introduced in [67], represents fuzzy subgroups defined by the flags of a group. Thus, in this section, we define flags and keychains, we also define some important features of keychains that make a difference to the equivalence of fuzzy subgroups, namely pins and the index of a keychain. We then define the notion of pinned-flags which arises from the pairing of flags and keychains. The terms maximal chains and flags will be used interchangeably in this dissertation.
The maximal chains of a group are created from the maximal subgroups of the group, thus we begin with the following definition.

**Definition 2.1.3.1.** *[88] A maximal subgroup $M$ of a group $G$, is a proper subgroup of the group $G$ such that no other proper subgroup $H$ of $G$ contains $M$.*

**Definition 2.1.3.2.** *[72] A flag $\zeta$ is a maximal chain of maximal subgroups of a group $G$ of the form*

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$$

*where $G_0 = \{e\}$ and $G_n = G$.*

The $G_i s$ are referred to as **components** of a flag $\zeta$.

**Definition 2.1.3.3.** *[70] An $(n+1)$-tuple $(1, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n)$ of real numbers in the interval $I = [0, 1]$ of the form*

$$1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n > 0$$

*is called a **keychain**, denoted by $\ell$*
*With the exception of the component 1, which can be found at the begining of every keychain, the real numbers $\lambda_1\lambda_2 \ldots \lambda_{n-1}\lambda_n$ are components of the keychain called **pins**. The length of a keychain $\ell$ is $n+1$.*

A keychain may contain pins $\lambda_i$ that are either distinct or exactly the same. Counting the number of repetitions of $\lambda_i$, for $i \in \{1, 2, \ldots, n\}$ in a keychain $\ell$ gives the following definition.



**Definition 2.1.3.4.** *[74] An unordered set of positive integers $i = (l_1, l_2, \ldots l_n)$, associated with a keychain $\ell$, where $n = l_1 + l_2 + \cdots + l_n$ and whose values depend on the equality sign between pins $\lambda_i$, is called the **index** of a keychain. For example.*
*The keychain $1\lambda\lambda \ldots \lambda\lambda$ is of index $(n)$ and referred to as an exceptional keycahain*
*The keychain $1\lambda\beta\alpha \ldots \theta\zeta$ is of index $(1, 1, 1, \ldots, 1)$ and referred to as a most stratified keychain*

The following characterise keychains in terms of their indices.

**Definition 2.1.3.5.** *[74] Two keychains $\ell_1 = 1\lambda_1\lambda_2 \ldots \lambda_n$ and $\ell_2 = 1\beta_1\beta_2 \ldots \beta_n$ are equivalent $\iff \ \forall\ 1 \leq i, j \leq n$*

*(i)* $\lambda_i > \lambda j \iff \beta_i > \beta j$

*(ii)* $\lambda_i = 0 \iff \beta_i = 0$

**Proposition 2.1.3.0.1.** *[74] If two keychains are equivalent then they have the same index.*

*Proof.* [74] □

**Proposition 2.1.3.0.2.** *[74] There are precisely four equivalent keychains of a given index, with the exception of keychains of index $(n)$, that give rise to only three equivalent keychains.*

*Proof.* [74] □

**Definition 2.1.3.6.** *[67] The number of keychains is determined by the formula $2^{n+1} - 1$, where $n + 1$ is the length of the flag.*

**Definition 2.1.3.7.** *[70] The pair $(\zeta, \ell)$, with a flag $\zeta$ on $G$, and a keychain $\ell$, from $I$, written as*

$$G_0^1 \subseteq G_1^{\lambda_1} \subseteq G_2^{\lambda_2} \subseteq \cdots \subseteq G_n^{\lambda_n}$$

*is called a* ***pinned flag****. $G_i^{\lambda_i}$ for $i = 0, 1, 2, \ldots, n$, is called the $(i+1)$-th component of the pinned-flag.*



The connection between pinned flags and equivalent classes of fuzzy subgroups is further clarified when we represent the pair $(\zeta, \ell)$ on $G$ as a fuzzy subgroup $\mu$ of $G$ as shown:

$$\mu(x) = \begin{cases} 1 & if \ x = e \\ \lambda_1 & if \quad x \in G_1 \backslash \{e\} \\ \lambda_2 & if \quad x \in G_2 \backslash G_1 \\ \vdots & \vdots \qquad \vdots \\ \lambda_n & if \quad x \in G_n \backslash G_{n-1} \end{cases}$$

where $G_n$ is the whole group.

Conversely, given any fuzzy subgroup $\mu$ of $G$, then $\mu$ can be decomposed into a pinned-flag on $G$ as in Definition 2.1.3.7.

# Chapter 3

# Fuzzy Equivalence, Fuzzy Isomorphism and Distinct Fuzzy Subgroups



## 3.1 Introduction

A number of researchers such as Bhattacharya [24] and [25], Das [39], Dixit [7], and Ray [101], characterised fuzzy subgroups of finite groups by their level subgroups. Fuzzy sets are an extension of the theory of crisp sets. Although it is easier to categorize objects in the crisp case as either striclty belonging or not belonging to a certain class of objects, in the fuzzy case, objects have a degree of belonging that requires the notion of an equivalence relation on fuzzy sets for categorization of similar objects that are related, based on the classes in which they belong. According to Murali and Makamba in [67], if there is no equivalence relation on fuzzy subsets of a set, the number of fuzzy subgroups of even the trivial group is infinite. In [110], Schmechel and Thiele defined the concepts of fuzzy equivalence relations and fuzzy partitions, while De Cock [40] stated the general unsuitability of fuzzy equivalence relations to model approximate equality, as is in the crisp case. Several more studies on fuzzy equivalence relations have also been conducted by Tarnauceanu [126], Recasens [103], and Murali and Makamba in [67], [68] and [69]. When drawing a comparison between the the notions of fuzzy isomorphism and an equivalence relation on fuzzy sets in [67], Murali and Makamba observed that the notion of fuzzy equivalence relation is finer than that of fuzzy isomorphism. In this chapter, we begin by giving a general definition of a fuzzy equivalence relation as well as the definition of an equivalence relation. The different notions of equivalence relation defined have given rise to a number of different enumeration techniques to compute the number of distinct fuzzy subgroups of finite groups. We give an overview of the two enumeration techniques introduced in [74] which we will utilize in our study.

### 3.1.1 Equivalence Relations



**Definition 3.1.1.1.** *[107] If $X$ and $Y$ are sets, then a relation from $X$ to $Y$ is a subset $\Re \subseteq X \times Y$.*
*We usually write $a\Re b$ to denote $(a, b) \in \Re$*
*If $X = Y$ we say $\Re$ is a relation on $X$.*

**Definition 3.1.1.2.** *[107] A relation $\Re$, denoted $\sim$, on a set $X$, is*

*(i)* ***Reflexive****, if $a \sim a$ $\forall a \in X$*

*(ii)* ***Symmetric****, if $a \sim b \Rightarrow b \sim a$ $\forall a, b \in X$*

*(iii)* ***Transitive****, if ($a \sim b$ and $b \sim c \Rightarrow a \sim c$ $\forall a, b, c \in X$*

*A relation that has all three of the above properties is an* ***equivalence relation****.*

**Definition 3.1.1.3.** *[107] Let $\sim$ be an equivalence relation on a set $X$. If $a \in X$ then the equivalence class of $a$ is the set $[a] = \{x \in X : a \sim x\} \subseteq X$*

**Proposition 3.1.1.0.1.** *[107] Let $\sim$ be an equivalence relation on $X$. Then $a \sim b \iff [a] = [b]$*

*Proof.* [107] $\square$

### 3.1.2 Fuzzy Relations

**Definition 3.1.2.1.** *[137] Let $X$ and $Y$ be sets. A fuzzy relation $\mu$ on $X \times Y$, is the mapping $\mu : X \times Y \mapsto I = [0, 1]$, where $\mu \subseteq X \times Y$, and is denoted by:*

$$\mu = \{((x, y), \mu(x, y)) \mid (x, y) \in X \times Y\}$$

*$\mu(x, y)$ is said to be the degree to which $x$ is related to $y$.*

**Definition 3.1.2.2.** *[137] A fuzzy relation that is reflexive, symmetric, and transitive is called a fuzzy equivalence relation.*
*Thus, $\forall (x, y) \in X \subseteq X \times Y$ and $(y, z) \in Y \times Z$, a fuzzy relation is:*

*(i)* ***Reflexive****, if* $\mu(x, x) = 1$



*(ii)* ***Symmetric****, if* $\mu(x, y) = \mu(y, x)$

*(iii)* ***Transitive****, if $\mu \circ \mu(x, y) \leq \mu$ where $\mu \circ \mu$ is defined by:*

$$\mu \circ \mu = \{(x, z), \vee(\mu(x, y) \wedge \mu(y, z))\}$$

### 3.1.3 Equivalence Relations

Murali and Makamba [67] define an equivalence relation on the set of all fuzzy subsets of a set $X$ as follows:

**Definition 3.1.3.1.** *[67] $\mu \sim \nu$ if and only if $\forall x, y \in X$*

*(i)* $\mu(x) \geq \mu(y) \iff \nu(x) \geq \nu(y)$

*(ii)* $\mu(x) = 0 \iff \nu(x) = 0$

**Note 3.1.3.1.** The condition $\mu(x) = 0 \ \Leftrightarrow \nu(x) = 0$ denotes the equality of the supports of $\mu$ and $\nu$, and does not follow from condition $(i)$. Also, when restricted to the two truth values $\{0, 1\}$, the equivalence relation $\mu \sim \nu$ coincides with equality of sets.

We use the particular case of $D_{12}$ to explain the above definition:

**Example 3.1.3.1.** Let $G = D_{12} = \langle a, b \mid a^{12} = e = b^2 = (ab)^2\rangle$. We define the fuzzy subgroups $\mu$ and $\nu$ as follows:

$$\mu(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in \langle a^6\rangle \setminus \{e\} \\ \beta & if \quad x \in \langle a^2\rangle \setminus \langle a^6\rangle \\ \alpha & if \quad x \in \langle a\rangle \setminus \langle a^2\rangle \\ 0 & otherwise \end{cases}$$

$$\nu(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in \langle a^6\rangle \setminus \{e\} \\ \beta & if \quad x \in \langle a^3\rangle \setminus \langle a^6\rangle \\ \alpha & if \quad x \in \langle a\rangle \setminus \langle a^3\rangle \\ 0 & otherwise, \end{cases}$$

for $1 > \lambda > \beta > \alpha > 0$.

We observe that $supp(\mu) = \langle a\rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, \dots, a^{11}\}$. Similarly, $supp(\nu) = \langle a\rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, \dots, a^{11}\}$. Hence $supp(\mu) = supp(\nu)$.

We also note that,

$\mu(a^6) > \mu(a^2) = \mu(a^4) = \mu(a^8) = \mu(a^{10}) > \mu(a) = \mu(a^3) = \mu(a^5) = \mu(a^7) = \mu(a^9) = \mu(a^{11})$

But, $\nu(a^6) > \nu(a^3) = \nu(a^9) > \nu(a) = \nu(a^2) = \nu(a^4) = \nu(a^5) = \nu(a^7) = \nu(a^8) = \nu(a^{10}) = \nu(a^{11})$,

$\Longrightarrow \forall x, y, z \in G$, where $x \in \langle a^6\rangle \setminus \{e\}$, $y \in \langle a^2\rangle \setminus \langle a^6\rangle = \langle a\rangle \setminus \langle a^3\rangle$ and $z \in \langle a\rangle \setminus \langle a^2\rangle = \langle a^3\rangle \setminus \langle a^6\rangle$ we have $\mu(x) > \mu(y) > \mu(z)$, but $\nu(x) > \nu(y) \ngtr \nu(z)$, since $\nu(z) > \nu(y)$. Thus $\mu \nsim \nu$.

**Proposition 3.1.3.0.1.** *[67] If $\mu \sim \nu$ then $Im|\mu| = Im|\nu|$*

*Proof.* [67] □

**Definition 3.1.3.2.** *[88] Two fuzzy subgroups $\mu$ and $\nu$ are said to be distinct $\Leftrightarrow$ $[\mu] \neq [\nu]$ where $[\mu]$ and $[\nu]$ are equivalence classes containing $\mu$ and $\nu$ respectively.*

To further clarify the idea of equivalence of fuzzy subgroups, in [67], Murali and Makamba use the following Proposition for the characterisation of a fuzzy equivalence relation.

**Proposition 3.1.3.0.2.** *[67] Suppose $\mu$ and $\nu$ are two equivalent fuzzy subsets of $X$. Then, for each $t \in [0,1]$ there is an $s \in [0,1]$ such that $\mu_t = \nu_s$.*

*Proof.* [67] □



### 3.1.4 Fuzzy Isomorphism

There are various notions of fuzzy isomorphism used in literature to characterise fuzzy subgroups of a group. However, in the case of classifying or counting the number of fuzzy subgroups of a group $G$, these notions have limitations. Murali and Makamba in [67], [68], [69] and Mordeson in [27] discerned the concept of a fuzzy equivalence as finer than that of fuzzy isomorphism since the concept of fuzzy isomorphism gives a generalised crisp equality of sets by replacing this equality with the equality of flags. This implies that although two equivalent fuzzy subgroups are fuzzy isomorphic, the converse is not true. We begin by defining a fuzzy homomorphism.

**Definition 3.1.4.1.** *[107] If $(G, \star)$ and $(H, \circ)$ are groups, then a function, $f : G \to H$ such that $f(x \star y) = f(x) \circ f(y), \quad \forall x, y \in G$, is called a homomorphism.*

**Definition 3.1.4.2.** *[107] If the homomorphism $f : G \to H$ is a bijection, then it is an isomorphism.*

**Definition 3.1.4.3.** *[69] If $f : G \to H$ is a homomorphism and $\mu$ is a fuzzy subset of $G$, then $f(\mu)$ is the image of $\mu$ and is a fuzzy subset of $H$ defined, for $h \in H$ by $(f(\mu))(h) = sup\{\mu(g) : g \in G,\ f(g) = h\}$ if $f^{-1}(h) \neq \varnothing$ and $f(\mu)(h) = 0$ if $f^{-1}(h) = \varnothing$.*
*Similarly, if $\nu$ is a fuzzy subset of $H$, then the pre-image of $\nu$, i.e. $f^{-1}(\nu)$ is a fuzzy subset of $G$ defined by $(f^{-1}(\nu))(g) = \nu(f(g))$.*

The following definition of isomorphic fuzzy subgroups was given by Murali and Makamba in [75].

**Definition 3.1.4.4.** *[75] Let $G$ be a group and let $\mu, \nu \in \mathcal{F}(G)$. We say $\mu$ is fuzzy isomorphic to $\nu$, denoted by $\mu \simeq \nu$, $\Leftrightarrow$ $\exists$ an isomorphism $f : G \to G$ such that*

*(i)* $\mu(a) > \mu(b) \Longleftrightarrow \mu(f(a)) > \nu(f(b))$

*(ii)* $\mu(a) = 0 \Longleftrightarrow \nu(f(a)) = 0$ *for* $a, b \in G$.



### 3.1.5 Counting Distinct Fuzzy Subgroups

The different concepts of fuzzy equivalence relation that have been studied and defined are a foundation for the number of different techniques developed for the enumeration of distinct fuzzy subgroups of a finite group. In this section, using the equivalence relation defined by Murali and Makamba in [67], we define the criss-cut and cross-cut counting techniques presented in [74]. Since fuzzy subgroups of a finite group $G$ are inherently linked to the lattice of subgroups of $G$, it is imperative that we characterise the flags of the group prior to utilising any method of enumeration.

**Criss-cut counting technique**

This enumeration technique requires that we list all the flags of a group $G$, in order to identify their distinguishing factors and hence compute the number of distinct fuzzy subgroups attributed to each flag. The listing order of flags is irrelevant and will not affect the results, so any numbering associated with the flags is purely based on the counting sequence. From the propositions stated and proved in [88], [90], and [83], we explain this counting technique as follows:

Let the first flag of a finite group $G$ be given as:

(1) $\{e\} \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$

By [67], the number of distinct fuzzy subgroups contributed by flag (1) is $2^{n+1} - 1$, where $n+1$ is the length of the flag.

Suppose we have a second flag,

(2) $\{e\} \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = G$



Where $K_i \neq H_i$, for any $i \in \{1, 2, 3, \ldots, n-1\}$

We call this $K_i$ a distinguishing factor of flag (2), as it distinguishes flag (2) from flag (1). The following proposition gives the number of distinct fuzzy subgroups contributed by flag (2).

**Proposition 3.1.5.0.0.1.** *[88] The number of distinct fuzzy subgroups of $G$ contributed by a flag with a single distinguishing factor is given by $\frac{2^{n+1}}{2} = 2^n$ for $n \geq 2$.*

*Proof.* [88] $\square$

If we consider a third flag of $G$

(3) $\{e\} \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = G$

where $J_i$ and $J_j$ is a pair of components with $i \neq j$, $J_i \neq K_i \neq H_i$ for any $i = \{1, 2, 3, \ldots n-1\}$ and $J_j \neq K_j \neq H_j$ for any $j = \{1, 2, 3, \ldots n-1\}$.

We have that the pair $J_i$ and $J_j$ distinguishes flag (3) from both flag (1) and flag (2), thus flag (3) has a pair of distinguishing factors. Hence, the number of distinct fuzzy subgroups contributed by a flag (3) is given by the following proposition.

**Proposition 3.1.5.0.0.2.** *[88] The number of distinct fuzzy subgroups of $G$ contributed by a flag with a pair of distinguishing factors is given by $\frac{2^{n+1}}{2^2} = 2^{n-1}$ for $n \geq 4$.*

*Proof.* [88] □

Now suppose we have another flag of $G$ with a triple of distinguishing factors that do not appear in any previous flags of $G$. Then the number of distinct fuzzy subgroups contributed by this chain is given by $\frac{2^{n+1}}{2^3} = 2^{n-2}$. In [88] an inductive process is applied to determine the number of distinct fuzzy subgroups contributed by each flag of $G$, based on the number of distinguishing factors in each flag. Thus, in general we have the following proposition.



**Proposition 3.1.5.0.0.3.** *[88] With the exception of the first flag that contributes $2^{n+1} - 1$ distinct fuzzy subgroups; the number of distinct fuzzy subgroups contributed by a flag of $G$ with length $n+1$ and $m < n+1$ distinguishing factors is given as $\frac{2^{n+1}}{2^m}$*

We now use a specific case of the dihedral group $D_{p^n q}$ to give an example of the criss-cut enumeration technique.

**Example 3.1.5.0.1.** If we let $p = 2$, $q = 3$ and $n = 2$, then for $D_{p^n q}$, we have $D_{12} = \langle a, b : a^{12} = b^2 = e = (ab)^2 \rangle$, $|G| = 24$. This group has subgroups of orders that divide 24. From the manual construction of flags of $D_{12}$, we obtain the following:

The cyclic maximal chains of $D_{12}$ are:

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle^\star \subseteq \langle a \rangle \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{12}$

Thus $D_{12}$ has 3 cyclic flags. To account for the distinctions between flags, we identify distinguishing factors in all flags by a star. Flag (1) is considered to contain subgroups that are all distinguishing factors and by [67], contributes $2^5 - 1$ distinct fuzzy subgroups, where 5 is the length of each flag. Each of the flags (2) and (3) has a single distinguishing factor and therefore the two flags contribute $2^4(2)$ distinct fuzzy subgroups.
The $d$-cyclic maximal chains are:

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^{ab\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^{ab\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^{a^2b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq D_6^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq D_6^{ab\star} \subseteq D_{12}$

These are the only $d$-cyclic flags of $D_{12}$. Hence we obtain 7 $d$-cyclic flags. We observe that five flags have single distinguishing factors and thus contribute $2^4(5)$ distinct fuzzy subgroups, and the two remaining flags have a pair of distinguishing factors, contributing $2^3(2)$ distinct fuzzy subgroups. Thus, the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags is $2^4(5) + 2^3(2)$

A listing of the $2d$-cyclic maximal chains yields:

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{ab\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^2b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^3b\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^4b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^5b\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{b\star} \subseteq D_4^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{ab\star} \subseteq D_4^{ab\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^2b\star} \subseteq D_4^{a^2b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^3b\star} \subseteq D_4^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^4b\star} \subseteq D_4^{a^2b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^5b\star} \subseteq D_4^{a^2b\star} \subseteq D_{12}$



$\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{ab\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{a^2b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{a^3b\star} \subseteq D_6^{ab} \subseteq D_{12}$

These are the only $2d$-cyclic flags of $D_{12}$. Thus, we obtain 16 $2d$-cyclic flags. There are ten flags, each with a single distinguishing factor, that contribute $2^4(10)$ distinct fuzzy subgroups. Each of the remaining six flags have a pair of distinguishing factors and thus contribute $2^3(6)$ distinct fuzzy subgroups. Hence, the $2d$-cyclic flags contribute $2^4(10) + 2^3(6)$ distinct fuzzy subgroups.

We have the $b$-cyclic maximal chains as:

$$\{e\} \subseteq \langle b\rangle^\star \subseteq D_2^{b} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle b\rangle^\star \subseteq D_2^{b} \subseteq D_4^{b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle b\rangle^\star \subseteq D_3^{b\star} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_2^{ab} \subseteq D_6^{ab} \subseteq D_{12}$$

$$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_2^{ab} \subseteq D_4^{ab\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_3^{ab\star} \subseteq D_6^{ab} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_4^{a^2b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_3^{a^2b\star} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_4^{b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_3^{a^3b\star} \subseteq D_6^{ab} \subseteq D_{12}$$



$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_4^{ab\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_3^{b\star} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_6^{ab} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_4^{a^2b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_3^{ab\star} \subseteq D_6^{ab} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{b} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{b} \subseteq D_4^{b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_3^{a^2b\star} \subseteq D_6^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_2^{ab} \subseteq D_6^{ab} \subseteq D_{12}$$

$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_2^{ab} \subseteq D_4^{ab\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_3^{a^3b\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_4^{a^2b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_3^{b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_4^{b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_3^{ab\star} \subseteq D_6^{ab} \subseteq D_{12}$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_4^{ab\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_3^{a^2b\star} \subseteq D_6^{b} \subseteq D_{12}$

$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_6^{ab} \subseteq D_{12}$



$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_4^{a^2b\star} \subseteq D_{12}$

$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_3^{a^3b\star} \subseteq D_6^{ab} \subseteq D_{12}$

These are the only $b$-cyclic flags of $D_{12}$, and we obtain 36. There are 12 flags, each with one distinguishing factor, that contribute $2^4(12)$ distinct fuzzy subgroups. Each of the remaining 24 flags has a pair of distinguishing factors, and yield $2^3(24)$ distinct fuzzy subgroups. Hence, the number of distinct fuzzy subgroups obtained from the $b$-cyclic flags is $2^4(12) + 2^3(24)$.

From the summation of the cyclic, $d$-cyclic, $2d$-cyclic and $b$-cyclic flags, we have that the number of flags of $D_{12}$ is given as

$$\mathcal{M}(D_{12}) = 3 + 7 + 16 + 36 = 62$$

and the number of distinct fuzzy subgroups contributed by the 62 flags of

$D_{12}$ is given as

$$\mathcal{F}(D_{12}) = 2^5 - 1 + 2^4(2) + 2^4(5) + 2^3(2) + 2^4(10) + 2^3(6) + 2^4(12) + 2^3(24) = 751$$

**Cross-cut counting technique**

In a similar way to the criss-cut counting technique, the cross-cut counting technique relies on the construction of maximal chains of a group $G$. For this method, we make use of the concept of pins and keychains. We begin by using the levels (length) of a flag to determine the number of keychains. We then use the components of the flag to determine the number of pins in each keychain, including 1, which is not considered a pin as it is a constant component in each keychain. From the definition in [67], each keychain defines a distinct equivalent class of fuzzy subgroups. The total number of these equivalence classes depends solely on the characteristics of these keychains, and the number of flags of the group. Hence we list all the flags of a group $G$ before we begin the enumeration process. This method is described as follows:



**STEP ONE** : From [67], we have that the number of keychains that represent flags of length $n$, is $2^n - 1$. Each of these keycahins is of the form $1\lambda\beta\cdots0$, where $1 > \lambda > \beta > \cdots > 0$.

**STEP TWO** : We let the first flag of a finite group $G$ be:

(1) $\{e\} \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$.

We select any keychain, say $1\lambda\beta\cdots0$, and "pin" it to the first flag to obtain the pinned flag $\{e\}^1 \subseteq {G_1}^\lambda \subseteq {G_2}^\beta \subseteq \cdots \subseteq {G_n}^0$. We then represent this pinned flag as the fuzzy subgroup:

$$\mu_1(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in G_1 \backslash \{e\} \\ \beta & if \quad x \in G_2 \backslash G_1 \\ \vdots & \vdots \qquad \vdots \\ 0 & otherwise \end{cases}$$

Suppose the second flag of $G$ is:

(2) $\{e\} \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$.

Where $H_i \neq G_i$ for any $i \in \{1, 2, \ldots, n-1\}$

We select the same keychain, $1\lambda\beta\cdots0$, and "pin" it to flag (2), to obtain the fuzzy subgroup:



$$\mu_2(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in H_1 \backslash \{e\} \\ \beta & if \quad x \in H_2 \backslash H_1 \\ \vdots & \vdots \qquad \vdots \\ 0 & otherwise \end{cases}$$

For $1 > \lambda > \beta > \cdots > 0$, we observe that:

$\forall x, y, z \in G$, $\mu_1(x) > \mu_1(y) > \mu_1(z)$ and $\mu_2(x) > \mu_2(y) > \mu_2(z)$, but $supp(\mu_1) \neq supp(\mu_2)$. Therefore the keychains $1\lambda\beta\cdots0$ represents two distinct fuzzy subgroups on flags (1) and (2). If $supp(\mu_1) = supp(\mu_2)$, then the fuzzy subgroups $\mu_1$ and $\mu_2$ belong to the same distinct equivalence class of fuzzy subgroups, and the keychain $1\lambda\beta\cdots0$ represents one distinct fuzzy subgroup on both flags (1) and (2). We continue the process of "pinning" the keychain $1\lambda\beta\cdots0$ to all the flags of $G$ to compute the number of distinct fuzzy subgroups it represents.

**STEP THREE** : From the list of $2^n - 1$, we again select a different keychain to pin on all the flags of $G$. We repeat this procedure, computing the number of distinct fuzzy subgroups each keychain represents on each flag, until we have exhausted the list of keychains.

**STEP FOUR** : Finally, we add the number of distinct fuzzy subgroups that all the $2^n - 1$ keychains represent on each flag, to obtain the total number of distinct of fuzzy subgroups of a finite group $G$.

**Note 3.1.5.0.1.** "Pinning" a keychain on all the flags of a group is a task that needs to be thoroughly performed to prevent instances of either overcounting or undercounting the number of fuzzy subgroups.

