Effect of a low density dust shell on the propagation of gravitational waves

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Abstract
Using the Bondi-Sachs formalism, the problem of a gravitational wave source surrounded by a spherical dust shell is considered. Using linearized perturbation theory, the geometry is found in the regions: in the shell, exterior to the shell, and interior to the shell. It is found that the dust shell causes the gravitational wave to be modified both in magnitude and phase, but without any energy being transferred to or from the dust.

Keywords Gravitational Waves · Bondi-Sachs · Spherical dust · Linearized perturbation theory

Contents

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1 Introduction

Calculations of gravitational waves (GWs), both analytical and numerical, normally assume that they propagate from source to a detector on Earth in a vacuum spacetime. Although the average cosmological density of baryonic plus dark matter is small, of order 10^{-29} g/cm^3, a detected GW event may be a considerable distance away from its source, up to order 1 Gpc, and the quantity of intervening matter is not negligible. Further, it is possible that the astrophysical environment of a source event could be such that the source is surrounded by a substantial amount of matter. Thus, as we move into an era of precision GW measurements, it is important to quantify any effects due to propagation of GWs through a non-vacuum spacetime.

These issues have been investigated previously. There is a simple physical argument that an ideal fluid should not extract energy from a GW, because there is no physical mechanism for it to do so; and this idea has been given a precise expression in the work of Esposito [1], and of Ehlers et al. [2,3]. However, if the matter is dissipative, e.g. through shear viscosity, then one would expect GWs to be attenuated. Hawking [4] investigated GWs in cosmological models and determined conditions for complete absorption; subsequently, the general theory of GW propagation through a viscous fluid was further developed [5–8]. More recently, Goswami et al. [9] have investigated whether the properties of dark matter can be constrained by the attenuation effect and GW observations. Baym et al. [10] have extended the hydrodynamics model of the cosmological fluid to a kinetic model with low collision rates and calculated the attenuation effect.

This work uses the Bondi-Sachs [11,12] formalism for the Einstein equations; see also the reviews [13,14]. We consider small perturbations about a fixed background, which topic was first considered some time ago and known as the "quasi-Netwonian" or "quasi-spherical" approximation [15,16]; there has also been previous work using this approach in which a dust cloud source was considered [17]. However, this work uses the method of separation of variables to construct, within a Bondi-Sachs framework, eigensolutions for linearized perturbations [18]; when the background is Minkowski spacetime, these solutions have a remarkably simple analytic form. This approach has given additional insights in another context, that of GWs propagating in de Sitter spacetime [19]; see also [20,21].

The paper considers the model problem of a GW source in a spacetime that is empty apart from matter contained in a thin shell around the source (then, results for a thick matter shell can be modelled by adding up, i.e. integrating, over thin shells). As a first step, the shell is given the simplest equation of state, i.e., that of dust. It is found that the effect of the shell is to modify the outgoing GWs in both phase and magnitude, although in a way that does not contradict previous results about energy transfer. The modification of the GW is small, and in a cosmological context is not expected to be measurable; but it is possible that a GW event could occur in which the local astrophysical environment is such that the effect would be measurable. Further, there is a view that LIGO data for black hole mergers may contain echoes, and explanations investigated, using numerical simulations, have included new physics near the event horizon, and the astrophysical environment such as a shell around the system; see,

e.g., [22,23]; this matter is discussed further in Sect. 5. In any case, the results are certainly of interest to the theory of GWs propagating in matter.

Section 2 specifies the problem, and constructs the background solution, i.e. when the geometry is spherically symmetric. The solution when the source is emitting GWs is then constructed in Sect. 3. The physical interpretation of the modified GWs is discussed in Sect. 4. The main part of the paper ends with a Conclusion (Sect. 5), which also includes a discussion of further work. The paper makes substantial use of computer algebra; the scripts used are available online, and are summarized in "Appendix A".

2 Background solution

The Bondi-Sachs formalism for the Einstein equations is well-known [11,12]. The coordinates are based on outgoing null hypersurfaces labelled by the coordinate $x^0 = u$. Let x^A ($A = 2, 3$) be angular coordinates (e.g. spherical polars (θ, ϕ)) that label the null ray generators of a hypersurface $u = $ constant, and let $x^1 = r$ be a surface area radial coordinate. The Bondi-Sachs metric describes a general spacetime, and here we write it as

$$ds^2 = -\left(e^{2\beta}\left(1 + \frac{W}{r}\right) - r^2 h_{AB}U^A U^B\right)du^2 - 2e^{2\beta}dudr$$
$$- 2r^2 h_{AB}U^B dudx^A + r^2 h_{AB}dx^A dx^B, \tag{1}$$

where $h^{AB}h_{BC} = \delta^A_C$, and the condition that r is a surface area coordinate implies $\det(h_{AB}) = \det(q_{AB})$ where q_{AB} is a unit sphere metric (e.g. $d\theta^2 + \sin^2\theta d\phi^2$). We represent q_{AB} by a complex dyad (e.g. $q^A = (1, i/\sin\theta)$) and introduce the complex differential angular operators $\eth, \bar{\eth}$ [24], with the operators defined with respect to the unit sphere as detailed in [14,25]. Then h_{AB} is represented by the complex quantity $J = q^A q^B h_{AB}/2$ (with $J = 0$ characterizing spherical symmetry), and we also introduce the complex quantity $U = U^A q_A$. Einstein's equations are

$$E_{ab} := R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right), \tag{2}$$

with: E_{ab} representing a set of **equations**, rather than tensor components; $T_{ab} = \rho V_a V_b$ for dust of density ρ and 4-velocity V_a; $T = -\rho$. Einstein's equations may be categorized as

$$\text{Hypersurface equations: } E_{11}, E_{1A}q^A, E_{AB}h^{AB},$$
$$\text{Evolution equation: } E_J = E_{AB}(q^A q^B - Jh^{AB}),$$
$$\text{Constraint equations: } E_{00}, E_{01}, E_{0A}q^A. \tag{3}$$

Much previous work uses $E_J = E_{AB}q^A q^B$; in the case that the background is not Minkowskian, the additional term introduced above leads to some simplification of

