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Geometry of Linear Subspace Arrangements with Connections to Matroid Theory

William Trok
University of Kentucky, billtrok@gmail.com
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William Trok, Student

Dr. Uwe Nagel, Major Professor

Dr. Peter Hislop, Director of Graduate Studies

Geometry of Linear Subspace Arrangements with Connections to Matroid Theory

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy in the
College of Arts and Sciences at the
University of Kentucky

By
William Trok
Lexington, Kentucky

Director: Uwe Nagel, Professor of Mathematics
Lexington, Kentucky
2020

ABSTRACT OF DISSERTATION

Geometry of Linear Subspace Arrangements with Connections to Matroid Theory

This dissertation is devoted to the study of the geometric properties of subspace configurations, with an emphasis on configurations of points. One distinguishing feature is the widespread use of techniques from Matroid Theory and Combinatorial Optimization. In part we generalize a theorem of Edmond's about partitions of matroids in independent subsets. We then apply this to establish a conjectured bound on the Castelnuovo-Mumford regularity of a set of fat points.

We then study how the dimension of an ideal of point changes when intersected with a generic fat subspace. In particular we introduce the concept of a "very unexpected hypersurface" passing through a fixed set of points Z. We show in certain cases these can be characterized via combinatorial data and geometric data from the Hyperplane Arrangement dual to Z. This generalizes earlier results on unexpected curves in the plane due to Faenzi, Vallés [FV14], Cook, Harbourne, Migliore and Nagel [CHMN18].

KEYWORDS: Mathematics, Algebraic Geometry, Commutative Algebra, Subspace Configurations, Fat Points

William Trok

July 15, 2020

Geometry of Linear Subspace Arrangements with Connections to Matroid Theory

By
William Trok

Uwe Nagel
Director of Dissertation

Peter Hislop
Director of Graduate Studies

July 15, 2020
Date

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Chapter 1 Introduction

This is a thesis at the intersection of algebra, geometry and combinatorics. In broad terms we study the algebro-geometric properties of configurations of linear subspaces with a special focus on configurations of points. A key feature of this thesis is the use of techniques from Matroid theory and Combinatorial Optimization.

Linear subspaces are some of the simplest objects in geometry. Yet despite their elementary nature many basic questions about their properties remain open even if we limit our study to finite sets of points. A classical problem in both pure and applied mathematics is the problem of polynomial interpolation. A polynomial interpolation problem gives data about the value of a function $f : \mathbb{C}^n \to \mathbb{C}$ at a finite set of points $\{P_1, .., P_k\}$ and asks to the find the polynomial $q \in \mathbb{C}[X_1, .., X_n]$ of minimal degree which fits that data, meaning that $q(P_i) = f(P_i)$. In general it can be difficult to determine q, and it's degree can depend heavily on the data as well as the position of the points in space. A closely related geometric problem is to determine the Hilbert Function of the graded ideal I_Z associated to a finite set of points in Z in the projective space $\mathbb{P}^n_{\mathbb{C}}$.

In the more general context of Hermite Polynomial interpolation, we are given data about the values of a function $f : \mathbb{C}^n \to \mathbb{C}$ and the values of some partial derivatives of f at a finite set of points $\{P_1, .., P_k\}$. We again want to find a polynomial $q \in \mathbb{C}[X_1, .., X_n]$ of minimal degree whose value and partial derivatives agree with f at each point $\{P_1, .., P_k\}$. The related geometric problem becomes determining the Hilbert Function of the graded ideal associated to a set of *fat points* in $\mathbb{P}^n_{\mathbb{C}}$. Here the ideal of a *fat point* P *of multiplicity* m is the ideal consisting of those polynomial which vanishing at P and where all partial derivatives of order less than m also vanish at P.

Chapter 2 provides an expository account of the connections between fat points and Hermite Interpolation problems. It also makes explicit a connection between the *Interpolation Degree* and the *Castelnuovo-Mumford regularity*, a well known geometric invariant(see theorem 2.3.7). This gives the Castelnuovo-Mumford regularity practical as well as theoretical interest.

In chapter 3, we recall some background from Matroid theory and collect the necessary results on Matroid Theory and Optimization that are used in the rest of the thesis. In particular, theorem 3.3.11 gives a generalization of the celebrated "Matroid Partition Theorem" due to Edmonds [Edm65]. This generalization is a key tool in the results of the next chapter.

Chapter 4 focuses on providing upper bounds on the Castelnuovo-Mumford regularity for fat point subschemes of $\mathbb{P}^n$. Namely, for our fat point scheme X, the value $\operatorname{reg}(X)$ is bounded above by $1 + \operatorname{Seg}(X)$ where

$$\operatorname{Seg}(X) := \max\left\{ \left\lceil \frac{-1 + \sum_{P_i \in L} m_i}{\dim L} \right\rceil \mid L \subseteq \mathbb{P}^n \text{ a linear subspace with } \dim L > 0 \right\}.$$

Theorem 4.3.3. *If* $X = \sum_{i=1}^{s} m_i P_i$ *is any fat point subscheme of* $\mathbb{P}^n$*, then* $r(X) = \operatorname{reg}(X) - 1 \leq \operatorname{Seg}(X)$*.*

This establishes a conjecture due to Trung ([Thi00]) and independently, Fatabbi and Lorenzini ([FL01]). A key tool in the proof is theorem 3.3.11. We then continue to provide a generalization of the Segre Bound, in the form of a series of bounds.

In chapter 5, we take a set of points $Z \subseteq \mathbb{P}^n$ and a general codimension 2 linear subspace $Q \subseteq \mathbb{P}^n$ and study how $\dim[I(Z) \cap I(Q)^{d-1}]_d$ relates to $\dim[I(Z)]$. In particular we introduce the concept of a *very unexpected hypersurface*. In rough terms a set of points Z admits *very unexpected* Q*-hypersurfaces* in degree d if the intersection of $[I(Z)]_d$ and $[I(Q)^{d-1}]_d$ is larger than a naive dimension count would suggest, and the difference in dimension is not "easily explained". A key technique is a new duality between $I(Z)$ and the derivation bundle $D_0(\mathcal{A}_Z)$ of the hyperplane arrangement $\mathcal{A}_Z$ dual to Z. This new duality builds upon a duality due to [FV14]. We show that the degree's in which a set of points admits very unexpected hypersurfaces can be determined from combinatorial information and basic information about $D_0(\mathcal{A}_Z)$. In particular to each set of points $Z \subseteq \mathbb{P}^n$ and degree $d \geq 1$ we define an integer $\operatorname{Ex.C}(Z, d)$ via a combinatorial optimization problem. Armed with this definition the following result holds.

Theorem 5.4.23. *Let* $Z \subseteq \mathbb{P}^n$ *be a finite set of points, and suppose that* $D_0(\mathcal{A}_Z)$ *has splitting type* $(a_1, .., a_n)$*. Then for a fixed integer* d*,*

$$\sum_{i=1}^{n} \max\{0, d - a_i\} \leq \quad nd + 1 - \operatorname{Ex.C}(Z, d)$$

and the inequality is strict if and only if Z *admits very unexpected hypersurfaces in degree* d*.*

This gives a direct generalization of much of [CHMN18], which characterizes the degrees of unexpected curves in $\mathbb{P}^2$.

We then continue establishing some further results on the structure of unexpected curves in $\mathbb{P}^2_{\mathbb{C}}$. We close by giving some applications of these structural results to Terao's Freeness Conjecture on Line Arrangements.

Chapter 2 The Basic Objects of Study

In this chapter we fix notation, and the basic objects which we will study. This thesis is largely devoted to geometric problems, namely the study of subspace configurations of projective space. However, we choose to use the algebraic language of graded modules over graded rings and graded modules as opposed to the geometric language of sheaves and schemes.

A recurring theme throughout this dissertation is the study of Castelnuovo-Mumford Regularity of fat points. In section 2.3 we give an expository account of the Castelnuovo-Mumford regularity for fat points and it's relationship with Hermite Interpolation Problems.

2.1 Notation and Conventions

In this section we fix some notation which will be used throughout this thesis.

Convention 2.1.1. All rings are commutative and Noetherian with identity. Here we let $\mathbb{N}$ denote the set of *non-negative* integers, in accordance with ISO standard 80000-2-7.1, this is the set of all positive integers and 0. If $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_k) \in \mathbb{N}^k$ is a integer vector we define

$$\|\boldsymbol{\alpha}\|_1 = \sum_{j=1}^{k} \alpha_j.$$

Definition 2.1.2 (Graded Rings and Modules)**.** A $\mathbb{Z}$-*graded ring* is a ring R together with a decomposition of R as a direct sum of abelian groups $R = \oplus_{i\in\mathbb{Z}}[R]_i$, so that $[R]_i \cdot [R]_j \subseteq [R]_{i+j}$.

For $f \in R$ we say that f is a *homogeneous element*, if $f \in [R]_d$ for some d. For any homogeneous $f \in R \setminus 0$, we define the degree $\deg(f)$ to be the integer d where $f \in [R]_d$.

Example 2.1.3. An example of a graded ring which we will use frequently is the polynomial ring over a field $\mathbb{K}$, namely $R = \mathbb{K}[X_0, .., X_n]$. In this case $[R]_d$ is the $\mathbb{K}$-vector space of homogeneous polynomials of degree d. This can be given a specific $\mathbb{K}$-basis which we construct now.

Given a vector $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ we define a monomial in R, via

$$X^{\boldsymbol{\alpha}} := X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n}.$$

Note that $\deg(X^{\boldsymbol{\alpha}}) = \|\boldsymbol{\alpha}\|_1 = \sum_{i=0}^{n} |\alpha_i|$. The set of monomials $\{X^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{N}^k \text{ with } \|\boldsymbol{\alpha}\|_1 = d\}$ forms a $\mathbb{K}$ basis for $[R]_d$. In particular, every $f \in [R]_d$ can be written uniquely as

$$f = \sum_{\|\boldsymbol{\alpha}\|_1 = d} c_{\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}.$$

Similarly if $S = R/I$ is a quotient ring of $\mathbb{K}[X_0, X_1, \dots, X_n]$, we define $\overline{X}^{\boldsymbol{\alpha}} \in R$ as the image of $X^{\boldsymbol{\alpha}}$ in R. More generally given $f \in R$ we use $\overline{f}$, to denote the image of f in S.

Definition 2.1.4 (Graded Modules)**.** A *graded module* over a graded ring R is an R-module M together with a decomposition of $M = \oplus_{j \in \mathbb{Z}} [M]_j$. We further require the action of R on M to respect the grading on M, meaning that for all $f \in [R]_i$ and $m \in [M]_j$ we have $fm \in [M]_{i+j}$.

We similarly say that $m \in M$ is *homogeneous of degree* d if $m \in [M]_d$.

Remark 2.1.5. It can be assumed for this thesis, that any module over a graded ring is graded unless we explicitly state otherwise.

Definition 2.1.6 (Homogeneous Ideals and Graded Submodules)**.** For a graded module M over R, a *graded submodule* $N \subseteq M$ is an R-submodule of M which is generated by its homogeneous elements. That is $N = \oplus_{i \in \mathbb{Z}} N \cap [M]_i$. In this case we set $[N]_d = N \cap [M]_d$.

A graded submodule of R is referred to as a *homogeneous ideal.*

Remark 2.1.7. In this thesis all graded rings R will have $[R]_0 = \mathbb{K}$ a field. We will further require that R is generated over $\mathbb{K}$ by $[R]_1$, which we require to be a finite dimensional $\mathbb{K}$ vector space. In particular this ensures that $\dim_{\mathbb{K}} [R]_d$ is finite. In this scenario there is a unique maximal homogeneous ideal $\mathfrak{m}$ which consists of the positively graded elements, namely $\mathfrak{m} = \oplus_{i \geq 1} [R]_i$.

Definition 2.1.8 (Hilbert Functions)**.** If R is a graded ring so $[R]_0 = \mathbb{K}$ is a field, then its *Hilbert Function* is given by

$$HF_R(d) := \dim_{\mathbb{K}} [R]_d.$$

More generally, if M is a graded R-module we define the *Hilbert Function* of M as

$$HF_M(d) := \dim_{\mathbb{K}} [M]_d.$$

If $[R]_1$ is finite dimensional and generates R as a $\mathbb{K}$-algebra then for any finitely generated module M there's a polynomial $HP_M(d)$ known as the *Hilbert Polynomial of* M so that for all sufficiently large d we have

$$HF_M(d) = HP_M(d).$$

Local Cohomology and in particular it's relationship with Castelnuovo-Mumford Regularity will be an important tool in some parts of this thesis. For this reason we recall these definitions now.

Definition 2.1.9 (Local Cohomology)**.** Let $R = \mathbb{K}[X_0, .., X_n]$ and $I \subseteq R$ a homogeneous ideal, then the j-th local cohomology module of a graded R-module M, denoted $H^j_I(M)$, is the j-th right derived functor of the I-torsion functor, Γ_I, defined on objects as

$$\Gamma_I(M) := \{m \in M \mid I^k m = 0 \text{ for some } k \geq 1\}.$$

Remark 2.1.10. A particularly important variant of local cohomology on $\mathbb{K}[X_0, .., X_n]$ is the local cohomology with respect to the maximal ideal $\mathfrak{m} = (X_0, .., X_n)$. This is in part due to it's relationship with sheaf cohomology. Namely for any finitely generated module M over $\mathbb{K}[X_0, .., X_n]$ we get a corresponding sheaf of modules $\widetilde{M}$ over the structure sheaf of $\mathbb{P}^n_{\mathbb{K}}$. Then there's an isomorphism natural in M for all $i > 1$

$$[H^i_{\mathfrak{m}}(M)]_d \cong H^{i-1}(\mathbb{P}^n, \widetilde{M}(d)).$$

The modules $H^0_{\mathfrak{m}}(M)$ and $H^1_{\mathfrak{m}}(M)$ are connected to $H^0(\mathbb{P}^n, \widetilde{M}(d))$ via the following exact sequence of graded modules

$$\begin{tikzcd} 0 \arrow[r] & {[H^0_{\mathfrak{m}}(M)]_d} \arrow[r] & {[M]_d} \arrow[r] & H^0(\mathbb{P}^n, \widetilde{M}(d)) \arrow[r] & {[H^1_{\mathfrak{m}}(M)]_d} \arrow[r] & 0 \end{tikzcd}.$$

Definition 2.1.11 (Castelnuovo-Mumford Regularity)**.** Let M be a finitely generated graded module over a graded ring R. The *Castelnuovo-Mumford Regularity* of M is the integer $\operatorname{reg}(M)$ defined as

$$\operatorname{reg}(M) := \max\{i + j : [H^i_{\mathfrak{m}}(M)]_j \neq 0\}$$

We recall a few results on local cohomology which will be used in the sequel.

Theorem 2.1.12. *If M is a finitely generated graded module over a graded local ring R, then*

$$H^i_{\mathfrak{m}}(M) = 0$$

for all $i < \operatorname{depth}(M)$ and all $i > \dim M$. Here $\dim(M)$ refers to krull dimension of $\operatorname{Spec}(R/\operatorname{Ann}(M))$. Furthermore,

$$H^j_{\mathfrak{m}}(M) \neq 0$$

for $j \in \{\operatorname{depth}(M), \dim M\}$.

Proof. See theorem 3.5.7 of [BH98]. ☐

Theorem 2.1.13 (Hirzeburch-Riemann-Roch for Graded Modules)**.** *If M is a graded module over the standard graded polynomial ring $R = \mathbb{K}[X_0, .., X_n]$ then*

$$HP_M(d) - \dim[M]_d = \sum_{i=0}^{n} (-1)^{i+1} \dim_{\mathbb{K}}[H^i_{\mathfrak{m}}(M)]_d.$$

Proof. See theorem 4.4.3 of [BH98]. ☐

2.2 Fat Linear Subspace Schemes

Here we recall the concept of a *fat* linear subspace of $\mathbb{P}^n$, and recall some basic objects which are associated to it.

Definition 2.2.1. A (nonempty) linear subspace $L \subseteq \mathbb{P}^n_{\mathbb{K}}$ is the image of a nonzero linear subspace $c(L) \subseteq \mathbb{K}^{n+1}$ under the quotient map $\mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

Definition 2.2.2 (Ideal Associated to a Linear Subspace)**.** Given a subspace $L \subseteq \mathbb{P}^n$, there is a corresponding homogeneous ideal $I_L \subseteq \mathbb{K}[X_0, ..X_n]$. $[I_L]_d$ consists of all homogeneous degree d forms f where $f(P) = 0$ for all $P \in L$.

Definition 2.2.3 (Fat Subspace Scheme)**.** A *Fat Subspace* of $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$ is a subspace $L \subseteq \mathbb{P}^n$ together with a multiplicity $m \in \mathbb{N}$. We denote this using multiplicative notation mL. To each fat subspace mL we associate a homogeneous ideal

$$I(mL) = I(L)^m.$$

More generally to a collection of Fat Subspaces, H, written using additive notation as $H = m_1L_1 + \ldots + m_kL_k$ we associate the homogeneous ideal

$$I\left(\sum_{i=1}^{k} m_iL_i\right) = \bigcap_{i=1}^{k} I(L_i)^{m_i}$$

Remark 2.2.4. $I(mL)$ has a well known geometric interpretation, namely it consists of those polynomials f which vanish at all points $p \in L$ with multiplicity m. If $\dim L = d$ and $S = \mathbb{K}[X_0, \ldots, X_n]$ is the projective coordinate ring we may up to change in coordinates assume that $I(L) = (X_{d+1}, \ldots, X_n)$, then for $f \in [S]_k$ we have that $f \in I(L)^m$ if and only if f can be written as

$$f = \sum_{\substack{\|\boldsymbol{\alpha}\|_1 = k \\ \sum_{j=d+1}^{n} \alpha_j \geq m}} c_{\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}.$$

If $\mathrm{Char}(\mathbb{K}) = 0$, then these are the polynomials f which vanish at p and where all derivatives of f of order $k < m$ also vanish at p.

As an illustration $f \in I(2L) \subseteq \mathbb{K}[X_0, .., X_n]$ if for all $p \in L$ we have $f(p) = 0$ and $\frac{\partial f}{\partial X_i}(p) = 0$ for all $i = 0, 1, .., n$.

Example 2.2.5. Consider the coordinate points $E_0, E_1, E_2 \subseteq \mathbb{P}^2_{\mathbb{K}}$, which have projective coordinates $(1 : 0 : 0), (0 : 1 : 0)$ and $(0 : 0 : 1)$. Then $I(E_j) \subseteq \mathbb{K}[X_0, X_1, X_2]$ is the ideal generated by $(X_i \mid i \in \{0, 1, 2\} \setminus \{j\})$, for instance $I(E_0) = (X_1, X_2)$. It can then be computed that

$$I(E_0 + E_1 + E_2) = (X_1X_2, X_0X_2, X_0X_1).$$

Similarly, $I(E_0)^2 = (X_1^2, X_1X_2, X_2^2)$ and so

$$I(2E_0 + 2E_1 + 2E_2) = (X_0X_1X_2, X_1^2X_2^2, X_0^2X_2^2, X_0^2X_1^2).$$

Definition 2.2.6 (Castelnuovo-Mumford Regularity of Subschemes)**.** If $X \subseteq \mathbb{P}^n$ is a closed subscheme we define the Castelnuovo-Mumford Regularity of X denoted $\mathrm{reg}(X)$, as the regularity of the ideal $I(X)$.

2.3 The Regularity of Fat Points

As the Castelnuovo-Mumford Regularity is important in this paper, and in fact the main focus of chapter 4 we give an elementary interpretation of $\operatorname{reg}(Z)$ for the case that $Z \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a fat point subscheme. This interpretation is well known to experts, though not mentioned much in the literature. One notable place were this appears is chapter 4 of [Eis05], however there the scope is limited to subschemes of simple points. To keep this as elementary as possible we commit ourselves to working over $\mathbb{C}$, though we will remark when results go through over any field.