**Example 3.1.5.0.2.** To illustrate this method, we use the dihedral group of the form $D_{p^n q}$, where $p = 2$, $q = 3$, and $n = 2$
and thus we have $D_{12} = \langle a, b \mid a^{12} = b^2 = e = (ab)^2 \rangle$. The subgroups of $D_{12}$ are:



$\{e\}$; $D_{12}$;

$\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, \cdots a^{11}\}$;

$\langle a^2 \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}\}$;

$\langle a^3 \rangle = \{e, a^3, a^6, a^9\}$;

$\langle a^4 \rangle = \{e, a^4, a^8\}$;

$\langle a^6 \rangle = \{e, a^6\}$;

$\langle b \rangle$; $\langle ab \rangle$; $\langle a^2 b \rangle$; $\ldots$ $\langle a^{11} b \rangle$

$D_2^b = \langle a^6, b \rangle = \{e, a^6, b, a^6 b\}$

$D_2^{ab} = \langle a^6, ab \rangle = \{e, a^6, ab, a^7 b\}$

$D_2^{a^2 b} = \langle a^6, a^2 b \rangle = \{e, a^6, a^2 b, a^8 b\}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$D_2^{a^5b} = \langle a^6, a^5b \rangle = \{e, a^6, a^5b, a^{11}b\}$

$D_3^{b} = \langle a^4, b \rangle = \{e, a^4, a^8, b, a^4b, a^8b\}$

$D_3^{ab} = \langle a^4, ab \rangle = \{e, a^4, a^8, ab, a^5b, a^9b\}$

$D_3^{a^2b} = \langle a^4, a^2b \rangle = \{e, a^4, a^8, a^2b, a^6b, a^{10}b\}$

$D_3^{a^3b} = \langle a^4, a^3b \rangle = \{e, a^4, a^8, a^3b, a^7b, a^{11}b\}$

$D_4^{b} = \langle a^3, b \rangle = \{e, a^3, a^6, a^9, b, a^3b, a^6b, a^9b\}$

$D_4^{ab} = \langle a^3, ab \rangle = \{e, a^3, a^6, a^9, ab, a^4b, a^7b, a^{10}b\}$

$D_4^{a^2b} = \langle a^3, a^2b \rangle = \{e, a^3, a^6, a^9, a^2b, a^5b, a^8b, a^{11}b\}$

$D_6^{b} = \langle a^2, b \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, b, a^2b, a^4b, a^6b, a^8b, a^{10}b\}$

$D_6^{ab} = \langle a^2, ab \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, ab, a^3b, a^5b, a^7b, a^9b, a^{11}b\}$



Thus $D_{12}$ has 62 flags shown below:

(1) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{12}$

(2) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq \langle a \rangle \subseteq D_{12}$

(3) $\{e\} \subseteq \langle a^4 \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{12}$

(4) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^b \subseteq D_{12}$

(5) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^{ab} \subseteq D_{12}$

(6) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^b \subseteq D_{12}$

(7) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^{ab} \subseteq D_{12}$

(8) $\{e\} \subseteq \langle a^6 \rangle \subseteq \langle a^3 \rangle \subseteq D_4^{a^2b} \subseteq D_{12}$

(9) $\{e\} \subseteq \langle a^4 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^b \subseteq D_{12}$

(10) $\{e\} \subseteq \langle a^4 \rangle \subseteq \langle a^2 \rangle \subseteq D_6^{ab} \subseteq D_{12}$

(11) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{b} \subseteq D_6^{b} \subseteq D_{12}$

(12) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(13) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(14) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(15) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^4b} \subseteq D_6^{b} \subseteq D_{12}$

(16) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^5b} \subseteq D_6^{ab} \subseteq D_{12}$

(17) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{b} \subseteq D_4^{b} \subseteq D_{12}$

(18) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_{12}$

(19) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(20) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^3b} \subseteq D_4^{b} \subseteq D_{12}$



(21) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^4b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(22) $\{e\} \subseteq \langle a^6 \rangle \subseteq D_2^{a^5b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(23) $\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{b} \subseteq D_6^{b} \subseteq D_{12}$

(24) $\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(25) $\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(26) $\{e\} \subseteq \langle a^4 \rangle \subseteq D_3^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(27) $\{e\} \subseteq \langle b \rangle \subseteq D_2^{b} \subseteq D_6^{b} \subseteq D_{12}$

(28) $\{e\} \subseteq \langle b \rangle \subseteq D_2^{b} \subseteq D_4^{b} \subseteq D_{12}$

(29) $\{e\} \subseteq \langle b \rangle \subseteq D_3^{b} \subseteq D_6^{b} \subseteq D_{12}$

(30) $\{e\} \subseteq \langle ab \rangle \subseteq D_2^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(31) $\{e\} \subseteq \langle ab \rangle \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_{12}$

(32) $\{e\} \subseteq \langle ab \rangle \subseteq D_3^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(33) $\{e\} \subseteq \langle a^2b \rangle \subseteq D_2^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(34) $\{e\} \subseteq \langle a^2b \rangle \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(35) $\{e\} \subseteq \langle a^2b \rangle \subseteq D_3^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(36) $\{e\} \subseteq \langle a^3b \rangle \subseteq D_2^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(37) $\{e\} \subseteq \langle a^3b \rangle \subseteq D_2^{a^3b} \subseteq D_4^{b} \subseteq D_{12}$

(38) $\{e\} \subseteq \langle a^3b \rangle \subseteq D_3^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(39) $\{e\} \subseteq \langle a^4b \rangle \subseteq D_2^{a^4b} \subseteq D_6^{b} \subseteq D_{12}$

(40) $\{e\} \subseteq \langle a^4b \rangle \subseteq D_2^{a^4b} \subseteq D_4^{ab} \subseteq D_{12}$

(41) $\{e\} \subseteq \langle a^4b \rangle \subseteq D_3^{b} \subseteq D_6^{b} \subseteq D_{12}$



(42) $\{e\} \subseteq \langle a^5b \rangle \subseteq D_2^{a^5b} \subseteq D_6^{ab} \subseteq D_{12}$

(43) $\{e\} \subseteq \langle a^5b \rangle \subseteq D_2^{a^5b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(44) $\{e\} \subseteq \langle a^5b \rangle \subseteq D_3^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(45) $\{e\} \subseteq \langle a^6b \rangle \subseteq D_2^{b} \subseteq D_6^{b} \subseteq D_{12}$

(46) $\{e\} \subseteq \langle a^6b \rangle \subseteq D_2^{b} \subseteq D_4^{b} \subseteq D_{12}$

(47) $\{e\} \subseteq \langle a^6b \rangle \subseteq D_3^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(48) $\{e\} \subseteq \langle a^7b \rangle \subseteq D_2^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(49) $\{e\} \subseteq \langle a^7b \rangle \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_{12}$

(50) $\{e\} \subseteq \langle a^7b \rangle \subseteq D_3^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(51) $\{e\} \subseteq \langle a^8b \rangle \subseteq D_2^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(52) $\{e\} \subseteq \langle a^8b \rangle \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(53) $\{e\} \subseteq \langle a^8b \rangle \subseteq D_3^{b} \subseteq D_6^{b} \subseteq D_{12}$

(54) $\{e\} \subseteq \langle a^9b \rangle \subseteq D_2^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$

(55) $\{e\} \subseteq \langle a^9b \rangle \subseteq D_2^{a^3b} \subseteq D_4^{b} \subseteq D_{12}$

(56) $\{e\} \subseteq \langle a^9b \rangle \subseteq D_3^{ab} \subseteq D_6^{ab} \subseteq D_{12}$

(57) $\{e\} \subseteq \langle a^{10}b \rangle \subseteq D_2^{a^4b} \subseteq D_6^{b} \subseteq D_{12}$

(58) $\{e\} \subseteq \langle a^{10}b \rangle \subseteq D_2^{a^4b} \subseteq D_4^{ab} \subseteq D_{12}$

(59) $\{e\} \subseteq \langle a^{10}b \rangle \subseteq D_3^{a^2b} \subseteq D_6^{b} \subseteq D_{12}$

(60) $\{e\} \subseteq \langle a^{11}b \rangle \subseteq D_2^{a^5b} \subseteq D_6^{ab} \subseteq D_{12}$

(61) $\{e\} \subseteq \langle a^{11}b \rangle \subseteq D_2^{a^5b} \subseteq D_4^{a^2b} \subseteq D_{12}$

(62) $\{e\} \subseteq \langle a^{11}b \rangle \subseteq D_3^{a^3b} \subseteq D_6^{ab} \subseteq D_{12}$



Since this method employs the concept of keychains on the flags of the group, our observation is that each one of the flags has 5 levels. Therefore, each equivalence class of fuzzy subgroups can be represented by a keychain with five components consisting of 4 pins, and 1. From [67] we have that the number of keychains is $2^5 - 1 = 31$, where $n = 5$ is the length of each flag, viz.

$11111$ $1111\lambda$ $11110$ $111\lambda\lambda$ $111\lambda\beta$ $111\lambda0$ $11100$ $11\lambda\lambda\lambda$ $11\lambda\lambda\beta$ $11\lambda\lambda0$ $11\lambda\beta\beta$ $11\lambda\beta\alpha$ $11\lambda\beta0$ $11\lambda00$ $11000$ $1\lambda\lambda\lambda\lambda$ $1\lambda\lambda\lambda\beta$ $1\lambda\lambda\lambda0$ $1\lambda\lambda\beta\beta$ $1\lambda\lambda\beta\alpha$ $1\lambda\lambda\beta0$ $1\lambda\lambda00$ $1\lambda\beta\beta\beta$ $1\lambda\beta\beta\alpha$ $1\lambda\beta\alpha\alpha$ $1\lambda\beta\alpha\theta$ $1\lambda\beta\alpha0$ $1\lambda\beta\beta0$ $1\lambda\beta00$ $1\lambda000$ $10000$

We now select any keychain and "pin" it on all the flags of the group, to determine the number of distinct fuzzy subgroups each represents. Each one of the three exceptional keychains $11111$, $1\lambda\lambda\lambda\lambda$ and $10000$ of index (4) will yield a count of one fuzzy subgroup, on all 62 flags. Hence, the number of equivalence classes of fuzzy subgroups represented by these keychains is $3 \times 1 = 3$. Suppose we let $\mu = 1\lambda\lambda\beta\beta$, a keychain of index $(2, 2)$ then pinning this keychain on each one of the maximal chains yields the following:

On flag (1), $\mu_1 = 1\lambda\lambda\beta\beta$ yields:

$$\mu_1(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in \langle a^6 \rangle \setminus \{e\} \\ \lambda & if \quad x \in \langle a^2 \setminus \langle a^6 \rangle \\ \beta & if \quad x \in \langle a \rangle \setminus \langle a^2 \rangle \\ \beta & if \quad x \in D_{12} \setminus \langle a \rangle \end{cases}$$

On flag (2), $\mu_2 = 1\lambda\lambda\beta\beta$ yields:



$$\mu_2(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in \langle a^6 \rangle \setminus \{e\} \\ \lambda & if \quad x \in \langle a^3 \setminus \langle a^6 \rangle \\ \beta & if \quad x \in \langle a \rangle \setminus \langle a^3 \rangle \\ \beta & if \quad x \in D_{12} \setminus \langle a \rangle \end{cases}$$

On flag (3), $\mu_3 = 1\lambda\lambda\beta\beta$ yields:

$$\mu_3(x) = \begin{cases} 1 & if\ x = e \\ \lambda & if \quad x \in \langle a^4 \rangle \setminus \{e\} \\ \lambda & if \quad x \in \langle a^2 \setminus \langle a^4 \rangle \\ \beta & if \quad x \in \langle a \rangle \setminus \langle a^2 \rangle \\ \beta & if \quad x \in D_{12} \setminus \langle a \rangle \end{cases}$$

From the above fuzzy subgroups, it is evident that $\mu_1$ and $\mu_3$ belong to the same distinct equivalence class of fuzzy subgroups. So $1\lambda\lambda\beta\beta$ represents one distinct fuzzy subgroups on both flags (1) and (3), while on flag (2), it represents one distinct fuzzy subgroup, $\mu_2$. Continuing in this fashion on the rest of the flags of $D_{12}$. $1\lambda\lambda\beta\beta$ yields the following:

On flag (4), $\mu_4 = 1\lambda\lambda\beta\beta$ yields:

$\mu_4(a^6) = \mu_4(a^2) = \mu_4(a^4) = \mu_4(a^8) = \mu_4(a^{10}) > \mu_4(b) = \mu_4(a^2b) = \mu_4(a^4b) = \mu_4(a^6b) = \mu_4(a^8b) = \mu_4(a^{10}b) = \mu_4(a) = \mu_4(a^3) = \mu_4(a^5) = \mu_4(a^7) = \mu_4(a^9) = \mu_4(a^{11}) = \mu_4(ab) = \mu_4(a^3b) = \mu_4(a^5b) = \mu_4(a^7b) = \mu_4(a^9b) = \mu_4(a^{11}b)$

On flag (5), $\mu_5 = 1\lambda\lambda\beta\beta$ yields:

$\mu_5(a^6) = \mu_5(a^2) = \mu_5(a^4) = \mu_5(a^8) = \mu_5(a^{10}) > \mu_5(ab) = \mu_5(a^3b) = \mu_5(a^5b) = \mu_5(a^7b) = \mu_5(a^9b) = \mu_5(a^{11}b) = \mu_5(a) = \mu_5(a^3) = \mu_5(a^5) = \mu_5(a^7) = \mu_5(a^9) = \mu_5(a^{11}) = \mu_5(b) = \mu_5(a^2b) = \mu_5(a^4b) = \mu_5(a^6b) = \mu_5(a^8b) = \mu_5(a^{10}b)$

On flag (6), $\mu_6 = 1\lambda\lambda\beta\beta$ yields:

$\mu_6(a^6) = \mu_6(a^3) = \mu_6(a^9) > \mu_6(b) = \mu_6(a^3b) = \mu_6(a^6b) = \mu_6(a^9b) = \mu_6(a) = \mu_6(a^2) = \mu_6(a^4) = \mu_6(a^5) = \mu_6(a^7) = \mu_6(a^8) = \mu_6(a^{10}) = \mu_6(a^{11}) = \mu_6(ab) = \mu_6(a^2b) = \mu_6(a^4b) = \mu_6(a^5b) = \mu_6(a^7b) = \mu_6(a^8b) = \mu_6(a^{10}b) = \mu_6(a^{11}b)$



On flag (7), $\mu_7 = 1\lambda\lambda\beta\beta$ yields:

$\mu_7(a^6) = \mu_7(a^3) = \mu_7(a^9) > \mu_7(ab) = \mu_7(a^4b) = \mu_7(a^7b) = \mu_7(a^{10}b) = \mu_7(a) = \mu_7(a^2) = \mu_7(a^4) = \mu_7(a^5) = \mu_7(a^7) = \mu_7(a^8) = \mu_7(a^{10}) = \mu_7(a^{11}) = \mu_7(b) = \mu_7(a^2b) = \mu_7(a^3b) = \mu_7(a^5b) = \mu_7(a^6b) = \mu_7(a^9b) = \mu_7(a^8b) = \mu_7(a^{11}b)$

On flag (8), $\mu_8 = 1\lambda\lambda\beta\beta$ yields:

$\mu_8(a^6) = \mu_8(a^3) = \mu_8(a^9) > \mu_8(a^2b) = \mu_8(a^5b) = \mu_8(a^8b) = \mu_8(a^{11}b) = \mu_8(a) = \mu_8(a^2) = \mu_8(a^4) = \mu_8(a^5) = \mu_8(a^7) = \mu_8(a^8) = \mu_8(a^{10}) = \mu_8(a^{11}) = \mu_8(b) = \mu_8(ab) = \mu_8(a^3b) = \mu_8(a^4b) = \mu_8(a^6b) = \mu_8(a^7b) = \mu_8(a^9b) = \mu_8(a^{10}b)$

On flag (9), $\mu_9 = 1\lambda\lambda\beta\beta$ yields:

$\mu_9(a^4) = \mu_9(a^8) = \mu_9(a^2) = \mu_9(a^6) = \mu_9(a^{10}) > \mu_9(b) = \mu_9(a^2b) = \mu_9(a^4b) = \mu_9(a^6b) = \mu_9(a^8b) = \mu_9(a^{10}b)\mu_9(a) = \mu_9(a^3) = \mu_9(a^5) = \mu_9(a^7) = \mu_9(a^9) = \mu_9(a^{11}) = \mu_9(ab) = \mu_9(a^3b) = \mu_9(a^7b) = \mu_{10}(a^9b) = \mu_9(a^5b) = \mu_9(a^{11}b)$

On flag (10), $\mu_{10} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{10}(a^4) = \mu_{10}(a^8) = \mu_{10}(a^2) = \mu_{10}(a^6) = \mu_{10}(a^{10}) > \mu_{10}(ab) = \mu_{10}(a^3b) = \mu_{10}(a^5b) = \mu_{10}(a^7b) = \mu_{10}(a^9b) = \mu_{10}(a^{11}b) = \mu_{10}(a) = \mu_{10}(a^3) = \mu_{10}(a^5) = \mu_{10}(a^7) = \mu_{10}(a^9) = \mu_{10}(a^{11}) = \mu_{10}(b) = \mu_{10}(a^2b) = \mu_{10}(a^4b) = \mu_{10}(a^6b) = \mu_{10}(a^8b) = \mu_{10}(a^{10}b)$

On flag (11), $\mu_{11} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{11}(a^6) = \mu_{11}(b) = \mu_{11}(a^6b) > \mu_{11}(a^2) = \mu_{11}(a^4) = \mu_{11}(a^8) = \mu_{11}(a^{10}) = \mu_{11}(a^2b) = \mu_{11}(a^4b) = \mu_{11}(a^8b) = \mu_{11}(a^{10}b) = \mu_{11}(a) = \mu_{11}(a^3) = \mu_{11}(a^5) = \mu_{11}(a^7) = \mu_{11}(a^9) = \mu_{11}(a^{11}) = \mu_{11}(ab) = \mu_{11}(a^3b) = \mu_{11}(a^5b) = \mu_{11}(a^7b) = \mu_{11}(a^9b) = \mu_{11}(a^{11}b)$

On flag (12), $\mu_{12} = 1\lambda\lambda\beta\beta$ yields:



$\mu_{12}(a^6) = \mu_{12}(ab) = \mu_{12}(a^7b) > \mu_{12}(a^2) = \mu_{12}(a^4) = \mu_{12}(a^8) = \mu_{12}(a^{10}) = \mu_{12}(a^3b) = \mu_{12}(a^5b) = \mu_{12}(a^7) = \mu_{12}(a^9b) = \mu_{12}(a^{11}b) = \mu_{12}(a) = \mu_{12}(a^3) = \mu_{12}(a^5) = \mu_{12}(a^9) = \mu_{12}(a^{11}) = \mu_{12}(b) = \mu_{12}(a^2b) = \mu_{12}(a^4b) = \mu_{12}(a^6b) = \mu_{12}(a^8b) = \mu_{12}(a^{10}b)$

On flag (13), $\mu_{13} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{13}(a^6) = \mu_{13}(a^2b) = \mu_{13}(a^8b) > \mu_{13}(b) = \mu_{13}(a^2) = \mu_{13}(a^4) = \mu_{13}(a^8) = \mu_{13}(a^{10}) = \mu_{13}(a^4b) = \mu_{13}(a^6b) = \mu_{13}(a^{10}b) = \mu_{13}(a) = \mu_{13}(a^3) = \mu_{13}(a^5) = \mu_{13}(a^7) = \mu_{13}(a^9) = \mu_{13}(a^{11}) = \mu_{13}(ab) = \mu_{13}(a^3b) = \mu_{13}(a^5b) = \mu_{13}(a^7b) = \mu_{13}(a^9b) = \mu_{13}(a^{11}b)$

On flag (14), $\mu_{14} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{14}(a^6) = \mu_{14}(a^3b) = \mu_{14}(a^9b) > \mu_{14}(ab) = \mu_{14}(a^2) = \mu_{14}(a^4) = \mu_{14}(a^8) = \mu_{14}(a^{10}) = \mu_{14}(a^5b) = \mu_{14}(a^7b) = \mu_{14}(a^{11}b) = \mu_{14}(a) = \mu_{14}(a^3) = \mu_{14}(a^5) = \mu_{14}(a^7) = \mu_{14}(a^9) = \mu_{14}(a^{11}) = \mu_{14}(b) = \mu_{14}(a^2b) = \mu_{14}(a^4b) = \mu_{14}(a^6b) = \mu_{14}(a^8b) = \mu_{14}(a^{10}b)$

On flag (15), $\mu_{15} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{15}(a^6) = \mu_{15}(a^4b) = \mu_{15}(a^{10}b) > \mu_{15}(b) = \mu_{15}(a^2) = \mu_{15}(a^4) = \mu_{15}(a^8) = \mu_{15}(a^{10}) = \mu_{15}(a^2b) = \mu_{15}(a^6b) = \mu_{15}(a^8b) = \mu_{15}(a) = \mu_{15}(a^3) = \mu_{15}(a^5) = \mu_{15}(a^7) = \mu_{15}(a^9) = \mu_{15}(a^{11}) = \mu_{15}(ab) = \mu_{15}(a^3b) = \mu_{15}(a^5b) = \mu_{15}(a^7b) = \mu_{15}(a^9b) = \mu_{15}(a^{11}b)$

On flag (16), $\mu_{16} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{14}(a^6) = \mu_{14}(a^5b) = \mu_{14}(a^{11}b) > \mu_{14}(ab) = \mu_{14}(a^2) = \mu_{14}(a^4) = \mu_{14}(a^8) = \mu_{14}(a^{10}) = \mu_{14}(a^3b) = \mu_{14}(a^7b) = \mu_{14}(a^9b) = \mu_{14}(a) = \mu_{14}(a^3) = \mu_{14}(a^5) = \mu_{14}(a^7) = \mu_{14}(a^9) = \mu_{14}(a^{11}) = \mu_{14}(b) = \mu_{14}(a^2b) = \mu_{14}(a^4b) = \mu_{14}(a^6b) = \mu_{14}(a^8b) = \mu_{14}(a^{10}b)$

On flag (17), $\mu_{17} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{17}(a^6) = \mu_{17}(b) = \mu_{17}(a^6b) > \mu_{17}(a^3) = \mu_{17}(a^9) = \mu_{17}(a^3b) = \mu_{17}(a^9b) = \mu_{17}(a) = \mu_{17}(a^2) = \mu_{17}(a^4) = \mu_{17}(a^5) = \mu_{17}(a^7) = \mu_{17}(a^8) = \mu_{17}(a^{10}) = \mu_{17}(a^{11}) = \mu_{17}(ab) = \mu_{17}(a^2b) = \mu_{17}(a^4b) = \mu_{17}(a^5b) = \mu_{17}(a^7b) = \mu_{17}(a^8b) = \mu_{17}(a^{10}b) = \mu_{17}(a^{11}b)$



On flag (18), $\mu_{18} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{18}(a^6) = \mu_{18}(ab) = \mu_{18}(a^7b) > \mu_{18}(a^3) = \mu_{18}(a^9) = \mu_{18}(a^4b) = \mu_{18}(a^{10}b) = \mu_{18}(a) = \mu_{18}(a^2) = \mu_{18}(a^4) = \mu_{18}(a^5) = \mu_{18}(a^7) = \mu_{18}(a^8) = \mu_{18}(a^{10}) = \mu_{18}(a^{11}) = \mu_{18}(b) = \mu_{18}(a^2b) = \mu_{18}(a^3b) = \mu_{18}(a^5b) = \mu_{18}(a^6b) = \mu_{18}(a^8b) = \mu_{18}(a^9b) = \mu_{18}(a^{11}b)$

On flag (19), $\mu_{19} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{19}(a^6) = \mu_{19}(a^2b) = \mu_{19}(a^8b) > \mu_{19}(a^3) = \mu_{19}(a^9) = \mu_{19}(a^5b) = \mu_{19}(a^{11}b) = \mu_{19}(a) = \mu_{19}(a^2) = \mu_{19}(a^4) = \mu_{19}(a^5) = \mu_{19}(a^7) = \mu_{19}(a^8) = \mu_{19}(a^{10}) = \mu_{19}(a^{11}) = \mu_{19}(b) = \mu_{19}(ab) = \mu_{19}(a^3b) = \mu_{19}(a^4b) = \mu_{19}(a^6b) = \mu_{19}(a^7b) = \mu_{19}(a^9b) = \mu_{19}(a^{10}b)$

On flag (20), $\mu_{20} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{20}(a^6) = \mu_{20}(a^3b) = \mu_{20}(a^9b) > \mu_{20}(b) = \mu_{20}(a^3) = \mu_{20}(a^9) = \mu_{20}(a^5b) = \mu_{20}(a^{11}b) = \mu_{20}(a) = \mu_{20}(a^2) = \mu_{20}(a^4) = \mu_{20}(a^5) = \mu_{20}(a^7) = \mu_{20}(a^8) = \mu_{20}(a^{10}) = \mu_{20}(a^{11}) = \mu_{20}(ab) = \mu_{20}(a^2b) = \mu_{20}(a^4b) = \mu_{20}(a^6b) = \mu_{20}(a^7b) =$

$\mu_{20}(a^8b) = \mu_{20}(a^{10}b)$

On flag (21), $\mu_{21} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{21}(a^6) = \mu_{21}(a^4b) = \mu_{21}(a^{10}b) > \mu_{21}(ab) = \mu_{21}(a^3) = \mu_{21}(a^9) = \mu_{21}(a^7b) = \mu_{21}(a) = \mu_{21}(a^2) = \mu_{21}(a^4) = \mu_{21}(a^5) = \mu_{21}(a^7) = \mu_{21}(a^8) = \mu_{21}(a^{10}) = \mu_{21}(a^{11}) = \mu_{21}(b) = \mu_{21}(a^2b) = \mu_{21}(a^3b) = \mu_{21}(a^6b) = \mu_{21}(a^5b) = \mu_{21}(a^8b) = \mu_{21}(a^{11}b)$

On flag (22), $\mu_{22} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{22}(a^6) = \mu_{22}(a^5b) = \mu_{22}(a^{11}b) > \mu_{22}(a^3) = \mu_{22}(a^9) = \mu_{22}(a^2b) = \mu_{22}(a^8b) = \mu_{22}(a) = \mu_{22}(a^2) = \mu_{22}(a^4) = \mu_{22}(a^5) = \mu_{22}(a^7) = \mu_{22}(a^8) = \mu_{22}(a^{10}) = \mu_{22}(b) = \mu_{22}(ab) = \mu_{22}(a^3b) = \mu_{22}(a^4b) = \mu_{22}(a^6b) = \mu_{22}(a^7b) = \mu_{22}(a^9b) = \mu_{22}(a^{11}) = \mu_{22}(a^{10}b)$

On flag (23), $\mu_{23} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{23}(a^4) = \mu_{23}(a^8) = \mu_{23}(b) = \mu_{23}(a^4b) = \mu_{23}(a^8b) > \mu_{23}(a^2) = \mu_{23}(a^6) = \mu_{23}(a^{10}) = \mu_{23}(a^2b) = \mu_{23}(a^6b) = \mu_{23}(a^{10}b) = \mu_{23}(a) = \mu_{23}(a^3) = \mu_{23}(a^5) = \mu_{23}(a^7) = \mu_{23}(a^9) = \mu_{23}(a^{11}) = \mu_{23}(ab) = \mu_{23}(a^3b) = \mu_{23}(a^5b) = \mu_{23}(a^7b) = \mu_{23}(a^9b) = \mu_{23}(a^{11}b)$



On flag (24), $\mu_{24} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{24}(a^4) = \mu_{24}(a^8) = \mu_{24}(ab) = \mu_{24}(a^5b) = \mu_{24}(a^9b) > \mu_{24}(a^2) = \mu_{24}(a^6) = \mu_{24}(a^{10}) = \mu_{24}(a^3b) = \mu_{24}(a^7b) = \mu_{24}(a^{11}b) = \mu_{24}(a) = \mu_{24}(a^3) = \mu_{24}(a^5) = \mu_{24}(a^7) = \mu_{24}(a^9) = \mu_{24}(a^{11}) = \mu_{24}(ab) = \mu_{24}(a^2b) = \mu_{24}(a^4b) = \mu_{24}(a^6b) = \mu_{24}(a^8b) = \mu_{24}(a^{10}b)$

On flag (25), $\mu_{25} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{25}(a^4) = \mu_{25}(a^8) = \mu_{25}(a^2b) = \mu_{25}(a^6b) = \mu_{25}(a^{10}b) > \mu_{25}(a^2) = \mu_{25}(a^6) = \mu_{25}(a^{10}) = \mu_{25}(b) = \mu_{25}(a^4b) = \mu_{25}(a^8b) = \mu_{25}(a) = \mu_{25}(a^3) = \mu_{25}(a^5) = \mu_{25}(a^7) = \mu_{25}(a^9) = \mu_{25}(a^{11}) = \mu_{25}(ab) = \mu_{25}(a^3b) = \mu_{25}(a^5b) = \mu_{25}(a^7b) = \mu_{25}(a^9b) = \mu_{25}(a^{11}b)$

On flag (26), $\mu_{26} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{26}(a^4) = \mu_{26}(a^8) = \mu_{26}(a^2b) = \mu_{26}(a^7b) = \mu_{26}(a^{11}b) > \mu_{26}(a^2) = \mu_{26}(a^6) = \mu_{26}(a^{10}) = \mu_{26}(ab) = \mu_{26}(a^5b) = \mu_{26}(a^9b) = \mu_{26}(a) = \mu_{26}(a^3) = \mu_{26}(a^5) =$

$\mu_{26}(a^6) = \mu_{26}(a^9) = \mu_{26}(a^{11}) = \mu_{26}(b) = \mu_{26}(a^3b) = \mu_{26}(a^4b) = \mu_{26}(a^7b) = \mu_{26}(a^8b) = \mu_{26}(a^{10}b)$

On the flags (2),(6), (7), and (8), the keychain $1\lambda\lambda\beta\beta$ yields identical fuzzy subgroups, hence we will group all maximal chains according to the nature of equivalence of their fuzzy subgroups. Our observation of the above fuzzy subgroups indicates that the fuzzy subgroups $\mu_2$, $\mu_6$, $\mu_7$ and $\mu_8$ represent one distinct equivalence class of fuzzy subgroup. Continuing these iterations on the rest of the maximal chains gives the following:

On flag (27), $\mu_{27} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{27}(b) = \mu_{27}(a^6) = \mu_{27}(a^6b) > \mu_{27}(a^2) = \mu_{27}(a^4) = \mu_{27}(a^8) = \mu_{27}(a^{10}) = \mu_{27}(a^2b) = \mu_{27}(a^4b) = \mu_{27}(a^8b) = \mu_{27}(a^{10}b) = \mu_{27}(a) = \mu_{27}(a^3) = \mu_{27}(a^5) = \mu_{27}(a^7) = \mu_{27}(a^9) = \mu_{27}(a^{11}) = \mu_{27}(ab) = \mu_{27}(a^3b) = \mu_{27}(a^5b) = \mu_{27}(a^7b) = \mu_{27}(a^9b) = \mu_{27}(a^{11}b)$

On flag (28), $\mu_{28} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{28}(b) = \mu_{28}(a^6) = \mu_{28}(a^6b) > \mu_{28}(a^3) = \mu_{28}(a^9) = \mu_{28}(a^3b) = \mu_{28}(a^9b) = \mu_{28}(a) = \mu_{28}(a^2) = \mu_{28}(a^4) = \mu_{28}(a^5) = \mu_{28}(a^7) = \mu_{28}(a^8) = \mu_{28}(a^{10}) = \mu_{28}(a^{11}) = \mu_{28}(ab) = \mu_{28}(a^2b) = \mu_{28}(a^4b) = \mu_{28}(a^5b) = \mu_{28}(a^7b) = \mu_{28}(a^8b) = \mu_{28}(a^{10}b) = \mu_{28}(a^{11}b)$



On flag (29), $\mu_{29} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{29}(b) = \mu_{29}(a^4) = \mu_{29}(a^8) = \mu_{29}(a^4b) = \mu_{29}(a^8b) > \mu_{29}(a^2) = \mu_{29}(a^6) = \mu_{29}(a^{10}) = \mu_{29}(a^2b) = \mu_{29}(a^6b) = \mu_{29}(a^{10}b) = \mu_{29}(a) = \mu_{29}(a^3) = \mu_{29}(a^5) = \mu_{29}(a^7) = \mu_{29}(a^9) = \mu_{29}(a^{11}) = \mu_{29}(ab) = \mu_{29}(a^3b) = \mu_{29}(a^5b) = \mu_{29}(a^7b) = \mu_{29}(a^9b) = \mu_{29}(a^{11}b)$