Fig. 1 Schematic representation: The spacetime is empty apart from a GW source at the origin, and a shell of mass M_S located between $r = r_0$ and $r = r_0 + \Delta$. The spacetime thus comprises three regions as shown: Interior (I), Shell (S) and Exterior (E)
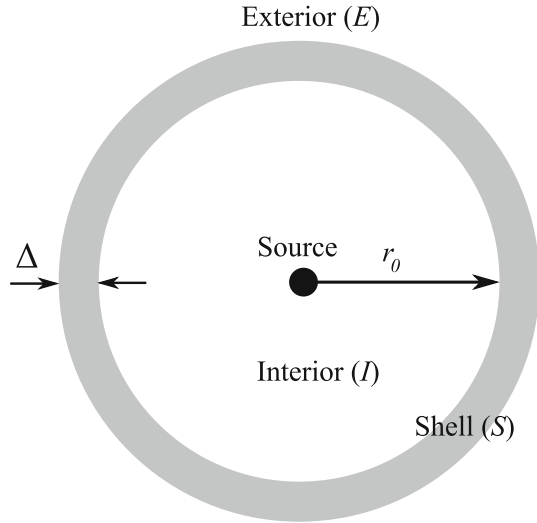
the equation. A characteristic initial value problem may be formulated for the above with J specified on a null hypersurface $u = $ constant [26].

We consider the physical problem of a spacetime that is empty except in a shell located at $r_0 < r < r_0 + \Delta$. where there is a spherically symmetric distribution of dust with a density profile that vanishes at $r = r_0$ and $r = r_0 + \Delta$; see Fig. 1. An example density profile is

$$\rho = \rho_c \left(\frac{1}{r^3} - \frac{r_0}{r^4} \right) \left(\frac{r_0 + \Delta}{r^4} - \frac{1}{r^3} \right), \tag{4}$$

and other density profiles tested are in Eq. (28) in "Appendix A". There is also a source of quadrupolar GWs at the origin, but the first problem that needs to be solved is the background solution for which this source is neglected. The problem is spherically symmetric so $J = U = 0$. Further, the shell density is small and terms of $\mathcal{O}(\rho^2)$ are neglected. The collapse of the shell under its own gravity is an effect with acceleration $\mathcal{O}(\rho^2)$ and is therefore ignored. Thus the shell is treated as static, and the only non-zero metric coefficients are $\beta(r)$, $W(r)$. The Einstein equations for E_{11}, $E_{AB}h^{AB}$ then simplify to [26,27]

$$\partial_r \beta = 2\pi r \rho (V_1)^2, \qquad \partial_r W = e^{2\beta} - 1 - 4\pi e^{2\beta} \rho r^2. \tag{5}$$

Here, to $\mathcal{O}(\rho)$, $\rho(V_1^{[B]})^2 = \rho$ and $\beta = \mathcal{O}(\rho)$, so we have

$$\partial_r \beta = 2\pi r \rho, \qquad \partial_r W = 2\beta - 4\pi \rho r^2, \tag{6}$$

which are solved subject to the boundary conditions $\beta \to 0$ as $r \to \infty$ (so that the background coordinates are asymptotically Minkowskian), and $W = \mathcal{O}(r)$ as $r \to 0$

(so that the W/r is regular at the origin). We find:

$$r < r_0 : \ \beta^{[B]} = B_0, \qquad W^{[B]} = 2r B_0$$
$$r_0 + \Delta < r : \ \beta^{[B]} = 0, \qquad W^{[B]} = -2M_S, \tag{7}$$

with the superfix $[B]$ indicating a quantity in the background depending only on r, and where B_0 is a constant and M_S is the mass of the shell (see below). The solution for $\beta^{[B]}, W^{[B]}$ in $r_0 \le r \le r_0 + \Delta$ is lengthy and is in the Supplementary Material, see "Appendix A". In the case of Eq. (4) we find,

$$M_S = \int_{r_0}^{r_0+\Delta} 4\pi \rho r^2 dr = \rho_c \frac{\pi \Delta^3 (10r_0^2 + 10r_0 \Delta + 3\Delta^2)}{15r_0^4 (r_0 + \Delta)^4},$$

$$B_0 = -\rho_c \frac{\pi \Delta^3 (2r_0 + \Delta)(5r_0^2 + 5r_0 \Delta + 2\Delta^2)}{30r_0^5 (r_0 + \Delta)^5} = -\frac{M_S}{2r_0} + \mathcal{O}(\Delta), \tag{8}$$

with the result $B_0 = -M_S/2r_0 + \mathcal{O}(\Delta)$ applying to all density profiles tested. The solution in $r > r_0 + \Delta$ is Schwarzschild spacetime in Eddington-Finkelstein coordinates. The solution in $r < r_0$ is Minkowski spacetime (which in standard form has $\beta = W = 0$), and this can be seen upon applying the coordinate transformation

$$u \to u' = (1 + 2B_0)u. \tag{9}$$

We check, using computer algebra, that for a given density profile, β and W and their first derivatives are continuous at the interfaces at $r = r_0$ and $r = r_0 + \Delta$; and also that the solution in the shell satisfies all 10 Einstein equations.

3 Perturbed solution

Having determined the background solution, which can be regarded as Minkowski plus small spherically symmetric corrections of $\mathcal{O}(\rho)$, we perturb the solution by writing the metric quantities (β, U, W, J) as

$$\beta = \beta^{[B]} + \beta^{[e]}, \ U = U^{[e]}, \ W = W^{[B]} + W^{[e]}, \ J = J^{[e]}, \tag{10}$$

where the superfix $[e]$ indicates the deviation of a quantity about its background value. The metric perturbations are functions of (u, r, x^A) and are treated as $\mathcal{O}(\epsilon)$ with $\epsilon \ll 1$, and with the smallness parameters ρ_c, ϵ independent of each other. We also introduce the matter field perturbations

$$\rho = \rho^{[B]} + \rho^{[e]}, \ V_a = V_a^{[B]} + V_a^{[e]}, \tag{11}$$

with $V_a^{[e]}$ treated as $\mathcal{O}(\epsilon)$ and $\rho^{[e]}$ treated as $\mathcal{O}(\rho_c \epsilon)$. Linearization about certain backgrounds was described in [18]; using a similar procedure we retain only terms of

order $\mathcal{O}(\rho_c, \epsilon, \rho_c\epsilon)$ and find that the hypersurface and evolution equations are

$$E_{11} : \frac{4}{r}\left(\partial_r \beta^{[e]} + \partial_r \beta^{[B]}\right) = 8\pi T_{11},$$

