2020

Large N Fields and Holography

Animik Ghosh
University of Kentucky, animikghosh@gmail.com
Author ORCID Identifier:
https://orcid.org/0000-0002-4934-2955
Digital Object Identifier: https://doi.org/10.13023/etd.2020.325

STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained needed written permission statement(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine) which will be submitted to UKnowledge as Additional File.

I hereby grant to The University of Kentucky and its agents the irrevocable, non-exclusive, and royalty-free license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless an embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's thesis including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Animik Ghosh, Student

Dr. Sumit R. Das, Major Professor

Dr. Christopher Crawford, Director of Graduate Studies

LARGE N FIELDS AND HOLOGRAPHY

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Animik Ghosh
Lexington, Kentucky

Director: Dr. Sumit R. Das, Professor of Physics & Astronomy

Lexington, Kentucky

https://orcid.org/0000-0002-4934-2955

ABSTRACT OF DISSERTATION

LARGE N FIELDS AND HOLOGRAPHY

We study (nearly) AdS/CFT holography within the context of the Sachdev-Ye-Kitaev (SYK) model. We present a systematic procedure to extract the dynamics of the low energy Schwarzian mode in SYK type models with a single energy scale J and emergent reparametrization symmetry in the infrared within the framework of perturbation theory. We develop a systematic approach using Feynman diagrams in bilocal theory to obtain a formal expression for the enhanced and $O(1)$ corrections to the bilocal propagator and apply this general technique to large q SYK. We show that the Schwarzian theory describes a sector of a general class of two dimensional CFTs using conformal bootstrap techniques. We also provide a gravitational interpretation of this fact by showing that the dynamics in the near horizon throat of a specific class of BTZ black holes is described very well by Jackiw-Teitelboim (JT) gravity which gives rise to the Schwarzian action. We study the highly nontrivial conformal matter spectrum of the SYK model at arbitrary q. We provide a three dimensional bulk interpretation and carry out Kaluza-Klein (KK) reduction to reproduce the SYK spectrum and more significantly, the conformal bilocal propagator. We provide an interpretation of the two dimensional dual spacetime of the conformal sector of the SYK model using techniques from bilocal holography. We present a resolution of a conundrum about the signature of the dual spacetime using nonlocal integral transformations.

KEYWORDS: AdS/CFT correspondence, $1/N$ expansion, Sachdev-Ye-Kitaev, collective field theory, BTZ black holes, conformal bootstrap

Animik Ghosh

July 27, 2020

LARGE N FIELDS AND HOLOGRAPHY

By
Animik Ghosh

Sumit R. Das
Director of Dissertation

Christopher Crawford
Director of Graduate Studies

July 27, 2020

Dedicated to my parents and sister.

ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Sumit Das for inculcating in me a passion to understand physics deeply. I consider myself very fortunate to have been provided with the opportunity to be mentored by him, especially because of his infectious enthusiasm for physics, constant encouragement and love for precision. I started learning from Sumit when I took a course on statistical mechanics with him in my first year of graduate school, and I cannot see the top yet.

I am also grateful to my other collaborators Antal Jevicki, Henry Maxfield, Kenta Suzuki and Joaquin Turiaci, for their guidance and inspiration. I am especially thankful to Joaquin for innumerable discussions and doubt clarifying sessions on the sixth floor of Broida, which made my time in Santa Barbara very productive. My interaction with David Gross was limited, but inspiring nevertheless.

I consider myself extremely fortunate to have been taught by excellent teachers like Misha Eides, Ribhu Kaul, Ganpathy Murthy, Al Shapere and Joe Straley (during my time in graduate school), and Patrick Dasgupta, Abhinav Gupta, Sanjay Jain, Bikram Phookun, Swarnendu Sarkar, T. R. Seshadri and Vikram Vyas (during my undergraduate days). I am also grateful to Shiraz Minwalla for supervising me during my time in TIFR, in spite of his extremely busy schedule.

I would like to thank Ankur Das and Amartya Saha for their friendship, and keeping company during several all nighters slogging through homework in the first year of graduate school, and Ashish Kakkar, Allic Sivaramakrishnan, Pranjal Nayak, Shaun Hampton, Sotaro Sugishita, Si-nong Liu, Sumiran Pujari and Nisheeta Desai for their camaraderie over the last five years.

I was fortunate to spend a good amount of time on the west coast even though I did not go to school there and make good friends. I would like to thank all the other

Graduate Fellows at the KITP who shared office with me, especially Ryan Farber and Etiénne Lantagne-Hurtubise for several interesting discussions and fun experiences. I would also like to thank Sridip Pal, Ahanjit Bhattacharya, Soumen Ghosh, Shouvik Ganguly and Brato Chakraborti for making my trips to San Diego a lot of fun.

I was encouraged to pursue a career in physics by my high school english teacher Ms. Sudeshna Chakraborti, so this dissertation would be incomplete without mentioning my gratitude towards her.

Finally, I would like to thank my parents and sister for their unwavering support and encouragement.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1
Introduction

Two of the major developments in theoretical physics in the twentieth century have been quantum mechanics (whose relativistic version is described by quantum field theory) and general relativity. These constitute the two pillars on which most of modern theoretical physics is based on. The former describes the physics of microscopic particles while the latter becomes relevant at the cosmological scale which describes the largest entities in the universe like stars and galaxies. Both theories are extremely successful within their regime of validity, but they have resisted unification.

One of the major recent developments in theoretical physics has been to view quantum theory and general relativity as equivalent descriptions of the same physics, a lesson that string theory has taught us via various dualities. This has sparked a lot of interest in trying to develop a theory for quantum gravity which would describe physics at *all* scales.

Even though these discussions are in principle well motivated, one can ask why should we even study quantum gravity. After all, achieving the energy scales at which quantum gravitational effects become relevant ($\sim 10^{19}$ GeV) seems to be ridiculous in comparison to the energy scales that present day colliders can achieve ($\sim 10^{4}$ GeV). However, nature has already given us such high energy colliders for free: the Big Bang. So if we can rewind the cosmological clock, we should encounter an era when it is unavoidable to ignore quantum gravity effects. This is a very interesting and complicated problem.

Another area where quantum gravity effects become important are black holes. Apart from being interesting from an astronomical point of view (since even light cannot escape from their event horizon), a naive application of quantum mechanics and general relativity to them leads to various contradictions. The most famous of these was first noticed by Hawking [1]. He discovered that black holes have a temperature T_H associated with them, which depends on their mass M as

$$T_H = \frac{M_{Pl}}{8\pi M} T_{Pl}, \tag{1.1}$$

where $M_{Pl} \sim 10^{-8}$ kg and $T_{Pl} \sim 10^{32}$ K denote the Planck mass and Planck temperature respectively. This implies that black holes emit thermal radiation which contains no information about the initial state of the black hole when it was formed. This is in contradiction with the basic principle of unitarity which says that information can never be destroyed, and leads to the famous *information paradox*. Thus, resolving this paradox is crucial for understanding quantum gravity. One of the major triumphs of string theory was to provide a microscopic understanding of the black hole entropy [2], which eventually led to a non-perturbative understanding of quantum gravity by means of gauge/gravity duality [3]. It is of fundamental importance in modern theoretical physics since it provides new links between quantum mechanics and gravity based on string theory. Among its many advantages, the duality maps strongly coupled quantum field theories which are generically hard to study to more tractable classical gravity theories.

Gauge/gravity duality is a concrete realization of the *holographic principle*, which was first suggested by Susskind [4] (see also [5]), and is therefore also referred to as holography. The holographic principle states that the entire information content of a theory of quantum gravity in a given volume can be encoded in an effective theory at the boundary surface of this volume. The theory describing the boundary degrees of freedom thus encodes all information about the bulk degrees of freedom and their dynamics, and vice versa. The holographic principle is of a very general nature and is expected to be realized in many examples. In many of these cases, however, the precise form of the boundary theory is unknown, so that it cannot be used to describe the bulk dynamics.

The most concrete realization of gauge/gravity duality is the AdS/CFT correspondence, proposed by Maldacena in 1997 [3]. AdS refers to anti-deSitter space which arises as a solution to Einstein's equations with a negative cosmological constant and CFT refers to conformal field theory. This consisted of a duality between type IIB string theory in an $AdS_5 \times S^5$ background with a $SU(N)$ gauge theory, $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM), living in the four dimensional boundary of AdS (without gravity). The rank of the gauge group $N \sim g_s^{-1}$ (for fixed L/ℓ_s) where g_s refers to the string coupling constant, L refers to the AdS radius and ℓ_s refers to the string length. Since string theory is currently well understood only in the perturbative regime ($g_s \ll 1$), the $\mathcal{N} = 4$ SYM theory is tractable only in the large N limit, sometimes also referred to as the *'t Hooft limit*. In this limit, a $1/N$ expansion on the field theory side corresponds to a genus expansion (in g_s) on the string theory side.

There have been several attempts to generalize the above conjecture to claim that any d-dimensional (conformal) field theory on the *boundary* is dual to a $(d+1)$-dimensional theory of quantum gravity in the *bulk*. One of the extra dimensions in the bulk corresponds to a 'radial' coordinate which is interpreted as an energy scale of the CFT living on the boundary. The deep interior of AdS corresponds to the IR while the boundary corresponds to the UV. Radial evolution in AdS corresponds to a renormalization group (RG) flow in the CFT.

Another important application of AdS/CFT duality is to provide a non-perturbative description of black holes in the bulk, which are considered to be dual to a thermal state in the boundary. The extended geometry of a black hole in AdS is believed to be dual to two identical, non-interacting copies of the conformal field theory (denoted by $\mathrm{CFT_L}$ and $\mathrm{CFT_R}$) and picking a particular entangled state [6]

$$|\mathrm{TFD}\rangle \simeq \sum_n e^{-\beta E_n/2} |n\rangle_{\mathrm{L}} \otimes |n\rangle_{\mathrm{R}}, \tag{1.2}$$

which is called the *thermofield double state*. It purifies the thermal state on one of the boundaries. The Hawking temperature and Bekenstein entropy of the black hole are equal to the boundary temperature and entropy respectively.

To extend the validity of AdS/CFT, we can ask the following question: given a CFT, does the dual gravity theory support black holes? In order to answer this question, we need to understand some basic facts about quantum chaos.

Classical chaos gives a measure of the sensitivity of a dynamical many body system to initial conditions. In a chaotic system, this sensitivity is very high and is quantified by an exponential growth of the derivative of a phase space trajectory $q(t)$ with respect to the initial condition $q(0)$ (which is related to the Poisson bracket) with time

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\} \sim e^{\lambda_L t}, \tag{1.3}$$

where λ_L is called the *Lyapunov exponent* and is a measure of chaos.

The classical discussion can be extended to quantum systems by replacing the phase space by a Hilbert space, classical observables by operators acting on this Hilbert space and Poisson brackets by commutators. In particular, one can consider the thermal expectation value of the commutator squared of two operators $V(t)$ and $W(0)$ (in order to avoid phase cancellations) and see the Lyapunov growth as a characteristic of quantum chaos

$$\langle [V(t), W(0)]^2 \rangle_\beta \sim \frac{1}{N^2} e^{\lambda_L t}, \qquad \langle \mathcal{O} \rangle_\beta \equiv \frac{1}{Z} \mathrm{Tr}[e^{-\beta H} \mathcal{O}], \tag{1.4}$$

where Z denotes the partition function, N^2 roughly corresponds to the number of degrees of freedom (and would be characterized by the central charge of a CFT for example) and β denotes the inverse temperature of the system. The time during which the exponential growth is visible lies in the regime $t_d \ll t \ll t_s$ where $t_d \sim \beta$ denotes the *dissipation time* and $t_s \sim \beta \log N$ denotes the *scrambling time* of the system (when the commutator saturates to a constant value). Since we are usually interested in macroscopic systems, N is very large and there is a clear distinction between the dissipation and scrambling times.

We can expand (1.4) to get two kinds of correlators: time ordered correlators (TOC) or out-of-time-ordered correlators (OTOC). The OTOCs are the interesting ones which are responsible for the Lyapunov growth for chaotic systems. So, the following equation summarizes the statement of quantum chaos for large N systems

$$F(t) \equiv \frac{\langle V^\dagger(t) W^\dagger(0) V(t) W(0) \rangle}{\langle V(t) W(0) \rangle \langle W(t) W(0) \rangle} \sim f_0 - \frac{f_1}{N^2} e^{\lambda_L t} + \dots, \tag{1.5}$$

where f_0 and f_1 denote order one coefficients. In an important paper [7], Maldacena, Shenker and Stanford obtained the following bound on the Lyapunov exponent which under reasonable assumptions is valid for any quantum system

$$\lambda_L \leq \frac{2\pi}{\beta}. \tag{1.6}$$

The bulk interpretation of the exponential growth of the OTOCs was realized by Shenker and Stanford [8, 9], where they interpreted the OTOC as the inner product of two states $|\Psi_i\rangle$ and $|\Psi_f\rangle$ created by acting the operators V and W (both acting on the right CFT) on the thermofield double state (1.2) which is dual to an eternal AdS black hole as follows

$$|\Psi_f\rangle = W(t) V(-t) |\mathrm{TFD}\rangle, \qquad |\Psi_i\rangle = V(-t) W(t) |\mathrm{TFD}\rangle. \tag{1.7}$$

For times lying in between the dissipation and scrambling times, the bulk OTOC is given by

$$F(t) = 1 - \kappa G_N e^{\frac{2\pi}{\beta}t} + \dots \tag{1.8}$$

where κ is some order one positive constant. The Lyapunov behavior arises as a consequence of kinematics, which says that if we throw in a particle of energy E, then the invariant energy of a collision near the horizon is given by $s \sim e^{\frac{2\pi}{\beta}t}$. The holographic dictionary tells us that $G_N \sim 1/N^2$, and by comparing (1.5) and (1.8), we can easily see that black holes in Einstein gravity saturate the chaos bound.

From the above discussion, it is clear that OTOCs play a key role in answering the question of whether the bulk dual of a given CFT contains a black hole. If the OTOC for all operators in the CFT grows with a maximal Lyapunov exponent $\lambda_L = 2\pi/\beta$, we can expect the gravity dual to contain black holes.

However, studies of holography have been hampered by the lack of a model that is simple enough to solve as well as has a gravity dual which contains black holes. For example, matrix models are simple to solve but they do not lead to black holes [10]. On the other hand, $\mathcal{N} = 4$ super Yang Mills theory at strong 't Hooft coupling certainly leads to black holes, and exact results are known at large N for many anomalous dimensions and some vacuum correlation functions, but at finite temperature the theory is difficult to study.

In order to deal with these problems, Kitaev has recently proposed to study a quantum mechanical model of N Majorana fermions interacting with random interactions. This model, which goes by the name of the *Sachdev-Ye-Kitaev (SYK)* model is notable because of the confluence of the following interesting properties:

- **Solvable at strong coupling:** At large N one can sum all Feynman diagrams, and thereby compute correlation functions at strong coupling.

- **Maximally chaotic:** The OTOCs of this model grow exponentially at late times with a maximal value of the Lyapunov exponent. Thus, the dual theory is expected to contain black holes.

- **Emergent conformal symmetry:** There is an emergent reparametrization invariance in the infrared. This (combined with a maximal value for the Lyapunov exponent) implies that the model has some kind of dual description in terms of Einstein gravity.

The combination of these three items makes the SYK model a potential candidate for a solvable model of holography. Since we study this model in a large part of this dissertation, we provide an overview of this model now.

1.1 Overview of the SYK model

The SYK model [11, 12] is a quantum mechanical many body system with all-to-all interactions on fermionic N sites ($N \gg 1$), represented by the Hamiltonian

$$H = (i)^{\frac{q}{2}} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 i_2 \cdots i_q} \, \chi_{i_1} \, \chi_{i_2} \, \cdots \chi_{i_q} \, , \tag{1.9}$$

where q denotes the number of interacting fermions and needs to be even. χ_i denote Majorana fermions, which satisfy the following anticommutation relation

$$\{\chi_i, \chi_j\} = \delta_{ij} \tag{1.10}$$

The coupling constants $J_{i_1 i_2 \cdots i_q}$ are random with a Gaussian distribution with width J.

$$P(J_{i_1 i_2 \cdots i_q}) \propto \exp\left[-\frac{N^{q-1} J^2_{i_1 i_2 \cdots i_q}}{4(q-1)J^2}\right] \tag{1.11}$$

The mean is zero and the variance is given by

$$\langle J^2_{i_1 i_2 \cdots i_q} \rangle = \frac{J^2 (q-1)!}{N^{q-1}} \tag{1.12}$$

It is a simple variant of a model introduced by Sachdev and Ye [13], which was first discussed in relation to holography in [14].

We can perform a disorder averaging of the random couplings (which corresponds to annealed averaging at large N). There is an emergent $O(N)$ symmetry as a consequence of this disorder averaging. Subsequently, we can define a bilocal field $\Psi(\tau_1, \tau_2)$ as

$$\Psi(\tau_1, \tau_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1) \chi_i(\tau_2) \, , \tag{1.13}$$

which leads to the following effective bilocal action

$$S_{\text{col}}[\Psi] = \frac{N}{2} \int d\tau \left[\partial_\tau \Psi(\tau, \tau')\right]_{\tau'=\tau} + \frac{N}{2} \operatorname{Tr} \log \Psi - \frac{J^2 N}{2q} \int d\tau_1 d\tau_2 \left[\Psi(\tau_1, \tau_2)\right]^q , \tag{1.14}$$

which is of order N and thus has a systematic $1/N$ expansion. τ denotes the Euclidean time. It is useful for computing n-point connected correlators of the bilocal field which correspond to $2n$-point connected correlation functions of the original SYK fermions.

To reveal the holographic behavior we focus at low energies (the IR limit) which correspond to $1 \ll \beta J \ll N$ at finite temperature. The first term in (1.14) can be ignored in the IR limit, and this results is an emergent reparametrization symmetry in the IR where the saddle point solution of the bilocal theory transforms as follows under an arbitrary reparametrization $\tau \to f(\tau)$

$$\Psi(\tau_1, \tau_2) \to \Psi_f(\tau_1, \tau_2) \equiv \left| f'(\tau_1) f'(\tau_2) \right|^{\frac{1}{q}} \Psi(f(\tau_1), f(\tau_2)) \, . \tag{1.15}$$

It is easy to recognize that (1.15) looks similar to the transformation of a conformal two point function with scaling dimension $1/q$. This dictates the IR saddle point to have the form

$$\Psi^{(0)}(\tau_1, \tau_2) = b \, \frac{\operatorname{sgn}(\tau_{12})}{|\tau_{12}|^{\frac{2}{q}}} . \tag{1.16}$$

where b is some function of q and J. It is the breaking of this conformal symmetry at finite coupling which is responsible for the IR dynamics and the emergence of a gravitational mode, which has an action

$$S[f] = -\frac{\alpha N}{J}\int d\tau \left[\frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2\right], \tag{1.17}$$

where the quantity in square brackets denotes the Schwarzian derivative of the symmetry mode $f(\tau)$ and α is some order one number.

The connected four point function of the SYK fermions (which is of order $1/N$) is given by the two point function of the quantum fluctuations around the IR saddle point (1.16). This quantity receives contribution from an infinite number of modes. However, all but one of these modes lead to a finite contribution and represent the conformal sector of the SYK model. The "zero-mode" is special since it leads to a divergence in the bilocal two point function, which arises as a consequence of the conformal symmetry in the IR. In order to get a finite answer, we need to treat $(\beta J)^{-1}$ as a perturbation parameter and carry out systematic perturbation theory. This allows us to write the connected bilocal two point function as

$$\mathcal{G}_c(\tau_1,\tau_2;\tau_3,\tau_4) = \frac{\beta J}{N}\,\mathcal{G}^{(-1)}(\tau_1,\tau_2;\tau_3,\tau_4) + \frac{1}{N}\,\mathcal{G}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4) + O\left(\frac{1}{\beta J}\right), \tag{1.18}$$

where $\mathcal{G}^{(-1)}$ denotes the enhanced Schwarzian contribution and $\mathcal{G}^{(0)}$ denotes an $O(1)$ correction to it. Calculating (1.18) up to $O(1)$ will be one of the main goals of Chapter 2.

The above discussion makes it clear that the Schwarzian mode plays an important role in the IR limit, and controls the entropy, free energy and Lyapunov exponent of the model. So we now discuss how this mode emerges from a bulk perspective.

1.2 Holography in the SYK model

The dynamics of the Schwarzian mode $f(\tau)$ arises from the Jackiw-Teitelboim (JT) theory, which is a two dimensional theory of dilaton gravity with a negative cosmological constant. It has the following action (after coupling to matter)

$$S = \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} \Phi(R+2) + \int_{\partial\mathcal{M}} K\Phi_\partial\right] + S_M \tag{1.19}$$

where Φ denotes the dilaton, K denotes the trace of the extrinsic curvature on the boundary of some two dimensional manifold $\mathcal{M}$ (which we denote by $\partial\mathcal{M}$), and S_M denotes some matter action coupled to gravity. The size of the dilaton corresponds to the radius of the S^2 which arises as a consequence of dimensional reduction on $AdS_2 \times S^2$, which describes the near horizon geometry of a near extremal Reissner Nordstrom black hole.

At a classical level, the solution to the dilaton equation of motion sets the geometry to be AdS_2, which has the following metric in Euclidean signature

$$ds^2 = \frac{d\tau^2 + dz^2}{z^2}. \tag{1.20}$$

(1.20) has an isometry group of SL(2,R) and is a good description of the geometry deep in the IR, where the dilaton is a constant Φ_0. However, the asymptotic AdS_2 region suffers from a backreaction problem in the sense that the presence of any nonzero matter destroys the asymptotic geometry [15]. Therefore, we need to study small perturbations of the dilaton around its extremal value Φ_0,

$$\Phi = \Phi_0 + \phi \tag{1.21}$$

This breaks the conformal symmetry (similar to the pattern of breaking of reparametrization symmetry in SYK) and is thus expected to lead to the Schwarzian action (1.17).

In order to see how the Schwarzian action arises in JT gravity, we first consider a boundary trajectory of the form $(\tau, z) \to (f(\tau), z(\tau))$, where τ denotes the boundary time now. The fluctuation ϕ blows up linearly close to the boundary, which forces us to put a holographic cutoff $\epsilon \ll 1$ on the AdS geometry otherwise perturbation theory would break down. Subsequently, if we impose the following boundary conditions on the induced metric and the dilaton

$$g|_{\partial\mathcal{M}} = \epsilon^{-2}, \qquad\qquad \Phi_\partial = \frac{\phi_r}{\epsilon} \tag{1.22}$$

we get $z(\tau) = \epsilon f'(\tau)$ which implies that $f(\tau)$ is the only dynamical degree of freedom in the gravitational model. If we now evaluate the Gibbons-Hawking-York term in (1.19), we get the Schwarzian action

$$S = -\frac{\phi_r}{8\pi G_N} \int d\tau \, \left[\frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2 \right] \tag{1.23}$$

as expected.

The emergent $O(N)$ symmetry after performing a disorder average in SYK makes it analogous to $O(N)$ vector models, which have a holographic description in terms of higher spin gauge fields in an AdS background in one higher dimension [16–22]. This analogy can be used for an explicit implementation of *bilocal holography* in SYK. In this method, we interpret the center of mass and difference coordinates in bilocal space as the time and space coordinates in the Poincaré patch of AdS_2 respectively

$$t = \frac{\tau_1 + \tau_2}{2}, \qquad\qquad z = \frac{\tau_1 - \tau_2}{2}. \tag{1.24}$$

This identification is a consequence of the fact that SL(2,R) symmetry in bilocal space (in the IR limit) is the isometry group of AdS_2. A consequence of this is that the SL(2,R) Casimir in bilocal space is the AdS_2 Laplacian. This identification of symmetry allows us to interpret the bilocal fluctuation two point function as a scalar propagator in AdS_2. We will describe this identification in detail in Chapter 5.

In summary, although we have a bulk understanding of the Schwarzian action and the conformal sector, a complete understanding of the bulk dual of the SYK model is still lacking. The reason for this is that the $O(1)$ correction to the bilocal two point function receives contribution from non conformal matter interacting with the Schwarzian mode, whose bulk interpretation is still unknown.

1.3 Overview of the Dissertation

In Chapter 2, we will focus on the low energy soft mode and present a systematic procedure to extract the Schwarzian dynamics which govern it. This is given in the framework of a perturbative scheme based on a specific (off-shell) breaking of conformal invariance in the UV, adjusted to yield the exact large N saddle point. While this breaking term formally vanishes on-shell, it has a nontrivial effect on correlation functions and the effective action. In particular, it leads to the Schwarzian action with a specific coupling to bilocal matter. The method is applied to the evaluation of $O(1)$ corrections to the correlation function of bi-locals. As a byproduct we confirm precise agreement with the explicit, symmetry breaking procedure. We provide a verification in the large q limit (Liouville theory), where the correlators can be calculated exactly at all length scales. In this case, we can see how the enhanced $O(J)$ and the subleading $O(1)$ contributions originate from the Schwarzian dynamics of the soft mode and its interaction with non conformal bilocal matter.This chapter is based on a paper with S.R. Das, A. Jevicki and K. Suzuki [23].

In Chapter 3, we will show that an extremely generic class of two dimensional CFTs contains a sector described by the Schwarzian theory. This applies to theories with no additional symmetries and large central charge, but does not require a holographic dual. Specifically, we use bootstrap methods to show that in the grand canonical ensemble, at low temperature with a chemical potential sourcing large angular momentum, the density of states and correlation functions are determined by the Schwarzian theory, up to parametrically small corrections. In particular, we compute out-of-time-order correlators in a controlled approximation. For holographic theories, these results have a gravitational interpretation in terms of large, near-extremal rotating BTZ black holes, which have a near horizon throat with nearly $\mathrm{AdS}_2 \times S^1$ geometry. The Schwarzian describes strongly coupled gravitational dynamics in the throat, which can be reduced to Jackiw-Teitelboim (JT) gravity interacting with a $U(1)$ field associated to transverse rotations, coupled to matter. We match the physics in the throat to observables at the AdS_3 boundary, reproducing the CFT results. It is based on a paper with H. Maxfield and G.J. Turiaci [24].

In Chapter 4, we show that the conformal spectrum and bilocal propagator of the SYK model with q fermion interactions can be realized as a three dimensional model in $\mathrm{AdS}_2 \times S^1/Z_2$ with nontrivial boundary conditions in the additional dimension. The three dimensional realization is given on a space whose metric is conformal to $AdS_2 \times S^1/Z_2$ and is subject to a nontrivial potential in addition to a delta function at the center of the interval. In particular, we show that a Horava-Witten compactification reproduces the exact SYK spectrum and a nonstandard propagator between points which lie at the center of the interval exactly agrees with the bilocal propagator. As $q \to \infty$, the wave function of one of the modes at the center of the interval vanish as $1/q$, while the others vanish as $1/q^2$, in a way consistent with the fact that in the SYK model only one of the modes contributes to the bilocal propagator in this limit. This chapter is based on a paper with S.R. Das, A. Jevicki and K. Suzuki [25].

In Chapter 5, we will consider the question of identifying the bulk spacetime of the conformal sector of the SYK model. Focusing on the signature of emergent spacetime of the (Euclidean) model, we explain the need for nonlocal (Radon type) transformations on external legs of n point Green's functions. This results in a dual theory with Euclidean AdS signature with additional leg factors. We speculate that these factors incorporate the coupling of additional bulk states similar to the discrete states of two dimensional string theory.This chapter is based on a paper with S.R. Das, A. Jevicki and K. Suzuki [26].

Some technical results are relegated to the Appendices.

Chapter 2
Near Conformal Perturbation Theory in SYK type Models

In the large N dynamics of models of SYK type [11–13, 26–34] (including tensor models [35**?** –53]), a central role is played by the emergent Schwarzian mode which is dual to the gravitational mode in the dual theory. While the origin of this mode can be traced to (time) reparametrization symmetry of the critical theory, its dynamics and its couplings to the matter degrees of freedom emerges in the near-critical region.

Generally, the low energy dynamics of the soft modes arising from the spontaneous breaking of an approximate symmetry in a quantum field theory is an important problem which appears in many areas of physics. In quantum field theories with a set of global symmetries in many physical situations, one is interested in fluctuations around a classical solution which break some of these symmetries (e.g. a soliton solution [54, 55]). Then a naive perturbation theory around this solution will be infrared divergent because of a zero mode associated with the broken symmetry. The way to deal with this is well known: one needs to introduce collective coordinates whose dynamics provides the essential low energy physics.

In many cases, however, the symmetry is approximate, and there is a parameter λ such that only when $\lambda = 0$ the action has the symmetry. This would cause the saddle point to shift and the zero mode will be lifted to an eigenvalue of order λ. There is no need for a collective coordinate and the answer for e.g. the connected correlation function would be then proportional to λ^{-1} which is large for small λ, hence "enhanced". Nevertheless, we would like to have an understanding of this enhanced contribution in terms of an effective low energy description.

In the present case, the bilocal collective picture offers a framework to study in detail the emergent zero mode dynamics coupled to bilocal matter. In particular in [30] a framework was given for both evaluating the leading Schwarzian action, and also developing a systematic perturbation expansion near criticality. It relied on understanding the mechanism for (spontaneously) broken conformal symmetry in these nonlinear theories: which was introduced in [29] through an off-shell (source) mechanism. A related scheme was also used in [34].

In addition to the Schwarzian action, there is the action of fluctuations with conformal dimension $h = 2$ (which should not be confused with the infinite tower corresponding to operators with conformal dimensions $h_n > 2$) which we call ($h = 2$) "matter" and interaction between this fluctuation field and the Schwarzian mode. The interpretation of these fluctuations in the gravity dual is not clear at this moment. The importance of a full understanding of the emergence of the Schwarzian mode (and even more of its interaction with matter) lies in the fact that this relates to the dual description of the gravitational mode in Jackiw-Teitelboim gravity in two dimensions [56–61], and its various possible interactions with matter. Several possibilities for the dual gravity theory for the SYK model are investigated for example in [25, 62–72] In this chapter we discuss further and complete the scheme of [29, 30] developing fully the Schwarzian-matter coupled description. The key idea is to replace the explicit symmetry breaking term in the SYK action by a regularized $O(1/J)$ source term. A systematic diagrammatic picture is then established, which will allow evaluation of perturbative effects. The singular nature of interactions following from [29] makes the explicit evaluations of contributions nontrivial. One of the improvements provided by the present work will be a clear, systematic procedure for their evaluation (where each diagram will reduce to well defined matrix elements of perturbation theory). Specifically, we prove that once the regularized source is chosen such that one gets the correct large N saddle, this diagrammatic procedure yields the correct $1/J$ expansion of the exact two point correlation function. These results receive a verification as follows.

In the $q \to \infty$ limit, the SYK model can be solved with large N for any finite J. In this limit, the bilocal theory reduces to Liouville theory. The fact that the Dyson-Schwinger equations reduce to those which follow from a Liouville action is well known. Here we show that the action itself reduces to the Liouville action. Correlators in the large q limit are discussed in [12, 32, 73–75].

The situation here is a bit different from the finite q model where an explicit symmetry breaking term in the action for finite J is present. This breaking term plays an essential role in the large q limit. Nevertheless, as is well known, the resulting Liouville theory has an emergent conformal symmetry. The large-N saddle point breaks this symmetry. However, the finite J theory comes with a boundary condition - and the expansion around the saddle point which obeys this boundary condition does not lead to a zero mode since a candidate zero mode does not obey this boundary condition. Therefore a calculation for correlators of fluctuations around this saddle is well defined. This calculation was performed in [12] at finite temperature T and expanded in an expansion in powers of T/J. We perform this calculation at zero temperature in a different way and reproduce the zero temperature limit of the result of [12]. Instead of expanding around this saddle point one could have expanded around the saddle point appropriate to the large J (or long distance) limit. Now one gets a zero mode and the formalism developed above can be applied. We then compare an expansion of the exact result with this collective coordinate calculation and show a perfect agreement as expected.

This chapter is organized as follows: In Section 2.1, we review the bilocal formalism leading to the coupled Schwarzian/(bilocal) matter system and a corresponding diagrammatic scheme. For correlation functions we describe techniques for evaluation of contributing Feynman diagrams using various Schwarzian identities. We apply this to the $O(1)$ correction to the enhanced propagator and show agreement with a perturbative expansion around the exact saddle point solution of the theory. In Section 2.2 we consider the explicit example of large q SYK/ Liouville theory and perform explicit computations implementing the general method. We conclude with some open questions in Section 2.3.

2.1 Schwarzian Dynamics in SYK

2.1.1 The Method

In this subsection, we will give a brief review of our formalism [29, 30] which was introduced to exhibit the Schwarzian mode and generate a perturbative expansion around the conformal point. The Sachdev-Ye-Kitaev model [11] is a quantum mechanical many body system with all-to-all interactions on fermionic N sites ($N \gg 1$), represented by the Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \tag{2.1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The coupling constant J_{ijkl} are random with a Gaussian distribution,

$$P(J_{ijkl}) \propto \exp\left[-\frac{N^3 J_{ijkl}^2}{12 J^2}\right] \tag{2.2}$$

The original model is given by this four-point interaction, with a simple generalization to analogous q-point interacting model [11, 12]. In this chapter, we follow the more general q model, unless otherwise specified.

After the disorder averaging of the random couplings (which corresponds to annealed averaging at large N), the effective action is written as

$$S_q = -\frac{1}{2} \int d\tau \sum_{i=1}^{N} \sum_{a=1}^{n} \chi_i^a \partial_\tau \chi_i^a - \frac{J^2}{2qN^{q-1}} \int d\tau_1 d\tau_2 \sum_{a,b=1}^{n} \left(\sum_{i=1}^{N} \chi_i^a(\tau_1) \chi_i^b(\tau_2) \right)^q, \tag{2.3}$$

where a, b denote the replica indices. Here τ is treated as an Euclidean time. We do not expect a spin glass state in this model [27] and we can restrict to replica diagonal subspace [29]. Therefore, introducing a (replica diagonal) bi-local collective field:

$$\Psi(\tau_1, \tau_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1) \chi_i(\tau_2)\,, \tag{2.4}$$

the model is described by a path-integral

$$Z = \int \prod_{\tau_1,\tau_2} \mathcal{D}\Psi(\tau_1, \tau_2)\; \mu(\Psi)\, e^{-S_{col}[\Psi]}\,, \tag{2.5}$$

with an appropriate $\mathcal{O}(N^0)$ measure μ and the collective action:

$$S_{\rm col}[\Psi] = \frac{N}{2} \int d\tau \Big[\partial_\tau \Psi(\tau, \tau')\Big]_{\tau'=\tau} + \frac{N}{2} \operatorname{Tr}\log\Psi - \frac{J^2 N}{2q} \int d\tau_1 d\tau_2 \Big[\Psi(\tau_1, \tau_2)\Big]^q, \tag{2.6}$$

where the trace log term comes from a Jacobian factor due to the change of path-integral variable, and the trace is taken over the bi-local time. This action being of order N gives a systematic $1/N$ expansion, while the measure μ found as in [76] begins to contribute at one loop level (in $1/N$).

At zero temperature, one can redefine the time τ to get rid of the energy scale J. This also shows that in the IR the theory is strongly coupled and the first term linear in the bi-local field can be dropped. At finite temperature T this redefinition rescales the thermal circle to have dimensionless size J/T, which then becomes the coupling. In this paper we will deal with the zero temperature limit.

In the IR with the strong coupling $J|\tau| \gg 1$, the collective action reduces to the critical action

$$S_{\rm c}[\Psi] \;=\; \frac{N}{2}\,{\rm Tr}\log\Psi \;-\; \frac{J^2N}{2q}\int d\tau_1 d\tau_2 \Big[\Psi(\tau_1,\tau_2)\Big]^q\,, \tag{2.7}$$

which exhibits an emergent conformal reparametrization symmetry $\tau \to f(\tau)$ with

$$\Psi(\tau_1,\tau_2) \;\to\; \Psi_f(\tau_1,\tau_2) \;\equiv\; \Big|f'(\tau_1)f'(\tau_2)\Big|^{\frac{1}{q}}\,\Psi(f(\tau_1),f(\tau_2))\,. \tag{2.8}$$

The first term in (4.5) explicitly breaks this symmetry.

The saddle point solution of the effective action (2.7) gives the critical classical solution,

$$\Psi^{(0)}(\tau_1,\tau_2) \;=\; b\,\frac{{\rm sgn}(\tau_{12})}{|\tau_{12}|^{\frac{2}{q}}}\,, \qquad \Psi^{(0)}_f(\tau_1,\tau_2) \;=\; b\left(\frac{|f'(\tau_1)f'(\tau_2)|}{|f(\tau_1)-f(\tau_2)|^2}\right)^{\frac{1}{q}}\,, \tag{2.9}$$

where $\tau_{ij} \equiv \tau_i - \tau_j$ and b is given by

$$b^q \;=\; \left(\frac{1}{2}-\frac{1}{q}\right)\frac{\tan(\pi/q)}{\pi J^2}\,. \tag{2.10}$$

In general, for a bilocal primary field $\phi_h(\tau_1,\tau_2)$ with the conformal dimension h, the transformation is given by

$$\phi_h(\tau_1,\tau_2) \;\to\; \phi_{h,f}(\tau_1,\tau_2) \;\equiv\; |f'(\tau_1)f'(\tau_2)|^h\phi_h(f(\tau_1),f(\tau_2))\,. \tag{2.11}$$

For an infinitesimal transformation $f(\tau) = \tau + \varepsilon(\tau)$, we get the variation of the field

$$\delta\phi_h(\tau_1,\tau_2) \;=\; \int d\tau\,\varepsilon(\tau)\hat{d}_{h,\tau}(\tau_1,\tau_2)\phi_h(\tau_1,\tau_2) + \frac{1}{2}\int d\tau d\tau'\,\varepsilon(\tau)\varepsilon(\tau')\,\hat{d}^{(2)}_{h,\tau,\tau'}\phi_h(\tau_1,\tau_2) + \cdots\,, \tag{2.12}$$

where

$$\hat{d}_{h,\tau}(\tau_1,\tau_2) \;=\; h\Big(\delta'(\tau_1-\tau)+\delta'(\tau_2-\tau)\Big) + \delta(\tau_1-\tau)\partial_{\tau_1} + \delta(\tau_2-\tau)\partial_{\tau_2}\,, \tag{2.13}$$

and

$$\begin{aligned}
\hat{d}^{(2)}_{h,\tau,\tau'}(\tau_1,\tau_2) \;=&\; 2h^2\delta'(\tau_1-\tau)\delta'(\tau_2-\tau') \\
&+ h(h-1)\Big(\delta'(\tau_1-\tau)\delta'(\tau_1-\tau') + \delta'(\tau_2-\tau)\delta'(\tau_2-\tau')\Big) \\
&+ 2h\Big(\delta'(\tau_1-\tau) + \delta'(\tau_2-\tau)\Big)\Big(\delta(\tau_1-\tau')\partial_{\tau_1} + \delta(\tau_2-\tau')\partial_{\tau_2}\Big) \\
&+ \delta(\tau_1-\tau)\delta(\tau_1-\tau')\partial^2_{\tau_1} + \delta(\tau_2-\tau)\delta(\tau_2-\tau')\partial^2_{\tau_2} \\
&+ 2\delta(\tau_1-\tau)\delta(\tau_2-\tau')\partial_{\tau_1}\partial_{\tau_2}\,, \qquad (2.14)
\end{aligned}$$

The critical saddle point spontaneously breaks the conformal reparametrization symmetry, leading to the appearance of zero modes in the strict IR critical theory. This problem was addressed in [29] in analogy with the quantization of extended systems with symmetry modes [54]. The above symmetry mode representing time reparametrization can be elevated to a dynamical variable introduced according to [55] through the Faddeev-Popov method which we summarize below. We insert into the partition function (5.3), the functional identity:

$$\int \prod_{\tau} \mathcal{D}f(\tau)\; \prod_{\tau} \delta\big(\textstyle\int u\cdot\Psi_f\big) \left|\frac{\delta\left(\int u\cdot\Psi_f\right)}{\delta f}\right| \;=\; 1\,, \qquad (2.15)$$

so that after an inverse change of the integration variable, it results in a combined representation

$$Z \;=\; \int \prod_{\tau}\mathcal{D}f(\tau)\prod_{\tau_1,\tau_2}\mathcal{D}\Psi(\tau_1,\tau_2)\;\mu(\Psi_f)\,\delta\big(\textstyle\int u\cdot\Psi\big)\,e^{-S_{\rm col}[\Psi_f]}\,, \qquad (2.16)$$

with an appropriate Jacobian. Here

$$S_{\rm col}[\Psi_f] = S_c[\Psi] + \frac{N}{2}\int d\tau\,\left[\partial_\tau\Psi_f(\tau,\tau')\right]_{\tau=\tau'} \qquad (2.17)$$

where we have used the invariance of the critical action.

The symmetry breaking term implies a modification of the critical, conformal theory, which we would like to calculate perturbatively (in $1/J$) in a long distance expansion. However since the breaking term is singular and non-vanishing only at *short distances*, the corrected solution cannot be obtained by treating it as an ordinary perturbation. To see this more clearly, let the exact classical solution be

$$\Psi_{\rm cl} = \Psi^{(0)} + \Psi^{(1)} + \cdots \qquad (2.18)$$

Here $\Psi^{(1)}$ is the lowest order shift which should satisfy the linearized equation

$$\int d\tau_3 d\tau_4\,\mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\Psi^{(1)}(\tau_3,\tau_4) \;=\; \partial_1\delta(\tau_{12})\,, \qquad (2.19)$$

where the kernel is given by

$$\begin{aligned}
\mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4) \;=&\; \frac{\delta^2 S_c[\Psi_{(0)}]}{\delta\Psi_{(0)}(\tau_1,\tau_2)\delta\Psi_{(0)}(\tau_3,\tau_4)} \qquad (2.20)\\
=&\; \big[\Psi_{(0)}\big]^{-1}_{\star}(\tau_1,\tau_3)\big[\Psi_{(0)}\big]^{-1}_{\star}(\tau_2,\tau_4) \;+\; (q-1)J^2\,\delta(\tau_{13})\delta(\tau_{24})\,\Psi^{q-2}_{(0)}(\tau_1,\tau_2)\,,
\end{aligned}$$

where the inverse $\big[\Psi_{(0)}\big]_\star^{-1}$ is defined in the sense of the star product (i.e. matrix product): $\int d\tau' A(\tau_1,\tau')\big[A\big]_\star^{-1}(\tau',\tau_2) = \delta(\tau_{12})$ and explicitly given by

$$\big[\Psi_{(0)}\big]_\star^{-1}(\tau_1,\tau_2) \;=\; -\,J^2 b^{q-1}\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2-\frac{2}{q}}}\,. \tag{2.21}$$

Since the kernel transforms under scaling with a dimension $4-\frac{4}{q}$, i.e. $\mathcal{K}^{(0)} \sim |\tau|^{-4+4/q}$, one might expect $\Psi^{(1)}$ to have the form $\Psi^{(1)}(\tau_1,\tau_2) \propto \mathrm{sgn}(\tau_{12})|\tau_{12}|^{-4/q}$ which, however is in disagreement with the desired δ'-source of (2.19). Indeed, one sees that the exact δ'-source could only be matched at the non-perturbative level, where the $1/J$ corrections are all summed up.

Nevertheless, it was shown in [30] that a consistent $1/J$ perturbation theory around the critical solution is possible. The key idea is to *replace* the explicit symmetry breaking term in (4.5) by a regularized source term which is determined as follows. One considers an off-shell extension (of $\Psi^{(1)}$):

$$\Psi_s^{(1)}(\tau_1,\tau_2) \;=\; B_1\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{\frac{2}{q}+2s}}\,, \tag{2.22}$$

with a parameter $s > 0$, because the dimension of $\Psi^{(1)}$ needs to be less than the scaling dimension of $\Psi^{(0)}$. At the end of the calculation we will take the on-shell limit $s \to 1/2$. With this ansatz, we have

$$\int d\tau_3 d\tau_4\,\mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\Psi_s^{(1)}(\tau_3,\tau_4) \;=\; (q-1)B_1 b^{q-2}\gamma(s,q)\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2-\frac{2}{q}+2s}}\,. \tag{2.23}$$

The coefficient γ is uniquely specified by the integral with $\gamma(1/2,q) = 0$ [30], so that the ansatz (2.22) reduces to the homogeneous (on-shell) equation in the limit $s \to 1/2$. The overall coefficient B_1 obviously does not follow from the above, but must be taken from the numerical result found in [12]. In turn, this overall coefficient can be left arbitrary and its value is only fixed after summing the expansion.

The RHS in Eq.(2.23) defines an off-shell regularized source term, which takes the form

$$\begin{aligned} Q_s(\tau_1,\tau_2) \;&\equiv\; \int d\tau_3 d\tau_4\,\mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\Psi_s^{(1)}(\tau_3,\tau_4) \\ &\propto\; (s-\tfrac{1}{2})\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2-\frac{2}{q}+2s}} \;+\; \mathcal{O}\big((s-\tfrac{1}{2})^2\big)\,, \end{aligned} \tag{2.24}$$

and replaces the non-perturbative source in (2.17) through

$$\int\,\big[\Psi_f\big]_s \;\equiv\; -\lim_{s\to\frac{1}{2}}\int d\tau_1 d\tau_2\,\Psi_f(\tau_1,\tau_2)\,Q_s(\tau_1,\tau_2)\,. \tag{2.25}$$

We stress that on-shell (with the limit $s \to 1/2$) so that $\Psi_{1/2}^{(1)}(\tau_1,\tau_2)$ this vanishes. It is a highly nontrivial feature (of this breaking procedure) that it leads to systematic nonzero effects. In particular, separating the critical classical solution from the bilocal field: $\Psi = \Psi^{(0)} + \bar{\Psi}$ we get

$$S_{\rm col}[\Psi, f] \;=\; S[f] \,+\, \lim_{s\to\frac{1}{2}} \int \overline{\Psi}_f \cdot Q_s \,+\, S_{\rm c}[\Psi]\,, \tag{2.26}$$

where $S[f]$ is the action of the collective coordinate

$$S[f] \;=\; \lim_{s\to\frac{1}{2}} \int \, \Psi_f^{(0)} \cdot Q_s\,. \tag{2.27}$$

This can be evaluated by using the Q_s given above. Taking $s \to 1/2$ at the end of the calculation one gets the Schwarzian action [29, 30].

$$S[f] \;=\; -\,\frac{\alpha N}{J} \int d\tau \, \left[\frac{f'''(\tau)}{f'(\tau)} \,-\, \frac{3}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2 \right] , \tag{2.28}$$

where α is a dimensionless coefficient [30]. The limit $s \to 1/2$ is nontrivial because the evaluation of the integral (2.27) leads to a simple pole at $s = 1/2$. This cancels the overall $s - 1/2$ in Q_s to yield a finite result. In this sense we can view the source Q_s as an off-shell extension.

The above strategy can alternatively be described as follows. Suppose we knew the correction to the saddle point solution by e.g. solving the equation of motion following from the SYK action exactly and then performing a long distance expansion. Then we can *define* a source Q_s using (2.24), and replace the symmetry breaking term in the SYK action by the source term $\int d\tau_1 d\tau_2 \; Q_s(\tau_1,\tau_2)\Psi(\tau_1,\tau_2)$. In the rest of this section we will show that once this is done, a well defined perturbation expansion around the critical solution can be developed. The nontrivial aspect of this scheme is that a specification of this off-shell source is all that is needed to obtain a full near-conformal perturbation expansion. We will show for the example of sub-leading corrections for the two point correlation function that the method produces results which agree with a $1/J$ expansion of the exact answer.

2.1.2 Vertices

From (2.16) and (2.26), near the critical theory, we have the partition function

$$Z \;=\; \int \mathcal{D}f \int \mathcal{D}\Psi \, \mu(f,\Psi)\, \delta\big(\textstyle\int u \cdot \Psi_f \big)\, e^{-\left(S[f] \,+\, \lim_{s\to\frac{1}{2}} \int \overline{\Psi}_f \cdot Q_s \,+\, S_{\rm c}[\Psi] \right)} \,, \tag{2.29}$$

where $S[f]$ is given by the Schwarzian action (2.28). This leads to the interaction between the Schwarzian mode and the bilocal matter Ψ.

First taking an infinitesimal transformation $f(\tau) = \tau + \varepsilon(\tau)$, we expand the Schwarzian action (2.28) to obtain

$$S[\varepsilon] \;=\; \frac{\alpha N}{2J} \int d\tau \Big[(\varepsilon'')^2 \,+\, (\varepsilon')^2 \varepsilon''' \,+\, \mathcal{O}(\varepsilon^4) \Big]\,. \tag{2.30}$$

which gives the Schwarzian propagator

$$\big\langle \varepsilon(\tau_1)\varepsilon(\tau_2)\big\rangle \,=\, \frac{J}{12\alpha N}\,|\tau_{12}|^3\,, \qquad \big\langle \varepsilon(\omega)\varepsilon(-\omega)\big\rangle \,=\, \frac{J}{\alpha N}\frac{1}{\omega^4}\,, \tag{2.31}$$

and the cubic vertex

$$V^{(3)}(\tau_1,\tau_2,\tau_3) \,=\, \frac{\alpha N}{6J}\Big[\partial_1^3\partial_2\partial_3 \,+\, (\text{permutations})\Big]\,. \tag{2.32}$$

Because of the projector $\delta\big(\int u\cdot\Psi_f\big)$ which projects out the zero mode, the matter bilocal propagator is given by the non-zero mode propagator $\mathcal{D}_c$ (2.51).

Finally the interaction between the Schwarzian mode and the bilocal matter is given by the interaction term $\int \overline{\Psi}_f \cdot Q_s$. Taking an infinitesimal transformation $f(\tau) = \tau + \varepsilon(\tau)$ with (2.12), we have the interaction term

$$\lim_{s\to\frac{1}{2}}\int d\tau_1 d\tau_2\, Q_s(\tau_{12})\bigg[1 \,+\, \int d\tau\,\varepsilon(\tau)\hat{d}_\tau(\tau_1,\tau_2) \\ +\, \frac{1}{2}\int d\tau d\tau'\,\varepsilon(\tau)\varepsilon(\tau')\,\hat{d}^{(2)}_{\tau,\tau'} \,+\, \cdots\bigg]\overline{\Psi}(\tau_1,\tau_2)\,. \tag{2.33}$$

where we have used a shorthand notation $\hat{d}_\tau \equiv \hat{d}_{\frac{1}{q},\tau}$. This leads to the $\epsilon(\tau)\overline{\Psi}(\tau_1,\tau_2)$ vertex:

$$Q_s(\tau_{12})\hat{d}_\tau(\tau_1,\tau_2)\,, \tag{2.34}$$

the $\epsilon(\tau)\epsilon(\tau')\overline{\Psi}(\tau_1,\tau_2)$ vertex:

$$\frac{1}{2}\,Q_s(\tau_{12})\hat{d}^{(2)}_{\tau,\tau'}(\tau_1,\tau_2)\,, \tag{2.35}$$

and so on.

2.1.3 Critical SYK eigenvalue problem

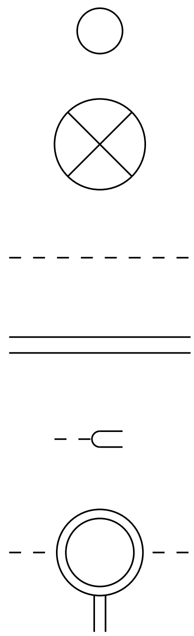
Now we focus on the eigenvalue problem of the critical theory. We have the conformal symmetry: $\tau \to f(\tau)$ in the critical theory (i.e. strict large J):

$$\frac{\delta S_c[\Psi^{(0)}]}{\delta f(\tau)} \,=\, 0\,, \tag{2.36}$$

where $S[\Psi^{(0)} + J^{-1}\Psi^{(1)} + \cdots] = S_c[\Psi^{(0)}] \,+\, \cdots$. This leads to a zero mode in the strict large J limit:

$$\chi^{(0)}_{0,\omega}(\tau_1,\tau_2) \,=\, \int d\tau\, e^{i\omega\tau}\chi^{(0)}_{0,\tau}(\tau_1,\tau_2)\,, \qquad \chi^{(0)}_{0,\tau}(\tau_1,\tau_2) \,=\, \big(N^{(0)}_0\big)^{-\frac{1}{2}}\,\frac{\delta\Psi^{(0)}_f(\tau_1,\tau_2)}{\delta f(\tau)}\bigg|_{f(\tau)=\tau}\,, \tag{2.37}$$

with $\lambda^{(0)}_{0,\omega} = 0$. The normalization factor $N^{(0)}_0$ for the second equation is fixed by the orthonormality condition of the eigenfunction $\chi^{(0)}$ (2.41). This zero mode leads to



$$= \chi_0^{(0)} \propto 1$$

$$= Q_s \propto N/J$$

$$= \langle \varepsilon\varepsilon \rangle \propto J/N$$

$$= \mathcal{D}_c \propto N^{-1}$$

$$= \hat{d} \propto 1$$

$$= \hat{d}^{(2)} \propto 1$$

Figure 2.1: Diagrammatic expressions for the Feynman rules.

the enhanced contribution of the Green's function $G^{(-1)}$, which will be discussed in Section 2.1.4 as well as in Appendix A. Also for the notation we will use below for the perturbation theory, readers are referred to Appendix A.