Definition 2.3.1 (Hermite Interpolation)**.** If $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{C}^k$ is a set of fat points, a *Hermite Interpolation Problem* on Z associates to each pair $(P_i, \boldsymbol{\alpha})$ where P_i is one of the points in Z and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$ is a sequence of non-negative integers which satisfy $\|\boldsymbol{\alpha}\|_1 = \sum_{j=1}^{k} \alpha_j < m_i$ a value $C_{P_i,\boldsymbol{\alpha}} \in \mathbb{C}$. It then asks to find a polynomial $f \in \mathbb{C}[X_1, .., X_k]$ of minimal degree so that for each pair $(P_i, \boldsymbol{\alpha})$ we have

$$\frac{\partial^{\|\boldsymbol{\alpha}\|_1} f}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2} \ldots \partial X_k^{\alpha_k}}(P_i) = C_{P_i,\boldsymbol{\alpha}}.$$

In order to simplify notation we use the shorthand $\partial_{\boldsymbol{\alpha}} f$ for $\dfrac{\partial^{\|\boldsymbol{\alpha}\|_1} f}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2} \ldots \partial X_k^{\alpha_k}}$. The *interpolation degree* of Z is the smallest integer $\operatorname{int.deg}(Z)$ so that every Hermite Interpolation problem on Z has a solution f with $\deg(f) \leq \operatorname{int.deg}(Z)$.

We note that given any hyperplane H such as $X_0 = 0$ in $\mathbb{P}^n$ we have that there is an isomorphism of varieties $\mathbb{C}^n \cong \mathbb{P}^n_{\mathbb{C}} \setminus H$. Since for any fat point subscheme $Z \subseteq \mathbb{P}^n$ we can find a hyperplane $H \subseteq \mathbb{P}^n$ which avoids the points in Z, this gives a correspondence between fat point subschemes of $\mathbb{C}^n$ and $\mathbb{P}^n$ (though only up to projective equivalence). For instance if H is the hyperplane defined by $X_0 = 0$, then ι_H is the map

$$\iota_H(a_1, a_2, \ldots, a_n) = (1 : a_1 : a_2 : \ldots : a_n).$$

This allows us to include $\mathbb{C}^n \subseteq \mathbb{P}^n$. Which allows us to think of fat points $Z \subseteq \mathbb{C}^n$ as a subscheme of $\mathbb{P}^n$. Furthermore for every fat points scheme $Z \subseteq \mathbb{P}^n$ there is a hyperplane H where $H \cap Z = \emptyset$ and this allows us to take $Z \subseteq \mathbb{P}^n \setminus H \cong \mathbb{C}^n$. Essentially, we can think of a fat point scheme $Z \subseteq \mathbb{P}^n$ as equally lying in $\mathbb{C}^n$ and vice versa.

Definition 2.3.2 (Filtered Rings)**.** Let $R = \mathbb{C}[X_1, .., X_n]$ thought of as a standard graded ring. If $I \subseteq R$ is a not necessarily homogeneous ideal, then R/I inherits a *filtration* from the graded structure of R.

Let $[R]_{\leq d}$ denote the vector space of polynomials of degree at most d. Given a not necessarily homogeneous ideal $I \subseteq R$ we let $[R/I]_{\leq d}$ denote the image of $[R]_{\leq d}$ in $[R/I]$ under the canonical quotient map $R \to R/I$. We note that since $[R/I]_{\leq d} \cdot [R/I]_{\leq e} \subseteq [R/I]_{\leq d+e}$ this gives R/I the structure of what is often called a *filtered algebra*.

This filtration $[R/I(Z)]_{\leq d}$ allows us to give a new interpretation of the interpolation degree, $\operatorname{int.deg}(Z)$, of a set of fat points $Z \subseteq \mathbb{C}^n$.

Proposition 2.3.3. *Let* $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{C}^n$ *be a set of fat points. Then the vector space dimension of* $[R/I(Z)]$ *is equal to* $\sum_{i=0}^{s} \binom{n+m_i-1}{n}$*. Furthermore,*

$$\operatorname{int.deg}(Z) = \min\left\{ d \in \mathbb{Z} \,\middle|\, \dim_{\mathbb{C}}[R/I(Z)]_{\leq d} = \sum_{i=0}^{s} \binom{n+m_i-1}{n} \right\}.$$

Proof. We first claim that $f \in I(Z)$ if and only if $f(P_i) = 0$ for all $i \in \{0, 1, \ldots, s\}$ and for each P_i and all nonnegative integer vectors $\boldsymbol{\alpha} \in \mathbb{Z}^n$ with $\|\boldsymbol{\alpha}\|_1 < m_i$ we have $\partial_{\boldsymbol{\alpha}} f(P_i) = 0$. Where we use the notation $\partial_{\boldsymbol{\beta}}$ as shorthand for $\dfrac{\partial^{\|\boldsymbol{\beta}\|_1}}{\partial X_1^{\beta_1} \ldots \partial X_n^{\beta_n}}$. To establish this fix a point $P_i = (P_{i,1}, \ldots, P_{i,n})$ and write write

$$f = \sum_{\|\boldsymbol{\gamma}\|_1 \leq \deg(f)} c_{\boldsymbol{\gamma}} (X_1 - P_{i,1})^{\gamma_1} (X_2 - P_{i,2})^{\gamma_2} \ldots (X_n - P_{i,n})^{\gamma_n}.$$

When written in this form we note that $\partial_{\boldsymbol{\beta}} f(P_i) = c_{\boldsymbol{\beta}} \prod_{i=1}^{n} (\beta_i!)$, and that $f \in I(P_i)^d$ if and only if $\boldsymbol{\beta}$.

We now proceed to establish the claim that $\dim_{\mathbb{C}}[R/I(Z)] = \sum_{i=0}^{s} \binom{n+m_i-1}{n}$. We proceed by induction on $d = \sum_{i=0}^{s} m_i$. The case $d = 1$ is an easy computation. Given our $Z = \sum_{i=0}^{s} m_i P_i$ we suppose the result holds for $Z' = (m_t - 1)P_t + \sum_{\substack{i=0 \\ i \neq j}}^{s} m_i P_i$. We then construct for each $d \in \{0, \ldots, s\}$ and each $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}\|_1 \leq m_t - 1$ a polynomial $F_{t,\boldsymbol{\alpha}}$ where for each pair $(P_i, \boldsymbol{\beta})$ with $\|\boldsymbol{\beta}\|_1 < m_i$ and $P_i \neq P_t$ we have

$$\partial_{\boldsymbol{\beta}} F_{t,\boldsymbol{\alpha}}(P_i) = 0$$

and where for all $\boldsymbol{\beta}$ with $\|\boldsymbol{\beta}\|_1 \leq \|\boldsymbol{\alpha}\|_1$ we have

$$\partial_{\boldsymbol{\beta}} F_{t,\boldsymbol{\alpha}}(P_t) = \delta_{\boldsymbol{\beta},\boldsymbol{\alpha}}$$

where δ is the Kronecker delta. Once the existence of the $F_{t,\boldsymbol{\alpha}}$ is established we note that the representative of $F_{t,\boldsymbol{\alpha}}$ necessarily form a basis of $[I(Z')/I(Z)]$ of cardinality $\binom{n+m_t-2}{n-1}$. This is because by our characterization in terms of partial derivatives we have that the $F_{t,\boldsymbol{\alpha}}$ are linear independent mod $I(P_t)^{m_t}$ and for arbitrary $g \in I(Z')$ we have $g - \sum_{|\boldsymbol{\alpha}|=m_t-1} F_{t,\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} g(P_t) \in I(P_t)^{m_t}$. Hence, it follows then that $\dim[R/I(Z)] = \binom{n+m_t-2}{n-1} + \dim[R/I(Z')] = \binom{n+m_t-2}{n-1} + \binom{n+m_t-2}{n} + \sum_{i \neq j} \binom{n+m_i-1}{n} = \sum_{i=0}^{s} \binom{n+m_i-1}{n}$. Thus establishing the result, once we have the existence of the $F_{t,\boldsymbol{\alpha}}$.

Fixing P_t and $\boldsymbol{\alpha}$ we proceed to constructing $F_{t,\boldsymbol{\alpha}}$. For any point P_j let $p_{j,k}$ denote the K-th coordinate of P_j, so $p_{j,k} = X_k(P_j)$. Now for each point $P_j \neq P_t$ we can find some index k so that $p_{t,k} \neq p_{j,k}$ then setting $\ell_j = \dfrac{X_k - p_{j,k}}{p_{t,k} - p_{j,k}}$ we see that $\ell_j(P_j) = 0$ and $\ell_j(P_t) = 1$.

We now claim that

$$F_{t,\boldsymbol{\alpha}} := \left(\prod_{i=1}^{n} \frac{(X_i - p_{t,i})^{\alpha_i}}{(\alpha_i)!}\right)\left(\prod_{\substack{j=0\\ j\neq t}}^{s} \ell_j^{m_j}\right)$$

has the desired property. First we note that for all $P_i \neq P_t$ that as $\ell_i^{m_i}$ divides $F_{t,\boldsymbol{\alpha}}$ we have letting $G = F_{t,\boldsymbol{\alpha}}/(\ell_i^{m_i})$ that by the product rule for derivatives that

$$\partial_{\boldsymbol{\beta}} F_{t,\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}+\boldsymbol{\lambda}=\boldsymbol{\beta}} (\partial_{\boldsymbol{\gamma}} G)(\partial_{\boldsymbol{\lambda}} \ell_i^{m_i})$$

where the summation is over all nonegative integer vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ with $\boldsymbol{\gamma} + \boldsymbol{\lambda} = \boldsymbol{\beta}$. Using this expression we see that for all relevant $\boldsymbol{\lambda}$ that $\partial_{\boldsymbol{\lambda}} \ell_i^{m_i}(P_i) = 0$ and so $\partial_{\boldsymbol{\beta}} F_{t,\boldsymbol{\alpha}}(P_i) = 0$.

We similarly note that for any $\boldsymbol{\beta}$ with $\|\boldsymbol{\beta}\|_1 \leq \|\boldsymbol{\alpha}\|_1$ we have

$$\partial_{\boldsymbol{\beta}} F_{t,\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}+\boldsymbol{\lambda}=\boldsymbol{\beta}} \partial_{\boldsymbol{\gamma}} \left(\prod_{i=1}^{n} \frac{(X_i - p_{t,i})^{\alpha_i}}{(\alpha_i)!}\right) \partial_{\boldsymbol{\lambda}} \left(\prod_{\substack{j=0\\ j\neq t}}^{s} \ell_j^{m_j}\right).$$

Evaluating the above expression at P_t we see the term $\partial_{\boldsymbol{\gamma}}\left(\prod_{i=1}^{n} \frac{(X_i-p_{t,i})^{\alpha_i}}{(\alpha_i)!}\right)$ evaluates to 0 unless $\boldsymbol{\gamma} = \boldsymbol{\alpha}$, as $\|\boldsymbol{\gamma}\|_1 \leq \|\boldsymbol{\beta}\|_1 \leq \|\boldsymbol{\alpha}\|_1$ this occurs if and only if $\boldsymbol{\beta} = \boldsymbol{\gamma} = \boldsymbol{\alpha}$. As $\ell_i^{m_i}(P_t) = 1$ the rest now follows.

We now continue to establishing the statement about $\operatorname{int.deg}(Z)$. Note that if $\dim[R/I(Z)]_{\leq d} < \sum_{i=0}^{s} \binom{n+m_i-1}{n}$ then since $\dim R/I(Z) = \sum_{i=0}^{s} \binom{n+m_i-1}{n}$ there exists some nonzero $f \in R/I(Z)$ of minimial degree so that $f \notin \dim[R/I(Z)]_{\leq d}$. Then for every $g \in [R]_{\leq d}$ we have $f - g \notin I(Z)$ hence there is no polynomial of degree at most d so that

$$\partial_{\boldsymbol{\beta}} g(P_i) = \partial_{\boldsymbol{\beta}} f(P_i)$$

for all pairs $(P_i, \boldsymbol{\beta})$ with $\|\boldsymbol{\beta}\|_1 < m_i$. In particular this says that any solution Hermite Interpolation problem with values $C_{i,\boldsymbol{\alpha}} = \partial_{\boldsymbol{\beta}} f(P_i)$ has necessarily has degree larger than d, and so $\operatorname{int.deg}(Z) > d$.

Conversely, assume that $\dim[R/I(Z)]_{\leq d} = \sum_{i=0}^{s} \binom{n+m_i-1}{n}$, then by dimension counting $[R/I(Z)]_{\leq d} = R/I(Z)$. Then given a Hermite Interpolation Problem we know that there is a solution $f \in R$. Since for our chosen d we have $\dim[R/I(Z)]_{\leq d} = R/I(Z)$ it follows that theres a polynomial $g \in [R]_{\leq d}$ with $g - f \in I(Z)$ which implies that g is also a solution to the Hermite Interpolation Problem. Therefore, $\operatorname{int.deg}(Z) \leq d$. Together with the previous result this establishes the stated equality. $\square$

Definition 2.3.4 (Homogenization)**.** Let $S = \mathbb{K}[X_0, X_1, \ldots, X_n]$ then there are $\mathbb{K}$-linear maps $\operatorname{Hmg}_d : [R]_{\leq d} \to [S]_d$ which maps a polynomial $F(X_1, \ldots, X_n)$ to $X_0^d F(X_1/X_0, X_2/X_0, ..., X_n/X_0)$. Alternatively, given a monomial $X_1^{e_1} \ldots X_n^{e_n}$ with $e_1 + \ldots + e_n \leq d$, we define

$$\operatorname{Hmg}_d(X_1^{e_1} \ldots X_n^{e_n}) := X_0^t X_0^{e_1} X_2^{e_2} \ldots X_n^{e_n}$$

where $t := d - \sum_{i=1}^{n} e_i$, and extend Hmg_d linearly to all polynomials.

Given our ideal I the *homogenization* of I is denoted ${}^hI$, and is the homogeneous ideal where $[{}^hI]_d = \mathrm{Hmg}_d(I \cap [R]_{\le d})$.

Proposition 2.3.5. *Using the notation above, and given a nonhomogeneous ideal $I \subseteq R$ we have*

$$\dim_{\mathbb{K}}[R/I]_{\le d} = \dim[S/{}^hI]_d.$$

Proof. As the map $\mathrm{Hmg}_d : [R]_{\le d} \to [S]_d$ is bijective, and we note that by definition $[{}^hI]_d$ is the image of $[I]_{\le d}$ under Hmg_d. It follows that Hmg_d induces an isomorphism $[R/I]_{\le d} \to [S/{}^hI]_d$. □

One corollary of the preceding proposition is that the Hilbert function $\mathrm{HF}_{S/I(Z)}(d)$ is non-decreasing.

Corollary 2.3.6. *If $I \subseteq R$ is any ideal, and ${}^hI$ it's homogenization. Then*

$$\dim[S/{}^hI]_d \ge \dim[S/{}^hI]_{d-1}.$$

Proof. This follows since by proposition 2.3.5 we have

$$\dim[S/{}^hI]_d = \dim[R/I]_{\le d} \ge \dim[R/I]_{\le d-1} = \dim[S/{}^hI]_{d-1}.$$

□

Theorem 2.3.7. *Let $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{C}^n$ be a fat point scheme and $\iota : \mathbb{C}^n \to \mathbb{P}^n_{\mathbb{C}}$ the inclusion of $\mathbb{C}^n$ as the complement of some coordinate hyperplane $X_i = 0$. Then*

$$\mathrm{int.deg}(Z) = \mathrm{reg}(\iota(Z)) - 1.$$

Proof. Using proposition 2.3.5 and proposition 2.3.3, we see that it suffices to show

$$\mathrm{reg}(Z) + 1 = \min\left\{ r \;\middle|\; \dim[R/I(Z)]_r = \sum_{i=0}^{s} \binom{n+m_i-1}{n} \right\}.$$

Furthermore by corollary 2.3.6 we see the above equality is equivalent to

$$\mathrm{reg}(Z) = \max\left\{ r \;\middle|\; \dim[R/I(Z)]_r \ne \sum_{i=0}^{s} \binom{n+m_i-1}{n} \right\}.$$

Applying the local cohomology functor to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I(Z) \arrow[r] & R \arrow[r] & R/I(Z) \arrow[r] & 0 \end{tikzcd},$$

we get a long exact sequence in local cohomology. From this and the fact that $\mathrm{reg}(R) = 0$ we can conclude that that $[H^i_{\mathfrak{m}}(I(Z))]_d \cong [H^{i-1}_{\mathfrak{m}}(R/I(Z))]_d$ for all $d \ge -n$ and all $i \ge 1$. It then follows that $\mathrm{reg}(I(Z)) = \mathrm{reg}(R/I(Z)) + 1$.

Applying theorem 2.1.13 and theorem 2.1.12, we conclude that

$$\sum_{i=0}^{s} \binom{n+m_i-1}{n} = \dim[R/I(Z)]_d + [H^1_{\mathfrak{m}}(R/I(Z))]_d.$$

By the definition of regularity and theorem 2.1.12 we have $\operatorname{reg}(R/I(Z)) = \max\{r \mid [H^1_{\mathfrak{m}}(M)]_{r-1} \neq 0\}$ from the above formula we conclude then that

$$\operatorname{reg}(R/I(Z)) = \min\left\{ r \,\middle|\, \dim[R/I(Z)]_r = \sum_{i=0}^{s} \binom{n+m_i-1}{n} \right\}.$$

We conclude then that $\operatorname{reg}(I(Z))-1 = \operatorname{reg}(R/I(Z)) = \operatorname{int.deg}(Z)$ by proposition 2.3.3. □

We close this subsection, examining the well known Lagrange Interpolation formula in this context.

Proposition 2.3.8 (Lagrange Interpolation Formula). *Given a set of simple points $Z = \sum_{i=0}^{s} P_i \subseteq \mathbb{C}^1$, and a Interpolation problem with values $\{C_{i,0}\}$. Then the polynomial of minimal degree interpolating the data is given by*

$$f = \sum_{i=0}^{s} C_{i,0} \prod_{\substack{j=0 \\ j\neq i}}^{s} \frac{x-p_j}{p_i-p_j}.$$

Consequently, for any set of points $Z \subseteq \mathbb{P}^1_{\mathbb{C}}$ we have $\operatorname{reg}(Z) = |Z|$.

Proof. By direct evaluation we see that $f(P_i) = C_{i,0}$. Moreover, if g is any other polynomial with $\deg(g) < \deg(f)$ and $g(P_i) = C_{i,0}$ then $f-g$ is a degree $\leq s$ polynomial which vanishes on $s+1$ points. Since the only polynomial vanishing on $s+1$ points of degree $\leq s$ is the 0 polynomial, it follows that f is the unique polynomial of degree $\leq s$ with $f(P_i) = C_{i,0}$.

We see then that $\operatorname{int.deg}(Z) \leq |Z|-1$, if we take $C_{0,0} = 1$ and $c_{i,0} = 0$ for $i > 1$ we see that $\deg(f) = |Z|-1$ and so it follows that $\operatorname{int.deg}(Z) = |Z|-1$. By theorem 2.3.7 we conclude that for $Z \subseteq \mathbb{P}^1$ that $\operatorname{reg}(Z) = |Z|$. □

2.4 Bounds on Regularity of Fat Point Schemes

Given the many interpretations of the regularity of a fat point scheme, the particular value of $\operatorname{reg}(Z)$ has both theoretical and practical mathematical interest. Unfortunately known methods for computing the regularity of an ideal typically involve Gröbner basis computation which have poor computational complexity. In the context of Hermite interpolation, $\operatorname{reg}(Z)$, is related to the computational complexity of the Hermite Interpolation problem.

Remark 2.4.1. In this section we work over a fixed field $\mathbb{K}$ of arbitrary characteristic.

We note that over an arbitrary field the interpretation of $\operatorname{reg}(Z)$ for a fat point scheme can no longer be stated in terms Hermite Interpolation, since in particular derivatives are poorly behaved in characteristic p. The regularity of fat point schemes over arbitrary fields can still be stated in terms of their Hilbert Functions.

Proposition 2.4.2. *If $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^n_{\mathbb{K}}$ is a fat point scheme then the Hilbert Polynomial of $R/I(Z)$ is a constant*

$$HP_{R/I(Z)}(d) := \sum_{i=0}^{s} \binom{n + m_i - 1}{n}.$$

Furthermore, $\operatorname{reg}(Z)$ *is equal to the integer*

$$\operatorname{reg}(Z) = \min \left\{ r+1 \,\middle|\, \dim[R/I(Z)]_r = \sum_{i=0}^{s} \binom{n + m_i - 1}{n} \right\}.$$

We omit the proof as it is identical to the proof over $\mathbb{C}$.