$$q^A E_{1A} : \frac{1}{2r^2}\left[-2\,r^4\partial_r\left(\frac{\eth\beta^{[e]}}{r^2}\right) - 2\,\partial_r\left(r^4\,\beta^{[B]}\partial_r U^{[e]}\right)\right.$$

$$\left. +\partial_r\left(r^4\partial_r U^{[e]}\right) + r^2\bar{\eth}\partial_r J^{[e]}\right]$$

$$= 8\pi q^A T_{1A},$$

$$h^{AB} E_{AB} : 4\,\beta^{[B]} - 2\,\eth\bar{\eth}\beta^{[e]} - 2\,\partial_r W^{[B]} + 4\,\beta^{[e]}\partial_r W^{[B]}$$

$$-\left(2\,\beta^{[B]} - 1\right)\left[4\,\beta^{[e]} - 2\partial_r W^{[e]} + \frac{1}{r^2}\,\partial_r\left(r^4\eth\bar{U}^{[e]} + r^4\bar{\eth}U^{[e]}\right)\right]$$

$$+\frac{1}{2}\left(\bar{\eth}^2 J^{[e]} + \eth^2\bar{J}^{[e]}\right)$$

$$= 8\pi\left(h^{AB}T_{AB} - r^2\,T\right),$$

$$E_J : -2\,\eth^2\beta^{[e]} - \partial_r\left(r\,\partial_r J^{[e]} W^{[B]}\right)$$

$$+\left(2\,\beta^{[B]} - 1\right)\left[-\partial_r\left(r^2\eth U^{[e]}\right) + \partial_r\left(r^2\partial_r J^{[e]}\right) - 2\,r\,\partial_r\left(r\partial_u J^{[e]}\right)\right]$$

$$= 0. \tag{12}$$

The above equations are tackled using the method of separation of variables, rather than formulation as a characteristic initial value problem. The required ansatz for the quantities $\beta^{[e]}$, $U^{[e]}$, $W^{[e]}$, $J^{[e]}$, $\rho^{[e]}$, $V_a^{[e]}$ is

$$\beta^{[e]} = \Re(\beta^{[2,2]}(r)e^{i\nu u})\,_0Z_{2,2}, \quad U^{[e]} = \Re(U^{[2,2]}(r)e^{i\nu u})\,_1Z_{2,2},$$

$$W^{[e]} = \Re(W^{[2,2]}(r)e^{i\nu u})\,_0Z_{2,2},$$

$$J^{[e]} = \Re(J^{[2,2]}(r)e^{i\nu u})\,_2Z_{2,2},$$

$$\rho^{[e]} = \Re(\rho^{[2,2]}(r)e^{i\nu u})\,_0Z_{2,2}, \quad V_0^{[e]} = \Re(V_0^{[2,2]}(r)e^{i\nu u})\,_0Z_{2,2},$$

$$V_1^{[e]} = \Re(V_1^{[2,2]}(r)e^{i\nu u})\,_0Z_{2,2}, \quad q^A V_A^{[e]} = \Re(V_{ang}^{[2,2]}(r)e^{i\nu u})\,_1Z_{2,2}, \tag{13}$$

with the superfix $[2,2]$ indicating a coefficient of $_sZ_{2,2}$. The perturbations oscillate in time with frequency $\nu/(2\pi)$. The quantities $_sZ_{\ell,m}$ are spin-weighted spherical harmonic basis functions related to the usual $_sY_{\ell,m}$ as specified in [14,18]. They have the property that $_0Z_{\ell,m}$ are real, enabling the description of the metric quantities β, W (which are real) without mode-mixing; however, for $s \neq 0$ $_sZ_{2,2}$ is, in general, complex. A general solution may be constructed by summing over the (ℓ, m) modes, but that is not needed here, since we are considering a source that is continuously emitting GWs at constant frequency dominated by the $\ell = 2$ (quadrupolar) components (Of course, the wave frequency changes with inspiral, and we are assuming that this timescale is much longer than the wave period). We substitute the ansatz Eq. (13) into

Eq. (12) with $\ell = 2$ to obtain: (1) On integrating over the sphere, only the $\ell = 0$ part survives and Eq. (6) for the background are obtained; (2) Multiplication by $_s Z_{2,2}^*$ (where $*$ denotes complex conjugate) followed by integration over the sphere kills the spherically symmetric part, and Eq. (12) transform to a system of ordinary differential equations in r:

$$E_{11} : \frac{4}{r} \frac{d}{dr} \beta^{[2,2]} = 8\pi \left(\rho^{[2,2]} - 2 \rho^{[B]} V_1^{[2,2]} \right),$$

$$q^A E_{1A} : \frac{1}{r^2} \left[\sqrt{6} r^4 \frac{d}{dr} \left(\frac{\beta^{[2,2]}}{r^2} \right) + \frac{1}{r^2} \frac{d}{dr} \left(r^4 \beta^{[B]} \frac{d}{dr} U^{[2,2]} \right) \right.$$
$$\left. - \frac{1}{2} \frac{d}{dr} \left(r^4 \frac{d}{dr} U^{[2,2]} \right) + r^2 \frac{d}{dr} J^{[2,2]} \right]$$
$$= 8\pi \rho^{[B]} V_{ang}^{[2,2]},$$

$$h^{AB} E_{AB} : 12 \beta^{[2,2]} + 4 \beta^{[2,2]} \frac{d}{dr} W^{[B]} + 2\sqrt{6} J^{[2,2]}$$
$$- \left(2 \beta^{[B]} - 1 \right) \left[4 \beta^{[2,2]} - 2\sqrt{6} r \, U^{[2,2]} \right.$$
$$\left. - \sqrt{6} \frac{d}{dr} \left(r^2 U^{[2,2]} \right) - 2 \frac{d}{dr} W^{[2,2]} \right]$$
$$= 8\pi r^2 \rho^{[2,2]},$$

$$E_J : 4\sqrt{6} \beta^{[2,2]} + \partial_r \left(r \, W^{[B]} \frac{d}{dr} J^{[2,2]} \right)$$
$$+ \left(2 \beta^{[B]} - 1 \right) \left[2 \frac{d}{dr} \left(r^2 U^{[2,2]} \right) - \frac{d}{dr} \left(r^2 \frac{d}{dr} J^{[2,2]} \right) \right.$$
$$\left. + 2 i \, \nu r \frac{d}{dr} \left(r \, J^{[2,2]} \right) \right]$$
$$= 0. \tag{14}$$