For general modes, the second term in (2.20) is already diagonal, but contains an extra factor $\Psi_{(0)}^{q-2}$. Because of this extra factor, the eigenvalue problem of this kernel must be like

$$\int d\tau_3 d\tau_4 \, \mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\, \chi_{n,\omega}^{(0)}(\tau_3,\tau_4) \;=\; \lambda_{n,\omega}^{(0)}\, \Psi_{(0)}^{q-2}(\tau_1,\tau_2)\, \chi_{n,\omega}^{(0)}(\tau_1,\tau_2)\,. \tag{2.38}$$

The extra factor appearing in the RHS can be compensated by defining new eigenfunctions by $\chi^{(0)} = \Psi_{(0)}^{1-q/2}\, \tilde{\chi}^{(0)}$ and a new kernel [12, 28, 29] by

$$\widetilde{\mathcal{K}}(\tau_1,\tau_2;\tau_3,\tau_4) \;=\; \Psi_{(0)}^{1-\frac{q}{2}}(\tau_1,\tau_2)\, \mathcal{K}(\tau_1,\tau_2;\tau_3,\tau_4)\, \Psi_{(0)}^{1-\frac{q}{2}}(\tau_3,\tau_4)\,, \tag{2.39}$$

so that the new kernel and eigenfunctions obey the standard eigenvalue problem

$$\int d\tau_3 d\tau_4 \, \widetilde{\mathcal{K}}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\, \tilde{\chi}_{n,\omega}^{(0)}(\tau_3,\tau_4) \;=\; \lambda_{n,\omega}^{(0)}\, \tilde{\chi}_{n,\omega}^{(0)}(\tau_1,\tau_2)\,. \tag{2.40}$$

We note that the eigenvalue does not change under this new definition. Since the new eigenfunctions $\tilde{\chi}^{(0)}$ obey the standard eigenvalue problem, we impose the orthonormality

$$\int d\tau_1 d\tau_2 \, \tilde{\chi}_{n,\omega}^{(0)}(\tau_1,\tau_2) \tilde{\chi}_{n',\omega'}^{(0)}(\tau_1,\tau_2) \;=\; \delta(n-n')\delta(\omega-\omega')\,, \tag{2.41}$$

and the completeness

$$\sum_{n,\omega} \tilde{\chi}_{n,\omega}^{(0)}(\tau_1,\tau_2) \tilde{\chi}_{n,\omega}^{(0)}(\tau_3,\tau_4) \;=\; \delta(\tau_{13})\delta(\tau_{24})\,. \tag{2.42}$$

For the eigenvalue problem (2.38), we denote that $\chi^{(0)}_{n,\omega} = \chi^{(0)}_{h,\omega}$ and $\lambda^{(0)}_{n,\omega} = \lambda^{(0)}_q(h)$ by using the fact that the complete eigenfunctions are spanned by the conformal dimension h containing the continuous modes ($h = 1/2 + ir$, $0 < r < \infty$) and the discrete modes ($h = 2n$, $n = 0, 1, 2, \cdots$) with ω ($-\infty < \omega < \infty$) [12, 28]. The eigenvalue is known to be independent of ω and we can parametrize it as

$$\lambda^{(0)}_q(h) \;=\; (q-1)J^2\big(1 - \widetilde{g}(q;h)\big)\,, \tag{2.43}$$

which excludes the trivially diagonalized second term in (2.20). Therefore, the nontrivial eigenvalue problem becomes

$$\begin{aligned}&\int d\tau_3 d\tau_4 \,\big[\Psi_{(0)}\big]^{-1}_{\star}(\tau_1,\tau_3)\big[\Psi_{(0)}\big]^{-1}_{\star}(\tau_2,\tau_4)\,\chi^{(0)}_{h,\omega}(\tau_3,\tau_4)\\ &= -\,(q-1)J^2\,\widetilde{g}(q;h)\,\Psi^{q-2}_{(0)}(\tau_1,\tau_2)\,\chi^{(0)}_{h,\omega}(\tau_1,\tau_2)\,,\end{aligned} \tag{2.44}$$

where the explicit form of the inverse is given in (2.21).

Next we note that the eigenvalue problem of the following form was solved in [12]:

$$-\frac{1}{\alpha_0(q)}\int d\tau_3 d\tau_4\,\frac{\mathrm{sgn}(\tau_{13})\mathrm{sgn}(\tau_{24})}{|\tau_{13}|^{\frac{2}{q}}|\tau_{24}|^{\frac{2}{q}}|\tau_{34}|^{2-\frac{4}{q}}}\;\tilde{\tilde{\chi}}^{(0)}_{h,\omega}(\tau_3,\tau_4) \;=\; k_c(q;h)\,\tilde{\tilde{\chi}}^{(0)}_{h,\omega}(\tau_1,\tau_2)\,, \tag{2.45}$$

where the eigenfunctions are independent of q with

$$\alpha_0(q) \;=\; \frac{1}{(q-1)J^2 b^q}\,, \tag{2.46}$$

and the eigenvalue

$$k_c(q;h) \;=\; -(q-1)\,\frac{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})\Gamma(\frac{1}{q}+\frac{h}{2})\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})\Gamma(1-\frac{1}{q}+\frac{h}{2})}\,. \tag{2.47}$$

We note that this eigenvalue has a symmetry

$$k_c\big(\tfrac{q}{q-1};h\big) \;=\; \frac{1}{k_c(q;h)}\,. \tag{2.48}$$

Therefore, from (2.45) transforming $q \to q/(q-1)$, we obtain

$$-\frac{1}{\alpha_0(\frac{q}{q-1})}\int d\tau_3 d\tau_4\,\frac{\mathrm{sgn}(\tau_{13})\mathrm{sgn}(\tau_{24})}{|\tau_{13}|^{2-\frac{2}{q}}|\tau_{24}|^{2-\frac{2}{q}}|\tau_{34}|^{\frac{4}{q}-2}}\;\tilde{\tilde{\chi}}^{(0)}_{h,\omega}(\tau_3,\tau_4) \;=\; \frac{1}{k_c(q;h)}\,\tilde{\tilde{\chi}}^{(0)}_{h,\omega}(\tau_1,\tau_2)\,. \tag{2.49}$$

This is related to our eigenvalue problem (2.44) by $\tilde{\tilde{\chi}}^{(0)} = \Psi^{q-2}_{(0)}\,\chi^{(0)}$. Therefore comparing these expressions, we find

$$\widetilde{g}(q;h) \;=\; \frac{\alpha_0(\frac{q}{q-1})}{(q-1)^2\alpha_0(q)}\,\frac{1}{k_c(q;h)} \;=\; \frac{1}{k_c(q;h)}\,. \tag{2.50}$$

After determining the eigenfunctions [12, 28, 29], one can obtain the non-zero mode bilocal propagator $\mathcal{D}_c$. This propagator contains three types of contributions:

$$\mathcal{D}_c \;=\; \mathcal{D}' \,+\, \mathcal{D}'' \,+\, \sum_{m=1}^{\infty} \mathcal{D}_m \,. \tag{2.51}$$

The $h = 2$ contributions consist of a non vanishing single-pole $\mathcal{D}'$ and a gauge-dependent $\mathcal{D}''$ contribution. Detailed evaluation of these are given in Section 2.2.4 and Appendix D. The former is given by

$$\mathcal{D}'(t,z;t',z') \;=\; \frac{|zz'|^{\frac{1}{2}}}{N\tilde{g}'(\frac{3}{2})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \frac{3\pi}{2}\, J_{-\frac{3}{2}}(|\omega z^{>}|) J_{\frac{3}{2}}(|\omega| z^{<})\,, \tag{2.52}$$

while the latter contribution is given by

$$\mathcal{D}''(t,z;t',z') \;=\; -\,\frac{|zz'|^{\frac{1}{2}}}{N\tilde{g}'(\frac{3}{2})} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \left(3\partial_\nu + 2 - \frac{3\tilde{g}''(\frac{3}{2})}{2\tilde{g}'(\frac{3}{2})}\right) \Big(J_\nu(|\omega z|) J_\nu(|\omega z'|)\Big)_{\nu=\frac{3}{2}}\,,$$

where we used the change of variables in (2.106). The other contributions are given by single-poles located at $h = p_m + 1/2 > 2$. [29]

$$\mathcal{D}_m(t,z;t',z') \;=\; -\,\frac{|zz'|^{\frac{1}{2}}}{N\tilde{g}'(p_m)} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \frac{\pi p_m}{\sin(\pi p_m)}\, Z_{-p_m}(|\omega z^{>}|) J_{p_m}(|\omega| z^{<})\,. \tag{2.53}$$

2.1.4 Enhanced contribution: $G^{(-1)}$

In this and next subsections, we consider the bilocal two-point function:

$$\Big\langle \Psi(\tau_1,\tau_2)\Psi(\tau_3,\tau_4) \Big\rangle\,, \tag{2.54}$$

where the expectation value is evaluated by the path integral (5.3). After the Faddeev-Popov procedure and changing the integration variable as we discussed in Section **??**, this two-point function becomes

$$\Big\langle \Psi_f(\tau_1,\tau_2)\Psi_f(\tau_3,\tau_4) \Big\rangle\,, \tag{2.55}$$

where now the expectation value is evaluated by the gauged path integral (2.16). We expand the bilocal field around the critical saddle-point solution $\Psi = \Psi^{(0)} + \overline{\Psi}$, so that the two-point function is now

$$\Big\langle \Psi_f(\tau_1,\tau_2)\Psi_f(\tau_3,\tau_4) \Big\rangle \;=\; \Big\langle \big(\Psi_f^{(0)} + \overline{\Psi}_f\big)(\tau_1,\tau_2)\big(\Psi_f^{(0)} + \overline{\Psi}_f\big)(\tau_3,\tau_4) \Big\rangle\,. \tag{2.56}$$

Taking an infinitesimal transformation $f(\tau) = \tau + \varepsilon(\tau)$ with (2.12), we find the order $\mathcal{O}(J)$ contribution

$$G^{(-1)}(\tau_1,\tau_2;\tau_3,\tau_4) \;=\; \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\, \big(N_{0,\omega}^{(0)} N_{0,\omega'}^{(0)}\big)^{\frac{1}{2}}\, \chi_{0,\omega}^{(0)}(\tau_1,\tau_2)\, \big\langle \epsilon(\omega)\epsilon(\omega') \big\rangle\, \chi_{0,\omega'}^{(0)}(\tau_3,\tau_4)\,, \tag{2.57}$$

where we used the zero mode expression (2.37) With the explicit form of the Schwarzian propagator this leading contribution is of order $\mathcal{O}(J)$.

Following the standard perturbation theory, (which is described in detail in Appendix A), the exact Green's function can be written as

$$\widetilde{G}_{(\mathrm{ex})} = \sum_n \frac{\tilde{\chi}_n^{(\mathrm{ex})}\tilde{\chi}_n^{(\mathrm{ex})}}{\lambda_n^{(\mathrm{ex})}} \equiv \Psi_{\mathrm{cl}}^{\frac{q}{2}-1}\, G_{(\mathrm{ex})}\, \Psi_{\mathrm{cl}}^{\frac{q}{2}-1}. \tag{2.58}$$

In the following, we will be interested in the contribution coming from the zero mode of the lowest order kernel, $n=0$. Since $\lambda_0^{(0)}=0$, we have

$$G^{(-1)} = \frac{\chi_0^{(0)}\chi_0^{(0)}}{\lambda_0^{(1)}} \tag{2.59}$$

$$G^{(0)} = \frac{\chi_0^{(0)}\chi_0^{(1)}}{\lambda_0^{(1)}} + \frac{\chi_0^{(1)}\chi_0^{(0)}}{\lambda_0^{(1)}} - \frac{\lambda_0^{(2)}\chi_0^{(0)}\chi_0^{(0)}}{(\lambda_0^{(1)})^2} + \mathcal{D}_c, \qquad \mathcal{D}_c = \sum_{n\neq 0} \frac{\chi_n^{(0)}\chi_n^{(0)}}{\lambda_n^{(0)}}. \tag{2.60}$$

Here we suppressed all τ (and ω) dependence since they don't play any crucial role here. The expression of the perturbative kernels are obtained as (see Appendix A)

$$\mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4) = S_c^{(2)}(\tau_{1,2};\tau_{3,4}), \tag{2.61}$$

$$\mathcal{K}^{(1)}(\tau_1,\tau_2;\tau_3,\tau_4) = \int d\tau_5 d\tau_6\, S_c^{(3)}(\tau_{1,2};\tau_{3,4};\tau_{5,6})\,\Psi^{(1)}(\tau_{56}), \tag{2.62}$$

$$\begin{aligned}\mathcal{K}^{(2)}(\tau_1,\tau_2;\tau_3,\tau_4) &= \frac{1}{2}\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\, S_c^{(4)}(\tau_{1,2};\tau_{3,4};\tau_{5,6};\tau_{7,8})\,\Psi^{(1)}(\tau_{56})\Psi^{(1)}(\tau_{78})\\ &\quad + \int d\tau_5 d\tau_6\, S_c^{(3)}(\tau_{1,2};\tau_{3,4};\tau_{5,6})\,\Psi^{(2)}(\tau_{56}),\end{aligned} \tag{2.63}$$

where we used the short-hand notation for $S_c^{(n)}$ defined in (A.17). The first and second order eigenvalue and eigenfunction corrections are given by

$$\lambda_0^{(1)} = \int \chi_0^{(0)}\cdot\mathcal{K}^{(1)}\cdot\chi_0^{(0)}, \tag{2.64}$$

$$\chi_0^{(1)} = -\sum_{k\neq 0}\frac{\chi_k^{(0)}}{\lambda_k^{(0)}}\int \chi_k^{(0)}\cdot\mathcal{K}^{(1)}\cdot\chi_0^{(0)}, \tag{2.65}$$

$$\lambda_0^{(2)} = -\sum_{k\neq 0}\frac{1}{\lambda_k^{(0)}}\left|\int \chi_k^{(0)}\cdot\mathcal{K}^{(1)}\cdot\chi_0^{(0)}\right|^2 + \int \chi_0^{(0)}\cdot\mathcal{K}^{(2)}\cdot\chi_0^{(0)}, \tag{2.66}$$

where for the first order shift of the zero-mode eigenfunction $\chi_0^{(1)}$ we used (A.22).

In the rest of this subsection, we relate the diagrammatic expression of $G^{(-1)}$ in (2.57) to the eigenvalue perturbation expression (2.59).

Now we note that the Schwarzian action is given by

$$S_{\mathrm{Sch}}[f] = -\lim_{s\to 1/2}\int d\tau_1 d\tau_2\, Q_s(\tau_1,\tau_2)\,\Psi_f^{(0)}(\tau_1,\tau_2) \tag{2.67}$$

Expanding $f(\tau) = \tau + \epsilon(\tau)$ we know that in the $s \to 1/2$ limit, the first nonzero term in an expansion in $\epsilon(\tau)$ is the term which is quadratic in $\epsilon(\tau)$ and given by

$$S^{(2)}_{\rm Sch}[\epsilon] = \int d\tau_1 d\tau_2 \; Q_s(\tau_1,\tau_2) \int \frac{d\omega}{2\pi}\, \epsilon(\omega)\epsilon(-\omega)\hat{d}^{(2)}_{\omega,-\omega}\Psi^{(0)} \,. \tag{2.68}$$

For the zero mode (2.37) with $\lambda^{(0)}_{0,\omega} = 0$, the eigenvalue problem is written as

$$0 \;=\; \int d\tau_3 d\tau_4 \, \frac{\delta^2 S_c[\Psi^{(0)}_f]}{\delta\Psi^{(0)}_f(\tau_1,\tau_2)\delta\Psi^{(0)}_f(\tau_3,\tau_4)} \, \frac{\delta\Psi^{(0)}_f(\tau_3,\tau_4)}{\delta f(\tau)} \,. \tag{2.69}$$

Taking one more variation respect to $f(\tau')$, we obtain

$$\begin{aligned} &\int d\tau_3 d\tau_4 d\tau_5 d\tau_6 \, \frac{\delta^3 S_c[\Psi^{(0)}]}{\delta\Psi^{(0)}(\tau_1,\tau_2)\delta\Psi^{(0)}(\tau_3,\tau_4)\delta\Psi^{(0)}(\tau_5,\tau_6)} \, \chi^{(0)}_{0,\omega}(\tau_3,\tau_4)\chi^{(0)}_{0,\omega'}(\tau_5,\tau_6) \\ &= -\left(N^{(0)}_{0,\omega}N^{(0)}_{0,\omega'}\right)^{-\frac{1}{2}} \int d\tau_3 d\tau_4 \, \mathcal{K}^{(0)}(\tau_1,\tau_2;\tau_3,\tau_4)\hat{d}^{(2)}_{\omega,\omega'}\Psi^{(0)}(\tau_3,\tau_4) \,, \end{aligned} \tag{2.70}$$

where after the variation we set $f(\bullet) = \bullet$ and then Fourier transformed τ to ω (and τ' to ω'). Multiplying $\Psi^{(1)}(\tau_1,\tau_2)$ and integrating, the first term becomes $\mathcal{K}^{(1)}$ (2.62). Therefore, now we find

$$\frac{1}{J} \int \chi^{(0)}_{0,\omega} \cdot \mathcal{K}^{(1)} \cdot \chi^{(0)}_{0,\omega'} \;=\; -\left(N^{(0)}_{0,\omega}N^{(0)}_{0,\omega'}\right)^{-\frac{1}{2}} \int Q_s \cdot \left(\hat{d}^{(2)}_{\omega,\omega'}\Psi^{(0)}\right) . \tag{2.71}$$

Then, the first order zero-mode eigenvalue correction $\lambda^{(1)}_{0,\omega}$ (2.64) is written as

$$\frac{\lambda^{(1)}_{0,\omega}}{J}\,\delta(\omega+\omega') \;=\; -\left(N^{(0)}_{0,\omega}\right)^{-1} \int d\tau_1 d\tau_2 \, Q_s(\tau_1,\tau_2)\left(\hat{d}^{(2)}_{\omega,-\omega}\Psi^{(0)}\right)(\tau_1,\tau_2) \,. \tag{2.72}$$

Now this equation immediately shows that

$$\frac{J}{\lambda^{(1)}_{0,\omega}}\,\delta(\omega+\omega') \;=\; \frac{N^{(0)}_{0,\omega}}{2\pi}\left\langle \epsilon(\omega)\epsilon(\omega')\right\rangle . \tag{2.73}$$

Therefore, $\lambda^{(1)}_0$ is the quadratic kernel of the Schwarzian action.

Hence, using this relationship, we relate the diagrammatic expression (2.57) to the perturbation expression (2.59) as

$$G^{(-1)}(\tau_1,\tau_2;\tau_3,\tau_4) \;=\; J \int \frac{d\omega}{2\pi} \, \frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,-\omega}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}} \,. \tag{2.74}$$

2.1.5 Diagrams for $G^{(0)}$

Next we consider the subleading $O(J^0)$ contribution $G^{(0)}$. Besides the non-zero mode contribution $\mathcal{D}_c$, the contributions to $G^{(0)}$ given by four diagrams(in Fig 2)

$b_1 =$

$b_2 =$

$b_3 =$

$b_4 =$

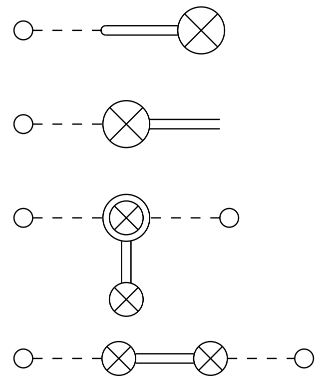


Figure 2.2: Diagrammatic expressions for the $\mathcal{O}(J^0)$ contributions of the Green's function.

that arise from interactions between the Schwarzian mode and the bilocal field, and also the Schwarzian redefinition of the field itself:

$$G^{(0)} = b_1 + b_2 + b_3 + b_4 + \mathcal{D}_c\,. \tag{2.75}$$

The corresponding Feynman diagrams are shown in Figure 2.1 and they are given by

$$\begin{aligned}
b_1 &\equiv \left\langle \int \frac{d\omega}{2\pi}\,\varepsilon(\omega)\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\int \frac{d\omega'}{2\pi}\,\varepsilon(\omega')\hat{d}_{\frac{1}{q},\omega'}(\tau_3,\tau_4)\int d\tau_5 d\tau_6\,\mathcal{D}_c(\tau_3,\tau_4;\tau_5,\tau_6)\,Q_s(\tau_{56})\right\rangle,\\
b_2 &\equiv \left\langle \int \frac{d\omega}{2\pi}\,\varepsilon(\omega)\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\int d\tau_5 d\tau_6\,Q_s(\tau_{56})\int \frac{d\omega'}{2\pi}\,\varepsilon(\omega')\hat{d}_{\frac{1}{q},\omega'}(\tau_5,\tau_6)\mathcal{D}_c(\tau_3,\tau_4;\tau_5,\tau_6)\right\rangle,\\
b_3 &\equiv \frac{1}{2}\left\langle \int \frac{d\omega_a}{2\pi}\,\varepsilon(\omega_a)\chi^{(0)}_{0,\omega_a}(\tau_1,\tau_2)\int \frac{d\omega_b}{2\pi}\frac{d\omega_c}{2\pi}\,\varepsilon(\omega_b)\varepsilon(\omega_c)\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\,Q_s(\tau_{56})\right.\\
&\qquad\left.\times\,\hat{d}^{(2)}_{\frac{1}{q},\omega_b,\omega_c}(\tau_5,\tau_6)\mathcal{D}_c(\tau_5,\tau_6;\tau_7,\tau_8)\,Q_s(\tau_{78})\int \frac{d\omega_d}{2\pi}\,\varepsilon(\omega_d)\chi^{(0)}_{0,\omega_d}(\tau_3,\tau_4)\right\rangle, \qquad (2.76)\\
b_4 &\equiv \left\langle \int \frac{d\omega_a}{2\pi}\,\varepsilon(\omega_a)\chi^{(0)}_{0,\omega_a}(\tau_1,\tau_2)\int d\tau_5 d\tau_6\,Q_s(\tau_{56})\int \frac{d\omega_b}{2\pi}\,\varepsilon(\omega_b)\hat{d}_{\frac{1}{q},\omega_b}(\tau_5,\tau_6)\right\rangle\\
&\quad\times\left\langle \int \frac{d\omega_c}{2\pi}\,\varepsilon(\omega_c)\chi^{(0)}_{0,\omega_c}(\tau_3,\tau_4)\int d\tau_7 d\tau_8\,Q_s(\tau_{78})\int \frac{d\omega_d}{2\pi}\,\varepsilon(\omega_d)\hat{d}_{\frac{1}{q},\omega_d}(\tau_7,\tau_8)\right\rangle\mathcal{D}_c(\tau_5,\tau_6;\tau_7,\tau_8)\,,
\end{aligned}$$

where we defined $\hat{d}_{h,\omega}$ is the Fourier transform of $\hat{d}_{h,\tau}$ defined in Eq.(2.13). In this subsection, we always omit the normalization factor $N^{(0)}_{0,\omega}$ introduced through (2.37) in order to simplify the notation. This factor does not play any crucial role here and the final expressions are independent of this factor. For b_1 and b_2, there are also $(\tau_{1,2} \leftrightarrow \tau_{3,4})$ contributions we did not write explicitly above. We note that the integration by parts of this operator is given by a shadow transform:

$$\int d\tau_1 d\tau_2\,A(\tau_1,\tau_2)\hat{d}_{h,\tau}(\tau_1,\tau_2)B(\tau_1,\tau_2) = -\int d\tau_1 d\tau_2\,B(\tau_1,\tau_2)\hat{d}_{1-h,\tau}(\tau_1,\tau_2)A(\tau_1,\tau_2)\,. \tag{2.77}$$

From the definition of this operator (2.13), also the action of this operator onto a product can be written as

$$\begin{aligned}&\hat{d}_{h_A+h_B,\tau}(\tau_1,\tau_2)\Big(A(\tau_1,\tau_2)B(\tau_1,\tau_2)\Big)\\ &= B(\tau_1,\tau_2)\hat{d}_{h_A,\tau}(\tau_1,\tau_2)A(\tau_1,\tau_2) + A(\tau_1,\tau_2)\hat{d}_{h_B,\tau}(\tau_1,\tau_2)B(\tau_1,\tau_2)\,. \end{aligned}\tag{2.78}$$

Now we evaluate the diagrams. For b_1, using (2.24) and (2.73), we find

$$\begin{aligned} b_1 &= \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\,\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\,\big\langle\varepsilon(\omega)\varepsilon(\omega')\big\rangle\,\hat{d}_{\frac{1}{q},\omega'}(\tau_3,\tau_4)\Psi^{(1)}(\tau_3,\tau_4)\\ &= J\int\frac{d\omega}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)}{\lambda^{(1)}_{0,\omega}}\,\big(\hat{d}_{\frac{1}{q},-\omega}\Psi^{(1)}\big)(\tau_3,\tau_4)\,. \end{aligned}\tag{2.79}$$

For b_2, first using (2.60) and (2.73) we have

$$b_2 = J\int\frac{d\omega}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)}{\lambda^{(1)}_{0,\omega}}\int\frac{d\omega'}{2\pi}\sum_{k\neq 0}\frac{\chi^{(0)}_{k,-\omega'}(\tau_3,\tau_4)}{\lambda^{(0)}_{k,\omega'}}\int d\tau_5 d\tau_6\,Q_s(\tau_{56})\big(\hat{d}_{\frac{1}{q},-\omega}\chi^{(0)}_{k,\omega'}\big)(\tau_5,\tau_6)\,.\tag{2.80}$$

Now following the same manipulation as in (2.69) - (2.71), but now for non-zero mode, we find

$$\frac{1}{J}\int\chi^{(0)}_{0,\omega}\cdot\mathcal{K}^{(1)}\cdot\chi^{(0)}_{k,\omega'} + \int Q_s\cdot\Big(\hat{d}_{\frac{1}{q},\omega}\chi^{(0)}_{k,\omega'}\Big) = \lambda^{(0)}_{k,\omega'}\int\Psi^{(1)}\cdot\hat{d}_{1-\frac{1}{q},\omega}\Big(\Psi^{q-2}_{(0)}\chi^{(0)}_{k,\omega'}\Big)\,.\tag{2.81}$$

For the RHS, we can move $\hat{d}$ onto $\Psi_{(1)}$ by the shadow transform (2.77). Using this relation, b_2 is now written as

$$\begin{aligned} b_2 = &-J\int\frac{d\omega}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)}{\lambda^{(1)}_{0,\omega}}\,\big(\hat{d}_{\frac{1}{q},-\omega}\Psi_{(1)}\big)(\tau_3,\tau_4)\\ &-\int\frac{d\omega}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)}{\lambda^{(1)}_{0,\omega}}\int\frac{d\omega'}{2\pi}\sum_{k\neq0}\frac{\chi^{(0)}_{k,-\omega'}(\tau_3,\tau_4)}{\lambda^{(0)}_{k,\omega'}}\int\chi^{(0)}_{k,\omega'}\cdot\mathcal{K}^{(1)}\cdot\chi^{(0)}_{0,-\omega}\,, \end{aligned}\tag{2.82}$$

where for the first term, we used the completeness of $\tilde{\chi}^{(0)}$ (2.42). The first term cancels with b_1, while the second term combines into the first order shift of the eigenfunctions (2.65). Therefore, we find

$$b_1 + b_2 = \int\frac{d\omega}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(1)}_{0,-\omega}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}} + (\tau_{1,2}\leftrightarrow\tau_{3,4})\,,\tag{2.83}$$

where we implemented the explicit symmetrization of the external legs.
For b_3, using (2.24) and (2.73), we find

$$b_3 = \frac{J^2}{2}\int\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\,\frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,\omega'}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\lambda^{(1)}_{0,\omega'}}\,S^{(2)}_c\Big[\Psi^{(1)};\hat{d}^{(2)}_{h,-\omega,-\omega'}\Psi^{(1)}\Big]\,.\tag{2.84}$$

Now we consider the equation of motion of $\Psi^{(1)}$ (2.24) with the on-shell limit ($s \to 1/2$): $\int \mathcal{K}_f^{(0)} \cdot \Psi_f^{(1)} = 0$, where we transformed $\Psi \to \Psi_f$. Taking variations respect to $f(\tau)$ and $f(\tau')$, then multiplying another $\Psi^{(1)}$, we find

$$\begin{aligned} \int \Psi^{(1)} \cdot \mathcal{K}^{(0)} \cdot \left(\hat{d}^{(2)}_{\frac{1}{q},\tau,\tau'} \Psi^{(1)} \right) &= -\frac{1}{J} \int \chi^{(0)}_{0,\tau} \cdot \mathcal{K}^{(1)} \cdot \left(\hat{d}_{\frac{1}{q},\tau'} \Psi^{(1)} \right) - (\tau \leftrightarrow \tau') \\ &\quad - \frac{2}{J^2} \int \chi^{(0)}_{0,\tau} \cdot \mathcal{K}^{(2)} \cdot \chi^{(0)}_{0,\tau'} \,. \end{aligned} \tag{2.85}$$

Therefore, we can write b_3 as

$$\begin{aligned} b_3 &= -\int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,\omega'}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\,\lambda^{(1)}_{0,\omega'}} \\ &\quad \times \left[J \int \chi^{(0)}_{0,-\omega} \cdot \mathcal{K}^{(1)} \cdot \left(\hat{d}_{\frac{1}{q},-\omega'} \Psi^{(1)} \right) + \int \chi^{(0)}_{0,\tau} \cdot \mathcal{K}^{(2)} \cdot \chi^{(0)}_{0,\tau'} \right] . \end{aligned} \tag{2.86}$$

For b_4, first using (2.59) and (2.73) we have

$$\begin{aligned} b_4 &= J^2 \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,\omega'}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\,\lambda^{(1)}_{0,\omega'}} \int d\tau_5 d\tau_6 d\tau_7 d\tau_8 \, Q_s(\tau_{56}) Q_s(\tau_{78}) \\ &\quad \times \int \frac{d\omega''}{2\pi} \sum_{k\neq 0} \frac{1}{\lambda^{(0)}_{k,\omega''}} \left(\hat{d}_{h,-\omega} \chi^{(0)}_{k,\omega''} \right)(\tau_5,\tau_6) \left(\hat{d}_{h,-\omega'} \chi^{(0)}_{k,-\omega''} \right)(\tau_7,\tau_8) \,. \end{aligned} \tag{2.87}$$

Again using the relation (2.81), this is written as

$$\begin{aligned} b_4 &= \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,\omega'}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\,\lambda^{(1)}_{0,\omega'}} \int \frac{d\omega''}{2\pi} \sum_{k\neq 0} \frac{1}{\lambda^{(0)}_{k,\omega''}} \\ &\quad \times \left[J \lambda^{(0)}_{k,\omega''} \int \Psi_{(1)} \cdot \hat{d}_{1-\frac{1}{q},-\omega} \left(\Psi^{q-2}_{(0)} \, \chi^{(0)}_{k,\omega''} \right) - \int \chi^{(0)}_{0,-\omega} \cdot \mathcal{K}^{(1)} \cdot \chi^{(0)}_{k,\omega''} \right] \\ &\quad \times \left[\omega \to \omega' \,,\ \omega'' \to -\omega'' \right] . \end{aligned} \tag{2.88}$$

For the first term, we use the shadow transform (2.77) to move $\hat{d}$ onto $\Psi_{(1)}$. After using the completeness of $\tilde{\chi}^{(0)}$ (2.42), we obtain

$$\begin{aligned} b_4 &= \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{\chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,\omega'}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\,\lambda^{(1)}_{0,\omega'}} \\ &\quad \times \left[J \int \chi^{(0)}_{0,-\omega} \cdot \mathcal{K}^{(1)} \cdot \left(\hat{d}_{\frac{1}{q},-\omega'} \Psi^{(1)} \right) + \int \frac{d\omega''}{2\pi} \sum_{k\neq 0} \frac{1}{\lambda^{(0)}_{k,\omega''}} \left| \int \chi^{(0)}_{k} \cdot \mathcal{K}^{(1)} \cdot \chi^{(0)}_{0} \right|^2 \right] . \end{aligned} \tag{2.89}$$

The first term precisely cancels with b_3, while the second term combines with the other term in b_3 to give the second order eigenvalue shift. Therefore, we finally obtain

$$b_3 + b_4 = -\int \frac{d\omega}{2\pi} \frac{\lambda^{(2)}_{0,\omega}\, \chi^{(0)}_{0,\omega}(\tau_1,\tau_2)\chi^{(0)}_{0,-\omega}(\tau_3,\tau_4)}{\lambda^{(1)}_{0,\omega}\, \lambda^{(1)}_{0,-\omega}} \,. \tag{2.90}$$

The first and second order eigenvalue and eigenfunction corrections which will be needed below are given by (2.64) - (2.66).
We now found that the diagrammatic expressions denoted by b_1 - b_4 shown in Figure 2.2 are given by the matrix elements (2.83) and (2.90). These are in agreement with the standard perturbative evaluations summarized in Appendix A. In the next section, we will explicitly evaluate these matrix elements in the large q SYK model as an example.
The total $O(J^0)$ contribution to the correlation function is given by the following

$$G^{(0)} = b_1 + b_2 + b_3 + b_4 + \mathcal{D}_c \,, \tag{2.91}$$

It is seen that the gauge-dependent contribution $\mathcal{D}''$ is precisely cancelled by the diagrams $b_3 + b_4$ and part of $b_1 + b_2$. The result is therefore given by the non-vanishing single pole propagator $\mathcal{D}'$ and the remaining piece of the diagrams $b_1 + b_2$ which, as we have shown above involves the first order correction to the zero mode eigenfunction. We evaluate this (and the other matrix elements) in the large q case in the following section. In this limit (with higher massive modes decoupling) the result reads:

$$\begin{aligned}\langle \eta(t,z)\eta(t',z')\rangle \simeq \frac{(zz')^{\frac{1}{2}}}{2}\int_{-\infty}^{\infty} d\omega\; e^{i\omega(t-t')}\Bigg[& J_{-\frac{3}{2}}(|\omega|z^{>})J_{\frac{3}{2}}(|\omega|z^{<}) \\ & + \frac{1}{|\omega|}\left(\frac{1}{2z}+\frac{1}{2z'}+\partial_z+\partial_{z'}\right) J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z')\Bigg] \,.\end{aligned} \tag{2.92}$$

This result consists of two terms: the first one due to Schwarzian interaction with the bilocal and the second representing the contribution of the $h = 2$ matter. For general q one would have the further contribution of massive mode propagators. We also mention that a similar evaluation in [34] gave a different result. We are not sure why the method of [34] disagrees at $O(J^0)$, however we have compared the above result in the large q limit, where a finite J evaluation is possible.

2.2 Large q: Liouville Theory

The scheme described in the previous section ensures that once we have an expression for the first correction to the critical saddle point, $\Psi^{(1)}$, an unambiguous perturbation theory (which is really a derivative expansion) can be developed by replacing the actual symmetry breaking source term by a new source term which would give rise to this $\Psi^{(1)}$. This procedure is necessary since the symmetry breaking term is singular and nonzero only at short distances, which implies that this cannot be used as a perturbation in a long distance expansion. This means one has to make an ansatz for $\Psi^{(1)}$ given in (2.22).

In the large q limit, the SYK model simplifies considerably and at the leading order the action is that of Liouville theory on the bilocal space. At large N this model can be now solved exactly at all scales, i.e. for any finite J. Even though the symmetry breaking term of the SYK model needs to be included to obtain this large q limit, the resulting Liouville action acquires an emergent reparametrization symmetry. The large N saddle point breaks this symmetry. However, this does *not* lead to a zero mode, and a calculation of the exact bilocal function proceeds without any obstruction. In particular, in this exact calculation there is no emergent Schwarzian dynamics.

Nevertheless, we would like to understand the long distance derivative expansion in this limit, and extract the dynamics of the soft mode, since the Schwarzian action is the most direct link to a dual description in terms of JT gravity. In this section we develop this expansion using the scheme of the previous section. However, now we can simply use the leading correction to the critical saddle point to determine the regularized source Q_s. The result of the previous section then ensures that the scheme using this source reproduces the large J expansion of the exact answer for correlation functions.

2.2.1 Bilocal theory at large q

Let us begin with the collective action for SYK model (4.5)

$$S_{\text{col}}[\Psi] = -\frac{N}{2}\int d\tau\, \Big[\partial_\tau \Psi(\tau,\tau')\Big]_{\tau'\to\tau} + \frac{N}{2}\text{Tr}\log\Psi - \frac{2^{q-2}\mathcal{J}^2 N}{q^2}\int d\tau_1 d\tau_2\, \big[\Psi(\tau_1,\tau_2)\big]^q\,, \tag{2.93}$$

where we have defined

$$\mathcal{J}^2 \equiv \frac{q}{2^{q-1}}\, J^2\,. \tag{2.94}$$

This $\mathcal{J}$ is kept fixed as $q\to\infty$. At large q, we can do the following field redefinition

$$\Psi(\tau_1,\tau_2) = \frac{\text{sgn}(\tau_{12})}{2}\left[1 + \frac{\Phi(\tau_1,\tau_2)}{q}\right]. \tag{2.95}$$

Using (2.95) in (2.93), and performing a $1/q$ expansion we get a field independent $O(1)$ term and the next contribution is $O(1/q^2)$, given by the Liouville action

$$\begin{aligned} S_{\text{L}}[\Phi] = -\frac{N}{16q^2}\int d\tau_1 d\tau_2\, \Big[&\partial_1\big(\text{sgn}(\tau_{12})\Phi(\tau_1,\tau_2)\big)\partial_2\big(\text{sgn}(\tau_{21})\Phi(\tau_2,\tau_1)\big) \\ &+ 4\mathcal{J}^2\, e^{\Phi(\tau_1,\tau_2)}\Big] + O(q^{-3})\,. \end{aligned} \tag{2.96}$$

The details of the derivation of (2.96) is given in Appendix B. Note that there is no $O(1/q)$ term in this expansion. The kinetic term of SYK - the first term of (2.93)- provides a $1/q$ piece which cancels with a $1/q$ piece coming from the second term. The inclusion of the symmetry breaking term is crucial.

Nevertheless the action $S_{\rm L}[\Phi]$ has an emergent reparametrization symmetry for $\tau_i \to f(\tau_i)$,

$$\Phi(\tau_1,\tau_2) \;\;\to\;\; \Phi(f(\tau_1),f(\tau_2)) + \log|f'(\tau_1)f'(\tau_2)| \tag{2.97}$$

At finite temperature and finite $\mathcal{J}$, we need to impose the physical requirement that the expectation value of the bilocal field $\langle\Psi(\tau_1,\tau_2)\rangle$ should be equal to the free fermion two point function $\frac{1}{2}{\rm sgn}(\tau_{12})$ in the short distance limit. This means we need to impose a boundary condition

$$\Phi(\tau,\tau) \;=\; 0\,. \tag{2.98}$$

At zero temperature the expansion is really in $\mathcal{J}|\tau_{12}|$: this means that we cannot really access the point $\tau_1=\tau_2$. However the zero temperature theory should be really thought of as a limit of the finite temperature theory. Accordingly we should impose the condition (2.98) even at zero temperature. In fact, as is well known, the Liouville action on an infinite plane has a symmetry which has two copies of Virasoro, i.e. τ_1 and τ_2 can be reparametrized by different functions. The restriction to the same function as in (2.97) comes from this boundary condition (2.98).

The above derivation is in a $1/q$ expansion. The fact that the resulting action is a standard two derivative action signifies that to leading order of this expansion there is a single pole of the two point correlation function of the bilocal field. This appears to be in conflict with the well known fact that even for $q=\infty$ there are an infinite number of poles in the conformal limit, i.e. an infinite number of solutions of $\tilde{g}(q;h)=1$ where $\tilde{g}(q;h)$ is defined in (2.50). In fact at large q the solutions to this equation are given by $h=2$ and the tower [32]

$$h = 2n+1+\frac{2}{q}\frac{2n^2+n+1}{2n^2+n-1}+O(1/q^2), \qquad\qquad n=1,2,\ldots \tag{2.99}$$

However the residues of these poles all vanish as $q\to\infty$ except for $h=2$ [25]. The $1/q$ expansion of the function $\tilde{g}(q;h)$ is given by

$$\begin{aligned}\tilde{g}(q;h) \;=\; \frac{h(h-1)}{2}\Bigg[1+\frac{1}{q}\bigg(&\frac{2}{h(h-1)}+3\\ &-2\left[2\gamma+\log 4+\psi\left(\frac{1}{2}-\frac{h}{2}\right)+\psi\left(\frac{h}{2}\right)\right]\bigg)+O(q^{-2})\Bigg]\,,\end{aligned} \tag{2.100}$$

where $\psi(x)$ denotes the digamma function and γ is the Euler-Mascheroni constant. Since $\psi(-n)$ for integer n has a pole, the coefficient of the $1/q$ term is singular for $h=2n+1$. This is the signature of the infinite number of solutions of the spectral equation in a $1/q$ expansion. In the following, however, we will restrict our attention to values of h close to 2. Therefore these other solutions will not be relevant for us. Note, however, on-shell modes corresponding to these infinite tower of solutions do have nontrivial higher point correlation functions [32].

2.2.2 Quantum fluctuations at leading order

The equation of motion which follows from the action (2.96) is given by

$$\partial_1\partial_2\Big(\mathrm{sgn}(\tau_{12})\Phi(\tau_1,\tau_2)\Big) \;=\; -\,2\mathcal{J}^2\,\mathrm{sgn}(\tau_{12})\,e^{\Phi(\tau_1,\tau_2)}\,. \tag{2.101}$$

We should really view the zero temperature theory as a limit of the finite temperature theory. At finite temperature, periodicity in both τ_1 and τ_2, together with the boundary condition (2.98) determines the solution uniquely [12] and its zero temperature limit is given by

$$\Phi_{\rm cl}(\tau_{12}) \;=\; -\,2\log\Big(\mathcal{J}|\tau_{12}|+1\Big)\,, \tag{2.102}$$

where $\tau_{12}\equiv\tau_1-\tau_2$. Let us now consider quantum fluctuations around the saddle point solution (2.102) by defining

$$\Phi(\tau_1,\tau_2) \;=\; \Phi_{\rm cl}(\tau_{12}) \,+\, \sqrt{\frac{2}{N}}\,\eta(\tau_1,\tau_2)\,, \tag{2.103}$$

where η denotes the quantum fluctuations. The fluctuation must also obey the boundary condition

$$\eta(\tau,\tau) \;=\; 0\,. \tag{2.104}$$

Substituting (2.103) in (2.96), we get the quadratic action for quantum fluctuations as

$$S_{(2)}[\eta] \;=\; -\,\frac{1}{8q^2}\int d\tau_1 d\tau_2\;\eta(\tau_1,\tau_2)\left[\partial_1\partial_2 \,+\, 2\mathcal{J}^2\,e^{\Phi_{\rm cl}(\tau_{12})}\right]\eta(\tau_1,\tau_2)\,. \tag{2.105}$$

Using the bilocal map

$$t \;\equiv\; \frac{\tau_1+\tau_2}{2}\,, \qquad z \;\equiv\; \frac{\tau_1-\tau_2}{2} \tag{2.106}$$

we can write, using (2.102)

$$S^{(2)}[\eta] = \frac{1}{16q^2}\int_{-\infty}^{\infty}dt\int_0^{\infty}dz\;\eta(t,z)\left[-\partial_t^2+\partial_z^2-\frac{8\mathcal{J}^2}{(2\mathcal{J}z+1)^2}\right]\eta(t,z) \tag{2.107}$$

The operator which appears in the square bracket is the expression for $\mathcal{K}_{\rm ex}$ which is defined in (A.2).
In the following it will be convenient to define

$$\tilde{z} \;\equiv\; z+\frac{1}{2\mathcal{J}}\,. \tag{2.108}$$

It is convenient to define a conformally covariant operator $\widetilde{\mathcal{K}}_L$ (similar to(2.39))

$$\widetilde{\mathcal{K}}_{\rm L} \;=\; \tilde{z}\left[-\partial_t^2+\partial_z^2-\frac{2}{\tilde{z}^2}\right]\tilde{z}\,, \tag{2.109}$$

and redefine the field

$$\tilde{\eta}(t,z) \equiv \frac{\eta(t,z)}{\tilde{z}}\,. \tag{2.110}$$

The eigenvalue problem we need to solve is

$$\widetilde{\mathcal{K}}_{\rm L}\,\tilde{\chi}^{\rm (ex)}_{\nu}(t,z) = \lambda^{\rm (ex)}_{\nu}\tilde{\chi}^{\rm (ex)}_{\nu}(t,z) \tag{2.111}$$

These eigenfunctions are given by

$$\tilde{\chi}^{\rm (ex)}_{\nu,\omega}(t,z) = \frac{e^{i\omega t}\,\tilde{z}^{-\frac{1}{2}}}{\sqrt{2\pi\hat{N}_{\nu,\omega}}}\hat{Z}_{\nu}(|\omega|\tilde{z})\,, \tag{2.112}$$

where

$$\hat{Z}_{\nu}(|\omega|\tilde{z}) = J_{\nu}(|\omega|\tilde{z}) + \xi_{\omega}(\nu)J_{-\nu}(|\omega|\tilde{z}), \tag{2.113}$$

$$\xi_{\omega}(\nu) = -\frac{\Gamma\left(\frac{1}{4}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}. \tag{2.114}$$

and the eigenvalue $\lambda^{\rm (ex)}_{\nu}$ is given by

$$\lambda^{\rm (ex)}_{\nu} \equiv \left[\nu^2 - \frac{9}{4}\right] \tag{2.115}$$

The eigenfunctions satisfy the following orthonormality condition which follows from self-adjointness of the operator $\widetilde{\mathcal{K}}_{\rm L}$ in the interval $[0,\infty)$ with the boundary condition $\tilde{\chi}^{\rm ex}_{\nu,\omega}(t,0) = 0$

$$\int_{-\infty}^{\infty} dt \int_{0}^{\infty} dz\ \tilde{\chi}^{\rm (ex)}_{\nu,\omega}(t,z)\tilde{\chi}^{\rm (ex)}_{\nu',\omega'}(t,z) = \delta(\nu-\nu')\delta(\omega+\omega') \tag{2.116}$$

Here $\hat{N}_{\nu,\omega}$ appears in the orthonormality relation

$$\int_{0}^{\infty}\frac{dz}{\tilde{z}}\ \hat{Z}_{\nu_1}(|\omega|\tilde{z})\hat{Z}_{\nu_2}(|\omega|\tilde{z}) = \hat{N}_{\nu_1,\omega}\ \delta(\nu_1-\nu_2) \tag{2.117}$$

We do not have an analytic expression for $\hat{N}_{\nu,\omega}$, but we can determine it perturbatively in $\omega/\mathcal{J}$ to get for the discrete modes

$$\hat{N}_{\nu} = \frac{1}{2\nu} + \mathcal{O}\left(\frac{|\omega|}{2\mathcal{J}}\right)^3 \tag{2.118}$$

For the continuous modes, we will not need to know the $\omega/\mathcal{J}$ corrections since there is no enhancement. So, we use

$$\hat{N}_{\nu} = \frac{2\sin\pi\nu}{\nu} + \mathcal{O}\left(\frac{|\omega|}{2\mathcal{J}}\right) \tag{2.119}$$

We now perform a mode expansion of the fluctuation field in terms of these exact eigenfunctions as follows

$$\begin{aligned} \eta(t,z) &= \tilde{z}\int_{-\infty}^{\infty} d\omega \int d\nu\, \widetilde{\Phi}_{\nu,\omega}\, \tilde{\chi}^{(\mathrm{ex})}_{\nu,\omega}(t,z) \\ &= \int_{-\infty}^{\infty} d\omega \int d\nu\, \widetilde{\Phi}_{\nu,\omega}\, \chi^{(\mathrm{ex})}_{\nu,\omega}(t,z) \end{aligned} \tag{2.120}$$

The operator $\mathcal{K}_{\mathrm{ex}}$ is the zero temperature version of the operator considered in [12], where the finite temperature exact eigenfunctions were determined. So our exact eigenfunctions should arise as the zero temperature limit of those eigenfunctions. We discuss details of how to take this zero temperature limit and the derivation of various properties of the exact eigenfunctions in Appendix C.

The boundary condition (2.104), i.e. $\eta(t,z=0)=0$ determines the allowed values ν. These are of the form

$$\begin{aligned} &\nu = ir + \mathcal{O}\left(\frac{\omega}{\mathcal{J}}\right), && r \in \mathrm{Real} \\ &\nu = 2n + \frac{3}{2} + a_1\left(\frac{\omega}{\mathcal{J}}\right) + a_2\left(\frac{\omega}{\mathcal{J}}\right)^2 + \ldots, && n = 0,1,2,\ldots \end{aligned} \tag{2.121}$$

In the following we will not need the correction to the continuous series. The coefficients a_1 and a_2 for the $n=0$ case are given by $1/\pi$ and zero respectively (see Appendix C for details). Using (2.117) – (2.120) in (2.107), we get

$$S^{(2)} = \frac{1}{16q^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d\nu\, \widetilde{\Phi}_{\nu,\omega}\lambda^{(\mathrm{ex})}_{\nu}\widetilde{\Phi}_{\nu,-\omega} \tag{2.122}$$

Note that the eigenvalue $\lambda^{(\mathrm{ex})}_{\nu}$ arises from the large q limit of the eigenvalue of the finite q bilocal kernel (see (2.47), (2.50)), with $h=\nu+1/2$

$$\widetilde{g}(\nu,\infty) = \frac{1}{2}\left(\nu^2 - \frac{1}{4}\right). \tag{2.123}$$

From (2.122), we can read off the momentum space bilocal correlator

$$\langle \widetilde{\Phi}_{\nu_1,\omega_1}\widetilde{\Phi}_{\nu_2,\omega_2}\rangle \simeq \frac{8q^2}{\lambda^{(\mathrm{ex})}_{\nu_1}}\,\delta(\nu_1-\nu_2)\delta(\omega_1+\omega_2) \tag{2.124}$$

Since the solution (2.102) breaks the symmetry (2.97), one might expect that there is a zero mode given by its variation $\delta\Phi_{\mathrm{cl}}$. This would be of the form

$$\eta_{(0)}(t,z) \sim \tilde{z}^{1/2}\int d\omega\, e^{i\omega t}\epsilon(\omega)\left[\cos(|\omega|/\mathcal{J})J_{\frac{3}{2}}(|\omega|\tilde{z}) + \sin(|\omega|/\mathcal{J})J_{-\frac{3}{2}}(|\omega|\tilde{z})\right] \tag{2.125}$$

As expected this solves the equation of motion which follows from (2.107), but does not satisfy the boundary condition (2.104) at any finite J. Indeed, (2.121) shows that

there is no eigenfunction with $\nu = \frac{3}{2}$. Since there is no zero mode, a calculation of the bilocal two point function proceeds in a straightforward fashion.
Using (2.120) and (2.124), we can write down the exact bilocal two point correlator in position space as

$$\begin{aligned} \langle \eta(t,z)\eta(t',z')\rangle &= 8q^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d\nu \, \frac{\chi^{(\mathrm{ex})}_{\nu,\omega}(t,z)\chi^{(\mathrm{ex})}_{\nu,-\omega}(t',z')}{\lambda^{(\mathrm{ex})}_{\nu}} \\ &= 4q^2(\tilde{z}\tilde{z}')^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t-t')} \int \frac{d\nu}{\hat{N}_\nu} \, \frac{\hat{Z}^*_\nu(|\omega|\tilde{z})\hat{Z}_\nu(|\omega|\tilde{z}')}{\widetilde{g}(\nu,\infty)-1} \end{aligned} \tag{2.126}$$

2.2.3 Perturbative expansion

In this subsection, we calculate the various ingredients needed to obtain a perturbative expansion of the bilocal propagator, which are the matrix elements given by (2.64) – (2.66) for Liouville theory. The full (or "exact") kernel $\widetilde{\mathcal{K}}_{\mathrm{L}}$ can be expanded in powers of $(|\omega|\mathcal{J}^{-1})$ as follows. This is the analog of the expansion (A.7).

$$\widetilde{\mathcal{K}}_{\mathrm{L}} = \widetilde{\mathcal{K}}^{(0)} + \frac{1}{\mathcal{J}}\widetilde{\mathcal{K}}^{(1)} + \frac{1}{\mathcal{J}^2}\widetilde{\mathcal{K}}^{(2)} + \dots \tag{2.127}$$

where

$$\widetilde{\mathcal{K}}^{(0)} = z^2(-\partial_t^2 + \partial_z^2) + 2z\partial_z - 2 \tag{2.128}$$

$$\widetilde{\mathcal{K}}^{(1)} = z(-\partial_t^2 + \partial_z^2) + \partial_z \tag{2.129}$$

$$\widetilde{\mathcal{K}}^{(2)} = \frac{-\partial_t^2 + \partial_z^2}{4} \tag{2.130}$$

The kernel $\widetilde{\mathcal{K}}^{(0)}$ now has a zero mode,

$$\lambda_0^{(0)} = 0 \tag{2.131}$$

$$\tilde{\chi}^{(0)}_{0,\omega}(t,z) = \sqrt{\frac{3}{2\pi}}\, z^{-\frac{1}{2}} e^{i\omega t} J_{\frac{3}{2}}(|\omega|z) \tag{2.132}$$

The eigenvalue and eigenfunction for this mode will be corrected by the perturbations. The corresponding exact eigenvalue is $\lambda_{0,\omega}$ which has the expansion

$$\lambda_{0,\omega} = \lambda_0^{(0)} + \frac{1}{\mathcal{J}}\lambda_0^{(1)} + \left(\frac{1}{\mathcal{J}}\right)^2 \lambda_0^{(2)} + \cdots \tag{2.133}$$

analogous to (A.10). The corrections to the eigenvalues can be read off from (2.121) and (2.115),

$$\lambda_0^{(1)} = \frac{3|\omega|}{\pi}\,. \tag{2.134}$$

Similarly, for the first order correction to the eigenfunction, we get

$$\tilde{\chi}^{(1)}_{0,\omega}(t,z) \;=\; \sqrt{\frac{3}{2\pi}\frac{|\omega|}{2\mathcal{J}}}\, z^{-\frac{1}{2}} e^{i\omega t}\left(\frac{1}{|\omega|}\,\partial_z - \frac{1}{2|\omega|z} + \frac{2}{3\pi} + \frac{2}{\pi}\,\partial_\nu\right) J_\nu(|\omega|z)\bigg|_{\nu=\frac{3}{2}}\,, \tag{2.135}$$

and

$$\lambda^{(2)}_0 \;=\; \frac{\omega^2}{\pi^2}\,. \tag{2.136}$$

While we have written these down using the expansion of the known exact eigenvalue and eigenfunction, one can of course calculate these directly in perturbation theory. The first order correction follows easily from (2.129) and (2.132) in (2.64). For the first order correction to the eigenfunction, we were unable to perform the sum analytically in (2.66). The expression (2.135) can be used to verify that standard perturbation theory indeed leads to the correct second order correction to the eigenvalue. The details of this calculation have been included in Appendix F. These calculations are in agreement with the results of [12].