Often it is more useful in formulas to refer to $\operatorname{reg}(Z) - 1$, as opposed to $\operatorname{reg}(Z)$. We introduce a piece of notation to refer to exactly this.

Definition 2.4.3 (Regularity Index)**.** Let $Z = \sum_{i=0}^{s} m_i P_i$ be a fat point scheme in $\mathbb{P}^n$. Let $R = \mathbb{K}[X_0, \dots, X_n]$ be the projective coordinate ring of $\mathbb{P}^n$. We define the *regularity index of Z*, as the integer, $\mathrm{r}(Z)$ where

$$\mathrm{r}(Z) := \operatorname{reg}(R/I(Z)).$$

From the previous proposition we also get a good interpretation of $\mathrm{r}(Z)$ in terms of the Hilbert function.

Corollary 2.4.4. *If $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^n_{\mathbb{K}}$ is a fat point scheme then* $\mathrm{r}(Z)$ *is equal to the integer*

$$\mathrm{r}(Z) = \min \left\{ r \,\middle|\, \dim[R/I(Z)]_r = \sum_{i=0}^{s} \binom{n + m_i - 1}{n} \right\}.$$

We discussed the case of simple points in $\mathbb{P}^1_{\mathbb{C}}$, in which case $\operatorname{reg}(Z) = |Z|$. If we instead consider arbitrary fat points in $\mathbb{P}^1_{\mathbb{K}}$ the situation is not much more complicated.

Theorem 2.4.5 (Regularity of Fat Points in $\mathbb{P}^1$)**.** *Given a fat point scheme $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^1_{\mathbb{K}}$ we have*

$$\operatorname{reg}(Z) = \sum_{i=0}^{s} m_i.$$

Equivalently $\mathrm{r}(Z) = \sum_{i=0}^{s} m_i - 1$.

Proof. Let $R = \mathbb{K}[X_0, X_1]$ be the projective coordinate ring of $\mathbb{P}^1$. Then if $P_i = (a_i : b_i)$ we have that $I(P_i) = (b_i X_0 - a_i X_1)$, and more generally $I(P_i)^{m_i} = ((b_i X_0 - a_i X_1)^{m_i})$. Then $I(Z)$ is generated by the polynomial $Q_Z = \prod_{i=0}^{s}(b_i X_0 - a_i X_1)^{m_i}$. Therefore,

$$\begin{aligned}\dim[R/I(Z)]_d = \dim[R]_d - \dim[I(Z)]_d &= \dim[R]_d - \dim Q_z \cdot [R]_{d-\deg(Q_Z)} \\ &= \dim[R]_d - \dim[R]_{d-\deg(Q_Z)}.\end{aligned}$$

For $0 \leq d < \deg(Q_Z) = \sum_{i=0}^{s} m_i$ we have $\dim[R]_{d-\deg(Q_Z)} = 0$ and so

$$\dim[R/I(Z)]_d = \dim[R]_d = d + 1 \leq \deg(Q_Z).$$

If $d \geq \deg(Q_Z)$, then

$$\dim[R/I(Z)]_d = \dim[R]_d - \dim[R]_{d-\deg(Q_Z)} = (d+1) - (d+1-\deg(Q_Z)) = \deg(Q_Z).$$

We note that $d = \deg(Q_z) - 1$ is the smallest integer where $\dim[R/I(Z)]_d = \sum_{i=0}^{s} m_i$. Hence applying proposition 2.4.2 we see that $\operatorname{reg}(Z) = \deg(Q_Z) = \sum_{i=0}^{s} m_i$ as desired. $\square$

Chapter 3 Matroids and Optimization

This chapter recalls the concepts from combinatorics that are needed for the rest of the dissertation. A unifying theme is the study of non-decreasing submodular functions $f : 2^E \to \mathbb{Z}$. These are functions $f : 2^E \to \mathbb{Z}$ where $f(A) \leq f(B)$ if $A \subseteq B$ and for subsets $X, Y \subseteq E$ we have

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y).$$

In section 3.1, we recall the concept of a matroid, a combinatorial object which abstracts the concept of linear independence in a vector space. In particular any finite set of vectors in a vector space has an associated matroid. More generally any increasing submodular function $f : 2^E \to \mathbb{Z}$ defines a matroid M_f, a class of examples which is important in this dissertation.

Section 3.2 studies partitions $\{A_1, .., A_n\}$ of E where $\sum_{i=1}^{n} f(A_i)$ achieves a minimum for $f : 2^E \to \mathbb{R}$ a submodular function. This section largely follows [Nar91].

Lastly section 3.3 focuses on a subclass of submodular functions. Namely those of the form $f(X) = k\,\mathrm{rk}_M(X) - p$, where rk_M is a the rank function of a matroid and k and p are positive integers. It is here that we give our generalization of Edmond's Matroid Partition theorem, which first appeared in [NT20].

3.1 Matroids and Submodular Functions

We recall some definitions from matroid theory and collect the necessary results in the area needed for the remaining sections of this thesis. Matroids are known for the vast number of seemingly different axiomatizations, which define the same concept. The correspondences between these are colloquially referred to as *cryptomorphisms.*

We recall two of these cryptomorphic definitions below.

Definition 3.1.1 (Matroids)**.** A *matroid* M is a finite set $E = E(M)$ called the *base set* or *edge set* along with a rank function $\mathrm{rk}_M : 2^E \to \mathbb{Z}$, which satisfies the following 3 conditions for subsets $A, B \subseteq E$.

(Rk. 1) $0 \leq \mathrm{rk}_M(A) \leq |A|$

(Rk. 2) If $A \supseteq B$, then $\mathrm{rk}_M(A) \leq \mathrm{rk}_M(B)$

(Rk. 3) $\mathrm{rk}_M(A) + \mathrm{rk}_M(B) \geq \mathrm{rk}_M(A \cup B) + \mathrm{rk}_M(A \cap B)$

We note a function satisfying only (Rk. 3) is a *submodular function.*

Equivalently, a matroid may be defined as a nonempty collection of subsets $\mathcal{I}$ of E, which satisfy

(Ind. 1) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$

(Ind. 2) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists some $b \in B$ so that $A \cup \{b\} \in \mathcal{I}$.

We refer to [Oxl11] for definitions and proofs that the stated axiomatic formulations are equivalent.

Example 3.1.2. Every finite set of points $Z \subseteq \mathbb{P}(V)$ defines a matroid, $M(Z)$. Namely for every nonempty $A \subseteq Z$, we set

$$\operatorname{rk}_{M(Z)}(A) = 1 + \dim \operatorname{Span}(A).$$

Here $\operatorname{Span}(A)$ is the smallest linear subvariety of $\mathbb{P}(V)$ containing all the points of A.

An independent set I in $M(Z)$ is a subset $I = \{i_1, .., i_k\} \subseteq Z$ so that for every linear subspace $L \subseteq \mathbb{P}^n$ we have that $|L \cap I| \leq \dim L + 1$. In particular, taking $L = \mathbb{P}^n$ we see $|I| \leq n + 1$.

Matroids of this type are referred to as *representable matroids*, and are in some sense the prototypical example of a matroid.

Remark 3.1.3. An abuse of notation common in the literature is to identify a matroid M with it's base set $E(M)$. We will use this convention when convenient. Be warned that there will be situations where we have two matroids M_1 and M_2 both defined on the same edge set E.

We now recall a few more pieces of related terminology. We refer to [Oxl92] for definitions.

- A subset $I \subseteq M$ is *Independent* if and only if $|I| = \operatorname{rk}_M(I)$. Conversely for $A \subseteq M$, $\operatorname{rk}_M(A)$ is equal to the largest size of an independent $I \subseteq A$.
- A maximal independent set is a *Basis* of M. Every basis has the same size namely $\operatorname{rk}_M(M)$, and every independent subset is contained in some basis.
- A subset $D \subseteq M$ is *dependent* if it is not independent. A *circuit* of M is minimal *dependent set*, meaning a dependent set $C \subseteq M$ so that for all $C' \subseteq C$ with $C' \neq C$ we have that C' is independent.
- A *flat* of rank r is a subset $F \subseteq M$, which is maximal among subsets of M with rank r. Every subset A of M is contained in a unique flat, F, with $\operatorname{rk}_M(A) = \operatorname{rk}_M(F)$, this flat F is called the *closure* or *span* of A and is often denoted $\operatorname{Cl}_M(A)$.

  For $M(Z)$ a flat, is any set of the form $Z \cap L$ where $L \subseteq \mathbb{P}(V)$ is a linear subspace. The closure of a subset $A \subseteq Z$ is $\operatorname{Cl}_{M(Z)}(A) = \operatorname{Span}(A) \cap Z$.

We mention one more example of matroids those that arise from submodular set functions.

Definition 3.1.4 (Submodular Functions)**.** A *submodular set function* or simply *submodular function* on a finite set X is a function $f : 2^X \to \mathbb{R}$ satisfying either one of the following equivalent conditions

(I) For all subsets $A, B \subseteq X$ we have

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

(II) For all subsets $A \subseteq X$ and all $x, y \in X \setminus A$ with $x \neq y$ we have

$$f(A \cup \{x, y\}) + f(A) \leq f(A \cup \{x\}) + f(A \cup \{y\})$$

A submodular set function is *non-decreasing* if for all $A, B \subseteq X$ with $A \subseteq B$ we have

$$f(A) \leq f(B).$$

Given the axioms above and those appearing in definition 3.1.1, we see that rank functions of matroids gives one class of examples of increasing submodular functions. This map from matroids to non-decreasing submodular functions has a left inverse, which associates to every submodular function an underlying matroid. This construction is important in the sequel.

Proposition 3.1.5. *If $f : 2^E \to \mathbb{Z}$ is a non-decreasing submodular function, then there is a matroid $M(f)$ on E whose independent subsets, $\mathcal{I}(f)$ are those $I \subseteq E$ where for all nonempty $J \subseteq I$ we have*

$$|J| \leq f(J).$$

Proof. First, note that $\emptyset \in \mathcal{I}(f)$ even if $f(\emptyset) < 0$ since there are no nonempty subsets of $\emptyset$, and so $\emptyset$ trivially satisfies the condition. Furthermore, if $J \subseteq I$ and $I \in \mathcal{I}$ then for any nonempty $A \subseteq J$ we have that $A \subseteq I$ and so by assumption $|A| \leq f(A)$, hence $J \in \mathcal{I}(f)$ and $\mathcal{I}(f)$ satisfies (IND 1).

Finally, we must show it satisfies the axiom (IND 2). Suppose that $I, J \in \mathcal{I}(f)$ and $|J| < |I|$. Let S be the subset of $I \setminus J$ consisting of those a with $J \cup \{a\} \notin \mathcal{I}(f)$. We note it suffices to show that $|S| \leq |J \setminus I|$.

For each $a \in S$, there exists some nonempty subset $C_a \subseteq J \cup \{a\}$ so that $|C_a| > f(C_a)$. As $\mathrm{Cl}\, I(f)$ is closed under inclusion we can conclude that $a \in C_a$ and that $J_a := C_a \cap J$ is not contained in $I \cap J$. As f in increasing and J_a is independent we have

$$|J_a| \leq f(J_a) \leq f(C_a) < |C_a| = |J_a| + 1.$$

Hence, $|J_a| = f(J_a) = f(C_a) = |C_a| - 1$.

Furthermore, if $b \in I \setminus J$ with $b \neq a$ we have that if $C_a \cap C_b \neq \emptyset$ then

$$f(C_a \cup C_b) + f(C_a \cap C_b) \leq f(C_a) + f(C_b) = |J_a| + |J_b|$$

and so

$$|J_a \cup J_b| \leq f(C_a \cup C_b) \leq |J_a| + |J_b| - f(C_a \cap C_b) \leq |J_a| + |J_b| - |C_a \cap C_b| = |J_a \cup J_b|.$$

We can then build a partition $\{S_1, .., S_t\}$ of S which is generated by the equivalence relation $a \equiv b$ if $C_a \cap C_b \neq \emptyset$. Letting $C_i = \bigcup_{a \in S_i} C_a$ it follows by induction that $f(C_i) = f(C_i \cap J) = |C_i \cap J| = |C_i| - |S_i|$. Furthermore, $C_i \cap I$ is nonempty and contained in $\mathcal{I}(f)$ and so we have

$$|C_i \cap I| \leq f(C_i \cap I) = |C_i| - |S_i|.$$

Rearranging yields $|S_i| \leq |C_i| - |C_i \cap I| = |C_i \setminus I|$. Hence $|S| \leq \sum_{i=1}^{t} |C_i \setminus I| \leq |J \setminus I|$. $\square$

3.2 Submodular Functions and the Partition Lattice

Given any finite set X and a function $f : 2^X \to \mathbb{R}$. We can extend f to its *partition associate*, $\hat{f} : 2^{2^X} \to \mathbb{R}$. This is defined on any collection of subsets $\boldsymbol{\chi} = \{X_0, .., X_s\}$ of X where we set

$$\hat{f}(C) := \sum_{i=0}^{s} f(X_i).$$

In this section we study the case where f is a non-decreasing submodular set function and $\boldsymbol{\chi}$ is a partition of X (or possibly some subset of X). We do not claim originality for the results in this section, and largely follow [Nar91]. The main difference between this section and [Nar91], is that we focus almost entirely on the case that f is a non-decreasing submodular function. Even in the few cases our results are stronger, we note the proofs are straightforward extensions of those appearing in [Nar91].

Definition 3.2.1 (Partition Lattice)**.** Let X be a finite set. A *partition of* X is a collection of nonempty subsets $\boldsymbol{\pi} = \{P_0, P_1, \ldots, P_\ell\}$ of X, so that $P_i \cap P_j = \emptyset$ for $i \neq j$ and

$$X = \bigcup_{i=0}^{\ell} P_i.$$

The elements of a partition $\boldsymbol{\pi}$ are called *blocks*.

The collection of partitions of X can be given the structure of a lattice, called the *partition lattice of* X and denoted Π^X. If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Pi^X$ we say that α is *finer* than $\boldsymbol{\beta}$ and write $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ if for every block $A \in \boldsymbol{\alpha}$ there is some block $B \in \boldsymbol{\beta}$ so that $A \subseteq B$. We dually say $\boldsymbol{\alpha}$ is *coarser* than $\boldsymbol{\beta}$ if $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$.

The *meet* of two partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is denoted $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and is the partition consisting of all blocks of the form $A \cap B$, with $A \in \boldsymbol{\alpha}$, $B \in \boldsymbol{\beta}$ and $A \cap B \neq \emptyset$.

The *join* of two partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is denoted $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$. $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ is the partition of X where $x, y \in X$ are in the same block if and only if there is a sequences of blocks $A_1, ..., A_k \in \boldsymbol{\alpha}$ and $B_1, \ldots, B_k \in \boldsymbol{\beta}$ so $x \in A_1$, $y \in B_k$ and for each i we have $A_i \cap B_i \neq \emptyset$ and for all $i > 1$ we have $A_i \cap B_{i-1} \neq \emptyset$.

Remark 3.2.2.

- Note that if $A, B \subseteq X$ are disjoint sets, and $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ is a partition of A and $\boldsymbol{\beta}$ is a partition of B. Then $\boldsymbol{\beta} \cup \boldsymbol{\alpha}$ is a partition of $A \cup B$. This is distinct from the $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ which is not defined in this context.

- We note that that each partition $\boldsymbol{\alpha}$ defines an equivalence relation $\sim_{\boldsymbol{\alpha}}$ on the set X. Where $x_1 \sim_{\boldsymbol{\alpha}} x_2$ if x_1 and x_2 lie in the same block of $\boldsymbol{\alpha}$. Conversely to each equivalence relation R on X we can associate a partition where each block consists of the elements in a single equivalence class of R. These maps give bijections between partitions and equivalence relations.

  Viewed in these terms the *meet* of two partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, corresponds to the equivalence relation $\sim_{\boldsymbol{\alpha} \wedge \boldsymbol{\beta}}$. Where $x \sim_{\boldsymbol{\alpha} \wedge \boldsymbol{\beta}} y$ if and only if $x \sim_{\boldsymbol{\alpha}} y$ and $x \sim_{\boldsymbol{\beta}} y$.

  The *join* is slightly more complicated. If $\sim_1$ and $\sim_2$ are equivalence relations. Then relation R where xRy if and only if $x \sim_1 y$ or $x \sim_2 y$ is reflexive and symmetric but is not transitive in general. For this reason, $\sim_{\boldsymbol{\alpha}\vee\boldsymbol{\beta}}$ is instead the equivalence relation generated by the relations $x \sim_1 y$ and $x \sim_2 y$. Meaning $x \sim_{\boldsymbol{\alpha}\vee\boldsymbol{\beta}} y$ if and only if there is a sequence of elements $x_0, x_1, .., x_n$ with

$$x = x_0 \sim_{\boldsymbol{\alpha}} x_1 \sim_{\boldsymbol{\beta}} x_2 \ldots \sim_{\boldsymbol{\alpha}} x_{n-1} \sim_{\boldsymbol{\beta}} x_n = y$$

We define a few special partitions, and note a few basic properties of the partition lattice.

Remark 3.2.3. For a set X we introduce some notation to refer to certain special partitions of X.

- Given any set X we define $\boldsymbol{\pi}_0^X$ as the partition consisting of the singletons of X. We note that $\boldsymbol{\pi}_0^X \preceq \boldsymbol{\alpha}$ for all other partitions $\boldsymbol{\alpha} \in \Pi^X$.
- If $A \subseteq X$ we set $\boldsymbol{\pi}_A^X := \{A\} \cup \boldsymbol{\pi}_0^{X\setminus A}$. If $A \subseteq B \subseteq X$ then note that $\boldsymbol{\pi}_A^X \preceq \boldsymbol{\pi}_B^X$. In particular we note that if $E \subseteq X$ with E either empty or a singleton, then $\boldsymbol{\pi}_E^X = \boldsymbol{\pi}_0^X$.
- If $\boldsymbol{\alpha} = \{A_0, A_1, \ldots, A_s\}$ is a partition of X, then $\boldsymbol{\alpha} = \bigvee_{i=0}^s \boldsymbol{\pi}_{A_i}^X$.
- If $A, B \subseteq X$, then $\boldsymbol{\pi}_A^X \wedge \boldsymbol{\pi}_B^X = \boldsymbol{\pi}_{A\cap B}^X$, and $\boldsymbol{\pi}_A^X \vee \boldsymbol{\pi}_B^X = \boldsymbol{\pi}_{A\cup B}^X$.
- More generally if $A \subseteq X$ and $\boldsymbol{\alpha}$ is a partition of A then we set $\boldsymbol{\pi}_\alpha^X = \boldsymbol{\alpha} \cup \boldsymbol{\pi}_0^X$.

If $f : 2^X \to \mathbb{R}$ is a submodular function, and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are partitions of X. Then it is not necessarily true that the partition associate $\hat{f}$ satisfies the analogous inequality

$$\hat{f}(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) + \hat{f}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \leq \hat{f}(\boldsymbol{\alpha}) + \hat{f}(\boldsymbol{\beta}).$$

However, this property does hold for some partitions of X.

Proposition 3.2.4. *If $f : 2^X \to \mathbb{R}$ is a submodular set function then every $A \subseteq X$ and every partition $\boldsymbol{\beta}$ of X we have*

$$\hat{f}(\boldsymbol{\pi}_A^X \vee \boldsymbol{\beta}) + \hat{f}(\boldsymbol{\pi}_A^X \wedge \boldsymbol{\beta}) \leq \hat{f}(\boldsymbol{\pi}_A^X) + \hat{f}(\boldsymbol{\beta}).$$

Proof. We first establish the following claim.