On flag (30), $\mu_{30} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{30}(ab) = \mu_{30}(a^6) = \mu_{30}(a^7b) > \mu_{30}(a^2) = \mu_{30}(a^4) = \mu_{30}(a^8) = \mu_{30}(a^{10}) = \mu_{30}(a^3b) = \mu_{30}(a^5b) = \mu_{30}(a^9b) = \mu_{30}(a^{11}b) = \mu_{30}(a) = \mu_{30}(a^3) = \mu_{30}(a^5) = \mu_{30}(a^7) = \mu_{30}(a^9) = \mu_{30}(a^{11}) = \mu_{30}(b) = \mu_{30}(a^2b) = \mu_{30}(a^4b) = \mu_{30}(a^6b) = \mu_{30}(a^8b) = \mu_{30}(a^{10}b)$

On flag (31), $\mu_{31} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{31}(ab) = \mu_{31}(a^6) = \mu_{31}(a^7b) > \mu_{31}(a^3) = \mu_{31}(a^9) = \mu_{31}(a^4b) = \mu_{31}(a^{10}b) = \mu_{31}(a) = \mu_{31}(a^2) = \mu_{31}(a^4) = \mu_{31}(a^5) = \mu_{31}(a^7) = \mu_{31}(a^8) = \mu_{31}(a^{10}) = \mu_{31}(a^{11}) = \mu_{31}(b) = \mu_{31}(a^2b) = \mu_{31}(a^3b) = \mu_{31}(a^5b) = \mu_{31}(a^6b) = \mu_{31}(a^8b) = \mu_{31}(a^9b) = \mu_{31}(a^{11}b)$

On flag (32), $\mu_{32} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{32}(ab) = \mu_{32}(a^4) = \mu_{32}(a^8) = \mu_{32}(a^5b) = \mu_{32}(a^9b) > \mu_{32}(a^2) = \mu_{32}(a^7) = \mu_{32}(a^{11}) = \mu_{32}(a^3b) = \mu_{32}(a^6b) = \mu_{32}(a^{10}b) = \mu_{32}(a) = \mu_{32}(a^3) = \mu_{32}(a^5) = \mu_{32}(a^7) = \mu_{32}(a^9) = \mu_{32}(a^{11}) = \mu_{32}(ab) = \mu_{32}(a^2b) = \mu_{32}(a^4b) = \mu_{32}(a^6b) = \mu_{32}(a^8b) = \mu_{32}(a^{10}b)$

On flag (33), $\mu_{33} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{33}(a^2b) = \mu_{33}(a^6) = \mu_{33}(a^8b) > \mu_{33}(a^2) = \mu_{33}(a^4) = \mu_{33}(a^8) = \mu_{33}(a^{10}) = \mu_{33}(b) = \mu_{33}(a^2b) = \mu_{33}(a^4b) = \mu_{33}(a^{10}b) = \mu_{33}(a) = \mu_{33}(a^3) = \mu_{33}(a^5) = \mu_{33}(a^7) = \mu_{33}(a^9) = \mu_{33}(a^{11}) = \mu_{33}(ab) = \mu_{33}(a^3b) = \mu_{33}(a^5b) = \mu_{33}(a^7b) = \mu_{33}(a^9b) = \mu_{33}(a^{11}b)$

On flag (34), $\mu_{34} = 1\lambda\lambda\beta\beta$ yields:



$\mu_{34}(a^2b) = \mu_{34}(a^6) = \mu_{34}(a^8b) > \mu_{34}(a^3) = \mu_{34}(a^9) = \mu_{34}(a^5b) = \mu_{34}(a^{11}b) = \mu_{34}(a) = \mu_{34}(a^2) = \mu_{34}(a^4) = \mu_{34}(a^5) = \mu_{34}(a^7) = \mu_{34}(a^8) = \mu_{34}(a^{10}) = \mu_{34}(a^{11}) = \mu_{34}(b) = \mu_{34}(ab) = \mu_{34}(a^3b) = \mu_{34}(a^4b) = \mu_{34}(a^6b) = \mu_{34}(a^7b) = \mu_{34}(a^9b) = \mu_{34}(a^{11}b)$

On flag (35), $\mu_{35} = 1\lambda\lambda\beta\beta$ yields:

$\mu_{35}(a^2b) = \mu_{35}(a^4) = \mu_{35}(a^8) = \mu_{35}(a^6b) = \mu_{35}(a^{10}b) > \mu_{35}(a^2) = \mu_{35}(a^6) = \mu_{35}(a^{10}) = \mu_{35}(b) = \mu_{35}(a^4b) = \mu_{35}(a^8b) = \mu_{35}(a) = \mu_{35}(a^3) = \mu_{35}(a^5) = \mu_{35}(a^7) = \mu_{35}(a^9) = \mu_{35}(a^{11}) = \mu_{35}(ab) = \mu_{35}(a^3b) = \mu_{35}(a^5b) = \mu_{35}(a^7b) = \mu_{35}(a^9b) = \mu_{35}(a^{11}b)$

When we pin the keychain $1\lambda\lambda\beta\beta$ on all 62 maximal chains we observe that, the fuzzy subgroups $\mu_{11},\ \mu_{17},\ \mu_{27},\ \mu_{28}, \mu_{45}$ and $\mu_{46}$ represent one disitinct equivalence class of fuzzy subgroups, since the keychain $1\lambda\lambda\beta\beta$ yields identical fuzzy subgroups when we pin it on the flags (11), (17), (27), (28), (45) and (46). Now, taking into account all the distinct equivalence classes of

fuzzy subgroups we obtain from the 62 flags of $D_{12}$, we get the total number of distinct fuzzy subgroups yielded by $1\lambda\lambda\beta\beta$ as 12. Similarly, the keychains $1\lambda\lambda00$, $111\lambda\lambda$, and 11100, each gives a count of 12 distinct fuzzy subgroups. We note that each of these keychains is of index $(2,2)$ and therefore contributes the same number of distinct fuzzy subgroups. Continuing in this fashion for each flag, we are thus able to calculate the total number of distinct fuzzy subgroups for $D_{12}$. The following table provides these results.
It is clear from the table that keychains of the same index will contribute the same number of distinct fuzzy subgroups.

Table 3.1: The number of Keychains and Distinct Equivalence Classes of Fuzzy Subgroups of $D_{12}$

| **Keychains** | **Index of keychain** | **Fuzzy subgroup count in all flags** |
|---|---|---|
| 11111 | $(4)$ | 1 |
| $1\lambda\lambda\lambda\lambda$ | $(4)$ | 1 |
| 10000 | $(4)$ | 1 |
| $1111\lambda$ | $(3,1)$ | 6 |
| 11110 | $(3,1)$ | 6 |
| $1\lambda\lambda\lambda\beta$ | $(3,1)$ | 6 |
| $1\lambda\lambda\lambda0$ | $(3,1)$ | 6 |
| $111\lambda\lambda$ | $(2,2)$ | 12 |
| 11100 | $(2,2)$ | 12 |
| $1\lambda\lambda\beta\beta$ | $(2,2)$ | 12 |
| $1\lambda\lambda00$ | $(2,2)$ | 12 |
| $11\lambda\lambda\lambda$ | $(1,3)$ | 14 |
| $1\lambda\beta\beta\beta$ | $(1,3)$ | 14 |
| $1\lambda000$ | $(1,3)$ | 14 |
| 11000 | $(1,3)$ | 14 |

| **Keychains** | **Index of keychain** | **Fuzzy subgroup count in all flags** |
|---|---|---|
| $111\lambda\beta$ | $(2,1,1)$ | 23 |
| $111\lambda 0$ | $(2,1,1)$ | 23 |
| $1\lambda\lambda\beta\alpha$ | $(2,1,1)$ | 23 |
| $1\lambda\lambda\beta 0$ | $(2,1,1)$ | 23 |
| $11\lambda\lambda\beta$ | $(1,2,1)$ | 33 |
| $11\lambda\lambda 0$ | $(1,2,1)$ | 33 |
| $1\lambda\beta\beta\alpha$ | $(1,2,1)$ | 33 |
| $1\lambda\beta\beta 0$ | $(1,2,1)$ | 33 |
| $11\lambda\beta\beta$ | $(1,1,2)$ | 37 |
| $1\lambda\beta\alpha\alpha$ | $(1,1,2)$ | 37 |
| $1\lambda\beta 00$ | $(1,1,2)$ | 37 |
| $11\lambda 00$ | $(1,1,2)$ | 37 |
| $11\lambda\beta\alpha$ | $(1,1,1,1)$ | 62 |
| $11\lambda\beta 0$ | $(1,1,1,1)$ | 62 |
| $1\lambda\beta\alpha\theta$ | $(1,1,1,1)$ | 62 |
| $1\lambda\beta\alpha 0$ | $(1,1,1,1)$ | 62 |
| **Total** | | 751 |



Thus the total number of distinct equivalence classes of fuzzy subgroups for $D_{12}$ is 751.

# Chapter 4

# The Dihedral Group $D_n$ for $n \in \mathbb{N}$

## 4.1 Introduction



The geometric intricacies involved in symmetric groups has always held an appeal to mathematics researchers. Although the feasibility of classifying all finite groups is improbable, research has shown that most finite groups that occur naturally, closely resemble the structures of either simple groups or dihedral groups. The dihedral group of order $2n$, denoted by $D_n$ for $n \geq 3$, can thus be defined as the symmetry group of the regular $n$-gon, with both rotations and reflections. The fascinating geometric properties of dihedral groups have deemed them an important aspect in a variety of mathematical disciplines, hence their crisp characterisation by numerous researchers that include Al-Hasanat et al, [12], Conrad [35] and [36], Dummit and Foote [44], Erfanian, Omer and Sarmin [45], Feng, Kwak and Kwon [47], Gallian [51], Herstein [55], Lenz [62] and Zhang [135], amongst others. Since our classification problem relies entirely on the construction of the maximal chains of a group, this chapter briefly outlines the general properties of the dihedral group $D_n$, for $n \in \mathbb{N}$ in the crisp case. Furthermore, as an introduction to Chapter 5, we briefly characterise the dihedral groups $D_{p^n}$ and $D_{pq}$ where $p$ is a prime and $n \in \mathbb{N}$, and $p$ and $q$ are distinct primes. We also characterise maximal chains according to the terms introduced in [93]. To classify dihedral groups, it is always useful to move from their geometric, to their algebraic setting. Thus we have that:

$$D_n = \langle a, b : a^n = e = b^2 = (ab)^2 \rangle$$

where, $a^n = e = b^2$ and $bab^{-1} = a^{-1}$
In $D_n$, since we can obtain $a$ from $b$ and $ab$, this implies that the reflections $ab$ and $b$ can be used as generators for $D_n$, so:

$$D_n = \langle a, b \rangle = \langle ab, a \rangle$$

Thus $D_n$ is generated by two non-commutative elements of order 2. And hence we have the following theorems:



**Theorem 4.1.0.1.** *[36] If a finite group $G$ of order $2n$, is generated by two elements, such that $G = \langle x, y \rangle$, where $x^n = e$, for some $n \geq 3$, $y^2 = e$ and $yxy^{-1} = x^{-1}$, then $G$ is isomorphic to a dihedral group $D_n$.*

*Proof.* [36] □

**Theorem 4.1.0.2.** *[36] Let $G$ be a finite non-abelian group generated by two elements of order 2. Then $G$ is isomorphic to a dihedral group.*

*Proof.* [36] □

### 4.1.1 Properties of the Dihedral group $D_n$, $n \in \mathbb{N}$

The dihedral groups are characterised by the following Theorems and propositions.

**Theorem 4.1.1.1.** *[35] The group $D_n$ has $2n$ elements, listed as*

$$D_n = \{e, a, a^2, a^3, \cdots, a^{n-1}, b, ab, a^2b, a^3b, \cdots, a^{n-1}b\}$$

**Note 4.1.1.1.** All elements of $D_n$ with orders greater than 2, are powers of the rotation $a$, however, one should not falsely assume that the only elements of order 2 are the reflections $b$. Also, the relationship between the rigid motions $a$ and $b$ is given as $bab^{-1} = a^{-1}$, hence for any $k \in \mathbb{Z}$ we have that $(a^k b)^2 = (a^k b)(a^k b) = a^k a^{-1} bb = b^2 = e$

**Theorem 4.1.1.2.** *[129] $D_n = \{a^i : i \in \mathbb{Z}_n\} \cup \{a^i b : i \in \mathbb{Z}_n\}$ Is the irredundant list of elements of $D_n$*

*Proof.* [129] □

**Proposition 4.1.1.2.1.** *[36] Every subgroup of $D_n$ is either dihedral or cyclic, and listed as:*

*i The unique cyclic subgroups $\langle a^d \rangle \cong \mathbb{Z}_n$ where $d|n$ and index $2d$.*

*ii the dihedral subgroups $\langle a^d,\ a^i b\rangle$ , where $d|n$, $0 \leq i \leq d-1$ and index $d$.*



Every subgroup of $D_n$ described above occurs no more than once in this listing.

*Proof.* [36] □

**Example 4.1.1.1.** The dihedral subgroup $D_s^b$ of $D_n$, where $s|n$, defines the dihedral subgroup $D_s^b = \langle a^r,\ b \mid (a^r)^s = b^2 = e = (a^r b)^2 \rangle$. If we let $n = 12$, we have the dihedral group $D_{12}$, then for $s = 3$, and $r = 4$, we obtain the dihedral subgroup $D_3^b = \langle a^4, b\rangle = \{e, a^4, a^8, b, a^4 b, a^8 b\}$

**Definition 4.1.1.1.** *[88] A subgroup that separates a flag $\zeta$ from other flags on a group $G$, is called a distinguishing factor. A collection of these subgroups, if more than one exists, are called distinguishing factors*

In [93], the authors extensively characterized the flags of dihedral groups, according to the terms listed in the following definitions which are imperative for our classification problem.

**Definition 4.1.1.2.** *[93] In an ascending maximal chain of subgroups of $G = D_{p_1 p_2} \dots p_n$, a dihedral subgroup of $G$ can only be followed by a dihedral subgroup.*

**Definition 4.1.1.3.** *[93] A cyclic maximal chain of subgroups of a group $G = D_{p_1 p_2 \cdots p_n}$ is any maximal chain that contains only cyclic proper subgroups of the form $\langle a^i \rangle$, $i \geq 1$.*

The dihedral group $D_{28}$ has the following cyclic maximal chains:

(1) $\{e\} \subseteq \langle a^{14} \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{28}$

(2) $\{e\} \subseteq \langle a^{14} \rangle \subseteq \langle a^7 \rangle \subseteq \langle a \rangle \subseteq D_{28}$

(3) $\{e\} \subseteq \langle a^4 \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{28}$



**Definition 4.1.1.4.** *[93] A maximal chain of subgroups of $G = D_{p_1 p_2 \cdots p_m} = \langle a, b : a^{p_1 p_2 \cdots p_m} = b^2 = e = (ab)^2 \rangle$, for $m \geq 2$ is d-cyclic if:*

*i It contains cyclic subgroups $\langle a^i \rangle$, $i > 1$ and*

*ii It contains exactly one proper non-trivial dihedral subgroup, where all the $p_i$ are distinct primes.*

$D_{28}$ has the following three clusters of $d$-cyclic maximal chains.

(1) $\{e\} \subseteq \langle a^{14} \rangle \subseteq \langle a^2 \rangle \subseteq D_{14}^{a^m b} \subseteq D_{28}$ for $m \in \{0, 1\}$

(2) $\{e\} \subseteq \langle a^{14} \rangle \subseteq \langle a^7 \rangle \subseteq D_4^{a^r b} \subseteq D_{28}$ for $r \in \{0, 1, 2, \dots, 6\}$

(3) $\{e\} \subseteq \langle a^4 \rangle \subseteq \langle a^2 \rangle \subseteq D_{14}^{a^m b} \subseteq D_{28}$ for $m \in \{0, 1\}$

**Definition 4.1.1.5.** *[93] A maximal chain of subgroups of* $G = D_{p_1p_2\dots p_m} = \langle a, b \mid a^{p_1p_2\dots p_m} = e = b^2 = (ab)^2 > $ *for* $m \geq 2$ *is* $2d$*-cyclic if*

$i$ *It contains cyclic subgroups* $\langle a^i \rangle, i > 1$ *and*

$ii$ *It contains exactly two non-trivial proper dihedral groups, where all the* $p_i$ *are distinct primes*

We have the following three clusters of $2d$-cyclic maximal chains of $D_{28}$

(1) $\{e\} \subseteq \langle a^{14} \rangle \subseteq D_2^{a^s b} \subseteq D_{14}^{a^m b} \subseteq D_{28}$ for $s \in \{0, 1, 2, \dots, 13\}$

(2) $\{e\} \subseteq \langle a^{14} \rangle \subseteq D_2^{a^s b} \subseteq D_4^{a^r b} \subseteq D_{28}$ for $s \in \{0, 1, 2, \dots, 13\}$

(3) $\{e\} \subseteq \langle a^{4} \rangle \subseteq D_7^{a^t b} \subseteq D_{14}^{a^m b} \subseteq D_{28}$ for $t \in \{0, 1, 2, 3\}$

**Definition 4.1.1.6.** *[93] A maximal chain of subgroups of* $G = D_{p_1p_2\dots p_m} = \langle a, b \mid a^{p_1p_2\dots p_n} = e = b^2 = (ab)^2 \rangle$ *for* $n \geq 2$ *is* $3d$*-cyclic if:*

$i$ *It contains cyclic subgroups* $\langle a^i \rangle, i > 1$ *and*

$ii$ *It contains exactly three non-trivial proper dihedral groups, where all the* $p_i$ *are distinct primes*



For this instance, we use the dihedral group $D_{40}$, which yields the following four clusters of $3d$-cyclic maximal chains:

(1) $\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^n b} \subseteq D_{10}^{a^s b} \subseteq D_{20}^{a^t b} \subseteq D_{40}$ for $n \in \{0, 1, 2, \dots, 19\}$

(2) $\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^n b} \subseteq D_4^{a^r b} \subseteq D_{20}^{a^t b} \subseteq D_{40}$ for $n \in \{0, 1, 2, \dots, 19\}$

(3) $\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^n b} \subseteq D_4^{a^r b} \subseteq D_8^{a^i b^\star} \subseteq D_{40}$ for $n \in \{0, 1, 2 \dots, 19\}$

(4) $\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^k b} \subseteq D_{10}^{a^s b} \subseteq D_{20}^{a^t b} \subseteq D_{40}$ for $k \in \{0, 1, 2, \dots, 7\}$

**Definition 4.1.1.7.** *[93] A maximal chain of subgroups of* $G = D_{p_1p_2\dots p_m} = \langle a, b \mid a^{p_1p_2\dots p_m} = e = b^2 = (ab)^2 \rangle$ *for* $n \geq 2$ *is* $b$*-cyclic if it contains exactly one non-trivial proper subgroup of the form* $\langle a^i b \rangle$*, for* $0 \leq i \leq p_1p_2 \dots p_m - 1$*, where all the* $p_i$ *are distinct primes.*

$D_{28}$ has the following clusters of 3 $b$-cyclic maximal chains:

(1) $\{e\} \subseteq \langle b \rangle \subseteq D_2^b \subseteq D_{14}^b \subseteq D_{28}$

(2) $\{e\} \subseteq \langle b \rangle \subseteq D_2^b \subseteq D_4^b \subseteq D_{28}$

(3) $\{e\} \subseteq \langle b \rangle \subseteq D_7^b \subseteq D_{14}^b \subseteq D_{28}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

(1) $\{e\} \subseteq \langle a^i b \rangle \subseteq D_2^{a^s b} \subseteq D_{14}^{a^m b} \subseteq D_{28}$

(2) $\{e\} \subseteq \langle a^i b \rangle \subseteq D_2^{a^s b} \subseteq D_4^{a^r b} \subseteq D_{28}$

(3) $\{e\} \subseteq \langle a^i b \rangle \subseteq D_7^{a^t b} \subseteq D_{14}^{a^m b} \subseteq D_{28}$

For every $i \in \{1, 2, \ldots, 27\}$



**Definition 4.1.1.8.** *[93] If two or more maximal chains are of identical length and contain corresponding subgroups that are isomorphic, then the maximal chains are isomorphic.*

### 4.1.2 The Dihedral Group $D_{p^n}$ for prime $p$ and $n \in \mathbb{N}$

To aid in the classification of distinct fuzzy subgroups of $D_{p^n q}$, we list down useful propositions that characterize maximal chains of subgroups of the dihedral groups $D_{p^n}$ and $D_{pq}$. Detailed studies on these groups can be found in [90], [91] and [92].

Let $D_{p^n} = \langle a, b \mid a^{p^n} = b^2 = e = (ab)^2 \rangle$. Therefore

**Proposition 4.1.2.0.1.** *[90] The only non-abelian proper subgroups of $D_{p^n}$, for $p$ and prime, and $n \in \mathbb{N}$, are the dihedral subgroups $D_{p^k}$ for $k < n$*

*Proof.* [90] $\square$

**Proposition 4.1.2.0.2.** *[92] In $Dp^n$, there are $p^{n-k}$ dihedral subgroups of order $2p^k$ for $1 \leq k \leq n$ and $p$ a prime.*

*Proof.* [92] $\square$

**Proposition 4.1.2.0.3.** *[92] In a flag of $D_{p^n}$, where $n \geq 1$, the dihedral subgroup $D_{p^k}^{a^{p^s}b}$ of $D_{p^n}$ can only precede a unique dihedral subgroup if the flag is arranged in an increasing order of subgroups.*

*Proof.* [92] $\square$

From the construction of flags in [92], we obtain the following formulae for the number of maximal chains and fuzzy subgroups of $D_{p^n}$.

**Proposition 4.1.2.0.4.** *[92] The number of cyclic maximal chains of $D_{p^n}$ is $\mathcal{M}(D_{p^n}) = 1$*

**Proposition 4.1.2.0.5.** *[92] The number of d-cyclic maximal chains of $D_{p^n}$ is $\mathcal{M}(D_{p^n})_d = p$.*



**Proposition 4.1.2.0.6.** *[92] The number of 2d-cyclic maximal chains of $D_{p^n}$ is $\mathcal{M}(D_{p^n})_{2d} = p^2$.*

**Proposition 4.1.2.0.7.** *[92] The number of 3d-cyclic maximal chains of $D_{p^n}$ is $\mathcal{M}(D_{p^n})_{3d} = p^3$.*

**Proposition 4.1.2.0.8.** *[92] The number of md-cyclic maximal chains, $1 \leq m \leq n$ of $D_{p^n}$ is $\mathcal{M}(D_{p^n})_{md} = p^m$.*

**Proposition 4.1.2.0.9.** *[92] The number of b-cyclic maximal chains of $D_{p^n}$ is $\mathcal{M}(D_{p^n})_b = p^n$.*

**Theorem 4.1.2.1.** *[92] The number of maximal chains of $D_{p^n}$ for a prime $p \geq 2$ and $n \in \mathbb{N}$ is*

$$\mathcal{M}(D_{p^n}) = \sum_{i=0}^{n} p^i$$

*Proof.* [92] □

**Theorem 4.1.2.2.** *[92] The number of distinct fuzzy subgroups of $D_{p^n}$ for a prime $p \geq 2$ and $n \in \mathbb{N}$ is*

$$\mathcal{F}(D_{p^n}) = 2^{n+2} - 1 + 2^{n+1} \times \sum_{i=1}^{n} p^i$$

*Proof.* [92] □

**Theorem 4.1.2.3.** *[92] The number of non-isomorphic classes of fuzzy subgroups of $D_{p^n}$, for $n \geq 1$ is*

$$2^{n+2} - 1 + n \times 2^{n+1}$$

*Proof.* [92] □

### 4.1.3 The Dihedral Group $D_{pq}$, for distinct primes $p$ and $q$

The classification of fuzzy subgroups of the dihedral group $D_{p^n q}$, where $n = 1$ was established in [90] from the classification of fuzzy subgroups of $D_{pq}$. The following is a list of propositions obtained in that study, which forms the basis for our classification problem.



We let $Dpq = \langle a, b \mid a^{pq} = b^2 = e = (ab)^2 \rangle$. Thus

**Proposition 4.1.3.0.1.** *[90] The number of cyclic maximal chains of $D_{pq}$ is $\mathcal{M}(D_{pq})_c = 2$.*

**Proposition 4.1.3.0.2.** *[90] The number of d-cyclic maximal chains of $D_{pq}$ is $\mathcal{M}(D_{pq})_d = (2-1)![p+q]$.*

**Proposition 4.1.3.0.3.** *[90] The number of b-cyclic maximal chains of $D_{pq}$ is $\mathcal{M}(D_{pq})_b = 2![pq]$.*

**Theorem 4.1.3.1.** *[90] The number of maximal chains of $D_{pq}$, where $p$ and $q$ are distinct primes is $\mathcal{M}(D_{pq}) = 2 + (p+q) + 2pq$.*

**Theorem 4.1.3.2.** *[90] The number of distinct fuzzy subgroups of $D_{pq}$, for $p$ and $q$ distinct primes is $\mathcal{F}(D_{pq}) = 23 + 8(p+q) + 12pq$.*

# Chapter 5

# On the Dihedral Group $D_{p^nq}$, for $p$ and $q$ distinct primes, $n \in \mathbb{N}$



## 5.1 Introduction

In this chapter we establish general formulae for the number of flags and distinct fuzzy subgroups of $D_{p^nq}$ for $p$ and $q$ distinct primes and $n \in \mathbb{N}$. It is important that we first observe specific cases of $D_{p^nq}$, where $n = 2$ and 3 because the results we obtain will aid us in the generalization of this group. We again use the characterization of flags introduced in [93] to classify the flags of these specific groups $D_{p^2q}$ and $D_{p^3q}$ and the general group $D_{p^nq}$, as cyclic, $md$-cyclic, for $1 \leq m \leq n$, and $b$-cyclic maximal chains. We use the criss-cut counting technique to compute the number of distinct fuzzy subgroups attributed to each of the flags of these groups. Finally, we obtain a formula for the number of distinct fuzzy subgroups of $D_{p^nq}$.

## 5.2 The Dihedral group $D_{p^2q}$ for $p$ and $q$ distinct primes

We know by La Granges Theorem [55], that the order of a proper subgroup $H$ of a finite group $G$ is a divisor of the order of $G$. We therefore use this theorem to construct the maximal subgroups of $D_{p^2q}$.
If we let $n = 2$, we have $D_{p^2q} = \langle a, b \mid a^{p^2q} = b^2 = e = (ab)^2\rangle$. The subgroups of $D_{p^2q}$ are:

The trivial subgroup, $\{e\}$ of order 1 ; $D_{p^2q}$ of order $2p^2q$

The cyclic subgroups of $D_{p^2q}$ are:

$\langle a\rangle$ of order $p^2q$

$\langle a^p\rangle$ of order $pq$

$\langle a^{p^2}\rangle$ of order $q$



$\langle a^{pq}\rangle$ of order $p$

$\langle a^q\rangle$ of order $p^2$

The subgroups of order 2 are:

$\langle b\rangle$; $\langle ab\rangle$; $\langle a^2b\rangle$; $\langle a^3b\rangle$; $\ldots$; $\langle a^{p^2q-1}b\rangle$

We also have the dihedral subgroups:

$D_p^{a^kb}$ for $k \in \{0, 1, 2, \ldots, pq-1\}$ of order $2p$

$D_{p^2}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, q-1\}$ of order $2p^2$

$D_{pq}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p-1\}$ of order $2pq$

$D_q^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p^2-1\}$ of order $2q$.

The following propositions are a characterization of the subgroups of $D_{p^2q}$, hence we are able to easily construct the flags of the group.

**Proposition 5.2.0.1.** *Let $D_{p^2q} = \langle a, b : a^{p^2q} = b^2 = e = (ab)^2 \rangle$. Then $D_{p^2q}$ has subgroups with orders that divide $2p^2q$.*

*Proof.* Based on the Theory of La Grange in [55] □

**Proposition 5.2.0.2.** *The number of dihedral subgroups of $D_{p^2q}$ of order:*

*(i)* $2p$ *is* $pq$

*(ii)* $2p^2$ *is* $q$

*(iii)* $2pq$ *is* $p$

*(iv)* $2q$ *is* $p^2$

*Proof.* From the manual construction of subgroups of $D_{p^2q}$ □

Using the propositions 5.2.0.1 and 5.2.0.2, we obtain the following maximal chains of $D_{p^2q}$.



**(a) Cyclic maximal chains**

(1) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^2q}$

(2) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle^\star \subseteq \langle a \rangle \subseteq D_{p^2q}$

(3) $\{e\} \subseteq \langle a^{p^2} \rangle^\star \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^2q}$

These are the only cyclic maximal chains of $D_{p^2q}$. Therefore, $D_{p^2q}$ has 3 cyclic maximal chains.

All the components of the first flag (1) are distinguishing factors. Flag (2) has a single distinguishing factor $\langle a^q \rangle$. This is because this component of flag (2) appears for the first time in our listing and does not appear in the previous flag (1). The subgroup $\langle a^{p^2} \rangle$ is a distinguishing factor for flag (3) since it appears for the first time in our listing and and does not appear in any of the previous flags (1) or (2).

For the rest of this dissertation we will indicate the distinguishing factors of each flag by a star.