2.2.4 Evaluation of bilocal two point function

We now have all the necessary ingredients to evaluate the bilocal two point function (2.126) perturbatively in $\left(|\omega|\mathcal{J}^{-1}\right)$ using the formalism of Section 2.1 and Appendix A. First, we note that the values of ν to be integrated over are given by (2.121). The imaginary and discrete values give rise to a continuous and discrete contribution respectively. The discrete sum receives an enhancement from the zero mode so it needs to be treated separately. We separate it from the non zero modes in order to write

$$\langle \eta(t,z)\eta(t',z')\rangle \;=\; G^{(-1)}(t,z;t',z') \;+\; G^{(0)}(t,z;t',z')\,, \tag{2.137}$$

where $G^{(-1)}$ and $G^{(0)}$ are given by equations (2.59) and (2.60) respectively. The perturbative corrections to the zero mode eigenvalue and eigenfunction can be substituted in these expressions to write down the bilocal propagator explicitly. Using (2.134) and (2.132) in (2.59), we get the enhanced propagator

$$\begin{aligned} G^{(-1)}(t,z;t',z') \;&=\; 8q^2\mathcal{J}\int_{-\infty}^{\infty} d\omega\; \frac{\chi^{(0)}_{0,\omega}(t,z)\chi^{(0)}_{0,-\omega}(t',z')}{\lambda^{(1)}_0} \\ &=\; 4q^2\mathcal{J}(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{d\omega}{|\omega|}\; e^{i\omega(t-t')}J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z') \end{aligned} \tag{2.138}$$

Next, let us consider the $\mathcal{O}(1)$ contribution $G^{(0)}$. It receives a contribution from $\mathcal{D}_c$, which for the Liouville case is given by

$$\mathcal{D}_c(t,z;t',z') \;=\; 8q^2\int_{-\infty}^{\infty} d\omega\left[\sum_{n=1}^{\infty}\frac{\chi^{(0)}_{\nu_n,\omega}(t,z)\chi^{(0)}_{\nu_n,\omega}(t',z')}{\lambda^{(0)}_{\nu_n}} + \int d\nu\;\frac{\chi^{(0)*}_{\nu,\omega}(t,z)\chi^{(0)}_{\nu,\omega}(t',z')}{\lambda^{(0)}_{\nu}}\right]. \tag{2.139}$$

The details of evaluation of $\mathcal{D}_c$ is given in Appendix D. The result is

$$\mathcal{D}_c(t,z;t',z') \;=\; 4q^2(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\Bigg[\frac{J_{-\frac{3}{2}}(|\omega|z^{>})J_{\frac{3}{2}}(|\omega|z^{<})}{2} - \frac{1}{\pi}\left(\frac{d}{d\nu}+\frac{1}{3}\right)J_\nu(|\omega|z)J_\nu(|\omega|z')\Bigg|_{\nu=\frac{3}{2}}\Bigg]. \tag{2.140}$$

We identify the first term in (2.140) with the non-vanishing single pole term $\mathcal{D}'$ and the second term with the gauge-dependent piece $\mathcal{D}''$ in (2.51) respectively. The remaining terms in (2.60) can also be evaluated using (2.131) – (2.135) to get

$$\begin{aligned}
&8q^2\int_{-\infty}^{\infty} d\omega\left[\frac{\chi^{(0)}_{0,\omega}(t,z)\chi^{(1)}_{0,-\omega}(t',z')}{\lambda^{(1)}_0}+\frac{\chi^{(1)}_{0,\omega}(t,z)\chi^{(0)}_{0,-\omega}(t',z')}{\lambda^{(1)}_0}-\frac{\lambda^{(2)}_0\chi^{(0)}_{0,\omega}(t,z)\chi^{(0)}_{0,-\omega}(t',z')}{(\lambda^{(1)}_0)^2}\right]\\
=&4q^2(zz')^{\frac{1}{2}}\int d\omega\, e^{i\omega(t-t')}\left[\frac{1}{\pi}\left(\frac{d}{d\nu}+\frac{1}{3}\right)+\frac{1}{2|\omega|}\left(\partial_z+\partial_{z'}+\frac{1}{2z}+\frac{1}{2z'}\right)\right]\\
&\times J_\nu(|\omega|z)J_\nu(|\omega|z')\Bigg|_{\nu=\frac{3}{2}}.
\end{aligned} \tag{2.141}$$

From (2.140) and (2.141), we see that the gauge-dependent piece $\mathcal{D}''$ precisely cancels. This is an explicit illustration of the claim made in Section 2.1 that the final answer for the bilocal propagator receives contribution only from the simple pole in $\mathcal{D}_c$ (which corresponds to $h=2$ matter) and from the b_1+b_2 diagrams. Collecting all the terms, we get the bilocal propagator for Liouville theory up to $\mathcal{O}(1)$ to be

$$\begin{aligned}
\langle\eta(t,z)\eta(t',z')\rangle \;=&\,2q^2(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty} d\omega\; e^{i\omega(t-t')}\Bigg[\frac{2\mathcal{J}}{|\omega|}J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z')\\
&+J_{-\frac{3}{2}}(|\omega|z^{>})J_{\frac{3}{2}}(|\omega|z^{<})\\
&+\frac{1}{|\omega|}\left(\partial_z+\partial_{z'}+\frac{1}{2z}+\frac{1}{2z'}\right)J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z')\Bigg].
\end{aligned} \tag{2.142}$$

2.2.5 Comparison with zero temperature limit of four-point function

We now show that the zero temperature limit of the finite temperature four point function calculated in [12] agrees with (2.142). The SYK four point function at finite temperature up to $\mathcal{O}(1)$ is given by

$$\begin{aligned}
\mathcal{F}(x,y;x',y') \;=&\; \left[\beta\mathcal{J}-2\left[-1+\left(y-\frac{\pi}{2}\right)\partial_y+(x-\pi)\partial_x+(x'-\pi)\partial_{x'}\right]\right]\\
&\times\sum_{|n|\geq 2} e^{-in(y-y')}\frac{f_n(x)f_n(x')}{\pi^2n^2(n^2-1)}
\end{aligned} \tag{2.143}$$

It should be noted that the explicit calculation of the $O(1)$ contribution in [12] is obtained by writing the propagator in terms of eigenfunctions of

$$-\partial_{\tilde{x}}^2 + \frac{1}{2\sin^2\frac{\tilde{x}}{2}}$$

rather than

$$-\frac{\sin^2\frac{\tilde{x}}{2}}{v^2}\partial_y^2 + 4\sin^2\frac{\tilde{x}}{2}\partial_{\tilde{x}}^2 + \frac{1}{4}$$

which is the finite temperature version of the Bessel operator we consider. They, of course, lead to the same result since at finite temperature the periodicity conditions ensure a unique Green's function.

Let us define

$$y = \frac{2\pi t}{\beta} \qquad x = \frac{4\pi z}{\beta} \qquad \omega = \frac{2\pi n}{\beta} \tag{2.144}$$

In the zero temperature limit the $h = 2$ eigenfunctions become Bessel functions

$$\begin{aligned} f_n(x) &= \frac{\sin\frac{nx}{2}}{\tan\frac{x}{2}} - n\cos\frac{nx}{2} \\ &= \frac{\beta\omega}{2\pi}\sqrt{\frac{\pi\omega z}{2}}\, J_{\frac{3}{2}}(\omega z) \end{aligned} \tag{2.145}$$

Using (2.145) in (2.143) and replacing the sum over the discrete Fourier index n by a continuous integral over ω, we get

$$\begin{aligned} \mathcal{F}(t,z;t',z') &= \frac{1}{2}(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t-t')}\Bigg[\frac{2\mathcal{J}}{|\omega|}\, J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z') \\ &\quad + \frac{1}{|\omega|}\left(\partial_z + \partial_{z'} + \frac{1}{2z} + \frac{1}{2z'}\right) J_{\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z')\Bigg] \\ &\quad - (zz')^{\frac{1}{2}}\int_0^{\infty} d\omega\ \sin\omega(t-t')\ J_{\frac{3}{2}}(\omega z)J_{\frac{3}{2}}(\omega z')\,. \end{aligned} \tag{2.146}$$

Now, we use the following integrals expressed in terms of the quantity $\xi \equiv \frac{-(t-t')^2+z^2+z'^2}{2zz'}$

$$(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t-t')}J_{-\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z') = \begin{cases} 0, & \text{for } |\xi| > 1 \\ P_1(-\xi), & \text{for } |\xi| < 1 \end{cases}$$

and

$$(zz')^{\frac{1}{2}}\int_0^{\infty} d\omega\ \sin\omega(t-t')\ J_{\frac{3}{2}}(\omega z)J_{\frac{3}{2}}(\omega z') = \begin{cases} 0, & \text{for } |\xi| > 1 \\ \frac{1}{2}P_1(\xi), & \text{for } |\xi| < 1 \end{cases}$$

The details of the derivation of these integrals are given in Appendix E. Using these integrals, we see from (2.142) and (2.146),

$$\langle \eta(t,z)\eta(t',z')\rangle = 4q^2\,\mathcal{F}(t,z;t',z') \tag{2.147}$$

The proportionality factor of $4q^2$ arises as a consequence of the field redefinition in (2.95).

2.2.6 Expansion around the critical saddle point

In the $\mathcal{J}|\tau_{12}| \gg 1$ limit (2.102) becomes

$$\Phi^{(0)} \;=\; -2\log(\mathcal{J}|\tau_{12}|) \tag{2.148}$$

We will call this the "critical saddle point". This is in fact what follows from the large q limit of $\Psi^{(0)}$ in (2.9) and plays its role. There is, however, an important difference. For finite q the critical solution $\mathcal{J}|\tau_{12}|$ is not a solution of the full theory because of the term which breaks the reparametrization symmetry. In this case (2.148) is also a solution of the classical equations of motion. $\Phi^{(0)}$, however, does not satisfy the boundary condition (2.98). In fact, in an expansion around the critical limit, there is no reason to impose this boundary condition. Keeping this point in mind we will call $\Phi_{\rm cl}$ the "exact" solution and develop a perturbative expansion around the critical solution following the same steps as in the finite q SYK model.

An expansion around this solution would in fact lead to a normalizable zero mode given by (2.132) which needs to be treated properly in precisely the same way as the finite q SYK model. In order to deal with this, we introduce a source term given by

$$\frac{N}{8q^2\mathcal{J}}\int d\tau_1 d\tau_2\; Q_s^{\rm L}(\tau_1,\tau_2)\Phi(\tau_1,\tau_2) \tag{2.149}$$

where the source is again a regularized version of (2.24). The fact that this source term goes as $\mathcal{J}^{-1}$ is reminiscent of the conformal breaking term in finite q SYK. It should be noted that if we add other source terms which are suppressed by higher powers of $\mathcal{J}$, that would be inconsistent with the finite q picture. In this case, the kernel $\mathcal{K}_{\rm L}^{(0)}$ can be read off from the first line of (2.107) as

$$\mathcal{K}_{\rm L}^{(0)}(t,z) \;=\; -\partial_t^2+\partial_z^2-\frac{2}{z^2} \tag{2.150}$$

The source is then related to the $O(1/\mathcal{J})$ correction to the classical solution by

$$Q_s^{\rm L}(t,z)=\mathcal{K}_{\rm L}^{(0)}\Phi_s^{(1)}(t,z) \tag{2.151}$$

The $\mathcal{O}(1/\mathcal{J})$ correction to the classical solution is given by

$$\Phi_{\rm cl}-\Phi_0\equiv\Phi^{(1)}(z) \;=\; -\frac{1}{z} \tag{2.152}$$

We have stripped off the power of $\mathcal{J}$ in the expression for $\Psi^{(1)}$ in order to maintain consistency in notation with Section 2.1. Also, since the domain of integration of the z coordinate is always positive, we drop the absolute sign. As expected, the source in (2.24) with this non-regularized $\Phi^{(1)}(z)$ vanishes. Following the treatment in finite q SYK, we therefore introduce a regulator s to define

$$\Phi_s^{(1)}(z) \;\equiv\; -\lim_{s\to\frac{1}{2}}\frac{1}{z^{2s}} \tag{2.153}$$

This results in a regularized version of the source $Q_s^{\rm L}$ given by

$$Q_s^{\rm L}(\tau_1,\tau_2) \;\simeq\; -\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\frac{s+1}{|\tau_{12}|^{2s+2}} \tag{2.154}$$

Since the exact classical solution is now known, the coefficient of the resulting Schwarzian action for the soft mode can be now determined unambiguously. The Schwarzian action is given by

$$S^{\rm L}_{\rm Sch}[f] \;=\; -\frac{N}{8q^2\mathcal{J}}\lim_{s\to\frac{1}{2}}\int d\tau_1 d\tau_2\; Q_s^{\rm L}(\tau_1,\tau_2)\Phi_0^f(\tau_1,\tau_2) \tag{2.155}$$

where Φ_0^f corresponds to the transformed classical solution

$$\Phi_0^f(\tau_1,\tau_2) \;=\; \log\left[\frac{f'(\tau_1)f'(\tau_2)}{|f(\tau_1)-f(\tau_2)|^2}\right] \tag{2.156}$$

To determine the coefficient one could proceed by writing $f(\tau)=\tau+\varepsilon(\tau)$, evaluate (2.155) in an expansion in $\varepsilon(t)$, perform the limit $s\to 1/2$ and resum to obtain the full expression. Since we know that the answer should be proportional to the Schwarzian it is adequate to expand to second order in $\varepsilon(t)$. We have performed this calculation, but will omit the details here.

Alternatively we can calculate the propagator of the Fourier transform of $\varepsilon(\tau)$, which we call $\varepsilon(\omega)$ given in (2.73) which we reproduce below

$$\frac{\mathcal{J}}{\lambda^{(1)}_{0,\omega}}\,\delta(\omega+\omega') \;=\; \frac{N^{(0)}_{0,\omega}}{2\pi}\,\langle\varepsilon(\omega)\varepsilon(\omega')\rangle \tag{2.157}$$

The main ingredient is contained in the set of conformal Ward identities which are described for the arbitrary q model in equations (2.69)-(2.71). These equations can be explicitly verified in our case with the expressions for $\mathcal{K}^{(0)}_{\rm L}$ given in (2.150) and $\mathcal{K}^{(1)}_{\rm L}$ given by

$$\mathcal{K}^{(1)}_{\rm L} \;=\; -\frac{2}{z^2}\Phi^{(1)} \tag{2.158}$$

For our case $\lambda^{(1)}_0$ is given in (2.134). To determine the normalization $N^{(0)}_{0,\omega}$, we need

$$\chi_{{\rm zero},\omega}(\tau_1,\tau_2) = \int d\tau\; e^{i\omega\tau}\left[\frac{\delta\Phi^{(0)}_f(\tau)}{\delta f(\tau')}\right]_{f(\tau)=\tau} \tag{2.159}$$

It is straightforward to see that

$$\chi_{{\rm zero},\omega}(\tau_1,\tau_2) = -\sqrt{2\pi}\; i\omega^{\frac{3}{2}}z^{\frac{1}{2}}J_{\frac{3}{2}}(|\omega|z) \tag{2.160}$$

Comparing with (2.132), we get (from the definition)

$$N^{(0)}_{0,\omega} = \frac{4\pi^2|\omega|^3}{3} \tag{2.161}$$

To apply (2.73) we need to note that while the source term for finite q was defined by (2.24) while in the Liouville theory we defined the source term by (2.149). Thus in Liouville theory we have

$$\langle \varepsilon(\omega)\varepsilon(\omega')\rangle = \frac{8q^2\mathcal{J}}{N}\frac{2\pi\delta(\omega+\omega')}{\lambda^{(1)}_{0,\omega}N^{(0)}_{0,\omega}} = \frac{4q^2\mathcal{J}}{N\omega^4} \tag{2.162}$$

where we have used (2.161) and (2.134). This propagator can be easily seen to follow from the Schwarzian action

$$S^{\mathrm{L}}_{\mathrm{Sch}}[f] = -\frac{N}{4q^2\mathcal{J}}\int d\tau\ \{f(\tau),\tau\} \tag{2.163}$$

The coefficient is in precise agreement with the large q limit of the action given in [12]. The calculation of the bilocal two point function now follows the diagrammatic technique of Section 2.1.

2.3 Conclusion

In this chapter, we concentrated on the development of a complete understanding of systematic near conformal perturbation expansion in SYK type models. It develops further the initial work of [29, 30] where the soft mode with Schwarzian dynamics is extracted from the bilocal field in a systematic fashion and arises as an emergent degree of freedom. This mode interacts with the remaining "matter" degrees of freedom which include a component dual to an operator with conformal dimension $h \sim 2$, in a manner which is completely determined. The nontriviality of exhibiting this interacting representation lies in the fact that, as we explain, the symmetry breaking effects responsible both for the Schwarzian and Schwarzian-matter interactions comes from a very subtle off-shell regularization which produces non-zero effects when removed in the limiting procedure. This leaves a series of interacting vertices that are determined explicitly. Representing these corrections in a diagrammatic picture provides a complete and transparent scheme. This allows for a concrete perturbation calculation of corrections to leading (conformal) correlation functions and other physical quantities. We do this for the bilocal two point functions by considering the evaluation of the first correction (in a low energy expansion) to the leading (enhanced) answer. The evaluation of these is facilitated by a series of conformal identities.

We applied this formalism to the large q limit, where the model is reduced to the Liouville theory which is exactly solvable. In this case, expansion around the correct saddle-point does not result in a zero mode and the correlators can be calculated exactly. We can nevertheless expand around a saddle point appropriate for a long distance expansion and apply the perturbation scheme described above. In addition, it is instructive to see the workings of the method in this example of a prototype conformal field theory. We note that it is of interest to consider the complex SYK model already, since issues related to applicability of perturbation theory have been observed recently [77]. Indeed we believe that the applicability of the scheme goes beyond the SYK model, applying generally to perturbations in conformal field theory and more generally quantum field theory.

Returning to the case of SYK theories , the present systematic reformulation of the model as a bilocal matter coupled Schwarzian theory might offer the needed insight into the outstanding question of its gravity dual. The Liouville theory in particular provides the simplest limiting case. In this limit the higher h modes decouple and we find an interacting picture of the soft mode with bilocal $h = 2$ matter. We emphasize this fact since most studies of the gravity dual focus on the dilaton gravity sector of the dual theory. In [58–61], the enhanced part of the bilocal two point function has been reproduced in a dual theory which contains bulk fields which can be thought to be dual to the SYK fermions. Our preliminary results indicate that some parts of the subleading correction may be obtainable by considering the effect of a coupling to the nontrivial dilaton background [78].

However, additional ingredients, related to the contribution of $h = 2$ bilocal matter, are probably necessary to understand the low energy sector completely. We hope to return to this problem in future studies.

Chapter 3
A Universal Schwarzian Sector in two dimensional Conformal Field Theories

Our understanding of near-extremal black holes has been recently revolutionized by the improved understanding of a universal dynamics which emerges at low temperature [27, 58–61, 79]. It has long been known that black holes near extremality develop a long AdS_2 throat near the event horizon, which behaves rather differently from the analogous higher-dimensional AdS regions near black branes [15, 80]. The underlying reason for this is that the AdS_2 region does not completely decouple from the physics far from the black hole in the low-temperature limit. Instead, there is a single mode which becomes increasingly important at low temperature, specifying the relationship between the time in AdS_2 and the time far from the black hole. This mode is governed by the Schwarzian theory (described gravitationally inside the AdS_2 throat by Jackiw-Teitelboim gravity [56, 57]), with action

$$I_{\mathrm{Schw}} = -C \int_0^{\beta} dt_E \left\{ \tan\left(\tfrac{\pi}{\beta} f(t_E)\right), t_E \right\}, \tag{3.1}$$

where t_E is the asymptotic (Euclidean) time and $f(t_E)$ a time measured in the AdS_2 region, with $\{\cdot, \cdot\}$ denoting the Schwarzian derivative. The coefficient C, with dimensions of time, marks the inverse temperature β at which the Schwarzian dynamics becomes strongly coupled, with quantum fluctuations of f unsuppressed. This is a theory of a pseudo-Goldstone mode, determined by the nature of the symmetry breaking, which therefore appears in more general circumstances, not least the SYK model [12, 13, 28–30, 34].

In this chapter, we show that, in extremely generic circumstances, two-dimensional conformal field theories (CFTs) contain a sector described by the Schwarzian theory. This sector has a gravitational description in terms of a near-horizon AdS_2 region of near-extremal rotating BTZ black holes, but exists even in theories without a local weakly coupled AdS_3 dual!

Since our results are very general, we will not make use of details of a particular theory to derive the Schwarzian. The discussion in this chapter is orthogonal to previous constructions of SYK-like models in two dimensions [81–83]. Rather, we will use conformal bootstrap methods to study observables in states that enhance the effects of the Schwarzian sector, with very large angular momentum and low temperature. More specifically, we explicitly construct correlation functions in an appropriate limit of the grand canonical ensemble for angular momentum, requiring only a theory with a large central charge $c \gg 1$ and no conserved currents besides those of local conformal symmetry. With these general assumptions, we will show that the density of states and all correlators are dictated by the Schwarzian theory, with parametrically small corrections. To achieve this, we rigorously demonstrate that the correlators are dominated by a Virasoro identity block in an appropriate channel, before showing that the block reduces to the Schwarzian correlator in the appropriate limit. The methods for the latter calculation are much the same as used in [84] to compute correlation functions of the Schwarzian at strong coupling from Liouville theory, though the interpretation in this chapter is rather different. This result is a striking example of the universality of gravity as a description of chaotic quantum systems.

This chapter is organized as follows. For the remainder of the introduction, we will summarise our main results. In section 3.1, we discuss the partition function and spectrum of BTZ black holes and of CFTs in the limit of interest, which illustrates the main ideas while avoiding too many technical details. The main results appear in section 3.2, where we show how correlation functions of CFTs reduce to those of the Schwarzian. Finally, in section 3.3 we study near-extremal rotating BTZ black holes, to obtain a detailed gravitational interpretation of our CFT calculations.

3.0.1 A near-extremal limit of CFT$_2$

For our main results, we study correlation functions of two-dimensional, compact unitary CFTs with large central charge $c \gg 1$ in the grand canonical ensemble. Instead of using a temperature and chemical potential for angular momentum J, this ensemble can be described by independent temperatures T_L, T_R for left- and right-moving conformal dimensions $h, \bar{h}$. We study a regime which we call the near-extremal limit, where T_L is of order c^{-1}, and T_R is taken to be very large. This ensures that the physics is dominated by states with very large angular momentum $J \gg c$ (controlled by the large T_R) and low temperature T of order c^{-1} (controlled by the small T_L). These states are related to CFT operators with a very large right-moving scaling dimension $\bar{h} \sim J$, and left-moving dimension close to $\frac{c-1}{24}$, with $h - \frac{c-1}{24}$ of order c^{-1}.

The physics in this regime is exemplified by the simplest correlator, namely a two-point function of light operators in the grand canonical ensemble. In our near-extremal limit, we find that the real-time two-point function (using light-cone coordinates $z = \varphi - t$ and $\bar{z} = \varphi + t$) is given by

$$G^{(\mathrm{CFT})}(z, \bar{z}; \beta_L, \beta_R) \sim \left(\frac{c}{6}\right)^{-2h} \left[\frac{1}{\pi T_R} \sinh\left(\pi T_R \bar{z}\right)\right]^{-2\bar{h}} G_h^{(\mathrm{Schw})}(-z), \tag{3.2}$$

where $G_h^{(\mathrm{Schw})}(-z)$ is the exact (that is, strong coupling) real-time Schwarzian two point function at a temperature proportional to T_L. This is valid for large time separations t of order the inverse temperature T_L^{-1} (and for $\bar{z}$ of order one, but not too large, as described in section (3.2.1)). Note that the Schwarzian appears most naturally in real time, not Euclidean time. There is another regime where the Euclidean time Schwarzian appears, related by a modular transformation. Taking very low or zero right-moving temperature, and left-moving temperature of order c, the left-moving block gives a Schwarzian evaluated at Euclidean time z.

To show this, we first use the limit of large T_R and modular invariance: in a modular S-dual channel, this becomes very small right-moving temperature, which effectively projects the sum over intermediate states onto the smallest value of $\bar{h}$, corresponding to the vacuum state and its left-moving descendants. This is where the restrictions on our theory are important, namely that it is unitary (so all operators have $h, \bar{h} \geq 0$), compact (having a unique $\mathfrak{sl}(2)$-invariant vacuum state, with $h = \bar{h} = 0$), and has a twist gap (so no primary states besides the vacuum have $\bar{h} = 0$). With mild kinematic restrictions on $\bar{z}$, we conclude that the correlation function is given in this limit by a single conformal block with the vacuum state exchanged. This explains why the result (3.2) has factorised dependence on left- and right-moving variables z, T_L and $\bar{z}, T_R$.

Next, we show that the left-moving Virasoro vacuum block in the appropriate limit becomes the Schwarzian correlation function, with a calculation closely related to [84]. At low left-moving temperature, the conformal blocks are simplest in a 'direct' channel, where the OPE coefficients can be interpreted as matrix elements of intermediate states in the thermal trace, but we must compute the vacuum block in a dual channel related by the modular transformation. To relate these two channels, we use the sequence of modular S and fusion transformations pictured in figure 3.2, and the associated kernels can then be interpreted in terms of the density of states and matrix elements of light operators in the Schwarzian theory. These Schwarzian data in fact follow from a particular limit of the universal density of states and OPE coefficients recently discussed in [85].

These considerations extend to higher-point functions and general time-orderings, discussed in section 3.2.5, giving formulas analogous to (3.2) in appropriate ranges of kinematics. In particular, correlation functions of partial waves are always given by the Schwarzian. This applies to out-of-time order four-point functions (OTOC), implying that the Lyapunov exponent in this limit saturates the chaos bound of [7]. This is the first calculation of OTOC in 2d CFT in a fully controlled approximation, thanks to the large T_R limit.

In this work, we assume the absence of any conserved current beyond Virasoro. With knowledge of the appropriate representation theoretic objects (the modular S-matrix and fusion kernel), our methods could be generalized to extended symmetry algebras (including supersymmetric CFTs). This would enhance the Schwarzian theory by an extra mode associated to the additional symmetry. For the case of an extra $U(1)$, this is analogous to the IR theory that appears in complex SYK [31]. One could also apply the same methods to study near-extremal states with a large charge associated to this extra symmetry. We leave such generalizations for future work.

3.0.2 Near-extremal BTZ black holes

In $\mathrm{AdS}_3/\mathrm{CFT}_2$, our results have a dual interpretation in terms of large, near-extremal rotating BTZ black holes. In section 3.3 we will perform the bulk analysis for Einstein gravity coupled to light matter fields. As stressed in the previous section, our 2d CFT results are valid even beyond holographic theories. The underlying reason is that, while the Schwarzian mode is strongly interacting, the other bulk interactions are suppressed by an additional scale, namely the very large horizon radius. Therefore, even an effective bulk dual with a cutoff on the AdS scale or larger (for example, from strings with AdS scale tension) is useful for describing physics at these very long distances. This is very much in the spirit of previous results recovering AdS long-distance physics from generic CFTs [86–89], but rather more striking because a strongly interacting mode remains.

We discuss the details of this gravitational dual in section 3.3. We study this from the perspective of a Kaluza-Klein (KK) reduction of three dimensional gravity (initially pure gravity, later adding matter), which leads to an Einstein-Maxwell-dilaton theory. One may initially wonder whether the KK gauge field has a significant effect on the physics; we deal with it by working with boundary conditions describing an ensemble with fixed angular momentum (the charge dual to the KK gauge field), in which case the gauge field can (in the absence of charged matter) be integrated out to give a *local* effective action, adding a term to the dilaton potential. These are rather unconventional boundary conditions from the three-dimensional point of view, since they allow a parameter of the boundary metric to fluctuate, but the usual boundary conditions are recovered simply by changing back to the ensemble with chemical potential for angular momentum. Charged matter (that is, with nonzero angular momentum, which includes KK modes) does not significantly affect this if the angular momentum carried is small compared to that of the black hole.

The resulting theory of dilaton gravity (first written down by [90]) is in the class of Almheiri-Polchinski models [58], and admits a nAdS_2 JT gravity regime, which describes the near-horizon region of rotating BTZ black holes. At low temperature, this gravitational physics becomes strongly coupled, and is described by the Schwarzian theory [60]. The near-horizon region governed by JT gravity has a parametrically large overlap with a region far from the horizon where gravity is classical, so the physics is well-described by propagation on a fixed background. Matching these two regimes allows us to recover correlation functions of the dual CFT from the asymptotic boundary of AdS_3.

Importantly, interactions (including those with KK modes) are suppressed in the limit of a large black hole, even for matter which is originally strongly interacting in AdS_3. This is the reason why the reduction to two-dimensional gravitational physics is useful: the KK modes become decoupled, independent free fields. Note that this is different from the usual case of small transverse dimensions in KK reductions, where the higher KK modes can be ignored because they become very massive.

We work in a second order metric formalism due to the presence of matter, in analogy to the higher dimensional cases studied recently in [66, 91–93]. For pure three dimensional Einstein gravity, there is a more direct route to the Schwarzian, making use of the Chern-Simons formulation [94], which describes perturbations around a given geometry. In this case, it is possible to completely reduce the bulk theory to a set of left- and right-moving boundary modes [95]. The dynamics of these two modes is described by left- and right-moving copies of the Alekseev-Shatashvili action [96]. In the near extremal limit, the Alekseev-Shatashvili theory is known to reduce to the Schwarzian [84, 95, 97]. This boundary theory was previously proposed as a Goldstone mode of spontaneously broken conformal invariance for any chaotic 2d CFT [98]. From this perspective, the Schwarzian emerges from a reparameterization mode of CFT_2, with dynamics governed by the conformal anomaly [99].

While our results bear some resemblance to previous considerations of near-extremal BTZ such as [100, 101], there are crucial differences. In particular, the small but nonzero temperature is necessary; at very low temperatures, the Schwarzian becomes increasingly strongly coupled, and at exponentially low temperatures, when there is no longer a parametrically large number of available states, nonperturbative corrections from different topologies become uncontrolled [102].

3.1 Invitation: the near-extremal spectrum of BTZ and CFT$_2$

We begin by studying the partition function of near-extremal rotating BTZ black holes, and compare to the Schwarzian theory. We then obtain the corresponding spectrum directly from a very general class of CFTs, using an argument which we will later generalize to Schwarzian correlation functions. This simple example illustrates the main ideas used for the more technical calculations in section 3.2.

3.1.1 The BTZ partition function

Our starting point is the BTZ black hole in three-dimensional pure Einstein gravity. In Euclidean signature, this is a solution whose asymptotic boundary is a torus parameterized by angle φ and Euclidean time t_E, where we periodically identify t_E with inverse temperature $\beta = T^{-1}$ and twist angle θ:

$$ds^2 = dt_E^2 + d\varphi^2, \qquad (t_E, \varphi) \sim (t_E, \varphi + 2\pi) \sim (t_E + \beta, \varphi + \theta) \tag{3.3}$$

We have chosen units such that the spatial circle has unit radius, and a corresponding dimensionless time t_E, so β and energy will also be dimensionless. The Euclidean BTZ black hole is a saddle-point contribution to the dual CFT partition function on this torus; this is a grand canonical partition function, where θ plays the role of an imaginary chemical potential for the angular momentum J:

$$\begin{aligned} Z(\beta, \theta) &= \mathrm{Tr}\left[e^{2\pi i\tau\left(L_0 - \frac{c}{24}\right) - 2\pi i\bar{\tau}\left(\bar{L}_0 - \frac{c}{24}\right)}\right] \\ &= \mathrm{Tr}\left[e^{-\beta H - i\theta J}\right] \end{aligned} \tag{3.4}$$

$$\tau = \frac{\theta + i\beta}{2\pi}, \quad \bar{\tau} = \frac{\theta - i\beta}{2\pi} \tag{3.5}$$

$$H = L_0 + \bar{L}_0 - \tfrac{c}{12}, \quad J = \bar{L}_0 - L_0 \tag{3.6}$$

The 2π periodicity of θ is equivalent to integer quantization of J.

Now, the Euclidean BTZ black hole is a solid torus, where the Euclidean time circle is contractible in the bulk. The one-loop partition function of this geometry [103] (which is in fact exact to all loops, up to possible renormalisation of c, see [104] and more recently [95]) can be written as

$$Z_{\mathrm{BTZ}} = \chi_0(-1/\tau)\chi_0(1/\bar{\tau}) \tag{3.7}$$

where χ is the Virasoro character of the vacuum representation,

$$\chi_0(\tau) = \frac{(1-q)q^{-\frac{c-1}{24}}}{\eta(\tau)}, \quad q = e^{2\pi i\tau}. \tag{3.8}$$

To explain this result, we first note that the modular transform $\tau \mapsto -1/\tau$ swaps space and Euclidean time directions, after which we can interpret the BTZ solution as empty AdS_3, periodically identified with a twist. After this reinterpretation, the partition function counts perturbative excitations of AdS_3, which are the boundary gravitons, with dual CFT interpretation as the Virasoro descendants of the vacuum state. The central charge c appears as the Casimir energy of the vacuum, which (at tree level) takes the Brown-Henneaux [105] value $c = \frac{3\ell_3}{2G_N}$, where ℓ_3 is the AdS length (the subscript distinguishing it from the two-dimensional AdS length which we will encounter later).

3.1.2 The near-extremal limit

To recover the Schwarzian theory, we now wish to take an extremal limit, which requires low temperature (of order c^{-1}), with spin J of order c at least (and, as we will see later, much larger still for the simplest two-dimensional description). This requires a real chemical potential for the spin, corresponding to imaginary θ; the corresponding Euclidean solution is then complex, while the Lorentzian solution (given explicitly in equation (3.97)) is real. For such a situation, we parameterize the partition function using separate left- and right-moving temperatures,

$$\tau = \frac{i\beta_L}{2\pi}, \quad \bar{\tau} = -\frac{i\beta_R}{2\pi} \implies Z = \mathrm{Tr}\left[e^{-\beta_L\left(L_0 - \frac{c}{24}\right) - \beta_R\left(\bar{L}_0 - \frac{c}{24}\right)}\right], \tag{3.9}$$

so that $\beta = \frac{1}{2}(\beta_L + \beta_R)$, $\theta = \frac{1}{2i}(\beta_R - \beta_L)$.

To approach extremality, we will take low left-moving temperature, with β_L of order c and $c \gg 1$. We will also see that it is simplest to take a very large black hole, which here means very high right-moving temperature, $\beta_R \ll 1$. In particular, for the black hole to dominate over the vacuum in the grand canonical ensemble, we require $\beta_L \beta_R < (2\pi)^2$, which constrains β_R to be of order c^{-1} or smaller.

$$\text{Near extremal limit (grand canonical):} \quad c \gg 1, \quad \beta_L \text{ of order } c, \quad \beta_R \lesssim c^{-1} \tag{3.10}$$

We can now evaluate the BTZ partition function in this limit, simply taking low temperature for the left-moving character and high temperature for right-moving:

$$\chi_0\left(\frac{2\pi i}{\beta_L}\right) \sim 2\pi \left(\frac{2\pi}{\beta_L}\right)^{\frac{3}{2}} \exp\left[\frac{\beta_L}{24} + \frac{c}{24}\frac{(2\pi)^2}{\beta_L}\right] \tag{3.11}$$

$$\chi_0\left(\frac{2\pi i}{\beta_R}\right) \sim \exp\left[\frac{c}{24}\frac{(2\pi)^2}{\beta_R}\right] \tag{3.12}$$

For this, we use $\eta(\tau) \sim e^{\frac{i\pi}{12}\tau}$ as $\tau \to i\infty$, for the left-movers after the modular transform $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$.

We already see the Schwarzian partition function appearing in the left-moving half, but to compare it is most convenient to first pass to a different ensemble, where we fix the temperature and spin:

$$Z_J(\beta) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta J} Z(\beta, \theta) \tag{3.13}$$

This is a canonical ensemble in the Hilbert space of spin J states, in which we can take an equivalent near-extremal limit:

$$\text{Near extremal limit (canonical):} \quad c \gg 1, \quad \beta \text{ of order } c, \qquad J \gg c, \tag{3.14}$$

where taking the spin to be this large is not strictly necessary for now, but simplifies things more generally.

Inserting the BTZ partition function, we evaluate the integral taking us to fixed spin by saddle-point at large J.The BTZ partition function is not periodic in θ, so does not have a quantised spectrum in J. To fix this, one can sum over the restricted family of '$SL(2,\mathbb{Z})$' black holes [106] related by the modular transforms that take $\theta \mapsto \theta + 2n\pi$. Summing over this family is equivalent to extending the range of integration in (3.13) to all $\theta \in \mathbb{R}$. This makes no difference in the saddle-point approximation.In the near-extremal approximation, this is equivalent to fixing $\beta_L = 2\beta$, and doing an inverse Laplace transform in the right-moving sector, in the variable $\beta_R = \beta + i\theta$. We can evaluate this at the saddle point $\beta_R = 2\pi\sqrt{\frac{c}{24J}}$, to get

$$Z_J(\beta) \sim \frac{\pi}{2\sqrt{2}} \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}} \left(\frac{c}{6J^3}\right)^{1/4} \exp\left[2\pi\sqrt{\frac{c}{6}J} - \beta J + \frac{\beta}{12} + \frac{c}{12}\frac{\pi^2}{\beta}\right]. \tag{3.15}$$

We can match this precisely with the Schwarzian partition function:

$$Z_{\text{Schw}}(\tilde{\beta}) = \left(\frac{\pi}{\tilde{\beta}}\right)^{\frac{3}{2}} e^{\pi^2/\tilde{\beta}} \tag{3.16}$$

Our notation for Schwarzian correlation functions reflects the fact that they depend on temperature only through the combination $\tilde{\beta} = \frac{\beta}{2C}$ (and later, time $\tilde{t} = \frac{t}{2C}$). We can now write (3.15) as follows:

$$Z_J(\beta) = e^{S_0 - \beta E_0} Z_{\text{Schw}}(\tilde{\beta} = \tfrac{1}{2C}\beta) \tag{3.17}$$
$$C = \frac{c}{24} \tag{3.18}$$
$$S_0 \sim 2\pi\sqrt{\frac{c}{6}J} \tag{3.19}$$
$$E_0 = J - \frac{1}{12} \tag{3.20}$$

Recall that the Schwarzian coupling C has dimensions of time, and the scale here is set by the radius of the circle on which we put the CFT. For large c, the characteristic time of the Schwarzian theory is therefore parametrically long compared to the time it takes to encircle the spatial circle.

The (temperature independent part of the) prefactor contributes a logarithmic correction to the entropy S_0, which for large spin can be written as $S_0 \to S_0 - \frac{3}{2}\log S_0$. Precisely this logarithmic corrections was previously studied in references [107, 108]. As stressed in [109] this correction is important for a precise comparison between microscopic and macroscopic black hole entropy calculation.

So far we matched the partition function of a near extremal BTZ at fixed angular momentum with the Schwarzian partition function. We could try to do the same at fixed chemical potential. In this case, as we will see later, a $U(1)$ gauge mode besides the Schwarzian is relevant (the boundary conditions corresponding to fixed J will eliminate the dynamics of the gauge field). Including this mode one could do the match directly in the grand canonical ensemble. In any case, the difference only arises in one-loop corrections to the partition function, and for correlation functions the ensembles are equivalent to leading order at large J.

Finally, it is possible to extend this analysis to the case of a gravity theory with different left and right moving central charges $c_L \neq c_R$. The argument above works out in the same way, and the partition function will be the same as the Schwarzian theory with coupling $C = c_L/24$, while the extremal entropy goes as $S_0 \sim \sqrt{c_R J}$. This can be reproduced by considering a three dimensional black hole with an additional gravitational Chern Simons term in the bulk action [110]. A similar setup, and its relation to the Schwarzian theory, was recently considered in [111].

3.1.3 General irrational CFTs

Having recovered the density of states of the Schwarzian theory from near-extremal BTZ, we will now see that it is simple to recover this result for a very general class of CFTs. We give a two step argument, presented in a way that will later generalize to correlation functions. First, we show that the near-extremal partition function reduces to a vacuum character in a modular transformed expansion. Secondly, we will show that this character can be rewritten in the original channel in terms of the Schwarzian density of states.

Dominance of the dual vacuum character

For our first step, we use modular invariance of the theory, writing the partition function as a trace over a Hilbert space quantizing on a Euclidean time circle (rather than spatial circle) of the torus:

$$\begin{aligned} Z(\beta_L, \beta_R) &= Z\left(\frac{(2\pi)^2}{\beta_L}, \frac{(2\pi)^2}{\beta_R}\right) \\ &= \chi_0\left(\frac{2\pi i}{\beta_L}\right)\chi_0\left(\frac{2\pi i}{\beta_R}\right) + \sum_{\text{primaries}} \chi_h\left(\frac{2\pi i}{\beta_L}\right)\chi_{\bar{h}}\left(\frac{2\pi i}{\beta_R}\right) \end{aligned} \tag{3.21}$$

In the second line, we have written the trace as a sum over representations of the Virasoro symmetry, introducing the characters

$$\chi_h(q) = \frac{q^{h-\frac{c-1}{24}}}{\eta(\tau)} \tag{3.22}$$

of nondegenerate representations of lowest weight h. To write this, we have assumed that there are no currents in the theory (that is, operators with $h = 0$ or $\bar{h} = 0$)

besides the identity representation: if there are currents, we should classify representations according to the extended algebra, and would expect to recover a Schwarzian theory with corresponding additional symmetries. For simplicity, we assume something slightly stronger, namely that there is a 'twist gap' $\bar{h}_{\text{gap}}$, which is a positive lower bound on $\bar{h}$ for all non-vacuum primaries.

Given such a theory with large central charge and a (not necessarily large) twist gap, we can take the near-extremal limit (3.10), and find that the vacuum representation (in the modular transformed channel) dominates the partition function:

$$\frac{Z(\beta_L,\beta_R)}{Z_{\text{BTZ}}(\beta_L,\beta_R)} = 1 + O\Big(e^{-\frac{(2\pi)^2}{\beta_R}\bar{h}_{\text{gap}}}\Big) \tag{3.23}$$

The exponential suppression comes from the ratio of non-vacuum and vacuum right-moving characters, $\chi_{\bar{h}}\left(\frac{2\pi i}{\beta_R}\right)/\chi_0\left(\frac{2\pi i}{\beta_R}\right)$. While this is the parametric suppression in β_R (guaranteed for any given theory by uniform convergence of the partition function), it should be borne in mind that it can be accompanied by a prefactor which is parametrically large in c, so we need to take β_R correspondingly small. We give two examples.

For the contribution of a single primary operator, the ratio of left-moving characters $\chi_h\left(\frac{2\pi i}{\beta_L}\right)/\chi_0\left(\frac{2\pi i}{\beta_L}\right)$ contributes a factor of β_L, which is of order c, arising because the factor of $1-q$ in (3.8) subtracting null states from the vacuum. For a single state to be suppressed relative to the vacuum, we therefore must take $\beta_R \ll \frac{\bar{h}}{\log c}$. A gravitational explanation for this is that the BTZ extremality bound receives a one-loop correction of order $\exp\left(-\frac{(2\pi)^2}{\beta_R}\bar{h}\right)$ [89][112], which to be ignored must be much smaller than the typical energies (of order c^{-1}) that we are interested in. One could perhaps account for such corrections by absorbing them into a shift of E_0.

For pure gravity, the twist gap $\bar{h}_{\text{gap}}$ is of order c, but there are exponentially many states close to the twist gap, so corrections in (3.23) are accompanied by an exponentially large prefactor. These states are black holes in the modular dual channel we are using for our expansion; in the direct channel, they come from thermal AdS. This is simply the Hawking-Page transition we have already encountered, requiring us to take $\beta_R \lesssim c^{-1}$ for black holes to dominate the grand canonical ensemble.

We emphasize that, while these two examples may not exhaust all possible corrections from non-vacuum characters, for any given theory there is always some sufficiently small β_R to guarantee the dominance of the vacuum character. Our conclusions will apply to any CFT with large c and a twist gap, even if its bulk dual description (if any exists) is stringy or nonlocal on the AdS scale; we may just have to take the spin of the states we study to be very large.

Schwarzian spectrum from modular S matrix

We have already shown how to recover the Schwarzian partition function from the modular transform of the vacuum character, phrased as the BTZ partition function. However, our method, which required a simple closed form expression for the vacuum character, will not be available to us for the generalization to correlation functions. We therefore find it instructive to recover the Schwarzian in a different way.

Firstly, we can treat the right-moving sector, which is very simple. We are taking a very high right-moving temperature, which corresponds to very low temperature in the modular transformed channel. This simply projects us onto the vacuum, and we are sensitive only to the Casimir energy:

$$\chi_0\left(\frac{2\pi i}{\beta_R}\right) \sim \exp\left(\frac{c}{24}\frac{(2\pi)^2}{\beta_R}\right) \tag{3.24}$$

As we saw earlier, when we go to an ensemble of fixed spin this provides the zero temperature entropy S_0 and shifts the ground state energy E_0.

The interesting part, where the Schwarzian theory lives, is in the left-moving sector. Since the left-moving temperature is very low, the characters are simple in the 'direct' expansion rather than the modular transform; they simply become the Boltzmann weights of the lowest-weight state, since the low temperature suppresses all descendants:

$$\chi_h\left(\frac{i\beta_L}{2\pi}\right) \sim e^{-\beta_L\left(h-\frac{c}{24}\right)} \tag{3.25}$$

We therefore directly get the density of states of the Schwarzian theory if we can decompose the modular transformed vacuum character $\chi_0\left(\frac{2\pi i}{\beta_L}\right)$ into 'direct' characters $\chi_h\left(\frac{i\beta_L}{2\pi}\right)$ (even though the context is slightly different, this argument is much the same as the one used in [84] to compute the Schwarzian partition function).

This operation is the definition of the modular S-matrix, which we introduce presently. For this, we will use alternative parameters for central charge c and operator dimension h:

$$c = 1 + 6Q^2, \quad Q = b^{-1} + b \tag{3.26}$$

$$h = \frac{c-1}{24} + P^2 \quad \text{or} \quad h = \alpha(Q-\alpha), \text{ where } \alpha = \tfrac{Q}{2} + iP \tag{3.27}$$

These parameters are perhaps most familiar from Coulomb gas or Liouville theory, where Q is a background charge and b the Liouville coupling, and P is a target space momentum, but their appearance here is explained by something more universal, namely that they are the natural parameters for Virasoro representation theory. Note that there is a degeneracy of these new labels, since h is invariant under the reflection $P \to -P$.

We are interested in theories at large c, and two ranges of operator dimensions will turn out to be important for us:

$$\text{Limit } b \to 0 \implies c \to \infty \tag{3.28}$$

$$\text{Schwarzian states:} \quad k = b^{-1}P \text{ fixed} \implies h - \frac{c-1}{24} \sim \frac{6}{c}k^2 \tag{3.29}$$

$$\text{Schwarzian operators:} \quad h \text{ fixed} \implies \alpha \sim bh \tag{3.30}$$

As indicated, dimensions corresponding to fixed k will turn out to correspond to energies k^2 in the Schwarzian limit, while fixed h (which means imaginary momentum $P \sim i\frac{Q}{2}$) will correspond to operators we can insert in that limit.

As a side comment, in the analysis of Ponsot and Teschner [113] the parameter P labels unitary continuous representations of $\mathcal{U}_q(\mathfrak{sl}(2))$. As we will see below, in the Schwarzian limit the parameter $k = b^{-1}P$ will become a label of principal unitary series representations of $\mathfrak{sl}(2)$ with spin $j = \frac{1}{2} + ik$, while the dimension of these operators is related to the Casimir of these representations. We will see below how the Schwarzian limit of Virasoro is controlled by classical $\mathfrak{sl}(2)$ quantities. This limit is different than finite $L \sim bP$ with $b \to 0$ which corresponds to the classical limit of quantized Teichmuller space in the length basis. In CFT language this is called sometimes the semiclassical limit with large c and h/c finite.

Coming back to our calculation, with the notation introduced above, we can write the desired decomposition of the modular transformed vacuum block, now labeling the representations using P instead of h:

$$\chi_0\left(-1/\tau\right) = \int_{-\infty}^{\infty} \frac{dP}{2} \chi_P(\tau) \mathbb{S}_{P0} \tag{3.31}$$

$$\mathbb{S}_{P0} = 4\sqrt{2}\sinh(2\pi bP)\sinh(2\pi b^{-1}P) \tag{3.32}$$

The factor of $\frac{1}{2}$ in the measure here cancels the double counting of momenta P and $-P$. For completeness, we note the similar decomposition of a nondegenerate block. This relation is useful to compute the exact path integral of the Schwarzian theory over different Virasoro coadjoint orbits [114].

$$\chi_{P'}\left(-1/\tau\right) = \int_{-\infty}^{\infty} \frac{dP}{2} \chi_P(\tau) \mathbb{S}_{PP'} \tag{3.33}$$

$$\mathbb{S}_{PP'} = 2\sqrt{2}\cos(4\pi P'P) \tag{3.34}$$

The vacuum result can be recovered from this by subtracting the null states descending from the $h = 1$ state ($\chi_0 = \chi_{P=\frac{i}{2}(b^{-1}+b)} - \chi_{P=\frac{i}{2}(b^{-1}-b)}$, and similarly for the S-matrix). The characters are simple enough that these relations can be verified directly, but for the more complicated cases encountered later we will have access to the analogue of the S-matrix, but not the analogue of the characters (the conformal blocks). This is not surprising if we take the perspective that the S-matrix and its analogues are natural representation theoretic objects. Here, the identity S-matrix $\mathbb{S}_{P0}$ is the Plancherel measure of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$ closely associated with Virasoro [113].

We now finally take the near-extremal limit $c \to \infty$ with β_L of order c. The characters $\chi_P(\tau)$ become Boltzmann weights as discussed earlier, and the integral is dominated by weights with P of order b, labeled by k:

$$\chi_0\left(\frac{2\pi i}{\beta_L}\right) \sim e^{\frac{1}{24}\beta_L} 2^{\frac{3}{2}}(2\pi b^3) \int_0^{\infty} d(k^2)\sinh(2\pi k)\, e^{-\beta_L b^2 k^2} \tag{3.35}$$

In this integral, we recognize the Schwarzian density of states, going as sinh of the square root of energy. The prefactors contribute to S_0 and E_0. Doing the integral explicitly we can check that

$$\chi_0\left(\frac{2\pi i}{\beta_L}\right) \sim 4\pi\sqrt{2}b^3 e^{\frac{1}{24}\beta_L} Z^{(\mathrm{Schw})}\left(\tilde{\beta} = b^2\beta_L\right), \tag{3.36}$$

which obviously matches with the calculation done in the previous section.

From this method, the modular S-matrix $\mathbb{S}_{P0}$ very directly gives us the spectral data of the Schwarzian theory. As anticipated, this happens to be the Plancherel measure of the universal cover of classical $\mathfrak{sl}(2)$ if k is interpreted as the label of the principal series. To understand this, we recall the close relation between the Schwarzian and $\mathfrak{sl}(2)$ through a BF formulation of JT gravity [115][116] (see also [117] for another interpretation of the relation with $\mathfrak{sl}(2)$). Besides the principal series, the discrete series representation of $\mathfrak{sl}(2)$ will make an appearance when computing correlators.

We could perform a similar analysis for the right-moving sector, noting in particular that for large $\bar{P}$ we recover the 'Cardy' density of states $S_0 = 2\pi\sqrt{\frac{c}{6}J}$ from that limit of $\mathbb{S}_{0\bar{P}}$. However, this is not simple or natural, because the descendant states are very important at the high right-moving temperatures of interest. In this example, we cannot read S_0 directly from the modular S-kernel, because it counts only primaries, and is insensitive to the contamination from descendants, leading to a discrepancy in the logarithmic corrections. Our logic will always be to 'project onto the vacuum' in the right-moving sector, and then perform the modular transform to find the spectral data of the left-moving sector, where the Schwarzian resides.

3.2 CFT correlation functions

In this section, we discuss the correlation functions of CFTs in the near-extremal limit of the grand canonical ensemble $\beta_R \to 0$ with β_L of order c, under the previously introduced conditions of a twist gap and large c. Note that we do not require the CFT to be 'holographic', because additional assumptions of a sparse light spectrum or 't Hooft factorization are unnecessary (though our results could be strengthened under such assumptions). We obtain Lorentzian correlation functions at time separations t of order c, including dependence on angular separations φ. For comparison to the two-dimensional dual gravitational physics in the next section, we extract the S-wave correlation functions, where the operators are averaged over the circle.

We focus mainly on the simplest correlation function of interest, namely the two-point function of identical operators in the near-extremal limit (3.10) of the grand canonical ensemble. This will be sufficient to illustrate the main ideas, and we will discuss the general case in section 3.2.5. Our method parallels that of section 3.1.3: we first choose a conformal block decomposition of the correlation function such that the large spin limit $\beta_R \to 0$ selects only the vacuum block. The right-moving block is then simple to evaluate, but the left-moving block is more complicated. We follow the methods of [84] to show that this block contains the Schwarzian correlation function in the appropriate limit, by reexpressing it in a new channel which makes manifest the matrix elements of the operators with intermediate states.

For a given primary operator $\mathcal{O}$ with conformal weights $(h, \bar{h})$, we consider its two-point Wightman function (we will discuss time ordering and extract the retarded correlators later):

$$\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0,0)\rangle_{\beta_L,\beta_R} = \mathrm{Tr}\left[\mathcal{O}(z,\bar{z})\mathcal{O}(0,0)e^{-\beta_L\left(L_0-\frac{c}{24}\right)-\beta_R\left(\bar{L}_0-\frac{c}{24}\right)}\right] \tag{3.37}$$

For Lorentzian kinematics, the coordinates $z, \bar{z}$ are lightcone coordinates on the Lorentzian cylinder

$$z = \varphi - t, \quad \bar{z} = \varphi + t, \tag{3.38}$$

so in particular are 2π periodic $(z, \bar{z}) \sim (z+2\pi, \bar{z}+2\pi)$. In imaginary time $t_E = it$ they become complex coordinates on the torus, so additionally have the 'KMS' periodicity $(z, \bar{z}) \sim (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau})$, where $\tau = i\frac{\beta_L}{2\pi}$ and $\bar{\tau} = -i\frac{\beta_R}{2\pi}$. Since we are taking β_R very small and β_L very large this identification makes z approximately periodic and is reminiscent of the DLCQ limit [101]. We will not use this perspective in this chapter though. The operator ordering is determined by ordering in Euclidean time, provided in Lorentzian kinematics by an appropriate $i\epsilon$ prescription giving an imaginary part to t.

3.2.1 Right-moving sector: dominance of a vacuum block

The high right-moving temperature allows us to simplify the correlation function in the right-moving sector, by decomposing the amplitude with respect to quantization by spatial translations, instead of time evolution. Translations then suppress any intermediate state with $\bar{h} > 0$. We can phrase this as doing a modular transform to make the right-moving temperature very low, before writing the thermal trace as a sum over states, as well as inserting a complete set of intermediate states between the two operator insertions.

This means we are decomposing the correlation function into conformal blocks by inserting a complete set of states along a pair of Euclidean time cycles, separating the two insertions of $\mathcal{O}$. This is a 'necklace' channel decomposition of the correlation function, which we label by $\widetilde{N}$, with the tilde indicating that the intermediate states are inserted on Euclidean time circles, not spatial circles. Explicitly, we can write the conformal block expansion in this '$\widetilde{\text{Necklace}}$' channel as follows:

$$\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0,0)\rangle_{\beta_L,\beta_R} = \sum_{\substack{\text{primaries}\\ \mathcal{O}_1,\mathcal{O}_2}} |C_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}|^2 \mathcal{F}^{(\tilde{N})}_{h_1,h_2}(z,\beta_L)\bar{\mathcal{F}}^{(\tilde{N})}_{\bar{h}_1,\bar{h}_2}(\bar{z},\beta_R) \tag{3.39}$$

$$\begin{aligned}\bar{\mathcal{F}}^{(\tilde{N})}_{\bar{h}_1,\bar{h}_2}(\bar{z},\beta_R) = \left(\frac{2\pi}{\beta_R}\right)^{2\bar{h}} \sum_{N_1,N_2} &\langle \bar{h}_2, N_2|\mathcal{O}|\bar{h}_1, N_1\rangle\langle \bar{h}_1, N_1|\mathcal{O}|\bar{h}_2, N_2\rangle \\ &\times \exp\left[-\bar{z}\tfrac{2\pi}{\beta_R}(\bar{h}_1 + |N_1| - \tfrac{c}{24}) - (2\pi - \bar{z})\tfrac{2\pi}{\beta_R}(\bar{h}_2 + |N_2| - \tfrac{c}{24})\right]\end{aligned}$$

We have written the block decomposition in the 'barred' right-moving sector, which is most relevant for our present considerations; there is a similar expression for the left-moving blocks. The states $|\bar{h}, N\rangle$ are an orthonormal basis of descendants (at level $|N|$) of a primary state with weight $\bar{h}$. In the expression for the conformal block, we implicitly change the normalization of $\mathcal{O}$ to set $\langle \bar{h}_2|\mathcal{O}|\bar{h}_1\rangle$ to unity; this absorbs OPE coefficients, and also gives the prefactor in the block coming from the rescaling of the Euclidean time circle from length 2π to β_R. The exponential factors implement translation (generated by $\frac{2\pi}{\beta_R}(\bar{L}_0 - \frac{c}{24})$, where the factor comes from the same rescaling of lengths), first by $\bar{z}$ between the operator insertions, and then by $2\pi - \bar{z}$ to complete the spatial circle.