Equations (14) contain terms of order $\mathcal{O}(\epsilon, \rho\epsilon)$; terms of $\mathcal{O}(\rho^2)$ may be larger than those of $\mathcal{O}(\rho\epsilon)$, but are excluded by the procedure that kills the spherically symmetric part of Eq. (12). In changing from Eqs. (12) to (14), formulas were used for the effect of the $\eth$ operator on $_s Z_{\ell,m}$ [14,18]

$$\eth_{-1} Z_{2,2} = -\sqrt{6} \, _0 Z_{2,2}, \quad \eth_0 Z_{2,2} = \sqrt{6} \, _1 Z_{2,2}, \quad \bar{\eth}_1 Z_{2,2} = -\sqrt{6} \, _0 Z_{2,2},$$
$$\eth_1 Z_{2,2} = 2 \, _2 Z_{2,2},$$
$$\bar{\eth}_2 Z_{2,2} = -2 \, _1 Z_{2,2}, \quad \eth^2 {}_{-1} Z_{2,2} = -6 \, _1 Z_{2,2}, \quad \eth^2 {}_0 Z_{2,2} = 2\sqrt{6} \, _1 Z_{2,2},$$
$$\eth\bar{\eth}_0 Z_{\ell,m} = -6 \, _0 Z_{\ell,m},$$
$$\eth\bar{\eth}_1 Z_{\ell,m} = -6 \, _1 Z_{\ell,m}, \quad \bar{\eth}^2 {}_2 Z_{2,2} = 2\sqrt{6} \, _0 Z_{2,2}, \quad \eth^2 {}_{-2} Z_{2,2} = 2\sqrt{6} \, _0 Z_{2,2}, \tag{15}$$

where the above formulas are specialized to the case $\ell = 2$. As well as the hypersurface Eq. (14), we will also need the constraints E_{0a}, which are

$$
E_{00} : - \frac{1}{2r^3} \Bigg[12\, r\beta^{[2,2]} + 24\, r\beta^{[B]}\beta^{[2,2]} - \frac{2}{r}\frac{d}{dr}\left(r^2\frac{d}{dr}\beta^{[2,2]}\right)
$$
$$
- \sqrt{6}\, r\, U^{[2,2]}\left(\frac{d}{dr}\left(\frac{W^{[B]}}{r}\right) + 2\frac{d}{dr}\beta^{[B]}\right)
$$
$$
- 6\, r^3\, W^{[B]}\frac{d}{dr}\left(\frac{\beta^{[2,2]}}{r^2}\right) - 2\, r^2\left(2\frac{d^2}{dr^2}\beta^{[2,2]}W^{[B]} + \frac{d}{dr}\beta^{[2,2]}\frac{d}{dr}W^{[B]}\right)
$$
$$
- 6\, r^3\, W^{[2,2]}\frac{d}{dr}\left(\frac{\beta^{[B]}}{r^2}\right) - 2\, r^2\left(2\frac{d^2}{dr^2}\beta^{[B]}W^{[2,2]} + \frac{d}{dr}\beta^{[B]}\frac{d}{dr}W^{[2,2]}\right)
$$
$$
+ 6\, W^{[2,2]} - r^2\frac{d^2}{dr^2}W^{[2,2]}
$$
$$
- r\, W^{[2,2]}\frac{d^2}{dr^2}W^{[B]} - r\, W^{[B]}\frac{d^2}{dr^2}W^{[2,2]} - 2\, i\nu\, r\, W^{[2,2]}
$$
$$
+ i\,\frac{4\nu}{r^2}\frac{d}{dr}\left(r\,\beta^{[2,2]}\right) + 4\, i\,\nu\, r\,\beta^{[2,2]}W^{[B]}
$$
$$
+ 4\, i\,\nu\, r^2\frac{d}{dr}\beta^{[2,2]}W^{[B]} - 2\sqrt{6}\, i\,\nu\, r^3\, U^{[2,2]}\Bigg]
$$
$$
= 4\pi\left(\rho^{[2,2]} + \rho^{[B]}\frac{2\beta^{[2,2]}r + W^{[2,2]}}{r}\right),
$$
$$
E_{01} : - \frac{1}{2r^2}\Bigg[24\,\beta^{[B]}\beta^{[2,2]} + 12\,\beta^{[2,2]} - r\frac{d^2}{dr^2}W^{[2,2]} + 4\, i\nu r^2\frac{d}{dr}\beta^{[2,2]}
$$
$$
- \frac{d}{dr}\left(2\, r^2\frac{d}{dr}\beta^{[2,2]} + 2\, r\, W^{[B]}\frac{d}{dr}\beta^{[2,2]} + 2\, r\, W^{[2,2]}\frac{d}{dr}\beta^{[B]} + r^2\sqrt{6}U^{[2,2]}\right)\Bigg]
$$
$$
= 4\pi\left(\rho^{[2,2]} - 2\rho^{[B]}V_1^{[2,2]} + \rho^{[B]}\frac{W^{[2,2]}}{r}\right),
$$
$$
q^A E_{0A} : \frac{1}{2r^2}\Bigg[4\, r^2\beta^{[B]}U^{[2,2]} + 2\frac{d}{dr}\left(r^4\beta^{[B]}\frac{d}{dr}U^{[2,2]}\right) - 2\, i\,\nu\, r^4\beta^{[B]}\frac{d}{dr}U^{[2,2]}
$$
$$
- \sqrt{6}\left(2\, r\, W^{[2,2]}\frac{d}{dr}\beta^{[B]} + r^2\frac{d}{dr}\left(\frac{W^{[2,2]}}{r}\right) - 2\, i\,\nu\, r^2\,\beta^{[2,2]}\right)
$$
$$
- 2\, r^2\frac{d}{dr}W^{[B]}U^{[2,2]} - \frac{1}{r}\frac{d}{dr}\left(r^4 W^{[B]}\frac{d}{dr}U^{[2,2]}\right)
$$
$$
- 2\, r^2\, U^{[2,2]} - \frac{d}{dr}\left(r^4\, U^{[2,2]}\right) + i\,\nu\, r^4\frac{d}{dr}U^{[2,2]} + 2\, i\nu\, r^2\, J^{[2,2]}\Bigg]
$$
$$
= 4\pi\,\rho^{[B]}\left(2\, V_{ang}^{[2,2]} + U^{[2,2]}r^2\right). \tag{16}
$$

The procedure for solving Eqs. (14) and (16) was given in [18], and in outline is as follows. Firstly, E_{11} is integrated to find $\beta^{[2,2]}(r)$; then $q^A E_{1A}$ and E_J are solved together to give $J^{[2,2]}(r)$, $U^{[2,2]}(r)$; then $h^{AB}E_{AB}$ may be integrated to give $W^{[2,2]}(r)$. The solution obtained has 6 constants of intergration, 2 of which are fixed on applying the constraints E_{00}, $q^A E_{0A}$.