Claim 3.2.5. If $\boldsymbol{\beta} = \{B_1, .., B_n\}$ is a collection of nonempty disjoint subsets of X and then for any $A \subseteq X$

$$f\left(A \cup \bigcup_{i=1}^n B_i\right) + \sum_{i=1}^n f(B_i \cap A) \leq f(A) + \sum_{i=1}^n f(B_i).$$

The case $n = 1$ follows immediately as f is submodular. If the statement holds for $n = k - 1$. The consider the case when $|\boldsymbol{\beta}| = n = k$, by inductive hypothesis we have

$$f(B_k) + f\left(A \cup \bigcup_{i=1}^{k-1} B_i\right) + \sum_{i=1}^{k-1} f(B_i \cap A) \leq f(A) + \sum_{i=1}^{k} f(B_i).$$

As f is submodular we have $f\left(A \cup \bigcup_{i=1}^{k} B_i\right) + f\left(B_k \cap \left(A \cup \bigcup_{i=1}^{k-1} B_i\right)\right) \leq f(B_K) + f\left(A \cup \bigcup_{i=1}^{k-1} B_i\right)$. Yet $B_K \cap \left(A \cup \bigcup_{i=1}^{k-1} B_i\right) = B_k \cap A$ since the elements in $\boldsymbol{\beta}$ are pairwise disjoint. Therefore,

$$\begin{aligned} f\left(A \cup \bigcup_{i=1}^{k} B_i\right) + f(B_k \cap A) + \sum_{i=1}^{k-1} f(B_i \cap A) \leq & f(B_k) + f\left(A \cup \bigcup_{i=1}^{k-1} B_i\right) \\ & + \sum_{i=1}^{k-1} f(B_i \cap A) \\ & \leq f(A) + \sum_{i=1}^{k} f(B_i). \end{aligned}$$

Establishing our desired claim.

Continuing with the proof of the proposition, we fix a partition $\beta = \{B_1, .., B_n\}$. Up to relabeling we may assume that $A \cap B_i \neq \emptyset$ for precisely those i with $1 \leq i \leq k$. We have that

$$f\left(A \cup \bigcup_{i=1}^{k} B_i\right) + \sum_{i=1}^{k} f(B_i \cap A) \leq f(A) + \sum_{i=1}^{k} f(B_i).$$

Adding $\left(\sum_{j=k+1} f(B_j)\right) + \sum_{x \in X \setminus A} f(\{x\})$ to both sides we get

$$\begin{aligned} \hat{f}(\boldsymbol{\pi}_A^X \vee \boldsymbol{\beta}) + \hat{f}(\boldsymbol{\pi}_A^X \wedge \boldsymbol{\beta}) \leq & \left(f\left(\bigcup_{i=1}^{k} B_i\right) + \sum_{j=k+1} f(B_j)\right) + \\ & \left(\sum_{x \in X \setminus A} f(\{x\}) + \sum_{i=1}^{k} f(B_i \cap A)\right) \\ \leq & \left(f(A) + \sum_{x \in X \setminus A} f(\{x\})\right) + \sum_{i=1}^{n} f(B_i) \\ \leq & \hat{f}(\boldsymbol{\pi}_A^X) + \hat{f}(\boldsymbol{\beta}). \end{aligned}$$

□

As we will see one consequence of the preceding proposition is that the collection of partitions $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ of E which minimize $\sum_{i=1}^{n} f(A_i)$ form a sublattice of π^E. Before we give this proof we introduce a convenient piece of notation.

Definition 3.2.6 (Lower Dilworth Truncation)**.** If $f : 2^X \to \mathbb{R}$ is a submodular function, then the *Lower Dilworth Truncation* of f is the function $f_* : 2^X \to \mathbb{R}$ defined

$$f_*(A) = \min\left\{ \sum_{i=1}^{S} f(A_i) \,\middle|\, \{A_1, \ldots, A_s\} \text{ is a partition of } A \right\}.$$

Where $f_*(\emptyset) = 0$ by convention.

Proposition 3.2.7. *If $f : 2^X \to \mathbb{R}$ is a submodular function, then the collection of partitions $\boldsymbol{\alpha} \in \Pi^X$ with*

$$\hat{f}(\boldsymbol{\alpha}) = f_*(X)$$

forms a sublattice of Π^X which we denote Π^f.

In particular, there is a unique finest, $\boldsymbol{\pi}_0^f$, and a unique coarsest, $\boldsymbol{\pi}_1^f$, partition of X with

$$\hat{f}(\boldsymbol{\pi}_0^f) = f_*(X) = \hat{f}(\boldsymbol{\pi}_1^f).$$

Proof. First, note that if $\boldsymbol{\chi} = \{A_1, .., A_s\}$ is any partition of X with $\hat{f}(\boldsymbol{\chi}) = f_*(X)$. Then if $\boldsymbol{\alpha}_i$ is any partition of A_i we necessarily have that $\hat{f}(\boldsymbol{\alpha_i}) \geq f(A_i)$ since otherwise $\boldsymbol{\chi}' = \boldsymbol{\chi} \setminus \{A_i\} \cup \boldsymbol{\alpha}_i$, would be a partition with $\hat{f}(\boldsymbol{\chi}') < \hat{f}(\boldsymbol{\chi})$. Consequently, if $\boldsymbol{\gamma}$ is any partition of X we must have that $\hat{f}(\boldsymbol{\gamma} \wedge \boldsymbol{\pi}^X_{A_i}) \geq \hat{f}(\boldsymbol{\pi}^X_{A_i})$.

Applying this, we see that if $\boldsymbol{\alpha} = \{A_1, .., A_s\}$ and $\boldsymbol{\beta} = \{B_1, \ldots, B_t\}$ are in Π^f, we see by proposition 3.2.4 that $\hat{f}(\boldsymbol{\beta} \vee \boldsymbol{\pi}^X_{A_i}) + \hat{f}(\boldsymbol{\beta} \wedge \boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\beta}) + \hat{f}(\boldsymbol{\pi}^X_{A_i})$. From this we conclude the following two inequalities

$$\hat{f}(\boldsymbol{\beta}) \leq \hat{f}(\boldsymbol{\beta} \vee \boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\beta}) + \hat{f}(\boldsymbol{\pi}^X_{A_i}) - \hat{f}(\boldsymbol{\beta} \wedge \boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\beta}) \tag{3.2.7.1}$$

and

$$\hat{f}(\boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\beta} \wedge \boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\beta}) + \hat{f}(\boldsymbol{\pi}^X_{A_i}) - \hat{f}(\boldsymbol{\beta} \vee \boldsymbol{\pi}^X_{A_i}) \leq \hat{f}(\boldsymbol{\pi}^X_{A_i}). \tag{3.2.7.2}$$

The first inequality allows us to conclude that $\boldsymbol{\beta} \vee \boldsymbol{\pi}^X_{A_i} \in \Pi^f$ for any $\boldsymbol{\beta} \in \Pi^f$. Hence, $\boldsymbol{\beta} \vee \boldsymbol{\alpha} = \boldsymbol{\beta} \vee \left(\bigvee_{i=1}^s \boldsymbol{\pi}^X_{A_i}\right) \in \Pi^f$.

Similarly, for each A_i let $\boldsymbol{\lambda}_i = \{B_j \cap A_i \,|\, \text{for all } B_j \in \boldsymbol{\beta} \text{ with } B_j \cap A_i \neq \emptyset\}$, so that $\boldsymbol{\beta} \wedge \boldsymbol{\pi}^X_{A_i} = \boldsymbol{\pi}_0^{X \setminus A_i} \cup \boldsymbol{\lambda}_i$. Then as $\hat{f}(\boldsymbol{\beta} \wedge \boldsymbol{\pi}^X_{A_i}) = \hat{f}(\boldsymbol{\pi}^x_{A_i}) = \hat{f}(\boldsymbol{\pi}_0^{X \setminus A_i}) + f(A_i)$ we conclude that $\hat{f}(\boldsymbol{\lambda}_i) = f(A_i)$. Then

$$\hat{f}(\boldsymbol{\beta} \wedge \boldsymbol{\alpha}) = \sum_{i=1}^s \hat{f}(\boldsymbol{\lambda}_i) = \sum_{i=1}^s f(A_i) = f_*(X)$$

so $\boldsymbol{\beta} \wedge \boldsymbol{\alpha} \in \Pi^f$ as well. □

An important class of non-decreasing submodular functions are the *integer polymatroids.*

Definition 3.2.8. An *integer polymatroid* on a set E is a non-decreasing submodular function $f : 2^E \to \mathbb{Z}$, so that $f(\emptyset) = 0$.

Proposition 3.2.9. *Let $f : 2^X \to \mathbb{Z}$ be a non-decreasing submodular function, with $f(A) \geq 0$ for all non empty $A \subseteq X$. Then f_* is an integer polymatroid where $M_{f_*} = M_f$.*

Proof. First, we show that f_* is increasing. If $A \subseteq B \subseteq X$ and $\boldsymbol{\pi}_B = \{B_1, .., B_k\}$ is any partition of B so that $\hat{f}(\pi_B) = f_*(B)$. Then up to reordering we may assume there's an index ℓ so that that $B_i \cap A \neq \emptyset$ if and only if $i \leq \ell$. Setting $A_i = B_i \cap A$ for

$1 \leq i \leq \ell$ we get a partition $\boldsymbol{\pi}_A = \{A_1, .., A_\ell\}$. Then as f is increasing and $f(B_j) \geq 0$ all j we have

$$f_*(A) \leq \sum_{i=1}^{\ell} f(A_i) \leq \sum_{i=1}^{\ell} f(B_j) \leq \sum_{j=1}^{k} f(B_j) = f_*(B).$$

To show that f_* is submodular we use condition (II) of definition 3.1.4 namely for any set $A \subseteq X$ and distinct $x_1, x_2 \in X \setminus A$ that

$$f_*(A \cup \{x_1, x_2\}) + f_*(A) \leq f_*(A \cup \{x_1\}) + f_*(A \cup \{x_2\}).$$

To establish this, suppose that $\boldsymbol{\tau} = \{T_0, \ldots, T_m\}$ is a partition of $A \cup \{x_1\}$ and $\boldsymbol{\eta}$ is a partition of $A \cup \{x_2\}$ so that $\hat{f}(\boldsymbol{\tau}) = f_*(A \cup \{x_1\})$ and $\hat{f}(\boldsymbol{\eta}) = f_*(A \cup \{x_2\})$. We may without loss of generality assume that $x_1 \in T_0$. Set $\boldsymbol{\tau'} = \{T_0', .., T_m'\}$ where $T_i' = T_i$ for $i > 0$ and $T_0' = T_0 \setminus \{x_1\}$. Extending $\boldsymbol{\eta}$ to a partition $\tilde{\boldsymbol{\eta}}$ of $A \cup \{x_1, x_2\}$ by $\boldsymbol{\eta} \cup \{x_1\}$ we get by proposition 3.2.4,

$$\hat{f}(\tilde{\boldsymbol{\eta}} \vee \boldsymbol{\pi}_{T_0}^{A \cup \{x_1, x_2\}}) + \hat{f}(\tilde{\boldsymbol{\eta}} \wedge \boldsymbol{\pi}_{T_0}^{A \cup \{x_1, x_2\}}) \leq \hat{f}(\boldsymbol{\pi}_{T_0}^{A \cup \{x_1, x_2\}}) + \hat{f}(\tilde{\boldsymbol{\eta}})$$

adding $(\sum_{i=1}^m f(T_i)) - f(x_0) - f(x_1) - \hat{f}(\boldsymbol{\pi}_0^{A \setminus T_0'})$ to both sides gives

$$\begin{aligned} f_*(A \cup \{x_1, x_2\}) + f_*(A) &\leq \hat{f}(\tilde{\boldsymbol{\eta}} \vee \boldsymbol{\pi}_{T_0}^{A \cup \{x_1, x_2\}}) + \hat{f}(\boldsymbol{\tau'}) \\ &\leq \hat{f}(\boldsymbol{\tau}) + \hat{f}(\boldsymbol{\eta}) = f_*(A \cup \{x_1\}) + f_*(A \cup \{x_2\}) \end{aligned}$$

establishing the desired claim.

□

Proposition 3.2.10. *If $\rho : 2^E \to \mathbb{Z}$ is an integer polymatroid and* rk_ρ *the rank function of the induced matroid M_ρ then for any $X \subseteq E$ we have*

$$\mathrm{rk}_\rho(X) = \min \{|A| + \rho(X \setminus A) \mid A \subseteq X\}$$

Proof. Let rk_ρ denote the rank function of M_ρ and define $r : 2^E \to \mathbb{Z}$ via the proposed formula $r(X) := \min \{|A| + \rho(X \setminus A) \mid A \subseteq X\}$. From the definition we see for any subset $I \subseteq E$ that $r(I) = |I|$ if and only if $I \in \mathcal{I}(\rho)$.

Furthermore, we see that r is increasing, since if $X \subseteq Y$ and $r(Y) = |Y \setminus B| + \rho(B)$ then $r(X) \leq |X \setminus B| + \rho(X \cap B) \leq r(Y)$. Lastly by definition we have $0 \leq r(A) \leq |A|$ for all subsets A Hence, if we show that r is submodular we see that r is the rank function of a matroid on E call it M_r. Since M_r and M_ρ would necessarily have the same independence sets, we would conclude that $M_r = M_\rho$ and so $r = \mathrm{rk}_\rho$ as desired.

The proof of submodularity is relatively straightforward, though it requires the following set theoretic identities whose proofs we omit. Given sets X, Y, A, B with $A \subseteq X$ and $B \subseteq Y$ the following identities hold where $\sqcup$ denotes disjoint union

(Id. 1) $(X \setminus A) \cup (Y \setminus B) = [(X \cup Y) \setminus (A \cup B)] \sqcup [A \cap (Y \setminus B)] \sqcup [B \cap (X \setminus A)]$

(Id. 2) $(X \cap Y) \setminus (A \cap B) = [(X \setminus A) \cap (Y \setminus B)] \sqcup [A \cap (Y \setminus B)] \sqcup [B \cap (X \setminus A)]$

Using these identities we see that

$$\begin{aligned}|X \setminus A| + |Y \setminus B| &= |(X \setminus A) \cup (Y \setminus B)| + |(X \setminus A) \cap (Y \setminus B)| \\ &= |(X \cup Y) \setminus (A \cup B)| + |(X \cap Y) \setminus (A \cap B)|.\end{aligned}$$

Now for subsets $X, Y \subseteq E$ find $A \subseteq X$ and $B \subseteq Y$ so that $r(X) = |X \setminus A| + \rho(A)$ and $r(Y) = |Y \setminus B| + \rho(B)$. Then by submodularity of ρ we have $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$. Adding $|(X \cup Y) \setminus (A \cup B)| + |(X \cap Y) \setminus (A \cap B)| = |X \setminus A| + |Y \setminus B|$ to both sides gives

$$\begin{aligned}r(X \cup Y) + r(X \cap Y) &\leq |(X \cup Y) \setminus (A \cup B)| + |(X \cap Y) \setminus (A \cap B)| + \\ &\qquad \rho(A \cup B) + \rho(A \cap B) \\ &\leq |X \setminus A| + |Y \setminus B| + \rho(A) + \rho(B) \\ &\leq r(X) + r(Y)\end{aligned} .$$

Establishing that r is submodular and the proof of the theorem. $\square$

The previous two theorems combine to give the following result.

Corollary 3.2.11. *Let $f : 2^E \to \mathbb{Z}$ be a non-decreasing submodular function, and let $\mathrm{rk}_f : 2^E \to \mathbb{Z}$ denote the rank function of the matroid $M(f)$. Then for any $X \subseteq E$ we have that*

$$\mathrm{rk}_f(X) = \min \left\{ |X_0| + \sum_{i=1}^{s} f(Y_i) \right\}$$

where the minimum is taken over all collections of subsets where $\{Y_1, .., Y_s\}$ is a partition of $X \setminus X_0$.

Remark 3.2.12. Note X_0 may be empty so $\{X_0, Y_1, .., Y_s\}$ does not necessarily form a partition of X.

Fix a non-decreasing submodular $f : 2^E \to \mathbb{Z}$. It turns out that any collection of subsets $E_0, A_1, .., A_n \subseteq E$ where $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ forms a partition of $E \setminus A_0$ and

$$\mathrm{rk}_f(E) = |E_0| + \hat{f}(\boldsymbol{\alpha})$$

contains lots of structural information about the induced matroid M_f. In particular, one can achieve the following characterization of basis.

Proposition 3.2.13. *Let $f : 2^E \to \mathbb{Z}$ be a non-decreasing submodular function and $E_0 \subseteq E$ a subset and $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ a partition of $E \setminus E_0$ with*

$$\mathrm{rk}_f(E) = |E_0| + \hat{f}(\boldsymbol{\alpha}).$$

Then $B \subseteq E$ is a basis of M_f if and only if $E_0 \subseteq B$ and for each $i \in \{1, .., n\}$ the set $B \cap A_i$ is independent in M_f with

$$|B \cap A_i| = f(B \cap A_i) = f(A_i).$$

Proof. [$\Rightarrow$] First suppose $B \subseteq E$ is a basis of M_f. Define $B_0 = |E_0 \cap B|$ and $B_i = B \cap A_i$ for $1 \leq i \leq n$. Then $|B_0| \leq |E_0|$ as each B_i is independent and f is non-decreasing we have $|B_i| \leq f(B_i) \leq f(A_i)$. Then

$$|B| = |B_0| + \sum_{i=1}^{n} |B_i| \leq |B_0| + \sum_{i=1}^{n} f(B_i) \leq |E_0| + \sum_{i=1}^{n} f(A_i) = \mathrm{rk}_f(E) = |B|.$$

We conclude that $B_0 = E_0$ and $|B_i| = f(B_i) = f(A_i)$.

[$\Leftarrow$] For this direction suppose $B \subseteq E$ with $E_0 \subseteq B$ and assume each $B_i := B \cap A_i$ is independent with $f(B_i) = f(A_i) = |B_i|$. Then $|B| = |E_0| + \sum_{i=1}^{n} |B_i| = \mathrm{rk}_f(E)$, so it suffices to show that B is independent. Note it suffices to show that $\mathrm{rk}_f(E) = \mathrm{rk}_f(B)$. If note there would be some $e \in E \setminus B$ so that $\mathrm{rk}_f(B \cup \{e\}) > \mathrm{rk}_f(B)$. However, each such e is contained in some A_i. Noting that $|B_i| = \mathrm{rk}_f(B_i) < \mathrm{rk}_f(B \cup \{e\}) \leq \mathrm{rk}_f(A_i) \leq f(A_i) = |B_i|$ gives our contradiction. Thereby establishing the result. ☐

In order to give a stronger interpretation of the pairs $(E_0, \boldsymbol{\alpha})$ with $\mathrm{rk}_f(E) = |E_0| + \hat{f}(\boldsymbol{\alpha})$. We recall another piece of terminology from matroid theory.

Definition 3.2.14 (Direct Sums)**.** If M is a matroid and $\boldsymbol{\mu} = \{M_1, .., M_n\}$ is a partition of the ground set. Giving each M_i the matroid structure induced from M, we say that M is a *direct sum* of $M_1, .., M_n$ and write

$$M = \oplus_{i=1}^{n} M_i$$

if either of the following equivalent conditions holds.

(1) A subset $I \subseteq M$ is independent if and only if each $I \cap M_i$ is independent.

(2) Given $A \subseteq M$, $\mathrm{rk}_M(A) = \sum_{i=1}^{n} \mathrm{rk}_M(A \cap M_i)$.

Proposition 3.2.15. *Let $f : 2^E \to \mathbb{Z}$ be a non-decreasing submodular function. Given $E_0 \subseteq E$ and $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ a partition of $E \setminus E_0$ so $\mathrm{rk}_f(E) = |E_0| + \hat{f}(\boldsymbol{\alpha})$. We have that each $e \in E_0$ is a* ***coloop*** *of M_f meaning for each $X \subseteq E$ not containing e we have $\mathrm{rk}_f(X \cup \{e\}) = 1 + \mathrm{rk}_f(X)$. Furthermore,*

$$M_f = E_0 \oplus \left(\bigoplus_{i=1}^{n} A_i \right).$$

Proof. Let $X \subseteq E$ be any set not containing e. Let $I \subseteq X$ be an independent subset of X which spans X. Extending I to a basis $B \supseteq I$ of M_f we know by proposition 3.2.13 that $e \in I$ and so $\{e\} \cup I$ is independent. Hence,

$$\mathrm{rk}_f(X) + 1 \geq \mathrm{rk}_f(\{e\} \cup X) \geq \mathrm{rk}_f(\{e\} \cup I) = |I \cup \{e\}| = \mathrm{rk}_f(X) + 1.$$

Establishing that $\mathrm{rk}_f(\{e\} \cup X) = \mathrm{rk}_f(X) + 1$.