Now, when we take the $\beta_R \to \infty$ limit, choosing $0 < \bar{z} < 2\pi$ as we always may by the periodicity $(z, \bar{z}) \sim (z + 2\pi, \bar{z} + 2\pi)$, it is manifest in this decomposition that intermediate primaries $\mathcal{O}_1, \mathcal{O}_2$ with large $\bar{h}_1, \bar{h}_2$ and right-moving descendants are suppressed. Explicitly, the descendants drop out of the right-moving blocks, so we have

$$\bar{\mathcal{F}}^{(\tilde{N})}_{\bar{h}_1,\bar{h}_2}(\bar{z}, \beta_R) \sim \left(\frac{2\pi}{\beta_R}\right)^{2\bar{h}} \exp\left[-\frac{2\pi}{\beta_R}\left(\bar{z}\bar{h}_1 + (2\pi - \bar{z})\bar{h}_2 - 2\pi\tfrac{c}{24}\right)\right]. \tag{3.40}$$

We can guarantee that the infinite sum of terms is suppressed since the sum converges uniformly for any range of β_R bounded away from zero. From this, the dominant contribution comes from intermediate operators that minimize $\bar{z}\bar{h}_1 + (2\pi - \bar{z})\bar{h}_2$, under the condition that the OPE coefficient $C_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}$ is nonzero. In particular, this condition means that we cannot choose both $\mathcal{O}_1$ and $\mathcal{O}_2$ to be the identity. For sufficiently small $\bar{z}$, the dominant choice is $\mathcal{O}_1 = \mathcal{O}$, $\mathcal{O}_2 = 0$; this remains true in a finite (β_R-independent) range $0 < \bar{z} < \bar{z}_*$, where $\bar{z}_*$ is lower bounded by $2\pi\frac{\bar{h}_{\text{gap}}}{\bar{h}}$ (or $\bar{z}_* = \pi$ if that is smaller, from $\mathcal{O}_1 = 0$, $\mathcal{O}_2 = \mathcal{O}$). For now, we will fix $\bar{z}$ in this range; this is the most important region (in particular, dominating partial waves for any fixed angular momentum ℓ), since the correlator is exponentially suppressed when $|\bar{z}|$ is not small.

This behavior has a simple gravitational interpretation, which is easiest to state for spacelike Wightman functions. The two-point function gives an amplitude for a spacelike propagation of some particle between the insertion points, going a long distance because there is a very large black hole in the way. The particle can choose to go either way around the black hole, corresponding to the choices $\mathcal{O}_1 = \mathcal{O}$ and $\mathcal{O}_2 = 0$, or $\mathcal{O}_1 = 0$ and $\mathcal{O}_2 = \mathcal{O}$. However, the amplitude may be larger for the particle to split into two lighter particles (dual to $\mathcal{O}_1, \mathcal{O}_2$), each going a different way around the black hole, before rejoining on the far side. The corresponding amplitude with both particles on the same side is not relevant, since it has already been absorbed as a 'vacuum polarization' renormalization of $\mathcal{O}$, by choosing $\mathcal{O}$ to be a primary of definite scaling dimension: since BTZ is locally isometric to AdS_3, the only physical corrections come from configurations that make use of the nontrivial topology of the spacetime.

For $0 < \bar{z} < \bar{z}_*$, we therefore have dominance of this particular vacuum block, just as the vacuum character dominated the partition function in equation (3.23). The right-moving block becomes very simple in the $\beta_R \to 0$ limit, reducing to the exponential we have seen already if we fix $\bar{z}$ of order one. In fact, we can do a little better, giving an answer that works also for small $\bar{z}$ of order β_R or less, when the descendants of $\mathcal{O}$ in the intermediate states are not negligible. The descendants are suppressed in the sum over N_2, so the only relevant state is the vacuum, which is equivalent to replacing the torus with an infinite cylinder. The block therefore becomes the well-known result for the two-point function at finite temperature on an infinite line.

We can summarize the results of this section as follows:

$$\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0,0)\rangle_{\beta_L,\beta_R} \sim \mathcal{F}^{(\tilde{N})}_{\mathcal{O},0}(z,\beta_L)\bar{\mathcal{F}}^{(\tilde{N})}_{\mathcal{O},0}(\bar{z},\beta_R) \tag{3.41}$$

$$\bar{\mathcal{F}}^{(\tilde{N})}_{\mathcal{O},0}(\bar{z},\beta_R) \sim e^{\frac{(2\pi)^2}{\beta_R}\frac{c}{24}} \left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}\right)\right]^{-2\bar{h}} \tag{3.42}$$

So far, this applies for any theory with a twist gap (with any central charge) in the $\beta_R \to 0$ limit, either scaling $\bar{z} \propto \beta_R$ or with fixed $0 < \bar{z} < \bar{z}_*$. Our remaining task is to determine the left-moving identity block $\mathcal{F}^{(\tilde{N})}_{\mathcal{O},0}(z,\beta_L)$ in the limit of interest, taking $c \to \infty$ while keeping z and β_L proportional to c.

3.2.2 Left-moving sector: the Schwarzian

Much like in the previous section, there is a preferred channel in which the left-moving blocks simplify, now because we are taking low left-moving temperature $\beta_L \propto c \gg 1$. This corresponds to performing the usual quantization by time evolution, inserting complete sets of states in the thermal trace and between operator insertions. Just as in (3.39), we can write the blocks in this 'direct' necklace channel, which we label by N, as follows:

$$\begin{aligned}\mathcal{F}^{(N)}_{h_1,h_2}(z,\beta_L) = e^{-i\pi h}\sum_{N_1,N_2}&\langle h_2,N_2|\mathcal{O}|h_1,N_1\rangle\langle h_1,N_1|\mathcal{O}|h_2,N_2\rangle\\ &\times\exp\left[iz(h_1+|N_1|-\tfrac{c}{24})-(\beta_L+iz)(h_2+|N_2|-\tfrac{c}{24})\right]\end{aligned}$$

The phase arises because the z coordinate is rotated by $\frac{\pi}{2}$ relative to the time evolution, giving a factor $e^{i\frac{\pi}{2}h}$ for each operator insertion. Now, for $\operatorname{Im} z > 0$, which corresponds to the time-ordering where the insertion of $\mathcal{O}(z,\bar{z})$ comes after the insertion of $\mathcal{O}(0,0)$, the descendants are suppressed in the limit of interest. This is a limit $c \to \infty$ with $\beta_L \propto c$, but also (as we will see later) where we take

$$h_{1,2} = \frac{c-1}{24} + \frac{6}{c}k_{1,2}^2, \quad c\to\infty, \quad k_{1,2} \text{ fixed} \tag{3.43}$$

as in (3.29). The OPE coefficients of descendants are then suppressed by factors of $\frac{(h_1-h_2)^2}{c}$ and $\frac{h^2}{c}$ [87, 88], and we have

$$\mathcal{F}^{(N)}_{h_1,h_2}(z,\beta_L) \sim e^{-i\pi h}e^{iz(h_1-h_2)-\left(h_2-\frac{c}{24}\right)\beta_L} \tag{3.44}$$

$$\sim e^{-i\pi h}e^{\frac{1}{24}\beta_L}e^{i\frac{6z}{c}(k_1^2-k_2^2)-\frac{6\beta_L}{c}k_2^2}. \tag{3.45}$$

However, unlike for the right-movers, no single operator dominates in this necklace channel N where the blocks are simple. Instead, the correlation function is given by a vacuum block in the $\widetilde{N}$ channel, in which the blocks are more complicated. Our strategy is to evaluate the $\widetilde{N}$ identity block by decomposing it in terms of N channel blocks, which we can write simply. This conversion between different channels is implemented by fusion and modular S transformations (and, relevant later for higher-point out of time order correlators, braiding), generalizing the modular S transform we used in section 3.1.3. Fortunately, these transformations are known in relatively simple, explicit closed forms.

Warm-up: four-point identity block

As a warm-up, we first tackle a slightly simpler problem, finding a 'Schwarzian limit' of the Virasoro four-point identity block. This is a microcanonical version of the calculation we are interested in, where instead of taking a low temperature limit to take us near extremality, we fix a primary state $|\psi\rangle$ with dimension in the Schwarzian range $h_\psi = \frac{c-1}{24} + \frac{6}{c}k_\psi^2$, where k_ψ is fixed in the $c \to \infty$ limit. The cylinder kinematics we are using are then related to the usual four-point cross-ratio x by $x = e^{iz}$, since in radial quantization, we insert the operator $\mathcal{O}_\psi$ creating the state $|\psi\rangle$ at the origin and infinity, and $\mathcal{O}$ at 1 and $x = e^{iz}$. We then wish to compute the identity block in the 'T-channel', where we take the OPE of the two insertions of $\mathcal{O}$. Note that, unlike for the torus two-point function, we do not have a general argument that this identity block always dominates the four-point correlation function, which would require conditions on the OPE coefficients $C_{\psi\mathcal{O}t}$. Such a result would follow from the canonical result under the assumption of a version of the eigenstate thermalization hypothesis [118], applied to near-extremal large spin states.

To compute the T-channel identity, we will reexpress it in terms of the 'S-channel' blocks, where we take the OPE between $\mathcal{O}_\psi(0)$ and $\mathcal{O}(x)$, which are simple:

$$\mathcal{F}_t\begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \psi & \psi \end{bmatrix}(1-x) = \sum_N \langle\mathcal{O}|\mathcal{L}_{-N}\mathcal{O}_{h_t}|\mathcal{O}\rangle\langle\psi|\mathcal{L}_{-N}\mathcal{O}_{h_t}|\psi\rangle\ (1-x)^{-2h+h_t+|N|} \tag{3.46}$$

$$\begin{aligned}\mathcal{F}_s\begin{bmatrix} \mathcal{O} & \psi \\ \mathcal{O} & \psi \end{bmatrix}(x) &= \sum_N \langle\psi|\mathcal{O}|h_s, N\rangle\langle h_s, N|\mathcal{O}|\psi\rangle\ x^{-h_\psi-h+h_s+|N|} \\ &\sim x^{-h+\frac{6}{c}(k_s^2-k_\psi^2)}\end{aligned} \tag{3.47}$$

In the S-channel block, we have taken the appropriate 'Schwarzian' limit of operator dimensions and kinematics, in which case the descendants drop out for the same reason as before.

Now, we can evaluate the T-channel blocks if we can decompose them into S-channel blocks, since these become simple power laws in the limit of interest. Fortunately, there is an object that does precisely this, namely the fusion kernel $\mathbb{F}$ [119], which has the defining property

$$\mathcal{F}_t\begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \psi & \psi \end{bmatrix}(1-x) = \int_{-\infty}^{\infty} \frac{dP_s}{2}\mathcal{F}_s\begin{bmatrix} \mathcal{O} & \psi \\ \mathcal{O} & \psi \end{bmatrix}(x)\ \mathbb{F}_{P_sP_t}\begin{bmatrix} P & P_\psi \\ P & P_\psi \end{bmatrix}, \tag{3.48}$$

where we have used the variable P introduced in (3.26) to label operator dimensions $h = \frac{c-1}{24} + P^2$. The fusion kernel $\mathbb{F}$ is a kinematic object associated to Virasoro

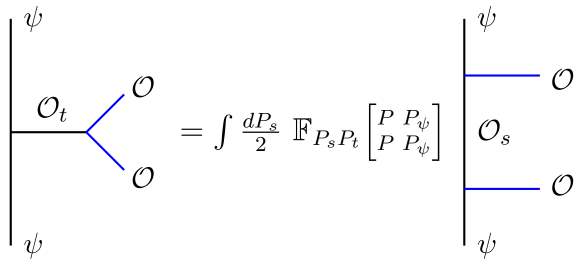


Figure 3.1: Diagram of the fusion transformation that was used to compute the left moving vacuum block in the appropriate limit. The blue lines correspond to the two operator insertions.

symmetry and it was computed explicitly in [113]. We will not quote the most general formula since we will only need certain special cases, but it can be found for example in equations (2.10)-(2.12) of [88].

Here, we need a special case of the fusion kernel, where the T-channel representation is the identity, denoted by 0. In this case, the fusion kernel simplifies [88, 120], and can be written as [85]

$$\mathbb{F}_{P_s0}\begin{bmatrix} P & P_\psi \\ P & P_\psi \end{bmatrix} = \rho_0(P_s)C_0(P_s, P, P_\psi), \tag{3.49}$$

where ρ_0 and C_0 are universal functions, appearing as densities of states and averaged OPE coefficients in a very general class of theories and a variety of limits [85]. We saw ρ_0 already in equation (3.32) as the decomposition of the identity character into modular transformed characters; C_0 can be written in closed form in terms of a special 'deformed Gamma-function' Γ_b:

$$\rho_0(P) = \mathbb{S}_{P0}[0] = 4\sqrt{2}\sinh(2\pi bP)\sinh(2\pi b^{-1}P) \tag{3.50}$$

$$C_0(P_1, P_2, P_3) = \frac{1}{\sqrt{2}}\frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3}\frac{\prod_{\pm\pm\pm}\Gamma_b\left(\frac{Q}{2} \pm iP_1 \pm iP_2 \pm iP_3\right)}{\prod_{k=1}^{3}\Gamma_b(Q+2iP_k)\Gamma_b(Q-2iP_k)} \tag{3.51}$$

In the numerator, we take the product over all eight possible combinations of sign choices. These objects simplify further when we take a 'Schwarzian limit' of large c, with either $P = bk$ corresponding to near-extremal operators, or h fixed. The two relevant limits are as follows:

$$\rho_0(bk) \sim 8\sqrt{2}\pi b^2 k \sinh(2\pi k) \tag{3.52}$$

$$C_0(bk_1, bk_2, i(\tfrac{Q}{2} - bh)) \sim \frac{b^{4h}}{\sqrt{2}(2\pi b)^3}\frac{\prod_{\pm\pm}\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)} \tag{3.53}$$

This is straightforward to derive using the identities

$$\frac{\Gamma_b(nQ+by)}{\Gamma_b(nQ)} \sim \left(\frac{\sqrt{2\pi}b^{n-1/2}}{\Gamma(n)}\right)^y \qquad (n>0), \qquad \frac{\Gamma_b(by)}{\Gamma_b(Q)} \sim \frac{(2\pi b^3)^{y/2}}{2\pi b}\Gamma(y) \tag{3.54}$$

valid in the $b \to 0$ limit with fixed y and integer n.

Those who have studied the Schwarzian theory will immediately find these formulas somewhat familiar [84, 117, 121]. In Appendix G we make the connection with Liouville theory and the calculation of [84]. First, as we have already seen, ρ_0 is proportional to the density of states of the Schwarzian theory, so we can write integrals over P (if they are dominated by integrating over P of order b) as

$$\int_{-\infty}^{\infty} \frac{dP}{2} \rho_0(P) f(P) \sim 4\pi\sqrt{2} b^3 \int_0^{\infty} d\mu(k) f(bk) \tag{3.55}$$

$$d\mu(k) = 2k dk \sinh(2\pi k), \tag{3.56}$$

where we have introduced the measure $d\mu(k)$ encoding the Schwarzian density of states:

$$Z^{(\mathrm{Schw})}(\tilde{\beta}) = \int d\mu(k) e^{-\tilde{\beta} k^2} = \left(\tfrac{\pi}{\tilde{\beta}}\right)^{\frac{3}{2}} e^{\pi^2/\tilde{\beta}} \tag{3.57}$$

Secondly, the result for C_0 is proportional to matrix elements appearing in Schwarzian correlation functions.

We can now apply these results to the four-point identity block, using (3.48). First, we can justify focusing on P_s of order b in the intermediate channel because the fusion kernel is exponentially suppressed for larger intermediate values. Putting everything together, we find our result for the T-channel vacuum block:

$$\begin{aligned} \mathcal{F}_t\left[\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \psi & \psi \end{smallmatrix}\right](1-x) &= \int_{-\infty}^{\infty} \frac{dP_s}{2} \rho_0(P_s) C_0(P_s, P, P_\psi)\, \mathcal{F}_s\left[\begin{smallmatrix} \mathcal{O} & \psi \\ \mathcal{O} & \psi \end{smallmatrix}\right](x) \\ &\sim \frac{2b^{4h} x^{-h}}{(2\pi)^2} \int_0^{\infty} d\mu(k_s) \frac{\prod_{\pm\pm} \Gamma(h \pm ik_s \pm ik_\psi)}{\Gamma(2h)} x^{b^2(k_s^2 - k_\psi^2)} \end{aligned} \tag{3.58}$$

As a check, we can evaluate this in the limit $x \to 1$, in which case the integral is dominated by large k_s; we find that $\mathcal{F}_t \sim (1-x)^{-2h}$, giving the expected short distance behavior of the block, with the usual normalization.

Before returning to the torus block, we discuss a limit of our result for the identity block, taking $k_\psi \gg 1$. In that case, the integral is dominated by k_s close to k_ψ, with $k_s - k_\psi = \delta$ of order one:

$$\begin{aligned} \mathcal{F}_t\left[\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \psi & \psi \end{smallmatrix}\right](1-x) &\sim \frac{x^{-h}}{2\pi} (2b^2 k_\psi)^{2h} \int d\delta e^{\pi\delta} \frac{\Gamma(h+i\delta)\Gamma(h-i\delta)}{\Gamma(2h)} x^{2b^2 k_\psi \delta} \\ &= x^{-h} \left(\frac{x^{-ib^2 k_\psi} - x^{ib^2 k_\psi}}{2ib^2 k_\psi} \right)^{-2h} \end{aligned} \tag{3.59}$$

This is the same result found for the vacuum block in the 'heavy-light' limit [87, 122], where c was taken to infinity with h_ψ/c fixed (but different from $\frac{1}{24}$). In fact, in [88] this result was derived with precisely the method used here. Our result thus interpolates smoothly to this different regime.

As anticipated, the T-channel vacuum block found in equation (3.58) is equal to the microcanonical two-point function of the Schwarzian theory between energy eigenstates labeled by k_ψ [123]. Equation (3.59) can be understood as a statement of equivalence of canonical and microcanonical ensembles in the thermodynamic limit of the Schwarzian theory, since

$$\mathcal{F}_t\begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \psi & \psi \end{bmatrix}(1-x) \propto \left(\frac{x^{-ib^2k_\psi} - x^{ib^2k_\psi}}{2b^2k_\psi}\right)^{-2h} = \left[\frac{\beta_\psi}{\pi}\sinh\left(\frac{\pi}{\beta_\psi}z\right)\right]^{-2h} \tag{3.60}$$

is the thermal correlator with temperature $\beta_\psi = \frac{\pi}{b^2k_\psi}$ (the proportionality factor arising from the conformal map between the plane and the cylinder).

Note that (3.60) has a thermal KMS periodicity, which in particular leads to a singularity at $z = i\beta_\psi$ as a thermal image of the short-distance singularity. This was dubbed a 'forbidden singularity' in [124], since it cannot appear in the exact four-point block or correlation function. As noted in [88], it is associated with a divergence of the integral in (3.59) as $\delta \to -\infty$, but this is an artifact of the approximation we are making in the integrand, which is valid only for $e^{-\beta_\psi} < |x| < 1$ (with $k_\psi \log x$ fixed in the limit $k_\psi \to \infty$). The exact Schwarzian integral (3.58) does not have such a divergence, so provides a regularization which resolves the forbidden singularity.

Returning to the torus

We now return to our discussion of the torus two-point function, for which we would like to compute the identity block in the $\widetilde{N}$ channel in (3.41). The blocks are simple in the N channel (3.45), so we would like to decompose the $\widetilde{N}$ identity block in terms of N blocks, with an expression analogous to (3.48) which expanded a T-channel block in terms of S-channel blocks for the four-point function. The idea is much the same, but we must go through some intermediate steps, following the Moore-Seiberg construction [119], recently reviewed in the current context of irrational theories in [85], to which we refer the reader for details. The sequence of moves we use is illustrated in figure 3.2.

First, we note that the $\widetilde{N}$ identity block is in fact equal to an identity block in a different channel, denoted $\widetilde{OPE}$. For the block decomposition in this channel, we take the OPE between the two insertions of $\mathcal{O}$, and insert a single complete set of states propagating in the spatial circle. In general, this is related to the $\widetilde{N}$ decomposition (3.39) by a fusion move, with external operators $\mathcal{O}, \mathcal{O}, \mathcal{O}_2, \mathcal{O}_2$ and internal operator $\mathcal{O}_1$ (using the labeling following (3.39)). However, for the vacuum block, we have $\mathcal{O}_2 = 0$, so this move is trivial: in the fusion of 0 with itself, only 0 appears, so the $\widetilde{N}$ vacuum block equals the $\widetilde{OPE}$ vacuum block. For the next step, we want to replace the complete set of intermediate states propagating in the spatial circle (the $\widetilde{OPE}$ channel) with states propagating in the Euclidean time circle (the OPE channel). This is a modular S-transform, with kernel $\mathbb{S}$. In general, this would be the S-transform appropriate for a torus one-point function, where the external operator is determined by the representation appearing in the OPE. For us, this representation is the identity, so the modular S kernel is the same one we used for the partition function, given in (3.32) and (3.50). Our final move takes us from the OPE channel

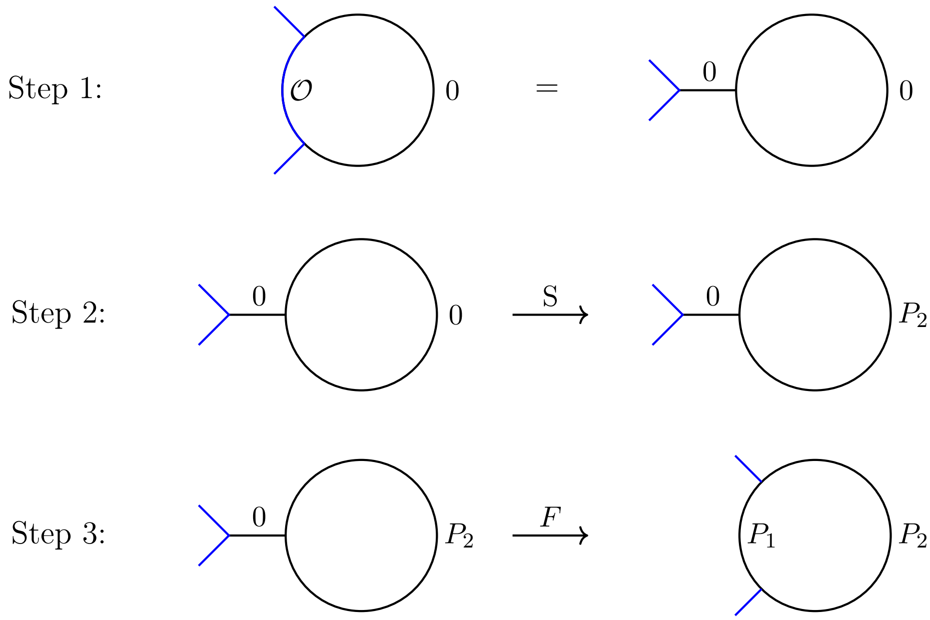


Figure 3.2: Diagram of the fusion and modular transformation that was used to compute the left moving vacuum torus block in the appropriate limit. In the top diagrams, the circle is a spatial circle, and in the bottom diagrams it is a Euclidean time circle; these are swapped by the S-transform in step 2.

to the necklace N channel, and is the same fusion move we just used for the four-point function; the only difference is that the external operator $\mathcal{O}_\psi$ is now whichever intermediate operator appears in the thermal sum over states, labeled here by P_2.

In equations, the moves we have just described are as follows:

$$\begin{aligned}
\mathcal{F}^{(\tilde{N})}_{h,0} &= \mathcal{F}^{(\widetilde{OPE})}_{0,0} && (3.61)\\
&= \int \frac{dP_2}{2} \mathbb{S}_{0P_2} \mathcal{F}^{(OPE)}_{0,P_2} && (3.62)\\
&= \int \frac{dP_1}{2}\frac{dP_2}{2} \mathbb{F}_{0P_1}\begin{bmatrix} P & P_2 \\ P & P_2 \end{bmatrix} \mathbb{S}_{0P_2} \mathcal{F}^{(N)}_{P_1,P_2} && (3.63)\\
&= \int \frac{dP_1}{2}\frac{dP_2}{2} \rho_0(P_1)\rho_0(P_2) C_0(P_1,P_2,P) \mathcal{F}^{(N)}_{P_1,P_2} && (3.64)
\end{aligned}$$

We have written the final expression in terms of our universal functions (3.50), (3.51).

This expression holds in complete generality, but we can now take a limit to extract an explicit expression for $\mathcal{F}^{(\tilde{N})}_{h,0}$. For this, we simply substitute (3.45) for $\mathcal{F}^{(N)}_{P_1,P_2}$, along with (3.52) and (3.53) for the Schwarzian limits of ρ_0 and C_0 to obtain our final result for the left-moving part of the correlation function:

$$\mathcal{F}_{h,0}^{(\tilde{N})} \sim e^{-i\pi h} b^{4h} \chi_0 \left(\frac{2\pi i}{\beta_L}\right) G_h^{(\mathrm{Schw})} \left(\tilde{t}_E = -ib^2 z, \tilde{\beta} = b^2 \beta_L\right) \tag{3.65}$$

$$G_h^{(\mathrm{Schw})} \left(\tilde{t}_E, \tilde{\beta}\right) = \frac{1}{2\pi^2 Z^{(\mathrm{Schw})}(\tilde{\beta})} \int d\mu(k_1) d\mu(k_2) \frac{\prod_{\pm\pm} \Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)} e^{-\tilde{t}_E(k_1^2 - k_2^2) - \tilde{\beta} k_2^2} \tag{3.66}$$

We have here extracted a normalizing factor of the vacuum character

$$\chi_0 \left(\frac{2\pi i}{\beta_L}\right) \sim 4\pi\sqrt{2} b^3 e^{\frac{1}{24}\beta_L} Z^{(\mathrm{Schw})} \left(\tilde{\beta} = b^2 \beta_L\right), \tag{3.67}$$

where $Z^{(\mathrm{Schw})}$ is the Schwarzian partition function given in (3.57).

$G_h^{(\mathrm{Schw})}$ is the result in [84] for the Schwarzian thermal two-point function, expressed in Euclidean time $\tilde{t}_E$, with $\mathrm{Re}\, \tilde{t}_E > 0$. The prefactor $e^{-i\pi h} b^{4h}$ appears naturally in the Schwarzian when we rescale time and analytically continue to Lorentzian signature, since the operators transform nontrivially under scaling, but we find it more convenient for the discussion below to separate it. It is normalized to give a pure power law in the limit of small $\tilde{t}_E$, namely $G_h^{(\mathrm{Schw})} \sim \tilde{t}_E^{-2h}$. Writing this in terms of z and combining with the factor $e^{-i\pi h} b^{4h}$, the block $\mathcal{F}_{h,0}^{(\tilde{N})}$ has the usual short-distance behavior z^{-2h}. This result is valid for $0 < \mathrm{Im}\, z < \beta_L$, and the result for real z (with the current operator ordering) is obtained in the limit $\mathrm{Im}\, z \to 0^+$. In fact, if we take real z, by expanding C_0 to higher orders we find that the integral over k_1 is rendered convergent by an '$i\epsilon$' appearing automatically with the correct sign: the relevant correction to $\log C_0$ is proportional to $(b^2 \log b)\, k_1^2$.

Just as in (3.59), we can evaluate the canonical two-point function $G^{(\mathrm{Schw})}$ in a semiclassical limit, which here means $\tilde{\beta}, \tilde{t}_E \ll 1$. The calculation is much the same as led to (3.59), with the integral dominated by the region where both k_1 and k_2 are close to the value $\pi/\tilde{\beta}$ corresponding to the thermodynamic energy, and $k_1 - k_2$ is of order one

$$G_h^{(\mathrm{Schw})} \left(\tilde{t}_E, \tilde{\beta}\right) \sim \left(\frac{\tilde{\beta}}{\pi} \sin\left(\frac{\pi}{\tilde{\beta}} \tilde{t}_E\right)\right)^{-2h} \tag{3.68}$$

When taking this limit we assumed that h is order one. One can also consider a semiclassical limit with large $h \sim c$ which is more complicated [125] but encodes some simple bulk backreaction effects. Finally, a nice aspect of the expression (3.65) (and similarly (3.58)) is that it interpolates between the semiclassical behavior of equation (3.68) and the late time behavior. At late times the approximation leading to (3.68) fails signaling that in this regime strong coupling Schwarzian effects are important. This happens in our context for times $\tilde{t}_E \gg c$. Expression (3.65) gives the asymptotics

$$G_h^{(Schw)}(\tilde{t}_E, \tilde{\beta}) \sim \tilde{t}_E^{-3}, \qquad \tilde{t}_E \gg c \tag{3.69}$$

where we omitted a time independent prefactor that depends on c, $\tilde{\beta}$ and h. In the $\tilde{\beta} \to \infty$ limit this power changes to $G_h^{(Schw)} \sim \tilde{t}_E^{-\frac{3}{2}}$. In the context of the Schwarzian theory this was observed in [84, 126]. In the context of 2d CFT this behavior of conformal blocks was observed numerically in [127] and analytically [88]. The advantage of (3.65) is then that it interpolates between different regimes.

3.2.3 Time ordering and retarded two-point function

Including both left- and right-moving pieces, we have our result for the normalized grand-canonical two-point function in the near-extremal limit:

$$\frac{\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0,0)\rangle_{\beta_L,\beta_R}}{Z_{\beta_L,\beta_R}} \sim e^{-i\pi h} b^{4h} \left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}\right)\right]^{-2\bar{h}} G_h^{(\mathrm{Schw})}\left(-ib^2 z, b^2\beta_L\right) \quad (3.70)$$

Note that the ordering is important here, with the insertion of $\mathcal{O}(z,\bar{z})$ coming after that of $\mathcal{O}(0,0)$, and furthermore that this is valid for $0<\bar{z}<\bar{z}_*\leq\pi$. The time-ordering appeared in our derivation through the choice of necklace channel N, since the simplification (3.45) of the blocks occurs only with the given time ordering. Here, the other ordering differs only by a phase from swapping the operator insertions in the left-moving block (though we will see that things get slightly trickier for out of time order correlators with more operator insertions):

$$\frac{\langle \mathcal{O}(0,0)\mathcal{O}(z,\bar{z})\rangle_{\beta_L,\beta_R}}{Z_{\beta_L,\beta_R}} \sim e^{+i\pi h} b^{4h} \left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}\right)\right]^{-2\bar{h}} G_h^{(\mathrm{Schw})}\left(+ib^2 z, b^2\beta_L\right) \quad (3.71)$$

Alternatively, we could take the original result and set $(z,\bar{z})\to(-z,-\bar{z})$, so the right-moving block produces a phase $e^{2\pi i\bar{h}}$. This is equivalent for integer spin, $\bar{h}-h\in\mathbb{Z}$. It will be useful to consider also the retarded correlator

$$G_R(z,\bar{z}) = -i\langle[\mathcal{O}(z,\bar{z}),\mathcal{O}(0,0)]\rangle\,\Theta\big(t=\tfrac{1}{2}(\bar{z}-z)\big)\,, \quad (3.72)$$

which for $0<\bar{z}<\bar{z}_*$ and $z=-b^{-2}\tilde{t}<0$ is given by

$$G_R(z,\bar{z}) \sim b^{4h}\left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}\right)\right]^{-2\bar{h}} 2\,\mathrm{Im}\left[e^{-i\pi h}G_h^{(\mathrm{Schw})}\left(-ib^2 z, b^2\beta_L\right)\right]. \quad (3.73)$$

Note in particular that with this definition of $G_h^{(\mathrm{Schw})}$, the retarded correlator is not simply proportional to the retarded correlator in the Schwarzian theory, due to the additional phase in our choice of normalization. On the opposite side of the lightcone, with $-\bar{z}_*<\bar{z}<0$ and again $z=-b^{-2}\tilde{t}<0$, we have

$$G_R(z,\bar{z}) \sim b^{4h}\left[\frac{\beta_R}{\pi}\sinh\left(-\frac{\pi}{\beta_R}\bar{z}\right)\right]^{-2\bar{h}} 2\,\mathrm{Im}\left[e^{+i\pi h}G_h^{(\mathrm{Schw})}\left(-ib^2 z, b^2\beta_L\right)\right]. \quad (3.74)$$

In the semiclassical limit of the Schwarzian (3.68), the phase is precisely $e^{-i\pi h}$, so this piece vanishes to leading order in that limit. Away from the lightcone, when $|\bar{z}|\gg\beta_R$, the correlation functions are exponentially suppressed.

We should note that the full correlator also contains lightcone singularities when z is an integer multiple of 2π, and our result is not strictly valid parametrically close to these lightcones. However, the strength of the singularity decays exponentially in time, and their contribution becomes negligible in the Schwarzian limit after smearing the operators by any fixed amount. We can smear either by slightly displacing the insertions in Euclidean time, or for retarded correlators by integrating against a more general smooth function, for example by taking partial waves as in the next section. In the conformal block expansion, the singularities arise from an infinite sum over double-twist exchanges; our argument for dominance of the vacuum block applies in any compact region bounded away from lightcone singularities, where the sum over blocks converges uniformly.

3.2.4 Partial waves

For direct comparison with a gravitational two-dimensional nAdS_2 dual, we should consider the partial waves, which are correlation functions of the Fourier modes of operators:

$$\mathcal{O}_\ell(t) = \int_0^{2\pi} \frac{d\varphi}{2\pi}\, e^{i\ell\varphi}\mathcal{O}(z=\varphi-t, \bar{z}=\varphi+t) \tag{3.75}$$

For this, we cannot consider time-ordered correlators, because the lightcone singularity at $\bar{z}=0$ is not integrable (at least, for $h>\frac{1}{2}$). However, the retarded correlator does not suffer from the same problem, when we interpret the singularity as a distribution. The expectation value of the commutator can be thought of as a discontinuity across a branch cut in the Euclidean time plane, and we can define the partial waves by deforming the integral to a contour passing below the cut, around the branch point, and back above it, while avoiding the singularity at the branch point itself. In practice, since this prescription for the integral is analytic, we can simply perform the integral for $h<\frac{1}{2}$ when it converges, and analytically continue to general h.

Since the typical scale on which the correlator varies in the z direction is of order c, the integral over φ at fixed t can, to good approximation, be replaced by an integral over $\bar{z}$ at fixed $z=-t$:

$$G_\ell(t) = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, e^{i\ell\varphi}G_R(z=\varphi-t, \bar{z}=\varphi+t) \tag{3.76}$$

$$\sim e^{-i\ell t}\int_{-\pi}^{\pi} \frac{d\bar{z}}{2\pi}\, e^{i\ell\bar{z}}G_R(z=-2t, \bar{z}) \tag{3.77}$$

The dominant contribution comes from close to the lightcone, with $\bar{z}$ of order β_R, both from $\bar{z} > 0$ (3.73) and $\bar{z} < 0$ (3.74):

$$G_\ell(t) \sim e^{-i\ell t}\mathcal{N}_\ell\, 2\,\mathrm{Im}\left[G_h^{(\mathrm{Schw})}\left(2ib^2 t, b^2\beta_L\right)\right] \tag{3.78}$$

$$\begin{aligned}\mathcal{N}_\ell &= \left(\frac{2\pi}{\beta_R}\right)^{2\bar{h}-1}\frac{\Gamma\left(\bar{h}+\frac{\ell\beta_R}{2\pi i}\right)\Gamma\left(\bar{h}-\frac{\ell\beta_R}{2\pi i}\right)}{\Gamma(2\bar{h})}\frac{\sin\left(\pi\bar{h}+\frac{1}{2}i\ell\beta_R\right)}{\sin\left(\pi\bar{h}\right)} \\ &\sim \left(\frac{2\pi}{\beta_R}\right)^{2\bar{h}-1}\frac{\Gamma\left(\bar{h}\right)^2}{\Gamma(2\bar{h})}\end{aligned} \tag{3.79}$$

We see that each individual partial wave is proportional to the retarded correlator of the Schwarzian. The approximation to $\mathcal{N}_\ell$ in the second line is valid when $\ell\beta_R \ll 1$; it is spin independent because such modes cannot resolve the details of the angular dependence, effectively seeing a delta-function on the lightcone.

To quantify the error we introduced in our approximation of the integral, we can Taylor expand the correlation function:

$$\begin{aligned}G_R(z=\varphi-t,\bar{z}=\varphi+t) &= G_R(\bar{z}-2t,\bar{z}) \\ &= G_R(-2t,\bar{z}) - \tfrac{1}{2}\bar{z}\partial_t G_R(-2t,\bar{z}) + \cdots\end{aligned} \tag{3.80}$$

The factor of $\bar{z}$ results in an additional factor of β_R after integrating, while the time derivative results in a factor of c^{-1}. The neglected corrections are therefore suppressed by a relative factor of β_R/c. We will later see that this is characteristic of interactions with graviton Kaluza-Klein modes.

3.2.5 Higher-point functions and OTOC

We now discuss some salient points for the generalization of the above results to higher point functions. This is fairly straightforward, but introduces some new ingredients: specifically, the choice of operator ordering becomes more important, and there are new kinematic regimes where an identity block need not dominate. To illustrate these new ideas, we discuss the computation of the four point function of pairwise identical operators.

In the canonical ensemble we want to compute

$$\langle\mathcal{O}_A(z_1,\bar{z}_1)\mathcal{O}_A(z_2,\bar{z}_2)\mathcal{O}_B(z_3,\bar{z}_3)\mathcal{O}_B(z_4,\bar{z}_4)\rangle_{\beta_L,\beta_R} \tag{3.81}$$

for operators with dimensions $(h_A,\bar{h}_A)$ and $(h_B,\bar{h}_B)$ (considering other time orderings later). As before, we use lightcone coordinates for the locations of the operators, $z_i = \varphi_i - t_i$ and $\bar{z}_i = \varphi_i + t_i$ for $i = 1,\ldots 4$. We will take all times to be large of order $t \sim c$ and choose the angles such that $0 < \bar{z} < 2\pi$ for all insertions.

As in section 3.2.1, we must first identify the relevant blocks in the $\beta_R \to 0$ limit by considering the right moving sector. Once again, we insert complete sets of states at circles of constant angle between every operator insertion, generalizing the dual necklace ($\widetilde{N}$) channel in equation (3.39), here with four sets of states. We consider first a configuration with $0 < \bar{z}_1 < \bar{z}_2 \ldots < \bar{z}_4 < 2\pi$ where all $\bar{z}_i$ and $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$ are fixed as we take the $\beta_R \to 0$ limit. Descendants then drop out and the right moving dual necklace block is proportional to

$$\bar{\mathcal{F}}^{(\tilde{N})}_{\bar{h}_1,\ldots,\bar{h}_4} \propto \exp\left[-\frac{2\pi}{\beta_R}\left(\bar{z}_{21}\bar{h}_1 + \bar{z}_{32}\bar{h}_2 + \bar{z}_{43}\bar{h}_3 + (2\pi - \bar{z}_{41})\bar{h}_4\right)\right]. \tag{3.82}$$

As in section 3.2.1, we find the dominant block by minimizing the kinematic combination in the exponent over values of $\bar{h}_i$ allowed by the fusion rules.

Since the right-moving block is exponentially suppressed for fixed $\bar{z}_{ij}$, these configurations will not in fact be relevant for correlation functions of partial waves. Instead, we must consider kinematic regimes where some $\bar{z}_{ij}$ scale proportionally to β_R, which we can split into three cases:

Case 1: Identical operators $\mathcal{O}_A$ and $\mathcal{O}_B$ are close together in pairs, but the separation between pairs is of order one. Concretely, we could have $\bar{z}_{12}, \bar{z}_{43}$ each of order β_R, with $\bar{z}_{32}$ order one.

Case 2: Each $\mathcal{O}_A$ is close to one of the $\mathcal{O}_B$ and the pairs are order one separated. For example, we take $\bar{z}_{32}, 2\pi - \bar{z}_{41}$ small, with $\bar{z}_{21}, \bar{z}_{43}$ order one.

Case 3: All operators are close to each other with all $\bar{z}_{ij}$ small.

Case 1: There is a unique choice of intermediate operators so the block is not exponentially suppressed in β_R, with the identity propagating when the $\bar{z}$ separation is order one. Namely, we must take $\mathcal{O}_1 = \mathcal{O}_A$, $\mathcal{O}_2 = 0$, $\mathcal{O}_3 = \mathcal{O}_B$ and $\mathcal{O}_4 = 0$. As in section 3.2.1 we can write a formula for the right-moving blocks which is valid when $\bar{z}_{12}, \bar{z}_{43} \sim \beta_R$, by including descendants of $\mathcal{O}_1 = \mathcal{O}_A$ and $\mathcal{O}_3 = \mathcal{O}_B$:

$$\bar{\mathcal{F}}^{(\tilde{N})}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(\bar{z}_{ij},\beta_R) \sim e^{\frac{(2\pi)^2}{\beta_R}\frac{c}{24}}\left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}_{12}\right)\right]^{-2\bar{h}_A}\left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}_{34}\right)\right]^{-2\bar{h}_B} \tag{3.83}$$

This is just the product of separate vacuum two-point functions on an infinite line, along with a term encoding the Casimir energy of the vacuum. The four point function is then given by the $\widetilde{N}$ identity blocks

$$\langle \mathcal{O}_A\mathcal{O}_A\mathcal{O}_B\mathcal{O}_B\rangle_{\beta_L,\beta_R} \sim \mathcal{F}^{(\tilde{N})}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(z_{ij},\beta_L)\bar{\mathcal{F}}^{(\tilde{N})}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(\bar{z}_{ij},\beta_R) \tag{3.84}$$

up to exponentially small corrections

Case 2: The blocks are exponentially suppressed unless the identity appears in the intermediate channels with finite $\bar{z}$ separation, namely $\mathcal{O}_1 = 0$ and $\mathcal{O}_3 = 0$. But the fusion rules for the identity would then simultaneously demand that $\mathcal{O}_2 = \mathcal{O}_A$ and $\mathcal{O}_2 = \mathcal{O}_B$ (and similarly for $\mathcal{O}_4$), which cannot both be satisfied in the same block. The leading contribution is therefore suppressed by either $e^{-\frac{2\pi}{\beta_R}\bar{z}_{21}\bar{h}_{\text{gap}}}$ or $e^{-\frac{2\pi}{\beta_R}\bar{z}_{43}\bar{h}_{\text{gap}}}$, and this region does not contribute to partial waves.

Case 3: When all operators are separated by $\bar{z}_{ij}$ of order β_R, there is no suppression in the $\beta_R \to 0$ limit as long as $\mathcal{O}_4 = 0$. This means we have $\mathcal{O}_1 = \mathcal{O}_A$

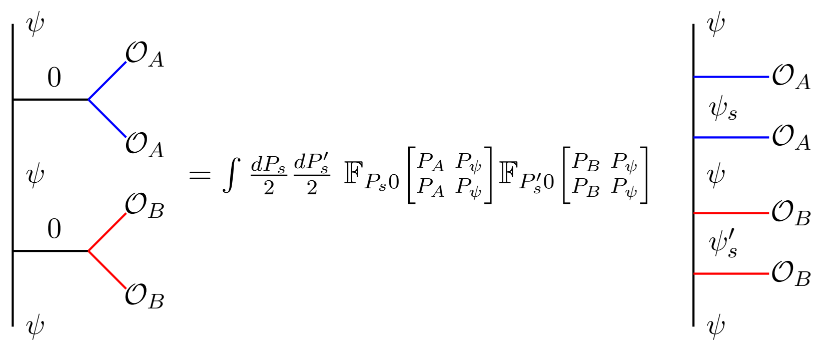


Figure 3.3: Diagram of the fusion transformation that was used to compute the four-point left moving vacuum block in the appropriate limit. The blue (red) line corresponds to the external $\mathcal{O}_A$ ($\mathcal{O}_B$) insertion.

and $\mathcal{O}_3 = \mathcal{O}_B$ (for the given the ordering of operators in $\bar{z}$), but we must include all intermediate operators in the sum over $\mathcal{O}_2$. At this point, we could make an additional assumption that we have a holographic 2d CFT, where 't Hooft factorization applies, which suppresses non-vacuum channels by small OPE coefficients of order c^{-1}. Nonetheless, as we will see in more detail later, in partial waves the contribution from case 3 is parametrically small regardless of such factorization, since it involves a small kinematic regime in the integral over angles.

The conclusion is that **Case 1** will give the dominant contribution to the correlators of partial waves, through the identity block (3.84), with right-moving piece given by (3.83). We next need to compute the left moving torus block by transforming back to the direct necklace channel, as in section 3.2.2. The only difference is that now the fusion transformation has to be applied twice, one to each pair $\mathcal{O}_A\mathcal{O}_A$ and $\mathcal{O}_B\mathcal{O}_B$. Instead of spelling out the details, we show the relevant fusion transformation in figure 3.3 for the case of the microcanonical ensemble calculation. The torus block is obtained similarly, with an additional integral over the state $|\psi\rangle$, weighted by a Boltzmann factor and Schwarzian density of states. Taking the appropriate limits and using the expression (3.49) we obtain

$$\mathcal{F}^{(\tilde{N})}_{h,0} \sim e^{-i\pi h_A - i\pi h_B} b^{4h_A+4h_B} \chi_0\left(\frac{2\pi i}{\beta_L}\right) \; G^{(\text{Schw})}_{h_A,h_B}\left(\tilde{t}_i = -ib^2 z_i, \tilde{\beta} = b^2\beta_L\right) \tag{3.85}$$

where anticipating its interpretation we defined the function appearing in the right hand side as

$$\begin{aligned} G^{(\text{Schw})}_{h_A,h_B} \;\; &= \;\; \frac{1}{2\pi^2 Z^{(\text{Schw})}(\tilde{\beta})} \int d\mu(k_1) d\mu(k_2) d\mu(k_3) e^{-\tilde{t}_{21}k_1^2 - \tilde{t}_{43}k_2^2 - (\tilde{\beta}-\tilde{t}_{21}-\tilde{t}_{43})k_3^2} \\ &\times \frac{\prod_{\pm\pm}\Gamma(h_A \pm ik_1 \pm ik_3)}{\Gamma(2h)} \frac{\prod_{\pm\pm}\Gamma(h_B \pm ik_2 \pm ik_3)}{\Gamma(2h)} \end{aligned} \tag{3.86}$$

with $\tilde{t}_{21} \equiv -ib^2(z_2 - z_1)$ and $\tilde{t}_{43} \equiv -ib^2(z_4 - z_3)$. Comparing with the notation in figure 3.3 we defined $P_s = bk_1$, $P'_s = bk_2$ and $P_\psi = bk_3$. This function that appears in the torus block is exactly the same as the Schwarzian time ordered four point function, analogously to the previous result (3.65). Putting all together we obtain the full four point function in the near extremal CFT limit as

$$\frac{\langle \mathcal{O}_A \mathcal{O}_A \mathcal{O}_B \mathcal{O}_B \rangle_{\beta_L,\beta_R}}{Z_{\beta_L,\beta_R}} \sim e^{-i\pi h_A - i\pi h_B} b^{4h_A+4h_B} G^{(\text{Schw})}_{h_A,h_B}\left(\tilde{t}_i = -ib^2 z_i, \tilde{\beta} = b^2\beta_L\right)$$
$$\times \left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}_{12}\right)\right]^{-2\bar{h}_A}\left[\frac{\beta_R}{\pi}\sinh\left(\frac{\pi}{\beta_R}\bar{z}_{34}\right)\right]^{-2\bar{h}_B} \tag{3.87}$$

for a choice of $\bar{z}_{1,2,3,4}$ that falls under Case 1 above. Note that the choice of operator ordering is important, as we will discuss more in a moment.

From the diagrammatic rules defined in [84] to compute Schwarzian correlators, and the fact that the fusion transformation of the block is always done in pairs, it is clear that this connection between Virasoro blocks and the Schwarzian theory will generalize to higher point correlators.

To summarize, the important configurations have identical operators in pairs, separated in $\bar{z}$ by order β_R (Case 1), but with finite separation between each pair. In such a case, the correlator is dominated by an appropriate identity block, which gives the Schwarzian correlator in the left-moving sector, times a product of cylinder two-point functions in the right-moving sector. The vacuum dominance can fail when we do not have nearby pairs of identical operators (such as Case 2), when the correlator is in any case exponentially suppressed. It can also fail when two or more pairs of identical operators come within a $\bar{z}$ separation of order β_R (Case 3), unless we make the additional assumption of factorization.

Partial waves

As anticipated above, the situation improves when we integrate over angles (which is equivalent to leading order in β_R to integrating over $\bar{z}$). When computing partial waves correlators as in section 3.2.4 we can fix the position of one insertion ($\bar{z}_1 = 0$ for example) and integrate over the remaining coordinates. In Case 1 the dominant contribution comes from fixing two other coordinates, $\bar{z}_{21}$ and $\bar{z}_{43}$ to accuracy β_R. In Case 3 the dominant contribution comes from all coordinates being of order β_R. Therefore the angular integral will produce, schematically, factors of

$$\int_{Case\ 1} \prod_{i=1}^{4} d\bar{z}_i(\ldots) \sim \beta_R^2(\ldots), \qquad vs. \qquad \int_{Case\ 3} \prod_{i=1}^{4} d\bar{z}_i(\ldots) \sim \beta_R^3(\ldots), \tag{3.88}$$

where the additional factor of β_R simply comes from the small region of $\bar{z}_{32}$ for which vacuum dominance fails. This implies that any non vacuum block, that due to the $\beta_R \to 0$ limit can only contribute for configurations of type 3, will come with an extra factor of β_R. Therefore we conclude that the partial wave correlators are dominated only by the vacuum blocks that produce the Schwarzian answer, namely

$$\langle \mathcal{O}^{\ell_A}_A(t_1)\mathcal{O}^{\ell_A}_A(t_2)\mathcal{O}^{\ell_B}_B(t_3)\mathcal{O}^{\ell_B}_B(t_4)\rangle_{\beta_L,\beta_R} \propto \mathcal{N}_{\ell_A}\mathcal{N}_{\ell_B}\, G^{(\text{Schw})}_{h_A,h_B}\left(2ib^2 t_i, b^2\beta_L\right), \tag{3.89}$$

where we have taken a 'retarded' combination of time-orderings for each pair of operators to make sense of the partial wave integral. The prefactors $\mathcal{N}_\ell$ are given by (3.78) for each pair of operators.

There are several types of corrections to our formula (3.89). The first are corrections to the vacuum block itself, arising from the angular integral as explained around equation (3.80), and (new for higher-point functions) also from ignoring right-moving descendants of the vacuum, most relevant for 'Case 3' configurations, where four operators have $\bar{z}$ separation of order β_R. Both sources of error result in corrections of order β_R/c, and admit a bulk interpretation as exchanges of graviton KK modes. In addition, we have corrections from non-vacuum blocks, most importantly from 'Case 3' configurations, which are of order β_R times the relevant OPE coefficients. In a gravitational description, these arise from interactions (including, but not limited to, interactions with matter KK modes), as will be explained in section 3.3.3. In a truly holographic theory with a weakly interacting bulk dual, the OPE coefficients multiplying the non-vacuum blocks are small, giving an additional suppression.

Out-of-time-order correlators (OTOC)

For the arguments above, it was important that we were considering a time-ordered four-point function. The simplification of left-moving 'direct' necklace channel blocks generalizing (3.45) is only valid when the order of operators in the choice of channel corresponds to the correct time ordering. On the other hand, the dominance and simplification of the vacuum block in the right-moving sector relies on choosing the $\widetilde{\text{Necklace}}$ channel corresponding to 'spatial' ordering, meaning that we place operators in order of $\bar{z}$ (though this is not so important for the nearby pairs of identical operators, since we have included the relevant intermediate descendants to give the sinh in (3.83), for example). We already saw a version of this in the difference between (3.70) and (3.71), though the effect of this was simple to account for since we needed only exchange the order of nearby identical operators.

For the time-ordered four-point function in the configurations such as those (Case 1) which dominate the partial wave correlators, we can always arrange for the ordering in time to match the ordering in $\bar{z}$ (up to exchanging the pairs of nearby identical operators). However, this is no longer true for out-of time ordered (OTOC) orderings such as

$$\langle \mathcal{O}_A(z_1, \bar{z}_1)\mathcal{O}_B(z_3, \bar{z}_3)\mathcal{O}_A(z_2, \bar{z}_2)\mathcal{O}_B(z_4, \bar{z}_4)\rangle_{\beta_L,\beta_R}, \tag{3.90}$$

when pairs of identical operators do not appear consecutively. We need to add an additional 'braiding' move to exchange operator order before inserting the simplified necklace block.

The argument in the right moving sector is unchanged for the OTOC, so we need only consider the left-moving vacuum block $\mathcal{F}^{(\tilde{N})}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(z, \beta_L)$, where the $\widetilde{N}$ channel arranges the operators in $\bar{z}$ ordering: $AABB$. First, we apply the same sequence of moves as before to rewrite the $\widetilde{N}$ block as a direct necklace channel block. However, this block inherits the $AABB$ ordering, but the necklace block only simplifies if the channel matches the $ABAB$ time ordering. The change of order of two of the

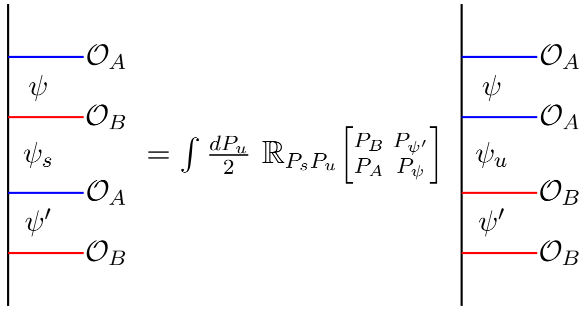


Figure 3.4: Diagram of the braiding transformation that is need to compute out-of-time-ordered correlators.

operators is achieved using the 'R-matrix' as shown in figure 3.4. After braiding, the new necklace channel block with $ABAB$ ordering becomes a scaling block, and we have new z dependence coming from integrating over the internal index in the R-matrix. This is a building block that we can now use for any higher point OTOC, constructing any time ordering with repeated applications. The R-matrix is simply related to the fusion matrix and therefore we can use again the formula derived by Ponsot and Teschner [113].