**(b) $d$-cyclic maximal chains**

To obtain the number of $d$-cyclic maximal chains, we replace the cyclic subgroup $\langle a \rangle$ in (1) by the dihedral subgroup $D_{pq}^{a^k b}$ for $k \in \{0, 1, 2, \ldots, p-1\}$ that contains the cyclic subgroup $\langle a^p \rangle$, and thus we have the following flags:

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{b\star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{ab\star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^2b\star} \subseteq D_{p^2q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^{p-1}b\star} \subseteq D_{p^2q}$



Thus, we obtain one cluster of $p$ $d$-cyclic maximal chains, since there are $p$ dihedral subgroups of order $2pq$ by Proposition 5.2.0.2. The dihedral subgroup $D_{pq}^{a^k b}$ for $k \in \{0, 1, 2, \ldots, p-1\}$, is a distinguishing factor for each flag.

In (2), we substitute the cyclic subgroup $\langle a \rangle$ with the dihedral subgroup $D_{p^2}^{a^s b}$, $s \in \{0, 1, 2, \ldots, q-1\}$, that contains the cyclic subgroup $\langle a^q \rangle$, to form the following flags:

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^2}^{b\star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^2}^{ab\star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^2}^{a^2b\star} \subseteq D_{p^2q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^2}^{a^{q-1}b\star} \subseteq D_{p^2q}$

This yields one cluster of $q$ $d$-cyclic maximal chains, since there are $q$ dihedral subgroups of order $2p^2$ by Proposition 5.2.0.2. The dihedral subgroup $D_{p^2}^{a^sb}$ for $s \in \{0, 1, 2, \ldots, q-1\}$, is a distinguishing factor for each flag.

In (3), we replace the cyclic subgroup $\langle a \rangle$ by the dihedral subgroup $D_{pq}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p-1\}$, that contains the cyclic subgroup $\langle a^p \rangle$, to obtain the flags:

$\{e\} \subseteq \langle a^{p^2} \rangle^\star \subseteq \langle a^p \rangle \subseteq D_{pq}^{b\ \star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{p^2} \rangle^\star \subseteq \langle a^p \rangle \subseteq D_{pq}^{ab\star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{p^2} \rangle^\star \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^2b\star} \subseteq D_{p^2q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$



$\{e\} \subseteq \langle a^{p^2} \rangle^\star \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^{p-1}b\star} \subseteq D_{p^2q}$

Thus we have one cluster of $p$ $d$-cyclic maximal chains, since there are $p$ dihedral subgroups of order $2pq$ by Proposition 5.2.0.2. The cyclic subgroup $\langle a^{p^2} \rangle$ and the dihedral subgroups $D_{pq}^{a^kb}$ for $k \in \{0, 1, 2 \ldots, p-1\}$ appear for the first time together in a single flag and thus are a pair of distinguishing factors for each flag.

These are the only $d$-cyclic flags of $D_{p^2q}$. Hence we obtain a total of $p + q + p = 2p + q$ $d$-cyclic flags.

**(c) $2d$-cyclic maximal chains**

To obtain the $2d$-cyclic maximal chains, we substitute the next cyclic subgroup with the appropriate dihedral subgroup. Thus, in (1), we replace

$\langle a^p \rangle$ by the dihedral subgroup $D_p^{a^t b}$ for $t \in \{0, 1, 2, \dots, pq-1\}$, that contains the cyclic subgroup $\langle a^{pq} \rangle$, and we get the following flags:

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{b^\star} \subseteq D_{pq}^{b} \subseteq D_{p^2 q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{ab^\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^2 b^\star} \subseteq D_{pq}^{a^2 b} \subseteq D_{p^2 q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^{pq-1} b^\star} \subseteq D_{pq}^{a^{p-1} b} \subseteq D_{p^2 q}$

This results in one cluster of $pq$ $2d$-cyclic flags, by Proposition 5.2.0.2. The dihedral subgroup $D_p^{a^t b}$ for $t \in \{0, 1, 2, \dots, pq-1\}$ is a distinguishing factor for each flag.

In (2), we substitute the cyclic subgroup $\langle a^q \rangle$ with the dihedral subgroup $D_p^{a^t b}$ for $t \in \{0, 1, 2, \dots, pq-1\}$ that contains $\langle a^{pq} \rangle$, to obtain the following flags:



$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{b^\star} \subseteq D_{p^2}^{b}{}^\star \subseteq D_{p^2 q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{ab^\star} \subseteq D_{p^2}^{ab^\star} \subseteq D_{p^2 q}$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^2 b^\star} \subseteq D_{p^2}^{a^2 b^\star} \subseteq D_{p^2 q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^{pq-1} b^\star} \subseteq D_{p^2}^{a^{q-1} b^\star} \subseteq D_{p^2 q}$

Thus we have one cluster of $pq$ $2d$-cyclic flags, by Proposition 5.2.0.2. The dihedral subgroups $D_p^{a^t b}$ for $t \in \{0, 1, 2, \dots, pq-1\}$, and $D_{p^2}^{a^s b}$ for $s \in \{0, 1, 2, \dots, q-1\}$ appear together for the first time in a single flag and are thus a pair of distinguishing factors for each flag.

In (3), we substitute the cyclic subgroup $\langle a^p \rangle$ with the dihedral subgroup $D_q^{a^r b}$ for $r \in \{0, 1, 2, \ldots, p^2 - 1\}$, that contains the cyclic subgroup $\langle a^{p^2} \rangle$, to obtain the flags:

$\langle e \rangle \subseteq \langle a^{p^2} \rangle \subseteq D_q^{b\star} \subseteq D_{pq}^{b} \subseteq D_{p^2q}$

$\langle e \rangle \subseteq \langle a^{p^2} \rangle \subseteq D_q^{ab\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2}$

$\langle e \rangle \subseteq \langle a^{p^2} \rangle \subseteq D_q^{a^2b\star} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\langle e \rangle \subseteq \langle a^{p^2} \rangle \subseteq D_q^{a^{p^2-1}b^\star} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}$

We get one cluster of $p^2$ $2d$-cyclic maximal chains, by Proposition 5.2.0.2. The dihedral subgroup $D_q^{a^r b}$ for $r \in \{0, 1, 2, \ldots, p^2 - 1\}$, is a distinguishing factor for each flag.

These are all the $2d$-cyclic flags of $D_{p^2q}$. A summation of these yields a total of $pq + pq + p^2 = 2pq + p^2$ $2d$-cyclic flags



**(d) $b$-cyclic maximal chains**

To obtain the $b$-cyclic maximal chains, we now replace the cyclic subgroups $\langle a^{pq} \rangle$ and $\langle a^{p^2} \rangle$ by the subgroup $\langle b \rangle$, to get the following flags:

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_p^b \subseteq D_{pq}^b \subseteq D_{p^2q}$

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_p^b \subseteq D_{p^2}^{b\ \star} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_q^{b\star} \subseteq D_{pq}^b \subseteq D_{p^2q}$

This results in one cluster of three flags. The first flag in the cluster has the subgroup $\langle b \rangle$ as a single distinguishing factor, since this group appears for the first time in any of the flags of $D_{p^2q}$. The second flag is distinguished by $\langle b \rangle$ and $D_{p^2}^b$, and the third, by $\langle b \rangle$ and $D_q^b$. Therefore both flags (2) and (3) have a pair of distinguishing factors each.

The subgroup $\langle b \rangle$ may be replaced by the subgroups $\langle a^m b \rangle$ for $m \in \{0, 1, 2, \ldots, p^2q - 1\}$, of order two, to obtain the following clusters of $b$-cyclic flags:

$\{e\} \subseteq \langle ab \rangle^{\star} \subseteq D_p^{ab} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle ab \rangle^{\star} \subseteq D_p^{ab} \subseteq D_{p^2}^{ab^{\star}} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle ab \rangle^{\star} \subseteq D_q^{ab^{\star}} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^2b \rangle^{\star} \subseteq D_p^{a^2b} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^2b \rangle^{\star} \subseteq D_p^{a^2b} \subseteq D_{p^2}^{a^2b^{\star}} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^2b \rangle^{\star} \subseteq D_q^{a^2b^{\star}} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q-1}b \rangle^{\star} \subseteq D_p^{a^{pq-1}b} \subseteq D_{pq}^{a^{p-1}b} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{p^2q-1}b \rangle^{\star} \subseteq D_p^{a^{pq-1}b} \subseteq D_{p^2}^{a^{q-1}b^{\star}} \subseteq D_{p^2q}$

$\{e\} \subseteq \langle a^{p^2q-1}b \rangle^{\star} \subseteq D_q^{a^{p^2-1}b^{\star}} \subseteq D_{pq}^{a^{p-1}b} \subseteq D_{p^2q}$



These are the only $b$-cyclic flags of $D_{p^2q}$, which result in $p^2q - 1$ clusters of 3 flags that are isomorphic to the flags that contain the subgroup $\langle b \rangle$. In each cluster, the subgroup $\langle a^m b, \rangle$, for $m \in \{1, 2, 3, \ldots, p^2q - 1\}$ is a single distinguishing factor for the first flag. The second flag has a pair of distinguishing factors, $\langle a^m b, \rangle$, for $m \in \{1, 2, 3, \ldots, p^2q - 1\}$ and $D_{p^2}^{a^s b}$, for $s \in \{1, 2, 3, \ldots, q - 1\}$. The third flag also has a pair of distinguishing factors, $\langle a^m b, \rangle$, for $m \in \{1, 2, 3, \ldots, p^2q - 1\}$ and $D_q^{a^r b}$, for $r \in \{1, 2, 3, \ldots, p^2 - 1\}$. This is an exact replica of the first case involving $\langle b \rangle$. Thus we have $3p^2q$ $b$-cyclic flags, where $p^2q$ have single distinguishing factors, while $2p^2q$ have a pair of distinguishing factors.

From these manual constructions of flags of $D_{p^2q}$, we obtain the following.

**Proposition 5.2.0.3.** *The number of cyclic maximal chains of* $D_{p^2q}$ *is,* $M(D_{p^2q})_c = 3$.

*Proof.* The construction of cyclic maximal chains of $D_{q^2q}$ yields 3 cyclic maximal chains. viz.

(1) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^2q}$

(2) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq \langle a \rangle \subseteq D_{p^2q}$

(3) $\{e\} \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^2q}$

□

**Proposition 5.2.0.4.** *The number of d-cyclic maximal chains of* $D_{p^2q}$ *is* $M(D_{p^2q})_d = 2p + q$.



*Proof.* By using the three cyclic maximal chains in Proposition 5.2.0.3, we get the following 3 clusters of $d$-cyclic maximal chains:

(a) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^kb} \subseteq D_{p^2q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$

(b) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^2}^{a^kb} \subseteq D_{p^2q}$ for $k \in \{0, 1, 2, \ldots, q-1\}$

(c) $\{e\} \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{pq}^{a^kb} \subseteq D_{p^2q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$

These are the only $d$-cyclic maximal chains of $D_{p^2q}$. Cluster $(a)$ yields $p$ $d$-cyclic flags, cluster $(b)$ yields $q$ $d$-cyclic flags, and cluster $c$ yields $p$ $d$-cyclic flags. Hence we get a total of $2p + q$ $d$-cyclic flags. □

**Proposition 5.2.0.5.** *The number of* $2d$*-cyclic maximal chains of* $D_{p^2q}$ *is* $M(Dp^2q)_{2d} = 2pq + p^2$.

*Proof.* From the cyclic maximal chains in Proposition 5.2.0.3, we obtain the following three clusters of $2d$-cyclic maximal chains:

(a) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^s b} \subseteq D_{pq}^{a^k b} \subseteq D_{p^2 q}$ for $s \in \{0, 1, 2, \ldots, pq-1\}$

(b) $\{e\} \subseteq \langle a^{pq} \rangle \subseteq D_p^{a^s b} \subseteq D_{p^2}^{a^k b} \subseteq D_{p^2 q}$ for $s \in \{0, 1, 2, \ldots, pq-1\}$

(c) $\{e\} \subseteq \langle a^{p^2} \rangle \subseteq D_q^{a^s b} \subseteq D_{pq}^{a^k b} \subseteq D_{p^2 q}$ for $s \in \{0, 1, 2, \ldots, p^2-1\}$

These are the only $2d$-cyclic maximal chains of $D_{p^2q}$. Cluster $(a)$ yields $pq$ $2d$-cyclic flags, cluster $(b)$ yields $pq$ $2d$-cyclic flags, and cluster $(c)$ yields $p^2$ $2d$-cyclic flags. This results in $2pq + p^2$ $2d$-cyclic maximal chains.

$\square$

**Proposition 5.2.0.6.** *The number of b-cyclic maximal chains of $D_{p^2q}$ is $M(D_{p^2q})_b = 3p^2q$.*

*Proof.* The construction of flags of $D_{p^2q}$ yields the following $p^2q$ clusters of 3 $b$-cyclic maximal chains

$$\{e\} \subseteq \langle b \rangle \subseteq D_p^b \subseteq D_{pq}^b \subseteq D_{p^2q}$$



$$\{e\} \subseteq \langle b \rangle \subseteq D_p^b \subseteq D_{p^2}^b \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle b \rangle \subseteq D_q^b \subseteq D_{pq}^b \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle ab \rangle \subseteq D_p^{ab} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle ab \rangle \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle ab \rangle \subseteq D_q^{ab} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\{e\} \subseteq \langle a^{p^2q-1}b\rangle \subseteq D_p^{a^{pq-1}b} \subseteq D_{pq}^{a^{p-1}b} \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle a^{p^2q-1}b\rangle \subseteq D_p^{a^{pq-1}b} \subseteq D_{p^2}^{a^{q-1}b} \subseteq D_{p^2q}$$

$$\{e\} \subseteq \langle a^{p^2q-1}b\rangle \subseteq D_q^{a^{p^2-1}b} \subseteq D_{pq}^{a^{p-1}b} \subseteq D_{p^2q}$$

All three flags in each cluster contain the subgroup $\langle a^m b\rangle$, for $m \in \{0, 1, 2, \ldots, p^2q-1\}$, of order two. There are $p^2q$ subgroups of this form in $D_{p^2q}$, hence we obtain $3p^2q$ $b$-cyclic maximal chains.

□

A summation of the above propositions results in the following theorem

**Theorem 5.2.1.** *The total number of maximal chains of subgroups for the dihedral group $D_{p^2q}$ is given by:*



$$M(D_{p^2q}) = 3 + (2p+q) + (2pq+p^2) + 3p^2q$$

*Proof.* This Theorem is a result of the number of cyclic, $d$-cyclic, $2d$-cyclic, and $b$-cyclic maximal chains of $D_{p^2q}$, in propositions 5.2.0.3, 5.2.0.4, 5.2.0.5, and 5.2.0.6. □

**Theorem 5.2.2.** *Let $G = D_{p^2q} = \langle a, b : a^{p^2q} = b^2 = e = (ab)^2\rangle$. The number of distinct fuzzy subgroups of $D_{p^2q}$ is:*

$F(D_{p^2q}) = 2^5 - 1 + 2^4 \times 2 + 2^4(p+q) + 2^4(pq+p^2) + 2^4(p^2q) + 2^3(pq) + 2^3(p) + 2^3(2p^2q)$

$= 2^5 - 1 + 2^4 \times 2 + 2^4(pq+p^2+p+q) + 2^3(pq+p) + 2^4(p^2q) + 2^3(2p^2q)$.

*Proof.* From our construction of maximal chains of $D_{p^2q}$, we have 3 cyclic flags, $(2p+q)$ $d$-cyclic flags, $(2pq+p^2)$ $2d$-cyclic flags, and $3p^2q$ $b$-cyclic flags. All the flags are of length $n = 5$, and we use the criss-cut counting technique to calculate the number of distinct fuzzy subgroups contributed by each flag.

Using the cyclic flags as a starting point, by [67], we know the first flag contributes $2^5 - 1$ distinct fuzzy subgroups. The remaining two cyclic maximal chains, each with a single distinguishing factor, contribute $2^4 \times 2$ distinct fuzzy subgroups. Next, we count the number of distinct fuzzy subgroups contributed by the $(2p+q)$ $d$-cyclic maximal chains. $p$ and $q$ $d$-cyclic flags have single distinguishing factors and contribute $2^4(p+q)$ distinct fuzzy subgroups. Each of the remaining $p$ $d$-cyclic flags has a pair of distinguishing factors, and hence contribute $2^3(p)$ distinct fuzzy subgroups. Therefore, the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags is $2^4(p+q) + 2^3(p)$. To count the number of distinct fuzzy subgroups contributed by $(2pq+p^2)$ $2d$-cyclic flags, we have that $pq$ and $p^2$ flags have single distinguishing factors and contribute $2^4(pq+p^2)$ distinct fuzzy subgroups. The remaining $pq$ have a pair of distinguishing factors each, and thus contribute $2^3(pq)$ distinct fuzzy subgroups. Hence the $2d$-cyclic flags contribute $2^4(pq+p^2) + 2^3(pq)$ distinct fuzzy subgroups. From $3p^2q$ $b$-cyclic flags we have $p^2q$ $b$-cyclic flags, each with one distinguishing factor, that contribute $2^4(p^2q)$ distinct fuzzy subgroups, while the remaining $2p^2q$ $b$-cyclic flags have pairs of distinguishing factors and contribute $2^3(2p^2q)$ distinct fuzzy subgroups. Hence, the $b$-cyclic flags contribute $2^4(p^2q) + 2^3(2p^2q)$ distinct fuzzy subgroups. Thus the sum of all distinct fuzzy subgroups contributed by each of the flags yields the result. □

We now examine a specific example of a dihedral group of order $2p^2q$, to verify the results obtained above.

**Example 5.2.1.** If we let $p = 2$, $q = 5$ and $n = 2$, then for $D_{p^nq}$, we have $D_{20} = \langle a, b : a^{20} = b^2 = e = (ab)^2$, $|G| = 40$. This group has subgroups of orders that divide 40. A complete listing of the subgroups of $D_{20}$ is:

$$\{e\};\, D_{20} = \{e, a, a^2, a^3, a^4, a^5, a^6, \ldots, a^{19}, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, \ldots, a^{19}b\}$$

$$\langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, \ldots, a^{19}\}$$

$$\langle a^2 \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, a^{18}\}$$

$$\langle a^4 \rangle = \{e, a^4, a^8, a^{12}, a^{16}\}$$

$$\langle a^5 \rangle = \{e, a^5, a^{10}, a^{15}\}$$

$$\langle a^{10} \rangle = \{e, a^{10}\}$$

$$\langle b \rangle = \{e, b\};\ \langle ab \rangle = \{e, ab\};\ \langle a^2b \rangle = \{e, ab\};\ \ldots;\ \langle a^{19}b \rangle = \{e, a^{19}b\}$$

The dihedral subgroups are:

$$D_2^b = \langle a^{10}, b \rangle = \{e, a^{10}, b, a^{10}b\}$$



$$D_2^{ab} = \langle a^{10}, ab \rangle = \{e, a^{10}, ab, a^{11}b\}$$

$$D_2^{a^2b} = \langle a^{10}, a^2b \rangle = \{e, a^{10}, a^2b, a^{12}b\}$$

$$D_2^{a^3b} = \langle a^{10}, a^3b \rangle = \{e, a^{10}, a^3b, a^{13}b\}$$

$$D_2^{a^4b} = \langle a^{10}, a^4b \rangle = \{e, a^{10}, a^4b, a^{14}b\}$$

$$D_2^{a^5b} = \langle a^{10}, a^5b \rangle = \{e, a^{10}, a^5b, a^{15}b\}$$

$$D_2^{a^6b} = \langle a^{10}, a^6b \rangle = \{e, a^{10}, a^6b, a^{16}b\}$$

$$D_2^{a^7b} = \langle a^{10}, a^7b \rangle = \{e, a^{10}, a^7b, a^{17}b\}$$

$$D_2^{a^8b} = \langle a^{10}, a^8b \rangle = \{e, a^{10}, a^8b, a^{18}b\}$$

$$D_2^{a^9b} = \langle a^{10}, a^9b \rangle = \{e, a^{10}, a^9b, a^{19}b\}$$

$$D_4^{b} = \langle a^5, b \rangle = \{e, a^5, a^{10}, a^{15}, b, a^5b, a^{10}b, a^{15}b\}$$

$$D_4^{ab} = \langle a^5, ab \rangle = \{e, a^5, a^{10}, a^{15}, ab, a^6b, a^{11}b, a^{16}b\}$$

$$D_4^{a^2b} = \langle a^5, a^2b \rangle = \{e, a^5, a^{10}, a^{15}, a^2b, a^7b, a^{12}b, a^{17}b\}$$

$$D_4^{a^3b} = \langle a^5, a^3b \rangle = \{e, a^5, a^{10}, a^{15}, a^3b, a^8b, a^{13}b, a^{18}b\}$$

$$D_4^{a^4b} = \langle a^5, a^4b \rangle = \{e, a^5, a^{10}, a^{15}, a^4b, a^9b, a^{14}b, a^{19}b\}$$

$$D_5^{b} = \langle a^4, b \rangle = \{e, a^4, a^8, a^{12}, a^{16}, b, a^4b, a^8b, a^{12}b, a^{16}b\}$$



$$D_5^{ab} = \langle a^4, ab \rangle = \{e, a^4, a^8, a^{12}, a^{16}, ab, a^5b, a^9b, a^{13}b, a^{17}b\}$$

$$D_5^{a^2b} = \langle a^4, a^2b \rangle = \{e, a^4, a^8, a^{12}, a^{16}, a^2b, a^6b, a^{10}b, a^{14}b, a^{18}b\}$$

$$D_5^{a^3b} = \langle a^4, a^3b \rangle = \{e, a^4, a^8, a^{12}, a^{16}, a^3b, a^7b, a^{11}b, a^{15}b, a^{19}b\}$$

$$D_{10}^{b} = \langle a^2, b \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, a^{12}, \dots, a^{18}, b, a^2b, a^4b, a^6b, a^8b, a^{10}b, a^{12}b, \dots, a^{18}b\}$$

$$D_{10}^{ab} = \langle a^2, ab \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, a^{12}, \dots, a^{18}, ab, a^3b, a^5b, a^7b, a^9b, a^{11}b, a^{13}b, \dots, a^{19}b\}$$

Using the aforementioned characterisation of flags, we obtain the following:

The cyclic maximal chains of $D_{20}$ are as follows:

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq \langle a \rangle \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^2 \rangle^\star \subseteq \langle a \rangle \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq \langle a \rangle \subseteq D_{20}$$

Thus $D_{20}$ has 3 cyclic flags, and $3 = (2+1)$. Two of the cyclic flags have single distinguishing factors.

The $d$-cyclic maximal chains are listed as:

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq D_4^{b\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq D_4^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq D_4^{a^2b\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq D_4^{a^3b\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^5 \rangle \subseteq D_4^{a^4b\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^2 \rangle \subseteq D_{10}^{b\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq \langle a^2 \rangle \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq D_{10}^{b\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^4 \rangle^\star \subseteq \langle a^2 \rangle \subseteq D_{10}^{ab\star} \subseteq D_{12}$$

The number of $d$-cyclic flags of $D_{20}$ is 9 and $9 = 5 + 2 \times 2$, thus showing that the formula $q + 2p$ holds for $p = 2$ and $q = 5$. We observe that seven flags have single distinguishing factors, and the remaining two flags have a pair of distinguishing factors.

The $2d$-cyclic maximal chains are listed as:

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{b\star} \subseteq D_4^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{ab\star} \subseteq D_4^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^2b\star} \subseteq D_4^{a^2b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^3b\star} \subseteq D_4^{a^3b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^4b\star} \subseteq D_4^{a^4b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^5b\star} \subseteq D_4^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^6b\star} \subseteq D_4^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^7b\star} \subseteq D_4^{a^2b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^8b\star} \subseteq D_4^{a^3b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^9b\star} \subseteq D_4^{a^4b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{b\star} \subseteq D_{10}^{b}{}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{ab\star} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^2b\star} \subseteq D_{10}^{b}{}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^3b^\star} \subseteq D_{10}^{ab^\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^4b^\star} \subseteq D_{10}^{b}{}^\star \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^5b^\star} \subseteq D_{10}^{ab^\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^6b^\star} \subseteq D_{10}^{b}{}^\star \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^7b^\star} \subseteq D_{10}^{ab^\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^8b^\star} \subseteq D_{10}^{b}{}^\star \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{10} \rangle \subseteq D_2^{a^9b^\star} \subseteq D_{10}^{ab^\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{4} \rangle \subseteq D_5^{b^\star} \subseteq D_{10}^{b} \subseteq D_{20}$$



$$\{e\} \subseteq \langle a^{4} \rangle \subseteq D_5^{ab^\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{4} \rangle \subseteq D_5^{a^2b^\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{4} \rangle \subseteq D_5^{a^3b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

Thus the number of $2d$-cyclic flags is 24, and $24 = 2(2 \times 5) + 4$, showing that the formula $2pq + p^2$ holds for $p = 2$ and $q = 5$. We observe that there are 14 flags with a single distinguishing factor, and the remaining 10 flags have a pair of distinguishing factors.

The $b$-cyclic maximal chains are listed below as:

$$\{e\} \subseteq \langle b \rangle^\star \subseteq D_2^{b} \subseteq D_4^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_2^{b} \subseteq {D_{10}^{b}}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_5^{b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_2^{ab} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq {D_{10}^{b}}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$



$$\{e\} \subseteq \langle a^3b\rangle^{\star} \subseteq D_2^{a^3b} \subseteq D_4^{a^3b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^3b\rangle^{\star} \subseteq D_2^{a^3b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^3b\rangle^{\star} \subseteq D_5^{a^3b\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^4b\rangle^{\star} \subseteq D_2^{a^4b} \subseteq D_4^{a^4b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^4b\rangle^{\star} \subseteq D_2^{a^4b} \subseteq {D_{10}^{b}}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^4b\rangle^{\star} \subseteq D_5^{b\star} \subseteq D_{10}^{b} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^5b\rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_4^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{a^6b} \subseteq D_4^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{a^6b} \subseteq D_{10}^{b\ \star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_5^{a^2b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_2^{a^7b} \subseteq D_4^{a^2b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_2^{a^7b} \subseteq D_{10}^{ab\star} \subseteq D_{12}$$

$$\{e\} \subseteq \langle a^7b\rangle^\star \subseteq D_5^{a^3b\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$



$$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_2^{a^8b} \subseteq D_4^{a^3b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_2^{a^8b} \subseteq D_{10}^{b\ \star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^8b\rangle^\star \subseteq D_5^{b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_2^{a^9b} \subseteq D_4^{a^4b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_2^{a^9b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^9b\rangle^\star \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_2^b \subseteq D_4^b \subseteq D_{20}$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_2^b \subseteq D_{10}^{b}{}^{\star} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{10}b\rangle^\star \subseteq D_5^{a^2b\star} \subseteq D_{10}^b \subseteq D_{20}$

$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_2^{ab} \subseteq D_{10}^{ab\star} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{11}b\rangle^\star \subseteq D_5^{a^3b\star} \subseteq D_{10}^{ab} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{12}b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{12}b\rangle^\star \subseteq D_2^{a^2b} \subseteq D_{10}^{b}{}^{\star} \subseteq D_{20}$



$\{e\} \subseteq \langle a^{12}b\rangle^\star \subseteq D_5^{b\star} \subseteq D_{10}^b \subseteq D_{20}$

$\{e\} \subseteq \langle a^{13}b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_4^{a^3b} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{13}b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{13}b\rangle^\star \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{14}b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_4^{a^4b} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{14}b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_{10}^{b}{}^{\star} \subseteq D_{20}$

$\{e\} \subseteq \langle a^{14}b\rangle^\star \subseteq D_5^{a^2b\star} \subseteq D_{10}^b \subseteq D_{20}$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_4^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_5^{a^3b\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_2^{a^6b} \subseteq D_4^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_2^{a^6b} \subseteq {D_{10}^{b}}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_5^{b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_2^{a^7b} \subseteq D_4^{a^2b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_2^{a^7b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$



$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_2^{a^8b} \subseteq D_4^{a^3b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_2^{a^8b} \subseteq {D_{10}^{b}}^{\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_2^{a^9b} \subseteq D_4^{a^4b} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_2^{a^9b} \subseteq D_{10}^{ab\star} \subseteq D_{20}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_5^{a^3b^{\star}} \subseteq D_{10}^{ab} \subseteq D_{20}$$

We count 60 $b$-cyclic flags and $60 = 3(2^2 \times 5)$. Hence, the formula $3p^2q$ holds for $p = 2$ and $q = 5$. We observe that there are 20 flags with single distinguishing factors, and the remaining 40 flags have a pair of distinguishing factors.

A summation of the above constructed flags yields the following result for the number of maximal chains of subgroups of the dihedral group $D_{20}$: $\mathcal{M}(D_{20}) = 3 + 9 + 24 + 60 = 96$ and

$96 = 3 + (3 + 2(2)) + (2(10) + 2^2) + 3(20)$
Therefore, this shows that the formula derived in Theorem 5.2.1 holds for $p = 2$, $q = 5$ and $n = 2$

To compute the number of distinct fuzzy subgroups of $D_{20}$ we calculate the number of fuzzy subgroups contributed by each flag. Starting from the cyclic maximal chains, by [67], we let the first flag contribute $2^5 - 1$ distinct fuzzy subgroups, where 5 is the length of each flag. The remaining two cyclic maximal chains contribute $2^4(2)$ distinct fuzzy subgroups. Next, we count the number of distinct fuzzy subgroups contributed by the $d$-cyclic maximal chains and using proposition 5.2.0.4 we obtain $2^4(2 + 5) + 2^3(2)$. The $2d$-cyclic maximal chains contribute $2^4((2 \times 5) + 2^2) + 2^3(2 \times 5)$ distinct fuzzy subgroups by proposition 5.2.0.5, and the $b$-cyclic maximal chains contribute $2^4(2^2 \times 5) + 2^3(2 \times 2^2 \times 5)$ distinct fuzzy subgroups by proposition 5.2.0.6. A summation of all distinct fuzzy subgroups contributed by each of the flags yields the result:

$$2^5 - 1 + 2^4(2) + 2^4(5+2) + 2^3(2) + 2^4(10+4) + 2^3(10) + 2^4(20) + 2^3(40) = 1135$$

Now, using the formula derived in Theorem 5.2.2 for $p = 2$, $q = 5$ and $n = 2$, we have that the number of distinct fuzzy subgroups of $D_{20}$ is:

$$\mathcal{F}(D_{20}) = 63 + 2^4(2 + 5 + 10 + 4 + 20) + 2^3(2 + 10 + 40) = 1135$$

.