We will see later that the matter terms can be expressed as functions of $\rho^{[B]}$ and the metric, so that Eq. (14) take the form

$$(\mathcal{M} + \rho_c \mathcal{R})(f) = 0, \tag{17}$$

where f is a multi-vector containing $\beta^{[2,2]}(r)$, $J^{[2,2]}(r)$, $U^{[2,2]}(r)$, $W^{[2,2]}(r)$, and $\mathcal{M}, \mathcal{R}$ are linear differential operators with $\mathcal{M}$ being the operator when $\rho_c = 0$, i.e. when the background is Minkowski. Let f_M be the solution to the homogeneous problem $\mathcal{M}(f_M) = 0$. f_M was derived in Eq. (56) of [18] (but note that the reference used a different notation, and used $\eth^s \, _0 Z_{\ell,m}$ rather than $_s Z_{\ell,m}$ as angular basis), and is

$$\begin{aligned}
\beta^{[2,2]} &= b_0, \\
W^{[2,2]} &= 6i\nu r^2 C_{10} + r(12C_{10} - 10b_0) + C_{50} \\
&\quad - \frac{12i\nu C_{40}}{r} - \frac{6C_{40}}{r^2} - C_{in0} \exp(2ir\nu)\frac{3}{r^2}, \\
U^{[2,2]} &= -\sqrt{6}i\nu C_{10} + \frac{2\sqrt{6}b_0}{r} + \frac{2\sqrt{6}C_{30}}{r^2} - \frac{4i\nu\sqrt{6}C_{40}}{r^4} - \frac{3\sqrt{6}C_{40}}{r^4} \\
&\quad - C_{in0} \exp(2ir\nu)\sqrt{6}\left(i\frac{\nu}{r^3} - \frac{3}{2r^4}\right), \\
J^{[2,2]} &= 2\sqrt{6}C_{10} + \frac{2\sqrt{6}C_{30}}{r} + \frac{2\sqrt{6}C_{40}}{r^3} + C_{in0} \exp(2ir\nu)\sqrt{6}\left(\frac{1}{r^3} - 2i\frac{\nu}{r^2} - \frac{\nu^2}{r}\right).
\end{aligned} \tag{18}$$

The solution includes constants of integration $b_0, C_{in0}, C_{10}, C_{30}, C_{40}, C_{50}$, two of which are fixed on applying the constraints $E_{00}, q^A E_{0A}$ giving

$$C_{50} = 12\nu^2 C_{40}, \quad C_{10} = \frac{2b_0 + i\nu C_{30} + i\nu^3 C_{40}}{3}. \tag{19}$$

The remaining Einstein equation E_{01} is known as the trivial equation, since it is automatically satisfied provided all the other Einstein equations are satisfied [11]. The solution Eq. (18) subject to Eq. (19) will be denoted by f_M; and this may be specialized to the case of no incoming radiation by setting $C_{in0} = 0$ with the solution denoted by f_{M0}.

The gravitational news $\mathcal{N}$ is defined in a coordinate system that satisfies the Bondi gauge conditions $\lim_{r\to\infty} J, U, \beta, W/r = 0$, and is calculated on making the required coordinate transformation. The procedure in the general case was described in [28], which was then simplified for the linearized approximation in [18] (Sec.3.3); an explicit expression for the news was given in [29], Eq. (16). Allowing for the conventions used here, we find that for f_{M0}, $\mathcal{N}_{M0} = -\sqrt{6}\nu^3 \Re(iC_{40} \exp(i\nu u)) \, _2 Z_{2,2}$. The gravitational wave strain and news are related (see Eq. (276) in [14]) $\mathcal{H}_{M0} = h_+ + ih_\times = 2\int \mathcal{N}_{M0} du$, giving

$$\mathcal{H}_{M0} = \Re(H_{M0} \exp(i\nu u)) \, _2 Z_{2,2} \text{ with } H_{M0} = -2\sqrt{6}\nu^2 C_{40}. \tag{20}$$

C_{40} is determined by the physical problem being modelled, and b_0, C_{30} represent gauge freedoms; e.g., for an equal mass m binary with orbital radius r_o, $C_{40} = 2mr_o^2\sqrt{\pi/15}$ [30].

We regard the solution f_{M0} as applying when no matter shell is present, i.e. when $\rho_c = 0$, with all the constants of order $\mathcal{O}(\epsilon)$ and with C_{40} fixed by the physics of the source. Then the solution in the presence of the matter shell will be f_{M0} plus small corrections of order $\mathcal{O}(\epsilon\rho_c)$. We construct a solution in each of the three regions $r < r_0, r_0 < r < r_0 + \Delta, r_0 + \Delta < r$, and then apply matching conditions at the boundaries $r = r_0, r = r_0 + \Delta$.

3.1 Solution inside the shell, i.e. $r < r_0$

Perturbations in this region are on a background that is explicitly Minkowskian in (u', r, θ, ϕ) coordinates; thus the solution f_M applies, although the effect of boundary conditions may be to change the values of the constants by $\mathcal{O}(\epsilon\rho_c)$ from their values in f_{M0}. We therefore make the substitutions in Eq. (18) $b_0 \rightarrow b_0 + b_{0I}, C_{in0} \rightarrow C_{inI}, C_{10} \rightarrow C_{10} + C_{11}, C_{30} \rightarrow C_{30} + C_{3I}, C_{40} \rightarrow C_{40} + C_{4I}, C_{50} \rightarrow C_{50} + C_{5I}$ with b_{0I} etc. of order $\mathcal{O}(\epsilon\rho_c)$, and denote the solution as $f_{I'}$. The solution f_I in global (u, r, θ, ϕ) coordinates is obtained on applying the coordinate transformation Eq. (9) to $f_{I'}$; using computer algebra, we have evaluated f_I explicitly and give the formulas in the Supplementary Material, see "Appendix A". Using computer algebra, the solution f_I has been substituted into Eq. (14) with $\beta^{[B]}, W^{[B]}$ taking the values for $r < r_0$ in Eqs. (7) and (8), and we have confirmed that the equations are satisfied. Values of the constants C_{1I}, C_{5I} have been determined by substituting f_I into Eq. (16), and then the constraints have been re-evaluated to confirm that they are satisfied.