The limit of the R-matrix required in the Schwarzian limit was computed in Appendix B of [84]. In the notation of figure 3.4 we take all intermediate states to be near extremal $P_\psi = bk$, $P_{\psi'} = bk'$, $P_{s/u} = bk_{s/u}$ and the external operators h_A and h_B to be light. Then the R-matrix becomes

$$\mathbb{R}_{P_sP_u}\begin{bmatrix} P_B & P_{\psi'} \\ P_A & P_\psi \end{bmatrix} = \frac{\rho_0(bk_u)}{2\pi b^3}\left|\frac{\Gamma(h_A+ik\pm ik_u)\Gamma(h_B+ik'\pm ik_u)}{\Gamma(h_A+ik'\pm ik_s)\Gamma(h_B+ik\pm ik_s)}\right| \begin{Bmatrix} h_A & k' & k_s \\ h_B & k & k_u \end{Bmatrix} \tag{3.91}$$

where we defined

$$\begin{aligned}\begin{Bmatrix} h_A & k' & k_s \\ h_B & k & k_u \end{Bmatrix} &= \mathbb{W}(k_s, k_u; h_A+ik, h_A-ik, h_B-ik', h_B+ik') \\ &\times \sqrt{\Gamma(h_A\pm ik'\pm ik_s)\Gamma(h_B\pm ik\pm ik_s)\Gamma(h_B\pm ik'\pm ik_u)\Gamma(h_A\pm ik\pm ik_u)}.\end{aligned} \tag{3.92}$$

This object that we obtain as a limit of the Ponsot-Teschner general formula is the 6j-symbol of the classical $\mathfrak{sl}(2)$ computed by Groenevelt [128]. $\mathbb{W}(\alpha,\beta;a,b,c,d)$ is the Wilson function also defined by Groenevelt. The definition we are using is

$$\mathbb{W}(\alpha,\beta;a,b,c,d) \equiv \frac{\Gamma(d-a)\,{}_4F_3\left[\begin{matrix} a+i\beta & a-i\beta & \tilde{a}+i\alpha & \tilde{a}-i\alpha \\ a+b & a+c & 1+a-d & \end{matrix};1\right]}{\Gamma(a+b)\Gamma(a+c)\Gamma(d\pm i\beta)\Gamma(\tilde{d}\pm i\alpha)} + (a\leftrightarrow d), \tag{3.93}$$

where $\tilde{d} = (b+c+d-a)/2$ and $\tilde{a} = (a+b+c-d)/2$. The prefactor of $(2\pi b^3)^{-1}$ combines with the factor of b^2 in (3.52) and the factor of b in the measure $dP_u = bdk_u$ to ensure that the braiding does not change the factors of b, so the OTOC is of the same order as the TOC.

When this transformation is applied to the left moving block, the final answer is proportional to a function $G^{OTOC}_{h_A,h_B}\left(\tilde{t}_i = -ib^2 z_i, \tilde{\beta} = b^2\beta_L\right)$ that reproduces again the Schwarzian OTOC. We will not write it down here but the final expression can be found in equation (1.28) of [84].The appearance of the 6j-symbol in Schwarzian OTOC found in [84] was reproduced using the BF formalism in [115, 116] and using the boundary particle approach in [129]. Moreover, it was verified in [123] that the semiclassical limit of this kernel gives the Dray-'t Hooft shockwave S-matrix, reproducing the semiclassical calculation of the OTOC of [60].

The full OTOC at fixed angles (with a configuration such as Case 1 where the identity block dominates, and spatial ordering is $AABB$) including left- and right-moving blocks is given by

$$\frac{\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_A \mathcal{O}_B \rangle_{\beta_L,\beta_R}}{Z_{\beta_L,\beta_R}} \propto G^{OTOC}_{h_A,h_B}\left(\tilde{t}_i = -ib^2 z_i, \tilde{\beta} = b^2 \beta_L\right) \bar{\mathcal{F}}^{(\tilde{N})}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(\bar{z}_i). \tag{3.94}$$

We can use this formula to obtain correlators of partial waves by integrating over angles and the result is that the OTOC for the CFT is proportional to the Schwarzian OTOC, similarly to all previous cases. We can also use this for the OTOC without integrating over angles, where the two insertions of $\mathcal{O}_A$ are at the same spacetime location (up to small shifts to regulate), and likewise the two insertions of $\mathcal{O}_B$.

We would like to stress that this is the first derivation of OTOC in 2d CFT where the vacuum dominance approximation is justified, thanks to the small parameter β_R. This is in contrast to [130], for example, for which there is no clear justification for vacuum dominance [131]. This shows in particular that the gravitational S-matrix of three dimensional gravity coupled to matter is exactly given in the near extremal limit by the 6j-symbol of $\mathfrak{sl}(2)$. This can be thought of as a controlled derivation, in a certain limit, of the universal gravitational scattering proposed in [132].

As a final application we can compute the semiclassical limit of the (out-of-time-ordered) left moving Virasoro block using the braiding matrix (3.91). The block is proportional to the Schwarzian correlator and the semiclassical limit of the exact OTOC was derived in [123] giving

$$\frac{\mathcal{F}^{OTOC}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(z,\beta_L)}{\mathcal{F}^{TO}_{\mathcal{O}_A,0,\mathcal{O}_B,0}(z,\beta_L)} \sim \eta^{-2h_A} U(2h_A, 1+2h_A-2h_B, \eta^{-1}), \tag{3.95}$$

$$\eta \equiv \frac{i\tilde{\beta}}{2\pi} \frac{e^{-i\frac{\pi}{\tilde{\beta}}(\tilde{t}_3+\tilde{t}_4-\tilde{t}_1-\tilde{t}_2)}}{\sin\frac{\pi \tilde{t}_{12}}{\tilde{\beta}} \sin\frac{\pi \tilde{t}_{34}}{\tilde{\beta}}} \tag{3.96}$$

with $\tilde{t} = -ib^2 t$ and $\tilde{\beta} = b^2\beta$ and $U(a,b,c)$ being the confluent hypergeometric function. The semiclassical limit corresponds to late times with η small (but larger than other small parameters). The analog for the time ordered four point block gives simply a product of (3.68) two point functions. This matches with the semiclassical calculation done in [60] and our derivation of this expression from a conformal block explains the origin of equation (4.14) and (4.17) of [133]. When this formula is expanded at small η it gives the maximal $\lambda_L = \frac{2\pi}{\beta}$ Lyapunov exponent saturating the chaos bound of [7]. The fixed angle OTOC grows with time exponentially with rate $\lambda_L = 2\pi/\beta_L \approx \pi/\beta$ while the s-wave correlator grows with rate $\lambda_L = 2\pi/\beta$. This is consistent with the bounds derived in [134] (see section 5).

Finally, since we have control over the OTOC calculation (thanks to the $\beta_R \to 0$ limit), we could also attempt to compute the quantum Lyapunov spectrum [135] which is given by OTOC between four arbitrary operators. This is related to inelastic scattering in the bulk and an analogous chaos bound applies [136]. We can compute OTOC between different operators in the near extremal limit, which picks the intermediate channel with lower twist. The bound implies their spin s has to satisfy $s \leq 2$ and also puts a bound on their OPE coefficients, but we leave a detailed analysis for future work.

3.3 Near extremal rotating BTZ black holes

3.3.1 The BTZ black hole and its AdS_2 throat

We begin our analysis of the dual gravitational physics with a look at the classical BTZ black hole and its thermodynamics. This is a solution to Einstein gravity in three dimensions with negative cosmological constant $\Lambda = -\ell_3^{-2}$ (the subscript on ℓ_3 distinguishing it from the two-dimensional AdS length encountered later), with metric

$$ds^2 = -f(r)dt^2 + \ell_3^2 \frac{dr^2}{f(r)} + r^2\Big(d\varphi - \frac{r_- r_+}{r^2}dt\Big)^2, \tag{3.97}$$

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}. \tag{3.98}$$

We are using a dimensionless time coordinate t such that the asymptotic metric is proportional to $-dt^2 + d\varphi^2$, and f has dimensions of length squared. The inner and outer horizons at $r = r_\pm$, with $0 < r_- < r_+$ (for $J > 0$), are related to the energy and angular momentum of the black hole by

$$M = \frac{r_+^2 + r_-^2}{8G_N\ell_3}, \qquad J = \frac{r_+ r_-}{4G_N\ell_3}. \tag{3.99}$$

The mass here is the dimensionless energy conjugate to the time t, with zero energy defined such that empty AdS_3 has $M = -\frac{\ell_3}{8G_N}$, corresponding to the Casimir energy of the dual.

Using the Brown-Henneaux relation $c = \frac{3\ell_3}{2G_N}$ for the central charge of the dual CFT in the classical limit, and including one-loop corrections to the energy from the Casimir energy of gravitons [89], the horizons can be related very simply to the CFT parameters introduced in (3.26) and (3.27):

$$P = Q\frac{r_+ - r_-}{2\ell_3}, \qquad \bar{P} = Q\frac{r_+ + r_-}{2\ell_3} \tag{3.100}$$

Applying the standard gravitational thermodynamics, we find the classical black hole temperature (from surface gravity), angular potential (from horizon angular velocity) and entropy (from the area):

$$T = \beta^{-1} = \frac{r_+^2 - r_-^2}{2\pi\ell_3 r_+}, \quad \Omega = \frac{r_-}{r_+}, \quad S = \frac{2\pi r_+}{4G_N} \tag{3.101}$$

The angular potential is the chemical potential for angular momentum, related to the parameter θ used in section 3.1 by $\theta = i\beta\Omega$. Writing this in terms of left- and right-moving temperatures $\beta_L = (1+\Omega)\beta$, $\beta_R = (1-\Omega)\beta$, we have

$$\beta_L = \frac{2\pi\ell_3}{r_+ - r_-}, \quad \beta_R = \frac{2\pi\ell_3}{r_+ + r_-}. \tag{3.102}$$

The BTZ solution we have given is smooth outside an event horizon as long as $r_\pm$ are real, which imposes the extremality bound $M \geq |J|$. We are interested in near-extremal black holes, close to saturating this bound, so the difference between r_+ and r_- goes to zero. This is a low temperature limit, in which the thermodynamic quantities approach the following:

$$M - J \sim \frac{\pi^2\ell_3}{8G_N}T^2, \qquad S \sim \pi\sqrt{\frac{J\ell_3}{G_N}} + \frac{\ell_3\pi^2}{4G_N}T \tag{3.103}$$

This can be compared with the thermodynamics of the Schwarzian theory, for which the energy and entropy are given by

$$E - E_0 = 2\pi^2 C T^2, \qquad S = S_0 + 4\pi^2 C T. \tag{3.104}$$

Defining the central charge using the Brown-Henneaux relation $c = \frac{3\ell_3}{2G_N}$, we recover the values $C = \frac{c}{24}$, $E_0 = J$ and $S_0 = 2\pi\sqrt{\frac{c}{6}J}$ found in (3.15) of section 3.1.

If we simply take a near-extremal limit of (3.97), fixing r_+ and taking $r_- \to r_+$ with fixed r, we find the extremal BTZ metric:

$$ds^2 \sim -\frac{(r^2 - r_+^2)^2}{r^2}dt^2 + \ell_3^2\frac{r^2}{(r^2 - r_+^2)^2}dr^2 + r^2\left(d\varphi - \tfrac{r_+^2}{r^2}dt\right)^2 \tag{3.105}$$

However, in this limit we obscure the most physically important and interesting region of the spacetime. We expect a large class of black holes in a near-extremal limit to develop a near-horizon AdS_2 throat region [66, 91–93], and this is no exception. To zoom in on this throat, we must scale the radius with the temperature, introducing a new coordinate ρ via

$$r = \tfrac{1}{2}(r_+ + r_-) + \tfrac{1}{2}(r_+ - r_-)\rho, \quad T \sim \frac{r_+ - r_-}{\pi\ell_3} \to 0, \tag{3.106}$$

and fix ρ as we go to low temperature. We then find the geometry

$$ds^2 \sim \frac{\ell_3^2}{4}\left[-(\rho^2-1)(2\pi T)^2 dt^2 + \frac{d\rho^2}{(\rho^2-1)}\right] + r_+^2\left(d\varphi - dt + \tfrac{\ell_3}{r_+}\rho\,\pi T\,dt\right)^2, \tag{3.107}$$

which is a fibration of a circle over AdS_2 of radius $\ell_2 = \frac{1}{2}\ell_3$, as found in [90]. This itself is a solution of pure three dimensional gravity, sometimes called the 'self-dual orbifold' [101, 137], and is the BTZ analog of the near horizon extremal Kerr geometry [138]. The metric has an enhanced isometry group $SL(2,\mathbb{R}) \times U(1)$. In this near-horizon region, the effects of finite temperature are still visible, and for temperatures of order c^{-1} quantum effects become of leading importance, as we now briefly review.

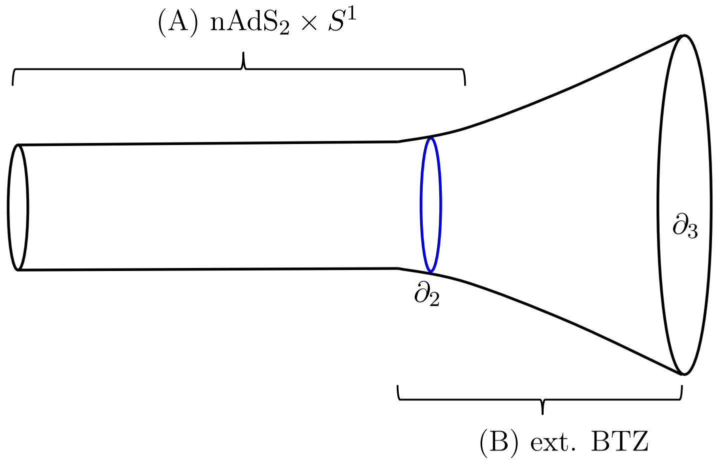


Figure 3.5: The near extremal BTZ geometry at fixed time, from the horizon (leftmost circle) to the asymptotically AdS$_3$ boundary (rightmost circle ∂_3). In region (A) the geometry is approximately nearly AdS$_2 \times S^1$ (described by JT gravity), and in (B) the geometry is approximately extremal BTZ and the physics is classical. In the overlap between (A) and (B), we introduce the boundary ∂_2 (the blue line) where the Schwarzian mode lives.

If we ignore corrections to the AdS$_2$ throat geometry, an infinite-dimensional symmetry emerges, of diffeomorphisms relating the boundary of AdS$_2$ to the physical time, which is spontaneously broken to $SL(2,\mathbb{R})$ by the choice of AdS$_2$ vacuum. We therefore expect the low-energy physics to be described by a Goldstone mode parameterizing $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$.

The simplest option would be for the theory of the Goldstone mode to preserve the $SL(2,\mathbb{R})$ and $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ symmetries, but no such theory exists. It was shown in [139] it is not possible to have a quantum mechanical system with an exact $SL(2,\mathbb{R})$ symmetry. Accordingly, in [80] it is shown that gravity in AdS$_2$ cannot support finite energy excitations (see also [58]). We are therefore forced to include the leading order explicit breaking of this symmetry; the resulting theory of the pseudo-Goldstone is precisely the Schwarzian theory [60]. For low temperatures $T \sim c^{-1}$ (the 'gap temperature' of [15] at which the thermodynamic description of near extremal black holes was thought to break down), this mode becomes strongly coupled, so quantum effects are of leading importance.

This Schwarzian theory will give a good description of the physics in the region where corrections to the AdS$_2$ fibration (3.107) are small. This is true as long as $r - r_+ \ll r_+$, corresponding to $\rho \ll \frac{r_+}{\ell_3 T}$, which is the region (A) illustrated in figure 3.5. In another region, labeled by (B) in figure 3.5, quantum fluctuations are suppressed, and we can accurately describe the physics classically, on a fixed background, which is approximately extremal BTZ. This is the region $\rho \gg 1$, corresponding to $r - r_+ \gg \ell_3 T$. The two regions (A), where JT gravity is useful, and (B), where the geometry is classical, have a parametrically large region of overlap. Somewhere in this region, we can place an artificial boundary ∂_2 of the AdS$_2$ region. The physics inside this boundary will be described by JT gravity, which induces a Schwarzian theory living on ∂_2. This will be matched to the asymptotic boundary ∂_3 of AdS$_3$, where the dual CFT lives, by the classical physics between ∂_2 and ∂_3.

3.3.2 A two-dimensional theory

To describe the dynamics in the AdS$_2$ throat described above, we first analyze the two-dimensional theory arising on dimensional reduction of three-dimensional gravity.

Dimensional reduction

Our 'parent' theory is simply three-dimensional Einstein-Hilbert gravity, with action (here in Euclidean signature)

$$I_{\text{EH}} = -\frac{1}{16\pi G_N}\left[\int_M d^3x\,\sqrt{g_3}\Big(R_3 + \tfrac{2}{\ell_3^2}\Big) + 2\int_{\partial M} d^2x\sqrt{\gamma}\Big(\kappa_3 - \tfrac{1}{\ell_3}\Big)\right], \tag{3.108}$$

where the metric g_3, scalar curvature R_3 and boundary extrinsic curvature κ_3 carry subscripts to distinguish them from the two-dimensional quantities introduced presently. The boundary conditions are those standard in AdS/CFT, and we have included the Gibbons-Hawking term and counterterm (γ is the induced metric on ∂M) to render the action finite.

We will study configurations of this theory with a $U(1)$ symmetry, with Killing field ∂_φ. We use coordinates x^a and φ, where a runs over two indices (which will be identified with the time and radial coordinates in the BTZ configuration), and the coordinate φ in the symmetry direction is periodically identified as $\varphi \sim \varphi + 2\pi$. With this restriction, a general metric can be written in Kaluza-Klein form

$$g_3 = g_2 + \Phi^2(d\varphi + A)^2, \tag{3.109}$$

where the two-dimensional metric g_2, gauge field one-form A and scalar dilaton Φ are independent of φ, depending only on the two-dimensional coordinates x^a.

With this ansatz, the three-dimensional quantities are written in terms of two-dimensional fields ($\Box_2$ is the Laplacian corresponding to g_2) as

$$\begin{aligned} R_3 &= R_2 - \tfrac{1}{4}\Phi^2 F_{ab}F^{ab} - 2\Phi^{-1}\Box_2\Phi \\ \sqrt{g_3} &= \Phi\sqrt{g_2} \\ d^2x\sqrt{\gamma} &= \Phi dt_E d\varphi \\ \kappa_3 &= \kappa_2 + \Phi^{-1}\partial_n\Phi. \end{aligned} \tag{3.110}$$

For the boundary terms, we also assume that the location of the cutoff ∂M is independent of φ. The equation for the measure on the boundary $d^2x\sqrt{\gamma}$ is then simply a definition of the proper time coordinate t_E parameterizing the boundary of the two-dimensional manifold. ∂_n denotes the unit normal derivative at the boundary.

Inserting this in the three-dimensional action and integrating over φ, we find the two-dimensional Einstein-Maxwell-dilaton action:

$$I_{\text{EH}} = -\frac{2\pi}{16\pi G_3}\left[\int d^2x\sqrt{g_2}\Phi\Big(R_2 - \tfrac{1}{4}\Phi^2 F_{ab}F^{ab} + \tfrac{2}{\ell_3^2}\Big) + 2\int_\partial ds\Phi(\kappa_2 - \tfrac{1}{\ell_3})\right] \tag{3.111}$$

Here, $F = dA$ is the field strength of the Kaluza-Klein gauge field, and all indices are contracted with g_2. Note in particular that a total derivative term $\Box_2\Phi$ exactly cancels with the extra term in the Gibbons-Hawking boundary action. The three-dimensional diffeomorphisms become two-dimensional diffeomorphisms, as well as gauge transformations arising from x^a-dependent translations in the φ direction, so that A is a compact $U(1)$ gauge field.

It remains only to discuss the boundary conditions of the reduced theory. The three-dimensional boundary conditions are that the induced metric γ is proportional (with holographic renormalisation parameter ϵ) to a chosen metric on which the CFT dual lives; we may always choose a flat metric $ds^2 = dt_E d\varphi$, so the boundary condition is $\gamma = \epsilon^{-2} dt_E d\varphi$. Reducing to two dimensions, this implies that the dilaton is just the holographic renormalization constant, $\Phi = \epsilon^{-1}$. We can think of this condition as defining the location of the cutoff surface, from which we subsequently determine t_E. The metric boundary condition then gives the period of t_E in terms of the inverse temperature β; the proper length of the boundary is $L_\partial = \epsilon^{-1}\beta$.

The gauge field is a little more subtle. At first sight, it looks like we should impose $A = 0$ on the boundary, but in fact we can only do this locally. The boundary value of A acts as a background gauge field for the one-dimensional dual quantum mechanics, which by gauge transformations we can always set to be trivial locally; however, this is not true globally when the boundary is a circle, since our gauge transformation must be single-valued around the circle. This leaves a single piece of gauge invariant data, the holonomy $\theta = \int A$. This is an angle, defined up to shifts by 2π, since we can take gauge transformations that wind an integer number of times around $U(1)$ as we go round the circle. Tracing the holonomy back to its higher-dimensional origin, we find that it is precisely the twist angle we encountered in section 3.1, from the periodicity condition $(t_E, \varphi) \sim (t_E, \varphi + 2\pi) \sim (t_E + \beta, \varphi + \theta)$. In summary, the reduction of the standard AdS/CFT boundary conditions becomes the 'Dirichlet boundary conditions',

$$\text{Dirichlet BC:} \qquad \Phi|_\partial = \epsilon^{-1}, \quad L_\partial = \epsilon^{-1}\beta, \quad \int_\partial A = \theta\,, \tag{3.112}$$

imposed on the asymptotically AdS_3 boundary.

Integrating out the gauge field

We can simplify this theory still further, due to the simplicity of gauge fields in two dimensions, in particular the absence of a propagating photon. Since our gauge field is abelian, its action is quadratic, so we could simply integrate it out; however, we will nonetheless find it convenient to first rewrite the Maxwell action in a first-order BF formalism. This approach becomes extremely useful for near extremal black holes in higher dimensions with non-abelian gauge fields [140][141], often used for the solution of nonabelian gauge theories in two dimensions [142, 143].

$$I_{\text{Maxwell}} = \frac{1}{32G_3}\int d^2x\sqrt{g_2}\Phi^3 F_{ab}F^{ab} \to \int \left[i\mathcal{J}F + \tfrac{1}{2}\mu\mathcal{J}^2\right] \tag{3.113}$$

$$\mu = 8G_3^2 x\sqrt{g_2}\Phi^{-3} \tag{3.114}$$

We have here introduced the auxiliary scalar field $\mathcal{J}$, and the Maxwell theory now manifestly depends on the metric and dilaton only through the measure two-form μ. Integrating out the auxiliary field, we find that it is related to the field strength by

$$F = i\mathcal{J}\mu, \tag{3.115}$$

and using this to substitute for $\mathcal{J}$ we return to the original second-order Maxwell action. Note that for real Euclidean geometries, A is real and $\mathcal{J}$ is imaginary, but this is reversed for solutions with a continuation to real Lorentzian geometries.

Now, we may instead integrate out the gauge field. We find that it imposes the constraint that $\mathcal{J}$ is a constant, which by examining the expansion of the gauge field near the boundary we can identify as the charge in the dual CFT. The three-dimensional origin of this charge is the ADM angular momentum J. The first term in the action then is a total derivative, which becomes the holonomy on the boundary via $\int F = \theta$, so we find

$$\mathcal{J} = J, \quad I_{\text{Maxwell}} = iJ\theta + \tfrac{1}{2}J^2 \int \mu. \tag{3.116}$$

The last step is to sum over spins J, which are forced to be integers by the periodicity of θ. The result is a partition function of the Maxwell piece:

$$Z_{\text{Maxwell}} = \sum_{J=-\infty}^{\infty} e^{-iJ\theta - \frac{1}{2}J^2 \int \mu} \tag{3.117}$$

As an alternative way to see this result, we can quantize the theory 'radially', and fix a static gauge; we find that the theory becomes the quantum mechanics of a free particle propagating on a circle parameterized by the holonomy θ, and J labels the momentum modes, which are eigenstates of the radial Hamiltonian with energy proportional to J^2. We then compute Z_{Maxwell} by preparing the ground state with $J = 0$ at the origin, evolving for Euclidean time proportional to $\int \mu$, and evaluating the wavefunction at fixed θ. See [97, 144] for more details, and the nonabelian generalization of a particle propagating on a group manifold. By Poisson resummation, we can also express the result (3.117) as a sum over winding numbers for the particle propagating on a circle, in which different terms are related by $\theta \to \theta + 2n\pi$. Different values of the index n correspond to different three-dimensional topologies, which are the same set of $SL(2,\mathbb{Z})$ black holes.

From this result, we see that each term in the sum over J has the very nice property that it contributes a local effective action for the dilaton and metric. Since the sum over terms does not retain this property, it is most natural to perform the path integral not with fixed holonomy 'Dirichlet' boundary conditions, but with fixed angular momentum 'Neumann' boundary conditions. This is just the change of ensemble in the partition function we saw already in section 3.1, where we pick out a Fourier mode of θ by an integral $\int d\theta e^{iJ\theta} Z_{\text{Maxwell}}$. From the bulk point of view, we can describe this by adding a local boundary counterterm to the action,

$$I \longrightarrow I + iJ \int_{\partial} \theta \tag{3.118}$$

and using fixed J boundary conditions; this counterterm is precisely what is required to make the variational problem with the new boundary conditions well-defined. Integrating out the Maxwell field then gives a local effective action, giving rise to an Einstein-dilaton theory:

$$I_J = -\frac{2\pi}{16\pi G_N}\left[\int d^2x\sqrt{g_2}\left(\Phi R_2 - U(\Phi)\right) + 2\int_\partial ds\,\Phi(\kappa_2 - \tfrac{1}{\ell_3})\right] \tag{3.119}$$

$$U(\Phi) = \tfrac{1}{2}(8G_N J)^2\Phi^{-3} - \tfrac{2}{\ell_3^2}\Phi \tag{3.120}$$

For calculations at fixed holonomy, corresponding to the grand canonical ensemble with fixed chemical potential, we can now simply compute at fixed spin J with this local effective action, before summing over J. Here we assumed the absence of matter charged under the gauge field, which are inevitably present from Kaluza-Klein modes breaking the $U(1)$ symmetry. In the presence of such fields, the gauge field can play a more nontrivial role.

The action in equation (3.119) was originally derived by Achucarro and Ortiz [90], but we hope that our presentation clarifies the role of the $U(1)$ gauge field and its boundary condition (see also [91] and [67] for a different approach and [145] for a recent more thorough analysis of JT gravity coupled to 2d Yang Mills theory).

The near-extremal limit and the Schwarzian theory

The description we have given so far did not require a near-extremal limit. Our next step is to take such a limit, and extract the dynamics of the near-AdS$_2$ region from the action given in (3.119), which is in the class of models studied by [58].

We first look for solutions where the metric g_2 is exactly AdS$_2$. This requires Φ to be a constant $\Phi = \Phi_0$ at a zero of the potential $U(\Phi_0) = 0$, and the AdS$_2$ radius ℓ_2 is determined by $U'(\Phi_0)$:

$$U(\Phi_0) = 0 \implies \Phi_0 = \sqrt{4G_N\ell_3 J} \tag{3.121}$$

$$U'(\Phi_0) = -\frac{2}{\ell_2^2} \implies \ell_2 = \tfrac{1}{2}\ell_3 \tag{3.122}$$

Reinstating the gauge field using

$$F = 8iJG_N\Phi^{-3}d^2x\sqrt{g_2} = \frac{i}{\sqrt{\ell_3^3 JG_N}}\mathrm{vol}_{\mathrm{AdS}_2}, \tag{3.123}$$

where $\mathrm{vol}_{\mathrm{AdS}_2}$ is the volume form (area element) of AdS$_2$, and Wick rotating to Lorentzian signature, we find that this is precisely the 'self-dual orbifold' geometry (3.107) we found in the near-horizon of near-extremal rotating BTZ.

To incorporate the leading order fluctuations away from AdS$_2$, we expand the dilaton around its extremal value, writing

$$\Phi = \Phi_0 + 4G_N\phi, \tag{3.124}$$

and find the terms in the action linear in ϕ:

$$I_J = -S_0\chi + I_{\rm JT}[g_2,\phi] + \cdots \tag{3.125}$$

$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{g_2}R_2 + \frac{1}{2\pi}\int_{\partial_2}\kappa_2, \quad S_0 = 2\pi\frac{\Phi_0}{4G_N} = 2\pi\sqrt{\frac{\ell_3 J}{4G_N}} \tag{3.126}$$

$$I_{\rm JT} = -\frac{1}{2}\int d^2x\sqrt{g_2}\,\phi\left(R_2+\frac{2}{\ell_2^2}\right) - \int_{\partial_2}\phi(\kappa_2 - \tfrac{1}{\ell_2}) \tag{3.127}$$

The leading order term is the two-dimensional Einstein-Hilbert action, which is topological, proportional to the Euler characteristic χ of spacetime. The next term is the action for Jackiw-Teitelboim (JT) gravity. Subsequent terms are suppressed in the limit $\phi \ll S_0$.

In writing the boundary terms here, we have implicitly introduced a new boundary, denoted ∂_2 (indicated by the blue line in figure 3.5), because the JT approximation is only valid deep in the near-horizon region where Φ is close to Φ_0. We choose to place this new, artificial boundary ∂_2 at a curve of constant dilaton $\phi = \phi_\partial$, where $1 \ll \phi_\partial \ll S_0$. We have introduced boundary terms in the JT action so that this boundary condition is a good variational problem, with an intrinsic counterterm so that the action has a finite limit when we take $\phi_\partial \to \infty$; these boundary terms are not physical, so we must add equal and opposite terms to the action for the exterior of the throat. For the metric, we would like a boundary condition that fixes the proper length L_2 of ∂_2, but this is not freely chosen; rather, it is determined by solving the theory classically outside the AdS_2 throat region, between ∂_2 and ∂_3 where the physical boundary conditions are imposed.

Solving the theory outside the throat

When we are far enough from the black hole so that ϕ is no longer much smaller than S_0, we must keep the full dilaton potential, but can solve the theory classically; we can think of ϕ^{-1} as a running coupling, so quantum fluctuations are large in the throat region where ϕ is of order one, but small outside the artificial boundary ∂_2 where $\phi \gg 1$. To good approximation, we can therefore simply solve the equations of motion of the theory with appropriate boundary conditions at the boundary of AdS_3 ∂_3, find the boundary conditions induced at ∂_2, and finally evaluate the on-shell action between ∂_2 and ∂_3.

The solution to our two-dimensional Einstein-dilaton theory (3.119) is simply the dimensional reduction of the extremal BTZ metric (3.105). In the coordinates of section 3.3.1, the dilaton is given by the radial coordinate $\Phi = r$, with $\Phi_0 = r_+$, and the boundary ∂_2 is in the region $\ell_3 T \ll r - r_+ \ll r_+$.

First, we can read off the boundary conditions at ∂_2 from the metric (3.105), by evaluating the proper length of the Euclidean time circle $L_2 = \frac{r^2-r_+^2}{r}\beta \sim 2(r-r_+)\beta$ at fixed $r = r_+ + 4G_N\phi_\partial$. The physical content of this boundary condition, independent of our choice of ϕ_∂, is the ratio of the length L_2 to the boundary dilaton:

$$\frac{L_2}{\phi_\partial} = 8G_N\beta = \frac{24}{c}\ell_2\beta \tag{3.128}$$

Our second task is to evaluate the on-shell action between ∂_2 and ∂_3, where we include boundary terms at ∂_2 to cancel the terms we added to the JT action (3.127). We can compute this from the extremal BTZ solution with ∂_2 located at constant radius, even though the configurations are perturbations of this classical solution. Outside the throat, we can treat the deviations to linear order, since higher orders do not contribute finite action. Because we are perturbing around a classical solution, the linearized deviation of the bulk action is a total derivative. But the boundary actions at ∂_3 and at ∂_2 (to cancel the boundary terms in the Euler characteristic and JT actions) have been chosen to make the variational problem well-posed, which means that the variation of the boundary action precisely cancels the variation of the bulk action after integrating by parts.

The terms contributing finite action in the limit of interest are the bulk action between ∂_2 and ∂_3, the boundary term at ∂_3, and the boundary term at ∂_2 from the Euler characteristic topological action:

$$\begin{aligned} -\frac{2\pi}{16\pi G_N}\int_{\partial_2}^{\infty} d^2x\sqrt{g_2}\left(\Phi R_2 - U(\Phi)\right) &\sim 2J\beta - 8\sqrt{\ell_3^{-1}G_N J}\beta\phi_\partial + \cdots \\ -\frac{1}{4G_N}\int_{\partial_3} ds\,\Phi(\kappa_2 - \tfrac{1}{\ell_3}) &\sim -J\beta \\ -\frac{S_0}{2\pi}\int_{\partial_2}\kappa_2 &\sim 8\sqrt{\ell_3^{-1}G_N J}\beta\phi_\partial \end{aligned} \tag{3.129}$$

Adding these up, we find a total action from the outside of

$$I_{\text{outside}} \sim J\beta, \tag{3.130}$$

which acts only to shift the zero of energy to the BTZ extremality bound. This is a concrete example of a general result derived in Appendix A of [92].

The Schwarzian theory

In the previous section we argued that the dynamics of BTZ can be reduced to JT gravity living in the throat. JT gravity is a very simple theory and can be completely reduced to a boundary mode [58] with the Schwarzian action [59–61].

To see this, we can first integrate out the dilaton, which acts as a Lagrange multiplier imposing $R_2 = -\frac{2}{\ell_2^2}$, so the metric is locally AdS$_2$, which we can write in Poincaré coordinates (u, z) as

$$ds^2 = \ell_2^2\frac{du^2 + dz^2}{z^2}. \tag{3.131}$$

The bulk action vanishes when the constraint $R_2 = -\frac{2}{\ell_2^2}$ is imposed, leaving only the boundary term. This nonetheless leaves nontrivial dynamics, arising from the location of the boundary, which is determined by the reparameterization f relating the coordinate u to the physical time t_E, as $u = \tan\frac{\pi}{\beta}f(t_E)$. This is the pseudo-Goldstone mode referred to earlier, taking values in the coset $f \in \text{Diff}(S^1)/SL(2,\mathbb{R})$, where the quotient removes physically equivalent configurations obtained by the isometries of AdS_2. The Euclidean time t_E is proportional to the proper length along ∂_2, up to a factor of L_2/β relating the coordinate periodicity β with the proper length L_2 of the curve. Since the length of the boundary is large $L_2 \gg \ell_2$, we can choose it to lie at small z, and obtain the extrinsic curvature in that approximation [60]:

$$\kappa_2 - \ell_2^{-1} \sim \ell_2 \frac{\beta^2}{L_2^2} \Big\{ \tan \frac{\pi}{\beta} f(t_E), t_E \Big\} \tag{3.132}$$

Integrating this, we recover the Schwarzian action

$$I_{\rm JT} = -C \int_0^\beta dt_E \Big\{ \tan \frac{\pi}{\beta} f(t_E), t_E \Big\}, \tag{3.133}$$

with coefficient C given by

$$C = \ell_2 \beta \frac{\phi_\partial}{L_2} = \frac{c}{24}, \tag{3.134}$$

where we finally made use of (3.128), and the result is independent of the arbitrary choice of L_2 determining the location of the cutoff surface ∂_2.

When JT gravity is coupled to free matter the boundary correlators are simply given by Schwarzian expectation values of the Schwarzian bilocal as shown in [60]. This expectation value is the quantity computed in [84]. The origin of the free matter approximation will be explained below in the next section.

Finally, we note that, while we have obtained JT gravity from a near-extremal limit of pure three dimensional gravity, this is not quite the end of the story once we consider nonperturbative corrections (though these will not be relevant for this work). Additional topologies, beyond those visible in pure JT gravity considered in [102], become relevant, and are particularly important at very low temperature. This is the subject of work in progress [146].

3.3.3 Matter, correlation functions and Kaluza-Klein modes

To study correlation functions in the BTZ background for comparison to the results of section 3.2, we now consider the effect of adding matter.

The classical limit

First, we study the classical limit, where the temperature is high enough ($\beta \ll c$) that backreaction is unimportant, so we are simply studying correlation functions in a fixed BTZ background. Since this geometry is a quotient of pure AdS_3, we can use the method of images to construct the two-point function of a free field from the corresponding result on the plane, which is determined by conformal symmetry. In the lightcone coordinates $(z, \bar{z})$, where we may choose $0 < \bar{z} < 2\pi$, the retarded correlator G_R (for $z < 0$, required so that G_R is nonzero) is given by

$$\begin{aligned} G_R(z,\bar{z}) = &- 2\sin(2\pi h) \sum_{n=0}^{\lfloor -z/2\pi \rfloor} \left(\frac{\beta_L}{\pi} \sinh\left(\frac{\pi}{\beta_L}(-z - 2n\pi) \right) \right)^{-2h} \\ &\times \left(\frac{\beta_R}{\pi} \sinh\left(\frac{\pi}{\beta_R}(\bar{z} + 2n\pi) \right) \right)^{-2\bar{h}} \end{aligned} \tag{3.135}$$

The image sum over n runs over the finite number of images lying in the future lightcone $\bar{z} > 0, z < 0$ of the origin. The $\sin(2\pi h)$ appearing in the prefactor does not break the left-right symmetry because the spin $\ell = \bar{h} - h$ is an integer, so it may also be written as $\sin(2\pi\bar{h})$ or $(-1)^\ell \sin(\pi\Delta)$.

For small β_R (and $\bar{h} > 0$), the $n > 0$ terms in the image sum are exponentially suppressed relative to the $n = 0$ term. This dominant term precisely reproduces the retarded correlator (3.73), where we evaluate the Schwarzian correlator in the semiclassical limit (3.68). As already mentioned, the result (3.74), which would usually dominate for $\bar{z}$ close to 2π, is zero in the semiclassical limit.

If we were to extract the Fourier modes of this result, we would of course reproduce the partial waves (3.78), which are given by a normalization factor times the Schwarzian correlation function.

Dimensional reduction of matter

To explain the results in a more general context, we perform a dimensional reduction of matter fields. Suppose our three-dimensional theory contains a massive scalar field χ, with action

$$I_\chi = -\frac{1}{2} \int d^3x \sqrt{g_3} \left[g_3^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + V(\chi) \right] \tag{3.136}$$

for some potential $V(\chi) = m^2\chi^2 +$ (interactions) (where we have ignored boundary terms). This is dual to a CFT operator $\mathcal{O}$ with dimensions $h = \bar{h} = \frac{\Delta}{2}$, with $m^2\ell_3^2 = \Delta(\Delta - 2)$.

Given our Kaluza-Klein form (3.109) for the three dimensional metric g_3, we can write the inverse metric in terms of two-dimensional fields as follows:

$$g_3^{\mu\nu} = \begin{pmatrix} g_2^{ab} & -g_2^{ab} A_b \\ -g_2^{ab} A_b & \Phi^{-2} + A_a g_2^{ab} A_b \end{pmatrix} \tag{3.137}$$

We write the matter field χ in terms of two-dimensional Kaluza-Klein modes, mirroring the mode decomposition (3.75) of the operator $\mathcal{O}$:

$$\chi(t, r, \varphi) = \sum_l e^{-il\varphi} \chi_l(t, r) \tag{3.138}$$

The three-dimensional action for χ then becomes an action for the complex scalars χ_l (with $\chi_{-l} = \chi_l^*$), which have charge l under the Kaluza-Klein gauge field A:

$$I_\chi = -\frac{1}{2}\int d^2x\sqrt{g_2}\, 2\pi\Phi \left[\sum_l \left(g_2^{ab}D_a\chi_l D_b\chi_{-l} + (m^2 + l^2\Phi^{-2})\chi_l\chi_{-l}\right) + \text{interactions}\right] \tag{3.139}$$

The covariant derivative is $D_a = \partial_a - A_a\partial_\varphi = \partial_a + ilA_a$, and the interactions involve products of three or more χ_l fields with total charge adding to zero.

Deep in the AdS_2 throat, we can replace the dilaton Φ by the constant value $\Phi_0 = r_+$, so the Kaluza-Klein modes have mass $m_l^2 = m^2 + l^2\Phi_0^{-2}$ in the AdS_2 region. For Φ_0 of order ℓ_3, these shifts of the mass are important for finite values of l, as are interactions if the field χ was originally strongly interacting in AdS_3. However, in the limit of very large black holes $\beta_R \ll 1$ studied in section 3.2, so Φ_0 is the largest parameter, we have many simplifications. For any fixed l, the effective mass m_l^2 of χ_l in the AdS_2 region becomes close to the original mass m^2. The charge of such Kaluza-Klein modes is also negligible, because the electric field (3.123) in the AdS_2 region is small at very large J, of order $(JG_N)^{-1/2}$ in AdS units. Most importantly, the action is multiplied by an overall factor Φ_0, which suppresses all interactions. For example, a cubic vertex $\lambda\chi^3$ in the potential is suppressed by a factor $\Phi_0^{-1/2}\lambda$ in the AdS_2 region. The same result should also hold for interactions with Kaluza-Klein modes of the graviton, so their corrections are suppressed by a factor of $\frac{G_N}{\Phi_0}$, or $\frac{\beta_R}{c}$ in the CFT variables of section 3.2.

As a result, to leading order in the large Φ_0 limit, we can treat the Kaluza-Klein modes as independent free fields of equal mass, and neglect their charge. Note that this is entirely different from the usual situations in Kaluza-Klein compactifications, where we can ignore the KK modes because they are heavy; here, they are instead light, but decoupled.

The parametric suppression of interactions found from this gravitational calculations reproduces the corrections to partial wave correlation functions found in section 3.2. First, the corrections from graviton Kaluza-Klein modes arise from the slight smearing of the Schwarzian time when we integrate over φ, as explained at the end of section 3.2.4. Our CFT calculations give us a specific prediction of the full dependence on t and φ, which should arise from these metric KK modes. For higher-point functions, we also have corrections from interactions of bulk fields. As explained more fully in section 3.2.5, these are important only in the limited regime of kinematics when four operators (identical in pairs) approach within β_R in the $\bar{z}$ coordinate, so are suppressed by a factor of β_R times the coupling when we integrate over angles to compute partial waves. This is the same parametric suppression deduced from the gravitational argument.

In conclusion, to describe matter in the AdS_2 region, to leading order in the limit of interest we can use correlation functions of free matter coupled to JT gravity. The only ingredient left is to compute the effective 'IR' conformal dimension Δ_S in this region, determining the dimension of the dual operator appearing in the Schwarzian correlation function. For this, we only need to know the relationship $\ell_3 = 2\ell_2$ between three- and two-dimensional AdS lengths:

$$m^2\ell_3^2 = \Delta(\Delta - 2), \quad m^2\ell_2^2 = \Delta_S(\Delta_S - 1) \implies \Delta_S = \frac{\Delta}{2} = h \tag{3.140}$$

We have written Δ_S in terms of the left-moving dimension h, which is the result we expect to generalise if we were to consider matter with spin.

Interpolating boundary conditions from AdS_3 to AdS_2

The dimensional reduction demonstrates that we can treat matter in the AdS_2 throat as free fields coupled to JT gravity. To compare with the 2d CFT results, we now need to propagate these correlators from the boundary of the throat to the boundary of the AdS_3 spacetime. We can neglect all interactions (including back-reaction) in this region, so it suffices to study matter wave equations on a fixed background.

From the usual AdS/CFT dictionary, CFT_2 correlators can be read off from the ratio between normalizable and non-normalizable modes of the field at the asymptotic boundary ∂_3 (at least for the correlators we consider with pairs of identical operators). The Schwarzian correlators are similarly related to the modes at the boundary of the AdS_2 region ∂_2. The wave equation between ∂_3 and ∂_2 relates these modes, providing a map from the Schwarzian correlators to the CFT_2 correlators. Here, we will solve this mapping between ∂_2 and ∂_3 in frequency space, and show that for the relevant frequencies (ω of order c^{-1}) the map is trivial, giving a rescaling independent of ω. This means that the Schwarzian correlators are directly imprinted on the asymptotic boundary of AdS_3. Physically, this occurs because the time to propagate from the edge of the AdS_2 throat to the asymptotic boundary is only of order ℓ_3, which is very short compared to the characteristic timescale c of the interactions in the deep AdS_2 region.

We consider a scalar field χ of mass m^2, and write $\Delta_\pm$ for the two roots of $\Delta(\Delta-2) = \ell_3^2 m^2$, so Δ_+ is the dimension of the dual CFT operator, and $\Delta_- = d - \Delta_+$. The asymptotic expansion of χ (in frequency space, so χ is a function of the radial coordinate r times $e^{-i\omega t - i\ell\varphi}$) is

$$\chi|_{\partial AdS_3} = A_\ell(\omega)\left(\frac{r}{\ell_3}\right)^{-\Delta_-} + B_\ell(\omega)\left(\frac{r}{\ell_3}\right)^{-\Delta_+} + \cdots , \tag{3.141}$$

and we can read off the correlation function from the ratio of B and A coefficients (in more detail, this could be achieved by deriving an effective action, as done in a similar context in [66]). For example, for a two-point function we have

$$G_\ell(\omega) = \frac{\pi}{\Delta - 1}\frac{B_\ell(\omega)}{A_\ell(\omega)}, \tag{3.142}$$

where the prefactor comes from applying the normalization standard in CFT_2, rather than the 'natural' normalization in AdS, which comes from taking AdS propagators with unit strength delta-function source to the boundary. Similarly, in the asymptotically AdS_2 region the field will behave as

$$\chi|_{\partial AdS_2} = \tilde{A}_\ell(\omega)\left(\frac{4}{\ell_3}(r-r_+)\right)^{-\frac{\Delta_-}{2}} + \tilde{B}_\ell(\omega)\left(\frac{4}{\ell_3}(r-r_+)\right)^{-\frac{\Delta_+}{2}} + \cdots, \tag{3.143}$$

where we have chosen the radial coordinate to give a canonical AdS_2 metric in the region $r_+ - r_- \ll r - r_+ \ll r$. This means that for two-point functions we have

$$G_\ell^{\mathrm{AdS}_2}(\omega) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\Delta-1}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)}\frac{\tilde{B}_\ell(\omega)}{\tilde{A}_\ell(\omega)}. \tag{3.144}$$

By solving the wave equation in the fixed BTZ background, we can express A, B in terms of $\tilde{A}, \tilde{B}$. The details are given in Appendix H, and we find the trivial rescaling

$$\tilde{A}_\ell(\omega) = \left(\frac{2}{\pi}\beta_R\right)^{1-\frac{\Delta}{2}} A_\ell(\omega), \qquad \tilde{B}_\ell(\omega) = \left(\frac{2}{\pi}\beta_R\right)^{\frac{\Delta}{2}} B_\ell(\omega) \tag{3.145}$$

This neither mixes normalizable and non-normalizable modes, nor adds any new frequency dependence. The result is a simple ω- and ℓ-independent rescaling factor between AdS_3 and AdS_2 correlators.

Putting this together with the ratio of the normalizing factors in (3.142) and (3.144), we find

$$G_\ell(\omega) \sim \left(\frac{2\pi}{\beta_R}\right)^{\Delta-1}\frac{\Gamma\left(\frac{\Delta}{2}\right)^2}{\Gamma(\Delta)}2^{-\Delta}G_\ell^{\mathrm{AdS}_2}(\omega). \tag{3.146}$$

This reproduces precisely the normalizing factor we found in (3.78), excepting the $2^{-\Delta}$, which arises from the factor of 2 between the 'lightcone' time $-z$ used to define the Schwarzian correlators and the asymptotic time t used here.

Chapter 4
A Three Dimensional view of arbitrary q SYK Model

The SYK hamiltonian (4.1) involves a q fermion interaction with a random coupling which has a gaussian probability distribution with width J. Averaging over the couplings gives rise to a theory with $2q$ Fermi interactions with coupling J^2 and an emergent $O(N)$ symmetry, where N is the number of fermions, making this similar to other vector models. Like other vector models, this is most easily solved at large N by making a change of variables to bilocal fields [147]. For such $O(N)$ models it was proposed in [18] that these bilocal fields in fact provide a bulk construction of the dual higher spin theory [16], with the pair of coordinates in the bilocal combining to provide the coordinates of the emergent AdS spacetime. In $d \geq 2$, the proposal of [18] was implemented, with additional nonlocal transformations on external legs [19–21] providing a map between the bilocal and conventional Vasiliev higher spin fields in AdS_4. For the $d = 1$ case (as in the SYK model), the simplest identification of the center of mass coordinate and the relative coordinate of the two points of the bilocal indeed provides coordinates of a Poincare patch of *Lorentzian* AdS_2. In Section 2.1.1, we developed the collective field theory of the bilocals, providing a transparent way of obtaining both the bilocal propagator and interactions as well as the Schwarzian theory of the low energy mode.

The precise bulk dual of the SYK model and its q fermion coupling generalization are still not well understood. It has been conjectured in [59–61, 148] that the gravity sector of this model is the Jackiw-Teitelboim model [56, 149] of dilaton-gravity with a negative cosmological constant, studied in [58], while [62] provides strong evidence that it is actually Liouville theory. (See also [64, 79, 150, 151]).

The spectrum of the SYK model is highly nontrivial. The matter sector of these theories contains an infinite tower of particles [12, 28, 29]. This is clear from the quadratic action for the bilocal fluctuations and the resulting bilocal propagator. The kinetic term in the action contains *all powers of the AdS_2 laplacian* which gives rise to a rather complicated form of the residues at the poles of the propagator. The couplings of these particles are likewise very complicated, as is clear from the higher point functions computed in [32, 33].

In [63], it was shown for the $q = 4$ model that the exact spectrum and the bilocal propagator follows from a three dimensional model. In this three dimensional realization (where the additional third dimension is used to parametrize the spectrum as in the Kaluza Klein scheme and Higher Spin theories), a scalar field with a conventional kinetic energy term is defined on $AdS_2 \times I$, where $I = S^1/Z_2$ is a finite interval with a suitable size. The mass of the scalar field is at the Breitenlohner-Freedman bound [152] of AdS_2. The scalar field satisfies Dirichlet boundary conditions at the ends and feels an external delta function potential at the middle of the interval. However, as we will see below, the odd parity modes do not play any role in our construction. This means that one can consider half of the interval with Dirichlet condition at one end, and a nontrivial boundary condition determining the derivative of the field at the other end. The background can be thought of as coming from the near-horizon geometry of an extremal charged black hole which reduces the gravity sector to Jackiw-Teitelboim model with the metric in the third direction becoming the dilaton of the latter model [60]. We discussed this in detail for the BTZ black hole in Chapter 3. The strong coupling limit of the SYK model corresponds to a trivial metric in the third direction, while at finite coupling this acquires a dependence on the AdS_2 spatial coordinate. At strong coupling a Horava-Witten compactification then leads to a spectrum of masses in AdS_2 which is in exact agreement with the SYK spectrum. More significantly, a *non-standard* propagator with the end points at the location of the delta function exactly reproduces the SYK bilocal propagator, if the two coordinates in the Poincare patch AdS_2 are identified with the center of mass and the relative coordinate of the two points of the bilocal. The nontrivial factors which appear in the SYK propagator from residues at the poles now appear as the values of the wave function at the center of I.

The strong coupling propagator is divergent due to the divergent contribution of a mode which can be identified with a reparametrization invariant zero mode in the SYK model. At finite coupling the zero mode is lifted and gives rise to an "enhanced contribution" proportional to J. In the three dimensional model [63], the proposal of [60, 61] was adopted and it was shown that to order $1/J$, the poles of the propagator shift in a manner consistent with the explicit results in [12] and the enhanced propagator was reproduced as well for the $q = 4$ case.

In this chapter, we show that such a three dimensional picture holds for generalizations of the SYK model with arbitrary q. We show that the three dimensional metric on which the scalar lives is now conformal to $AdS_2 \times I$. The scalar field is subject to a nontrivial potential in addition to a delta function at the center of the interval. This reproduces the spectrum exactly. Furthermore, the three dimensional propagator whose points lie at the center of I reproduce the arbitrary q SYK propagator up to a factor which depends only on q. We also discuss the large q limit in this picture. In the SYK model the spectrum becomes evenly spaced in this limit. However, only the zero mode contributes to the propagator since the residues at the other poles vanish. In the three dimensional picture, the different modes appear as Kaluza Klein (KK) modes. However, in the large q limit we find that the normalized wave function at the center of the interval is nonzero for only one of these modes, in a way consistent with the SYK result.

The chapter is organized as follows: In Section 4.1 we discuss how to derive the SYK spectrum for arbitrary q within the setup of bilocal theory. In Section 4.2 we describe our three dimensional model. Section 4.3 contains the comparison of the propagator of the three dimensional model with the SYK bilocal propagator. In Section 4.4 we comment on the large q limit. Section 4.5 contains some concluding remarks.

4.1 q Fermion SYK Model

The model is defined by a hamiltonian, with any even q,

$$H = (i)^{\frac{q}{2}} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 i_2 \cdots i_q}\, \chi_{i_1}\, \chi_{i_2} \cdots \chi_{i_q}\,, \tag{4.1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The random coupling has a gaussian distribution with

$$\langle J^2_{i_1 i_2 \cdots i_q} \rangle = \frac{J^2 (q-1)!}{N^{q-1}} \tag{4.2}$$

One way to perform the averaging is to use the replica trick. One does not expect a spin glass state in this model [27] so that we can restrict to the replica diagonal subspace [29]. At large N this model is efficiently solved by rewriting the theory in terms of replica diagonal bilocal collective fields [29, 30].

$$\Psi(\tau_1, \tau_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1) \chi_i(\tau_2)\,, \tag{4.3}$$

where we have suppressed the replica index. The corresponding path-integral is

$$Z = \int \prod_{\tau_1, \tau_2} \mathcal{D}\Psi(\tau_1, \tau_2)\; \mu(\Psi)\, e^{-S_{\rm col}[\Psi]}\,, \tag{4.4}$$

where $S_{\rm col}$ is the collective action:

$$S_{\rm col}[\Psi] = \frac{N}{2} \int d\tau \left[\partial_\tau \Psi(\tau, \tau') \right]_{\tau' = \tau} + \frac{N}{2} \operatorname{Tr} \log \Psi - \frac{J^2 N}{2q} \int d\tau_1 d\tau_2\, \Psi^q(\tau_1, \tau_2)\,. \tag{4.5}$$

Here the second term comes from a Jacobian factor due to the change of path-integral variable, and the trace is taken over the bi-local time. One also has an appropriate order $\mathcal{O}(N^0)$ measure μ. This action being of order N gives a systematic $G = 1/N$ expansion, while the measure μ found as in [76] begins to contribute at one-loop level (in $1/N$). As is well known, in the IR, i.e. at strong coupling the kinetic term can be ignored. There is now an emergent reparametrization invariance. In this limit the saddle point equation which follow from $S_{\rm col}[\Psi]$ has the solution

$$\Psi^{(0)}(\tau_1, \tau_2) = \frac{b}{|\tau_{12}|^{\frac{2}{q}}} \operatorname{sgn}(\tau_{12}) \qquad b^q = \frac{\tan(\frac{\pi}{q})}{J^2 \pi} \left(\frac{1}{2} - \frac{1}{q} \right) \tag{4.6}$$

where we defined $\tau_{ij} \equiv \tau_i - \tau_j$.