### 5.2.1 Isomorphic Classes of Fuzzy Subgroups of $D_{p^2q}$

We now employ the method outlined below, to compute the number of non-isomorphic fuzzy subgroups of $D_{p^2q}$, using the non-isomorphic maximal chains of the group.

**STEP 1 : Cyclic maximal chains**

Since all three cyclic maximal chains of $D_{p^2q}$ are non-isomorphic, they result in $2^5 - 1 + 2^4 \times 2 = 63$ non-isomorphic fuzzy subgroups.

**STEP 2 : $d$-cyclic maximal chains**

Any cluster of isomorphic $d$-cyclic flags counts as a single flag. Thus, in the formula $2^4(p + q) + 2^3(p)$ for the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags, $p$ and $q$ indicate the number of isomorphic flags in a cluster of flags. For instance, $p + q$ indicates the sum of one cluster of $p$ and one cluster of $q$ flags. Since all the $p$ flags are isomorphic, the whole cluster will give a count of one flag. The same argument applies for $q$. Thus the formula yields: $2^4(1 + 1) + 2^3(1) = 2^4(2) + 2^3 = 40$ non-isomorphic fuzzy subgroups.



**STEP 3 : $2d$-cyclic maximal chains**

The numbers $pq$ and $p^2$ also represent the number of isomorphic flags in a cluster of flags. As described in the above step, each cluster then counts as a single flag. Hence the formula $2^4(pq + p^2) + 2^3(pq)$ gives the number of non-isomorphic fuzzy subgroups contributed by the $2d$-cyclic maximal chains as: $2^4(1 + 1) + 2^3(1) = 2^4(2) + 2^3 = 40$ non-isomorphic fuzzy subgroups

**STEP 4 : $b$-cyclic maximal chains**

The number $p^2q$ also represents a cluster of isomorphic flags that gives a count of one, thus the number of non-isomorphic fuzzy subgroups contributed by the $b$-cyclic maximal chains is: $2^4(1) + 2^3 \times 2(1) = 2^4 + 2^3(2) = 32$.

The total sum of non-isomorphic fuzzy subgroups of $D_{p^2q}$ is thus given by the following proposition.

**Proposition 5.2.1.0.1.** *The number of non-isomorphic fuzzy subgroups of* $G = D_{p^2q}$ *is* $63 + 40 + 40 + 32 = 175$

## 5.3 The Dihedral group $D_{p^3q}$ for $p$ and $q$ distinct primes

If we let $n = 3$, then $D_{p^3q} = \langle a, b : a^{p^3q} = b^2 = e = (ab)^2 \rangle$. The subgroups of $D_{p^3q}$ are:

The trivial subgroup, $\langle e \rangle$ of order 1; $D_{p^3q}$ of order $2p^3q$



The cyclic subgroups:

$\langle a \rangle$ of order $p^3q$

$\langle a^p \rangle$ of order $p^2q$

$\langle a^{p^2} \rangle$ of order $pq$

$\langle a^{p^3} \rangle$ of order $q$

$\langle a^q \rangle$ of order $p^3$

$\langle a^{pq} \rangle$ of order $p^2$

$\langle a^{p^2q} \rangle$ of order $p$

Subgroups of order two:

$\langle b \rangle = \{e, b\}$; $\langle ab \rangle = \{e, ab\}$; $\langle a^2b \rangle = \{e, a^2b\}$; $\langle a^3b \rangle = \{e, a^3b\}$;

$\langle a^4b \rangle = \{e, a^4b\}$; $\ldots$; $\langle a^{p^3q-1}b \rangle = \{e, a^{p^3q-1}b\}$

The dihedral subgroups:

$D_p^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p^2q - 1\}$ of order $2p$

$D_{p^2}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, pq - 1\}$ of order $2p^2$

$D_{p^3}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, q - 1\}$ of order $2p^3$

$D_q^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p^3 - 1\}$ of order $2q$.

$D_{pq}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p^2 - 1\}$ of order $2pq$



$D_{p^2q}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, p - 1\}$ of order $2p^2q$

The following propositions classify subgroups of $D_{p^3q}$ and thus we can easily construct the flags of the group.

**Proposition 5.3.0.1.** *Let* $D_{p^3q} = \langle a, b : a^{p^3q} = b^2 = e = (ab)^2 \rangle$. *Then* $D_{p^3q}$ *has subgroups with orders that divide* $2p^3q$.

*Proof.* Based on the Theorem of La Grange in [55] □

**Proposition 5.3.0.2.** *The number of dihedral subgroups of* $D_{p^3q}$ *of order:*

*(i)* $2p$ *is* $p^2q$

*(ii)* $2p^2$ *is* $pq$

*(iii)* $2p^3$ *is* $q$

*(iv)* $2q$ *is* $p^3$

*(v)* $2pq$ *is* $p^2$ *and*

*(vi)* $2p^2q$ *is* $p$

*Proof.* From the manual construction of subgroups of $D_{p^3q}$ $\square$

Using the propositions 5.3.0.1 and 5.3.0.2, we obtain the following flags of $D_{p^3q}$.

**(a) Cyclic maximal chains**

(1) $\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle \subseteq \langle a^p\rangle \subseteq \langle a\rangle \subseteq D_{p^3q}$

(2) $\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle^{\star} \subseteq \langle a^p\rangle \subseteq \langle a\rangle \subseteq D_{p^3q}$

(3) $\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^q\rangle^{\star} \subseteq \langle a\rangle \subseteq D_{p^3q}$

(4) $\{e\} \subseteq \langle a^{p^3}\rangle^{\star} \subseteq \langle a^{p^2}\rangle \subseteq \langle a^p\rangle \subseteq \langle a\rangle \subseteq D_{p^3q}$



These are the only cyclic maximal chains of $G = D_{p^3q}$. Thus, $D_{p^3q}$ has 4 cyclic flags.

Flag (1) has all its components as distinguishing factors. Flag (2) has a single distinguishing factor $\langle a^{pq}\rangle$, which appears for the first time in our listing and not in the previous flag (1). Flag (3) has $\langle a^q\rangle$ as a distinguishing factor, which does not appear in the previous flags (1) or (2) and flag (4) has a distinguishing factor $\langle a^{p^3}\rangle$ that does not appear in the previous three cyclic flags. We indicate each distinguishing factor by a star.

**(b) $d$-cyclic maximal chains**

To obtain the $d$-cyclic maximal chains, we replace the the cyclic subgroup $\langle a\rangle$ in (1) by the dihedral subgroup $D^{a^kb}_{p^2q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$ that contains the cyclic subgroup $\langle a^p\rangle$, and we get the following flags:

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D^{b}_{p^2q}{}^{\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D^{ab}_{p^2q}{}^{\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D^{a^2b\star}_{p^2q} \subseteq D_{p^3q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D^{a^{p-1}b\star}_{p^2q} \subseteq D_{p^3q}$

This results in one cluster of $p$ $d$-cyclic maximal chains, since there are $p$ dihedral subgroups of order $2p^2q$ by Proposition 5.3.0.2. The dihedral subgroup $D^{a^kb}_{p^2q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$, is a distinguishing factor for each flag.

In (2), we replace the cyclic subgroup $\langle a \rangle$ by the dihedral subgroup $D^{a^kb}_{p^2q}$, $k \in \{0, 1, 2, \ldots, p-1\}$, that contains $\langle a^p \rangle$ to obtain the following flags:



$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq D^{b}_{p^2q}{}^{\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq D^{ab}_{p^2q}{}^{\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq D^{a^2b\star}_{p^2q} \subseteq D_{p^3q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq D^{a^{p-1}b\star}_{p^2q} \subseteq D_{p^3q}$

We get one cluster of $p$ $d$-cyclic maximal chains, since there are $p$ dihedral subgroups of order $2p^2q$ by Proposition 5.3.0.2. The cyclic subgroup $\langle a^{pq} \rangle$ and the dihedral subgroup $D^{a^kb}_{p^2q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$ appear for the first time together in a single chain, thus are a pair of distinguishing factors for each flag.

In (3), we replace the cyclic subgroup $\langle a \rangle$ by the dihedral subgroup $D_{p^3}^{a^s b}$ for $s \in \{0, 1, 2, \ldots, q-1\}$, that contains $\langle a^q \rangle$ to get the flags:

$\{e\} \subseteq \langle a^{p^2 q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^3}^{b\,\star} \subseteq D_{p^3 q}$

$\{e\} \subseteq \langle a^{p^2 q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^3}^{ab\star} \subseteq D_{p^3 q}$

$\{e\} \subseteq \langle a^{p^2 q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^3}^{a^2 b\star} \subseteq D_{p^3 q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^2 q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq D_{p^3}^{a^{q-1} b\star} \subseteq D_{p^3 q}$

This yields one cluster of $q$ $d$-cyclic maximal chains, since there are $q$ dihedral subgroups of order $2p^3$ by Proposition 5.3.0.2. The dihedral subgroup $D_{p^3}^{a^s b}$ for $s \in \{0, 1, 2, \ldots, q-1\}$, is a distinguishing factor for each flag.

In (4), we replace the cyclic subgroup $\langle a \rangle$ by the dihedral subgroup $D_{p^2 q}^{a^k b}$ for $k \in \{0, 1, 2, \ldots, p-1\}$, that contains $\langle a^p \rangle$ to get the flags:



$\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2 q}^{b\,\star} \subseteq D_{p^3 q}$

$\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2 q}^{ab\,\star} \subseteq D_{p^3 q}$

$\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2 q}^{a^2 b\star} \subseteq D_{p^3 q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2 q}^{a^{p-1} b\star} \subseteq D_{p^3 q}$

This results in one cluster of $p$ $d$-cyclic maximal chains, since there are $p$ dihedral subgroups of order $2p^2 q$ by Proposition 5.3.0.2. The cyclic subgroup $\langle a^{p^3} \rangle$ and the dihedral subgroup $D_{p^2 q}^{a^k b}$ for $k \in \{0, 1, 2, \ldots, p-1\}$ appear for the first time together in a single chain, thus are a pair of distinguishing factors for each flag.

These are the only $d$-cyclic flags of $D_{p^3q}$, hence we obtain $p+p+q+p = 3p+q$ $d$-cyclic maximal chains.

**(c) $2d$-cyclic maximal chains**

To obtain the $2d$-cyclic maximal chains, we replace the next cyclic subgroup after $\langle a\rangle$ by the appropriate dihedral subgroup. Thus, in (1), we replace $\langle a^p\rangle$ with the dihedral subgroup $D_{pq}^{a^tb}$ for $t \in \{0, 1, 2, \ldots, p^2-1\}$, that contains $\langle a^{p^2}\rangle$ and we get the following flags:

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle \subseteq D_{pq}^{b}{}^{\star} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle \subseteq D_{pq}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle \subseteq D_{pq}^{a^2b\star} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle \subseteq D_{pq}^{a^{p^2-1}b\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$



We get one cluster of $p^2$ $2d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroup $D_{pq}^{a^tb}$ for $t \in \{0, 1, 2, \ldots, p^2-1\}$ is a distinguishing factor for each flag.

In (2), we replace the cyclic subgroup $\langle a^p\rangle$ by the dihedral subgroup $D_{p^2}^{a^rb}$ for $r \in \{0, 1, 2, \ldots, pq-1\}$, that contains $\langle a^{pq}\rangle$, to obtain the flags:

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq D_{p^2}^{b}{}^{\star} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq D_{p^2}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq D_{p^2}^{a^2b\star} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq D_{p^2}^{a^{pq-1}b\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

We have one cluster of $pq$ $2d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroup $D_{p^2}^{a^rb}$ for $r \in \{0, 1, 2, \ldots, pq-1\}$ is a distinguishing factor for each flag.

In (3), we replace the cyclic subgroup $\langle a^q \rangle$ by the dihedral subgroup $D_{p^2}^{a^rb}$ for $r \in \{0, 1, 2, \ldots, pq-1\}$, that contains $\langle a^{pq} \rangle$ to obtain the flags:

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq D_{p^2}^{b}{}^{\star} \subseteq D_{p^3}^{b}{}^{\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq D_{p^2}^{ab\star} \subseteq D_{p^3}^{ab\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq D_{p^2}^{a^2b\star} \subseteq \langle D_{p^3}^{a^2b} \rangle \subseteq D_{p^3q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq D_{p^2}^{a^{pq-1}b\star} \subseteq D_{p^3}^{a^{q-1}b\star} \subseteq D_{p^3q}$

This yields one cluster of $pq$ $2d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroups $D_{p^2}^{a^rb}$ for $r \in \{0, 1, 2, \ldots, pq-1\}$, and $D_{p^3}^{a^sb}$ for $s \in \{0, 1, 2, \ldots q-1\}$ appear together for the first time in a single chain, thus are a pair of distinguishing factors for each flag.



In (4), we replace the cyclic subgroup $\langle a^p \rangle$ by the dihedral subgroup $D_{pq}^{a^tb}$ for $t \in \{0, 1, 2, \ldots p^2-1\}$, that contains $\langle a^{p^2} \rangle$, to get the flags:

$\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq D_{pq}^{b}{}^{\star} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq D_{pq}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq \langle D_{pq}^{a^2b} \rangle \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq D_{pq}^{a^{p^2-1}b\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

We obtain one cluster of $p^2$ $2d$-cyclic maximal chains, by Proposition 5.3.0.2. The cyclic subgroup $\langle a^{p^3}\rangle$ and the dihedral subgroup $D_{pq}^{a^tb}$ for $t \in \{0, 1, 2, \ldots p^2 - 1\}$ appear together for the first time in a single chain, thus are a pair of distinguishing factors for each flag.
Hence, we have a total count of $p^2 + pq + pq + p^2 = 2pq + 2p^2$ $2d$-cyclic maximal chains.

**(d) $3d$-cyclic maximal chains**

We obtain the $3d$-cyclic maximal chains by replacing the next cyclic subgroup by the appropriate dihedral subgroup. Thus, in (1), we replace $\langle a^{p^2}\rangle$ with the dihedral subgroup $D_p^{a^jb}$ for $j \in \{0, 1, 2, \ldots, p^2q - 1\}$, that contains $\langle a^{p^2q}\rangle$ to get the following flags:

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{b\star} \subseteq D_{pq}^{b} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{ab\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{a^2b\star} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{a^{p^2q-1}b\star} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$



Thus we obtain one cluster of $p^2q$ $3d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroup $D_p^{a^jb}$ for $j \in \{0, 1, 2, \ldots, p^2q - 1\}$ is a distinguishing factor for each flag.

In (2), we replace the cyclic subgroup $\langle a^{pq}\rangle$ by the dihedral subgroup $D_p^{a^jb}$ for $j \in \{0, 1, 2, \ldots, p^2q - 1\}$, that contains $\langle a^{p^2q}\rangle$, to obtain the flags:

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{b\star} \subseteq D_{p^2}^{b\star} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{ab\star} \subseteq D_{p^2}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q}\rangle \subseteq D_p^{a^2b\star} \subseteq D_{p^2}^{a^2b\star} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_p^{a^{p^3-1}b^\star} \subseteq D_{p^2}^{a^{pq-1}b^\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

We obtain one cluster of $p^2q$ $3d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroups $D_p^{a^jb}$, $j \in \{0, 1, 2, \ldots, p^2q-1\}$ and $D_{p^2}^{a^rb}$ for $r \in \{0, 1, 2, \ldots pq-1\}$ appear together for the first time in a single chain, thus are a pair of distinguishing factors for each flag.

In (3), we replace the cyclic subgroup $\langle a^{pq} \rangle$ by the dihedral subgroup $D_p^{a^sb}$ for $s \in \{0, 1, 2, \ldots, p^2q-1\}$, that contains $\langle a^{p^2q} \rangle$, to obtain the flags:

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_p^{b^\star} \subseteq D_{p^2}^{b} \subseteq D_{p^3}^{b}{}^\star \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_p^{ab^\star} \subseteq D_{p^2}^{ab} \subseteq D_{p^3}^{ab^\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_p^{a^2b^\star} \subseteq D_{p^2}^{a^2b} \subseteq D_{p^3}^{a^2b^\star} \subseteq D_{p^3q}$



$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_p^{a^{p^2q-1}b^\star} \subseteq D_{p^2}^{a^{pq-1}b} \subseteq D_{p^3}^{a^{q-1}b^\star} \subseteq D_{p^3q}$

This yields one cluster of $p^2q$ $3d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroups $D_p^{a^sb}$, $s \in \{0, 1, 2, \ldots, p^2q-1\}$ and $D_{p^3}^{a^kb}$ for $k \in \{0, 1, 2, \ldots q-1\}$ appear together for the first time in a single chain, thus are a pair of distinguishing factors for each flag.

In (4), replace the cyclic subgroup $\langle a^{p^2} \rangle$ by the dihedral subgroup $D_q^{a^sb}$ for $s \in \{0, 1, 2, \ldots, p^3-1\}$, that contains $\langle a^{p^3} \rangle$ to get the flags:

$\{e\} \subseteq \langle a^{p^3} \rangle \subseteq D_q^{b^\star} \subseteq D_{pq}^{b} \subseteq D_{p^2q}^{b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3} \rangle \subseteq D_q^{ab^\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3} \rangle \subseteq D_q^{a^2b^\star} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^3} \rangle \subseteq D_q^{a^{p^3-1}b^\star} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

This results in one cluster of $p^3$ $3d$-cyclic maximal chains, by Proposition 5.3.0.2. The dihedral subgroup $D_q^{a^sb}$ for $s \in \{0, 1, 2, \ldots, p^3 - 1\}$ is a distinguishing factor for each flag.

Thus we have a total count of $3p^2q + p^3$ $3d$-cyclic maximal chains.

**(e) $b$-cyclic maximal chains**

To obtain the $b$-cyclic maximal chains, we replace the cyclic subgroups $\langle a^{p^3} \rangle$ and $\langle a^{p^2q} \rangle$ by the subgroups $\langle a^m b \rangle$ for $0 \leq m \leq p^3q - 1$ of order two, to get the flags:

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_p^b \subseteq D_{pq}^b \subseteq D_{p^2q}^b \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_p^b \subseteq D_{p^2}^{b\ \star} \subseteq D_{p^2q}^b \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_p^b \subseteq D_{p^2}^b \subseteq D_{p^3}^{b\ \star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b \rangle^\star \subseteq D_q^{b\star} \subseteq D_{pq}^b \subseteq D_{p^2q}^b \subseteq D_{p^3q}$



This yields one cluster of four flags. The first flag is distinguished by the subgroup $\langle b \rangle$, since this group appears for the first time in this listing. The second flag is distinguished by the pair $\langle b \rangle$ and $D_{p^2}^b$, flag (3) is distinguished by $\langle b \rangle$ and $D_{p^3}^b$ and the fourth flag, by $\langle b \rangle$ and $D_q^b$. The subgroup $\langle b \rangle$ may be replaced by the subgroups $\langle a^m b \rangle$ for $m \in \{1, 2, \ldots, p^3q - 1\}$, of order two, to get the following flags:

$\{e\} \subseteq \langle ab \rangle^\star \subseteq D_p^{ab} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab \rangle^\star \subseteq D_p^{ab} \subseteq D_{p^2}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab \rangle^\star \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq D_{p^3}^{ab\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab \rangle^\star \subseteq D_q^{ab\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_p^{a^2b} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_p^{a^2b} \subseteq D_{p^2}^{a^2b^\star} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_p^{a^2b} \subseteq D_{p^2}^{a^2b} \subseteq D_{p^3}^{a^2b^\star} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^2b\rangle^\star \subseteq D_q^{a^2b^\star} \subseteq D_{pq}^{a^2b} \subseteq D_{p^2q}^{a^2b} \subseteq D_{p^3q}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

$$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{p^2}^{a^{pq-1}b^\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{p^2}^{a^{pq-1}b} \subseteq D_{p^3}^{a^{q-1}b^\star} \subseteq D_{p^3q}$$

$$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_q^{a^{p^3-1}b^\star} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$$

Thus we have $p^3q-1$ clusters of 4 flags isomorphic to the ones involving $\langle b\rangle$. In each cluster, the subgroups $\langle a^mb\rangle$ for $m \in \{1, 2, \ldots p^3q-1\}$ are distinguishing factors for the first flag. Each of the remaining three flags has the pair of distinguishing factors $\langle a^mb\rangle$ for $m \in \{1, 2, \ldots p^3q-1\}$ and $D_{p^2}^{a^tb}$ for $t \in \{1, 2, \ldots pq-1\}$; $\langle a^mb\rangle$ for $m \in \{1, 2, \ldots p^3q-1\}$ and $D_{p^3}^{a^kb}$, for $k \in \{1, 2, \ldots, q-1\}$; also $\langle a^mb\rangle$ for $m \in \{1, 2, \ldots p^3q-1\}$ and $D_q^{a^sb}$, for $s \in \{1, 2, \ldots, p^3-1\}$. These flags are a carbon copy of the first case involving $\langle b\rangle$. Thus we have $4p^3q$ $b$-cyclic maximal chains, and $p^3q$ have single distinguishing factors, while $3p^3q$ have pairs of distinguishing factors.

From the above manual constructions, we obtain the following:

**Proposition 5.3.0.3.** *The number of cyclic maximal chains of $D_{p^3q}$ is $M(D_{p^3q})_c = 4$.*

*Proof.* As we have observed, $D_{q^3q}$ has 4 cyclic maximal chains. viz.

(1) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^3q}$

(2) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^3q}$

(3) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle^{\star} \subseteq \langle a \rangle \subseteq D_{p^3q}$

(4) $\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq \langle a \rangle \subseteq D_{p^3q}$

□

**Proposition 5.3.0.4.** *The number of $d$-cyclic maximal chains of $D_{p^3q}$ is $M(D_{p^3q})_d = 3p + q$.*

*Proof.* Using the four cyclic maximal chains in Proposition 5.3.0.3, and the appropriate subgroup substitutions, we obtain the following four clusters of $d$-cyclic maximal chains:

(a) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2q}^{a^kb} \subseteq D_{p^3q}$ for $\in \{0, 1, 2, \ldots, p-1\}$

(b) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^{\star} \subseteq \langle a^p \rangle \subseteq D_{p^2q}^{a^kb} \subseteq D_{p^3q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$



(c) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle^{\star} \subseteq D_{p^3}^{a^ib} \subseteq D_{p^3q}$ for $i \in \{0, 1, 2, \ldots, q-1\}$

(d) $\{e\} \subseteq \langle a^{p^3} \rangle^{\star} \subseteq \langle a^{p^2} \rangle \subseteq \langle a^p \rangle \subseteq D_{p^2q}^{a^kb} \subseteq D_{p^3q}$ for $k \in \{0, 1, 2, \ldots, p-1\}$

These are the only $d$-cyclic flags of $D_{p^3q}$. Cluster $(a)$ contains $p$ $d$-cyclic flags, cluster $(b)$ contains $p$ $d$-cyclic flags, cluster $(c)$ contains $q$ $d$-cyclic flags, and cluster $(d)$ contains $p$ $d$-cyclic flags. Thus, we have a total of $3p + q$ $d$-cyclic flags.

□

**Proposition 5.3.0.5.** *The number of $2d$-cyclic maximal chains of $D_{p^3q}$ is $M(D_{p^3q})_{2d} = 2pq + 2p^2$.*

*Proof.* Using the four cyclic maximal chains in Proposition 5.3.0.3, and subgroup substitutions, we obtain the following four clusters of $2d$-cyclic flags:

(a) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq D_{pq}^{a^s b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $s \in \{0, 1, 2, \ldots, p^2-1\}$

(b) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle^\star \subseteq D_{p^2}^{a^t b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $t \in \{0, 1, \ldots, pq - 1\}$

(c) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq D_{p^2}^{a^t b} \subseteq D_{p^3}^{a^i b} \subseteq D_{p^3q}$ for $t \in \{0, 1, 2, \ldots, pq-1\}$

(d) $\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq D_{pq}^{a^s b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $s \in \{0, 1, 2, \ldots, p^2-1\}$

These are the only $2d$-cyclic flags of $D_{p^3q}$. The first cluster, $(a)$, yields $p^2$ flags, cluster $(b)$ yields $pq$ flags, cluster $(c)$ yields $pq$ flags, and cluster $(d)$ yields $p^2$ flags. Thus the number of $2d$-cyclic flags is $2pq + 2p^2$.

□

**Proposition 5.3.0.6.** *The number of 3d-cyclic maximal chains of $D_{p^3q}$ is $M(D_{p^3q})_{3d} = 3p^2q + p^3$.*

*Proof.* The cyclic maximal chains in Proposition 5.3.0.3 and use of the appropriate replacement of subgroups yield the following four clusters of $3d$-cyclic flags:



(a) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_{p}^{a^r b} \subseteq D_{pq}^{a^s b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $r \in \{0, 1, 2, \ldots, p^2q - 1\}$

(b) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_{p}^{a^r b} \subseteq D_{p^2}^{a^t b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $r \in \{0, 1, 2, \ldots, p^2q - 1\}$

(c) $\{e\} \subseteq \langle a^{p^2q} \rangle \subseteq D_{p}^{a^r b} \subseteq D_{p^2}^{a^t b} \subseteq D_{p^3}^{a^i b} \subseteq D_{p^3q}$ for $r \in \{0, 1, 2, \ldots, p^2q - 1\}$

(d) $\{e\} \subseteq \langle a^{p^3} \rangle^\star \subseteq D_{q}^{a^j b} \subseteq D_{pq}^{a^s b} \subseteq D_{p^2q}^{a^k b} \subseteq D_{p^3q}$ for $j \in \{0, 1, 2, \ldots, p^3-1\}$

These are the only $3d$-cyclic flags of $D_{p^3q}$. Cluster $(a)$ yields $p^2q$ flags, cluster $(b)$ yields $p^2q$ flags, cluster $(c)$ yields $p^2q$ flags, and cluster $(d)$ yields $p^3$ flags. Thus the total number of $3d$-cyclic flags is $3p^2q + p^3$.

□

**Proposition 5.3.0.7.** *The number of $b$-cyclic maximal chains of $D_{p^3q}$ is $M(D_{p^3q})_b = 4p^3q$.*

*Proof.* As observed in the construction of flags of $D_{p^3q}$, we have the following $p^3q$ clusters of 4 $b$-cyclic flags:

$\{e\} \subseteq \langle b\rangle^\star \subseteq D_p^b \subseteq D_{pq}^b \subseteq D_{p^2q}^b \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b\rangle^\star \subseteq D_p^b \subseteq D_{p^2}^{b}{}^\star \subseteq D_{p^2q}^b \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b\rangle^\star \subseteq D_p^b \subseteq D_{p^2}^b \subseteq D_{p^3}^{b}{}^\star \subseteq D_{p^3q}$

$\{e\} \subseteq \langle b\rangle^\star \subseteq D_q^{b\star} \subseteq D_{pq}^b \subseteq D_{p^2q}^b \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_p^{ab} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_p^{ab} \subseteq D_{p^2}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq D_{p^3}^{ab\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle ab\rangle^\star \subseteq D_q^{ab\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq D_{p^3q}$



$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{p^2}^{a^{pq-1}b\star} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_p^{a^{p^2q-1}b} \subseteq D_{p^2}^{a^{pq-1}b} \subseteq D_{p^3}^{a^{q-1}b\star} \subseteq D_{p^3q}$

$\{e\} \subseteq \langle a^{p^3q-1}b\rangle^\star \subseteq D_q^{a^{p^3-1}b^\star} \subseteq D_{pq}^{a^{p^2-1}b} \subseteq D_{p^2q}^{a^{p-1}b} \subseteq D_{p^3q}$

These are the only $b$-cyclic flags of $D_{p^3q}$. All four flags in each cluster contain the subgroup $\langle a^m b\rangle$, for $m \in \{0, 1, 2, \ldots, p^3q - 1\}$ of order two. There are $p^3q$ subgroups of this form in $D_{p^3q}$, thus we obtain $4p^3q$ $b$-cyclic maximal chains.

$\square$

**Theorem 5.3.1.** *The number of maximal chains of subgroups of the dihedral group $D_{p^3q}$ is*

$$M(D_{p^3q}) = 4 + (3p + q) + (2pq + 2p^2) + (3p^2q + p^3) + 4p^3q$$

.

*Proof.* This Theorem is a result of the total sum of cyclic, $d$-cyclic, $2d$-cyclic, $3d$-cyclic, and $b$-cyclic maximal chains of $D_{p^3q}$. □

**Theorem 5.3.2.** *Let $G = D_{p^3q} = \langle a, b : a^{p^3q} = b^2 = e = (ab)^2\rangle$. Then, the number of distinct fuzzy subgroups of $D_{p^3q}$ is:*
$F(D_{p^3q}) = 2^6 - 1 + 2^5 \times 3 + 2^5(p+q) + 2^4(2p) + 2^5(pq+p^2) + 2^4(pq+p^2) + 2^5(p^2q+p^3) + 2^4(2p^2q) + 2^5(p^3q) + 2^4(3p^3q)$
$= 2^6 - 1 + 2^5 \times 3 + 2^5(p^2q+p^3+pq+p^2+p+q) + 2^4(2p^2q+pq+p^2+2p) + 2^5(p^3q) + 2^4(3p^3q)$.