3.2 Solution within the shell, i.e. $r_0 < r < r_0 + \Delta$

The metric perturbations introduce perturbations into the matter fields, and these act as source terms in the Einstein equations. We evaluate the condition that the energy momentum tensor should be divergence-free, $\nabla_c T_{ab} g^{bc} = 0$, with the metric and metric connection terms evaluated using f_{M0} (rather than f). The result is four equations for five unknowns $\rho^{[2,2]}(r), V_0^{[2,2]}(r), V_1^{[2,2]}(r), V_{ang}^{[2,2]}(r)$ (note that $V_{ang}^{[2,2]}(r)$ is complex with two components), and the system is closed on applying the condition that the 4-velocity has unit norm, i.e. $V_a V_b g^{ab} = 1$. The resulting formulas are given in the Supplementary Material, see "Appendix A", and we note that they are explicit functions of r depending on the parameters $b_0, C_{10}, C_{30}, C_{40}, C_{50}, r_0, \Delta, \rho_c$. We also consider terms on the left hand side of Eq. (14) of the form $(\beta^{[B]}$ or $W^{[B]})\times$(metric perturbation) and use f_{M0} rather than f to evaluate the metric perturbation; thus these terms also become explicit functions of r depending on known parameters.

The justification for using f_{M0} instead of f is that the error introduced is $\mathcal{O}(\rho_c\epsilon)$, and all such terms are multiplied by a term of $\mathcal{O}(\rho_c)$ so the total error is $\mathcal{O}(\rho_c^2\epsilon)$ which is ignorable. The result of these simplifications is that Eq. (17) becomes

$$\mathcal{M}f = -\rho_c \mathcal{R}_S(f_{M0}), \tag{21}$$

for which the solution to the homogeneous part has already been obtained, and there is just a particular integral that needs to be found; $\mathcal{R}_S$ denotes the specific form of the operator $\mathcal{R}$ within the shell. We denote the solution within the shell as f_S, and introduce additional constants $b_{0S}, C_{inS}, C_{1S}, C_{3S}, C_{4S}, C_{5S}$ similarly to what was done in Sect. 3.1 with f_I. The explicit solution for f_S is obtained by means of computer algebra; it is very long and is given in the Supplementary Material, see "Appendix A". Using computer algebra, we have evaluated the constraints to find expressions for C_{1S}, C_{5S}, and the solution f_S has been confirmed by checking that it satisfies Eqs. (14) and (16) to order $\mathcal{O}(\rho_c \epsilon)$.

3.3 Solution exterior to the shell, i.e. $r_0 + \Delta < r$

In this region, the background geometry is Schwarzschild and perturbations about this background are not known analytically, but a numerical solution has been obtained [31]. Here, however, we can use the same procedure as in the previous sub-section, that is for terms of the form ($\beta^{[B]}$ or $W^{[B]}$)×(metric perturbation) we use f_{M0} rather than f to evaluate the metric perturbation, and Eq. (17) becomes $\mathcal{M}f = -\rho_c \mathcal{R}_E(f_{M0})$. We denote the solution outside the shell as f_E, and introduce additional constants $b_{0E}, C_{inE}, C_{1E}, C_{3E}, C_{4E}, C_{5E}$ similarly to what was done in Sect. 3.1 with f_I. The explicit solution for f_E is obtained by means of computer algebra and is given in the Supplementary Material, see "Appendix A". Using computer algebra, we have evaluated the constraints to find expressions for C_{1E}, C_{5E}, and the solution f_E has been confirmed by checking that it satisfies Eqs. (14) and (16) to order $\mathcal{O}(\rho_c \epsilon)$.

Having determined the solution in the exterior region, we can now evaluate the gravitational news $\mathcal{N}^{[2,2]}$, and the formula is given in the Supplementary Material, see "Appendix A". It involves $C_{40}, C_{4E}, \rho_c, r_0, \Delta$ and we need to express C_{4E} in terms of the physical variables $C_{40}, \rho_c, r_0, \Delta$ so as to complete the calculation.

3.4 Matching conditions and the complete solution

After having used the constraints in each region, f_E, f_S, f_I contain 12 free constants. The condition of no incoming radiation in the exterior requires that the coefficient of $\exp(2ir\nu)/r$ in f_E be zero, leading to

$$C_{inE} = -4\nu\pi C_{40} M_S, \tag{22}$$

and continuity of the metric quantities β, W, J, U at $r = r_0$ and $r = r_0 + \Delta$ imposes 8 conditions

$$
\begin{aligned}
\beta^{[2,2,I]}(r = r_0) &= \beta^{[2,2,S]}(r = r_0), & \beta^{[2,2,S]}(r = r_0 + \Delta) &= \beta^{[2,2,E]}(r = r_0 + \Delta), \\
J^{[2,2,I]}(r = r_0) &= J^{[2,2,S]}(r = r_0), & J^{[2,2,S]}(r = r_0 + \Delta) &= J^{[2,2,E]}(r = r_0 + \Delta), \\
U^{[2,2,I]}(r = r_0) &= U^{[2,2,S]}(r = r_0), & U^{[2,2,S]}(r = r_0 + \Delta) &= U^{[2,2,E]}(r = r_0 + \Delta), \\
W^{[2,2,I]}(r = r_0) &= W^{[2,2,S]}(r = r_0), & W^{[2,2,S]}(r = r_0 + \Delta) &= W^{[2,2,E]}(r = r_0 + \Delta),
\end{aligned}
\tag{23}
$$