In the following it will be useful to use the center of mass and relative coordinates defined by (2.106). The conformal transformations on τ_1, τ_2 then give rise to transformations on t, z which are identical to the isometries of AdS_2 with a metric

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2)$$

Note that this could very well be dS_2 since it has the same isometry group. Fluctuations around this critical IR solution, $\Psi_0(t,z)$ defined by

$$\Psi(\tau_1,\tau_2) \;=\; \Psi^{(0)}(\tau_1,\tau_2) \;+\; \sqrt{\frac{2}{N}}\;\eta(t,z) \tag{4.7}$$

can be expanded as

$$\eta(t,z) = \int \frac{d\omega}{2\pi} \int \frac{d\nu}{N_\nu} \tilde{\Phi}_{\nu,\omega} u_{\nu,\omega}(t,z) \tag{4.8}$$

where

$$u_{\nu,\omega}(t,z) \;=\; \mathrm{sgn}(z)\, e^{i\omega t}\, Z_\nu(|\omega z|) \tag{4.9}$$

Z_ν are a complete set of modes which diagonalizes the quadratic kernel [28],

$$Z_\nu(x) \;=\; J_\nu(x) \;+\; \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}\,. \tag{4.10}$$

Their normalization and completeness relations are given by

$$\begin{aligned} \int_0^\infty \frac{dx}{x}\, Z^*_\nu(x)\, Z_{\nu'}(x) \;&=\; N_\nu\, \delta(\nu-\nu') \\ \int \frac{d\nu}{N_\nu}\, Z^*_\nu(|x|)\, Z_\nu(|x'|) \;&=\; x\, \delta(x-x') \end{aligned} \tag{4.11}$$

where the normalization factor N_ν is

$$N_\nu \;=\; \begin{cases} (2\nu)^{-1} & \text{for } \nu = \frac{3}{2}+2n \\ 2\nu^{-1}\sin\pi\nu & \text{for } \nu = ir\,, \end{cases} \tag{4.12}$$

In all of the above expressions the integral over ν is a shorthand for an integral over the imaginary axis and a sum over the discrete values $\nu = \frac{3}{2}+2n$. The necessity of both the continuous and the discrete spectrum follows from SL(2,R) representation theory [153]. The quadratic part of the fluctuation action then becomes

$$S_{(2)} \propto \int d\nu \int d\omega\; \tilde{\Phi}^\star_{\nu,\omega} \left[\tilde{k}_c(\nu,q) - 1\right] \tilde{\Phi}_{\nu,\omega} \tag{4.13}$$

where

$$\tilde{k}_c(\nu,q) = \frac{1}{k_c(h,q)}, \qquad h = \frac{1}{2}+\nu \tag{4.14}$$

and $k_c(h,q)$ is the eigenvalue of the bilocal kernel derived in [12],

$$k_c(h,q) = -(q-1)\frac{\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)\Gamma\left(1-\frac{1}{q}\right)\Gamma\left(\frac{h}{2}+\frac{1}{q}\right)\Gamma\left(\frac{1}{2}+\frac{1}{q}-\frac{h}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)\Gamma\left(\frac{3}{2}-\frac{1}{q}-\frac{h}{2}\right)\Gamma\left(1-\frac{1}{q}+\frac{h}{2}\right)} \tag{4.15}$$

The spectrum is then given by solving $k_c(h,q) = 1$. Note that $p_m = \frac{3}{2}$ is an exact solution for all q.

The bilocal propagator in (t, z) space can be now derived following the same steps as in [29, 63] by substituting the expansion (4.8) and using the (ν, ω) space propagator which follows from (4.13),

$$\mathcal{G}(t,z;t',z') \sim -\frac{1}{J}|zz'|^{\frac{1}{2}} \sum_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\nu}{N_\nu} \frac{Z_\nu^*(|\omega z|) Z_\nu(|\omega z'|)}{\nu^2 - p_m^2} (2p_m) R(p_m) \tag{4.16}$$

where p_m denote the solutions of the spectral equation $k_c(p_m + \frac{1}{2}, q) = 1$. The factor $R(p_m)$ is the residue of the propagator at the poles $\nu = p_m$,

$$R(p_m) = \frac{1}{\left[\frac{\partial \tilde{k}_c(\nu,q)}{\partial \nu}\right]_{\nu = p_m}} \tag{4.17}$$

and

$$\frac{\partial \tilde{k}_c(\nu,q)}{\partial \nu} = N_h \left[H_{-1+\frac{h}{2}+\frac{1}{q}} + H_{\frac{1}{2}-\frac{h}{2}-\frac{1}{q}} - H_{\frac{h}{2}-\frac{1}{q}} - H_{-\frac{1}{2}-\frac{h}{2}+\frac{1}{q}} \right] \tag{4.18}$$

where H_n denotes the Harmonic number, and

$$N_h = \frac{\left(\sin \pi h + \sin \frac{2\pi}{q}\right) \Gamma\left(\frac{2}{q}\right) \Gamma\left(2 - h - \frac{2}{q}\right) \Gamma\left(1 + h - \frac{2}{q}\right)}{\pi q \Gamma\left(3 - \frac{2}{q}\right)} \tag{4.19}$$

The symbol $\int d\nu$ is as usual a shorthand notation for an integral over the imaginary axis and a sum over discrete values $\nu = \frac{3}{2} + 2n$. As in the $q = 4$ case, when one performs the ν integral over the imaginary axis there are two sets of poles, the ones at $\nu = \pm p_m$ and at $\nu = \frac{3}{2} + 2n$. The contribution from those latter poles exactly cancel the contribution from the discrete values , and one is finally left with an expression

$$\mathcal{G}(t,z;t',z') \sim -\frac{1}{J}|zz'|^{\frac{1}{2}} \sum_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{Z_{-p_m}(|\omega|z^>) J_{p_m}(|\omega|z^<)}{N_{p_m}} R_{p_m} \tag{4.20}$$

where $z^<(z^>)$ is the smaller (larger) of z, z'.

As expected, the expression (4.20) is divergent since this is a strong coupling propagator. This comes from the mode at $p_m = \frac{3}{2}$ which is a solution for all q. At this value $Z_{-\frac{3}{2}}$ diverges because $\xi_{-\frac{3}{2}}$ diverges. For finite J this mode is corrected by a term which is $O(1/J)$ and this leads to a contribution to the propagator which is $O(J)$ compared to the contribution from the other solutions of $k_c(p_m + 1/2, q) = 1$.

4.2 The Three Dimensional Model

We will now write down a model which reproduces the above spectrum exactly and the above propagator up to a function of q. The model is that of a single scalar field Φ with an action

$$\frac{1}{2} \int dt dz dx \sqrt{-g} \left[-g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(x) \Phi^2 \right] \tag{4.21}$$

where the background metric is given by

$$ds^2 = |x|^{\frac{4}{q}-1}\left[\frac{-dt^2+dz^2}{z^2}+\frac{dx^2}{4|x|(1-|x|)}\right] \tag{4.22}$$

and the direction x lies in the interval $-1 < x < 1$. The spacetime is then *conformal* to $AdS_2 \times S^1/Z_2$. The potential which appears in (4.21) is given by

$$V(x) = \frac{1}{|x|^{\frac{4}{q}-1}}\left[4\left(\frac{1}{q}-\frac{1}{4}\right)^2 + m_0^2 + \frac{2V}{J(x)}\left(1-\frac{2}{q}\right)\delta(x)\right] \tag{4.23}$$

where V is a constant to be determined below and

$$J(x) = \frac{|x|^{\frac{2}{q}-1}}{2\sqrt{1-|x|}} \tag{4.24}$$

The action can be now rewritten as

$$\begin{aligned} S =&\frac{1}{2}\int dtdzdx\ J(x)\Bigg[(\partial_t\Phi)^2-(\partial_z\Phi)^2-\frac{m_0^2}{z^2}\Phi^2 \\ &-\frac{4}{z^2}\left\{|x|(1-|x|)(\partial_x\Phi)^2+\left(\frac{1}{q}-\frac{1}{4}\right)^2\Phi^2+\left(1-\frac{2}{q}\right)\frac{V}{2J(x)}\delta(x)\Phi^2\right\}\Bigg] \end{aligned} \tag{4.25}$$

We will impose Dirichlet boundary conditions at $x = \pm 1$,

$$\Phi(t,z,\pm 1) = 0 \tag{4.26}$$

while the delta function discontinuity in the potential determines the discontinuity at $x = 0$ to be

$$\lim_{\epsilon\to 0}\left[|x|^{2/q}\sqrt{1-|x|}\partial_x\Phi\right]_{-\epsilon}^{\epsilon} = \left(1-\frac{2}{q}\right)V\Phi(t,z,0) \tag{4.27}$$

In the following we will be interested in fields which are even under $x \to -x$. For such fields (4.27) implies

$$\left[x^{2/q}\partial_x\Phi\right]_{x=0} = \left(1-\frac{2}{q}\right)\frac{V}{2}\Phi(t,z,0) \tag{4.28}$$

Once we impose this we can restrict to $0 < x < 1$ and forget about the delta function. This is what we will do in the rest of the chapter.

Performing an integration by parts and ignoring the boundary term the action becomes

$$S = \frac{1}{2}\int_{-\infty}^{\infty} dt \int_0^{\infty} dz \int_0^1 dx\ J(x)\ \Phi\mathcal{D}_0\Phi \tag{4.29}$$

where

$$\mathcal{D}_0 = -\partial_t^2+\partial_z^2-\frac{m_0^2}{z^2}+\frac{4}{z^2}\left[x(1-x)\partial_x^2+\left[\frac{2}{q}-x\left(\frac{1}{2}+\frac{2}{q}\right)\right]\partial_x-\left(\frac{1}{q}-\frac{1}{4}\right)^2\right] \tag{4.30}$$

This operator is self adjoint with the measure $J(x)$.

4.2.1 The Spectrum

To diagonalize $\mathcal{D}_0$ we first find solve the eigenvalue problem for the operator inside the square bracket in (4.30) in the domain $0 < x < 1$,

$$\left[x(1-x)\partial_x^2 + \left[\frac{2}{q} - x\left(\frac{1}{2}+\frac{2}{q}\right)\right]\partial_x - \left(\frac{1}{q}-\frac{1}{4}\right)^2\right]\phi_k(x) = -\frac{k^2}{4}\phi_k(x) \tag{4.31}$$

The general solution of this equation is

$$\phi_k(x) = A\ {}_2F_1(a,b;c,x) + x^{1-c}B\ {}_2F_1(a-c+1,b-c+1;2-c;x) \tag{4.32}$$

where ${}_2F_1$ denotes the usual hypergeometric function and

$$a = \frac{1}{q} - \frac{1}{4} - \frac{k}{2} \qquad b = \frac{1}{q} - \frac{1}{4} + \frac{k}{2} \qquad c = \frac{2}{q} \tag{4.33}$$

Imposing the boundary condition (4.28) gives

$$B = \frac{V}{2}A \tag{4.34}$$

while imposing (4.26) gives

$$A_2F_1(a,b;c,1) + B_2F_1(a-c+1,b-c+1;2-c;1) = 0 \tag{4.35}$$

Using

$${}_2F_1(a,b;c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4.36}$$

and combining (4.35) and (4.34) we get

$$\frac{\Gamma\left(\frac{5}{4}-\frac{1}{q}-\frac{k}{2}\right)\Gamma\left(\frac{2}{q}\right)\Gamma\left(\frac{5}{4}-\frac{1}{q}+\frac{k}{2}\right)}{\Gamma\left(2-\frac{2}{q}\right)\Gamma\left(\frac{1}{4}+\frac{1}{q}-\frac{k}{2}\right)\Gamma\left(\frac{1}{4}+\frac{1}{q}+\frac{k}{2}\right)} = -\frac{V}{2} \tag{4.37}$$

where we have used the values of a, b, c in (4.33). Remarkably if we choose

$$V = 2(q-1)\frac{\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)\Gamma\left(1-\frac{1}{q}\right)\Gamma\left(\frac{2}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)\Gamma\left(2-\frac{2}{q}\right)\Gamma\left(\frac{1}{q}\right)} \tag{4.38}$$

and define $h = k + 1/2$ the condition (4.37) becomes

$$k_c(h,q) = 1 \tag{4.39}$$

where $k_c(h,q)$ is precisely the SYK spectrum for arbitrary q given by (4.15). The significant point of course is that V given by (4.38) depends only on q.

4.2.2 The two point function

Using the eigenfunctions in the previous subsection we can now expand the three dimensional field in terms of a complete basis as follows

$$\Phi(t,z,x) = \int \frac{dk d\nu d\omega}{N_\nu} e^{-i\omega t}|z|^{1/2} Z_\nu(|\omega z|)\varphi_k(x)\ \chi(\omega,\nu,k) \tag{4.40}$$

where the combinations of Bessel functions Z_ν have been defined in (4.10) and $\varphi_k(x)$ are

$$\varphi_k(x) = {}_2F_1(a,b;c,x) + x^{1-c}\frac{V}{2}\ {}_2F_1(a-c+1,b-c+1;2-c;x) \tag{4.41}$$

where a,b,c are given in (4.33) and V is given in (4.38). The functions $\varphi_k(x)$ are orthogonal with the measure factor $J(x)$

$$\int_0^1 dx J(x)\varphi_k(x)\varphi_{k'}(x) = C_1(k)\delta_{k,k'} \tag{4.42}$$

The action now becomes

$$S = \frac{1}{2}\int \frac{dk d\nu d\omega}{N_\nu} C_1(k)(\nu^2-\nu_0^2)\chi(\omega,\nu,k)\chi(-\omega,\nu,k) \tag{4.43}$$

where

$$\nu_0^2 = k^2 + m_0^2 + \frac{1}{4} \tag{4.44}$$

Let us now choose

$$m_0^2 = -1/4 \tag{4.45}$$

so that one finally has $\nu_0 = k$. The two point function of $\chi(\omega\nu k)$ is then given by

$$\langle\chi(\omega,\nu,k)\chi(-\omega,\nu,k)\rangle = \frac{N_\nu}{C_1(k)(\nu^2-k^2)} \tag{4.46}$$

The three dimensional propagator in position space is then given by

$$\langle\Phi(t,z,x)\Phi(t',z',x')\rangle = |zz'|^{\frac{1}{2}}\sum_m C(p_m,x,x')\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\nu}{N_\nu}\frac{Z_\nu^\star(|\omega z|)Z_\nu(|\omega z'|)}{\nu^2-p_m^2} \tag{4.47}$$

where

$$C(p_m,x,x') = \frac{\varphi_{p_m}(x)\varphi_{p_m}(x')}{C_1(p_m)} \tag{4.48}$$

This can be regarded as a sum of AdS_2 propagators. However it is important to note that these are non-standard propagators.

4.3 Comparison of the three dimensional and SYK propagators

We now show that the propagator (4.47) evaluated at $x = x' = 0$ agrees with the SYK propagator (4.16) up to an overall factor which depends on q. The values of p_m over which the two expressions need to be summed have been already seen to be identical, so we need to compare the coefficients which appear. To compare (4.16) and (4.47) we need to compute the quantity

$$\frac{C(p_m, 0, 0)}{2p_m R(p_m)} \tag{4.49}$$

and show that this is independent of p_m. Here

$$C_1(p_m) = \int_0^1 dx J(x) \varphi_{p_m}(x) \varphi_{p_m}(x) \tag{4.50}$$

We have not been able to evaluate this integral analytically, but have performed this numerically to high precision. In Table 4.1 we tabulate the values of the relevant quantities for various values of q and p_m which solve the spectrum equation, and compare them with the corresponding factors which appear in (4.16) For a given value of q the value of the ratio (4.49) is independent of p_m upto 13 decimal places. This shows that this ratio is only a function of q which we denote by $f(q)$. These results show that for any given q, the non-standard propagator of the three dimensional model with the two points at $x = 0$ is proportional to the SYK propagator. The data also shows that $f(q)$ decreases with q.

As for $q = 4$, the propagator (4.47) is actually divergent from the contribution of the $p_m = \frac{3}{2}$ mode, as expected from the SYK propagator at infinite coupling. We expect that a modification of the three dimensional background would reproduce the enhanced propagator of this mode similarly to the $q = 4$ case [63].

4.4 The large q limit

In the $q \to \infty$ limit, $p_m = \frac{3}{2}$ remains a solution, while the other solutions of the spectral equation (4.15) become very simple,

$$p_m = 2m + \frac{1}{2} + \frac{2}{q}\frac{2m^2 + m + 1}{2m^2 + m - 1} + \cdots \qquad m = 1, 2, \cdots \tag{4.51}$$

To calculate the contribution to these poles to the SYK propagator consider the residue $R(p_m)$ in (4.17). In a $1/q$ expansion we find that for $p_m = \frac{3}{2}$

$$R\left(\frac{3}{2}\right) = \frac{2}{3} - \frac{1}{q}\left(\frac{5}{2} + \frac{\pi^2}{3}\right) + O(1/q^2) \tag{4.52}$$

while for the other solutions we get

$$R(p_m) \to \frac{1}{q}\frac{4(2m^2 + m)}{(2m^2 + m - 1)^2} + O(1/q^2) \tag{4.53}$$

Table 4.1: Comparison of Factors Appearing in the three dimensional and SYK Propagators

q	p_m	$C(p_m,0,0)$	$2p_mR(p_m)$	$\frac{C(p_m,0,0)}{2p_mR(p_m)}$
6	1.5	0.415724	1.09987	0.377976
	3.07763	0.566693	1.49928	0.377976
	4.95427	0.474406	1.25512	0.377976
	6.90849	0.409177	1.08255	0.377976
	8.88613	0.366967	0.970874	0.377976
8	1.5	0.344227	1.27788	0.269374
	2.95416	0.357211	1.32608	0.269374
	4.835	0.241026	0.894764	0.269374
	6.79849	0.188249	0.698838	0.269374
	8.78225	0.159144	0.590793	0.269374
12	1.5	0.256089	1.47882	0.173171
	2.81505	0.175464	1.01324	0.173171
	4.71763	0.0925242	0.534294	0.173171
	6.69343	0.0660067	0.381165	0.173171
	8.68356	0.0529405	0.305712	0.173171
20	1.5	0.169337	1.66312	0.101819
	2.69405	0.0678495	0.666375	0.101819
	4.62734	0.0287883	0.28274	0.101819
	6.61348	0.0192516	0.189077	0.101819
	8.60817	0.0148617	0.145963	0.101819
50	1.5	0.0745898	1.85438	0.0402235
	2.57914	0.0115317	0.286691	0.0402235
	4.54971	0.00396415	0.0985529	0.0402235
	6.54453	0.00251532	0.0625335	0.0402235

Thus only the pole at $p_m = \frac{3}{2}$ has a non-vanishing residue in the large q limit. Of course the strong coupling propagator is infinite from contribution of the $p_m = \frac{3}{2}$ mode. However a finite J correction would lead to a nonzero contribution proportional to J [12].

In the three dimensional picture this happens because of the different large q behavior of the wave function at $x = 0$ for $p_m = \frac{3}{2}$ compared to the other values of p_m. For large enough q we can use $p_m = 2m + 1$ as a good approximation to the solution of the spectral equation. In Table 4.2 we tabulate the values of the square of the wave function at $x = x' = 0$, i.e. $C(p_m, 0, 0)$, the value of the quantity $2p_mR(p_m)$ which appears in the SYK propagator and the quantity $qf(q)$ where

$$f(q) = \frac{C(p_m,0,0)}{2p_mR(p_m)} \tag{4.54}$$

for large values of q, for different values of m. We have checked that for the values of q which we have used, $p_m = 2m + 1$ is indeed a very good approximation to the exact solution of $k_c(h,q) = 1$. We then tabulate this quantity for given m for various values of q. The results show that $qf(q)$ is a constant to very high accuracy. Using (4.52) and (4.53) we then conclude that the wave function at $x = 0$ for $p_m \neq \frac{3}{2}$ modes vanishes as $\frac{1}{q^2}$, while that for the $p_m = \frac{3}{2}$ this vanishes as $\frac{1}{q}$.

Table 4.2: Large q behavior of the wave function at $x = x' = 0$ for different values of p_m

$p_m = 2m + 1/2$	q	$C_1(p_m, 0, 0)$	$2p_m R(p_m)$	$\frac{qC(p_m,0,0)}{2p_m R(p_m)}$
$\frac{3}{2}$	500	125.908	1.98465	2.00093
	600	150.908	1.9872	2.00077
	800	200.908	1.99039	2.00058
	1000	250.908	1.9923	2.00046
5/2	500	74442.2	0. 003356	2.00093
	600	107330	0. 002794	2.00077
	800	191107	0. 002092	2.00058
	1000	298883	0. 00167	2.00046
9/2	500	137225	0. 00182	2.00093
	600	198038	0. 00151	2.00077
	800	352937	0.001133	2.00058
	1000	552280	0. 000905	2.00046
21/2	500	321462	0.00077	2.00093
	600	464316	0.000645	2.00077
	800	828595	0.000484	2.00058
	1000	1.279×10^6	0.000385	2.00084
41/2	500	625414	0. 000399	2.00093
	600	904123	0. 000331	2.00077
	800	1.615×10^6	0. 000247	2.00082
	1000	2.52×10^6	0. 000197	2.00065

A plot of $f(q)$ for $p_m = \frac{3}{2}$ is given in Figure (4.1). This behavior provides an understanding of the decoupling of the other modes in the tower at large q. Note that the other modes are still present, though they do not contribute to the propagator. In fact the large q limit is subtle. If we perform a $1/q$ expansion of the collective action for the bilocal field, (4.5), using the parametrization used in [12] one ends up with a Liouville theory in the (t, z) space for all values of the suitably rescaled coupling [154].The Dyson-Schwinger equation in the $q \to \infty$ limit is already known to be akin to the Liouville equation [12]. Here we are making a stronger statement about the bilocal field itself, not just about its saddle point value. This has a conventional kinetic term - so that one seems to get a single two dimensional field, the corresponding pole of the propagator being precisely the $p_m = \frac{3}{2}$ mode. The other modes are simply absent in this treatment, and seem to be recovered due to nonlocal $1/q$ interactions.

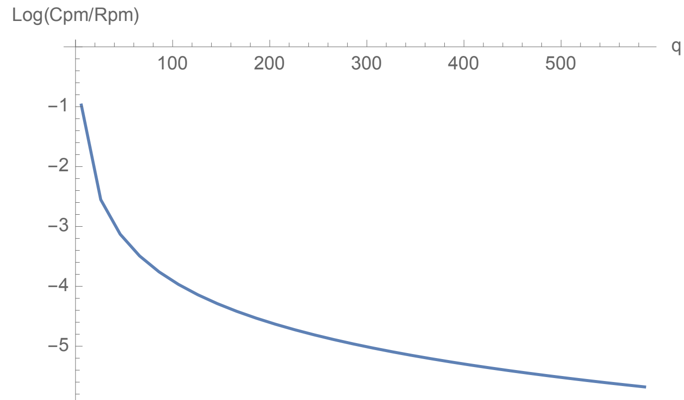


Figure 4.1: Plot of $\log f(q)$ for $p_m = \frac{3}{2}$. For large q the data fits well with the function $f(q) \sim \frac{2}{q}$

4.5 Conclusions

It is remarkable that the complicated tower of states which appear in the SYK model can be understood as a KK tower for arbitrary q. We want to emphasize that while reproducing the spectrum is already quite interesting, the agreement of the propagator with the bilocal propagator is highly nontrivial. This gives strong evidence that a three dimensional spacetime is an essential ingredient of the full dual to the SYK model.

Unlike the $q = 4$ case we do not yet have a natural understanding of the three dimensional background at arbitrary q in terms of a near horizon geometry of a black hole. We hope to be able to achieve this understanding. That will provide a natural way to understand the finite J correction and in particular the enhanced propagator of the $p_m = \frac{3}{2}$ mode.

In this chapter we have not addressed the question of interactions of the bilocals. These have been considered in [32, 33] and are expected to follow from the cubic and higher order terms in the collective field theory of [29]. It will be interesting to see what kind of interactions in the three dimensional model reproduce these and investigate their locality (or lack thereof) properties. In [33], a different three dimensional background is shown to reproduce the large q spectrum. An important aspect of the three dimensional picture is that while the propagator can be written as a sum of AdS_2 propagators, the latter are *non-standard* propagators. While they do vanish at the boundary, they have different boundary conditions at the Poincare horizon. A second unusual aspect is that the space of bilocals always gives rise to Lorentzian AdS_2 even if we start out with an euclidean theory. The issues raised above require a better understanding of the bulk theory. We will address them in the next chapter.

Chapter 5
Space Time in the SYK model

Detailed investigations of the Sachdev-Ye-Kitaev (SYK) model [11, 11–13, 27–34] have given an interesting, highly nontrivial example of the AdS/CFT duality and a potential framework for quantum black holes.

For all these theories, bilocal observables, as proposed in [18] in the context of $O(N)$ vector model/higher spin duality [16] provide a route to bulk construction of the dual theory with emergent spacetime [19–21]. For the SYK model the IR and the near-IR limit are solvable, with evaluations [11, 12, 30] of the invariant Schwarzian action representing the boundary gravity degrees of freedom possibly related to JT type [56, 149] dual theory [58–61, 148] (See also [62, 64, 65, 79, 84, 151, 155–158]).

From symmetry considerations, the center of mass and relative coordinates of the two points in the bilocal fields can be interpreted as the coordinates of the Poincare patch of AdS_2 or dS_2 [12, 28, 29] as in the simplest identification proposed in [18]. Indeed, the action of fluctuations of the bilocal is a non-polynomial function of the AdS_2 Laplacian, indicating that the theory contains an infinite tower of fields. The bilocal propagator can be expressed as a sum over poles, where each term in the sum is a *non-standard* AdS_2 propagator with nontrivial residues [29]. Remarkably, this tower can be realized as a Kaluza-Klein tower coming from an additional third dimension [25, 63]. This reproduces the spectrum *as well as the propagator*, including the enhanced propagator of [12]. The kinetic terms are now standard: the nontrivial residues at the poles in the SYK propagator now appear as nontrivial wave functions. However, one should not expect that higher point functions [32] can be reproduced by local interactions in this three dimensional picture [33]. The usefulness of an additional dimension also appears in the description of Higher Spin theories [21].

Despite all these successes, the actual emergent spacetime of the SYK model (or of any other similar SYK-type models) is not yet understood. There are several reasons why the AdS_2 or dS_2 on which the bilocals live should *not* be considered to be the bulk spacetime in the usual sense of AdS/CFT. Consider for concreteness the Euclidean partition function. Changing variables to bilocal fields one reaches a solution (the propagator and quadratic fluctuations) which features a Lorentzian signature, coming from the fact that the two points of the bilocal become coordinates of a Lorentizan signature. On the other hand, we expect that the dual theory should live in Euclidean spacetime $EAdS_2$ [12]. One issue which is detrimental to a potential Lorentzian identification associated with this data comes from the factors of " i " which inevitably appears in a Lorentzian dual theory, but absent in the SYK propagator. Secondly, the radial part of the AdS_2 wave functions which appear in the SYK propagator (whether or not we write this in the three dimensional language) are not the usual normalizable AdS wave functions, but satisfy different boundary conditions. These unusual wave functions are, however, required since these are the ones which diagonalize the SYK kernel [28, 29]. This suggests that they might be better thought of as dS_2 wave functions [12]

In this chapter, we provide a key step towards a resolution of both issues. We will show that a nonlocal transform relates the bilocal field to a field whose underlying dynamics is in Euclidean AdS_2. We will arrive at this transform following the same principles underlying the derivation of the corresponding transform for the $O(N)$ model in $d = 3$ [19, 20]. The idea is to find a canonical transformation in the four dimensional phase space of the two points in the bilocal such that the symmetries of EAdS_2 are realized correctly. This suggets a simple transformation kernel for the momentum space fields. It turns out that the corresponding position space kernel is a H^2 Radon transform. Radon transforms have appeared (explicitly or implicitly) in discussions of AdS/CFT, most notably in [159–161] where this is used to go from the bulk to the kinematic space of the boundary field theory on a time slice. Indeed the space on which the bilocals live is a version of kinematic space. However, unlike these papers we are not working on a time slice in the bulk - rather our transform takes unequal Euclidean time fields on EAdS_2 to bilocals. Though mathematically identical, our transform is conceptually somewhat different. The necessity of a Radon transform in this context has been in fact mentioned in [12].

This transformation takes the particular combinations of Bessel functions which appear in the SYK propagators to the modified Bessel functions which appear in the standard EAdS_2 propagator. In addition we find extra leg factors which resemble the leg pole factors of the $c = 1$ matrix model . In that case these were necessary to relate the collective field [162] to the tachyon field of the dual 2D string theory and reproduce the S-Matrix [163] (for a recent improved understanding see [164]). The leg poles represent discrete states of the 2D string and analogously it is tempting to suggest that the leg pole factors also arise from similar bulk degrees of freedom, which remain to be identified. An explicit correspondence between the SYK propagator and the propagator of macroscopic loop operators [163] that we establish supports this interpretation.

This chapter is organized as follows: In section 5.1, we review relevant aspects of the bilocal solution of the model and illuminate the dS_2 nature of the wave functions. In Section 5.2, we introduce the Leg transformations and their meaning. In Section 5.3, Leg factors and the Propagator is discussed. Section 5.4 is reserved for conclusions.

5.1 Question of Dual Spacetime

In this section, we clarify the question regarding the signature of the SYK dual gravity theory. The Sachdev-Ye-Kitaev model [11] is a quantum mechanical many body system with all-to-all interactions on fermionic N sites ($N \gg 1$), described by the Hamiltonian

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \tag{5.1}$$

where χ_i are Majorana fermions, which satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The coupling constant J_{ijkl} are random with a Gaussian distribution with width J. The generalization to analogous q-point interacting model is straightforward [11, 12]. After the disorder averaging for the random coupling J_{ijkl}, there is only one effective coupling J in the effective action. The model is usually treated by replica method. One does not expect a spin glass state in this model at least in the leading order of $1/N$ [27] so that we can restrict to the replica diagonal subspace [29]. The Large N theory is simply represented through a (replica diagonal) bi-local collective field:

$$\Psi(\tau_1,\tau_2) \equiv \frac{1}{N}\sum_{i=1}^{N}\chi_i(\tau_1)\chi_i(\tau_2)\,, \tag{5.2}$$

where we have suppressed the replica index. The corresponding path-integral is

$$Z = \int\prod_{\tau_1,\tau_2}\mathcal{D}\Psi(\tau_1,\tau_2)\;\mu[\Psi]\,e^{-S_{\rm col}[\Psi]}\,, \tag{5.3}$$

where $S_{\rm col}$ is the collective action:

$$S_{\rm col}[\Psi] = \frac{N}{2}\int dt\,\Big[\partial_t\Psi(t,t')\Big]_{\tau'=\tau} + \frac{N}{2}\,{\rm Tr}\log\Psi - \frac{J^2N}{2q}\int d\tau_1 d\tau_2\,\Psi^q(\tau_1,\tau_2)\,. \tag{5.4}$$

Here the trace term comes from a Jacobian factor due to the change of path-integral variable, and the trace is taken over the bi-local time. One also has an appropriate order $\mathcal{O}(N^0)$ measure $\mu[\Psi]$. There is another formulation with two bi-local fields: the fundamental fermion propagator $G(\tau_{12})$ and the self energy $\Sigma(\tau_{12})$. This is equivalent to the above formulation after elimination of $\Sigma(\tau_{12})$. In this chapter, we focus on this Euclidean time SYK model.

Fluctuations around the critical IR saddle point background $\Psi^{(0)}(\tau_1,\tau_2)$ can be studied by expanding the bilocal field as [29]

$$\Psi(\tau_1,\tau_2) = \Psi^{(0)}(\tau_1,\tau_2) + \frac{1}{\sqrt{N}}\,\overline{\Psi}(\tau_1,\tau_2)\,, \tag{5.5}$$

where $\Psi^{(0)}$ is the IR large N saddle-point solution and $\overline{\Psi}$ is the fluctuation. At the quadratic level, we have a quadratic kernel $\mathcal{K}$. The diagonalization of this quadratic kernel is done by the eigenfunctions $u_{\nu,\omega}$ and the eigenvalue $\widetilde{g}(\nu)$ as

$$\int d\tau_1' d\tau_2'\,\mathcal{K}(\tau_1,\tau_2;\tau_1',\tau_2')\,u_{\nu,\omega}(\tau_1',\tau_2') = \widetilde{g}(\nu)\,u_{\nu,\omega}(\tau_1,\tau_2)\,. \tag{5.6}$$

The quadratic kernel $\mathcal{K}$ is in fact a function of the bilocal $SL(2,\mathcal{R})$ Casimir

$$\begin{aligned} C_{1+2} &= \big(\hat{D}_1+\hat{D}_2\big)^2 - \frac{1}{2}\big(\hat{P}_1+\hat{P}_2\big)\big(\hat{K}_1+\hat{K}_2\big) - \frac{1}{2}\big(\hat{K}_1+\hat{K}_2\big)\big(\hat{P}_1+\hat{P}_2\big) \\ &= -\,(\tau_1-\tau_2)^2\,\partial_1\partial_2\,, \end{aligned} \tag{5.7}$$

with the $SL(2,\mathcal{R})$ generators $\hat{D} = -\tau\partial_\tau$, $\hat{P} = \partial_\tau$, and $\hat{K} = \tau^2\partial_\tau$. The common eigenfunctions of the bilocal $SL(2,\mathcal{R})$ Casimir (5.7) are, due to the properties of the conformal block, given by the three-point function of the form

$$|\tau_{12}|^{2\Delta}\Big\langle\mathcal{O}_h(\tau_0)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\Big\rangle = \frac{{\rm sgn}(\tau_{12})}{|\tau_{10}|^h|\tau_{20}|^h|\tau_{12}|^{-h}}\,, \tag{5.8}$$

where we defined $\tau_{ij} \equiv \tau_i - \tau_j$. Since the SYK quadratic kernel $\mathcal{K}$ is a function of this bilocal $SL(2,\mathcal{R})$ Casimir, this three-point function is also the eigenfunction of the SYK quadratic kernel. For the investigation of dual gravity theory, it is more useful to Fourier transform from τ_0 to ω by

$$\begin{aligned}\left\langle \widetilde{\mathcal{O}_h}(\omega)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\right\rangle \;&\equiv\; \int d\tau_0\, e^{i\omega\tau_0}\left\langle \mathcal{O}_h(\tau_0)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\right\rangle \\ &=\; -\sqrt{\pi}\,\cot(\pi\nu)\,\Gamma(\tfrac{1}{2}-\nu)\,|\omega|^\nu\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta-\frac{1}{2}}}\,e^{i\omega(\frac{\tau_1+\tau_2}{2})}Z_\nu(|\tfrac{\omega\tau_{12}}{2}|)\,,\end{aligned} \tag{5.9}$$

where we used $h=\nu+1/2$ and defined

$$Z_\nu(x) \;=\; J_\nu(x) \,+\, \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}\,. \tag{5.10}$$

The τ_0 integral in the Fourier transform can be performed by decomposing the integration region into three pieces. The complete set of ν can be understood from the representation theory of the conformal group, as discussed recently in [153]. We have the discrete modes $\nu = 2n+\frac{3}{2}$ with $(n=0,1,2,\cdots)$ and the continuous modes $\nu = ir$ with $(0<r<\infty)$. Adjusting the normalization, we define our eigenfunctions by

$$u_{\nu,\omega}(t,\hat{z}) \;\equiv\; \mathrm{sgn}(\hat{z})\,\hat{z}^{\frac{1}{2}}\,e^{i\omega t}\,Z_\nu(|\omega\hat{z}|)\,, \tag{5.11}$$

which have normalization condition

$$\int_{-\infty}^{\infty}\frac{dt}{2\pi}\int_0^\infty\frac{d\hat{z}}{\hat{z}^2}\;u^*_{\nu,\omega}(t,\hat{z})\,u_{\nu',\omega'}(t,\hat{z}) \;=\; N_\nu\,\delta(\nu-\nu')\delta(\omega-\omega')\,, \tag{5.12}$$

with

$$N_\nu \;=\; \begin{cases}(2\nu)^{-1} & \text{for } \nu=\frac{3}{2}+2n\\ 2\nu^{-1}\sin\pi\nu & \text{for } \nu = ir\,.\end{cases} \tag{5.13}$$

Here we used the change of the coordinates by

$$t \;\equiv\; \frac{\tau_1+\tau_2}{2}\,, \qquad \hat{z} \;\equiv\; \frac{\tau_1-\tau_2}{2}\,. \tag{5.14}$$

The bilocal $SL(2,\mathcal{R})$ Casimir can be seen to take the form of a Laplacian of Lorentzian two dimensional Anti de-Sitter or de-Sitter space (in this two dimensional case they are characterized by the same isometry group SO(2,1) or SO(1,2)). Under the canonical identification with AdS

$$ds^2 \;=\; \frac{-dt^2+d\hat{z}^2}{\hat{z}^2}\,, \tag{5.15}$$

it equals

$$C_{1+2} \;=\; z^2(-\partial_t^2+\partial_{\hat{z}}^2)\,. \tag{5.16}$$

Consequently the SYK eigenfunctions should be compared with known AdS_2 or dS_2 basis wave functions.

Note that the Bessel function Z_ν (5.10) are not the standard normalizable modes used in quantization of scalar fields in AdS_2: in particular they have rather different boundary conditions at the Poincare horizon. Another important property of this basis is that when viewed as a Schrodinger problem as in [28] it has a set of bound states, in addition to the scattering states. This will be discussed in detail in Section 5.2 (see the left picture of Figure 5.1).

This leads one to try an identification with de-Sitter basis functions As we will see, the bilocal SYK wave functions can be realized as a particular α-vacuum of Lorentzian dS_2 with a choice of $\alpha = i\pi h = i\pi(\nu + 1/2)$. This is seen as follows. We consider the dS_2 background with a metric given by

$$ds^2 \;=\; \frac{-d\eta^2 + dt^2}{\eta^2}\,. \tag{5.17}$$

This can be obtained by the coordinate change (5.14) by replacing $z \to \eta$. The Euclidean (Bunch-Davies [165]) wave function of a massive scalar field is given by

$$\phi^E_\omega(\eta)\, e^{i\omega t}\,, \tag{5.18}$$

with

$$\phi^E_\omega(\eta) \;=\; \eta^{\frac{1}{2}}\, H^{(2)}_\nu(|\omega|\eta)\,, \qquad \nu \;=\; \sqrt{\frac{1}{4} - m^2}\,, \tag{5.19}$$

where $H^{(2)}_\nu$ is the Hankel function of the second kind. Since the t-dependence is always like $e^{i\omega t}$, in the following we will focus only on the η dependence. The α-vacuum wave function is defined by Bogoliubov transformation from this Euclidean wave function [166, 167] as

$$\begin{aligned}\phi^\alpha_\omega(\eta) \;&\equiv\; N_\alpha\Big[\phi^E_\omega(\eta) \;+\; e^\alpha \phi^{E*}_\omega(\eta)\Big] \\ &=\; N_\alpha\, \eta^{\frac{1}{2}}\Big[H^{(2)}_\nu(|\omega|\eta) \;+\; e^\alpha H^{(1)}_\nu(|\omega|\eta)\Big]\,,\end{aligned} \tag{5.20}$$

where

$$N_\alpha \;=\; \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}}\,, \tag{5.21}$$

and α is a complex parameter. Now let us consider a possibility of α-vacuum with

$$\alpha \;=\; i\pi\left(\nu + \frac{1}{2}\right) \;=\; i\pi h\,. \tag{5.22}$$

With this choice of α, using the definition of the Hankel functions

$$H^{(1)}_\nu(x) \;=\; \frac{J_{-\nu}(x) - e^{-i\pi\nu} J_\nu(x)}{i\sin(\pi\nu)}\,, \qquad H^{(2)}_\nu(x) \;=\; \frac{J_{-\nu}(x) - e^{i\pi\nu} J_\nu(x)}{-i\sin(\pi\nu)}\,, \tag{5.23}$$

one can rewrite the α-vacuum wave function as

$$\phi^{\alpha}_{\omega}(\eta) \;=\; \left(\frac{2\,\eta^{\frac{1}{2}}}{1+\xi_\nu\,e^{-i\pi\nu}}\right) Z_\nu(|\omega|\eta)\,, \tag{5.24}$$

where Z_ν is defined in Eq.(5.10). After excluding the η-independent part of the wave function, we can write the η-dependent part as

$$\phi^{\alpha}_{\omega}(\eta) \;=\; \eta^{\frac{1}{2}}\,Z_\nu(|\omega|\eta)\,. \tag{5.25}$$

This wave function agrees with the eigenfunction of the SYK quadratic kernel (5.11) after the identifications of $\eta = (\tau_1 - \tau_2)/2$ and $t = (\tau_1 + \tau_2)/2$.

Due to this observation, one might attempt to claim that the dual gravity theory of the SYK model is given by Lorentzian dS_2 space. However, there is a critical issue in this claim. Apart from the Lorentzian signature in this metric (5.17), we still have a discrepancy in the exponent of the partition function (5.3) with a factor of "i". Namely, if the dual gravity theory (higher spin gravity or string theory) is Lorentzian dS_2, it must have

$$Z \;=\; \int \mathcal{D}h_n\,\mathcal{D}\Phi_m\,\exp\left[i\Big(S_{\mathrm{grav}}[h,\Phi] + S_{\mathrm{matter}}[h,\Phi]\Big)\right], \tag{5.26}$$

where we collectively denote the graviton and other "higher spin" gauge fields by h_n and the dilaton and other matter fields by Φ_m. Hence the agreement of the SYK bilocal propagator

$$\mathcal{D}_{\mathrm{SYK}}(\tau_1,\tau_2;\tau_1',\tau_2') \;=\; \Big\langle \overline{\Psi}(\tau_1,\tau_2)\overline{\Psi}(\tau_1',\tau_2')\Big\rangle \;=\; \sum_{m=0}^{\infty} G_{p_m}(\tau_1,\tau_2;\tau_1',\tau_2')\,, \tag{5.27}$$

with a dS_2 propagator

$$\mathcal{D}_{\mathrm{dS}}(\eta,t;\eta',t') \;=\; \frac{1}{i}\sum_{m=0}^{\infty}\Big\langle \Phi_m(\eta,t)\Phi_m(\eta',t)\Big\rangle \;=\; \frac{1}{i}\sum_{m=0}^{\infty} G_m(\eta,t;\eta',t')\,, \tag{5.28}$$

is only up to the factor i. Namely, even if we have a complete agreement of G_{p_m} with G_m by identifying the coordinates by (5.14) (with a replacement of $z \to \eta$), there is a problem with the signature (i.e. the discrepancy of the factor i). For higher point functions, the same i-problem proceeds due to the i factors coming from the propagator and each vertex.

To conclude, for the Euclidean SYK model under consideration, one needs a dual gravity theory to be in the hyperbolic plane H_2 (i.e. Euclidean AdS_2) for the matching of n-point functions. We will set the basis for the EAdS_2 realization in the next section.

5.2 Transformations and Leg Factors

As we have commented in the beginning of this chapter, in order to identify an Euclidean bulk dual description (rather than a Lorentzian), we will need a transformation which brings the SYK eigenfunctions (as given on bilocal spacetime) to the standard eigenfunctions of the EAdS_2 Laplacian. We will arrive at this transformation by considering the bilocal map described in [19, 21] for higher dimensional case. In our current $d = 1$ case, the map is even simpler. It will be seen to take the form of a H^2 Radon transform (a related suggestion was made in [12]). The need for a nonlocal transform on external legs appears to be characteristic of collective theory (which as a rule contains a minimal set of physical degrees of freedom). The first appearance of Radon type transforms in identifying holographic spacetime was seen in the $c = 1$ / $D = 2$ string correspondence. The transformation introduced in [168] from the collective to a two dimensional (black hole) spacetime took the form

$$T(u,v) \,=\, \int_{-\infty}^{\infty} dt \int_0^{\infty} dx\, \delta\left(\frac{ue^{-t} + ve^{t}}{2} - x^2\right) \gamma(i\partial_t)\, \phi(t,x)\,, \tag{5.29}$$

where $T(u,v)$ is the tachyon field in the Kruskal coordinates representing the target spacetime and $\phi(t,x)$ is related to the eigenvalue density field. Related maps from the collective field or fermions to fields in a black hole background have been proposed in [169] which are also possibly related to Radon transforms. This is seen precisely in the form of what is known as the regular Radon transform.

Let us describe procedure formulated in [19, 21] for constructing the bilocal to spacetime map. The method is based on construction of canonical transformations in phase space : bilocal (τ_1, p_1), (τ_2, p_2) and EAdS_2 (τ, p_τ), (z, p_z). We consider the Poincare coordinates for the Euclidean AdS_2 spacetime

$$ds^2 \,=\, \frac{d\tau^2 + dz^2}{z^2}\,. \tag{5.30}$$

One way to obtain the bilocal map is to equate the $SL(2,\mathcal{R})$ generators.

$$\hat{J}_{1+2} \,=\, \hat{J}_{\mathrm{EAdS}}\,. \tag{5.31}$$

The one-dimensional bilocal conformal generators are

$$\hat{D}_{1+2} \,=\, \tau_1\, p_1 + \tau_2\, p_2\,, \qquad \hat{P}_{1+2} \,=\, -p_1 - p_2\,, \qquad \hat{K}_{1+2} \,=\, -\,\tau_1^2\, p_1 \,-\, \tau_2^2\, p_2\,, \tag{5.32}$$

and the EAdS_2 generators are given by

$$\hat{D}_{\mathrm{EAdS}} \,=\, \tau\, p_\tau + z\, p_z\,, \qquad \hat{P}_{\mathrm{EAdS}} \,=\, -p_\tau\,, \qquad \hat{K}_{\mathrm{EAdS}} \,=\, (z^2 - \tau^2)\, p_\tau - 2\tau z\, p_z\,, \tag{5.33}$$

where we defined $p_1 \equiv -\partial_{\tau_1}$, $p_2 \equiv -\partial_{\tau_2}$, $p_\tau \equiv -\partial_\tau$, $p_z \equiv -\partial_z$. Equating the generators, we can determine the map. From the $\hat{P}$ generators, we have $p_\tau = p_1 + p_2$. Using this result for the other generators, we get two equations to solve:

$$\begin{aligned} z\, p_z \,&=\, (\tau_1 - \tau)p_1 \,+\, (\tau_2 - \tau)p_2 \\ -\, z^2\, p_\tau \,&=\, (\tau_1 - \tau)^2 p_1 \,+\, (\tau_2 - \tau)^2 p_2\,. \end{aligned} \tag{5.34}$$

These are solved by

$$\tau \;=\; \frac{\tau_1\, p_1 - \tau_2\, p_2}{p_1 - p_2}\,, \qquad p_\tau \;=\; p_1 + p_2\,, \qquad z^2 \;=\; -\left(\frac{\tau_1 - \tau_2}{p_1 - p_2}\right)^2 p_1 p_2\,, \qquad p_z^2 \;=\; -4p_1 p_2\,. \tag{5.35}$$

One can see that the canonical commutators are preserved under the transform (at least classically, i.e. in terms of the Poisson bracket). Namely, $[\tau, p_\tau] = [z, p_z] = 1$ and others vanish provided that $[t_i, p_j] = \delta_{ij}$, with $(i, j = 1, 2)$. Hence, we conclude the map is canonical transformation, which is also a point transformation in momentum space. For the kernel which implements this momentum space correspondence we can take

$$\mathcal{R}(p_1, p_2; p_\tau, p_z) \;=\; \frac{\delta(p_\tau - (p_1 + p_2))}{\sqrt{p_z^2 + 4p_1 p_2}}\,. \tag{5.36}$$

Through Fourier transforming all momenta to corresponding coordinates, the associated coordinate space kernel becomes

$$\mathcal{R}(\tau_1, \tau_2; \tau, z) \;=\; \delta(\eta^2 - (\tau - t)^2 - z^2)\,. \tag{5.37}$$

Here, we have ignored possible issues related to the range of variables. With an additional multiplicative factor of of η this is known as the Circular Radon transform (5.40) which has a simple relationship to Radon transform on H^2.

There is another construction of the Radon transform which is used in [159–161] and is based on integration over geodesics. For the Euclidean AdS_2 spacetime (5.30), a geodesic is given by a semicircle

$$(\tau - \tau_0)^2 \,+\, z^2 \;=\; \frac{1}{E^2}\,, \tag{5.38}$$

where $\tau \;=\; \tau_0$ is the center of the semicircle and $1/E$ is the radius. The Radon transform of a function of the bulk coordinates $f(\tau, z)$ is a function of the parameters of a geodesic (E, τ_0) defined by

$$\big[\mathcal{R}f\big](E, \tau_0) \;\equiv\; \int_\gamma ds\, f(\tau, z(\tau))\,, \tag{5.39}$$

where the integral is over the geodesic. From the geodesic equation (5.38), this transform is explicitly written as

$$\big[\mathcal{R}f\big](\eta, t) \;=\; 2\eta \int_{t-\eta}^{t+\eta} d\tau \int_0^\infty \frac{dz}{z}\, \delta\Big(\eta^2 - (\tau - t)^2 - z^2\Big)\, f\,(\tau, z)\;, \tag{5.40}$$

where we have used the identifications $1/E = \eta$ and $\tau_0 = t$; the resulting function $[\mathcal{R}f](\eta, t)$ is understood as a function on the Lorentzian dS_2 (5.17).

We will now explicitly evaluate the Radon transformation of (unit-normalized) EAdS_2 wave functions (see Appendix I)

$$\overline{\phi}_{\mathrm{EAdS}_2}(\tau, z) \;=\; \alpha_\nu\, z^{\frac{1}{2}}\, e^{-i\omega\tau}\, K_\nu(|\omega| z) \tag{5.41}$$

From the above formula of the Radon transform (5.40), we get

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; \alpha_\nu\,\eta\int_{t-\eta}^{t+\eta}\frac{d\tau}{\eta^2-(\tau-t)^2}\,(\eta^2-(\tau-t)^2)^{\frac{1}{4}}\,e^{-i\omega\tau}\,K_\nu(|\omega|\sqrt{\eta^2-(\tau-t)^2})\,. \tag{5.42}$$

Now shifting the integral variable $\tau\to\tau+t$ and using the symmetry of the integrand, one can rewrite this integral as

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; 2\,\alpha_\nu\,\eta\,e^{-i\omega t}\int_0^\eta d\tau\,\left(\frac{1}{\eta^2-\tau^2}\right)^{\frac{3}{4}}\cos(\omega\tau)\,K_\nu(|\omega|\sqrt{\eta^2-\tau^2})\,. \tag{5.43}$$

Further rewriting the $\cos(\omega\tau)$ in terms of $J_{-1/2}(\omega\tau)$ and changing the integration variable to $\tau=\eta\sin\theta$, we find

$$\begin{aligned}\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;&=\; \sqrt{\frac{\pi^3}{2}}\,\frac{\alpha_\nu|\omega|^{\frac{1}{2}}\eta}{\sin(\pi\nu)}\,e^{-i\omega t}\\ &\times\int_0^{\frac{\pi}{2}}d\theta\,(\tan\theta)^{\frac{1}{2}}\,J_{-\frac{1}{2}}(|\omega|\eta\sin\theta)\left[I_{-\nu}(|\omega|\eta\cos\theta)\;-\;I_\nu(|\omega|\eta\cos\theta)\right],\end{aligned} \tag{5.44}$$

where we decomposed the modified Bessel function of the second kind into two first kinds. This θ integral is indeed given in Eq.(4) of $12\cdot 11$ of [170], which leads to

$$\left[\mathcal{R}\overline{\phi}_{\mathrm{EAdS}_2}\right](\eta,t) \;=\; -2i\sqrt{\pi}\,\frac{\Gamma(\frac{1}{4}+\frac{\nu}{2})}{\Gamma(\frac{3}{4}+\frac{\nu}{2})}\,\beta_\nu\,\eta^{\frac{1}{2}}\,e^{-i\omega t}\left[J_\nu(|\omega|\eta)\;+\;\frac{\tan\frac{\pi\nu}{2}+1}{\tan\frac{\pi\nu}{2}-1}\,J_{-\nu}(|\omega|\eta)\right], \tag{5.45}$$

where we also used Eq.(I.9). The inside of the square bracket precisely agrees with the particular combination of Bessel functions, $Z_\nu(|\omega|\eta)$ function defined in Eq.(5.10).

When $\nu_n=\frac{3}{2}+2n$ the second term in this square bracket vanishes. As will be clear soon, we need the radon transform of the modified Bessel function I_{ν_n} with. This can be likewise evaluated to yield

$$\mathcal{R}[\alpha'_{\nu_n}z^{1/2}e^{-ik\tau}I_{\nu_n}(|k|z)] = (2\nu_n\eta)^{1/2}e^{-ikx}J_{\nu_n}(|k|\eta) \tag{5.46}$$

where

$$\alpha'_{\nu_n} = \left(\frac{2\nu_n}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma(\frac{1}{4}+\frac{\nu_n}{2})} \tag{5.47}$$

The extra ν-dependent factor in (5.45) which appears in front of the unit-normalized dS_2 wave function described in Appendix I should be understood as a leg factor (5.49). As we will see later, this is analogous to what happens in the $c=1$ matrix model [163, 171].