*Proof.* $D_{p^3q}$ has 4 cyclic flags, $(3p + q)$ $d$-cyclic flags, $(2pq + 2p^2)$ $2d$-cyclic flags, $(3p^2q + p^3)$ $3d$-cyclic flags, and $4p^3q$ $b$-cyclic flags. All the flags are of length $n = 6$. We use the criss-cut counting technique to calculate the number of distinct fuzzy subgroups contributed by each flag. Using the cyclic flags as our first point of enumeration, we know that by [67] the first flag contributes $2^6-1$ distinct fuzzy subgroups. The remaining three cyclic maximal chains, each with a single distinguishing factor, contribute $2^5 \times 3$, distinct fuzzy subgroups. Thus, the cyclic flags contribute $2^6 - 1 + 2^5 \times 3$ distinct fuzzy subgroups. We then count the number of distinct fuzzy subgroups contributed by the $(3p + q)$ $d$-cyclic maximal chains. $p$ and $q$ $d$-cyclic flags have single distinguishing factors and contribute $2^5(p + q)$ distinct fuzzy subgroups. Each of the remaining $2p$ $d$-cyclic flags has a pair of distinguishing factors and contribute $2^4(2p)$ distinct fuzzy subgroups. Therefore, the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags is

$2^5(p+q)+2^4(2p)$. From $(2pq+2p^2)$ $2d$-cyclic flags, $pq$ and $p^2$ flags have single distinguishing factors and contribute $2^5(pq+p^2)$ distinct fuzzy subgroups. The number of distinct fuzzy subgroups contributed by the remaining $pq$ and $p^2$ flags, each with a pair of distinguishing factors, is $2^4(pq+p^2)$. Thus the total count of the number of distinct fuzzy subgroups contributed by the $2d$-cyclic flags is $2^5(pq+p^2)+2^4(pq+p^2)$. Of the $(3p^2q+p^3)$ $3d$-cyclic flags, $p^2q$ and $p^3$ have single distinguishing factors and contribute $2^5(p^2q+p^3)$ distinct fuzzy subgroups. The remaining $2p^2q$, each witha pair of distinguishing factors, contribute $2^4(2p^2)$ distinct fuzzy subgroups. From $4p^3q$ $b$-cyclic flags we have $p^3q$ $b$-cyclic flags with single distinguishing factors that contribute $2^5(p^3q)$ distinct fuzzy subgroups. The remaining $3p^3q$ $b$-cyclic flags have pairs of distinguishing factors and contribute $2^4(3p^3q)$ distinct fuzzy subgroups. Thus, the number of distinct fuzzy subgroups contributed by the $b$-cyclic flags is $2^5(p^3q)+2^4(3p^3q)$. A summation of these fuzzy subgroups yields the result. □



We use the following example of a dihedral group of order $2p^3q$, to verify the results obtained above.

**Example 5.3.1.** We let $p=2$, $q=5$ and $n=3$. Thus, for $D_{p^nq}$, we have $D_{40}=\langle a,b : a^{40}=b^2=e=(ab)^2\rangle$, $|G|=80$. This group has subgroups of orders that divide 80. A complete listing of the subgroups of $D_{40}$results in the following.

$\{e\}$; $D_{40}=\{e,a,a^2,a^3,a^4,a^5,\dots a^{39},b,ab,a^2b,a^3b,a^4b,a^5b,\dots,a^{39}b\}$

$\langle a\rangle=\{e,a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},a^{13},a^{14},a^{15},a^{16}\dots,a^{39}\}$

$\langle a^2\rangle=\{e,a^2,a^4,a^6,a^8,a^{10},a^{12},a^{14},a^{16},a^{18},a^{20}\dots,a^{38}\}$

$\langle a^4\rangle=\{e,a^4,a^8,a^{12},a^{16},a^{20},a^{24},a^{28},a^{32},a^{36}\}$

$\langle a^5\rangle=\{e,a^5,a^{10},a^{15},a^{20},a^{25},a^{30},a^{35}\}$

$\langle a^8 \rangle = \{e, a^8, a^{16}, a^{24}, a^{32}\}$

$\langle a^{10} \rangle = \{e, a^{10}, a^{20}, a^{30}\}$

$\langle a^{20} \rangle = \{e, a^{20}\}$

The subgroups of order two are:

$\langle b \rangle = \{e, b\}$; $\langle ab \rangle = \{e, ab\}$; $\langle a^2b \rangle = \{e, a^2b\}$;

$\langle a^3b \rangle = \{e, a^3b\}$; $\ldots$; $\langle a^{39}b \rangle = \{e, a^{39}b\}$

The dihedral subgroups are:

$D_2^b = \langle a^{20}, b \rangle = \{e, a^{20}, b, a^{20}b, \}$

$D_2^{ab} = \langle a^{20}, ab \rangle = \{e, a^{20}, ab, a^{21}b\}$

$D_2^{a^2b} = \langle a^{20}, a^2b \rangle = \{e, a^{20}, a^2b, a^{22}b, \}$

$D_2^{a^3b} = \langle a^{20}, a^3b \rangle = \{e, a^{20}, a^3b, a^{23}b, \}$



$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$D_2^{a^{19}b} = \langle a^{20}, a^{19}b \rangle = \{e, a^{20}, a^{19}b, a^{39}b\}$

$D_4^b = \langle a^{10}, b \rangle = \{e, a^{10}, a^{20}, a^{30}, b, a^{10}b, a^{20}b, a^{30}b\}$

$D_4^{ab} = \langle a^{10}, ab \rangle = \{e, a^{10}, a^{20}, a^{30}, ab, a^{11}b, a^{21}b, a^{31}b\}$

$D_4^{a^2b} = \langle a^{10}, a^2b \rangle = \{e, a^{10}, a^{20}, a^{30}, a^2b, a^{12}b, a^{22}b, a^{32}b\}$

$D_4^{a^3b} = \langle a^{10}, a^3b \rangle = \{e, a^{10}, a^{20}, a^{30}, a^3b, a^{13}b, a^{23}b, a^{33}b, \}$

$\vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots$

$D_4^{a^9b} = \langle a^{10}, a^9b \rangle = \{e, a^{10}, a^{20}, a^{30}, a^9b, a^{19}b, a^{29}b, a^{39}b\}$

$D_5^b = \langle a^8, b \rangle = \{e, a^8, a^{16}, a^{24}, a^{32}, b, a^8b, a^{16}b, a^{24}b, a^{32}b\}$

$D_5^{ab} = \langle a^8, ab\rangle = \{e, a^8, a^{16}, a^{24}, a^{32}, ab, a^9b, a^{17}b, a^{25}b, a^{33}b\}$

$D_5^{a^2b} = \langle a^8, a^2b\rangle = \{e, a^8, a^{16}, a^{24}, a^{32}, a^2b, a^{10}b, a^{18}b, a^{26}b, a^{34}b\}$

$D_5^{a^3b} = \langle a^8, a^3b\rangle = \{e, a^8, a^{16}, a^{24}, a^{32}, a^3b, a^{11}b, a^{19}b, a^{27}b, a^{35}b, \}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$D_5^{a^7b} = \langle a^8, a^7b\rangle = \{e, a^8, a^{16}, a^{24}, a^{32}, a^7b, a^{15}b, a^{23}b, a^{31}b, a^{39}b\}$

$D_8^{b} = \langle a^5, b\rangle = \{e, a^5, a^{10}, a^{15}, \cdots, a^{30}, a^{35}, b, a^5b, a^{10}b, a^{15}b, \cdots, a^{30}b, a^{35}b\}$

$D_8^{ab} = \langle a^5, ab\rangle = \{e, a^5, a^{10}, \cdots, a^{30}, a^{35}, ab, a^6b, a^{11}b, a^{16}b, \cdots, a^{31}b, a^{36}b\}$

$D_8^{a^2b} = \langle a^5, a^2b\rangle = \{e, a^5, a^{10}, \cdots, a^{30}, a^{35}, a^2b, a^7b, a^{12}b, \cdots, a^{32}b, a^{37}b\}$

$D_8^{a^3b} = \langle a^5, a^3b\rangle = \{e, a^5, a^{10}, \cdots, a^{30}, a^{35}, a^3b, a^8b, a^{13}b, \cdots, a^{33}b, a^{38}b\}$

$D_8^{a^4b} = \langle a^5, a^4b\rangle = \{e, a^5, a^{10}, \cdots, a^{30}, a^{35}, a^4b, a^9b, a^{14}b, \cdots, a^{34}b, a^{39}b\}$

$D_{10}^{b} = \langle a^4, b\rangle = \{e, a^4, a^8, a^{12}, a^{16}, \ldots, a^{36}, b, a^4b, a^8b, a^{12}b, a^{16}b, \ldots a^{36}b\}$



$D_{10}^{ab} = \langle a^4, ab\rangle = \{e, a^4, a^8, a^{12}, a^{16}, \ldots, a^{36}, ab, a^5b, a^9b, a^{13}b, a^{17}b, \ldots a^{37}b\}$

$D_{10}^{a^2b} = \langle a^4, a^2b\rangle = \{e, a^4, a^8, a^{12}, a^{16}, \ldots, a^{36}, a^2b, a^6b, a^{10}b, a^{14}b, \ldots a^{38}b\}$

$D_{10}^{a^3b} = \langle a^4, a^3b\rangle = \{e, a^4, a^8, a^{12}, a^{16}, \ldots, a^{36}, a^3b, a^7b, a^{11}b, a^{15}b, \ldots a^{39}b\}$

$D_{20}^{b} = \langle a^2, b\rangle = \{e, a^2, a^4, a^6, a^8, \ldots, a^{38}, b, a^2b, a^4b, a^6b, a^8b, \ldots, a^{38}b\}$

$D_{20}^{ab} = \langle a^2, ab\rangle = \{e, a^2, a^4, a^6, a^8, \ldots, a^{38}, ab, a^3b, a^5b, a^7b, a^9b, \ldots, a^{39}b\}$

Now using the aforementioned characterisation of flags, we obtain the following:

The cyclic maximal chains of $D_{40}$ are as follows:

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^4\rangle \subseteq \langle a^2\rangle \subseteq \langle a\rangle \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle^{\star} \subseteq \langle a^2\rangle \subseteq \langle a\rangle \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle^{\star} \subseteq \langle a \rangle \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle^{\star} \subseteq \langle a^{4} \rangle \subseteq \langle a^{2} \rangle \subseteq \langle a \rangle \subseteq D_{40}$$

Thus $D_{40}$ has 4 cyclic maximal chains. Using a star on subgroups in a flag to establish distinguishing factors, we observe that three flags have single distinguishing factors.

The $d$-cyclic maximal chains are listed as:

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq \langle a^{2} \rangle \subseteq D_{20}^{b\,\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq \langle a^{2} \rangle \subseteq D_{20}^{ab\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle^{\star} \subseteq \langle a^{2} \rangle \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle^{\star} \subseteq \langle a^{2} \rangle \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle \subseteq D_{8}^{b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle \subseteq D_{8}^{ab\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle \subseteq D_{8}^{a^2b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle \subseteq D_{8}^{a^3b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq \langle a^{5} \rangle \subseteq D_{8}^{a^4b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle^{\star} \subseteq \langle a^{4} \rangle \subseteq \langle a^{2} \rangle \subseteq D_{20}^{b\,\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle^{\star} \subseteq \langle a^{4} \rangle \subseteq \langle a^{2} \rangle \subseteq D_{20}^{ab\star} \subseteq D_{40}$$

The number of $d$-cyclic maximal chains of $D_{40}$ is 11 and $11 = 3 \times 2 + 5$, showing that the formula $3p + q$ holds for $p = 3$ and $q = 5$. We observe that seven flags have single distinguishing factors, while each of four flags has a pair of distinguishing factors.

The 2d-cyclic maximal chains are listed as:

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq D_{10}^{b\,\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq D_{10}^{ab\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq D_{10}^{a^2b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{4} \rangle \subseteq D_{10}^{a^3b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{ab\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^2b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^3b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^4b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^5b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^6b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq \langle a^{10} \rangle \subseteq D_{4}^{a^7b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^8b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^9b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{b^\star} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{ab^\star} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^2b^\star} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^3b^\star} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^4b^\star} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^5b^\star} \subseteq D_8^{b^\star} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^6b^\star} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^7b^\star} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^8b^\star} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}\rangle \subseteq \langle a^{10}\rangle \subseteq D_4^{a^9b^\star} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8}\rangle^\star \subseteq \langle a^{4}\rangle \subseteq D_{10}^{b\ \star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8}\rangle^\star \subseteq \langle a^{4}\rangle \subseteq D_{10}^{ab^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8}\rangle^\star \subseteq \langle a^{4}\rangle \subseteq D_{10}^{a^2b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^8 \rangle^\star \subseteq \langle a^4 \rangle \subseteq D_{10}^{a^3b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

The number of 2d-cyclic maximal chains is 28, and $28 = 2(2 \times 5) + 2(2^2)$, showing that the formula $2pq + 2p^2$ holds for $p = 2$ and $q = 5$. We also observe that there are 14 flags with a single distinguishing factor, and 14 have a pair of distinguishing factors

The 3d-cyclic maximal chains are listed as:

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{ab^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^2b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^3b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^4b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^5b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^6b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^7b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^8b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^9b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{10}b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{11}b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{12}b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{13}b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{14}b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{15}b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{16}b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{17}b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{18}b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{19}b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{b^\star} \subseteq D_4^{b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{ab^\star} \subseteq D_4^{ab^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^2b^\star} \subseteq D_4^{a^2b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^3b^\star} \subseteq D_4^{a^3b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^4b^\star} \subseteq D_4^{a^4b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^5b^\star} \subseteq D_4^{a^5b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^6b^\star} \subseteq D_4^{a^6b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^7b^\star} \subseteq D_4^{a^7b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^8b^\star} \subseteq D_4^{a^8b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^9b^\star} \subseteq D_4^{a^9b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{10}b^\star} \subseteq D_4^{b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{11}b^\star} \subseteq D_4^{ab^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{12}b^\star} \subseteq D_4^{a^2b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{13}b^\star} \subseteq D_4^{a^3b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{14}b^\star} \subseteq D_4^{a^4b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{15}b^\star} \subseteq D_4^{a^5b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{16}b^\star} \subseteq D_4^{a^6b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{17}b^\star} \subseteq D_4^{a^7b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{18}b^\star} \subseteq D_4^{a^8b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{19}b^\star} \subseteq D_4^{a^9b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{b^\star} \subseteq D_4^{b} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{ab^\star} \subseteq D_4^{ab} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^2b^\star} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^3b^\star} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^4b^\star} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^5b^\star} \subseteq D_4^{a^5b} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^6b^\star} \subseteq D_4^{a^6b} \subseteq D_8^{ab^\star} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^7b^\star} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^8b^\star} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^9b^\star} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{10}b^\star} \subseteq D_4^{b} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{11}b^\star} \subseteq D_4^{ab} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{12}b^\star} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{13}b^\star} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{14}b^\star} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{15}b^\star} \subseteq D_4^{a^5b} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{16}b^\star} \subseteq D_4^{a^6b} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{17}b^\star} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{18}b^\star} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20} \rangle \subseteq D_2^{a^{19}b^\star} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{ab^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^2b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^3b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^4b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^5b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^6b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{8} \rangle \subseteq D_5^{a^7b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

We count 68 3d-cyclic maximal chains, and $68 = 3(2^2 5) + 2^3$, which shows that the formula $3p^2q + p^3$ holds for $p = 2$ and $q = 5$. We also note that there are 28 flags with a single distinguishing factor, and 40 that have a pair of distinguishing factors.

The b-cyclic maximal chains are listed below:

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_2^b \subseteq D_{10}^b \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_2^b \subseteq D_4^{b\star} \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_2^b \subseteq D_4^b \subseteq D_8^{b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D_5^{b\star} \subseteq D_{10}^b \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_2^{ab} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_2^{ab} \subseteq D_4^{ab\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_8^{ab\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_4^{a^2b\star} \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^2b\rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^b \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_4^{a^3b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_2^{a^3b} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^3b\rangle^\star \subseteq D_5^{a^3b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_4^{a^4b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_2^{a^4b} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^4b\rangle^\star \subseteq D_5^{a^4b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_4^{a^5b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_2^{a^5b} \subseteq D_4^{a^5b} \subseteq D_8^{b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^5b\rangle^\star \subseteq D_5^{a^5b^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{a^6b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{a^6b} \subseteq D_4^{a^6b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^6b\rangle^\star \subseteq D_2^{a^6b} \subseteq D_4^{a^6b} \subseteq D_8^{ab^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^6b \rangle^\star \subseteq D_5^{a^6b^\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^7b \rangle^\star \subseteq D_2^{a^7b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^7b \rangle^\star \subseteq D_2^{a^7b} \subseteq D_4^{a^7b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^7b \rangle^\star \subseteq D_2^{a^7b} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^7b \rangle^\star \subseteq D_5^{a^7b^\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^8b \rangle^\star \subseteq D_2^{a^8b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^8b \rangle^\star \subseteq D_2^{a^8b} \subseteq D_4^{a^8b^\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^8b \rangle^\star \subseteq D_2^{a^8b} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b^\star} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^8b \rangle^\star \subseteq D_5^{b^\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^9b \rangle^\star \subseteq D_2^{a^9b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^9b \rangle^\star \subseteq D_2^{a^9b} \subseteq D_4^{a^9b^\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^9b \rangle^\star \subseteq D_2^{a^9b} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b^\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^9b \rangle^\star \subseteq D_5^{ab^\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{10}b \rangle^\star \subseteq D_2^{a^{10}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{10}b\rangle^{\star} \subseteq D_2^{a^{10}b} \subseteq D_4^{b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{10}b\rangle^{\star} \subseteq D_2^{a^{10}b} \subseteq D_4^{b} \subseteq D_8^{b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{10}b\rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{11}b\rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{11}b\rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_4^{ab\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{11}b\rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_4^{ab} \subseteq D_8^{ab\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{11}b\rangle \subseteq D_5^{a^3b\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{12}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{12}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_4^{a^2b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{12}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{12}b\rangle^{\star} \subseteq D_5^{a^4b\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{13}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{13}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_4^{a^3b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{13}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{13}b\rangle \subseteq D_5^{a^5b\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{14}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{14}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_4^{a^4b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{14}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{14}b\rangle \subseteq D_5^{a^6b^{\star}} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_4^{a^5b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_4^{a^5b} \subseteq D_8^{b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{15}b\rangle^{\star} \subseteq D_5^{a^7b^{\star}} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_4^{a^6b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_4^{a^6b} \subseteq D_8^{ab^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{16}b\rangle^{\star} \subseteq D_5^{b^{\star}} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_4^{a^7b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{17}b\rangle^{\star} \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_4^{a^8b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{18}b\rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_4^{a^9b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{19}b\rangle^{\star} \subseteq D_5^{a^3b\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}b\rangle^{\star} \subseteq D_2^{b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}b\rangle^{\star} \subseteq D_2^{b} \subseteq D_4^{b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}b\rangle^{\star} \subseteq D_2^{b} \subseteq D_4^{b} \subseteq D_8^{b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{20}b\rangle^{\star} \subseteq D_5^{a^4b\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{21}b\rangle^{\star} \subseteq D_2^{ab} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{21}b\rangle^{\star} \subseteq D_2^{ab} \subseteq D_4^{ab^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{21}b\rangle^{\star} \subseteq D_2^{ab} \subseteq D_4^{ab} \subseteq D_8^{ab^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{21}b\rangle^{\star} \subseteq D_5^{a^5b^{\star}} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{22}b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{22}b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_4^{a^2b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{22}b\rangle^{\star} \subseteq D_2^{a^2b} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{22}b\rangle^{\star} \subseteq D_5^{a^6b^{\star}} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{23}b\rangle^{\star} \subseteq D_2^{a^3b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{23}b\rangle^{\star} \subseteq D_2^{a^3b} \subseteq D_4^{a^3b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{23}b\rangle \subseteq D_2^{a^3b} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{23}b\rangle^{\star} \subseteq D_5^{a^7b^{\star}} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{24}b\rangle^{\star} \subseteq D_2^{a^4b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{24}b\rangle^{\star} \subseteq D_2^{a^4b} \subseteq D_4^{a^4b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{24}b\rangle^{\star} \subseteq D_2^{a^4b} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{24}b \rangle \subseteq D_5^{b\star} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{25}b \rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{25}b \rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_4^{a^5b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{25}b \rangle^{\star} \subseteq D_2^{a^5b} \subseteq D_4^{a^5b} \subseteq D_8^{b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{25}b \rangle^{\star} \subseteq D_5^{ab\star} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{26}b \rangle^{\star} \subseteq D_2^{a^6b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{26}b \rangle^{\star} \subseteq D_2^{a^6b} \subseteq D_4^{a^6b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{26}b \rangle^{\star} \subseteq D_2^{a^6b} \subseteq D_4^{a^6b} \subseteq D_8^{ab\star} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{26}b \rangle^{\star} \subseteq D_5^{a^2b\star} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{27}b \rangle^{\star} \subseteq D_2^{a^7b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{27}b \rangle^{\star} \subseteq D_2^{a^7b} \subseteq D_4^{a^7b\star} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{27}b \rangle^{\star} \subseteq D_2^{a^7b} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b\star} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{27}b \rangle^{\star} \subseteq D_5^{a^3b\star} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{28}b \rangle^{\star} \subseteq D_2^{a^8b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{28}b \rangle^{\star} \subseteq D_2^{a^8b} \subseteq D_4^{a^8b\star} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{28}b \rangle^{\star} \subseteq D_2^{a^8b} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{28}b \rangle^{\star} \subseteq D_5^{a^4b^{\star}} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{29}b \rangle^{\star} \subseteq D_2^{a^9b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{29}b \rangle^{\star} \subseteq D_2^{a^9b} \subseteq D_4^{a^9b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{29}b \rangle^{\star} \subseteq D_2^{a^9b} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{29}b \rangle^{\star} \subseteq D_5^{a^5b^{\star}} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{30}b \rangle^{\star} \subseteq D_2^{a^{10}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{30}b \rangle^{\star} \subseteq D_2^{a^{10}b} \subseteq D_4^{b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{30}b \rangle^{\star} \subseteq D_2^{a^{10}b} \subseteq D_4^{b} \subseteq D_8^{b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{30}b \rangle^{\star} \subseteq D_5^{a^6b^{\star}} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{31}b \rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{31}b \rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_4^{ab^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{31}b \rangle^{\star} \subseteq D_2^{a^{11}b} \subseteq D_4^{ab} \subseteq D_8^{ab^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{31}b \rangle^{\star} \subseteq D_5^{a^7b^{\star}} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{32}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{32}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_4^{a^2b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{32}b\rangle^{\star} \subseteq D_2^{a^{12}b} \subseteq D_4^{a^2b} \subseteq D_8^{a^2b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{32}b\rangle^{\star} \subseteq D_5^{b^{\star}} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{33}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{33}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_4^{a^3b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{33}b\rangle^{\star} \subseteq D_2^{a^{13}b} \subseteq D_4^{a^3b} \subseteq D_8^{a^3b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{33}b\rangle^{\star} \subseteq D_5^{ab^{\star}} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{34}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{34}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_4^{a^4b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{34}b\rangle^{\star} \subseteq D_2^{a^{14}b} \subseteq D_4^{a^4b} \subseteq D_8^{a^4b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{34}b\rangle^{\star} \subseteq D_5^{a^2b^{\star}} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{35}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{35}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_4^{a^5b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{35}b\rangle^{\star} \subseteq D_2^{a^{15}b} \subseteq D_4^{a^5b} \subseteq D_8^{b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{35}b\rangle^{\star} \subseteq D_5^{a^3b^{\star}} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{36}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{36}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_4^{a^6b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{36}b\rangle^{\star} \subseteq D_2^{a^{16}b} \subseteq D_4^{a^6b} \subseteq D_8^{ab^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{36}b\rangle^{\star} \subseteq D_5^{a^4b^{\star}} \subseteq D_{10}^{b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{37}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{37}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_4^{a^7b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{37}b\rangle^{\star} \subseteq D_2^{a^{17}b} \subseteq D_4^{a^7b} \subseteq D_8^{a^2b^{\star}} \subseteq D_{40}$$



$$\{e\} \subseteq \langle a^{37}b\rangle^{\star} \subseteq D_5^{a^5b^{\star}} \subseteq D_{10}^{ab} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{38}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{38}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_4^{a^8b^{\star}} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{38}b\rangle^{\star} \subseteq D_2^{a^{18}b} \subseteq D_4^{a^8b} \subseteq D_8^{a^3b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{38}b\rangle^{\star} \subseteq D_5^{a^6b^{\star}} \subseteq D_{10}^{a^2b} \subseteq D_{20}^{b} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{39}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{39}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_4^{a^9b^{\star}} \subseteq D_{20}^{ab} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{39}b\rangle^{\star} \subseteq D_2^{a^{19}b} \subseteq D_4^{a^9b} \subseteq D_8^{a^4b^{\star}} \subseteq D_{40}$$

$$\{e\} \subseteq \langle a^{39}b\rangle^{\star} \subseteq D_5^{a^7b^{\star}} \subseteq D_{10}^{a^3b} \subseteq D_{20}^{ab} \subseteq D_{40}$$

We count 160 $b$-cyclic maximal chains and $160 = 4(2^3 \times 5)$. Hence, the formula $4p^3q$ holds for $p = 2$ and $q = 5$. We observe that there are 40 maximal chains with single distinguishing factors, and 120 have pairs of distinguishing factors.

Hence, the number of maximal chains of subgroups of the dihedral group $D_{40}$ is given by:

$\mathcal{M}(D_{40}) = 4 + 11 + 28 + 68 + 160 = 271$ and

$271 = 4 + (3(2) + 5) + (2(10) + 8) + (3(20) + 8) + 4(40)$



This shows that the formula derived in Theorem 5.3.1 holds for $p = 2$, $q = 5$ and $n = 3$.

To compute the number of distinct fuzzy subgroups of $D_{40}$ we calculate the number of equivalence classes of fuzzy subgroups contributed by each flag. Starting from the cyclic maximal chains, we let the first flag contribute $2^6 - 1$ distinct fuzzy subgroups. The remaining three cyclic maximal chains contribute $2^5 \times 3$ distinct fuzzy subgroups. Next, we count the contribution from the $d$-cyclic maximal chains using Proposition 5.3.0.4, and get $2^5(2 + 5) + 2^4(2(2))$ distinct fuzzy subgroups. The $2d$-cyclic maximal chains contribute $2^5(2 \times 5 + 2^2) + 2^4(2 \times 5 + 2^2)$ distinct fuzzy subgroups by proposition 5.3.0.5. The $3d$-cyclic maximal chains contribute $2^5(2^2 \times 5 + 2^3) + 2^4(2(2^2 \times 5))$ distinct fuzzy subgroups by proposition 5.3.0.6, and the $b$-cyclic maximal chains contribute

$2^5(2^3 \times 5) + 2^4(3(2^3 \times 5)$ distinct fuzzy subgroups by proposition 5.3.0.7.

The summation of all these contributions yields the result:

$$\begin{aligned}\mathcal{F}(D_{40}) &= 2^6 - 1 + 2^5 \times 3 + 2^5(2+5) + 2^4(4) + 2^5(10+4) + 2^4(10+4) \\ &+ 2^5(20+8) + 2^4(40) + 2^5(40) + 2^4(3(40)) \\ &= 159 + 2^5(40+20+8+10+4+2+5) + 2^4(120+40+10+4+4) \\ &= 5855\end{aligned}$$

### 5.3.1 Isomorphic Classes of Fuzzy Subgroups of $D_{p^3q}$

We calculate the number of non-isomorphic fuzzy subgroups of $D_{p^3q}$, using the non-isomorphic maximal chains of the group.

**STEP 1** : **Cyclic maximal chains**

Since all four cyclic maximal chains of $D_{p^3q}$ are non-isomorphic, they result in $2^6 - 1 + 2^5 \times 3 = 159$ non-isomorphic fuzzy subgroups.

**STEP 2** : $d$**-cyclic maximal chains**



Any cluster of isomorphic $d$-cyclic flags counts as a single flag. Thus, in the formula $2^5(p+q)+2^4(2p)$ for the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags, the numbers $p$ and $q$ indicate the number of isomorphic flags in a cluster. So the clusters of $p$ and $q$ flags count as 2 non-isomorphic flags. The same argument is applicable for $2p = (p+p)$. Therefore, the number of non-isomorphic fuzzy subgroups contributed by the $d$-cyclic maximal chains is: $2^5(1+1) + 2^4(1+1) = 2^5(2) + 2^4(2) = 96$ non-isomorphic fuzzy subgroups

**STEP 3** : $2d$**-cyclic maximal chains**

The numbers $pq$ and $p^2$ also represent the number of isomorphic flags in a cluster. Again, each number then counts as a single flag. Hence the formula $2^5(pq+p^2)+2^4(pq+p^2)$ gives the number of non-isomorphic fuzzy subgroups contributed by the $2d$-cyclic maximal chains as: $2^5(1+1) + 2^4(1+1) = 2^5(2) + 2^4(2) = 96$ non-isomorphic fuzzy subgroups

**STEP 4** : $3d$**-cyclic maximal chains**

In this case, the numbers $p^2q$ and $p^3$ in the formula $2^5(p^2q + p^3) + 2^4(p^2q \times 2$, each represents the number of flags that are isomorphic, in a cluster of flags. Therefore each cluster of $p^2q$ and $p^3$ will each count as one flag. Hence, the number of non-isomorphic fuzzy subgroups contributed by the $3d$-cyclic maximal chain is: $2^5(1+1)+2^4(1+1) = 2^5(2) + 2^4(2) = 96$ non-isomorphic fuzzy subgroups

**STEP 5** : $b$**-cyclic maximal chains**

The number $p^3q$ also represents a cluster of isomorphic flags that give a count of one flag, thus the formula $2^5(p^3q) + 2^4(p^3q \times 3)$ gives the number of non-isomorphic fuzzy subgroups contributed by the $b$-cyclic maximal chains as: $2^5(1) + 2^4(1 + 1 + 1) = 2^5 + 2^4(3) = 80$.