where the I, S, E within the superfix indicates a quantity in the interior, shell or exterior respectively. We first solve the continuity conditions at $r = r_0 + \Delta$ to find expressions for $b_{0E}, C_{3E}, C_{4E}, C_{inS}$, and then we solve the continuity conditions at $r = r_0$ to find expressions for $b_{0S}, C_{3S}, C_{4S}, C_{inI}$; all these expressions are given in the Supplementary Material, see "Appendix A". Since the Einstein hypersurface equations contain second derivatives of J and U, we also expect $\partial_r J$, $\partial_r U$ to be continuous at $r = r_0$ and $r = r_0 + \Delta$. Using computer algebra we have checked that, provided Eqs. (23) are satisfied, the following conditions are also satisfied

$$\partial_r J^{[2,2,I]}(r = r_0) = \partial_r J^{[2,2,S]}(r = r_0),$$
$$\partial_r J^{[2,2,S]}(r = r_0 + \Delta) = \partial_r J^{[2,2,E]}(r = r_0 + \Delta),$$
$$\partial_r U^{[2,2,I]}(r = r_0) = \partial_r U^{[2,2,S]}(r = r_0),$$
$$\partial_r U^{[2,2,S]}(r = r_0 + \Delta) = \partial_r U^{[2,2,E]}(r = r_0 + \Delta). \tag{24}$$

The outcome is that three degrees of freedom remain, b_{0I}, C_{3I}, C_{4I}, and the expression for $\mathcal{H}$ involves C_{40}, C_{4I}. As already discussed, the value of the constant C_{40} is determined by the physics of the GW source, and we need another physical condition in order to close the system and to fix C_{4I}. The calculation of the perturbations from an equal mass binary assumes no incoming radiation, but the back-reaction due to the matter distribution may introduce such a term. Thus, the computer algebra script for the calculation of GWs emitted by an equal mass binary [30] has been amended with $C_{in0} \neq 0$ but $\mathcal{O}(\rho_c)$. We find

$$C_{4I} = -\frac{C_{inI}}{2}. \tag{25}$$

This result applies independently of source properties, and in particular it applies when the mass of the orbiting binary is zero. Thus, it is reasonable to regard the result in Eq. (25) as general, representing the reflection at the origin of an incoming GW.

Then, finally, for all density profiles tested, the gravitational wave strain in terms of the mass of the shell M_S and as measured by an observer at future null infinity, was found to be

$$\mathcal{H} = \Re \left(H_{M0} \left(1 + \frac{2M_S}{r_0} + \frac{2i M_S}{r_0^2 v} + \frac{i M_S e^{-2i r_0 v}}{2r_0^2 v} \right. \right.$$
$$\left. \left. + \mathcal{O}\left(\frac{M_S \Delta}{r_0^2}, \frac{M_S}{r_0^3 v^2} \right) \right) \exp(i v u) \right) {}_2 Z_{2,2}, \tag{26}$$

where H_{M0} was given in Eq. (20). Each of the terms containing M_S in Eq. (26) represents a correction to the wave strain in the absence of the shell, and these correction terms are discussed in Sect. 4 below.

The formula Eq. (26) was derived using quite complicated computer algebra scripts, and it is important to investigate its reliability through consistency checks:

- In each of the three regions, we confirmed using computer algebra that all 10 Einstein equations are satisfied. The four constraint equations were not used in the

construction of the solution, although they were used to fix two (in each region) constants. Note that it was found that if an error were introduced into one of the hypersurface equations, then the formulas obtained for C_1, C_5 were not constant but were functions of r. Thus, the fact that the solutions satisfy the constraints amounts to a strong consistency test.

– The solution was constructed to ensure continuity of β, W, J, U at $r = r_0$ and $r = r_0 + \Delta$, but continuity of $\partial_r J$, $\partial_r U$ is also needed. We confirmed using computer algebra that these conditions are satisfied, which again amounts to a strong consistency test.

– As will be discussed in the next section, the formula Eq. (26) satisfies physical expectations, with one of the terms in the formula being derivable in an independent way.

4 Physical interpretation

We investigate the physical meaning of each of the correction terms in Eq. (26), in the light of the expectation that the dust shell cannot add or remove energy from the GWs:

– Correction 1, $2M_S/r_0$. In natural Minkowski coordinates u' and in the absence of a matter shell, $H' = -2\nu'^2\sqrt{6}C_{40}$. From Eq. (9), it follows that $\nu' = \nu(1-2B_0))$, and thus from Eq. (8), $\nu' = \nu(1+M_S/r_0))$. Thus $H' \approx -2\nu^2\sqrt{6}C_{40}(1+2M_S/r_0)$. This term is therefore a consequence of the coordinate transformation Eq. (8), i.e. it represents the gravitational red-shift effect of the shell.

– Correction 2, $2iM_S/(r_0^2\nu)$. The term is out of phase with the leading terms $1 + 2M_S/r_0$. Thus, to $\mathcal{O}(M_S)$, the magnitude of $\mathcal{H}$, and therefore of the energy of the GW, is not affected; but the shell does change the phase of the GW.

– Correction 3, $iM_Se^{-2ir_0\nu}/(r_0^2\nu)$. This term does affect the magnitude of $\mathcal{H}$. We have checked using computer algebra (see "Appendix A") that if $\mathcal{H}$ is calculated with C_{inI} set to zero the term disappears, and so it is interpreted as being an effect due to the shell generating an incoming GW. Such an incoming GW modifies the geometry near the source and affects the radiation reaction (or self-force) and thus the inspiral rate. The calculation of the self-force is a 2nd order effect which is beyond the scope of this work. The energy change in the GW at infinity can therefore be interpreted as being caused by the modification of the self-force, rather than by energy being transferred to or from the dust cloud.

In order for Correction 1 to be a measurable, M_S/r_0 would need to have a significant value (but note that the results obtained here also assume that $(M_S/r_0)^2$ is negligible), i.e. the shell would need to be close to the GW source. For Corrections 2 and 3 to be measurable, we would also need that $r_0\nu$ should not be large, where $r_0\nu = 2\pi r_0/\lambda$ with r_0/λ being the number of wavelengths in r_0; note that the only assumption that has been made about ν is $\nu \neq 0$. For example, if $M_S/r_0 = 10^{-2}$ (so that the linearized approximation can apply) and $r_0\nu = 10$, then the modification to the GW signal would be of order 1% by Correction 1, and of order 0.1% by Corrections 2 and 3.