Continuing to the proof that $M = E_0 \oplus (\bigoplus_{i=1}^{n} A_i)$, we use the independent subset criterion. If $I \subseteq E$ is a subset so that $I \cap A_i$ is independent for $1 \leq i \leq n$ then we can extend each $I \cap A_i$ to an independent subset $B_i \subseteq A_i$ with $\mathrm{rk}_f(B_i) = \mathrm{rk}_f(A_i) = f(A_i)$. Then by proposition 3.2.13 $B = E_0 \cup \bigcup_{i=1}^{n} B_i$ is a basis of M so in particular $I \subseteq B$ is independent. ☐

3.3 Special Submodular Functions coming from Matroids

Fix a matroid M on a base set E, with rank function $\mathrm{rk} : 2^E \to \mathbb{Z}$. In this section we study increasing submodular functions of the form

$$f_{k,p}(X) = k\,\mathrm{rk}(X) - p.$$

For the rest of the chapter we are interested in the case where k and p are nonnegative integers where $k > p$. In this section we take k and p to be arbitrary real numbers unless specified otherwise.

The main result of this section is a generalization of the following classical result of Edmonds.

Theorem 3.3.1. *Fix an integer $k \geq 0$. Then there exists a partition $I_1 \sqcup \ldots \sqcup I_K$ of a matroid M into independent k independent subsets if and only if for all $A \subseteq M$ we have*

$$|A| \leq k\,\mathrm{rk}(M).$$

Our generalization results in a characterization of those matroids M where for all nonempty $A \subseteq M$ we have

$$|A| \leq k\,\mathrm{rk}(A) - p.$$

If k and p are integers we denote by $M_{k,p}$ the induced matroid $M_{f_{k,p}}$. We note in this case that unless $k > p$ then $\mathrm{rank}(M_{k,p}) = 0$.

We similarly use the notation $\Pi^{k,p}$ to denote the sublattice $\Pi^{f_{k,p}}$ of the partition lattice Π^E.

Proposition 3.3.2. *Let k, p, λ be real numbers with $\lambda > 0$ then*

$$\Pi^{k,p} = \Pi^{\lambda k, \lambda p}.$$

In particular, if k and p are positive then $\Pi^{k,p} = \Pi^{1,p/k} = \Pi^{k/p,1}$.

Proof. This follows since $\lambda f_{k,p} = f_{\lambda k, \lambda p}$ and for any submodular function g

$$\min\left\{ \sum_{i=1}^{n} \lambda g(A_i) \,\middle|\, \{A_1, .., A_n\} \text{ is a partition of } E \right\}$$

$$= \lambda \min\left\{ \sum_{i=1}^{n} g(A_i) \,\middle|\, \{A_1, .., A_n\} \text{ is a partition of } E \right\}.$$

□

Proposition 3.3.3. *If r_1, r_2 are postive real numbers with $0 < r_2 < r_1$ then for any $\boldsymbol{\alpha}_1 \in \Pi^{r_1,1}$ and any $\boldsymbol{\alpha}_2 \in \Pi^{r_2,1}$ we have*

$$\boldsymbol{\alpha}_2 \preceq \boldsymbol{\alpha}_1.$$

Remark 3.3.4. We note that above result can be easily shown to follow from the results of [Nar91], which in fact proofs an analogous result replacing rk with any submodular function μ. We state it in this form for convenience of exposition.

Proof of proposition 3.3.3. Let $g_1 = f_{r_1,1}$ and $g_2 = f_{r_2,1}$. Given any block B of $\boldsymbol{\alpha}_2$ we know that $\{B\}$ minimizes $\hat{g}_2$ among partitions of B, and more generally

$$\hat{g}_2(\boldsymbol{\pi}_B^E) \leq \hat{g}_2(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\varepsilon})$$

for any partition $\boldsymbol{\varepsilon}$ of E. More specifically if $\boldsymbol{\beta} = \{B_1, .., B_k\}$ is a partition of B then

$$\left(\sum_{i=1}^{k} \mathrm{rk}(B_1)\right) - \mathrm{rk}(B) \geq \frac{k-1}{r_2}.$$

Continuing to the proof of the statement, still letting B be an arbitrary block of $\boldsymbol{\alpha}_2$. We note that it suffices to show that B is contained in some block of $\boldsymbol{\alpha}$. By proposition 3.2.4 we see that

$$\hat{g}_1(\boldsymbol{\alpha}_1) + \hat{g}_1(\boldsymbol{\pi}_B^E) \geq \hat{g}_1(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}) + \hat{g}_1(\boldsymbol{\pi}_B^E \vee \boldsymbol{\alpha}).$$

Adding $(\hat{g}_2 - \hat{g}_1)(\boldsymbol{\pi}^E B) = (r_2 - r_1)\hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E)$ to both sides we see

$$\hat{g}_1(\boldsymbol{\alpha}_1) + \hat{g}_2(\boldsymbol{\pi}_B^E) \geq \hat{g}_2(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}) + \hat{g}_1(\boldsymbol{\pi}_B^E \vee \boldsymbol{\alpha}) + (r_1 - r_2)\left(\hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E) - \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}_1)\right).$$

As $\hat{g}_1(\boldsymbol{\alpha}) \leq \hat{g}_1(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha})$ then we have

$$\hat{g}_2(\boldsymbol{\pi}_B^E) \geq \hat{g}_2(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}) + (r_1 - r_2)\left(\hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E) - \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}_1)\right).$$

As rk is subadditive and $r_1 > r_2$ the above inequality says that $\hat{g}_2(\boldsymbol{\pi}_B^E) \geq \hat{g}_2(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha})$ with equality if and only if $\hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E) - \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}_1) = 0$.

Yet from our earlier discussion $\hat{g}_2(\boldsymbol{\pi}_B^E) \leq \hat{g}_2(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha})$, so $\hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E) - \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}_1) = 0$. Letting $\boldsymbol{\beta}$ be the partition of B induced by $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\pi}_B^E$ we see that

$$0 = \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E) - \hat{\mathrm{rk}}(\boldsymbol{\pi}_B^E \wedge \boldsymbol{\alpha}_1) = \hat{\mathrm{rk}}(\boldsymbol{\beta}) - \mathrm{rk}(B) \geq \frac{|\boldsymbol{\beta}| - 1}{r_2}.$$

so $|\boldsymbol{\beta}| - 1 = 0$, and consequently B is contained in some block of $\boldsymbol{\alpha}$. □

Proposition 3.3.5. *For any positive integers k, p with $k > p$ if $\boldsymbol{\alpha} = \{A_1, .., A_n\} \in \Pi^{k,p}$, then each A_i is a flat in M. Furthermore for each $J \subseteq \{1, .., n\}$ we have*

$$(|J| - 1)\frac{p}{k} \geq \left(\sum_{j \in J} \mathrm{rk}(A_j)\right) - \mathrm{rk}\left(\bigcup_{j \in J} A_j\right).$$

Consequently, we have $\mathrm{rk}(A_i \cup A_j) = \mathrm{rk}(A_i) + \mathrm{rk}(A_j)$ for any distinct blocks A_i and A_j.

Proof. First for any collection of subsets $J \subseteq \{1, 2, \dots, n\}$ define $A_J = \bigcup_{j \in J} A_j$. As $\hat{f}_{k,p}(\boldsymbol{\alpha}) = f_*(E)$ we have $\hat{f}_{k,p}(\boldsymbol{\alpha}) \leq \hat{f}_{k,p}(\boldsymbol{\alpha} \vee \boldsymbol{\pi}^X_{A_J})$, cancelling like terms reveals $f_{k,p}(A_J) \geq \sum_{j \in J} f_{k,p}(A_j)$. Using the formula $f_{k,p}(A) = k \operatorname{rk}(A) - p$ we see that

$$(|J| - 1)\frac{p}{k} \geq \left(\sum_{j \in J} \operatorname{rk}(A_j) \right) - \operatorname{rk} \left(\bigcup_{j \in J} A_j \right).$$

If $J = \{i, j\}$, and $k > p$ then this becomes

$$1 > \frac{p}{k} \geq \operatorname{rk}(A_i) + \operatorname{rk}(A_j) - \operatorname{rk}(A_i \cup A_j).$$

As $\operatorname{rk}(A_i) + \operatorname{rk}(A_j) \geq \operatorname{rk}(A_i \cup A_j)$ we conclude that $\operatorname{rk}(A_i) + \operatorname{rk}(A_j) = \operatorname{rk}(A_i \cup A_j)$. $\square$

Proposition 3.3.6. *Let M be a matroid and fix integers $k > p > 0$. Let $M_{k,p}$ denote the matroid induced by $f_{k,p}(A) = k \operatorname{rk}_M(A) - p$. Then*

$$\operatorname{rank}(M_{k,p}) \leq \operatorname{rank}(M_{k-1,p-1}) + \operatorname{rk}_M(E) - 1.$$

Proof. Let $E_0, A_1, .., A_n \subseteq E$ be subsets so $\boldsymbol{\alpha} = \{A_1, .., A_n\}$ is a partition of $E \setminus E_0$ and

$$\operatorname{rk}_{M_{k-1,p-1}}(E) = |E_0| + \sum_{i=1}^{n} \left[k \operatorname{rk}_M(A_i) - p \right].$$

Let I be a basis of $M_{k-1,p-1}$ and extend it to a basis B of $M_{k,p}$. Letting $X = B \setminus I$ it suffices to show that $|X| \leq \operatorname{rk}_M(E) - 1$. By proposition 3.2.13, we have $E_0 \subseteq I \subseteq B$ and so $X \subseteq \bigcup_{i=1}^{n} A_i$. Hence, let $X_i = X \cap A_i$, then

$$|B \cap A_i| = |X_i| + |I \cap A_i| = |X_i| + f_{k-1,p-1}(A_i) \leq f_{k,p}(B \cap A_i).$$

Therefore, $|X_i| \leq f_{k,p}(B \cap A_i) - f_{k-1,p-1}(A_i) \leq \operatorname{rk}_M(A_i) - 1$.

Additionally, applying the inequality from the previous proposition we have

$$\sum_{i=1}^{n} (|X_i| + 1) - \operatorname{rk}_M(E) \leq \left(\sum_{i=1}^{n} \operatorname{rk}_M(A_i) \right) - \operatorname{rk}_M \left(\bigcup_{i=1}^{n} A_i \right) \leq (n-1)\frac{p}{k}.$$

So $|X| + n - \operatorname{rk}_M(E) \leq (n-1)\frac{p}{k}$ or $|X| + 1 \leq \operatorname{rk}_M(E) + (n-1)\left(\frac{p}{k} - 1\right)$. As $\frac{p}{k} < 1$ we conclude that $|X| \leq \operatorname{rk}_M(E) - 1$ as desired. $\square$

There is a generalization of theorem 3.3.1 due to Edmonds and Fulkerson [EF65, Theorem 1c], which we recall below.

Theorem 3.3.7. *Given matroids $M_1, \dots, M_k$ on a ground set E with rank functions $\operatorname{rk}_1, \dots, \operatorname{rk}_k$, there is a partition $E = I_1 \sqcup \dots \sqcup I_k$ such that each set I_j is independent in M_j if and only if, for each subset $A \subseteq E$, one has $|A| \leq \sum_{j=1}^{k} \operatorname{rk}_j(A)$.*

Given the simplicity of proposition 3.3.6, it natural to ask if their is a generalization of proposition 3.3.6 to the case where $f_{k,p}(A) = k\,\mathrm{rk}(A) - p$ is replaced with a function of the form $F(A) = -p + \sum_{i=1}^{k} \mathrm{rk}_k(A)$. If a generalization is possible then much of what follows could almost certainly be similarly generalized, including a version of Segre Bound to mixed degrees. We note however that a generalization of proposition 3.3.6 where f is an arbitrary increasing submodular function is not possible. Namely there exists increasing submodular functions $f, g : 2^E \to \mathbb{Z}$ with $f(A)$ and $g(A)$ both nonnegative where $\mathrm{rank}(M_{f+g}) > \mathrm{rank}(M_f) + g(E)$.

Example 3.3.8. Let g be the rank function of the uniform rank 1 matroid on a set E. Let $f(A) = \mathrm{rk}_M(A) - 1$ where rk_M is the rank function of a matroid having rank ≥ 2. Then for all nonempty $A \subseteq M$ we have $f(A) + g(A) = \mathrm{rk}_M(A)$. Yet $\mathrm{rank}(M_f) = 0$. Therefore,

$$\mathrm{rank}(M_{f+g}) = n > \mathrm{rank}(M_f) + g(E) = 1$$

.

Despite the above example we close noting, that $\mathrm{rank}(M_{f+g}) \geq \mathrm{rank}(M_f) + g(E)$ does hold if f and g are integer polymatroids. However, in [Oxl11][Exercise 12.3] it is shown that if f and g are integer polymatroids then $\mathrm{rank}(M_{f+g}) \leq \mathrm{rank}(M_f) + \mathrm{rank}(M_g) \leq \mathrm{rank}(M_f) + g(E)$. It is in fact further shown that $M_{f+g} = M_f \vee M_g$ where $\vee$ denote the operation of matroid union. Here $M_f \vee M_g$ is the matroid whose independence sets are those of the form $I = I_f \cup I_g$ where $I_f \in \mathcal{I}(f)$ and $I_g \in \mathcal{I}(g)$.

We now recall a few modifications which can be made to any matroid M.

Definition 3.3.9. Let M be a matroid on E.

(i) Suppose M is a submatroid of a matroid $\tilde{M}$ on $\tilde{E}$. For any $e \in \tilde{E} \setminus E$, define a matroid M/e on E by the rank function $\mathrm{rk}_{M/e}(A) = \mathrm{rk}_{\tilde{M}}(A + e) - 1$ for subsets $A \subseteq E$. It is called an *elementary quotient* of M. Note that the independent sets of M/e are the independent sets of M whose span does not contain e.

(ii) Let S be any subset of E. Realize the disjoint union $E \sqcup S$ as $(E, 0) \cup (S, 1)$. Denote by M_{+S} the matroid whose independent sets are of the form $(I_1, 0) \cup (I_2, 1)$ with $\mathrm{rk}_M(I_1 \cup I_2) = |I_1| + |I_2|$. The matroid M_{+S} is called the *parallel extension of M by S*.

Using theorem 3.3.7 we can obtain a corollary of proposition 3.3.6.

Corollary 3.3.10. *Let $\tilde{M}$ be a matroid on $\tilde{E} \neq \emptyset$, and let M be the submatroid induced on a subset $E \neq \emptyset$ of $\tilde{E}$. Assume that, for non-negative integers k and p and each non-empty subset $A \subseteq E$, one has*

$$|A| \leq (k+1) \cdot \mathrm{rk}\, A - (p+1).$$

Then, for any $e \in \tilde{E}$, there is an independent set $I \subset E$ such that $e \notin \mathrm{Cl}(I)$ and

$$|B| \leq k \cdot \mathrm{rk}(B) - p$$

for each non-empty subset $B \subseteq E - I$.

Proof. Consider the function $f : 2^E \to \mathbb{Z}$ defined by $f(A) = k \cdot \mathrm{rk}(A) - p$, and denote the submatroid of $\tilde{M}$ induced on E by M.

Let $A \neq \emptyset$ be any subset of E. Applying Proposition 3.3.6 to the submatroid of M induced on A, we get $\mathrm{rk}_{A(f)}(A) \geq |A| - \mathrm{rk}(A) + 1$, and so

$$|A| \leq \mathrm{rk}(A) + \mathrm{rk}_{A(f)}(A) - 1 \leq \mathrm{rk}(A) + \mathrm{rk}_{M(f)}(A) - 1. \tag{3.3.10.1}$$

We now consider two cases.

Case 1: Suppose e is not in E. Consider the elementary quotient M/e on E. By definition, for each subset $A \subseteq E$, one has $\mathrm{rk}_{M/e}(A) = \mathrm{rk}_{\tilde{M}}(A + e) - 1$. It follows that $\mathrm{rk}_{M/e}(A) \geq \mathrm{rk}(A) - 1$. Hence, Equation (3.3.10.1) gives

$$|A| \leq \mathrm{rk}_{M/e}(A) + \mathrm{rk}_{M(f)}(A).$$

Using Theorem 3.3.7, we conclude that there is a decomposition $E = I \sqcup J$ such that I is independent in M/e and J is independent in $M(f)$. Be definition of M/e, the span of I does not contain e. Therefore, $E = I \sqcup J$ is a partition with the required properties because, for each subset $B \neq \emptyset$ of J, one has

$$|B| \leq f(B) = k \cdot \mathrm{rk}(B) - p$$

as J is independent in $M(f)$.

Case 2: Suppose e is in E. Then consider first the parallel extension $M_{+\{e\}}$ of M on the set $(E, 0) \cup \{(e, 1)\}$. Second, passing to an elementary quotient of $M_{+\{e\}}$, we get a matroid $M_{+\{e\}}/(e, 1)$ on the ground set $(E, 0)$. To simplify notation, let us denote the latter matroid by M_{+e}/e and identify its ground set with E. Thus, we get for $A \subseteq E$ that

$$\mathrm{rk}_{M_{+e}/e}(A) = \mathrm{rk}_{M_{+\{e\}}}((A, 0) \cup \{(e, 1)\}) - 1 = \mathrm{rk}(A + e) - 1 \geq \mathrm{rk}(A) - 1.$$

Now we conclude as in Case 1, using M_{+e}/e in place of the matroid M/e. □

We are now ready to state the main theorem of this section. Before stating it we discuss it in the context of Edmond's Matroid Partition Theorem. If M is any matroid so that for all nonempty $A \subseteq M$ we have that $|A| \leq k\,\mathrm{rk}(M) - p$. Then it is a corollary of Edmond's Theorem, that if $N \supseteq M$ is another loopless Matroid so $|N \setminus M| \leq p$ then there is a partition of N into k independent sets.

It can further be shown using theorem 3.3.7 that if $N \setminus M = \{n_1, .., n_p\}$ then there is a partition of M into independent subsets sets $I_1, .., I_k$ so that $I_j \cup \{n_j\}$ is also independent for all $j \leq k$. However, this statement ends up not being strong enough for the algebraic applications we have in mind. In rough terms our statement below says we can in fact build this partition iteratively, so that for $t \leq p$ the sets $I_1, .., I_t$ depend only the matroid $M \cup \{n_1, .., n_t\}$. In particular the sets $I_1, .., I_t$ can be choosen independently of $n_{t+1}, .., n_k$.

Theorem 3.3.11. *Let $\tilde{M}$ be a matroid on $\tilde{E} \neq \emptyset$, and let k and p be non-negative integers. Assume there is a subset $E \neq \emptyset$ of $\tilde{E}$ such that*

$$|A| \leq k \cdot \mathrm{rk}_{\tilde{M}} A - p$$

for each non-empty subset $A \subseteq E$, and fix an integer q with $0 \leq q \leq p$. Then, for each q-tuple $(e_1, \ldots, e_q) \in \tilde{E}^q$, there are disjoint independent sets $\tilde{I}_1, \ldots, \tilde{I}_q$ of E with the following property: If $(a_1, \ldots, a_p) \in \tilde{E}^p$ is a p-tuple whose first q entries are $e_1, \ldots, e_q$, that is, $a_i = e_i$ if $1 \leq i \leq q$, then there is a partition $E = I_1 \sqcup \cdots \sqcup I_k$ into independent sets such that $a_j \notin \mathrm{Cl}(I_j)$ whenever $1 \leq j \leq p$ and $I_j = \tilde{I}_j$ for $j = 1 \ldots, q$.

Proof of Theorem 3.3.11. If $p = 0$, then the assertion is true by Edmond's criterion (theorem 3.3.1).

Let $p \geq 1$. First, we construct a suitable partition for a fixed p-tuple $(a_1, \ldots, a_p) \in \tilde{E}^p$ step by step. Consider $a_1 \in E$. By Corollary 3.3.10, there is a partition $E = I_1 \sqcup J_1$ such that I_1 is independent in M, $e_1 \notin \mathrm{Cl}(I_1)$, and $|B| \leq (k-1) \cdot \mathrm{rk}(B) - (p-1)$ for each non-empty subset $B \subseteq J_1$. Thus, we are done if $p = 1$. If $p \geq 2$, we apply Corollary 3.3.10 again, this time to $a_2 \in E$ and the submatroid of M induced on J_1. After p applications of Corollary 3.3.10, we obtain a partition $E = I_1 \sqcup \ldots \sqcup I_p \sqcup J_p$ such that $I_1, \ldots, I_p$ are independent in M, a_j is not in the span of I_j for each j, and $|B| \leq (k-p) \cdot \mathrm{rk}(B)$ for each non-empty subset $B \subseteq J_p$. Applying theorem 3.3.1 to the submatroid on J_p, we get a partition $J_p = I_{p+1} \sqcup \ldots \sqcup I_k$ into independent sets of M. This produces a desired partition for a fixed $(a_1, \ldots, a_p)$.