In summary, we have the Radon transform

$$\mathcal{R}\,\overline{\phi}^{(\mathrm{EAdS}_2)}_{\omega,\nu}(\tau,z)\;=\;L(\nu)\,\overline{\psi}^{(\mathrm{dS}_2)}_{\omega,\nu}(\eta,t)\,, \tag{5.48}$$

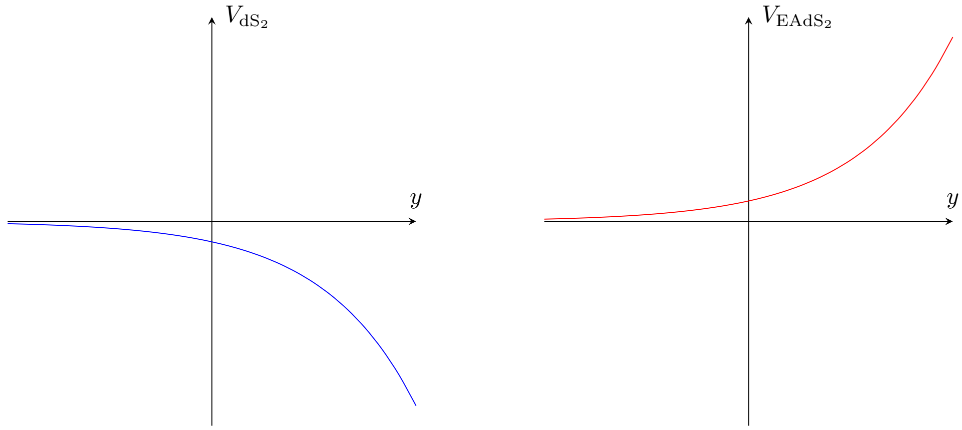


Figure 5.1: The de Sitter potential V_{dS_2} has bound states and scattering states. On the other hand, the Euclidean AdS potential V_{AdS_2} has only scattering modes.

where $\overline{\phi}_{\mathrm{EAdS}_2}$ and $\overline{\psi}_{\mathrm{dS}_2}$ are the unit-normlized wave functions defined in Eq.(I.1) and Eq.(I.6), respectively, while the leg factor is defined by

$$L(\nu) \,\equiv\, (\text{Leg Factor}) \,=\, -2i\sqrt{\pi}\,\frac{\Gamma(\frac{1}{4}+\frac{\nu}{2})}{\Gamma(\frac{3}{4}+\frac{\nu}{2})}\,. \tag{5.49}$$

The inverse transformations are

$$\mathcal{R}^{-1}\,\overline{\psi}^{(\mathrm{dS}_2)}_{\omega,\nu}(\eta,t) \,=\, L^{-1}(\nu)\,\overline{\phi}^{(\mathrm{EAdS}_2)}_{\omega,\nu}(\tau,z)\,. \tag{5.50}$$

for $\nu \neq \frac{3}{2}+2n$, while for $\nu = \frac{3}{2}+2n$ we have instead

$$\mathcal{R}^{-1}\,\overline{\psi}^{(\mathrm{dS}_2)}_{\omega,\nu_n}(\eta,t) \,=\, \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k|z) \tag{5.51}$$

Under the Radon transform $\mathcal{R}$, the Laplacian of Lorentzian dS_2 is transformed into that of Euclidean AdS_2:

$$\Box_{\mathrm{dS}_2}\,\psi_{\mathrm{dS}_2}(\eta,t) \,=\, -\,\mathcal{R}\,\Box_{\mathrm{EAdS}_2}\,\phi_{\mathrm{EAdS}_2}(\tau,z)\,, \tag{5.52}$$

with

$$\Box_{\mathrm{ds}_2} \,=\, \eta^2(-\partial_\eta^2+\partial_t^2)\,, \qquad \Box_{\mathrm{EAdS}_2} \,=\, z^2(\partial_\tau^2+\partial_z^2)\,. \tag{5.53}$$

Here, $\psi_{\mathrm{dS}_2} = \mathcal{R}\phi_{\mathrm{EAdS}_2}$. This role of the Radon transform was first suggested in citeBalasubramanian:2002zh.

In the rest of this section, we will show that the Radon transformation flips the sign of the potential appearing in the equivalent Schrodinger problem as formulated in [28]. We start from the Radon transformation (5.52). Expanding the wave functions by

$$\begin{aligned} \psi_{\mathrm{dS}_2}(\eta,t) \,&=\, \eta^{\frac{1}{2}}\sum_{\omega} e^{-i\omega t}\,\widetilde{\psi}_{\mathrm{dS}_2}(\eta;k)\,, \\ \phi_{\mathrm{EAdS}_2}(\tau,z) \,&=\, z^{\frac{1}{2}}\sum_{\omega} e^{-i\omega\tau}\,\widetilde{\phi}_{\mathrm{EAdS}_2}(\omega;z)\,, \end{aligned} \tag{5.54}$$

we have corresponding Bessel equations for $\widetilde{\psi}_{\mathrm{dS}_2}$ and $\widetilde{\phi}_{\mathrm{EAdS}_2}$. By changing the coordinates by $y \equiv \log(\omega\eta)$ or $y \equiv \log(\omega z)$, these Bessel equations are reduced to the Schrodinger equations as

$$\begin{aligned} \Big(-\partial_y^2 - e^y \Big) \widetilde{\psi}_{\mathrm{dS}_2} &= -\nu^2 \, \widetilde{\psi}_{\mathrm{dS}_2} \,, \\ \Big(-\partial_y^2 + e^y \Big) \widetilde{\phi}_{\mathrm{EAdS}_2} &= -\nu^2 \, \widetilde{\phi}_{\mathrm{EAdS}_2} \,. \end{aligned} \tag{5.55}$$

Therefore, the Radon transform flips the sign of the corresponding Schrodinger potential (see FIG. 5.1). The de Sitter potential $V_{\mathrm{dS}_2} = -e^y$ has bound states as well as scattering states. On the other hand, the Euclidean AdS potential $V_{\mathrm{AdS}_2} = e^y$ has only scattering modes. The difference is accounted by the leg pole factors.

5.3 Green's Functions and Leg Factors

In this section, we start from the SYK bilocal propagator [29, 63]. Applying the inverse Radon transformation (5.50), we will show that the resulting propagator can be written in terms of $EAdS_2$ wave-functions additional momentum space leg-factors.

The SYK bilocal propagator is given by

$$G(\tau_1, \tau_2; \tau_1', \tau_2') \;\propto\; J^{-1} \int_{-\infty}^{\infty} d\omega \sum_{\nu} \frac{u^*_{\nu,\omega}(\tau_1, \tau_2) u_{\nu,\omega}(\tau_1', \tau_2')}{N_\nu [\widetilde{g}(\nu) - 1]} \,, \tag{5.56}$$

where $u_{\nu,\omega}$ are the eigenfunctions defined in Eq.(5.11). Here the summation over ν is a short-hand notation denotes the discrete mode sum and the continuous mode sum. with the identification $\eta = (\tau_1 - \tau_2)/2$ and $t = (\tau_1 + \tau_2)/2$ the propagator is written in terms of the dS_2 wave functions as

$$\begin{aligned} G(\eta, t; \eta', t') \;=\; 2\pi J^{-1} \int_{-\infty}^{\infty} d\omega \Bigg\{ & \sum_{n=0}^{\infty} \frac{4 \sin \pi\nu_n}{\widetilde{g}(\nu_n) - 1} \; \overline{\psi}^{\,*}_{\omega,\nu_n}(\eta, t) \, \overline{\psi}_{\omega,\nu_n}(\eta', t') \\ & + \int_0^{\infty} dr \; \frac{\overline{\psi}^{\,*}_{\omega,\nu}(\eta, t) \, \overline{\psi}_{\omega,\nu}(\eta', t')}{\widetilde{g}(\nu) - 1} \Bigg|_{\nu = ir} \Bigg\} \,, \end{aligned} \tag{5.57}$$

where $\nu_n = 2n + \frac{3}{2}$. Next, we use the inverse Radon transform (5.50) to bring the dS wave functions into the EAdS wave functions.

$$\begin{aligned} G(\tau, z; \tau', z') \;=\; 2\pi J^{-1} \int_{-\infty}^{\infty} d\omega \Bigg\{ & \sum_{n=0}^{\infty} \frac{4 \sin \pi\nu_n}{\widetilde{g}(\nu_n) - 1} \; |L^{-1}(\nu_n)|^2 \; \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau, z) \, \overline{\phi}_{\omega,\nu_n}(\tau', z') \\ & + \int_0^{\infty} dr \; |L^{-1}(\nu)|^2 \; \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau, z) \, \overline{\phi}_{\omega,\nu}(\tau', z')}{\widetilde{g}(\nu) - 1} \Bigg|_{\nu = ir} \Bigg\} \,. \end{aligned} \tag{5.58}$$

Here we have denoted $\overline{\phi}_{\omega,\nu_n}(\tau, z)$ as

$$\overline{\phi}_{\omega,\nu_n}(\tau, z) = \alpha'_{\nu_n} z^{1/2} e^{-ik\tau} I_{\nu_n}(|k| z) \tag{5.59}$$

We will directly evaluate the continuous mode summation for the full Green's function with the leg factor contribution in the integrand. For clarity let us first formally feature the leg factors as Bessel differential operators,as

$$G(\tau,z;\tau',z') \;=\; 2\pi J^{-1}\left|L^{-1}(\hat{p}_{\mathrm{EAdS}_2})\right|^2 \int_{-\infty}^{\infty} d\omega \Bigg\{ \sum_{n=0}^{\infty} \frac{4\sin\pi\nu_n}{\widetilde{g}(\nu_n)-1}\, \overline{\phi}^{\,*}_{\omega,\nu_n}(\tau,z)\,\overline{\phi}_{\omega,\nu_n}(\tau',z') + \int_0^{\infty} dr\, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau,z)\,\overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir} \Bigg\}. \tag{5.60}$$

with

$$\hat{p}_{\mathrm{AdS}_2} \;\equiv\; \sqrt{\Box_{\mathrm{E}AdS_2}+\frac{1}{4}}\,, \tag{5.61}$$

where the Laplacian of $EAdS_2$ is defined in Eq.(5.53) and the factors are now acting on standard propagators $EAdS_2$. The above expression for the leg factor differential operators is slightly ambiguous. What we mean is that one of the leg factor differential operator is acting on (τ,z) and the other leg factor operator is acting on (τ',z').

We now proceed with our off-shell expression of the propagator (5.58) and evaluation of the continuous mode summation for the Green's function with leg factors in the integrand:

$$\mathcal{I}_{\mathrm{cont}} \;\equiv\; \int_0^{\infty} dr\; |L^{-1}(\nu)|^2\, \frac{\overline{\phi}^{\,*}_{\omega,\nu}(\tau,z)\,\overline{\phi}_{\omega,\nu}(\tau',z')}{\widetilde{g}(\nu)-1}\bigg|_{\nu=ir}. \tag{5.62}$$

We evaluate this integral as a contour integral as before. We note that since the modified Bessel function K_ν is regular on the entire ν-complex plane, we have two sets of poles: (i). $\nu = p_m$, with $(m = 0,1,2,\cdots)$. (ii). $\nu=\nu_n = 2n+\frac{3}{2}$, with $(n=0,1,2,\cdots)$ where $\Gamma(\frac{3}{4}-\frac{\nu}{2})=\infty$. After evaluating the residues at these poles, we find the integral as

$$\mathcal{I}_{\mathrm{cont}} \;=\; \frac{|zz'|^{\frac{1}{2}}}{4\pi^2}\, e^{-i\omega(\tau-\tau')} \Bigg\{ \sum_{m=0}^{\infty} \frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})}\,\frac{p_m}{\widetilde{g}'(p_m)}\, K_{p_m}(|\omega|z^>)I_{p_m}(|\omega|z^<) + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right) K_{\nu_n}(|\omega|z^>)I_{\nu_n}(|\omega|z^<) \Bigg\}. \tag{5.63}$$

The second line in the RHS looks similar to the discrete mode contribution to the propagator (5.58). However, these two contributions do not cancel each other. Hence there are two types of the contributions to the final result as

$$
\begin{aligned}
&G(\tau,z;\tau',z')\\
&= \frac{|zz'|^{\frac{1}{2}}}{2\pi J}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(\tau-\tau')}\left\{\sum_{m=0}^{\infty}\frac{\Gamma(\frac{3}{4}+\frac{p_m}{2})\Gamma(\frac{3}{4}-\frac{p_m}{2})}{\Gamma(\frac{1}{4}+\frac{p_m}{2})\Gamma(\frac{1}{4}-\frac{p_m}{2})}\,\frac{p_m}{\widetilde{g}'(p_m)}\,K_{p_m}(|\omega|z^>)I_{p_m}(|\omega|z^<)\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}\frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})}\left(\frac{\nu_n}{\widetilde{g}(\nu_n)-1}\right)\,I_{\nu_n}(|\omega|z^<)\Big[2I_{\nu_n}(|\omega|z^>)-I_{-\nu_n}(|\omega|z^>)\Big]\right\}.
\end{aligned}
\tag{5.64}
$$

Of course, here we still have the zero mode ($p_0=\frac{3}{2}$) problem coming from $\Gamma(\frac{3}{4}-\frac{p_0}{2})=\infty$. In this expression, the Bessel function part of the first contribution in the RHS is the standard form for EAdS propagator, while the extra factor coming from the leg-factors can be possibly understood as a contribution from the naively pure gauge degrees of freedom as in the $c=1$ model (c.f. [171]), in which case the second contribution in RHS represents the contribution from these modes as in [163].

In [25, 63], we presented at three dimensional picture of the SYK theory, based on the fact that the nontrivial spectrum predicted by the model, which are solutions of $\widetilde{g}(p_m)=1$ with $(m=0,1,2,\cdots)$ can be reproduced through Kaluza-Klein mechanism in one higher dimension. This picture is more natural in the AdS_2 interpretation of the bilocal space Now, we will point out a similarity between the three dimensional picture of the SYK model [25, 63] and the $c=1$ Liouville theory (2D string theory) [162, 163, 171]. In the three dimensional description we have a scalar field Φ

$$
S_{\mathrm{3D}} \;=\; \frac{1}{2}\int dx^3\sqrt{-g}\Big[-g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-m_0^2\Phi^2-V(y)\Phi^2\Big]\,, \tag{5.65}
$$

with a background metric

$$
ds^2 \;=\; \frac{-dt^2+d\hat{z}^2}{\hat{z}^2}\;+\;\left(1+\frac{a}{\hat{z}}\right)^2 dy^2\,, \tag{5.66}
$$

where $a\sim J^{-1}$, but here we only consider the leading in $1/J$ contribution and suppress the subleading contributions coming from the yy-component of the metric. The detail of the potential $V(y)$ depends on q and for that readers should refer to [25, 63]. The propagator for the scalar field in this background in the leading order of $1/J$ is given by

$$
G^{(0)}(\hat{z},t,y;\hat{z}',t',y') \;=\; |\hat{z}\hat{z}'|^{\frac{1}{2}}\sum_k f_k(y)f_k(y')\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\int\frac{d\nu}{N_\nu}\,\frac{Z^*_\nu(|\omega\hat{z}|)\,Z_\nu(|\omega\hat{z}'|)}{\nu^2-k^2}\,, \tag{5.67}
$$

where $f_k(y)$ is the wave function along the third direction y with momentum k. This is simply a rewriting the propagator (5.57) by treating the nonlocal kernel (eigenvalue) by an extra dimension. The identical procedure leads to the leg-factors. After the (inverse) Radon transform and the contour integral for the continuous mode sum, the propagator is reduced to

$$
\begin{aligned}
&G^{(0)}_{\omega;-\omega}(z,y;z',y') \\
&= \frac{|zz'|^{\frac{1}{2}}}{4\pi} \sum_k f_k(y) f_k(y') \Bigg\{ \frac{\Gamma(\frac{3}{4}+\frac{k}{2})\Gamma(\frac{3}{4}-\frac{k}{2})}{\Gamma(\frac{1}{4}+\frac{k}{2})\Gamma(\frac{1}{4}-\frac{k}{2})} K_k(|\omega| z^>) I_k(|\omega| z^<) \\
&\qquad + 2\sum_{n=0}^{\infty} \frac{\Gamma^2(\frac{3}{4}+\frac{\nu_n}{2})}{\Gamma^2(\frac{1}{4}+\frac{\nu_n}{2})} \left(\frac{\nu_n}{\nu_n^2-k^2}\right) I_{\nu_n}(|\omega| z^<) \Big[2I_{\nu_n}(|\omega| z^>) - I_{-\nu_n}(|\omega| z^>)\Big] \Bigg\} .
\end{aligned}
\tag{5.68}
$$

On the other hand, for the $c=1$ matrix model / 2D string duality, the Wilson loop operator is related to the matrix eigenvalue density field ϕ by

$$
W(t,\ell) \equiv \mathrm{Tr}\Big(e^{-\ell M(t)}\Big) = \int_0^\infty dx\, e^{-\ell x}\, \phi(t,x) . \tag{5.69}
$$

The corresponding propagator was found by Moore and Seiberg [163] as

$$
\Big\langle w(t,\varphi) w(t',\varphi')\Big\rangle = \int_{-\infty}^{\infty} dE \int_0^\infty dp\, \frac{p}{\sinh \pi p} \frac{\phi^*_{E,p}(t,\varphi)\phi_{E,p}(t',\varphi')}{E^2-p^2} , \tag{5.70}
$$

with $\ell = e^{-\varphi}$ and the normalized wave function

$$
\phi_{E,p}(t,\varphi) = \sqrt{p\sinh \pi p}\, e^{-iEt} K_{ip}(\sqrt{\mu} e^{-\varphi}) . \tag{5.71}
$$

After evaluating the p-integral as a contour integral, we obtain the propagator as

$$
\begin{aligned}
\Big\langle w(t,\varphi) w(t',\varphi')\Big\rangle = -\pi \int_{-\infty}^{\infty} dE\, e^{-iE(t-t')} \Bigg\{ &\frac{\pi E}{2\sinh \pi E} K_{iE}(\sqrt{\mu} e^{-\varphi^<})\, I_{iE}(\sqrt{\mu} e^{-\varphi^>}) \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{E^2+n^2} K_n(\sqrt{\mu} e^{-\varphi^<})\, I_n(\sqrt{\mu} e^{-\varphi^>}) \Bigg\} .
\end{aligned}
\tag{5.72}
$$

The point we want to make here is that this three dimensional picture is completely parallel to the $c=1$ Liouville theory (2D string theory) [162, 163, 171]. Namely, if we make a change of coordinate by $z=e^{-\varphi}$, then the φ-direction becomes the Liouville direction, while the y-direction (at least in the leading order of $1/J$) can be understood as the $c=1$ matter direction. In this comparison, the τ-direction serves as an extra direction. Finally, the ν appearing in the SYK model is realized as a momentum k along the y-direction in the three dimensional picture (5.66). Therefore, we have the following correspondence between the $c=1$ Liouville theory and the three dimensional picture of the SYK model.

$c=1$	three dimensional SYK
$ie^{-\varphi}$	z
$-it$	y
ip	ν
iE	k
$\sqrt{\mu}$	$\lvert\omega\rvert$

5.4 Conclusion

We have in this chapter addressed the question of what represents the bulk dual spacetime in the SYK model. At the outset the question seems simple since the small fluctuations of the (Euclidean) SYK model are completely given by a set of Lorentzian wave functions associated with the $SL(2,\mathcal{R})$ isometry group. With a simple identification of spacetime these are seen to associated with eigenfunctions in de-Sitter (or Anti de-Sitter) spacetime (as was discussed in [11, 12, 28, 29]). And as we have noted these wave functions are in correspondence with a particular α-vacuum wavefunctions of dS_2 spacetime. Likewise the propagator and higher point n-functions continue to feature this Lorentzian spacetime structure.

Even though the Lorentzian bulk dual interpretation seems to be straightforwardly associated with the SYK bilocal data, we have stressed that there is a problem with this interpretation. In the most naive sense one would essentially expect that an Euclidean CFT should lead to an Euclidean bulk dual. In the case of the SYK model there is a caveat since a role is played by random tensor couplings, whose bulk interpretation is even more unclear. However, concentrating on the effective bilocal large N version of the theory we have in this chapter provided a resolution, which follows from a further nonlocal redefinition of spacetime. This comes in terms of Leg transformations of Green's functions which place the theory in Euclidean AdS dual setting. Such transformations are actually characteristic of collective field representations of Large N theories. The leg transformations that we explicitly implement (apart from providing the EAdS_2 spacetime setting) also bring out the couplings of additional "discrete" states. Since this is implemented on all n point functions it represents a highly nonlinear effect (as was first understood by Natsuume and Polchinski). We expect that these additional features will play a central role in full identification of the bulk dual for the present theory.

Appendix A
Perturbative Expansions

In this appendix, we summarize several relations for the expansions (A.6) - (A.10) and introduce some short-hand notations. Formally we suppose to have the exact action S and the exact classical solution $\Psi_{\rm cl}$. Expanding the action around the exact solution, we have

$$S\big[\Psi_{\rm cl}+N^{-1/2}\eta\big] \;=\; S[\Psi_{\rm cl}] \,+\, \frac{1}{2}\int \eta\cdot\mathcal{K}_{(\rm ex)}\cdot\eta \,+\,\cdots\,, \tag{A.1}$$

where

$$\mathcal{K}_{(\rm ex)}(\tau_1,\tau_2;\tau_3,\tau_4) \;=\; \frac{\delta^2 S[\Psi_{\rm cl}]}{\delta\Psi_{\rm cl}(\tau_1,\tau_2)\delta\Psi_{\rm cl}(\tau_3,\tau_4)}\,. \tag{A.2}$$

Then, introducing $\widetilde{\mathcal{K}}_{(\rm ex)}$ from $\mathcal{K}_{(\rm ex)}$ using the analogous definition as in (2.39) where we replace all $\Psi_{(0)}$ by $\Psi_{\rm cl}$, formally we can consider the exact Green's function $\widetilde{G}_{(\rm ex)}$ which is determined by the Green's equation

$$\int d\tau_3 d\tau_4\,\widetilde{\mathcal{K}}_{(\rm ex)}(\tau_1,\tau_2;\tau_3,\tau_4)\widetilde{G}_{(\rm ex)}(\tau_3,\tau_4;\tau_5,\tau_6) \;=\; \delta(\tau_{15})\delta(\tau_{26})\,. \tag{A.3}$$

In order to invert the kernel $\widetilde{\mathcal{K}}_{(\rm ex)}$ in this Green's equation, we consider the eigenvalue problem of the kernel $\widetilde{\mathcal{K}}_{(\rm ex)}$:

$$\int d\tau_3 d\tau_4\,\widetilde{\mathcal{K}}_{(\rm ex)}(\tau_1,\tau_2;\tau_3,\tau_4)\,\tilde{\chi}^{(\rm ex)}_{n,\omega}(\tau_3,\tau_4) \;=\; \lambda^{(\rm ex)}_{n,\omega}\,\tilde{\chi}^{(\rm ex)}_{n,\omega}(\tau_1,\tau_2)\,, \tag{A.4}$$

with the eigenfunctions $\tilde{\chi}^{(\rm ex)}_{n,\omega}$ and the eigenvalues $\lambda^{(\rm ex)}_{n,\omega}$, where n,ω are some quantum numbers. We normalize the eigenfunctions by requiring

$$\int d\tau_1 d\tau_2\,\tilde{\chi}^{(\rm ex)}_{n,\omega}(\tau_1,\tau_2)\tilde{\chi}^{(\rm ex)}_{n',\omega'}(\tau_1,\tau_2) \;=\; \delta_{n,n'}\delta(\omega-\omega')\,. \tag{A.5}$$

Now we consider a perturbative expansion in $1/J$ of the above eigenvalue problem. We expand each of the quantities of interest as

$$\Psi_{\rm cl} \;=\; \Psi^{(0)} \,+\, \frac{1}{J}\,\Psi^{(1)} \,+\, \frac{1}{J^2}\,\Psi^{(2)} \,+\,\cdots\,, \tag{A.6}$$

$$\mathcal{K}_{(\rm ex)} \;=\; \mathcal{K}^{(0)} \,+\, \frac{1}{J}\,\mathcal{K}^{(1)} \,+\, \frac{1}{J^2}\,\mathcal{K}^{(2)} \,+\,\cdots\,, \tag{A.7}$$

$$G_{(\rm ex)} \;=\; J\,G^{(-1)} \,+\, G^{(0)} \,+\,\cdots\,, \tag{A.8}$$

$$\chi^{(\rm ex)} \;=\; \chi^{(0)} \,+\, \frac{1}{J}\,\chi^{(1)} \,+\,\cdots\,, \tag{A.9}$$

$$\lambda^{(\rm ex)} \;=\; \lambda^{(0)} \,+\, \frac{1}{J}\,\lambda^{(1)} \,+\, \frac{1}{J^2}\,\lambda^{(2)} \,+\,\cdots\,, \tag{A.10}$$

and similarly for the redefined kernel (2.39), and the bilocal propagator and eigenfunctions corresponding to it. In this chapter, the superscript in a round bracket denotes the order of $1/J$ expansion, while the subscript denotes quantum numbers.

The exact Green's function can be then written as

$$\widetilde{G}_{(\mathrm{ex})} = \sum_n \frac{\tilde{\chi}_n^{(\mathrm{ex})} \tilde{\chi}_n^{(\mathrm{ex})}}{\lambda_n^{(\mathrm{ex})}} \equiv \Psi_{\mathrm{cl}}^{\frac{q}{2}-1} G_{(\mathrm{ex})} \Psi_{\mathrm{cl}}^{\frac{q}{2}-1} . \tag{A.11}$$

In the following, we will be interested in the contribution coming from the zero mode of the lowest order kernel, $n = 0$. Since $\lambda_0^{(0)} = 0$, we have

$$G^{(-1)} = \frac{\chi_0^{(0)} \chi_0^{(0)}}{\lambda_0^{(1)}} \tag{A.12}$$

$$G^{(0)} = \frac{\chi_0^{(0)} \chi_0^{(1)}}{\lambda_0^{(1)}} + \frac{\chi_0^{(1)} \chi_0^{(0)}}{\lambda_0^{(1)}} - \frac{\lambda_0^{(2)} \chi_0^{(0)} \chi_0^{(0)}}{(\lambda_0^{(1)})^2} + \mathcal{D}_c , \qquad \mathcal{D}_c = \sum_{n \neq 0} \frac{\chi_n^{(0)} \chi_n^{(0)}}{\lambda_n^{(0)}} . \tag{A.13}$$

Here we suppressed all τ (and ω) dependence since they don't play any crucial role here.

The expression of the perturbative kernels are also found by expanding $\mathcal{K}_{\mathrm{ex}}$ (A.2)

$$\mathcal{K}^{(0)}(\tau_1, \tau_2; \tau_3, \tau_4) = S_c^{(2)}(\tau_{1,2}; \tau_{3,4}) , \tag{A.14}$$

$$\mathcal{K}^{(1)}(\tau_1, \tau_2; \tau_3, \tau_4) = \int d\tau_5 d\tau_6 \, S_c^{(3)}(\tau_{1,2}; \tau_{3,4}; \tau_{5,6}) \, \Psi^{(1)}(\tau_{56}) , \tag{A.15}$$

$$\begin{aligned} \mathcal{K}^{(2)}(\tau_1, \tau_2; \tau_3, \tau_4) = & \frac{1}{2} \int d\tau_5 d\tau_6 d\tau_7 d\tau_8 \, S_c^{(4)}(\tau_{1,2}; \tau_{3,4}; \tau_{5,6}; \tau_{7,8}) \, \Psi^{(1)}(\tau_{56}) \Psi^{(1)}(\tau_{78}) \\ & + \int d\tau_5 d\tau_6 \, S_c^{(3)}(\tau_{1,2}; \tau_{3,4}; \tau_{5,6}) \, \Psi^{(2)}(\tau_{56}) . \end{aligned} \tag{A.16}$$

where we used a short-hand notation

$$S_c^{(n)}(\tau_{1,2}; \cdots ; \tau_{2n-1,2n}) \equiv \frac{\delta^n S_c[\Psi^{(0)}]}{\delta \Psi^{(0)}(\tau_1, \tau_2) \cdots \delta \Psi^{(0)}(\tau_{2n-1}, \tau_{2n})} . \tag{A.17}$$

We will also use the following simplified notation for the contractions:

$$\begin{aligned} S_c^{(n)} \big[A_1; A_2; \cdots ; A_n \big] \equiv & \int d\tau_1 d\tau_2 \cdots d\tau_{2n-1} d\tau_{2n} \, S_c^{(n)}(\tau_{1,2}; \cdots ; \tau_{2n-1,2n}) \\ & \times A_1(\tau_1, \tau_2) \cdots A_n(\tau_{2n-1}, \tau_{2n}) . \end{aligned} \tag{A.18}$$

Finally expanding the eigenvalue equation (A.4), we can fix the perturbative eigenfunctions and eigenvalues

$$\lambda_n^{(0)} = \int \tilde{\chi}_n^{(0)} \cdot \widetilde{\mathcal{K}}^{(0)} \cdot \tilde{\chi}_n^{(0)} , \qquad \lambda_n^{(1)} = \int \tilde{\chi}_n^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_n^{(0)} , \tag{A.19}$$

$$\tilde{\chi}_n^{(1)} = \sum_{k \neq n} \frac{\tilde{\chi}_k^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}} \int \tilde{\chi}_k^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_n^{(0)} , \tag{A.20}$$

$$\lambda_n^{(2)} = \sum_{k \neq n} \frac{1}{\lambda_n^{(0)} - \lambda_k^{(0)}} \left| \int \tilde{\chi}_k^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_n^{(0)} \right|^2 + \int \tilde{\chi}_n^{(0)} \cdot \widetilde{\mathcal{K}}^{(2)} \cdot \tilde{\chi}_n^{(0)} . \tag{A.21}$$

where we used the normalization condition $\int \tilde{\chi}^{(0)} \cdot \tilde{\chi}^{(1)} = 0$. We note that the zero-mode ($n = 0$) first order eigenfunctions without tilde are obtained as

$$
\begin{aligned}
\chi_0^{(1)} &= -\sum_{k\neq 0} \frac{\chi_k^{(0)}}{\lambda_k^{(0)}} \int \tilde{\chi}_k^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_0^{(0)} + \left(1 - \frac{q}{2}\right) \Psi_{(0)}^{-1} \Psi_{(1)} \chi_0^{(0)} \\
&= -\sum_{k\neq 0} \frac{\chi_k^{(0)}}{\lambda_k^{(0)}} \int \chi_k^{(0)} \cdot \mathcal{K}^{(1)} \cdot \chi_0^{(0)} \,,
\end{aligned}
\tag{A.22}
$$

where the second term in the first line comes by expanding $\chi^{(\mathrm{ex})} = \Psi_{\mathrm{cl}}^{1-q/2}\, \tilde{\chi}^{(\mathrm{ex})}$ and for the second line we used

$$
\begin{aligned}
&\widetilde{\mathcal{K}}^{(1)}(\tau_1, \tau_2; \tau_3, \tau_4) \\
&= \int d\tau_5 d\tau_6\, S_c^{(3)}(\tau_{1,2}; \tau_{3,4}; \tau_{5,6})\, \Psi_{(0)}^{1-\frac{q}{2}}(\tau_{12}) \Psi_{(0)}^{1-\frac{q}{2}}(\tau_{34}) \Psi_{(1)}(\tau_{56}) \\
&- \left(\frac{q}{2} - 1\right) S_c^{(2)}(\tau_{1,2}; \tau_{3,4}) \left[\Psi_{(0)}^{-\frac{q}{2}}(\tau_{12}) \Psi_{(1)}(\tau_{12}) \Psi_{(0)}^{1-\frac{q}{2}}(\tau_{34}) + (\tau_{12} \leftrightarrow \tau_{34})\right] .
\end{aligned}
\tag{A.23}
$$

Appendix B
Details of Derivation of Liouville from Bilocal

In this Appendix, we give a detail derivation of the action (2.96) from (2.93). For the logarithm term, we use the following fact that

$$\frac{\mathrm{sgn}(\tau_{12})}{2} \;=\; \theta(\tau_{12}) - \frac{1}{2}\,, \qquad \left[\frac{\mathrm{sgn}}{2}\right]_\star^{-1}(\tau_{12}) \;=\; \partial_1\delta(\tau_{12})\,, \tag{B.1}$$

where the inverse $[\bullet]_\star^{-1}$ is defined in the sense of the star product (i.e. matrix product) $\int d\tau' A(\tau_1,\tau')[A]_\star^{-1}(\tau',\tau_2) = \delta(\tau_{12})$. By using this, one can rewrite the term as

$$\begin{aligned}\mathrm{Tr}\log\Psi \;&=\; \mathrm{Tr}\left[\log\left(\frac{\mathrm{sgn}}{2}\right) \;+\; \log\left(1+\frac{\partial(\mathrm{sgn}\times\Phi)}{2q}\right)\right]\\ &=\; \mathrm{Tr}\log\left(\frac{\mathrm{sgn}}{2}\right) \;-\; \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{-1}{2q}\right)^n\mathrm{Tr}\Big[\partial(\mathrm{sgn}\times\Phi)\Big]_\star^n\,,\end{aligned} \tag{B.2}$$

where the last term is defined for example for $n=2$ as

$$\mathrm{Tr}\Big[\partial(\mathrm{sgn}\times\Phi)\Big]_\star^2 \;\equiv\; \int d\tau_1 d\tau_2\,\partial_1\Big(\mathrm{sgn}(\tau_{12})\Phi(\tau_1,\tau_2)\Big)\partial_2\Big(\mathrm{sgn}(\tau_{21})\Phi(\tau_2,\tau_1)\Big)\,. \tag{B.3}$$

For the interaction term we have

$$\Big[\Psi(\tau_1,\tau_2)\Big]^q \;=\; \frac{1}{2^q}\left[1\;+\;\frac{\Phi(\tau_1,\tau_2)}{q}\right]^q\,. \tag{B.4}$$

We consider only $q=$ even integer case and the sign function disappeared here. We want to rewrite the RHS as the following form

$$\left[1\;+\;\frac{\Phi(\tau_1,\tau_2)}{q}\right]^q \;=\; e^{\Phi(\tau_1,\tau_2)}\left[1\;+\;\frac{c_1(\tau_1,\tau_2)}{q}\;+\;\frac{c_2(\tau_1,\tau_2)}{q^2}\;+\;\mathcal{O}(q^{-3})\right]\,, \tag{B.5}$$

where function c_1 and c_2 are to be determined. Taking logarithm of both-hand sides and expanding the logarithm, one finds

$$\begin{aligned}\log(\mathrm{L}HS) \;&=\; \Phi(\tau_1,\tau_2) \;-\; \frac{\Phi^2(\tau_1,\tau_2)}{2q} \;+\; \frac{\Phi^3(\tau_1,\tau_2)}{3q^2} \;+\; \mathcal{O}(q^{-3})\,,\\ \log(\mathrm{R}HS) \;&=\; \Phi(\tau_1,\tau_2) \;+\; \frac{c_1(\tau_1,\tau_2)}{q} \;+\; \frac{c_2(\tau_1,\tau_2)}{q^2} \;-\; \frac{c_1^2(\tau_1,\tau_2)}{2q^2} \;+\; \mathcal{O}(q^{-3})\,.\end{aligned} \tag{B.6}$$

Comparing the both sides, the unfixed functions are determined as

$$\begin{aligned}c_1(\tau_1,\tau_2) \;&=\; -\frac{1}{2}\,\Phi^2(\tau_1,\tau_2)\,,\\ c_2(\tau_1,\tau_2) \;&=\; \frac{1}{3}\,\Phi^3(\tau_1,\tau_2) \;+\; \frac{1}{8}\,\Phi^4(\tau_1,\tau_2)\,.\end{aligned} \tag{B.7}$$

Combining everything, the collective action is now written as

$$\begin{aligned} S_{\rm col}[\Phi] \;=\; & -\frac{N}{2}\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{-1}{2q}\right)^{n}\mathrm{Tr}\left[\partial\big(\mathrm{sgn}\times\Phi\big)\right]_{\star}^{n} \\ & -\frac{\mathcal{J}^2 N}{4q^2}\int d\tau_1 d\tau_2\, e^{\Phi(\tau_1,\tau_2)}\left[1+\frac{c_1(\tau_1,\tau_2)}{q}+\frac{c_2(\tau_1,\tau_2)}{q^2}+\cdots\right]. \end{aligned} \tag{B.8}$$

The $\mathcal{O}(q^0)$ and $\mathcal{O}(q^{-1})$ order terms are identically vanish due to the free field equation of motion. Therefore, the first nontrivial order is $\mathcal{O}(q^{-2})$. Lower order terms can be explicitly written down as

$$\begin{aligned} S_{\rm col}[\Phi] \;=\; & -\frac{N}{16q^2}\int d\tau_1 d\tau_2\,\partial_1\big(\mathrm{sgn}(\tau_{12})\Phi(\tau_1,\tau_2)\big)\partial_2\big(\mathrm{sgn}(\tau_{21})\Phi(\tau_2,\tau_1)\big) \\ & -\frac{\mathcal{J}^2 N}{4q^2}\int d\tau_1 d\tau_2\, e^{\Phi(\tau_1,\tau_2)} \\ & +\frac{N}{48q^3}\int d\tau_1 d\tau_2 d\tau_3\,\partial_1\big(\mathrm{sgn}(\tau_{12})\Phi(\tau_{12})\big)\partial_2\big(\mathrm{sgn}(\tau_{23})\Phi(\tau_{23})\big)\partial_3\big(\mathrm{sgn}(\tau_{31})\Phi(\tau_{31})\big) \\ & +\frac{\mathcal{J}^2 N}{8q^3}\int d\tau_1 d\tau_2\,\Phi^2(\tau_1,\tau_2)\, e^{\Phi(\tau_1,\tau_2)} \;+\;\mathcal{O}(q^{-4})\,. \end{aligned} \tag{B.9}$$

Appendix C
Details of Exact Eigenfunctions

In this Appendix, we elaborate on various properties of the exact eigenfunctions (2.113).

C.0.1 Zero temperature limit

Consider the exact eigenfunctions at finite temperature at large q [12] which can be written as

$$\chi^{(\mathrm{ex})}_{n,\nu}(x) \sim \left(\sin\frac{\tilde{x}}{2}\right)^{1/2}\left[P^{-\nu}_{\tilde{n}-\frac{1}{2}}\left(\cos\frac{\tilde{x}}{2}\right) + \kappa^{\mathrm{even/odd}}_{\tilde{n},\nu} P^{\nu}_{\tilde{n}-\frac{1}{2}}\left(\cos\frac{\tilde{x}}{2}\right)\right] \tag{C.1}$$

where

$$\kappa^{\mathrm{even}}_{\tilde{n},\nu} = -2^{-2\nu}\frac{\Gamma\left(\frac{1}{4}-\frac{\tilde{n}}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\tilde{n}}{2}-\frac{\nu}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\tilde{n}}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\tilde{n}}{2}+\frac{\nu}{2}\right)} \qquad \kappa^{\mathrm{odd}}_{\tilde{n},\nu} = -2^{-2\nu}\frac{\Gamma\left(\frac{3}{4}-\frac{\tilde{n}}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\tilde{n}}{2}-\frac{\nu}{2}\right)}{\Gamma\left(\frac{3}{4}-\frac{\tilde{n}}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\tilde{n}}{2}+\frac{\nu}{2}\right)} \tag{C.2}$$

and

$$\begin{aligned} \tilde{x} &= vx + (1-v)\pi \\ \tilde{n} &= \frac{n}{v} \\ v &= 1 - \frac{2}{\beta\mathcal{J}} + \mathcal{O}\left(\frac{1}{\beta^2\mathcal{J}^2}\right) \\ \omega &= \frac{2\pi n}{\beta} \end{aligned} \tag{C.3}$$

Let us define

$$x = \frac{4\pi z}{\beta} \tag{C.4}$$

This allows us to write

$$\tilde{x} = \frac{\omega}{n\mathcal{J}}\,(2\mathcal{J}z+1) + \mathcal{O}\left[\frac{1}{(n\mathcal{J})^2}\right] \tag{C.5}$$

Taking the zero temperature limit is equivalent to taking $n \to \infty$. In this limit,

$$\sin\frac{\tilde{x}}{2} \simeq \frac{\tilde{x}}{2} \tag{C.6}$$

and

$$\lim_{n\to\infty} P^{\pm\nu}_{\tilde{n}-\frac{1}{2}}\left(\cos\frac{\tilde{x}}{2}\right) \sim n^{\pm\nu} J_{\mp\nu}\left[\frac{\omega\,(2\mathcal{J}z+1)}{2\mathcal{J}}\right] \tag{C.7}$$

On doing so, we can write the exact eigenfunction in the zero temperature but finite coupling limit as

$$\chi^{(\mathrm{ex})}_{\omega,\nu}(z) \sim \left[\omega z + \frac{\omega}{2\mathcal{J}}\right]^{1/2} \left[J_\nu\left(\omega z + \frac{\omega}{2\mathcal{J}}\right) + \xi_\omega(\nu)\ J_{-\nu}\left(\omega z + \frac{\omega}{2\mathcal{J}}\right)\right] \tag{C.8}$$

where

$$\begin{aligned}\xi_\omega(\nu) &\equiv \lim_{n\to\infty} n^{2\nu}\kappa_{\tilde{n},\nu} \\ &= -\lim_{n\to\infty}\left(\frac{n}{2}\right)^{2\nu} \frac{\Gamma\left(\frac{1}{4}-\frac{n}{2}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}-\frac{n}{2}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)} \\ &= -\frac{\Gamma\left(\frac{1}{4}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}\end{aligned} \tag{C.9}$$

Equation (C.9) can be proved as follows.

$$\xi_\omega(\nu) = -\lim_{n\to\infty}\left(\frac{n}{2}\right)^{2\nu} \frac{\Gamma\left(\frac{1}{4}-\frac{n}{2}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}-\frac{n}{2}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)} \tag{C.10}$$

Now, we can use the identity

$$\lim_{n\to\infty}\left(\frac{n}{2}\right)^{\nu} \frac{\Gamma\left(\frac{1}{4}+\frac{n}{2}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}+\frac{n}{2}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)} = 1 \tag{C.11}$$

Using (C.11) in (C.10), we have

$$\begin{aligned}\xi_\omega(\nu) &= -\lim_{n\to\infty}\left(\frac{n}{2}\right)^{\nu} \frac{\Gamma\left(\frac{1}{4}-\frac{n}{2}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}-\frac{n}{2}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)} \\ &= -\frac{\Gamma\left(\frac{1}{4}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)} \lim_{n\to\infty}\left(\frac{n}{2}\right)^{\nu} \prod_{p=1}^{n/2} \frac{\Gamma\left(\frac{1}{4}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}-p\right)}{\Gamma\left(\frac{1}{4}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}-p\right)} \\ &= -\frac{\Gamma\left(\frac{1}{4}-\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}+\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}{\Gamma\left(\frac{1}{4}+\frac{\nu}{2}-\frac{\omega}{2\pi\mathcal{J}}\right)\Gamma\left(\frac{3}{4}-\frac{\nu}{2}+\frac{\omega}{2\pi\mathcal{J}}\right)}\end{aligned} \tag{C.12}$$

as required. We can expand $\xi_\omega(\nu)$ in a power series in $\frac{\omega}{2\mathcal{J}}$ to get

$$\xi_\omega(\nu) = \frac{\tan\frac{\pi\nu}{2}+1}{\tan\frac{\pi\nu}{2}-1} + \frac{\omega}{2\mathcal{J}}\frac{2\sin\pi\nu}{\sin\pi\nu-1} + \mathcal{O}\left(\frac{\omega^2}{\mathcal{J}^2}\right) \tag{C.13}$$

The leading order piece agrees with ξ_ν found in [28]. From (C.9), it is easy to see

$$\xi_\omega(\nu) = \frac{1}{\xi_\omega(-\nu)} \tag{C.14}$$

An alternative way to see that the exact eigenfunction should be a linear combination of Bessel functions is to notice that they are solutions to the following differential equation

$$\left[\omega^2 + \partial_z^2 - \frac{\nu^2 - 1/4}{\left(z + \frac{1}{2\mathcal{J}}\right)^2}\right] \chi^{(\mathrm{ex})}_{\omega,\nu}(z) = 0 \tag{C.15}$$

If we make a shift $z \to z - \frac{1}{2\mathcal{J}}$, we can see that (C.15) turns into a Bessel equation.

C.0.2 Quantization Condition

Let us impose the following boundary condition on the exact eigenfunctions

$$\chi^{(\mathrm{ex})}_{\omega,\nu}(z)\Big|_{z=0} = 0 \tag{C.16}$$

Using equation (C.8), imposing the above boundary condition requires us to solve the following equation in order to get the quantized values of ν

$$J_\nu\left(\frac{\omega}{2\mathcal{J}}\right) + \xi_\omega(\nu)\ J_{-\nu}\left(\frac{\omega}{2\mathcal{J}}\right) = 0 \tag{C.17}$$

We can solve this equation numerically and do a polynomial fit of order two for the curve given by Figure C.1 to get

$$\nu = \frac{3}{2} + a_1\frac{\omega}{\mathcal{J}} + a_2\left(\frac{\omega}{\mathcal{J}}\right)^2 \tag{C.18}$$

where $a_1 = 0.318048$ and $a_2 = 0.0046914$. This is in good agreement with the results obtained in [12]. This can be seen as follows

$$\begin{aligned} \nu_{\mathrm{MS}} &= \frac{3}{2} + n\left(\frac{1-v}{v}\right) \\ &= \frac{3}{2} + \frac{\omega}{\pi\mathcal{J}} + \mathcal{O}\left(\frac{\omega}{\mathcal{J}}\right)^3 \end{aligned} \tag{C.19}$$

We have used equation (2.31) of [12], to get from the first to second line of (C.19).

Another important observation is the following. The quantized values for ν one gets by solving the exact quantization equation (C.17) is in very good agreement with those obtained by solving the following equation

$$\xi_\omega(\nu) = 0 \implies \nu_n = 2n + \frac{3}{2} + \frac{\omega}{\pi\mathcal{J}} \tag{C.20}$$

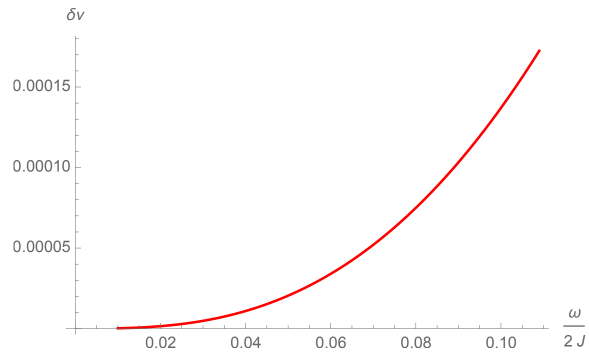


Figure C.1: Plot of the absolute difference between the allowed values of ν, where we have defined $\delta\nu = \nu_{\text{MS}} - \nu_{\text{exact}}$. We see that the difference is almost zero for small values of $\frac{\omega}{2\mathcal{J}}$, which corresponds to the conformal limit.

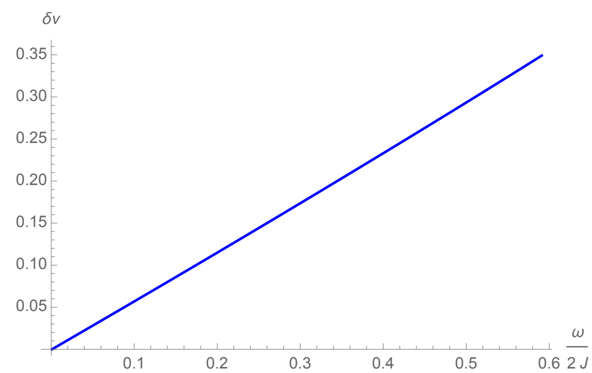


Figure C.2: Plot of the absolute difference between the allowed values of ν from two different methods, where we have defined $\delta\nu = \nu_{\text{approx}} - \nu_{\text{exact}}$ where ν_{approx} refers to the values of ν obtained by solving (C.20). We see that the difference grows with $\frac{\omega}{2\mathcal{J}}$.

for small values of $\frac{\omega}{2\mathcal{J}}$. It should be noted that the quantized values obtained by solving (C.20) hold true for any coupling. The discrepancy between the two methods arises only at order $\left(\frac{\omega}{2\mathcal{J}}\right)^{2\nu}$. This can be understood by recalling the asymptotic form of the Bessel function for small argument

$$J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(1+\nu)} \tag{C.21}$$

Using (C.21) in (C.17), requires one to solve the following equation

$$\xi_\omega(\nu) \sim \left(\frac{\omega}{2\mathcal{J}}\right)^{2\nu} \tag{C.22}$$

The following table gives the difference in quantized values for ν from the exact and approximate methods as a function of $\frac{\omega}{2\mathcal{J}}$. It provides numerical data in support of the above claim (see also Figure C.1)

Table C.1: Difference in quantized values for ν from the exact and approximate methods as a function of $\frac{\omega}{2\mathcal{J}}$.

$\frac{\omega}{2\mathcal{J}}$	$\delta\nu$
0.001	2.10064×10^{-10}
0.051	2.16037×10^{-5}
0.101	1.40694×10^{-4}
0.501	7.42506×10^{-3}

C.0.3 Perturbative determination of exact normalization

We derive (2.118) here. Let us consider $\nu = 2n + \frac{3}{2} + \frac{|\omega|}{\pi\mathcal{J}}$ where $\xi_\omega(\nu)$ vanishes. So, the integral we need to perform is

$$\int_0^\infty \frac{dz}{z + \frac{1}{2\mathcal{J}}} J_{\nu_1}(|\omega|\tilde{z}) J_{\nu_2}(|\omega|\tilde{z}) = \hat{N}_{\nu_1}\ \delta(\nu_1 - \nu_2) \tag{C.23}$$

We want to determine the normalization in (C.23). In order to do so, we perform a $1/\mathcal{J}$ expansion of the integrand. This gives

$$\int_0^\infty \frac{dz}{z + \frac{1}{2\mathcal{J}}} J_{\nu_1}(|\omega|\tilde{z}) J_{\nu_2}(|\omega|\tilde{z}) = I_1 + I_2 + I_3 \tag{C.24}$$

where

$$I_1 = \int_0^\infty \frac{dz}{z} J_{\nu_1}(|\omega|z) J_{\nu_2}(|\omega|z) = \frac{1}{2\nu_1}\delta(\nu_1 - \nu_2) \tag{C.25}$$

$$\begin{aligned} I_2 &= \frac{\omega}{2\mathcal{J}} \int_0^\infty \frac{dz}{z} \left[J_{\nu_1}(|\omega|z) J'_{\nu_2}(|\omega|z) + J'_{\nu_1}(|\omega|z) J_{\nu_2}(|\omega|z) - \frac{J_{\nu_1}(|\omega|z) J_{\nu_2}(|\omega|z)}{z} \right] \\ &= 0 \qquad\qquad (\text{for } \nu_1 + \nu_2 > 1) \end{aligned} \tag{C.26}$$

$$\begin{aligned} I_3 = &\left(\frac{\omega}{2\mathcal{J}}\right)^2 \int_0^\infty \frac{dz}{z} \left[\frac{1}{2} \left(J''_{\nu_2}(|\omega|z) + 2J'_{\nu_1}(|\omega|z) J'_{\nu_2}(|\omega|z) + J_{\nu_1}(|\omega|z) J''_{\nu_2}(|\omega|z) \right) \right. \\ &\left. - \frac{J_{\nu_1}(|\omega|z) J'_{\nu_2}(|\omega|z) + J'_{\nu_1}(|\omega|z) J_{\nu_2}(|\omega|z)}{z} + \frac{J_{\nu_1}(|\omega|z) J_{\nu_2}(|\omega|z)}{z^2} \right] \\ = &\ 0 \qquad\qquad (\text{for } \nu_1 + \nu_2 > 2) \end{aligned} \tag{C.27}$$

Based on the results of the integrals (C.25) – (C.27), we conclude that

$$\hat{N}_{\nu_1} = \frac{1}{2\nu_1} + \mathcal{O}\left(\frac{|\omega|}{2\mathcal{J}}\right)^3 \tag{C.28}$$

Appendix D
Detail Evaluation of Non-zero Mode bilocal Propagator $\mathcal{D}_c$

In this Appendix, we give details of the evaluation of $\mathcal{D}_c$ given by (2.140). Since there is no enhancement for this part, we do not need to keep track of the $\left(|\omega|\mathcal{J}^{-1}\right)$ corrections and we can evaluate it in the conformal limit as usual. Let us consider the continuous integral first in (2.139) which we denote by I_c

$$I_c = 4q^2(zz')^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t-t')} \int \frac{d\nu}{N_\nu}\, \frac{Z^*_\nu(|\omega|\tilde{z})Z_\nu(|\omega|\tilde{z}')}{\tilde{g}(\nu)-1}, \qquad \nu = ir \tag{D.1}$$

It can be written as

$$\begin{aligned} I_c &= 4q^2 \int_0^{\infty} dr\, \frac{ir}{2\sin \pi i r}\, \frac{Z_{-ir}(|\omega|z)Z_{ir}(|\omega|z')}{\tilde{g}(ir)-1} \\ &= \frac{4q^2}{i}\int_{-i\infty}^{i\infty} d\nu\, \frac{\nu}{2\sin\pi\nu}\, \frac{Z_{-\nu}(|\omega|z)J_\nu(|\omega|z')}{\tilde{g}(\nu)-1} \\ &= 4q^2(T_1+T_2) \end{aligned} \tag{D.2}$$

We have extended the limits over the entire imaginary axis and defined $\nu = ir$ in going from the first to second line of (D.2). For $z > z'$ we close the contour clockwise in the right half plane, which gives

$$T_1 = \frac{1}{i}\int_{-i\infty}^{i\infty} d\nu\, \frac{\nu}{2\sin\pi\nu}\, \frac{J_{-\nu}(|\omega|z)J_\nu(|\omega|z')}{\tilde{g}(\nu)-1} \tag{D.3}$$

and

$$T_2 = \frac{1}{i}\int_{-i\infty}^{i\infty} d\nu\, \frac{\nu}{2\sin\pi\nu}\, \frac{J_{\nu}(|\omega|z)J_\nu(|\omega|z')}{\tilde{g}(\nu)-1}\, \xi(-\nu) \tag{D.4}$$

T_1 only has a simple pole at $\nu = \frac{3}{2}$. We can evaluate the residue to get

$$T_1 = \frac{3\pi}{2}\frac{J_{-\frac{3}{2}}(|\omega|z)J_{\frac{3}{2}}(|\omega|z')}{\tilde{g}'\left(\frac{3}{2}\right)} \tag{D.5}$$

The T_2 term is a little more complicated. It has simple poles at $\nu = 2n+\frac{3}{2}$ where $\xi_\nu = 0$. We can evaluate the residue there to get

$$\begin{aligned} T_2 = &-\sum_{n=1}^{\infty}(4n+3)\, \frac{J_{2n+\frac{3}{2}}(|\omega|z)J_{2n+\frac{3}{2}}(|\omega|z')}{\tilde{g}(2n+\frac{3}{2})-1} \\ &+ \left[\frac{1}{i}\int_{-i\infty}^{i\infty} d\nu\, \frac{\nu}{2\sin\pi\nu}\, \frac{J_\nu(|\omega|z)J_\nu(|\omega|z')}{\tilde{g}(\nu)-1}\, \xi(-\nu)\right]_{\nu=\frac{3}{2}} \end{aligned} \tag{D.6}$$

Let us consider the integrand in the second term of (D.6). It has singularities coming from $\xi(-\nu)$ as well as $\tilde{g}(\nu)-1$. Let us expand the following quantity around $\nu=\frac{3}{2}$

$$\frac{\xi_\nu}{\tilde{g}(\nu)-1} = \frac{1}{\left(\nu-\frac{3}{2}\right)^2\tilde{g}'\left(\frac{3}{2}\right)\xi'\left(\frac{3}{2}\right)}\left[1-\frac{\left(\nu-\frac{3}{2}\right)}{2}\left(\frac{\xi''(\frac{3}{2})}{\xi'(\frac{3}{2})}+\frac{\tilde{g}''\left(\frac{3}{2}\right)}{\tilde{g}'\left(\frac{3}{2}\right)}\right)\right] \tag{D.7}$$

We have a double pole coming from the first term of (D.7) and a simple pole from the second term. Using (D.7) in (D.6), and evaluating the residues, we get

$$\begin{aligned} I_c \,=\, 8q^2(zz')^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega(t-t')}\Bigg[&\frac{3\pi}{2}\frac{J_{-\frac{3}{2}}(|\omega|z^>)J_{\frac{3}{2}}(|\omega|z^<)}{\tilde{g}'\left(\frac{3}{2}\right)}\,-\,I_d \qquad \text{(D.8)}\\ &+\frac{\pi}{\tilde{g}'\left(\frac{3}{2}\right)\xi'\left(\frac{3}{2}\right)}\left(\frac{3}{2}\partial_\nu+1-\frac{3\xi''(\frac{3}{2})}{4\xi'(\frac{3}{2})}-\frac{3\tilde{g}''(\frac{3}{2})}{4\tilde{g}'(\frac{3}{2})}\right)J_\nu(|\omega|z)J_\nu(|\omega|z')\Bigg|_{\nu=\frac{3}{2}}\Bigg], \end{aligned}$$

where

$$I_d \,=\, \sum_{n=1}^{\infty}(4n+3)\,\frac{J_{2n+\frac{3}{2}}(|\omega|z)J_{2n+\frac{3}{2}}(|\omega|z')}{\tilde{g}(2n+\frac{3}{2})-1}\,. \tag{D.9}$$

The first term in (2.139) exactly cancels with I_d in (D.8). Using $\tilde{g}'(\frac{3}{2})=\frac{3}{2},\tilde{g}''(\frac{3}{2})=1,\xi'\left(\frac{3}{2}\right)=-\frac{\pi}{2}$ and $\xi''\left(\frac{3}{2}\right)=0$ in (D.8), we get (2.140) We have dropped the ω subscript while writing ξ here because we are working in the conformal limit.