The sum of non-isomorphic fuzzy subgroups yields the following:



**Proposition 5.3.1.0.1.** *The number of non-isomorphic fuzzy subgroups of* $G = D_{p^3q}$ *is* $159 + 96 + 96 + 96 + 80 = 527$.

## 5.4 On the Dihedral group $D_{p^nq}$

In this section, we present the general formulae for the number of cyclic, $md$-cyclic, for $1 \leq m \leq n$, and $b$-cyclic maximal chains of the dihedral group $D_{p^nq}$. We also establish a formula for the number of distinct fuzzy subgroups of the group. The results are a generalizations of trends and patterns observed in the study of the specific groups. Thus we have the following propositions.

**Proposition 5.4.0.0.1.** *The length (levels) of flags of the dihedral group* $D_{p^nq}$ *is* $n + 3$.

### 5.4.1 The number of maximal chains of subgroups of $D_{p^nq}$

**Proposition 5.4.1.0.0.1.** *The number of cyclic maximal chains for the dihedral group $D_{p^nq}$ for $p$ and $q$ distinct primes and $n \in \mathbb{N}$ is*
$\mathcal{M}(D_{p^nq}) = (n+1)$.

*Proof.* Consider the cyclic maximal chain of powers of $p$

$$(1)\ \ \{e\} \subseteq \langle a^{p^n} \rangle \subseteq \langle a^{p^{n-1}} \rangle \subseteq \langle a^{p^{n-2}} \rangle \subseteq \cdots \subseteq \langle a^{p^2} \rangle \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$$

In (1), when we replace the cyclic subgroup $\langle a^p \rangle$ by $\langle a^q \rangle$, then $\langle a^{p^2} \rangle$ must be replaced by $\langle a^{pq} \rangle$, $\langle a^{p^3} \rangle$ must be replaced by $\langle a^{p^2q} \rangle$, and so forth, until we replace $\langle a^{p^n} \rangle$ by $\langle a^{p^{n-1}q} \rangle$. This gives us one cyclic flag from (1).
Next, in (1), we start with the cyclic subgroup $\langle a^{p^2} \rangle$ and replace it by $\langle a^{pq} \rangle$. Each $\langle a^{p^k} \rangle$ is then replaced by $\langle a^{p^{k-1}q} \rangle$ to produce another cyclic flag. We continue until only $\langle a^{p^n} \rangle$ is replaced by $\langle a^{p^{n-1}q} \rangle$.
Counting from the replacement of $\langle a^p \rangle$ by $\langle a^q \rangle = \langle a^{p^0q} \rangle$, we have $n$ cyclic flags coming from (1). Hence, the total number of cyclic maximal chains is equal to $n+1$.

□



The following table summarises the results obtained in sections 5.2 and 5.3.

| **Maximal Chains** | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $\cdots$ | $m \leq n$ |
|---|---|---|---|---|---|---|
| cyclic | 2 | 3 | 4 | 5 | $\cdots$ | $n + 1$ |
| $b$-cyclic | $2pq$ | $3p^2q$ | $4p^3q$ | $5p^4q$ | $\cdots$ | $(n + 1)p^nq$ |
| $d$-cyclic | $q + p$ | $q + 2p$ | $q + 3p$ | $q + 4p$ | $\cdots$ | $q + np$ |
| $2d$-cyclic | | $2pq + p^2$ | $2pq + 2p^2$ | $2pq + 3p^2$ | $\cdots$ | $2pq + (n - 1)p^2$ |
| $3d$-cyclic | | | $3p^2q + p^3$ | $3p^2q + 2p^3$ | $\cdots$ | $3p^2q + (n - 2)p^3$ |
| $4d$-cyclic | | | | $4p^3q + p^4$ | $\cdots$ | $4p^3q + (n - 3)p^4$ |
| $\vdots$ | | | | | $\cdots$ | $\vdots$ |
| $\vdots$ | | | | | $\cdots$ | $\vdots$ |
| $\vdots$ | | | | | $\cdots$ | $\vdots$ |
| $md$-cyclic | | | | | $\cdots$ | $mp^{m-1}q + (n - (m - 1))p^m$ |

Table 5.1: Number of Maximal Chains of $D_{p^nq}$ for varying $n$ values



We now present general formulae obtainable from the table above in the form of the following propositions.

**Proposition 5.4.1.0.0.2.** *The number of d-cyclic maximal chains of $D_{p^nq}$ for $n \geq 1$, is*

$$(M(D_{p^nq}))_d = np + q$$

*Proof.* $D_{p^nq}$ has maximal subgroups of the form $D_{p^n}^{a^kb} = \langle a^q, a^kb \rangle$ for $k = 0, 1, 2, \cdots, q - 1$, and $D_{p^{n-1}q}^{a^kb} = \langle a^p, a^kb \rangle$ for $k = 0, 1, 2, \cdots, p - 1$.

Consider the cyclic flag

$$(1)\ \{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^q \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$$

In (1), the cyclic subgroup $\langle a \rangle$ can only be replaced by the dihedral subgroup $D_{p^n}^{a^kb} = \langle a^q, a^kb \rangle$ for $k = 0, 1, 2, \cdots, q - 1$. This gives us $q$ $d$-cyclic flags. Since there are $n + 1$ cyclic flags, the component $\langle a \rangle$ in the remaining $n$ cyclic flags can only be replaced by the dihedral subgroup

$D^{a^kb}_{p^{n-1}q} = \langle a^p, a^kb\rangle$ for $k = 0, 1, 2, \cdots, p-1$. This replacement yields $np$ $d$-cyclic flags since there are $p$ values of $k$ and $n$ cyclic flags. Hence, the total number of $d$-cyclic maximal chains is equal to $np + q$.

□

**Proposition 5.4.1.0.0.3.** *The number of* $2d$*-cyclic maximal chains of* $D_{p^nq}$ *for* $n \geq 2$*, is*

$$(M(D_{p^nq}))_{2d} = 2pq + (n-1)p^2$$

*Proof.* Consider the cyclic flags

(1) $\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^q\rangle \subseteq \langle a\rangle \subseteq D_{p^nq}$

(2) $\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^p\rangle \subseteq \langle a\rangle \subseteq D_{p^nq}$

In (1) and (2), the cyclic subgroups $\langle a^q\rangle$ and $\langle a^p\rangle$ can only be replaced by the dihedral subgroup $D_{p^{n-1}} = \langle a^{pq}, a^tb\rangle$ for $t = 0, 1, 2, \cdots, pq-1$ (here, we assume that $\langle a\rangle$ has already been replaced by a suitable dihedral subgroup). Thus, each of the two flags yields $pq$ $2d$-cyclic flags. From a total of $n+1$ cyclic flags, the remaining $n-1$ cyclic flags have the component $\langle a^{p^2}\rangle$, so in each one of them, the cyclic subgroup $\langle a^p\rangle$ can only be replaced by the dihedral subgroup $D_{p^{n-2}q} = \langle a^{p^2}, a^tb\rangle$ for $t = 0, 1, 2, \cdots, p^2-1$. Thus, each of the $n-1$ cyclic flags yields $p^2$ $2d$-cyclic flags. Hence $D_{p^nq}$ has $2pq + (n-1)p^2$ $2d$-cyclic maximal chains.



□

**Proposition 5.4.1.0.0.4.** *The number of* $3d$*-cyclic maximal chains of* $D_{p^nq}$ *for* $n \geq 3$*, is*

$$(M(D_{p^nq}))_{3d} = 3p^2q + (n-2)p^3$$

*Proof.* Consider the cyclic flags:

(1) $\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^q\rangle \subseteq \langle a\rangle \subseteq D_{p^nq}$

(2) $\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{pq} \rangle \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

(3) $\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{p^2q} \rangle \subseteq \langle a^{p^2} \rangle \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

In (1), (2) and (3), the cyclic subgroups $\langle a^{pq} \rangle$ and $\langle a^{p^2} \rangle$ can only be replaced by the dihedral subgroup $D_{p^{n-2}} = \langle a^{p^2q}, a^s b \rangle$ for $s = 0, 1, 2, \cdots, p^2q - 1$ (here, we assume that the components $\langle a \rangle$, $\langle a^p \rangle$ and $\langle a^q \rangle$ have been replaced by the appropriate dihedral subgroups). Thus the three flags yield $3p^2q$ $3d$-cyclic flags. In the remaining $n-2$ cyclic flags there must be a component $\langle a^{p^3} \rangle$ preceded by $\langle a^{p^2} \rangle$ and this can only be replaced by the dihedral subgroup $D_{p^{n-3}q} = \langle a^{p^3}, a^s b \rangle$ for $s = 0, 1, 2, \cdots, p^3 - 1$. So, the $n-2$ remaining cyclic flags yield $(n-2)p^3$ $3d$-cyclic flags. Hence $D_{p^nq}$ has $3p^2q + (n-2)p^3$ $3d$-cyclic maximal chains.

□

Observing the patterns followed by the formulae in the above propositions determines that for any $n \geq m$, we obtain the following proposition.



**Proposition 5.4.1.0.0.5.** *The number of $md$-cyclic maximal chains of $D_{p^nq}$ for $1 \leq m \leq n$, is*

$$(M(D_{p^nq}))_{md} = mp^{m-1}q + (n-(m-1))p^m$$

*Proof.* Consider the following cyclic flags:

(1) $\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \cdots \subseteq \langle a^{p^{m-1}q} \rangle \subseteq \langle a^{p^{m-1}} \rangle \subseteq \cdots \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

(2) $\{e\} \subseteq \cdots \subseteq \langle a^{p^{m-1}q} \rangle \subseteq \langle a^{p^{m-2}q} \rangle \subseteq \langle a^{p^{m-2}} \rangle \subseteq \cdots \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

(3) $\{e\} \subseteq \cdots \subseteq \langle a^{p^{m-2}q} \rangle \subseteq \langle a^{p^{m-3}q} \rangle \subseteq \langle a^{p^{m-3}} \rangle \subseteq \cdots \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

(m-1) $\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \cdots \subseteq \langle a^{p^{m-1}q} \rangle \subseteq \cdots \subseteq \langle a^{pq} \rangle \subseteq \langle a^{p} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

(m) $\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \cdots \subseteq \langle a^{p^{m-1}q} \rangle \subseteq \cdots \subseteq \langle a^{pq} \rangle \subseteq \langle a^{q} \rangle \subseteq \langle a \rangle \subseteq D_{p^nq}$

We number the flags according to where $q$ is inserted for the first time. In (1), $q$ is inserted just after the cyclic subgroup $\langle a^{p^{m-1}}\rangle$, in (2), $q$ is inserted just after $\langle a^{p^{m-2}}\rangle$; in (3), $q$ is inserted just after the component $\langle a^{p^{m-3}}\rangle$. Continuing, we have that in flag $(m-1)$, $q$ is inserted just after the cyclic subgroup $\langle a^{p^1}\rangle$, and in flag $(m)$, $q$ comes right after $\langle a\rangle = \langle a^{p^0}\rangle$. Counting from 0 to $m-1$ yields $m$ cyclic flags. In each of these flags, the components $\langle a^{p^{m-1}}\rangle$ and $\langle a^{p^{m-2}q}\rangle$ can only be replaced by the dihedral subgroup $D^{a^r b}_{p^{n-m+1}} = \langle a^{p^{m-1}q}, a^r b\rangle$ for $r = 0, 1, 2, \cdots, p^{m-1}q - 1$. Therefore, the $m$ cyclic flags yield $mp^{m-1}q$ $md$-cyclic flags. In each of the remaining $n+1-m$ cyclic flags there is a component $\langle a^{p^m}\rangle$, preceded by $\langle a^{p^{m-1}}\rangle$, which can only be replaced by the dihedral subgroup $D^{a^r b}_{p^{n-m}} = \langle a^{p^m}, a^r b\rangle$ for $r = 0, 1, 2, \cdots, p^m - 1$. Note that we assume the components preceding $\langle a^{p^{m-1}}\rangle$ and $\langle a^{p^{m-2}q}\rangle$ have already been replaced by suitable dihedral subgroups. Hence, the total number of $md$-cyclic maximal chains of $D_{p^n q}$ is $mp^{m-1}q + (n+1-m)p^m$.



□

**Proposition 5.4.1.0.0.6.** *The number of $b$-cyclic maximal chains of $D_{p^n q}$ is*

$$(M(D_{p^n q}))_b = (n+1)p^n q$$

*Proof.* From [90], we know that for $n = 1$, the number of $b$-cyclic maximal chains of $D_{p^n q}$ is $2pq$. In propositions 5.2.0.6, and 5.3.0.7, we obtained $3p^2 q$ and $4p^3 q$ $b$-cyclic maximal chains for $D_{p^2 q}$ and $D_{p^3 q}$ respectively.

Thus for any $n \geq 1$, substituting the subgroup $\langle b\rangle$ of order 2 into each of the $n+1$ cyclic flags of $D_{p^n q}$ yields the following cluster of $n+1$ flags:

(1) $\{e\} \subseteq \langle b\rangle^\star \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq D^b_{p^{n-1}} \subseteq D^b_{p^n} \subseteq D_{p^n q}$

(2) $\{e\} \subseteq \langle b\rangle^\star \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq D^b_{p^{n-1}} \subseteq {D^b_{p^{n-1}q}}^\star \subseteq D_{p^n q}$

(3) $\{e\} \subseteq \langle b\rangle^\star \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq {D^b_{p^{n-2}q}}^\star \subseteq D^b_{p^{n-1}q} \subseteq D_{p^n q}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\text{(n)} \quad \{e\} \subseteq \langle b\rangle^{\star} \subseteq D_p^b \subseteq {D_{pq}^b}^{\star} \subseteq D_{p^2q}^b \subseteq \cdots \subseteq D_{p^{n-2}q}^b \subseteq D_{p^{n-1}q}^b \subseteq D_{p^nq}$$

$$\text{(n+1)} \quad \{e\} \subseteq \langle b\rangle^{\star} \subseteq D_q^{b\star} \subseteq D_{pq}^b \subseteq D_{p^2q}^b \subseteq \cdots \subseteq D_{p^{n-2}q}^b \subseteq D_{p^{n-1}q}^b \subseteq D_{p^nq}$$

When we substitute all subgroups $\langle a^m b\rangle$ for $m = 1, 2, \ldots, p^n q - 1$ into each of the $n+1$ cyclic flags we obtain the following $p^n q - 1$ clusters of $n+1$ flags:

$$\text{(1)} \quad \{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq \cdots \subseteq D_{p^{n-2}}^{ab} \subseteq D_{p^{n-1}}^{ab} \subseteq D_{p^n}^{ab} \subseteq D_{p^nq}$$

$$\text{(2)} \quad \{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq \cdots \subseteq D_{p^{n-2}}^{ab} \subseteq D_{p^{n-1}}^{ab} \subseteq {D_{p^{n-1}q}^{ab}}^{\star} \subseteq D_{p^nq}$$

$$\text{(3)} \quad \{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_p^{ab} \subseteq D_{p^2}^{ab} \subseteq \cdots \subseteq D_{p^{n-2}}^{ab} \subseteq {D_{p^{n-2}q}^{ab}}^{\star} \subseteq D_{p^{n-1}q}^{ab} \subseteq D_{p^nq}$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\text{(n)} \quad \{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_p^{ab} \subseteq D_{pq}^{ab\star} \subseteq D_{p^2q}^{ab} \subseteq \cdots \subseteq D_{p^{n-2}q}^{ab} \subseteq D_{p^{n-1}q}^{ab} \subseteq D_{p^nq}$$



$$\text{(n+1)} \quad \{e\} \subseteq \langle ab\rangle^{\star} \subseteq D_q^{ab\star} \subseteq D_{pq}^{ab} \subseteq D_{p^2q}^{ab} \subseteq \cdots \subseteq D_{p^{n-2}q}^{ab} \subseteq D_{p^{n-1}q}^{ab} \subseteq D_{p^nq}$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\text{(1)} \quad \{e\} \subseteq \langle a^{p^nq-1}b\rangle^{\star} \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq \cdots \subseteq D_{p^{n-1}}^{a^{pq-1}b} \subseteq D_{p^n}^{a^{q-1}b} \subseteq D_{p^nq}$$

$$\text{(2)} \quad \{e\} \subseteq \langle a^{p^nq-1}b\rangle^{\star} \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq \cdots \subseteq D_{p^{n-1}}^{a^{pq-1}b} \subseteq D_{p^{n-1}q}^{a^{p-1}b\star} \subseteq D_{p^nq}$$

$$\text{(3)} \quad \{e\} \subseteq \langle a^{p^nq-1}b\rangle^{\star} \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq \cdots \subseteq D_{p^{n-2}q}^{a^{p^2-1}b\star} \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^nq}$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\text{(n)} \quad \{e\} \subseteq \langle a^{p^nq-1}b\rangle^{\star} \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq D_{pq}^{a^{p^{n-1}-1}b\star} \subseteq \cdots \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^nq}$$

$$\text{(n+1)} \quad \{e\} \subseteq \langle a^{p^nq-1}b\rangle^{\star} \subseteq D_q^{a^{p^{n-1}-1}b\star} \subseteq D_{pq}^{a^{p^{n-1}-1}b} \subseteq \cdots \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^nq}$$

We observe that these clusters are isomorphic and are replicas of the cluster involving $\langle b \rangle$. A summation of these flags yields $(n+1)p^n q$ $b$-cyclic maximal chains. Which completes the proof.

$\square$

**Theorem 5.4.1.0.1.** *The number of maximal chains for the dihedral group $D_{p^n q}$, for $p$ and $q$ distinct primes and $n \in \mathbb{N}$ is*

$$\mathcal{M}(D_{p^n q}) = \sum_{i=0}^{n} [(n+1-i)p^i + (i+1)p^i q] \qquad \forall n \in \mathbb{N}$$

*Proof.* We obtain this formula from a combination of the cyclic, $md$-cyclic, for $1 \le m \le n$, and $b$-cyclic maximal chains of $D_{p^n q}$. $\square$

To validate the above formula, we use the following cases:

**CASE:** $n = 1$:

$$\mathcal{M}(D_{pq}) = \sum_{i=0}^{1} [(n+1-i)p^i + (i+1)p^i q] = (1+1)+q+p+2pq = 2+(p+q)+2pq$$



This is the same result obtained in [90].

**CASE:** $n = 2$ and $n = 3$:

$$\mathcal{M}(D_{p^2 q}) = 3 + 2(pq + p) + (p^2 + q) + 3p^3 q$$

and

$$\mathcal{M}(D_{p^3 q}) = 4 + (3p + q) + 2(pq + p^2) + (3p^2 q + p^3) + 4p^3 q$$

which is the same result obtained in Theorems 5.2.1 and 5.3.1.

### 5.4.2 General Formula for the number of distinct fuzzy subgroups of $D_{p^n q}$

To obtain the general formula for the total number of distinct fuzzy subgroups of $D_{p^n q}$, we use the criss-cut counting technique to calculate the number of distinct fuzzy subgroups obtainable from the cyclic, $md$-cyclic, for $1 \leq m \leq n$ and $b$-cyclic maximal chains of the group. Using the results obtained from the specific groups in sections 5.2 and 5.3 and identifying patterns, we obtain the following table:

| **Fuzzy subgroup contribution** | | | | | |
|---|---|---|---|---|---|
| **Flags** | $n = 1$ | $n = 2$ | $n = 3$ | $\cdots$ | $m \leq n$ |
| cyclic | $2^4 - 1 + 2^3(1)$ | $2^5 - 1 + 2^4(2)$ | $2^6 - 1 + 2^5(3)$ | $\cdots$ | $2^{n+3} - 1 + 2^{n+2}(n)$ |
| $b$-cyclic | $2^3(pq) + 2^2(pq)$ | $2^4(p^2q) + 2^3(2p^2q)$ | $2^5(p^3q) + 2^4(p^3q)$ | $\cdots$ | $2^{n+2}(p^nq) + 2^{n+1}(p^nq)$ |
| $d$-cyclic | $2^3(q + p)$ | $2^4(q + p) + 2^3(p)$ | $2^5(q + p) + 2^4(2p)$ | $\cdots$ | $2^{n+2}(q+p)+2^{n+1}((n-1)p)$ |
| $2d$-cyclic | | $2^4(pq+p^2)+2^3(p)$ | $2^5(pq + p^2) + 2^4(pq + p^2)$ | $\cdots$ | $2^{n+2}(pq + p^2) + 2^{n+1}(pq + (n-2)p^2)$ |
| $3d$-cyclic | | | $2^5(p^2q + p^3) + 2^4(2p^2q)$ | $\cdots$ | $2^{n+2}(p^2q + p^3)$ $+2^{n+1}(2pq + (n-3)p^3)$ |
| $\vdots$ | | | | $\cdots$ | $\vdots$ |
| $\vdots$ | | | | $\cdots$ | $\vdots$ |
| $\vdots$ | | | | $\cdots$ | $\vdots$ |
| $md$-cyclic | | | | $\cdots$ | $2^{n+2}(p^{m-1}q + p^m)$ $+2^{n+1}((m - 1)p^{m-1}q + (n-m)p^m)$ |



Table 5.2: Number of Distinct fuzzy subgroups contributed by the flags of $D_{p^nq}$ for varying $n$ values

It is from this table that we are able to arrive at the following propositions.

**Proposition 5.4.2.0.0.1.** *The number of distinct fuzzy subgroups contributed by the cyclic maximal chains of $D_{p^nq}$ is*

$$(F(D_{p^nq}))_c = 2^{n+3} - 1 + 2^{n+2} \times n$$

*Proof.* We have shown that the number of cyclic flags of $D_{p^nq}$ is $n+1$.

$$\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^{q}\rangle \subseteq \langle a\rangle \subseteq D_{p^nq} \quad (1)$$

$$\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{pq}\rangle \subseteq \langle a^{p}\rangle^{\star} \subseteq \langle a\rangle \subseteq D_{p^nq} \quad (2)$$

$$\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-2}q}\rangle \subseteq \cdots \subseteq \langle a^{p^2q}\rangle \subseteq \langle a^{p^2}\rangle^{\star} \subseteq \langle a^{p}\rangle \subseteq \langle a\rangle \subseteq D_{p^nq} \quad (3)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\{e\} \subseteq \langle a^{p^{n-1}q}\rangle \subseteq \langle a^{p^{n-1}}\rangle^{\star} \subseteq \cdots \subseteq \langle a^{p^3}\rangle \subseteq \langle a^{p^2}\rangle \subseteq \langle a^{p}\rangle \subseteq \langle a\rangle \subseteq D_{p^nq} \quad (n)$$

$$\{e\} \subseteq \langle a^{p^{n}}\rangle^{\star} \subseteq \langle a^{p^{n-1}}\rangle \subseteq \cdots \subseteq \langle a^{p^3}\rangle \subseteq \langle a^{p^2}\rangle \subseteq \langle a^{p}\rangle \subseteq \langle a\rangle \subseteq D_{p^nq} \quad (n+1)$$

These are the only cyclic maximal chains of $G = D_{p^nq}$. The length of each flag is $n+3$



All the components of flag (1) are distinguishing factors, and from [67] it contributes $2^{n+3} - 1$ distinct fuzzy subgroups. Flag (2) has one distinguishing factor $\langle a^p\rangle$ and thus contributes $\frac{2^{n+3}}{2} = 2^{n+2}$ distinct fuzzy subgroups. Flag (3) has the subgroup $\langle a^{p^2}\rangle$ as a distinguishing factor, and contributes $2^{n+2}$ distinct fuzzy subgroups. In general, the $i^{th}$ flag has a distinguishing factor $\langle a^{p^{i-1}}\rangle$ for $i = 2, 3, \ldots, n+1$, and contributes $\frac{2^{n+3}}{2}$ distinct fuzzy subgroups. Summing up, we obtain $2^{n+3} - 1 + 2^{n+2} \times n$ distinct fuzzy subgroups. This completes the proof.

□

**Proposition 5.4.2.0.0.2.** *The number of distinct fuzzy subgroups of $D_{p^nq}$ contributed by the d-cyclic maximal chains is*

$$(F(D_{p^nq}))_d = 2^{n+2}(q+p) + 2^{n+1}((n-1)p)$$

*Proof.* As shown previously, the number of $d$-cyclic maximal chains of $D_{p^nq}$ is $q + np$. Thus we have the following $n + 1$ clusters of flags.

$$\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{pq} \rangle \subseteq \langle a^{q} \rangle \subseteq D_{p^n}^{a^kb^\star} \subseteq D_{p^nq} \qquad (1)$$

for $k \in \{0, 1, 2, \ldots q - 1\}$

$$\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{pq} \rangle \subseteq \langle a^{p} \rangle \subseteq {D_{p^{n-1}q}^{a^ib}}^\star \subseteq D_{p^nq} \qquad (2)$$

for $i \in \{0, 1, 2, \ldots, p - 1\}$

$$\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-2}q} \rangle \subseteq \cdots \subseteq \langle a^{p^{2\star}} \rangle \subseteq \langle a^{p} \rangle \subseteq {D_{p^{n-1}q}^{a^ib}}^\star \subseteq D_{p^nq} \qquad (3)$$

for $i \in \{0, 1, 2, \ldots, p - 1\}$

$$\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \cdots \subseteq \langle a^{p^3} \rangle^\star \subseteq \langle a^{p^2} \rangle \subseteq \langle a^{p} \rangle \subseteq {D_{p^{n-1}q}^{a^ib}}^\star \subseteq D_{p^nq} \qquad (4)$$

for $i \in \{0, 1, 2, \ldots, p - 1\}$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\{e\} \subseteq \langle a^{p^{n-1}q} \rangle \subseteq \langle a^{p^{n-1}} \rangle^\star \subseteq \cdots \subseteq \langle a^{p^2} \rangle \subseteq \langle a^{p} \rangle \subseteq {D_{p^{n-1}q}^{a^ib}}^\star \subseteq D_{p^nq} \qquad (n)$$

for $i \in \{0, 1, 2, \ldots, p - 1\}$



$$\{e\} \subseteq \langle a^{p^{n}} \rangle^\star \subseteq \langle a^{p^{n-1}} \rangle \subseteq \cdots \subseteq \langle a^{p^2} \rangle \subseteq \langle a^{p} \rangle \subseteq {D_{p^{n-1}q}^{a^ib}}^\star \subseteq D_{p^nq} \qquad (n+1)$$

for $i \in \{0, 1, 2, \ldots, p - 1\}$

Each of the flags in cluster (1) has one distinguishing factor $D_{p^n}^{a^kb}$ for $k \in \{0, 1, 2, \ldots, q - 1\}$. Thus cluster (1) contributes $2^{n+2}(q)$ distinct fuzzy subgroups. Cluster (2) has flags, each with a single distinguishing factor $D_{p^{n-1}q}^{a^ib}$, for $i \in \{0, 1, 2, \ldots, p - 1\}$, that contribute $2^{n+2}(p)$ distinct fuzzy subgroups. In cluster (3), the subgroups $\langle a^{p^2} \rangle$ and $D_{p^{n-1}q}^{a^ib}$ are a pair of distinguishing factors for each of the flags, which contribute $\frac{2^{n+3}}{2^2}(p) = 2^{n+1}(p)$ distinct fuzzy subgroups. Each of the flags in cluster (4) has a pair of distinguishing factors $\langle a^{p^3} \rangle$ and $D_{p^{n-1}q}^{a^ib}$ for $i \in \{0, 1, 2, \ldots, p - 1\}$ and contribute $2^{n+1}(p)$ distinct fuzzy subgroups. In general, there is one cluster out of $n + 1$ with $q$ flags, each with one distinguishing factor $D_{p^n}^{a^k b}$, for $k \in \{0, 1, 2, \ldots, q-1\}$, that contribute $2^{n+2}(q)$ distinct fuzzy subgroups. In the remaining $n$ clusters, the first cluster has flags with one distinguishing factor $D_{p^{n-1}q}^{a^i b}$, for $i \in \{0, 1, 2, \ldots, p-1\}$, that contribute $2^{n+2}(p)$ distinct fuzzy subgroups. The $j^{th}$ cluster contains flags that have pairs of distinguishing factors $\langle a^{p^{j-1}} \rangle$ for $j = 3, 4, \ldots, n+1$ and $D_{p^{n-1}q}^{a^i b}$ for $i \in \{0, 1, 2, \ldots, p-1\}$, and thus contributes $2^{n+1}((n-1)p)$ distinct fuzzy subgroups. This completes the proof.