Second, we note that in the above construction the first p independent sets are obtained sequentially. Once the sets $I_1, \ldots, I_{j-1}$ have been found, the set I_j is determined in the complement of $I_1 \sqcup \ldots \sqcup I_{j-1}$. It depends on the choice of a_j, but not on the elements $a_{j+1}, \ldots, a_k$. This shows in particular that the sets $I_1, \ldots, I_q$ are independent of the elements $a_{q+1}, \ldots, a_k$. Thus, the argument is complete. □

Remark 3.3.12. (i) Using the notation of the proof of Corollary 3.3.10, the partition result in Theorem 3.3.11 can be also stated as follows: There is a partition $E = I_1 \sqcup \cdots \sqcup I_k$ such that $I_{p+1}, \ldots, I_k$ are independent in M and, for each $j = 1, \ldots, p$, the set I_j is independent in M/a_j if $a_j \notin E$ and independent in M_{+a_j}/a_j if $a_j \in E$, respectively.

(ii) If the ground set E of a matroid can be partitioned into k independent sets, then Edmond's criterion (theorem 3.3.1) implies that there is an independent set I such that $|A| \leq (k-1) \cdot \mathrm{rk}\, A$ for each subset A of $E \setminus I$. Thus, for a matroid satisfying the assumptions of Theorem 3.3.11, it is natural to wonder if there is an independent set I of E such that, for each $e \in I$ and each $A \subset (E \setminus I) + e$, one has $|A| \leq (k-1) \cdot \mathrm{rk}_{\tilde{M}} A - p$. However, this is not always possible, not even for representable matroids, see Example 4.2.8.

Chapter 4 The Segre Bound

In this section we give an application of the results from the previous section on Matroids to the study of fat points. Specifically we establish a conjectured bound on the regularity of an arbitrary fat point scheme. This chapter has appeared in [NT20].

For any set of fat points $Z = \sum_{i=1}^{s} m_i P_i \subseteq \mathbb{P}^1_{\mathbb{K}}$, we saw in proposition 2.4.2 that $\mathrm{r}(Z) = -1 + \sum_{i=1}^{s} m_i$. However, the case even for $\mathbb{P}^2$ is quite complicated. One of the earliest results in this area was a result due to Beniamino Segre

Theorem 4.0.1. *[Seg61] For $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^2$ a general set of fat points,*

$$\mathrm{r}(Z) \leq \max\left\{m_1 + m_2 - 1, \left\lceil \frac{-1 + \sum_{i=0}^{s} m_i}{2} \right\rceil\right\}.$$

This result was then subsequently generalized in a few directions. It was shown by Catalisano [Cat91] that the same bound holds as long as the points are in *linearly general position*, a concept we recall below.

Definition 4.0.2 (Linearly General Position)**.** We say $Z = \sum_{i=0}^{n} m_i P_i \subseteq \mathbb{P}^n$ is in *linearly general position* if every hyperplane $H \subseteq \mathbb{P}^n$ contains at most n points of $\mathrm{Supp}(Z)$.

In [CTV93] a generalized bound was given for the regularity fat points in linearly general position in $\mathbb{P}^n$. This bound was named the *Segre Bound* in honor of the original result.

Theorem 4.0.3. *[CTV93] Let $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^n$ be a set of points in linearly general position. Further suppose that $m_0 \geq m_1 \geq m_2 \geq \ldots \geq m_s$ then*

$$\mathrm{r}(Z) \leq \max\left\{m_0 + m_1 - 1, \left\lceil \frac{-1 + \sum_{i=0}^{s} m_i}{n} \right\rceil\right\}$$

Inspired by this result it was conjectured by Trung (as reported in [Thi00]) and, independently, by Fatabbi and Lorenzini in [FL01] that a generalized version of this bound holds for arbitrary sets of fat points. Namely, that $r(X) \leq \operatorname{Seg} X$, where $\operatorname{Seg} X$ is

$$\operatorname{Seg} X := \max\left\{\left\lceil \frac{-1 + \sum_{P_i \in L} m_i}{\dim L} \right\rceil \mid L \subseteq \mathbb{P}^n \text{ a linear subspace with } \dim L > 0\right\}.$$

The generalized conjecture had been shown in rather few cases, namely

- for any fat point subscheme of $\mathbb{P}^2$ in [94] and [Thi99], independently,
- for any fat point subscheme of $\mathbb{P}^3$ in [FL01] and [Thi00], independently, and
- if $s \leq n + 3$ and the s points span $\mathbb{P}^n$ in [BDP16].

Furthermore, there are partial results for certain fat point subschemes of $\mathbb{P}^4$ (see [Bal15a; Bal15b]) and for some fat point subschemes of $\mathbb{P}^n$ supported at at most $2n-1$ points (see [CFL16]). In the first section of this chapter we establish the conjecture in full generality, that is, we show $r(X) \leq \operatorname{Seg} X$ for each fat point subscheme X of some projective space. We then continue to give a further generalization which improves the bound in some cases, for instance in the case of general points. The Segre Bound cannot be improved in general (see Corollary 4.3.5).

4.1 Inductive Techniques

We now begin considering zero-dimensional subschemes of projective space. In this section we collect some facts that are used in subsequent parts of this chapter.

Let K be an arbitrary field, and let X be any projective subscheme of some projective space $\mathbb{P}^n = \mathbb{P}^n_K$. For short, we often write $H^1(\mathcal{I}_X(j))$ instead of $H^1(\mathbb{P}^n, \mathcal{I}_X(j))$ for the first cohomology of its ideal sheaf $\mathcal{I}_X$. We use $R = K[x_0, \ldots, x_n]$ to denote the coordinate ring of $\mathbb{P}^n$.

Lemma 4.1.1. *Let $X \subset \mathbb{P}^n$ be a zero-dimensional subscheme.*

(a) Then $r(X) = \min\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_X(j)) = 0\}$.

(b) For any zero-dimensional subscheme Z of X, one has that $r(Z) \leq r(X)$.

Proof. These results are known to specialists. We include a proof for the convenience of the reader. Part (a) is a consequence of

$$h_X(j) - \deg X = -\dim_K H^1(\mathcal{I}_X(j)).$$

This relation also shows that $h_X(j) \leq \deg X$ for all integers j and that equality is true if and only if $j \geq r(X)$. Hence, the exact sequence $0 \to I_Z/I_X \to R/I_X \to R/I_Z \to 0$ gives that $h_X(j) = \deg X$ implies $h_Z(j) = \deg Z$. Now (b) follows. □

A special case of lemma 4.1.1(b) has been shown in [Thi16, Proposition 3.2]. We also need the following fact about the Castelnuovo-Mumford regularity, which can be found, e.g., in [Eis05, Corollary 4.4].

Lemma 4.1.2. *If $A \neq 0$ is an artininan graded K-algebra, then one has*

$$\operatorname{reg}(A) = \max\{j \mid [A]_j \neq 0\}.$$

The following observation is an extension of [CTV93, Lemma 1].

Lemma 4.1.3. *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme, and let $P \in \mathbb{P}^n$ be a point that is not in the support of Z. Then one has, for every integer $m \geq 1$,*

$$r(Z + mP) = \max\{m - 1,\ r(Z),\ 1 + \operatorname{reg}(R/(I_Z + I_P^m))\}.$$

Proof. The argument is essentially given in [CTV93]. We recall it for the reader's convenience.

Consider the Mayer-Vietoris sequence

$$0 \to R/I_{Z+mP} \to R/I_Z \oplus R/I_{mP} \to R/(I_Z + I_P^m) \to 0.$$

Since $\deg(Z+mP) = \deg Z + \deg(mP)$ and $r(mP) = m-1$, it shows that $h_{Z+mP}(j) = \deg(Z+mP)$ if and only if $h_Z(j) = \deg Z,\quad h_{mP}(j) = \deg mP$, and $[R/(I_Z+I_P^m)]_j = 0$. Since $R/(I_Z + I_P^m)$ is artinian we conclude by Lemma 4.1.2. □

The following result follows from a standard residual sequence (see [FL01, Theorem 3.2] for a special case).

Lemma 4.1.4 (Inductive Technique 1)**.** *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme, and let $F \subset \mathbb{P}^n$ be a hypersurface defined by a form $f \in R$. Denote by $\emptyset \neq W \subset \mathbb{P}^n$ the residual of Z with respect to F (defined by $I_Z : f$). If $Z \cap F \neq \emptyset$, then one has*

$$r(Z) \leq \max\{r(W) + \deg F,\ r(Z \cap F)\}.$$

Proof. Let $d = \deg F$. Multiplication by f induces the following exact sequence of ideal sheaves

$$0 \to \mathcal{I}_W(-d) \to \mathcal{I}_Z \to \mathcal{I}_{Z\cap F} \to 0.$$

Its long exact cohomology sequence gives, for all integers j,

$$H^1(\mathcal{I}_W(j-d)) \to H^1(\mathcal{I}_Z(j)) \to H^1(\mathcal{I}_{Z\cap F}(j)).$$

Now the claim follows because $r(Z) = \min\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_Z(j)) = 0\}$ (see Lemma 4.1.1). □

If a hypersurface F is defined by a form f, then we also write $Res_f(Z)$ for $Res_F(Z)$.

For induction on the multiplicity of a point in the support of a fat point scheme, the statement below will be useful.

Lemma 4.1.5 (Inductive Technique 2)**.** *Let $Z = \sum_{i=1}^s m_i P_i \subset \mathbb{P}^n$ be a fat point scheme, and let $P \in \mathbb{P}^n$ be a point that is not in the support of Z. Fix integers b, k and m with $0 \leq k \leq m-1 \leq b$. Assume there are polynomials $g_1, .., g_t \in I_P^k$ and $f_1, .., f_t \in R$ so that $[I_P^k]_{b-m+k+1} = [(g_1, .., g_t) + I_P^{k+1}]_{b-m+k+1}$, $f_i(P) \neq 0$ and*

$$r(Res_{g_i f_i}(Z + mP)) \leq b - \deg(g_i f_i)$$

for all $i \in \{1, 2, \ldots, t\}$. If $r(Z + (m-1)P) \leq b$, then $r(Z + mP) \leq b$.

Proof. Note that it is enough to show $[R/(I_Z + I_P^m)]_b = 0$. Indeed, $[R/(I_Z + I_P^m)]_b = 0$ implies $1 + \operatorname{reg}(R/(I_Z + I_P^m)) \leq b$ by Lemma 4.1.2. Furthermore, the assumption $r(Z + (m-1)P) \leq b$ gives $r(Z) \leq b$ by Lemma 4.1.1(b). Since we also assume $m - 1 \leq b$, Lemma 4.1.3 shows $r(Z + mP) \leq b$.

In order to prove $[R/(I_Z + I_P^m)]_b = 0$ observe that

$$\dim_K [R/(I_Z + I_P^m)]_b = \sum_{j=0}^{m-1} \dim_K [(I_Z + I_P^j)/(I_Z + I_P^{j+1})]_b.$$

By assumption and Lemma 4.1.1, we know $r(Z + jP) \leq b$ if $0 \leq j < m$. Hence Lemma 4.1.3 gives $[I_Z + I_P^j]_b = [R]_b$. It follows that

$$\dim_K [R/(I_Z + I_P^m)]_b = \dim_K [(I_Z + I_P^{m-1})/(I_Z + I_P^m)]_b.$$

Thus, we are done once we have shown

$$[I_Z + I_P^{m-1}]_b = [I_Z + I_P^m]_b. \tag{4.1.5.1}$$

Let $\ell \in R$ be any linear form that does not vanish at P. Then $(x_0, \dots, x_n) = (\ell, I_P)$. Since I_P^{m-1} is generated by polynomials of degree $m-1$, it follows that Equality (4.1.5.1) is true if and only if

$$\ell^{b-m+1} \cdot [I_P^{m-1}]_{m-1} \subset I_Z + I_P^m. \tag{4.1.5.2}$$

Observe that, for each $i \in [t] = \{1, 2, \dots, t\}$, the scheme $W_i := Res_{g_i f_i}(Z + mP))$ is defined by $I_{Z+mP} : (g_i f_i)$ and has multiplicity $m - k$ at P because $f_i(P) \neq 0$ and g_i vanishes precisely to order k at P by assumption. Denote by J_i the homogeneous ideal of $W_i - (m-k)P$. Thus, $I_{W_i} = J_i \cap I_P^{m-k}$. Hence, Lemma 4.1.3 gives

$$r(W_i) = \max\{m - k - 1,\ r(W_i - (m-k)P),\ 1 + \operatorname{reg}(R/(J_i + I_P^{m-k}))\}.$$

Since $r(W_i) \leq b - d_i$ by assumption, where $d_i = \deg(g_i f_i)$, we get as above, for each $i \in [t]$,

$$0 = \dim_K [R/(J_i + I_P^{m-k})]_{b-d_i} = \sum_{j=0}^{m-k-1} \dim_K [(J_i + I_P^j)/(J_i + I_P^{j+1})]_{b-d_i}.$$

In particular, this yields $[J_i + I_P^{m-k-1}]_{b-d_i} = [J_i + I_P^{m-k}]_{b-d_i}$. Letting $D = b - m + k + 1$ we conclude

$$\ell^{D-d_i} \cdot [I_P^{m-k-1}]_{m-k-1} \subset J_i + I_P^{m-k} \tag{4.1.5.3}$$

because $D - d_i = b - d_i - m + k + 1 \geq 0$. This latter estimate follows from $(m-k)P \subset W_i$, which implies $0 \leq m - k - 1 = r((m-k)P) \leq r(W_i) \leq b - d_i$ (see Lemma 4.1.1).

Note that, for each $i \in [t]$, one has $J_i = I_Z : (g_i f_i)$. Using $g_i \in I_P^k$ this gives

$$g_i f_i \cdot (J_i + I_P^{m-k}) \subset I_Z + I_P^m.$$

Combined with Inclusion 4.1.5.3, we get

$$g_i f_i \ell^{D-d_i} \cdot [I_P^{m-k-1}]_{m-k-1} \subset I_Z + I_P^m.$$

Since $f(P_i) \neq 0$, possibly after rescaling, we may write $f_i = h_i + \ell^{\deg(f_i)}$ for some $h_i \in I_P$. Substituting, we obtain,

$$g_i(h_i + \ell^{\deg(f_i)})\ell^{D-d_i} \cdot [I_P^{m-k-1}]_{m-k-1} \subset I_Z + I_P^m.$$

Now $g_ih_i \in I_P^{k+1}$ yields $\ell^{D-\deg(g_i)}g_i \cdot [I_P^{m-k-1}]_{m-k-1} \subseteq I_Z + I_P^m$. Furthermore, as $(I_p^{k+1})(I_P^{m-k-1}) \subseteq I_P^m$ we can also conclude that

$$[(\ell^{D-\deg(g_i)}g_i) + I_P^{k+1}]_D \cdot [I_P^{m-k-1}]_{m-k-1} \subseteq I_Z + I_P^m, \quad \text{for each } i \in [t]. \qquad (4.1.5.4)$$

Now note that as a R-module $\operatorname{Ann}(I_P^k/I_P^{k+1}) = I_P$ so I_P^k/I_p^{k+1} is a R/I_P module. As $[R/I_p]_d$ is spanned by ℓ^d for all $d \geq 0$, and $g_i \in I_P^k$ then $[(g_i) + I_p^{k+1}]_d = [(\ell^{d-\deg(g_i)}g_i) + I_P^{k+1}]_d$ for $d \geq \deg(g_i)$. By assumption $g_1, \ldots, g_t$ generate a ideal with $[I_P^k]_D = [(g_1, \ldots, g_t) + I_p^{k+1}]_D$. Since

$$[(g_1, \ldots, g_t) + I_p^{k+1}]_D = [(\ell^{D-\deg(g_1)}g_1, \ldots, \ell^{D-\deg(g_t)}g_t) + I_P^{k+1}]_D,$$

we see in particular that $\ell^{b-m+1}[I_P^k]_k \subseteq [(\ell^{D-\deg(g_1)}g_1, \ldots, \ell^{D-\deg(g_t)}g_t) + I_P^{k+1}]_D$, combined with (4.1.5.4) this establishes the desired Containment (4.1.5.2). $\square$

4.2 Reduced Zero-dimensional Subschemes

We now establish the Segre bound for an arbitrary finite set of points. To this end we use suitable vector matroids.

Recall that a vector matroid or representable matroid M over a field K is given by an $m \times n$ matrix A with entries in K. Its ground set E is formed by the column vectors of A, and the rank of a subset of E is the dimension of the subspace of K^n they generate. Here we adapt this idea in order to use it in a projective space instead of an affine space.

Definition 4.2.1. (i) For a point P of $\mathbb{P}^n$ and an integer $m \geq 1$, denote by $[P]^m$ an $(n+1) \times m$ matrix whose m columns are all equal to a vector $v \in K^{n+1}$, where v is any representative of the point P.

(ii) Let $X = \sum_{i=1}^s m_iP_i \subset \mathbb{P}^n$ be a fat point scheme. We write $A_X := \oplus_{i=0}^s [P_i]^{m_i}$ for the concatenation of the matrices $[P_i]^{m_i}$. Define the *matroid of* X on the column set E_X of A_X, denoted M_X, as the vector matroid to the matrix A_X. Thus $|V_X| = \sum_{i=1}^s m_i$.

Remark 4.2.2. (i) Since we are only interested in the span of a subset of columns, the above definition does not depend on the choice of coordinate vectors for the points. Abusing notation slightly, we will identify a non-zero vector of K^{n+1} with a point in $\mathbb{P}^n$.

(ii) For consistency of notation, rk will always refer to rank in the matroid sense, that is, to a dimension of a subspace of K^{n+1}, and dim will always refer to dimension in $\mathbb{P}^n$. Hence, if S is a subset of the column set E_X, then $\operatorname{rk}(S) = 1 + \dim_{\mathbb{P}^n} \operatorname{Span}(S)$. Furthermore, we will use Cl to refer to the closure operator in a matroid and Span to refer to the span of the points in $\mathbb{P}^n$.

Recall that the Segre bound of $X = \sum_{i=1}^{s} m_i P_i$ is

$$\operatorname{Seg}(X) = \max\left\{ \left\lceil \frac{w_L(X) - 1}{\dim L} \right\rceil \mid L \subseteq \mathbb{P}^n \text{ a linear subspace with } \dim L > 0 \right\},$$

where $w_L(X) = \sum_{P_i \in L} m_i$ is the *weight* of $X|_L$.

Remark 4.2.3. In the literature the Segre bound has also been defined as

$$\operatorname{Seg}(X) = \max\left\{ \left\lfloor \frac{w_L(X) + \dim L - 2}{\dim L} \right\rfloor \mid L \subseteq \mathbb{P}^n \text{ a subspace with } \dim L > 0 \right\}.$$

Obviously, this is equivalent to our definition above.

Lemma 4.2.4. *If $X = \sum_{i=1}^{s} m_i P_i$ is a fat point scheme whose support consists of at least two distinct points, then $m_i \le \operatorname{Seg}(X)$ for all i and $\operatorname{Seg}(X) \ge m_i + m_j - 1$ whenever $i \ne j$.*

Proof. Let L be a line passing through two distinct points P_i and P_j in the support of X. Then $w_L(X) \ge m_i + m_j$, which implies $\operatorname{Seg}(X) \ge m_i + m_j - 1$. □

Remark 4.2.5. If $X = m_1 P_1$ is supported at a single point, then $r(X) = \operatorname{Seg} X = m_1 - 1$.

The following is the main result of this section.