Appendix E
Integrals of Products of Bessel Functions

In this appendix, we consider the following integrals involving a product of two Bessel functions:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\, J_{\pm\nu}(|\omega|z^{>})\, J_{\nu}(|\omega|z^{<}) \\
&= 2\int_{0}^{\infty} d\omega\, \cos(\omega(t-t'))\, J_{\pm\nu}(\omega z^{>})\, J_{\nu}(\omega z^{<})\,. \qquad \text{(E.1)}
\end{aligned}
$$

The plus sign case was already evaluated in Appendix D of [28]. The result is

$$
2\int_{0}^{\infty} d\omega\, \cos(\omega(t-t'))\, J_{\nu}(|\omega|z^{>})\, J_{\nu}(|\omega|z^{<}) \;=\; \frac{1}{\pi\sqrt{zz'}}\, \mathcal{I}_{\nu}^{(1)}(\xi)\,, \qquad \text{(E.2)}
$$

with

$$
\mathcal{I}_{\nu}^{(1)}(\xi) \;\equiv\; \begin{cases} 2Q_{\nu-1/2}(\xi) & (\xi > 1) \\ Q_{\nu-1/2}(\xi+i\varepsilon) + Q_{\nu-1/2}(\xi-i\varepsilon) & (-1<\xi<1) \\ -2\sin(\pi\nu)Q_{\nu-1/2}(-\xi), & (\xi<-1) \end{cases} \qquad \text{(E.3)}
$$

where

$$
\xi \;\equiv\; \frac{z^2+z'^2-(t-t')^2}{2zz'}\,. \qquad \text{(E.4)}
$$

This is related to the geodesic distance of AdS_2, which is given by $d(t,z;t',z') = \ln(\xi+\sqrt{\xi^2-1})$.

The minus sign integral is more involved. Let us start from Eq.(D.11) of [28], which reads

$$
\begin{aligned}
&\int_{0}^{\infty} dx\, e^{-\alpha x}\, J_{\nu}(\beta x)\, J_{-\nu}(\gamma x) \\
&= \frac{\left(-\frac{\beta\gamma}{\alpha^2}\right)^{\nu}}{\sqrt{\pi}\,\alpha\,\Gamma(\nu+\frac{1}{2})\Gamma(1-\nu)} \int_{0}^{\pi} d\phi\, (\sin\phi)^{2\nu} \left(-\frac{\overline{\omega}^2}{\alpha^2}\right)^{-\nu} {}_2F_1\left(\frac{1}{2}, 1; 1-\nu; -\frac{\overline{\omega}^2}{\alpha^2}\right)\,, \qquad \text{(E.5)}
\end{aligned}
$$

with $\overline{\omega} = \sqrt{\beta^2+\gamma^2-2\beta\gamma\cos\phi}$. In order to obtain the cosine integral (E.1), we need to evaluate the analytical continuation by $\alpha = ae^{i\theta}$ with $\theta = 0 \to \pm\pi/2$ as explained in Appendix D of [28]. One has to be careful about this analytical continuation since the hypergeometric function ${}_2F_1(a,b;c;z)$ has a branch cut along the real z axis through $1<z<\infty$. Anyway, one should evaluate the ϕ integral before this analytical continuation. For this integral it is convenient to rewrite the hypergeometric function in the integrand using the hypergeometric function identity (for example see 9.131.2 of [172]) as

$$
\begin{aligned}
&\left(-\frac{\overline{\omega}^2}{\alpha^2}\right)^{-\nu} {}_2F_1\left(\frac{1}{2}, 1; 1-\nu; -\frac{\overline{\omega}^2}{\alpha^2}\right) \\
&= \left(\frac{2\nu}{2\nu+1}\right) {}_2F_1\left(\nu+1, \nu+\frac{1}{2}; \nu+\frac{3}{2}; \frac{2\beta\gamma}{\alpha^2}(\xi-\cos\phi)\right) \\
&\quad - \frac{1}{\sqrt{\pi}}\left(\frac{2\nu}{2\nu+1}\right)\Gamma(\nu+\tfrac{3}{2})\Gamma(-\nu)\, {}_2F_1\left(\nu+1, \nu+\frac{1}{2}; \nu+1; -\frac{\overline{\omega}^2}{\alpha^2}\right) \\
&= \left(\frac{2\nu}{2\nu+1}\right) \sum_{n=0}^{\infty} \frac{(\nu+1)_n(\nu+\frac{1}{2})_n}{n!(\nu+\frac{3}{2})_n}\left(\frac{2\beta\gamma}{\alpha^2}\right)^n (\xi-\cos\phi)^n \\
&\quad - \frac{1}{\sqrt{\pi}}\left(\frac{2\nu}{2\nu+1}\right)\Gamma(\nu+\tfrac{3}{2})\Gamma(-\nu)\left[\frac{2\beta\gamma}{\alpha^2}(\xi-\cos\phi)\right]^{-\nu-\frac{1}{2}}, \qquad \text{(E.6)}
\end{aligned}
$$

where we assumed $|2\beta\gamma\zeta/\alpha^2| < 1$ and for the second equality, we expanded the hypergeometric function in the power series with the rising Pochhammer symbol $(x)_n$. We also defined

$$
\xi \equiv \frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}, \qquad \text{(E.7)}
$$

which after the analytical continuation agrees with Eq.(E.4). Therefore, now the ϕ integral can be performed as

$$
\begin{aligned}
&\int_0^{\pi} d\phi\,(\sin\phi)^{2\nu}(\xi-\cos\phi)^s \qquad \text{(E.8)} \\
&= \begin{cases}
4^{\nu}\,\pi\,\frac{(1+\xi)^s\Gamma(\nu+\frac{1}{2})}{\cos(\pi\nu)\Gamma(\frac{1}{2}-\nu)\Gamma(2\nu+1)}\,{}_2F_1\left(\nu+\frac{1}{2}, -s; 2\nu+1; \frac{2}{1+\xi}\right), & (\xi>1) \\
2^{\nu}\,\Gamma(\nu+\frac{1}{2})\Big[\frac{(-1)^s\pi\,2^{\nu+s}}{\Gamma(2\nu+s+1)\Gamma(\frac{1}{2}-\nu-s)\cos(\pi(\nu+s))}\,{}_2F_1\left(-2\nu-s, -s; \frac{1}{2}-\nu-s; \frac{1+\xi}{2}\right) & \\
\qquad + \frac{(1+\xi)^{\frac{1}{2}+\nu+s}\Gamma(1+s)[1-e^{i\pi s}\cos(\pi\nu)/\cos(\pi(\nu+s))]}{\sqrt{2}\Gamma(\frac{3}{2}+\nu+s)}{}_2F_1\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{3}{2}+\nu+s; \frac{1+\xi}{2}\right)\Big], & \\
 & (-1<\xi<1) \\
4^{\nu}(1-\xi)^s\,\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\,{}_2F_1\left(\nu+\frac{1}{2}, -s; 2\nu+1; \frac{2}{1-\xi}\right), & (\xi<-1)
\end{cases}
\end{aligned}
$$

where s is a general real number. One can see that for any value of ξ (and for any value of s), the analytical continuation $\alpha = ae^{i\theta}$ with $\theta = 0 \to \pm\pi/2$ does not hit the branch cut of the hypergeometric function. Therefore, we can perform a naive analytical continuation for this ϕ-integral part.

We note that for the cosine integral (E.1), the contribution from the first term in Eq.(E.6) vanishes due to the α^{-1} factor in Eq.(E.5) after the analytical continuation. Therefore, for the cosine integral (E.1), the contribution solely comes from the second term in Eq.(E.6), which is written as

$$
2\int_0^{\infty} dx\,\cos(ax)\,J_{\nu}(\beta x)\,J_{-\nu}(\gamma x) = \frac{1}{\pi\sqrt{\beta\gamma}}\,\mathcal{I}_{\nu}^{(2)}(\xi), \qquad \text{(E.9)}
$$

with

$$\mathcal{I}_\nu^{(2)}(\xi) \equiv \begin{cases} 2\cos(\pi\nu)\, Q_{\nu-\frac{1}{2}}(\xi)\,, & (\xi > 1) \\ \pi\, P_{\nu-\frac{1}{2}}(-\xi)\,, & (-1 < \xi < 1) \\ 2\cos(\pi\nu)\, Q_{\nu-\frac{1}{2}}(-\xi)\,, & (\xi < -1) \end{cases} \tag{E.10}$$

Appendix F
Details of Evaluation of the Second Order Eigenvalue Shift

In this Appendix, we explain how to get (2.136) from (2.66), which we can rewrite as

$$\lambda_0^{(2)} = \int \tilde{\chi}_k^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_0^{(1)} + \int \tilde{\chi}_0^{(0)} \cdot \widetilde{\mathcal{K}}^{(2)} \cdot \tilde{\chi}_0^{(0)} \tag{F.1}$$

The second term in (F.1) is given by

$$\int \tilde{\chi}_0^{(0)} \cdot \widetilde{\mathcal{K}}^{(2)} \cdot \tilde{\chi}_0^{(0)} = \frac{1}{10} \tag{F.2}$$

Using (2.135), the first term in (F.1) can be written as

$$\int \tilde{\chi}_k^{(0)} \cdot \widetilde{\mathcal{K}}^{(1)} \cdot \tilde{\chi}_0^{(1)} = I_1 + I_2 \tag{F.3}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} dt \int_0^{\infty} dz \, \left(\frac{\sqrt{3} e^{-i\omega t}}{\sqrt{z}} J_{\frac{3}{2}}(|\omega| z) \right) \widetilde{\mathcal{K}}^{(1)} \left(\frac{\sqrt{3} e^{i\omega t}}{2\sqrt{z}} \left[J'_{\frac{3}{2}}(|\omega| z) - \frac{1}{2z} J_{\frac{3}{2}}(|\omega| z) \right] \right) \\ &= -\frac{1}{10} \end{aligned} \tag{F.4}$$

and

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} dt \int_0^{\infty} dz \, \left[\frac{2\sqrt{3} e^{-i\omega t}}{\pi \sqrt{z}} \left[\frac{d}{d\nu} J_\nu(|\omega| z) \right]_{\nu = \frac{3}{2}} + \frac{e^{-i\omega t}}{\pi \sqrt{3z}} J_{\frac{3}{2}}(|\omega| z) \right] \widetilde{\mathcal{K}}^{(1)} u_{\omega,\nu}^{(0)}(t, z) \\ &= \frac{1}{\pi^2} \end{aligned} \tag{F.5}$$

$\widetilde{\mathcal{K}}^{(1)}$ to be used above is given by (2.129). Using (F.2) – (F.5) in (F.1), we get (2.136),up to a factor of two, which arises because of (2.115).

Appendix G
Connection with Liouville

In sections 3.1 and 3.2 we show that when a state of large charge (in our case angular momentum) and low temperature is considered, correlators (and the partition function itself) are dominated by vacuum blocks. Using explicit formulas for fusion matrix elements we computed the low temperature moving component of these blocks and got the Schwarzian correlators on the nose. A natural question to ask then is why did this work?

To simplify the problem we can focus on the two point function. We can rewrite the relation (3.49) in a suggestive way in terms of Liouville theory quantities. It is straightforward after some algebra to show that the fusion matrix element is equal to

$$\mathbb{F}_{P_s 0}\begin{bmatrix} P_1 & P_2 \\ P_1 & P_2 \end{bmatrix} = \frac{1}{2\pi} C(\alpha_1, \alpha_2, \alpha_s) \frac{\Psi_{\rm ZZ}(0)\Psi_{\rm ZZ}(\alpha_s^*)}{\Psi_{\rm ZZ}(\alpha_1)\Psi_{\rm ZZ}(\alpha_2)} \tag{G.1}$$

where the right hand side is written in terms of $\alpha_i = \frac{Q}{2} + iP_i$. The first factor is the DOZZ formula [173, 174] with the usual normalization

$$C(\alpha_1, \alpha_2, \alpha_s) = \frac{(\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{Q-\alpha_{1+2+s}}{b}} \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_s)}{\Upsilon(\alpha_{1+2+s} - Q)\Upsilon(\alpha_{1+2-s})\Upsilon(\alpha_{1-2+s})\Upsilon(-\alpha_{1-2-s})}. \tag{G.2}$$

where to shorten the notation we used $\alpha_{i\pm j\pm k} = \alpha_i \pm \alpha_j \pm \alpha_k$, and the second is the ZZ-brane boundary state wavefunction

$$\Psi_{ZZ}\Big(\alpha = \frac{Q}{2} + iP\Big) = \frac{2^{-1/4} 2\pi\ 2iP(\pi\mu\gamma(b^2))^{-iP/b}}{\Gamma(1-2ibP)\Gamma(1-2iP/b)} \tag{G.3}$$

derived in [175] from a modular bootstrap analysis. The left hand side of (G.1) is a Virasoro kinematical quantity (independent of the theory) while the right hand side is written in terms of objects appearing in a specific theory, Liouville. As a check that this relation makes sense, one can see that the only theory dependent quantity, the Liouville cosmological constant μ, cancels completely in the right hand side. This formula was proposed by analyzing the conformal bootstrap of Liouville theory with boundaries in [175] (see their equations 6.4 and 6.5). Starting from the exact expression of the fusion matrix, the identity (G.1) was derived by Teschner and Vartanov in Appendix D.2 of [120]. This has also an interesting interpretation in the context of AGT [176].

Using (G.1) it is easy to see that the torus vacuum block for the two point function used in (3.61) is equal to the one-point function of a primary operator of Liouville theory, living in an annulus between ZZ-brane boundary states. This is required by the consistency of the boundary CFT bootstrap of [175]. In the limit considered in this chapter, Liouville theory between ZZ-branes is equivalent to the Schwarzian theory and Liouville primary operators are equivalent to inserting a Schwarzian bilocal field. This fact was used in [84] to compute Schwarzian correlators using 2d Liouville

CFT techniques. We will leave the details of this relation for a future work, but this gives the underlying reason for the match between the results of section 3.2 and the Schwarzian theory.

This is also consistent with the fact that Liouville between ZZ-branes in equivalent to the Alekseev-Shatashvili coadjoint orbit action studied in [95, 98] which in the semiclassical limit reduces to the Schwarzian action.

Schwarzian Limit

The Schwarzian limit of the fusion matrix given in the form (G.1) was done in [84]. The relevant limit of the operator momentum is $P_1 = bk_1$, $\alpha_2 = bh$ and $P_s = bk_s$. The important relation here is $\Upsilon(bx) = \frac{b^{\frac{1}{2}-x}}{\Gamma(x)} f(b)$, and $\Upsilon'(0) = b^{-1/2} f(b)$, with $f(b)$ some known function that will cancel in the end. Then it is easy to see that

$$C(\alpha_1, \alpha_2, \alpha_s) \sim \frac{1}{b} \frac{\prod_{\pm\pm} \Gamma(h \pm ik_1 \pm ik_s)}{\Gamma(-2ik_1)\Gamma(2h)\Gamma(-2ik_s)}, \quad \frac{\Psi_{ZZ}(0)\Psi_{ZZ}(\alpha_s^*)}{\Psi_{ZZ}(\alpha_1)\Psi_{ZZ}(\alpha_2)} \sim b^{4h} \frac{\Gamma(-2ik_1)}{\Gamma(2ik_s)}$$

which gives the same expression for the fusion matrix element (3.53). This approximation for the DOZZ formula can also be obtained by using the minisuperspace approximation of Liouville theory without having to know the full expression involving special functions.

Appendix H
Details on matching boundary conditions

In this appendix, we solve the wave equation for a free massive scalar field in the fixed BTZ background and work out the proportionality factors between the sources on the AdS_3 boundary and the asymptotic AdS_2 boundary, as stated in 3.145.

We can carry out a mode decomposition of the scalar field as

$$\chi(t,r,\varphi)=\sum_{\ell}\int_{-\infty}^{\infty}d\omega\ e^{-i\omega t-i\ell\varphi}\chi_{\ell,\omega}(r), \tag{H.1}$$

The frequency space wave equation, written in terms of a new variable $z=\frac{r^2-r_+^2}{r^2-r_-^2}$, for near-extremal BTZ written in coordinates (3.97) is the following:

$$\begin{aligned}\Bigg[&z(1-z)\frac{d^2}{dz^2}+(1-z)\frac{d}{dz}\\&+\frac{1}{4}\left(\frac{(\omega r_+-\ell r_-)^2\ell_3^2}{z(r_+^2-r_-^2)^2}-\frac{(\omega r_--\ell r_+)^2\ell_3^2}{(r_+^2-r_-^2)^2}-\frac{\ell_3^2m^2}{1-z}\right)\Bigg]\chi_{\ell,\omega}(z)=0,\end{aligned} \tag{H.2}$$

See [177] for more details. In the following we will take the limit of small $T\sim r_+-r_-$ together with the low frequency limit $\omega\sim T$, which simplifies the equation considerably. Moreover if we focus on the region between the nAdS_2 and AdS_3 boundaries we are in the regime $r-r_+\gg r_+-r_-$, for which

$$1-z\sim\frac{2r_+(r_+-r_-)}{r^2-r_+^2}. \tag{H.3}$$

and from this we see that $1-z$ is small in the entire region outside the throat. This limit (low temperature, low frequency and (H.3)) makes both the frequency and angular momentum dependent terms in (H.2) negligible, and simplifies the solution to a simple power law proportional to $(1-z)^{\frac{\Delta_\pm}{2}}$. We can write these as

$$\chi\sim A\left(\frac{r^2-r_+^2}{\ell_3^2}\right)^{-\frac{\Delta_-}{2}}+B\left(\frac{r^2-r_+^2}{\ell_3^2}\right)^{-\frac{\Delta_+}{2}}, \tag{H.4}$$

with coefficients A,B depending on ω and ℓ; because we reduce to the simple power law solutions, this is the only dependence on those variables. For $r\to\infty$, these coefficients match those of the asymptotic expansion (3.141) near the AdS_3 boundary.

To match to the AdS_2 boundary, we now need only expand this solution for small $r-r_+$, and match coefficients of powers of the variable $\frac{4}{\ell_3}(r-r_+)$ in the expansion (3.143):

$$\tilde{A}=\left(\frac{r_+}{2\ell_3}\right)^{-\frac{\Delta_-}{2}}A,\quad \tilde{B}=\left(\frac{r_+}{2\ell_3}\right)^{-\frac{\Delta_+}{2}}B \tag{H.5}$$

Writing this in terms of the right-moving temperature $\beta_R \sim \frac{\pi \ell_3}{r_+}$, and using $\Delta_+ = \Delta$, $\Delta_- = 2 - \Delta$, we finally have

$$\tilde{A} = \left(\frac{2}{\pi}\beta_R\right)^{1-\frac{\Delta}{2}} A, \quad \tilde{B} = \left(\frac{2}{\pi}\beta_R\right)^{\frac{\Delta}{2}} B \tag{H.6}$$

as stated in (3.145). This relation is independent of where we put the cut-off between the 2d and three dimensional boundaries as expected.

Appendix I
Unit Normalized EAdS/dS Wave Functions

The unit-normalized Euclidean AdS_2 wave function is given by

$$\overline{\phi}_{\mathrm{EAdS}_2}(\tau, z) \;=\; \alpha_\nu \, z^{\frac{1}{2}} \, e^{-i\omega\tau} \, K_\nu(|\omega| z) \,, \tag{I.1}$$

where the normalization factor can be chosen as

$$\alpha_\nu \;=\; i \sqrt{\frac{\nu \sin(\pi\nu)}{\pi^3}} \,. \tag{I.2}$$

Then from the Bessel K_ν orthogonality condition

$$\int_0^\infty \frac{dx}{x} \, K_{i\nu}(x) \, K_{i\nu'}(x) \;=\; \frac{\pi^2}{2} \, \frac{\delta(\nu - \nu')}{\nu \sinh(\pi\nu)} \,, \tag{I.3}$$

where $x, y > 0$, the wave function is unit normalized:

$$\int_{-\infty}^\infty d\tau \int_0^\infty \frac{dz}{z^2} \, \overline{\phi}^{\,*}_{\omega,\nu}(\tau, z) \, \overline{\phi}_{\omega',\nu'}(\tau, z) \;=\; \delta(\omega - \omega')\delta(\nu - \nu') \,. \tag{I.4}$$

The completeness of modified Bessel function of the second kind:

$$\int_0^\infty d\nu \, \nu \sinh(\pi\nu) \, K_{i\nu}(x) \, K_{i\nu}(y) \;=\; \frac{\pi^2}{2} \, x \, \delta(x - y) \,, \tag{I.5}$$

is also used in section 5.3.

The Lorentzian dS_2 wave function is given by

$$\overline{\psi}_{\mathrm{dS}_2}(\eta, t) \;=\; \beta_\nu \, \eta^{\frac{1}{2}} \, e^{-i\omega t} \, Z_\nu(|\omega| \eta) \,. \tag{I.6}$$

Here, let us only consider the continuous modes ($\nu = ir$). Now choosing the normalization factor as

$$\beta_\nu \;=\; \sqrt{\frac{\nu}{4\pi \sin(\pi\nu)}} \,, \tag{I.7}$$

then the wave function is unit normalized for the continuous modes as

$$\int_{-\infty}^\infty dt \int_0^\infty \frac{d\eta}{\eta^2} \, \overline{\psi}^{\,*}_{\omega,\nu}(\eta, t) \, \overline{\psi}_{\omega',\nu'}(\eta, t) \;=\; \delta(\omega - \omega')\delta(\nu - \nu') \,, \qquad \text{for } (\nu = ir) \tag{I.8}$$

We note that

$$\alpha_\nu \;=\; 2i \, \frac{\sin \pi\nu}{\pi} \, \beta_\nu \,. \tag{I.9}$$

Bibliography

[1] S.W. Hawking. Breakdown of Predictability in Gravitational Collapse. *Phys. Rev. D*, 14:2460–2473, 1976.

[2] Andrew Strominger and Cumrun Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett. B*, 379:99–104, 1996.

[3] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999.

[4] Leonard Susskind. The world as a hologram. *Journal of Mathematical Physics*, 36(11):6377–6396, Nov 1995.

[5] Gerard 't Hooft. Dimensional reduction in quantum gravity. *Conf. Proc. C*, 930308:284–296, 1993.

[6] Juan Martin Maldacena. Eternal black holes in anti-de Sitter. *JHEP*, 04:021, 2003.

[7] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford. A bound on chaos. *JHEP*, 08:106, 2016.

[8] Stephen H. Shenker and Douglas Stanford. Black holes and the butterfly effect. *JHEP*, 03:067, 2014.

[9] Stephen H. Shenker and Douglas Stanford. Stringy effects in scrambling. *JHEP*, 05:132, 2015.

[10] Joanna L. Karczmarek, Juan Martin Maldacena, and Andrew Strominger. Black hole non-formation in the matrix model. *JHEP*, 01:039, 2006.

[11] A. Kitaev. Talks given at the Fundamental Physics Prize Symposium and KITP seminars.

[12] Juan Maldacena and Douglas Stanford. Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev. D*, 94(10):106002, 2016.

[13] Subir Sachdev and Jinwu Ye. Gapless spin fluid ground state in a random, quantum Heisenberg magnet. *Phys. Rev. Lett.*, 70:3339, 1993.

[14] Subir Sachdev. Holographic metals and the fractionalized Fermi liquid. *Phys. Rev. Lett.*, 105:151602, 2010.

[15] John Preskill, Patricia Schwarz, Alfred D. Shapere, Sandip Trivedi, and Frank Wilczek. Limitations on the statistical description of black holes. *Mod. Phys. Lett.*, A6:2353–2362, 1991.

[16] I.R. Klebanov and A.M. Polyakov. AdS dual of the critical O(N) vector model. *Phys. Lett. B*, 550:213–219, 2002.

[17] E. Sezgin and P. Sundell. Massless higher spins and holography. *Nucl. Phys. B*, 644:303–370, 2002. [Erratum: Nucl.Phys.B 660, 403–403 (2003)].

[18] Sumit R. Das and Antal Jevicki. Large N collective fields and holography. *Phys. Rev. D*, 68:044011, 2003.

[19] Robert de Mello Koch, Antal Jevicki, Kewang Jin, and Joao P. Rodrigues. AdS_4/CFT_3 Construction from Collective Fields. *Phys. Rev. D*, 83:025006, 2011.

[20] Robert de Mello Koch, Antal Jevicki, Joao P. Rodrigues, and Junggi Yoon. Holography as a Gauge Phenomenon in Higher Spin Duality. *JHEP*, 01:055, 2015.

[21] Robert de Mello Koch, Antal Jevicki, João P. Rodrigues, and Junggi Yoon. Canonical Formulation of $O(N)$ Vector/Higher Spin Correspondence. *J. Phys. A*, 48(10):105403, 2015.

[22] Eric Mintun and Joseph Polchinski. Higher Spin Holography, RG, and the Light Cone. 11 2014.

[23] Sumit R. Das, Animik Ghosh, Antal Jevicki, and Kenta Suzuki. Near Conformal Perturbation Theory in SYK Type Models. 6 2020.

[24] Animik Ghosh, Henry Maxfield, and Gustavo J. Turiaci. A universal Schwarzian sector in two-dimensional conformal field theories. *JHEP*, 05:104, 2020.

[25] Sumit R. Das, Animik Ghosh, Antal Jevicki, and Kenta Suzuki. Three Dimensional View of Arbitrary q SYK models. *JHEP*, 02:162, 2018.

[26] Sumit R. Das, Animik Ghosh, Antal Jevicki, and Kenta Suzuki. Space-Time in the SYK Model. *JHEP*, 07:184, 2018.

[27] Subir Sachdev. Bekenstein-Hawking Entropy and Strange Metals. *Phys. Rev. X*, 5(4):041025, 2015.

[28] Joseph Polchinski and Vladimir Rosenhaus. The Spectrum in the Sachdev-Ye-Kitaev Model. *JHEP*, 04:001, 2016.

[29] Antal Jevicki, Kenta Suzuki, and Junggi Yoon. Bi-Local Holography in the SYK Model. *JHEP*, 07:007, 2016.

[30] Antal Jevicki and Kenta Suzuki. Bi-Local Holography in the SYK Model: Perturbations. *JHEP*, 11:046, 2016.

[31] Richard A. Davison, Wenbo Fu, Antoine Georges, Yingfei Gu, Kristan Jensen, and Subir Sachdev. Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography. *Phys. Rev. B*, 95(15):155131, 2017.

[32] David J. Gross and Vladimir Rosenhaus. The Bulk Dual of SYK: Cubic Couplings. *JHEP*, 05:092, 2017.

[33] David J. Gross and Vladimir Rosenhaus. All point correlation functions in SYK. *JHEP*, 12:148, 2017.

[34] Alexei Kitaev and S. Josephine Suh. The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual. *JHEP*, 05:183, 2018.

[35] Edward Witten. An SYK-Like Model Without Disorder. *J. Phys. A*, 52(47):474002, 2019.

[36] Igor R. Klebanov and Grigory Tarnopolsky. Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models. *Phys. Rev. D*, 95(4):046004, 2017.

[37] Cheng Peng, Marcus Spradlin, and Anastasia Volovich. A Supersymmetric SYK-like Tensor Model. *JHEP*, 05:062, 2017.

[38] Chethan Krishnan, Sambuddha Sanyal, and P. N. Bala Subramanian. Quantum Chaos and Holographic Tensor Models. *JHEP*, 03:056, 2017.

[39] Tianlin Li, Junyu Liu, Yuan Xin, and Yehao Zhou. Supersymmetric SYK model and random matrix theory. *JHEP*, 06:111, 2017.

[40] Razvan Gurau. Quenched equals annealed at leading order in the colored SYK model. *EPL*, 119(3):30003, 2017.

[41] H. Itoyama, A. Mironov, and A. Morozov. Rainbow tensor model with enhanced symmetry and extreme melonic dominance. *Phys. Lett. B*, 771:180–188, 2017.

[42] Chethan Krishnan, K. V. Pavan Kumar, and Sambuddha Sanyal. Random Matrices and Holographic Tensor Models. *JHEP*, 06:036, 2017.

[43] Prithvi Narayan and Junggi Yoon. SYK-like Tensor Models on the Lattice. *JHEP*, 08:083, 2017.

[44] Robert de Mello Koch, David Gossman, and Laila Tribelhorn. Gauge Invariants, Correlators and Holography in Bosonic and Fermionic Tensor Models. *JHEP*, 09:011, 2017.

[45] Tatsuo Azeyanagi, Frank Ferrari, and Fidel I. Schaposnik Massolo. Phase Diagram of Planar Matrix Quantum Mechanics, Tensor, and Sachdev-Ye-Kitaev Models. *Phys. Rev. Lett.*, 120(6):061602, 2018.

[46] Simone Giombi, Igor R. Klebanov, and Grigory Tarnopolsky. Bosonic tensor models at large N and small ϵ. *Phys. Rev. D*, 96(10):106014, 2017.

[47] Frank Ferrari, Vincent Rivasseau, and Guillaume Valette. A New Large N Expansion for General Matrix–Tensor Models. *Commun. Math. Phys.*, 370(2):403–448, 2019.

[48] Dario Benedetti and Razvan Gurau. 2PI effective action for the SYK model and tensor field theories. *JHEP*, 05:156, 2018.

[49] Chethan Krishnan and K.V. Pavan Kumar. Complete Solution of a Gauged Tensor Model. 4 2018.

[50] Nicolas Delporte and Vincent Rivasseau. The Tensor Track V: Holographic Tensors. In *17th Hellenic School and Workshops on Elementary Particle Physics and Gravity*, 4 2018.

[51] P. Diaz and J.A. Rosabal. Spontaneous Symmetry Breaking in Tensor Theories. *JHEP*, 01:094, 2019.

[52] Robert De Mello Koch, David Gossman, Nirina Hasina Tahiridimbisoa, and Augustine Larweh Mahu. Holography for Tensor models. *Phys. Rev. D*, 101(4):046004, 2020.

[53] Dario Benedetti and Ilaria Costa. $SO(3)$-invariant phase of the $O(N)^3$ tensor model. *Phys. Rev. D*, 101(8):086021, 2020.

[54] Jean-Loup Gervais, A. Jevicki, and B. Sakita. Perturbation Expansion Around Extended Particle States in Quantum Field Theory. 1. *Phys. Rev. D*, 12:1038, 1975.

[55] Jean-Loup Gervais, A. Jevicki, and B. Sakita. Collective Coordinate Method for Quantization of Extended Systems. *Phys. Rept.*, 23:281–293, 1976.

[56] C. Teitelboim. Gravitation and Hamiltonian Structure in Two Space-Time Dimensions. *Phys. Lett. B*, 126:41–45, 1983.

[57] R. Jackiw. Lower Dimensional Gravity. *Nucl. Phys. B*, 252:343–356, 1985.

[58] Ahmed Almheiri and Joseph Polchinski. Models of AdS_2 backreaction and holography. *JHEP*, 11:014, 2015.

[59] Kristan Jensen. Chaos in AdS_2 Holography. *Phys. Rev. Lett.*, 117(11):111601, 2016.

[60] Juan Maldacena, Douglas Stanford, and Zhenbin Yang. Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space. *PTEP*, 2016(12):12C104, 2016.

[61] Julius Engelsöy, Thomas G. Mertens, and Herman Verlinde. An investigation of AdS_2 backreaction and holography. *JHEP*, 07:139, 2016.

[62] Gautam Mandal, Pranjal Nayak, and Spenta R. Wadia. Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models. *JHEP*, 11:046, 2017.

[63] Sumit R. Das, Antal Jevicki, and Kenta Suzuki. Three Dimensional View of the SYK/AdS Duality. *JHEP*, 09:017, 2017.

[64] Marika Taylor. Generalized conformal structure, dilaton gravity and SYK. *JHEP*, 01:010, 2018.

[65] Daniel Grumiller, Robert McNees, Jakob Salzer, Carlos Valcárcel, and Dmitri Vassilevich. Menagerie of AdS_2 boundary conditions. *JHEP*, 10:203, 2017.

[66] Pranjal Nayak, Ashish Shukla, Ronak M. Soni, Sandip P. Trivedi, and V. Vishal. On the Dynamics of Near-Extremal Black Holes. *JHEP*, 09:048, 2018.

[67] Adwait Gaikwad, Lata Kh Joshi, Gautam Mandal, and Spenta R. Wadia. Holographic dual to charged SYK from 3D Gravity and Chern-Simons. *JHEP*, 02:033, 2020.

[68] Arindam Lala and Dibakar Roychowdhury. SYK/AdS duality with Yang-Baxter deformations. *JHEP*, 12:073, 2018.

[69] Hernán A. González, Daniel Grumiller, and Jakob Salzer. Towards a bulk description of higher spin SYK. *JHEP*, 05:083, 2018.

[70] Shinji Hirano and Yang Lei. Nearly AdS_2 holography in quantum CGHS model. *JHEP*, 01:178, 2020.

[71] Hamid Afshar, Hernán A. González, Daniel Grumiller, and Dmitri Vassilevich. Flat space holography and the complex Sachdev-Ye-Kitaev model. *Phys. Rev. D*, 101(8):086024, 2020.

[72] Konstantin Alkalaev and Xavier Bekaert. Towards higher-spin AdS_2/CFT_1 holography. *JHEP*, 04:206, 2020.

[73] Grigory Tarnopolsky. Large q expansion in the Sachdev-Ye-Kitaev model. *Phys. Rev. D*, 99(2):026010, 2019.

[74] Alexandre Streicher. SYK Correlators for All Energies. *JHEP*, 02:048, 2020.

[75] Changha Choi, Márk Mezei, and Gábor Sárosi. Exact four point function for large q SYK from Regge theory. 11 2019.

[76] Antal Jevicki, Kewang Jin, and Junggi Yoon. 1/N and loop corrections in higher spin AdS_4/CFT_3 duality. *Phys. Rev. D*, 89(8):085039, 2014.

[77] Yingfei Gu, Alexei Kitaev, Subir Sachdev, and Grigory Tarnopolsky. Notes on the complex Sachdev-Ye-Kitaev model. *JHEP*, 02:157, 2020.

[78] S. R. Das, Animik Ghosh, Antal Jevicki, and Kenta Suzuki. Work in progress.

[79] Mirjam Cvetic and Ioannis Papadimitriou. AdS_2 holographic dictionary. *JHEP*, 12:008, 2016. [Erratum: JHEP01,120(2017)].

[80] Juan Martin Maldacena, Jeremy Michelson, and Andrew Strominger. Anti-de Sitter fragmentation. *JHEP*, 02:011, 1999.

[81] Micha Berkooz, Prithvi Narayan, Moshe Rozali, and Joan Simón. Higher Dimensional Generalizations of the SYK Model. *JHEP*, 01:138, 2017.

[82] Gustavo Turiaci and Herman Verlinde. Towards a 2d QFT Analog of the SYK Model. *JHEP*, 10:167, 2017.

[83] Jeff Murugan, Douglas Stanford, and Edward Witten. More on Supersymmetric and 2d Analogs of the SYK Model. *JHEP*, 08:146, 2017.

[84] Thomas G. Mertens, Gustavo J. Turiaci, and Herman L. Verlinde. Solving the Schwarzian via the Conformal Bootstrap. *JHEP*, 08:136, 2017.

[85] Scott Collier, Alexander Maloney, Henry Maxfield, and Ioannis Tsiares. Universal Dynamics of Heavy Operators in CFT_2. 2019.

[86] A. Liam Fitzpatrick, Jared Kaplan, David Poland, and David Simmons-Duffin. The Analytic Bootstrap and AdS Superhorizon Locality. *JHEP*, 12:004, 2013.

[87] A. Liam Fitzpatrick, Jared Kaplan, and Matthew T. Walters. Universality of Long-Distance AdS Physics from the CFT Bootstrap. *JHEP*, 08:145, 2014.

[88] Scott Collier, Yan Gobeil, Henry Maxfield, and Eric Perlmutter. Quantum Regge Trajectories and the Virasoro Analytic Bootstrap. 2018.

[89] Henry Maxfield. Quantum corrections to the BTZ black hole extremality bound from the conformal bootstrap. 2019.

[90] Ana Achucarro and Miguel E. Ortiz. Relating black holes in two-dimensions and three-dimensions. *Phys. Rev.*, D48:3600–3605, 1993.

[91] Ahmed Almheiri and Byungwoo Kang. Conformal Symmetry Breaking and Thermodynamics of Near-Extremal Black Holes. *JHEP*, 10:052, 2016.

[92] Upamanyu Moitra, Sunil Kumar Sake, Sandip P. Trivedi, and V. Vishal. Jackiw-Teitelboim Gravity and Rotating Black Holes. 2019.

[93] Alejandra Castro, Finn Larsen, and Ioannis Papadimitriou. 5D rotating black holes and the $nAdS_2/nCFT_1$ correspondence. *JHEP*, 10:042, 2018.

[94] Edward Witten. Three-Dimensional Gravity Revisited. 2007.

[95] Jordan Cotler and Kristan Jensen. A theory of reparameterizations for AdS_3 gravity. *JHEP*, 02:079, 2019.

[96] A. Alekseev and Samson L. Shatashvili. From geometric quantization to conformal field theory. *Commun. Math. Phys.*, 128:197–212, 1990. [,22(1990)].

[97] Thomas G. Mertens. The Schwarzian theory — origins. *JHEP*, 05:036, 2018.

[98] Gustavo Turiaci and Herman Verlinde. On CFT and Quantum Chaos. *JHEP*, 12:110, 2016.

[99] Felix M. Haehl and Moshe Rozali. Effective Field Theory for Chaotic CFTs. *JHEP*, 10:118, 2018.

[100] Vijay Balasubramanian, Asad Naqvi, and Joan Simon. A Multiboundary AdS orbifold and DLCQ holography: A Universal holographic description of extremal black hole horizons. *JHEP*, 08:023, 2004.

[101] Vijay Balasubramanian, Jan de Boer, M. M. Sheikh-Jabbari, and Joan Simon. What is a chiral 2d CFT? And what does it have to do with extremal black holes? *JHEP*, 02:017, 2010.

[102] Phil Saad, Stephen H. Shenker, and Douglas Stanford. JT gravity as a matrix integral. 2019.

[103] Simone Giombi, Alexander Maloney, and Xi Yin. One-loop Partition Functions of 3D Gravity. *JHEP*, 08:007, 2008.

[104] Alexander Maloney and Edward Witten. Quantum Gravity Partition Functions in Three Dimensions. *JHEP*, 02:029, 2010.

[105] J. David Brown and M. Henneaux. Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, 104:207–226, 1986.

[106] Juan Martin Maldacena and Andrew Strominger. AdS(3) black holes and a stringy exclusion principle. *JHEP*, 12:005, 1998.

[107] Romesh K. Kaul and Parthasarathi Majumdar. Logarithmic correction to the Bekenstein-Hawking entropy. *Phys. Rev. Lett.*, 84:5255–5257, 2000.

[108] Steven Carlip. Logarithmic corrections to black hole entropy from the Cardy formula. *Class. Quant. Grav.*, 17:4175–4186, 2000.

[109] Anthony M. Charles and Finn Larsen. A One-Loop Test of the near-AdS_2/near-CFT_1 Correspondence. 2019.

[110] Herman L. Verlinde. Conformal Field Theory, 2-D Quantum Gravity and Quantization of Teichmuller Space. *Nucl. Phys.*, B337:652–680, 1990.

[111] Alejandra Castro and Beatrix Muhlmann. Gravitational Anomalies in $nAdS_2/nCFT_1$. 2019.

[112] Nathan Benjamin, Hirosi Ooguri, Shu-Heng Shao, and Yifan Wang. Light-cone modular bootstrap and pure gravity. *Phys. Rev.*, D100(6):066029, 2019.

[113] B. Ponsot and J. Teschner. Liouville bootstrap via harmonic analysis on a noncompact quantum group. 1999.

[114] Thomas G. Mertens and Gustavo J. Turiaci. Defects in Jackiw-Teitelboim Quantum Gravity. 2019.

[115] Andreas Blommaert, Thomas G. Mertens, and Henri Verschelde. The Schwarzian Theory - A Wilson Line Perspective. *JHEP*, 12:022, 2018.

[116] Luca V. Iliesiu, Silviu S. Pufu, Herman Verlinde, and Yifan Wang. An exact quantization of Jackiw-Teitelboim gravity. 2019.

[117] Alexei Kitaev and S. Josephine Suh. Statistical mechanics of a two-dimensional black hole. *JHEP*, 05:198, 2019.

[118] Mark Srednicki. Chaos and quantum thermalization. *Physical Review E*, 50:888, 1994.

[119] Gregory W. Moore and Nathan Seiberg. Classical and Quantum Conformal Field Theory. *Commun. Math. Phys.*, 123:177, 1989.

[120] J. Teschner and G. S. Vartanov. Supersymmetric gauge theories, quantization of $\mathcal{M}_{\text{flat}}$, and conformal field theory. *Adv. Theor. Math. Phys.*, 19:1–135, 2015.

[121] Zhenbin Yang. The Quantum Gravity Dynamics of Near Extremal Black Holes. *JHEP*, 05:205, 2019.

[122] A. Liam Fitzpatrick, Jared Kaplan, and Matthew T. Walters. Virasoro Conformal Blocks and Thermality from Classical Background Fields. *JHEP*, 11:200, 2015.

[123] Ho Tat Lam, Thomas G. Mertens, Gustavo J. Turiaci, and Herman Verlinde. Shockwave S-matrix from Schwarzian Quantum Mechanics. *JHEP*, 11:182, 2018.

[124] A. Liam Fitzpatrick, Jared Kaplan, Daliang Li, and Junpu Wang. On information loss in AdS_3/CFT_2. *JHEP*, 05:109, 2016.

[125] Akash Goel, Ho Tat Lam, Gustavo J. Turiaci, and Herman Verlinde. Expanding the Black Hole Interior: Partially Entangled Thermal States in SYK. *JHEP*, 02:156, 2019.

[126] Dmitry Bagrets, Alexander Altland, and Alex Kamenev. Sachdev-Ye-Kitaev model as Liouville quantum mechanics. *Nucl. Phys.*, B911:191–205, 2016.

[127] Hongbin Chen, Charles Hussong, Jared Kaplan, and Daliang Li. A Numerical Approach to Virasoro Blocks and the Information Paradox. *JHEP*, 09:102, 2017.

[128] Wolter Groenevelt. Wilson function transforms related to racah coefficients, 2005.

[129] S. Josephine Suh. Dynamics of black holes in Jackiw-Teitelboim gravity. 2019.

[130] Daniel A. Roberts and Douglas Stanford. Two-dimensional conformal field theory and the butterfly effect. *Phys. Rev. Lett.*, 115(13):131603, 2015.

[131] Chi-Ming Chang, David M. Ramirez, and Mukund Rangamani. Spinning constraints on chaotic large c CFTs. *JHEP*, 03:068, 2019.

[132] Steven Jackson, Lauren McGough, and Herman Verlinde. Conformal Bootstrap, Universality and Gravitational Scattering. *Nucl. Phys.*, B901:382–429, 2015.

[133] Hongbin Chen, A. Liam Fitzpatrick, Jared Kaplan, Daliang Li, and Junpu Wang. Degenerate Operators and the $1/c$ Expansion: Lorentzian Resummations, High Order Computations, and Super-Virasoro Blocks. *JHEP*, 03:167, 2017.

[134] Mark Mezei and Gabor Sarosi. Chaos in the butterfly cone. *JHEP*, 01:186, 2020.

[135] Hrant Gharibyan, Masanori Hanada, Brian Swingle, and Masaki Tezuka. Quantum Lyapunov Spectrum. *JHEP*, 04:082, 2019.

[136] Gustavo J. Turiaci. An Inelastic Bound on Chaos. *JHEP*, 07:099, 2019.

[137] Olivier Coussaert and Marc Henneaux. Selfdual solutions of (2+1) Einstein gravity with a negative cosmological constant. In *The black hole: 25 years after*, pages 25–39, 1994. [,25(1994)].

[138] James M. Bardeen and Gary T. Horowitz. The Extreme Kerr throat geometry: A Vacuum analog of AdS(2) x S**2. *Phys. Rev.*, D60:104030, 1999.

[139] Kristan Jensen, Shamit Kachru, Andreas Karch, Joseph Polchinski, and Eva Silverstein. Towards a holographic marginal Fermi liquid. *Phys. Rev.*, D84:126002, 2011.

[140] Luca V. Iliesiu and Gustavo J. Turiaci. The statistical mechanics of near-extremal black holes. 2020.

[141] Juan Maldacena, Gustavo J. Turiaci, and Zhenbin Yang. Two dimensional Nearly de Sitter gravity. 2019.

[142] Matthias Blau and George Thompson. Quantum Yang-Mills theory on arbitrary surfaces. *Int. J. Mod. Phys.*, A7:3781–3806, 1992. [,28(1991)].

[143] Edward Witten. On quantum gauge theories in two-dimensions. *Commun. Math. Phys.*, 141:153–209, 1991.

[144] R. F. Picken. The Propagator for Quantum Mechanics on a Group Manifold From an Infinite Dimensional Analog of the Duistermaat-heckman Integration Formula. *J. Phys.*, A22:2285, 1989.

[145] Luca V. Iliesiu. On 2D gauge theories in Jackiw-Teitelboim gravity. 2019.

[146] Henry Maxfield and Gustavo J. Turiaci. *work in progress.*

[147] A. Jevicki and B. Sakita. The Quantum Collective Field Method and Its Application to the Planar Limit. *Nucl. Phys. B*, 165:511, 1980.

[148] Stefan Forste and Iris Golla. Nearly AdS_2 sugra and the super-Schwarzian. *Phys. Lett. B*, 771:157–161, 2017.

[149] R. Jackiw. *Quantum Theory of Gravity. (Hilger,1984).*

[150] Pablo Diaz, Saurya Das, and Mark Walton. Bilocal theory and gravity I. 9 2016.

[151] Márk Mezei, Silviu S. Pufu, and Yifan Wang. A 2d/1d Holographic Duality. 3 2017.

[152] Peter Breitenlohner and Daniel Z. Freedman. Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. *Phys. Lett. B*, 115:197–201, 1982.

[153] Alexei Kitaev. Notes on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ representations. 11 2017.

[154] Sumit R. Das, Animik Ghosh, Antal Jevicki, and Kenta Suzuki. Unpublished.

[155] Koji Hashimoto and Norihiro Tanahashi. Universality in Chaos of Particle Motion near Black Hole Horizon. *Phys. Rev. D*, 95(2):024007, 2017.

[156] Mike Blake and Aristomenis Donos. Diffusion and Chaos from near AdS_2 horizons. *JHEP*, 02:013, 2017.

[157] Douglas Stanford and Edward Witten. Fermionic Localization of the Schwarzian Theory. *JHEP*, 10:008, 2017.

[158] Gábor Sárosi. AdS_2 holography and the SYK model. *PoS*, Modave2017:001, 2018.

[159] Bartlomiej Czech, Lampros Lamprou, Samuel McCandlish, Benjamin Mosk, and James Sully. A Stereoscopic Look into the Bulk. *JHEP*, 07:129, 2016.

[160] Jan de Boer, Michal P. Heller, Robert C. Myers, and Yasha Neiman. Holographic de Sitter Geometry from Entanglement in Conformal Field Theory. *Phys. Rev. Lett.*, 116(6):061602, 2016.

[161] Samrat Bhowmick, Koushik Ray, and Siddhartha Sen. Bulk reconstruction in AdS and Gel'fand-Graev-Radon transform. *JHEP*, 10:082, 2017.

[162] Sumit R. Das and Antal Jevicki. String Field Theory and Physical Interpretation of $D = 1$ Strings. *Mod. Phys. Lett. A*, 5:1639–1650, 1990.

[163] Gregory W. Moore and Nathan Seiberg. From loops to fields in 2-D quantum gravity. *Int. J. Mod. Phys. A*, 7:2601–2634, 1992.

[164] Bruno Balthazar, Victor A. Rodriguez, and Xi Yin. The $c = 1$ string theory S-matrix revisited. *JHEP*, 04:145, 2019.

[165] T.S. Bunch and P.C.W. Davies. Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting. *Proc. Roy. Soc. Lond. A*, A360:117–134, 1978.

[166] E. Mottola. Particle Creation in de Sitter Space. *Phys. Rev. D*, 31:754, 1985.

[167] Bruce Allen. Vacuum States in de Sitter Space. *Phys. Rev. D*, 32:3136, 1985.

[168] Antal Jevicki and Tamiaki Yoneya. A Deformed matrix model and the black hole background in two-dimensional string theory. *Nucl. Phys. B*, 411:64–96, 1994.

[169] Sumit R. Das. Matrix models and black holes. *Mod. Phys. Lett. A*, 8:69–78, 1993.

[170] G. Watson. *A Treatise on the Theory of Bessel functions, Cambridge University Press*. 1944.

[171] David J. Gross, Igor R. Klebanov, and M.J. Newman. The Two point correlation function of the one-dimensional matrix model. *Nucl. Phys. B*, 350:621–634, 1991.

[172] I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series, and Products*, volume 7. 1994.

[173] Harald Dorn and H. J. Otto. Two and three point functions in Liouville theory. *Nucl. Phys.*, B429:375–388, 1994.

[174] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. Structure constants and conformal bootstrap in Liouville field theory. *Nucl. Phys.*, B477:577–605, 1996.

[175] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. Liouville field theory on a pseudosphere. pages 280–299, 2001.

[176] Bruno Le Floch and Gustavo J. Turiaci. AGT/$\mathbb{Z}_2$. *JHEP*, 12:099, 2017.

[177] Danny Birmingham, Ivo Sachs, and Sergey N. Solodukhin. Conformal field theory interpretation of black hole quasinormal modes. *Phys. Rev. Lett.*, 88:151301, 2002.

Vita

Name:
Animik Ghosh

Place of Birth:
Kolkata, West Bengal, India

Education:

- M.Sc. in Physics,
  August 2012 - July 2014
  Study focus: Theoretical Physics
- B.Sc. (Hons.) Physics
  St. Stephens College
  University of Delhi, New Delhi, India
  July 2009 - June 2012

Professional Experience:

- Research Assistant
  Department of Physics & Astronomy
  University of Kentucky, Lexington
  June 2018 - July 2019, May 2020-present
- Dean's Fellow for the College of Arts and Sciences
  Department of Physics & Astronomy
  University of Kentucky, Lexington
  August 2019 - December 2019
- Teaching Assistant
  Department of Physics & Astronomy
  University of Kentucky, Lexington
  August 2015 - May 2018, January 2020-May 2020
- Visiting Research Student
  Department of Theoretical Physics
  Tata Institute of Fundamental Research, Mumbai
  August 2014 - May 2015

Academic honors:

- Graduate Fellowship, Kavli Institute for Theoretical Physics, University of California, Santa Barbara, Fall 2019 (Advisor: David J. Gross).
- Dean's Competitive Fellowship, University of Kentucky, Fall 2019.

- Max Steckler Fellowship, University of Kentucky, Spring 2017.
- Virendra Kumar Scholarship for securing highest percentage of marks in M.Sc in St. Stephens College, University of Delhi, 2014.
- Tokyo Mitsubishi UFJ Scholarship for Academic Excellence, 2012.
- Usha (India) Ltd. Prize/medal for the best pre final physics (Hons.) student, St. Stephens College, University of Delhi, 2011.
- Shrimati Kamla Bajaj prize/medal for the best student of Physics, St. Stephens College, University of Delhi, 2011.
- Ramesh Goel Memorial prize/medal for best student in second year, St. Stephens College, University of Delhi, 2010.
- Tushar Nagia prize/medal for securing the highest percentage of marks in First year of B.Sc, St. Stephens College, University of Delhi, 2010.
- Sumitomo-St. Stephens Scholarship for Academic Excellence, 2009.
- INSPIRE Scholarship for being in the top 1 per cent bracket in board examinations throughout India, 2009.
- Samarpan Basu award for securing the highest marks in Physics, Chemistry and Mathematics in Class 12 board examinations, Calcutta Boys' School, 2009.

Publications:

- S.R. Das, **A. Ghosh**, A. Jevicki, and K. Suzuki. Duality in the Sachdev-Ye-Kitaev model. In: *Springer Proc.Math.Stat. 255 (2017) 43-61* , 2017. doi:10.1007/978-981-13-2179-5_4
- S.R. Das, **A. Ghosh**, A. Jevicki, and K. Suzuki. Three Dimensional view of arbitrary q SYK models. *Journal of High Energy Physics.* 2018. doi:10.1007/JHEP02(2018)162
- S.R. Das, **A. Ghosh**, A. Jevicki, and K. Suzuki. spacetime in the SYK model. *Journal of High Energy Physics.* 2018. doi:10.1007/JHEP07(2018)184
- **A. Ghosh**, H. Maxfield, and G.J. Turiaci. A universal Schwarzian sector in two-dimensional Conformal Field Theories. In: *Journal of High Energy Physics.* 2020. doi:10.1007/JHEP05(2020)104 (2020)
- S.R. Das, **A. Ghosh**, A. Jevicki, and K. Suzuki. Near Conformal Perturbation Theory in SYK Type Models. In: *arXiv e-print: 2006.13149* (2020)