We now consider the case of a thick shell comprising a dust cloud of constant density (ρ_0) extending from near the origin to an observer at $r = r_F$, so that $\delta M_S = 4\pi r_0^2 \rho_0 \delta r_0$; integration gives

$$
\mathcal{H} = \Re\left(H_{M0}\left(1 + 4\pi\rho_0 r_F^2 + 4i\rho_0 r_F \lambda + \mathcal{O}\left(\frac{\rho_0 \log(r_F)}{\nu^2}\right)\right)\exp(i\nu u)\right) {}_2 Z_{2,2},
$$
(27)

and the integral of Correction 3 is omitted since it is smaller than the order term. Applying Eq. (27) to cosmology, and taking the cosmological density as 10^{-29} g/cm^3 $\approx$ 0.7×10^{-8} Mpc^{-2} in geometric units, it is found that, apart from the gravitational redshift, no measurable effect is expected since the GW wavelength λ is at most $\mathcal{O}(10^4)$ km $\ll$ 1 Mpc.

5 Conclusion

Using the Bondi-Sachs formalism, this work has investigated within linearized perturbation theory, the effect of a spherical dust shell on GWs sourced from the center of the shell. It was found that the GWs were modified, although without any energy transfer between the GWs and the shell. This finding is novel. In the context of cosmology, the effects are too small to be measurable; but the effect would be measurable if a GW event were to occur with a source surrounded by a massive shell and with the radius of the shell and the wavelength of the GWs of the same order.

There are three avenues for further work. Firstly, the matter equation of state needs to be generalized beyond dust to include shear viscosity, and perhaps other forms of dissipation. Secondly, the background spacetime without the matter shell can be changed from Minkowski to something more appropriate to cosmology. Solutions are known for de Sitter spacetime [19]; and for Einstein-de Sitter [27], although in this case the algebraic complexity of the solution may make the construction of perturbative solutions problematic. Finally, we found that a shell of matter, in effect, reflects part of an outgoing GW, and so in the context of a burst (rather than a continuous) source it is possible that an echo would be the result. Now, it should be possible to model a burst source as a Fourier sum of the eigensolutions obtained here, i.e. as something of the form $\sum_n a_n f(\nu_n)$, and in this way to investigate GW echoes.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

A Description of computer algebra scripts

The computer algebra scripts used in this paper are written in Maple in plain text format, and are available as Supplementary Material. Note that the output files may be viewed using a plain text editor provided line-wrapping is switched off.

The scripts `gamma.out`, `initialize.map`, `lin.map`, `ProcsRules.map` are not used directly, but are called by the other scripts described below. `gamma.out` contains formulas for the Bondi-Sachs metric, its inverse and the metric connection coefficients. `lin.map` constructs the Einstein equations linearized about a given background. `ProcsRules.map` contains various procedures and rules that are used by other scripts.

The script `initialize.map` initializes various arrays etc., and sets the density profile of the matter as given in Eq. (4); other density profiles tested were

$$
\rho = \rho_c \left(\frac{1}{r^3} - \frac{r_0}{r^4} \right) \left(\frac{(r_0 + \Delta)^2}{r^5} - \frac{1}{r^3} \right), \qquad \rho = \rho_c \left(\frac{1}{r^4} - \frac{r_0}{r^5} \right) \left(\frac{r_0 + \Delta}{r^4} - \frac{1}{r^3} \right),
$$

$$
\rho = \rho_c \left(\frac{1}{r^2} - \frac{r_0}{r^3} \right) \left(\frac{(r_0 + \Delta)}{r^5} - \frac{1}{r^4} \right), \qquad \rho = \rho_c \left(\frac{1}{r^2} - \frac{r_0}{r^3} \right) \left(\frac{r_0 + \Delta}{r^6} - \frac{1}{r^5} \right).
$$

$$(28)$$

The script `backgroundShell.map` constructs the background (spherically symmetric) solution for the given density profile; it also checks that the metric functions are sufficiently smooth at the interface $r = r_0$, and that the solution satisfies all 10 Einstein equations. The output is in `backgroundShell.out`

The script `paperEqs.map`, with output in `paperEqs.out`, generates the formulas given in Eqs. (12), (14) and (16); note that some manual simplifications have been applied to the formulas generated by the computer algebra.

The script `shell.map` uses the divergence-free condition on the energy-momentum tensor, $\nabla_a T_{bc} g^{ac} = 0$, to determine the fluid properties, i.e. density and velocity perturbations. It then constructs the metric in $r < r_0, r_0 < r < r_0 + \Delta$ and $r > r_0$, and checks that the solutions obtained satisfy all 10 Einstein equations. Finally, it constructs and solves the continuity conditions at $r = r_0, r = r_0 + \Delta$, and then evaluates the gravitational wave strain. The output is in `shell.out`.

The script `shellCinI_0.map` is the same as `shell.map` except that: on line 399 `CinI` is hard-coded to be 0, and some output has been suppressed. The output is in `shellCinI_0.out`.

The script `regular_0_IncomingGW.map` is an adaptation of the script `regular_0.map` used in [30] to calculate the GWs emitted by an equal mass binary. The adaptations are: (a) an incoming wave, as a free parameter, is included; and (b) the coefficient names have been changed to be consistent with those used in `shell.map`. The output is in `regular_0_IncomingGW.out`.

The file `formulas.pdf` is in pdf format and contains the Maple output for

$\beta^{[B]}, W^{[B]}, \beta^{[2,2,I]}, J^{[2,2,I]}, U^{[2,2,I]}, W^{[2,2,I]}, C_{1I}, C_{5I}, \rho^{[2,2]}, V_0^{[2,2]}, V_1^{[2,2]}, V_{ang}^{[2,2]}$

$\beta^{[2,2,S]}, J^{[2,2,S]}, U^{[2,2,S]}, W^{[2,2,S]}, C_{1S}, C_{5S}, \beta^{[2,2,E]}, J^{[2,2,E]}, U^{[2,2,E]},$

$\quad W^{[2,2,E]}, C_{1E}, C_{5E},$

$\mathcal{N}^{[2,2]}, b_{0E}, C_{3E}, C_{4E}, C_{inS}, b_{0S}, C_{3S}, C_{4S}, C_{inI}.$

The formulas for the above are generated during the execution of `shell.map` and written to the file `formulas.out`.

The file `Eqs.pdf` is in pdf format and includes the content of `paperEqs.out` with annotations, together with the formulas for $\beta^{[B]}$, $W^{[B]}$ within the matter shell extracted from `backgroundShell.out`.

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