Theorem 4.2.6. *Let $Z \subset \mathbb{P}^n$ be a fat point scheme satisfying $r(Z) \le \operatorname{Seg}(Z)$. Then, for every point $P \in \mathbb{P}^n$ that is not in the support of Z, one has $r(Z+P) \le \operatorname{Seg}(Z+P)$.*

Proof. We want to use inductive technique 1. To this end, consider the matrix

$$A = A_Z \oplus [P]^B = \oplus_{i=1}^{s} [P_i]^{m_i} \oplus [P]^B,$$

where $B = \operatorname{Seg}(Z + P)$ and $Z = \sum_{i=1}^{s} m_i P_i$. Let M be the vector matroid on the column set V of A. Set $X = Z + P$.

Consider any subset S of V. If $P \notin \operatorname{Span}(S)$, then the definition of weight gives

$$|\operatorname{Cl}(S)| = w_{\operatorname{Span}(S)}(Z) = w_{\operatorname{Span}(S)}(X).$$

If $P \in \operatorname{Span}(S)$, then $w_{\operatorname{Span}(S)}(X) = 1 + w_{\operatorname{Span}(S)}(Z)$, and thus

$$|\operatorname{Cl}(S)| = w_{\operatorname{Span}(S)}(X) + B - 1.$$

In either case we have

$$|S| \le w_{\operatorname{Span}(S)}(X) + B - 1.$$

Using $\operatorname{rk}(S) = 1 + \dim_{\mathbb{P}^n} S$, the definition of $B = \operatorname{Seg}(X)$ yields, for any subset $S \subset V$ with $\operatorname{rk}(S) \ge 2$,

$$\frac{|S| - B}{\operatorname{rk}(S) - 1} \le \frac{w_{\operatorname{Span}(S)}(X) - 1}{\dim(\operatorname{Span}(S))} \le \operatorname{Seg}(X) = B.$$

It follows that

$$|S| \leq \mathrm{rk}(S) \cdot B.$$

This estimate is also true if $\mathrm{rk}(S) \leq 1$ as $B \geq m_i$ for all i (see Lemma 4.2.4). Therefore Corollary 3.3.1 gives that there is a partition of the column set V into B linearly independent subsets $I_1, \ldots, I_B$. Note that $P \in I_j$ for each $j \in \{1, 2, \ldots, B\}$ as B columns of the matrix A correspond to the point P. Thus, for each such j, there is a hyperplane H_j such that

$$\mathrm{Span}(I_j \setminus \{P\}) \subset H_j \quad \text{and} \quad P \notin H_j.$$

It follows that the hypersurface $F = H_1 + \cdots + H_B$ does not contain P. However, F does contain Z because any form defining F vanishes at each point P_j to order at least m_j as m_j columns of A correspond to P_j. Hence we get $Res_F(X) = P$ and $X \cap F = Z$. Now Lemma 4.1.4 gives $r(X) \leq \max\{B, r(Z)\} = B$, as desired. □

Corollary 4.2.7. *If X is any reduced zero-dimension subscheme of $\mathbb{P}^n$, then $r(X) \leq$ $\mathrm{Seg}(X)$.*

Proof. This is true if X consists of one point (see Remark 4.2.5). Thus, we conclude by induction on the cardinality of X using the above theorem. □

We conclude this section with an example as promised in Remark 3.3.12(ii).

Example 4.2.8. Consider any integers $k > p > 0$, and let K be an infinite field. Let $L_1, \ldots, L_t \subset K^{t-1}$ be t generic one-dimensional subspaces, where $t \geq \frac{k}{p} + 1$. On each of the lines choose generically $k-p$ points. Let M be the vector matroid on the set E of all these vectors. Then, one has for each non-empty subset $A \subset E$ that $|A| \leq k \cdot \mathrm{rk}\, A - p$. Indeed, if $A = E$ this follows because $|E| = t(k-p) \leq k \cdot \mathrm{rk}\, E - p = k \cdot (t-1) - p$ by the assumption on t. If the rank of A is at most $t-2$, then it contains at most $\mathrm{rk}\, A$ of the lines $L_1, \ldots, L_t$, which implies $|A| \leq \mathrm{rk}\, A \cdot (k-p) \leq k \cdot \mathrm{rk}\, A - p$, as desired.

Assume now there is an independent $I \subset E$ with at most $t-2$ elements such that for each non-empty subset $B \subset E \setminus I$ one has $|B| \leq (k-1) \cdot \mathrm{rk}\, B - p$. Thus, $|B| \leq k - 1 - p$ if B has rank one. Consider now $B = E \setminus I$. By assumption on I, we have $|B| \geq t(k-p) - (t-2) = t(k-p-1) + 2$. However, we also obtain $|B| = \sum_{i=1}^{t} |B \cap L_i| \leq t(k-p-1)$. This contradiction shows that M is a matroid as desired in Remark 3.3.12(ii).

4.3 Arbitrary Fat Point Schemes

The goal of this section is to establish the conjecture by Trung, Fatabbi, and Lorenzini. We also discuss the sharpness of the Segre bound and establish an alternate regularity estimate.

We need one more preparatory result on the matroid introduced in Definition 4.2.1.

Lemma 4.3.1. *Consider the vector matroid M to a fat point scheme $Z = \sum_{j=1}^{s} m_j P_j$ on the column set E_Z. Then, for every subset $S \subset E_Z$ with $\mathrm{rk}\, S \geq 2$, one has*

$$|S| \leq \mathrm{Seg}(Z) \cdot \{\mathrm{rk}(S) - 1\} + 1.$$

Proof. Recall that $\operatorname{rk}(S) = \dim(\operatorname{Span}(S)) + 1$ for any subset $S \subset E_Z$. Moreover, one has $|S| \leq |\operatorname{Cl}_M(S)| = w_L(Z)$, where $L = \operatorname{Span}(S)$. Hence, if $\operatorname{rk} S \geq 2$ we obtain

$$\frac{|S| - 1}{\operatorname{rk}(S) - 1} \leq \frac{w_L(Z) - 1}{\dim L} \leq \operatorname{Seg}(Z).$$

Now the claim follows. $\square$

The following result allows us to use induction on the cardinality of the support of a fat point scheme.

Proposition 4.3.2. *Let $Z \subset \mathbb{P}^n$ be a fat point scheme satisfying $r(Z) \leq \operatorname{Seg}(Z)$. Then, for every point $P \in \mathbb{P}^n$ that is not in the support of Z and every integer $m \geq 1$, one has $r(Z + mP) \leq \operatorname{Seg}(Z + mP)$.*

Proof. We want to apply Inductive Technique 2 to $X = Z + mP$, where $Z = \sum_{j=1}^{s} m_j P_j$. This requires some preparation. Consider the vector matroid associated to the matrix

$$A_Z = \oplus_{i=1}^{s} [P_i]^{m_i}$$

with column set E_Z. Define another matroid M on E_Z by setting the rank of any subset $S \subseteq E_Z$ as $\operatorname{rk}_M(S) = \operatorname{rk}(S + P) - 1 = \dim \operatorname{Span}(S + P)$. Thus, we get

$$\operatorname{rk}_M(S) \geq \dim \operatorname{Span}(S) = \operatorname{rk}(S) - 1.$$

In particular, a subset I of E_Z is independent in M if and only if $I + P$ is a linearly independent subset of $\mathbb{P}^n$. Notice that the matroid M is determined by Z and P only and independent of the multiplicity of P in X. We now argue that, for every subset $S \neq \emptyset$ of E_Z, one has

$$|S| \leq \operatorname{Seg}(X) \cdot \operatorname{rk}_M(S) - (m - 1). \tag{4.3.2.1}$$

Indeed, given any subset $S \neq \emptyset$ of E_Z, extend S by m copies of P to a subset S' of E_X. Then one has $\operatorname{rk} S' \geq 2$, and thus by applying Lemma 4.3.1 to S' we obtain

$$|S| + m = |S'| \leq \operatorname{Seg}(X) \cdot \{\operatorname{rk}(S') - 1\} + 1 = \operatorname{Seg}(X) \cdot \operatorname{rk}_M(S) + 1,$$

which completes the argument for Estimate (4.3.2.1).

We are now going to show the following key statement.

Claim: Given Z and P as above, suppose that there are integers σ and $m \geq 1$ such that, for every non-empty subset $S \subseteq E_Z$, one has

$$|S| \leq \sigma \cdot \operatorname{rk}_M(S) - (m - 1). \tag{4.3.2.2}$$

Then there are $t = \binom{n+m-2}{n-1}$ generators $g_1, \ldots, g_t$ of I_P^{m-1} and degree $\sigma - m + 1$ forms $f_1, \ldots, f_t$ with $f_j(P) \neq 0$ such that

$$g_j f_j \in I_{Z+(m-1)P} \quad \text{for } j = 1, \ldots, t. \tag{4.3.2.3}$$

To establish this claim, we use induction on $m \geq 1$. Let $m = 1$. Then Assumption (4.3.2.2) is also true for $S = \emptyset$. Hence Corollary 3.3.1 gives a partition $E_Z = I_1 \sqcup \ldots \sqcup I_\sigma$ into independent sets of M. Thus, P is not in any $\operatorname{Span}(I_j)$, and so there are σ linear forms ℓ_j such that $\ell_j(P) \neq 0$ and $I_j \subset H_j$, where H_j is the hyperplane defined by ℓ_j. It follows that $f = \ell_1 \cdots \ell_\sigma$ is in I_Z and $f(P) \neq 0$, as desired.

Let $m \geq 2$. Choose a point $Q_1 \in \mathbb{P}^n \setminus \{P\}$. Pass from the vector matroid to the matrix $A_Z \oplus [Q_1]$ to a matroid $\widetilde{M}$ on $E_Z \cup \{Q_1\}$ as for M above. That is, $\operatorname{rk}_{\widetilde{M}}(S) = \operatorname{rk}(S + P) - 1 = \dim \operatorname{Span}(S + P)$ for any subset $S \subseteq E_Z \cup \{Q_1\}$. Due to Assumption (4.3.2.2) we can apply Corollary 3.3.10 to obtain a partition

$$E_Z = I_1 \sqcup J_1,$$

where I_1 is independent in M, $Q_1 \notin \operatorname{Span}(I_1 + P)$, and

$$|B| \leq (\sigma - 1) \cdot \operatorname{rk}_M(B) - (m - 2) \tag{4.3.2.4}$$

for each subset $B \neq \emptyset$ of J_1. Let W_1 be the fat point scheme determined by J_1, that is, $W_1 = \sum_{j=1}^{s} n_j P_j$, where n_j is the number of column vectors in J_1 corresponding to the point P_j. Estimate (4.3.2.4) shows that the induction hypothesis applies to W_1. Hence, there are $u = \binom{n+m-3}{n-1}$ generators $h_1^{(1)}, \ldots, h_u^{(1)}$ of I_P^{m-2} and degree $\sigma - m + 1$ forms $q_1^{(1)}, \ldots, q_u^{(1)}$ with $q_j^{(1)}(P) \neq 0$ such that $h_j^{(1)} q_j^{(1)} \in I_{W_1 + (m-2)P}$ for each j.

Since Q_1 is not in the span of the linearly independent set $I_1 + P$, there is a linear form ℓ_1 such that $\ell_1(Q_1) \neq 0$ and $I_1 + P \subset H_1$, where H_1 is the hyperplane defined by ℓ_1. Taking into account that $E_Z = I_1 \sqcup J_1$, it follows that $\ell_1 h_j^{(1)} q_j^{(1)} \in I_{Z + (m-1)P}$ for each j.

Notice that the above construction of the forms $h_1^{(1)}, \ldots, h_u^{(1)}$, $q_1^{(1)}, \ldots, q_u^{(1)}$, and ℓ_1, depending on the choice of Q_1, works for any point in $\mathbb{P}^n \setminus \{P\}$. Repeating it $(n-1)$ more times by choosing alltogether points $Q_1, \ldots, Q_n \in \mathbb{P}^n \setminus \{P\}$, we obtain linear forms $\ell_1, \ldots, \ell_n \in I_P$ as well as n generating sets $\{h_1^{(i)}, \ldots, h_u^{(i)}\}$ of I_P^{m-2}, and degree $\sigma - m + 1$ forms $q_j^{(i)}$ with $q_j^{(i)}(P) \neq 0$ such that

$$\ell_i h_j^{(i)} q_j^{(i)} \in I_{Z+(m-1)P} \quad \text{for all } i = 1, \ldots, n, \; j = 1, \ldots, u. \tag{4.3.2.5}$$

The forms $h_1^{(i)}, \ldots, h_u^{(i)}$, $q_1^{(i)}, \ldots, q_u^{(i)}$, and ℓ_i depend on the choice of the point Q_i, $i = 1 \ldots, n$.

We now claim that by choosing the points $Q_2, \ldots, Q_n$ suitably we can additionally achieve that the linear forms $\ell_1, \ldots, \ell_n$ are linearly independent. We show this recursively. Let $2 \leq i \leq n$ and assume that points $Q_1, \ldots, Q_{i-1}$ have been found such that the linear forms $\ell_1, \ldots, \ell_{i-1}$ are linearly independent. Let H_j be the hyperplane defined by ℓ_j. Since $\dim(\bigcap_{j=1}^{i-1} H_j) \geq 1$, there is a point Q_i in $(\bigcap_{j=1}^{i-1} H_j) \setminus \{P\}$. By construction of H_i, the point Q_i is not contained in H_i. Thus, we get

$$\dim \bigcap_{j=1}^{i} H_j = \dim \bigcap_{j=1}^{i-1} H_j - 1 = n - (i-1) - 1 = n - i.$$

In particular, we have shown that $\dim(\bigcap_{j=1}^{n} H_j) = 0$. Since each of the hyperplanes H_j contains the point P, we conclude that the ideal of this point is $I_P = (\ell_1, \dots, \ell_n)$. Now it follows that $\{\ell_i h_j^{(i)} \mid 1 \le i \le n,\ 1 \le j \le u\}$ is a generating set of $I_P \cdot I_P^{m-2} = I_P^{m-1}$. It is not minimal. However, it contains a minimal generating set $\{f_1, \dots, f_t)$ of I_P^{m-1}, where each f_k is of the form $\ell_i h_j^{(i)}$. Setting $g_k = q_j^{(i)}$, Containment (4.3.2.5) implies the claim.

After these preparations we are ready to show $r(Z + mP) \le \operatorname{Seg}(Z + mP)$. We use induction on $m \ge 1$. If $m = 1$, then we are done by Theorem 4.2.6.

Let $m \ge 2$. Estimate (4.3.2.1) shows that we can apply the above claim with $\sigma = \operatorname{Seg}(X)$ and m being the multiplicity of P in $X = Z + mP$. Adopt the notation of this claim. Since each form g_j vanishes precisely to order $m - 1$ at P, it follows that $I_{Z+mP} : f_j g_j = I_P$, and thus

$$r(Res_{g_j f_j}(Z + mP)) = r(P) = 0$$

for each j. Since $Z + (m-1)P$ is a subscheme of $Z + mP$, the definition of the Segre bound implies $\operatorname{Seg}(Z + (m-1)P) \le \operatorname{Seg}(Z + mP) = \operatorname{Seg}(X)$. By the induction hypothesis on m, we know $r(Z + (m-1)P) \le \operatorname{Seg}(Z + (m-1)P)$, and so we get $r(Z + (m-1)P) \le \operatorname{Seg}(X)$. Thus, applying Lemma 4.1.5 we conclude that $r(Z + mP) \le \operatorname{Seg}(X)$, as desired. □

The regularity bound announced in the introduction follows now easily.

Theorem 4.3.3. *If $X = \sum_{i=1}^{s} m_i P_i$ is any fat point subscheme of $\mathbb{P}^n$, then $r(X) = \operatorname{reg}(X) - 1 \le \operatorname{Seg}(X)$.*

Proof. This is true if X consists of one point (see Remark 4.2.5). Thus, we conclude by induction on the cardinality of $\operatorname{Supp} X$ using the above proposition. □

We conclude by discussing a modification of the above Segre bound. To this end consider the d-th Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^N$, where $d \in \mathbb{N}$ and $N = \binom{n+d}{d} - 1$. We use it to compare the regularity indices of fat point schemes in $\mathbb{P}^n$ and $\mathbb{P}^N$, respectively.

Proposition 4.3.4. *Let $X = \sum_{i=1}^{s} m_i P_i$ be a fat point subscheme of $\mathbb{P}^n$. Define a fat point subscheme $\hat{X}$ of $\mathbb{P}^N$ by $\hat{X} = \sum_{i=1}^{s} m_i v_d(P_i)$. Then one has $\left\lceil \frac{r(X)}{d} \right\rceil \le r(\hat{X})$.*

Moreover, if both $n = 1$ and $d(m_j + m_k) \le 2d - 2 + \sum_{i=1}^{s} m_i$ for all integers j, k with $1 \le j < k \le s$, then this is an equality and $r(\hat{X}) = \left\lceil \frac{-1+\sum_{i=1}^{d} m_i}{d} \right\rceil$.

Proof. Let $S = \oplus_{j \in \mathbb{N}_0} [R]_{jd}$ be the d-th Veronese subring of $R = K[x_0, ..., x_n]$. It is a polynomial ring in variables y_a, where y_a corresponds to the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ of degree d. Consider the ring homomorphism $\varphi : S \to R$ that maps y_a onto x^a. Observe that, for each point $P \in \mathbb{P}^n$, one has $\varphi(I_{v_d(P)}) \subset I_P$. If follows that $\varphi(I_{\hat{X}}) \subset I_X$, and so $I_{\hat{X}} \subset \varphi^{-1}(I_X)$. Furthermore, the ideal $\varphi^{-1}(I_X)$ of S is saturated. Indeed, if $f \in S$ is a homogeneous polynomial that multiplies a power, say, the k-th power of the ideal generated by all the variables in S into $\varphi^{-1}(I_X)$, then $\varphi(f) \cdot (x_0, \ldots, x_n)^{kd} \subset I_X$. Since I_X is saturated, this implies $f \in \varphi^{-1}(I_X)$, as desired.

Thus, the ideal $\varphi^{-1}(I_X)$ is the homogenous ideal of a zero-dimensional subscheme $W \subset \mathbb{P}^N$, and one has

$$H^1(\mathbb{P}^n, \mathcal{I}_X(j)) \cong H^1(\mathbb{P}^N, \mathcal{I}_W(jd)).$$

Hence, Lemma 4.1.1(a) implies $r(W) = \left\lceil \frac{r(X)}{d} \right\rceil$. Since W is a subscheme of $\hat{X}$, Lemma 4.1.1(b) gives $r(W) \leq r(\hat{X})$, and now the first assertion follows.

In order to show the second claim, assume $n = 1$. Thus $N = d$, and $\operatorname{Supp} \hat{X}$ lies on a rational normal curve of $\mathbb{P}^d$. It follows that the support of $\hat{X}$ is in linearly general position, that is, any subset of $j + 1 \leq d + 1$ points span a j-dimensional linear subspace of $\mathbb{P}^d$. Therefore, a straightforward computation shows that the Segre bound of $\hat{X}$ is determined by the one-dimensional subspaces and $\mathbb{P}^d$, that is,

$$\operatorname{Seg} \hat{X} = \max \left\{ m_j + m_k - 1, \left\lceil \frac{-1 + \sum_{i=1}^{s} m_i}{d} \right\rceil \mid 1 \leq j < k \leq s \right\}.$$

Combining the assumption and Theorem 4.3.3, we obtain

$$r(\hat{X}) \leq \operatorname{Seg} \hat{X} = \left\lceil \frac{-1 + \sum_{i=1}^{s} m_i}{d} \right\rceil.$$

Since X is a subscheme of $\mathbb{P}^1$, its homogeneous ideal is a principal ideal of degree $\sum_{i=1}^{s} m_i$. Thus, $r(X) = -1 + \sum_{i=1}^{s} m_i$. Now the first assertion gives the desired equality. □

As a first consequence, we describe instances where the Segre bound in Theorem 4.3.3 is sharp. The result extends [CTV93, Proposition 7].

Corollary 4.3.5. *Let $X \subset \mathbb{P}^n$ be a fat point subscheme, and let $L \subset \mathbb{P}^n$ be a positive-dimensional linear subspace such that* $\operatorname{Seg} X = \left\lceil \frac{w_L(X)-1}{\dim L} \right\rceil$. *If the points of* $\operatorname{Supp} X$ *that are in L lie on a rational normal curve of L, then* $r(X) = \operatorname{Seg} X$.