□

If we use the same argument and manual construction as in the previous propositions, we are able to obtain the following:

**Proposition 5.4.2.0.0.3.** *The number of distinct fuzzy subgroups of $D_{p^n q}$ contributed by the 2d-cyclic maximal chains is*

$$(F(D_{p^n q}))_{2d} = 2^{n+2}(pq + p^2) + 2^{n+1}(pq + (n-2)p^2)$$



**Proposition 5.4.2.0.0.4.** *The number of distinct fuzzy subgroups of $D_{p^n q}$ contributed by the 3d-cyclic maximal chains is*

$$(F(D_{p^n q}))_{3d} = 2^{n+2}(p^2 q + p^3) + 2^{n+1}(2p^2 q + (n-3)p^3)$$

Thus, in generalizing the formulae from the above propositions, for any $n \geq m$ we obtain the following:

**Proposition 5.4.2.0.0.5.** *The number of distinct fuzzy subgroups of $D_{p^n q}$ contributed by the md-cyclic maximal chains, for $1 \leq m \leq n$ is*

$$(F(D_{p^n q}))_{md} = 2^{n+2}(p^{m-1}q + p^m) + 2^{n+1}((m-1)p^{m-1}q + (n-m)p^m)$$

**Proposition 5.4.2.0.0.6.** *The number of distinct fuzzy subgroups of $D_{p^n q}$ contributed by the b-cyclic maximal chains is*

$$(F(D_{p^n q}))_b = 2^{n+2}(p^n q) + 2^{n+1}(np^n q)$$

*Proof.* From proposition 5.4.1.0.0.6 we know that the number of $b$-cyclic maximal chains of $D_{p^nq}$ is $(n+1)p^nq$, obtained from

**Cluster** (1)

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq D^b_{p^{n-1}} \subseteq D^b_{p^n} \subseteq D_{p^nq} \qquad (1)$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq D^b_{p^{n-1}} \subseteq {D^b_{p^{n-1}q}}^{\star} \subseteq D_{p^nq} \qquad (2)$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^b_p \subseteq D^b_{p^2} \subseteq \cdots \subseteq D^b_{p^{n-2}} \subseteq {D^b_{p^{n-2}q}}^{\star} \subseteq D^b_{p^{n-1}q} \subseteq D_{p^nq} \qquad (3)$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^b_p \subseteq \cdots \subseteq D^b_{p^{n-3}} \subseteq {D^b_{p^{n-3}q}}^{\star} \subseteq D^b_{p^{n-2}q} \subseteq D^b_{p^{n-1}q} \subseteq D_{p^nq} \quad (4)$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^b_p \subseteq {D^b_{pq}}^{\star} \subseteq \cdots \subseteq D^b_{p^{n-3}q} \subseteq D^b_{p^{n-2}q} \subseteq D^b_{p^{n-1}q} \subseteq D_{p^nq} \qquad (n)$$

$$\{e\} \subseteq \langle b\rangle^{\star} \subseteq D^{b\star}_q \subseteq D^b_{pq} \subseteq \cdots \subseteq D^b_{p^{n-3}q} \subseteq D^b_{p^{n-2}q} \subseteq D^b_{p^{n-1}q} \subseteq D_{p^nq} \ (n+1)$$

**Cluster** (2)



$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D^{ab}_p \subseteq D^{ab}_{p^2} \subseteq \cdots \subseteq D^{ab}_{p^{n-2}} \subseteq D^{ab}_{p^{n-1}} \subseteq D^{ab}_{p^n} \subseteq D_{p^nq} \qquad (1)$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D^{ab}_p \subseteq D^{ab}_{p^2} \subseteq \cdots \subseteq D^{ab}_{p^{n-2}} \subseteq D^{ab}_{p^{n-1}} \subseteq {D^{ab}_{p^{n-1}q}}^{\star} \subseteq D_{p^nq} \qquad (2)$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D^{ab}_p \subseteq D^{ab}_{p^2} \subseteq \cdots \subseteq D^{ab}_{p^{n-2}} \subseteq {D^{ab}_{p^{n-2}q}}^{\star} \subseteq D^{ab}_{p^{n-1}q} \subseteq D_{p^nq} \qquad (3)$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq \cdots \subseteq D^{ab}_{p^{n-3}} \subseteq {D^{ab}_{p^{n-3}q}}^{\star} \subseteq D^{ab}_{p^{n-2}q} \subseteq D^{ab}_{p^{n-1}q} \subseteq D_{p^nq} \qquad (4)$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D^{ab}_p \subseteq D^{ab\star}_{pq} \subseteq \cdots \subseteq D^{ab}_{p^{n-3}q} \subseteq D^{ab}_{p^{n-2}q} \subseteq D^{ab}_{p^{n-1}q} \subseteq D_{p^nq} \qquad (n)$$

$$\{e\} \subseteq \langle ab\rangle^{\star} \subseteq D^{ab\star}_q \subseteq D^{ab}_{pq} \subseteq D^{ab}_{p^2q} \subseteq \cdots \subseteq D^{ab}_{p^{n-2}q} \subseteq D^{ab}_{p^{n-1}q} \subseteq D_{p^nq} \quad (n+1)$$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

**Cluster** $(p^nq)$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq \cdots \subseteq D_{p^{n-1}}^{a^{pq-1}b} \subseteq D_{p^n}^{a^{q-1}b} \subseteq D_{p^n q} \qquad (1)$$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq \cdots \subseteq D_{p^{n-1}}^{a^{pq-1}b} \subseteq D_{p^{n-1}q}^{a^{p-1}b^\star} \subseteq D_{p^n q} \qquad (2)$$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq \cdots \subseteq D_{p^{n-2}}^{a^{p^2 q-1}b} \subseteq D_{p^{n-2}q}^{a^{p^2-1}b^\star} \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^n q} \qquad (3)$$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq \cdots \subseteq D_{p^{n-3}q}^{a^{p^3-1}b^\star} \subseteq D_{p^{n-2}q}^{a^{p^2-1}b} \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^n q} \qquad (4)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq D_p^{a^{p^{n-1}q-1}b} \subseteq D_{pq}^{a^{p^{n-1}-1}b^\star} \subseteq \cdots \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^n q} \qquad (n)$$

$$\{e\} \subseteq \langle a^{p^n q-1}b\rangle^\star \subseteq D_q^{a^{p^n-1}b^\star} \subseteq D_{pq}^{a^{p^{n-1}-1}b} \subseteq \cdots \subseteq D_{p^{n-1}q}^{a^{p-1}b} \subseteq D_{p^n q} \qquad (n+1)$$

Thus we obtain $p^n q$ clusters of $n+1$ flags.

In cluster (1), the first flag has one distinguishing factor $\langle b\rangle$ and contributes $2^{n+2}$ distinct fuzzy subgroups. The second flag has a pair of distinguishing factors, $\langle b\rangle$ and $D_{p^{n-1}q}^{a^i b}$ for $i \in \{0, 1, 2, \ldots, p-1\}$ and thus contribute $2^{n+1}$ distinct fuzzy subgroups. The third flag has two distinguishing factors $\langle b\rangle$ and $D_{p^{n-2}q}^{a^r b}$ for $r \in \{0, 1, 2, \ldots, p^2-1\}$ and contributes $2^{n+1}$ distinct fuzzy subgroups. Thus, in general, the $k^{th}$ flag has a pair of distinguishing factors $\langle b\rangle$ and $D_{p^{k-1}}^{a^t b}$ for $t \in \{0, 1, 2, \ldots, p^{k-1}\}$ and contributes $\frac{2^{n+3}}{2^2}$ distinct fuzzy subgroups. Summing up we obtain $2^{n+2} + 2^{n+1}(n)$ distinct fuzzy subgroups for one cluster of $n+1$ flags. We now substitute the subgroups $\langle a^m b\rangle$ for $m \in \{1, 2, \ldots, p^n q\}$ of order two, into each of the remaining $p^n q - 1$ clusters and we obtain clusters that are a carbon copy of the one involving $\langle b\rangle$. Thus the number of distinct fuzzy subgroups obtainable from the $b$-cyclic flags is $2^{n+2}(p^n q) + 2^{n+1}(np^n q)$. This completes the proof.

□

**Theorem 5.4.2.0.1.** *The total number of distinct fuzzy groups of $D_{p^nq}$ is:*

$$
\begin{aligned}
F(D_{p^nq}) = {} & 2^{n+3} - 1 + 2^{n+2}(n) + 2^{n+2}(q+p) + 2^{n+1}((n-1)p) + 2^{n+2}(pq+p^2) \\
& + 2^{n+1}(pq + (n-2)p^2) + 2^{n+2}(p^2q + p^3) + 2^{n+1}(2p^2q + (n-3)p^3) \\
& + \cdots + 2^{n+2}((p^{m-1}q + p^m) + 2^{n+1}((m-1)p^{m-1}q + (n-m)p^m) \\
& = 2^{n+3} - 1 + 2^{n+2}(n) + 2^{n+2}[\sum_{i=1}^{n}(p^{i-1}q + p^i)] \\
& + 2^{n+1}[\sum_{i=1}^{n}((i-1)p^{i-1}q + (n-i)p^i)] + 2^{n+2}(p^nq) + 2^{n+1}(np^nq)
\end{aligned}
$$

*Proof.* This is a result of a combination of the number of distinct fuzzy subgroups contributed by the cyclic, $md$-cyclic, for $1 \leq m \leq n$ and $b$-cyclic flags of $D_{p^nq}$. □

# Chapter 6

# Conclusion

The primary objective of this dissertation was to classify distinct fuzzy subgroups of the dihedral group $D_{p^n q}$ for $p$ and $q$ distinct primes, and $n \in \mathbb{N}$. We used the natural equivalence relation defined in [67] and the two counting techniques introduced in [74] that were developed from this equivalence. We have successfully established and proved formulae that can be used to obtain the number of flags and distinct fuzzy subgroups of this group.



This study has highlighted the significance of the crisp characterization of dihedral groups as the first requisite step needed in the classification of distinct fuzzy subgroups, since we use flags to compute the number of distinct fuzzy subgroups of the group. We exploited the characterization of flags introduced in [93] and managed to successfully classify flags of $D_{p^n q}$ as either cyclic, $md$-cyclic, for $1 \leq m \leq n$, or $b$-cyclic. This study has shown that the notion of distinguishing factors for flags introduced in [88] is an integral part of the criss-cut counting technique and it helps avoid under counting and over counting the number of distinct fuzzy subgroups attributed to each flag. In addition, we observed that the Murali and Makamba definition of an equivalence relation is stronger than other notions of equivalence relations, as it results in an improvement in the number of distinct fuzzy subgroups obtained for the same group.

This is evidenced by comparing results for the number of distinct fuzzy subgroup of two specific dihedral groups, $D_{12}$ and $D_{40}$, using the general formula from Theorem 5.4.2.0.1 and formulae established by Sehgal and Sharma in [111], for the number of distinct fuzzy subgroups of the dihedral group $D_{p^n q^m}$, for $p$, $q$ distinct primes, $n \in \mathbb{N}$, and a fixed $m = 1$ and by Tarnauceanu in [123], for the number of distinct fuzzy subgroups of the dihedral group $D_{p^n q}$, for $p$, $q$ distinct primes, and $n \in \mathbb{N}$. Both [111] and [123] compute the number of distinct fuzzy subgroups of groups without utilizing the notion of distinguishing factors to classify flags, and obtain their formulae from the definition of an equivalence relation in [20].

Theorem 5.4.2.0.1 of our study states that the number of distinct fuzzy subgroups of $D_{p^n q}$ is given by

$$\begin{aligned} F(D_{p^n q}) &= 2^{n+3} - 1 + 2^{n+2}(n) + 2^{n+2}[\sum_{i=1}^{n}(p^{i-1}q + p^i)] \\ &\quad + 2^{n+1}[\sum_{i=1}^{n}((i-1)p^{i-1}q + (n-i)p^i)] + 2^{n+2}(p^n q) + 2^{n+1}(np^n q) \end{aligned}$$



In [123] the formula for the number of distinct fuzzy subgroups of $D_{p^n q}$ is

$$\begin{aligned} F(D_{p^n q}) &= \frac{2^n}{(p-1)^3}[(n+2)p^{n+3}q + 2p^{n+3} - (2n+5)p^{n+2}q - 3p^{n+2} + (n+3)p^{n+1}q \\ &\quad + p^{n+1} + (n+2)p^3 - p^2 q - (4n+9)p^2 + 3pq + (5n+11)p - 2q - (2n+4)] \end{aligned}$$

While using a recurrence relation, Sehgal and Sharma derive the formula for the number of distinct fuzzy subgroups of $D_{p^n q}$ as

$$\begin{aligned} F(D_{p^n q}) &= \frac{2^n}{(p-1)^2}[(n+2)p^{n+2}q - (n+3)p^{n+1}q + 2p^{n+2} \\ &\quad - p^{n+1} + (n+2)p^2 - (3n+7)p + (2n+4)] \end{aligned}$$

Now, suppose $n = 2$, $p = 2$, and $q = 3$. We have the dihedral group $D_{12}$, and if we let $n = 3$, $p = 2$ and $q = 5$, we have the dihedral group $D_{40}$.

Thus, our formula in Theorem 5.4.2.0.1 yields the following results

$$\begin{aligned}
F(D_{12}) &= 2^5 - 1 + 2^4(n) + 2^4(q + p + pq + p^2) + 2^3(p + pq) + 2^4(p^2q) + 2^3(2p^2q) \\
&= 2^5 - 1 + 2^4(2) + 2^4(3 + 2 + 6 + 4) + 2^3(2 + 6) + 2^4(12) + 2^3(24) \\
&= 63 + 240 + 64 + 192 + 192 \\
&= 751
\end{aligned}$$

$$\begin{aligned}
F(D_{40}) &= 2^6 - 1 + 2^5(n) + 2^5(q + p + pq + p^2 + p^2q + p^3) \\
&\quad + 2^4(2p + pq + p^2 + 2p^2q) + 2^5(p^3q) + 2^4(3p^2q) \\
&= 2^6 - 1 + 2^5(3) + 2^5(5 + 2 + 10 + 4 + 20 + 8) + 2^4(4 + 10 + 4 + 40) \\
&\quad + 2^5(40) + 2^4(120) \\
&= 159 + 1568 + 928 + 1280 + 1920 \\
&= 5855
\end{aligned}$$

With the formula from [123] we obtain the following results.



$$\begin{aligned}
F(D_{12}) &= \frac{2^2}{(2-1)^3}[(2 + 2)(2^5)(3) + 2(2^5) - (2(2) + 5)(2^4)(3) - 3(2^4) + (2 + 3)(2^3)(3) \\
&\quad + 2^3 + (2 + 2)(2^3) - 2^2(3) - (4(2) + 9)2^2 + 3(6) + (5(2) + 11)2 - 6 - (2(2) + 4)] \\
&= 4((4)(32)(3) + 64 - (9)(16)(3) - 3(16) + (5)(8)(3) + 8 + 32 - 12 \\
&\quad - (17)(4) + 18 + 42 - 6 - 8) \\
&= 4(94) \\
&= 376
\end{aligned}$$

$$\begin{aligned}
F(D_{40}) = \frac{2^3}{(2-1)^3}&[(3+2)(2^6)(5) + 2(2^6) - (2(3)+5)(2^5)(5) - 3(2^5) + (3+3)(2^4)(5)\\
&+ 2^4 + (3+2)2^3 - (2^2)5 - (4(3)+9)2^2 + 3(2)(5) + (5(3)+11)2\\
&- 2(3) - (2(3)+4)]\\
&= 8((5)(64)(5) + 128 - (11)(32)(5) - 96 + (9)(16)(5) + 16 + 40 - 20\\
&- (21)(4) + 30 + (26)2 - 8 - 10)\\
&= 8(366)\\
&= 2928
\end{aligned}$$

The formula from [111] results in the following

$$\begin{aligned}
F(D_{12}) = \frac{2^2}{(2-1)^2}&[(2+2)(2^4)(3) - (2+3)(2^3)(3) + 2(2^4)\\
&- 2^3 + (2+2)(2^2) - (3(2)+7)(2) + (2(2)+4)]\\
&= 4((4)(16)(3) - (5)(8)(3) + 32 - 8 + 16 - 26 + 8)\\
&= 4(94)\\
&= 376
\end{aligned}$$



$$\begin{aligned}
F(D_{40}) = \frac{2^3}{(2-1)^2}&[(3+2)(2^5)(5) - (3+3)(2^4)(5) + 2(2^4)\\
&- 2^4 + (3+2)(2^2) - (3(3)+7)(2) + (2(3)+4)]\\
&= 8((5)(16)(5) - (6)(16)(5) + 32 - 16 + 20 - 32 + 10)\\
&= 8(366)\\
&= 2928
\end{aligned}$$

The figures obtained in the above comparisons clearly show that the study has made an improvement in the results for the number of distinct fuzzy subgroups of the dihedral group $D_{p^nq}$ for $p$ and $q$ distinct primes and $n \in \mathbb{N}$. Finally, in light of this work, this study lays some groundwork and forms the basis for the extension of the dihedral group $D_{p^nq}$ for $p$ and $q$

distinct primes, and $n \in \mathbb{N}$, to the broader group of a larger size, which is $D_{p^n q^m}$, for $p$ and $q$ distinct primes, and $m, n \in \mathbb{N}$ which will be undertaken in our future work.

# Appendices

# Appendix A

# Tree diagrams

Figure A.1: For $n = 1$; number of Cyclic Maximal Chains = 2




Figure A.2: For $n = 2$; number of Cyclic Maximal Chains = 3

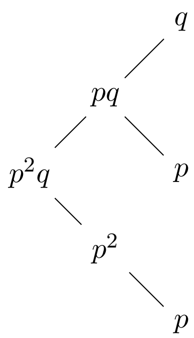


Figure A.3: For $n = 3$; Number of Cyclic Maximal chains = 4

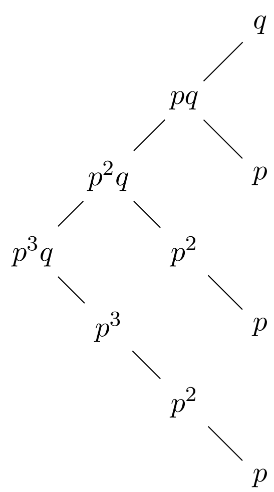



Figure A.4: For $n = 4$; Number of Cyclic Maximal chains = 5

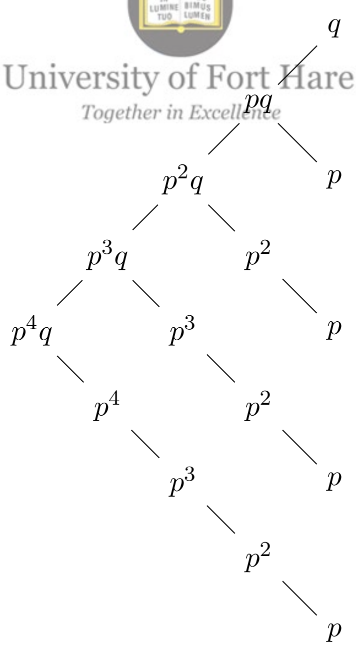
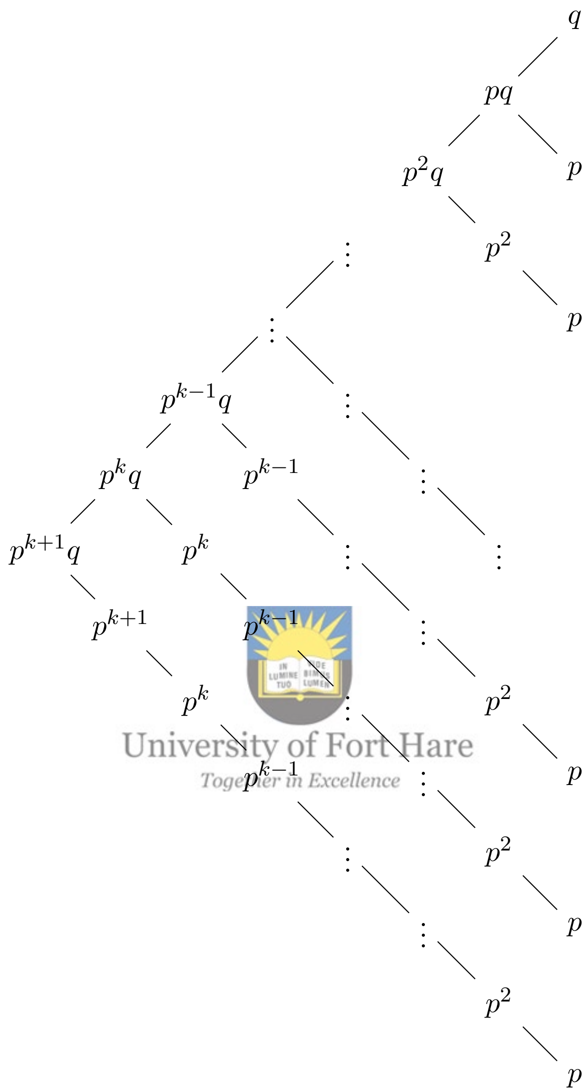


Figure A.5: For $n = k + 1$; Number of Cyclic Maximal chains $= k + 2$



# Appendix B

# The Dihedral group $D_{p^4q}$

If we let $n = 4$, then $D_{p^4q} = \langle a, b : a^{p^4q} = b^2 = e = (ab)^2 \rangle$. Similarly to the case where $n = 2$ and 3, we list all subgroups of $D_{p^4q}$ and manually construct the flags, to obtain the following:

**Proposition B.0.0.1.** *Let $D_{p^4q} = \langle a, b : a^{p^4q} = b^2 = e = (ab)^2 \rangle$. Then $D_{p^4q}$ has subgroups with orders that divide $2p^4q$.*



*Proof.* Based on the Theorem of La Grange in [55] □

**Proposition B.0.0.2.** *The number of dihedral subgroups of $D_{p^4q}$ of order:*

*(i)* $2p$ *is* $p^3q$

*(ii)* $2p^2$ *is* $p^2q$

*(iii)* $2p^3$ *is* $pq$

*(iv)* $2p^4$ *is* $q$

*(v)* $2q$ *is* $p^4$

*(vi)* $2pq$ *is* $p^3$

*(vi)* $2p^2q$ *is* $p^2$

*(vii)* $2p^3q$ *is* $p$

*Proof.* From the manual construction of subgroups of $D_{p^4q}$ $\square$

Using the propositions B.0.0.1 and B.0.0.2, we manually construct the flags of $D_{p^4q}$ and hence obtain a list of the following propositions and theorems:

**Proposition B.0.0.3.** *The number of cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_c = 5$.

**Proposition B.0.0.4.** *The number of d-cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_d = 4p + q$

**Proposition B.0.0.5.** *The number of* $2d$*-cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_{2d} = 2pq + 3p^2$

**Proposition B.0.0.6.** *The number of* $3d$*-cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_{3d} = 3p^2q + 2p^3$



**Proposition B.0.0.7.** *The number of* $4d$*-cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_{4d} = 4p^3q + p^4$

**Proposition B.0.0.8.** *The number of b-cyclic maximal chains of* $D_{p^4q}$ *is* $M(D_{p^4q})_b = 5p^4q$

**Theorem B.0.1.** *The number of maximal chains of subgroups for the dihedral group* $D_{p^4q}$ *is:*

$$M(D_{p^4q}) = 5 + (4p + q) + (2pq + 3p^2) + (3p^2q + 2p^3) + (4p^3q + p^4) + 5p^4q$$

*Proof.* This result is a combination of the sum of cyclic, $d$-cyclic, $2d$-cyclic, $3d$-cyclic, $4d$-cyclic, and $b$-cyclic maximal chains of $D_{p^4q}$, in propositions B.0.0.3, B.0.0.4, B.0.0.5, B.0.0.6, B.0.0.7 and B.0.0.8. $\square$

**Theorem B.0.2.** *The number of distinct fuzzy subgroups of* $D_{p^4q}$ *is:*
$F(D_{p^4q}) = 2^7 - 1 + 2^6 \times 4 + 2^6(p+q) + 2^5(3p) + 2^6(pq+p^2) + 2^5(pq+2p^2) +$

$2^6(p^2q + p^3) + 2^5(2p^2q + p^3) + 2^6(p^3q + p^4) + 2^5(3p^3q) + 2^6(p^4q) + 2^5(4p^4q)$
$= 2^7 - 1 + 2^6 \times 4 + 2^6(p + q + pq + p^2 + p^2q + p^3 + p^3q + p^4) + 2^5(3p + pq + 2p^2 + p^3 + 2p^2q + 3p^3q) + 2^6(p^4q) + 2^5(4p^4q)$.

*Proof.* From our construction of maximal chains of $D_{p^4q}$, we note that there are 5 cyclic flags, $(4p+q)$ $d$-cyclic flags, $(2pq+3p^2)$ $2d$-cyclic flags, $(3p^2q+2p^3)$ $3d$-cyclic flags, $(4p^3q+p^4)$ $4d$-cyclic flags, and $5p^4q$ $b$-cyclic flags. All the flags are of length $n = 7$, and we use the criss-cut counting technique to calculate the number of distinct fuzzy subgroups contributed by each flag. Using the cyclic flags as a starting point, the first flag contributes $2^7 - 1$ distinct fuzzy subgroups by [67]. The remaining four cyclic flags have single distinguishing factors and contribute $2^6 \times 4$ distinct fuzzy subgroups. Thus the number of distinct fuzzy subgroups attributed to the cyclic flags is $2^7 - 1 + 2^6 \times 4$. Next, we count the number of distinct fuzzy subgroups contributed by $(4p + q)$ $d$-cyclic maximal chains. $p$ and $q$ $d$-cyclic flags have single distinguishing factors and contribute $2^6(p + q)$ distinct fuzzy subgroups. The remaining $3p$ $d$-cyclic flags have pairs of distinguishing factors and contribute $2^5(3p)$ distinct fuzzy subgroups. Therefore, the $d$-cyclic flags contribute $2^6(p + q) + 2^5(3p)$ distinct fuzzy subgroups. From $(2pq + 3p^2)$ $2d$-cyclic flags, we have that $pq$ and $p^2$ flags have single distinguishing factors and contribute $2^6(pq + p^2)$ distinct fuzzy subgroups. The remaining $pq$ and $2p^2$ flags have pairs of distinguishing factors, and contribute $2^5(pq + 2p^2)$ distinct fuzzy subgroups. Thus the $2d$-cyclic flags contribute $2^6(pq + p^2) + 2^5(pq + 2p^2)$ distinct fuzzy subgroups. From $(3p^2q + 2p^3)$ $3d$-cyclic flags, $p^2q$ and $p^3$ flags have single distinguishing factors and contribute $2^6(p^2q + p^3)$ distinct fuzzy subgroups. The remaining $2p^2q$ and $p^3$ flags have pairs of distinguishing factors and contribute $2^5(2p^2 + p^3)$ distinct fuzzy subgroups. From $5p^4q$ $b$-cyclic flags we have $p^4q$ $b$-cyclic flags with single distinguishing factors that contribute $2^6(p^4q)$ distinct fuzzy subgroups, while the remaining $4p^4q$ $b$-cyclic flags have pairs of distinguishing factors and a contribute $2^5(4p^4q)$

distinct fuzzy subgroups. Thus, the number of distinct fuzzy subgroups attributed to the $b$-cyclic flags is $2^6(p^4q) + 2^5(4p^4q)$. A summation of all these contributions yields the result. □

## B.1 Isomorphic Classes of Fuzzy Subgroups of $D_{p^4q}$

We calculate the number of non-isomorphic fuzzy subgroups of $D_{p^4q}$, using the non-isomorphic maximal chains of the group.

**STEP 1** : **Cyclic maximal chains**

All five cyclic flags of $D_{p^4q}$ are non-isomorphic, hence, they result in $2^7 - 1 + 2^6 \times 4 = 383$ non-isomorphic fuzzy subgroups.

**STEP 2** : ***d*-cyclic maximal chains**

Any cluster of isomorphic $d$-cyclic flags counts as a single flag. Thus, in the formula $2^6(p+q)+2^5(3p)$ for the number of distinct fuzzy subgroups contributed by the $d$-cyclic flags, the numbers $p$ and $q$ indicate the number of isomorphic flags in a cluster. So the clusters of $p$ and $q$ flags count as 2 non-isomorphic flags. The same argument is applicable for $3p = (p + p + p)$ which gives a count of 3 non-isomorphic flags. Therefore, the number of non-isomorphic fuzzy subgroups contributed by the $d$-cyclic flags is: $2^6(1+1)+2^5(1+1+1) = 2^6(2)+2^5(3) = 224$ non-isomorphic fuzzy subgroups

**STEP 3** : **$2d$-cyclic maximal chains**

The numbers $pq$ and $p^2$ also represent the number of isomorphic flags in a cluster of flags. Each number then counts as a single flag. Hence, the formula $2^6(pq+p^2)+2^5(pq+2p^2)$ gives the number of non-isomorphic fuzzy subgroups contributed by the $2d$-cyclic maximal chains as: $2^6(1+1)+2^5(1+1+1) = 2^6(2)+2^5(3) = 224$

**STEP 4** : **$3d$-cyclic maximal chains**

The numbers $p^2q$ and $p^3$ in the formula $2^6(p^2q+p^3)+2^5(2p^2q+p^3)$, each represent the number of flags that are isomorphic, in a cluster of flags. Therefore each cluster of $p^2q$ and $p^3$ will each count as one flag. Hence, the number of non-isomorphic fuzzy subgroups contributed by the $3d$-cyclic maximal chain is: $2^6(1+1)+2^4(1+1+1) = 2^6(2)+2^5(3) = 224$ non-isomorphic fuzzy subgroups

**STEP 5** : $4d$**-cyclic maximal chains**

The numbers $p^3q$ and $p^4$ in the formula $2^6(p^3q+p^3)+2^5(3p^3q)$, each represent the number of flags that are isomorphic, in a cluster of flags. Therefore each cluster of $p^3q$ and $p^4$ flags will each count as one flag. Hence, the number of non-isomorphic fuzzy subgroups contributed by the $4d$-cyclic maximal chain is: $2^6(1+1)+2^5(1+1+1) = 2^6(2)+2^5(3) =$ 224 non-isomorphic fuzzy subgroups

**STEP 5** : $b$**-cyclic maximal chains**

The number $p^4q$ also represents a cluster of isomorphic flags that give a count of one flag, thus the formula $2^6(p^4q)+2^5(4p^4q)$ gives the number of non-isomorphic fuzzy subgroups contributed by the $b$-cyclic maximal chains as: $2^6(1)+2^5(1+1+1) = 2^6+2^5(3) = 160$ non-isomorphic fuzzy subgroups.

The sum of non-isomorphic fuzzy subgroups yields the following:

**Proposition B.1.0.0.1.** *The number of non-isomorphic fuzzy subgroups of* $G = D_{p^4q}$ *is* $383+224+224+224+224+160 = 1439$

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