Proof. Consider the fat point subscheme $Y = \sum_{P_i \in L} m_i P_i$ of X such that $w_L(X) = w_L(Y)$. If $\dim L = 1$, then $w_L(Y) - 1 = r(Y) \leq r(X) \leq w_L(X) - 1$, and thus the claim follows.

Assume $\dim L \geq 2$. Considering lines through any two points in the support of X, the assumption on L gives $m_j + m_k - 1 \leq \left\lceil \frac{w_L(Y)-1}{\dim L} \right\rceil$ for all $j < k$. Hence, applying Proposition 4.3.4 with $\hat{X} = Y$, we conclude $r(Y) = \left\lceil \frac{-1+\sum_{P_i \in L} m_i}{\dim L} \right\rceil = \left\lceil \frac{w_L(Y)-1}{\dim L} \right\rceil = \operatorname{Seg} X$. Since $r(Y) \leq r(X)$, the desired equality follows by Theorem 4.3.3. □

The second consequence of Proposition 4.3.4 is a generalized regularity bound. Notice that the following result specializes to Theorem 4.3.3 if $d = 1$.

Theorem 4.3.6. *Given any scheme of fat points* $X = \sum_{i=1}^{s} m_i P_i \subseteq \mathbb{P}^n$ *and any integer* $d \geq 1$*, the regularity index of* X *is subject to the bound*

$$r(X) \leq \max\left\{ d \cdot \left\lceil \frac{-1 + \sum_{P_i \in Y} m_i}{\dim_K [R/I_Y]_d - 1} \right\rceil \;\middle|\; Y \subseteq \operatorname{Supp} X \textit{ and } |Y| \geq 2 \right\}.$$

Proof. Consider the d-th Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^N$. As above, let R and S be the coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^N$, respectively. Notice that the Segre bound of $\hat{X} = \sum_{i=1}^{s} m_i v_d(P_i)$ is

$$\operatorname{Seg} \hat{X} = \max\left\{ \left\lceil \frac{-1 + \sum_{v_d(P_i) \in L} m_i}{\dim L} \right\rceil \;\middle|\; L \subseteq \mathbb{P}^N \text{ linear}, \dim L \geq 1 \right\}.$$

Consider a linear subspace $L \subset \mathbb{P}^N$ for which the right-hand side above is maximal. Set $Y = \{P_i \in \operatorname{Supp} X \mid v_d(P_i) \in L\}$. The assumption on L gives that $\hat{Y} = v_d(Y)$ is not contained in a proper subspace of L, that is, $\dim_K [S/I_{\hat{Y}}]_1 - 1 = \dim L$. Since $\dim_K [S/I_{\hat{Y}}]_1 = \dim_K [R/I_Y]_d$, Theorem 4.3.3 gives

$$r(\hat{X}) \leq \operatorname{Seg} \hat{X} = \left\lceil \frac{-1 + \sum_{P_i \in Y} m_i}{\dim_K [R/I_Y]_d - 1} \right\rceil.$$

Using $\frac{r(X)}{d} \leq r(\hat{X})$ due to Proposition 4.3.4, the claim follows. □

If one has information on subsets of the points supporting a fat point scheme, then the above result can be used to obtain a better regularity bound than the Segre bound of Theorem 4.3.3. We illustrate this by a simple example.

Example 4.3.7. Let $X = \sum_{i=1}^{s} m P_i \subset \mathbb{P}^n$ be a fat point scheme, where all points have the same multiplicity m. Suppose that the support of X consists of five arbitrary points and $\binom{d+n}{n}$ generic points for some $d \geq 5$. Thus, $s = 5 + \binom{d+n}{n}$. Let $L \subset \mathbb{P}^n$ be a linear subspace of dimension k with $1 \leq k < n$. Then $|L \cap \operatorname{Supp} X| \leq k + 4$. It follows that for sufficiently large d (or n)

$$\begin{aligned} \operatorname{Seg} X &= \max\left\{ \left\lceil \frac{(k+4)m - 1}{k} \right\rceil, \left\lceil \frac{[\binom{d+n}{n} + 5]m - 1}{n} \right\rceil \;\middle|\; 1 \leq k < n \right\} \\ &= \left\lceil \frac{\binom{d+n}{n} m + 5m - 1}{n} \right\rceil. \end{aligned}$$

Consider now any subset $Y \subset \operatorname{Supp} X$ of $t \geq 2$ points. Since $d \geq 5$, one gets

$$\dim_K [R/I_Y]_d = \begin{cases} t & \text{if } t \leq \binom{n+d}{n} \\ \binom{n+d}{n} & \text{otherwise.} \end{cases}$$

Hence, Theorem 4.3.6 and a straightforward computation give

$$\begin{aligned} r(X) &\leq d \cdot \max\left\{ \left\lceil \frac{tm - 1}{t - 1} \right\rceil, \left\lceil \frac{[\binom{d+n}{n} + 5]m - 1}{\binom{d+n}{n} - 1} \right\rceil \;\middle|\; 2 \leq t \leq \binom{d+n}{n} \right\} \\ &= d \cdot \max\left\{ 2m - 1, \left\lceil \frac{[\binom{d+n}{n} + 5]m - 1}{\binom{d+n}{n} - 1} \right\rceil \right\}. \end{aligned}$$

For sufficiently large d (or n), this implies $r(X) \leq d(2m-1)$. In comparison, $\operatorname{Seg} X$ is essentially a polynomial function in d of degree n.

4.4 Generalizations of Segre Bound

In this section we give a conjectural further extension of the Segre bound on regularity and prove it for fat point schemes consisting only of double and single points. We state this in parallel with a conjecture in matroid theory which would imply the regularity bound.

Conjecture 4.4.1 (Segre Type Regularity Bounds). *Let $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj}(R)$ be a fat point subscheme with $s \geq 1$. For any $A \subseteq \operatorname{Supp}(Z)$ and any integer $d > 0$ let $h_Y(d) := \dim_{\mathbb{K}}[R/I_Y]_d$. Then if $(d_1, d_2, .., d_k)$ is a k-tuple of positive integers and for every nonempty $A \subseteq \operatorname{Supp}(Z)$ with $|A| \geq 2$ we have*

$$-1 + \sum_{P_i \in A} m_i \leq \sum_{i=1}^{k} (h_A(d_i) - 1). \tag{4.4.1.1}$$

Then $\mathrm{r}(Z) \leq \sum_{i=1}^{k} d_i$.

We note that if $d_1 = d_2 = \ldots = d_k$ then we recover the statement of theorem 4.3.6. The analogous statement for matroids was discussed briefly following proposition 3.3.6, though we make the formal statement now.

For convenience we will introduce a piece of notation to refer to certain cases of the above conjecture.

Definition 4.4.2. Let ST_m denote the proposition that conjecture 4.4.1 holds for all $Z = \sum_{i=0}^{s} m_i P_i$ where $m_i \leq m$.

As tools from matroid theory worked to establish theorem 4.3.3, we discuss a possible matroid theoretic approach for establishing conjecture 4.4.1. We consider the following statement which is in someway a matroid theoretic analog of ST_m.

Definition 4.4.3. Let MS_p denote the following proposition: " *Let $\mathrm{rk}_1, .., \mathrm{rk}_k$ denote the rank functions of matroids $M_1, .., M_k$ on a finite set E, with $p < k$. Further suppose that there is a subset $F \subseteq E$ so that for all nonempty $A \subseteq F$ we have $|A| \leq \left(\sum_{i=1}^{k} \mathrm{rk}_i(A)\right) - p$. Then there is an integer $1 \leq j \leq k$ so that for every $e \in E$ there is a subset $I \subseteq F$ so that $I \cup \{e\}$ is independent in M_j and for every nonempty $B \subseteq (F \setminus I)$ we have $|B| \leq \left(\sum_{\substack{i=1 \\ i \neq j}}^{k} \mathrm{rk}_i(B)\right) - (p-1)$?*"

A natural question to ask is then the following:

Question 4.4.4. *Does MS_p hold for all p?*

We note that if we restrict the matroids in MS_p so that $M_1 = M_2 = \ldots = M_n$, then the answer to the above question is yes, as shown in corollary 3.3.10. We first note that while an affirmative answer to question 4.4.4 would imply conjecture 4.4.1 (see proposition 4.4.7), we only need the result for a very limited subset of all possible sequences of matroids. Namely, we only need the result in the case E is the set corresponding to $\sum_{i=1}^{k} m_i P_i$ (containing m_i copies of P_i) and $\mathrm{rk}_i(A) = h_{A+Q}(d_i) - h_Q(d_i)$ where Q is any point not in $\mathrm{Supp}(Z)$. Hence, one obvious restraint is that the matroids $M_1, .., M_n$ are representable over some fixed field $\mathbb{K}$.

A perhaps more useful constraint, assuming we have ordered the matroids so $\mathrm{rk}_i(A) = h_{A+Q}(d_i) - 1$ and $d_1 \leq d_2 \leq \ldots \leq d_k$ is that for $i \leq j$ the induced map $M_i \to M_j$ is a strong map of matroids (at least when restricted to F).

However, the general statement may be true in general. In fact we note below that the case $p = 1$ is true. Allowing us to establish ST_2 (see theorem 4.4.9)

Proposition 4.4.5. *Let* $\mathrm{rk}_1, .., \mathrm{rk}_k$ *denote the rank functions of matroids* $M_1, .., M_k$ *on a finite set* E*. Further suppose that there is an integer* $0 < p < k$ *and a subset* $F \subseteq E$ *so that for all nonempty* $A \subseteq F$ *we have*

$$|A| \leq \left(\sum_{i=1}^{k} \mathrm{rk}_i(A)\right) - 1.$$

Then for any $e \in E$ *and any integer* $1 \leq j \leq k$ *there is a subset* $I \subseteq F$ *so that* $I \cup \{e\}$ *is independent in* M_j *and for every nonempty* $B \subseteq (F \setminus I)$ *we have*

$$|B| \leq \sum_{\substack{i=1\\ i\neq j}}^{k} \mathrm{rk}_i(B).$$

Proof. This is in some sense a direct corollary of theorem 3.3.7. Namely for any $1 \leq j \leq k$ and any $e \in E \setminus F$ define $\hat{\mathrm{rk}}_i = \mathrm{rk}_i$ for $i \neq j$ and set $\hat{\mathrm{rk}}_j(A) = \mathrm{rk}_j(A \cup \{e\}) - \mathrm{rk}_j(\{e\})$. That is $\hat{\mathrm{rk}}_j$ is the rank function of the quotient matroid M_j/e. Then by assumption for every $A \subseteq F$ we have

$$|A| \leq \left(\sum_{i=1}^{k} \mathrm{rk}_i(A)\right) - \mathrm{rk}_j(\{e\}) \leq \sum_{i=1}^{k} \hat{\mathrm{rk}}_i(A).$$

Applying theorem 3.3.7 we can conclude that there is a partition $I_1 \sqcup I_2 \sqcup \ldots \sqcup I_k$ of F so that for I_ℓ is independent in M_ℓ for $\ell \neq j$ and I_j is independent in M_j/e. Note this means that $I_j \cup \{e\}$ is independent. Moreover, we note that $I_1 \sqcup \ldots \sqcup I_{j-1} \sqcup I_{j+1} \sqcup \ldots \sqcup I_k$ is a partition of $F \setminus I_j$ and so again by theorem 3.3.7 we have for every $B \subseteq F \setminus I$ that

$$|B| \leq \sum_{\substack{i=1\\ i\neq j}}^{k} \mathrm{rk}_i(B).$$

□

As mentioned one evidence for conjecture 4.4.1 is that it holds in some cases. One example is the case of reduced sets of points, which follows directly from theorem 3.3.7 similarly to how theorem 4.2.6 follows directly from theorem 3.3.1.

Theorem 4.4.6. *Let* $Z = \sum_{i=0}^{s} P_i \subseteq \mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj}(R)$ *be finite set of points subscheme* $s \geq 1$. *For any* $A \subseteq Z$ *any integer* $d > 0$ *let* $h_Y(d) := \dim_{\mathbb{K}}[R/I_Y]_d$. *Then for any* k*-tuple of positive integers* $(d_1, d_2, .., d_k)$ *if for every nonempty* $A \subseteq \operatorname{Supp}(Z)$ *we have*

$$-1 + |A| \leq \sum_{i=1}^{k} (h_A(d_i) - 1). \tag{4.4.6.1}$$

Then $\mathrm{r}(Z) \leq \sum_{i=1}^{k} d_i$.

Proof. Let $Z = \sum_{i=0}^{s} P_i \subseteq \mathbb{P}^n$ and let $0 \leq d_1 \leq \ldots \leq d_k$ be a sequence of integers so that Z satisfies eq. (4.4.6.1). It suffices by theorem 2.3.7 to find for every $P_i \in Z$ a polynomial f_i of degree $D = \sum_{j=1}^{k} d_j$ where $f_i(P_i) \neq 0$ but $f_i(P_j) = 0$ for $j \neq i$.

Fix arbitrary $P_i \in Z$, and for any integer $d \geq 0$ consider the matroid $M_{Z/P_i}(d)$ on the set $Z - P_i$ with rank function

$$\mathrm{rk}_{M_{Z/P_i}(d)}(A) = h_{A+P_i}(d) - h_{P_i}(d) = h_{A+P_i}(d) - 1.$$

By assumption for every nonempty $A \subseteq Z - P_i$ we have

$$|A| = -1 + |A + P_i| \leq \sum_{j=1}^{k} (h_{A+P_i}(d) - 1) = \sum_{j=1}^{k} \mathrm{rk}_{M_{Z/P_i}(d_j)}(A).$$

By theorem 3.3.7 we conclude that there is a partition $B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k$ of $Z - P_i$ so that B_j is independent in $M_{Z/P_i}(d_j)$. Note then $|B_j| = h_{B_j+P_i}(d_j) - 1$ and so $|B_j + P_i| = h_{B_j+P_i}(d_j)$. Consequently, there exists some $g_j \in [I(B_j)]_{d_j}$ with $g_j(P_i) \neq 0$.

We now define $f_i = \prod_{j=1}^{k} g_j$ and claim that f_i has the desired property. Note first that $\deg(f_i) = \sum_{j=1}^{k} \deg(g_j) = \sum_{j=1}^{k} d_j = D$. Moreover, since $g_j(P_i) \neq 0$ for each j we see that $f_i(P_i) \neq 0$. Lastly, given $P_j \in Z - P_i$ we know there exists some B_ℓ so $P_j \in B_\ell$. Then $g_\ell(P_j) = 0$ and consequently $f_i(P_j) = 0$ as well. Thus establishing this case. □

Proposition 4.4.7. *For all* $p > 0$ *we have that* MS_p *implies* ST_{p+1}.

Proof. We suppose that MS_p holds for some $p > 0$. Let $Z = \sum_{i=0}^{s} m_i P_i$ with $m_0 \geq m_1 \geq \ldots \geq m_s$ and suppose $(d_1, d_2, \ldots, d_k)$ is a k-tuple so that Z satisfies eq. (4.4.1.1). We proceed by induction on $\rho(Z) = \sum_{i=0}^{s}(m_i - 1)$, the base case where $\rho(Z) = 0$ is established by theorem 4.4.6.

The inductive step is a consequence of the claim below along with lemma 4.1.5.

Claim 4.4.8. Assuming MS_p holds. Fix an integer $1 \leq m \leq p$ and a point $Q \notin \operatorname{Supp}(Z)$. If for all $A \subseteq \operatorname{Supp}(Z)$ we have we have

$$m + \sum_{P_i \in A} m_i \leq \sum_{j=1}^{k} h_{Q+A}(d_j) - 1, \tag{4.4.8.1}$$

then there exist polynomials $g_1, .., g_t \in I_Q^m$ and $f_1, .., f_t \in R$ so letting $D = \sum_{j=1}^k d_j$

$$[(g_1, .., g_t) + I_Q^{m+1}]_D = [I_Q^m]_D$$

and for each $1 \leq i \leq t$, $f_i \notin I_Q$ and $g_i f_i \in I_{Z+mQ}$.

(*Proof of claim.*) Given our fat point scheme $Z = \sum_{i=0}^s m_i P_i \subseteq \mathbb{P}^n$, we define a set E_Z which contains m_i copies of each point $P_i \in X$. Given $A \subseteq E_Z$ we define $\overline{A} \subseteq \mathrm{Supp}(Z)$ to denote the underlying reduced set of points. With this we define for each integer $d \geq 0$ a matroid $M_{Z/Q}(d)$ on E_Z, via

$$\mathrm{rk}_d(A) = h_{\overline{A}+Q}(d) - 1.$$

We note that $M_{Z/Q}(d)$ is representable via the map sending each P_i to the linear form $\ell_{P_i} \in [I_Q]_d^*$ which sends each polynomial $f \in I_Q$ to $f(P_i)$ (for some choice of coordinates for P_i).

We now proceed by induction on m, the base case $m = 0$ is essentially identical to the proof of theorem 4.4.6 so we omit it. Continuing with the case $m \geq 1$ we next establish that there exists polynomials $\ell_1, .., \ell_s \in I_Q$ so that the following conditions are satisfied

(Cond. I) $[(\ell_1, .., \ell_s) + I_Q^2]_{D-m+1} = [I_Q]_{D-m+1}$.

(Cond. II) For each ℓ_i, we can associate an index $1 \leq j_i \leq k$ so that $\ell_i \in [I_Q]_{d_{j_i}}$, and letting $(Z : \ell_j) = \sum_{i=0}^s m_{i,j} P_i$ we have all $A \subseteq \mathrm{Supp}(Z)$ that

$$m - 1 + \sum_{P_i \in A} m_{i,j} \leq \sum_{\substack{j=1 \\ j \neq j_i}}^k \mathrm{rk}_{d_j}(A) \tag{4.4.8.2}$$

To establish the existence of the $(\ell_1, .., \ell_s)$, we assume we have $\ell_1, .., \ell_{s'-1} \in I_Q$ where each ℓ_i satisfies (Cond. II). We show that if (Cond. I) is not yet satisfied we can find some $\ell_{s'}$ satisfying (Cond. II) where

$$[(\ell_1, .., \ell_{s'}) + I_Q^2]_d \supsetneq [(\ell_1, .., \ell_{s'-1}) + I_Q^2]_d$$

for some $d < D - m + 1$.

First, note that for each $P_i \in Z$ we have by eq. (4.4.8.1) that $m + m_i \leq \sum_{j=1}^k \mathrm{rk}_{d_j}(P_i) = k$ so $m \leq k - m_i \leq k - 1$. Furthermore, as $1 \leq d_1 \leq \ldots d_k$ we see for each d_i that

$$d_i \leq d_i + \sum_{\substack{j=1 \\ j \neq i}}^k \mathrm{rk}_{d_i} = D - k + 1 \leq D - m + 1.$$

Hence, it suffices to find $\ell_{s'} \in [I_Q]_{d_i}$ so that $[(\ell_1, .., \ell_{s'}) + I_Q^2]_d \supsetneq [(\ell_1, .., \ell_{s'-1}) + I_Q^2]_d$.

Since by assumption $[(\ell_1, .., \ell_{s'-1}) + I_Q^2]_{D-m+1} \neq [I_Q]_{D-m+1}$, for each d_j we can find some $\alpha_{d_j} \in [I_Q]_{d_j}^*$ where $\alpha_{d_j}(h) = 0$ for all $h \in [(\ell_1, \ldots, \ell_{s'-1}) + I_Q^2]_{d_j}$. We may then extend, for each d_j, the matroid $M_{Z/Q}(d_j)$ to a matroid on $E_Z \sqcup \{\alpha\}$. We do this by declaring for $A \subseteq E_Z$ that $\operatorname{rk}_{d_j}(A \sqcup \{\alpha\})$ is the dimension of the $\mathbb{K}$-subspace of $[I_Q]^*_{d_j}$ spanned by $\{\ell_{P_i} \mid P_i \in A\} \cup \{\alpha_{d_j}\}$.

By MS_p we have that there exists some $I \subseteq E_Z$ and an index d_i, so that $I \sqcup \{\alpha\}$ is independent in $M_{Z/Q}(d_i)$, and for all $B \subseteq (E_Z \setminus I)$ we have

$$|B| \leq \left(\sum_{\substack{j=1 \\ j\neq i}}^{k} \operatorname{rk}_{d_i}(B) \right) - m + 1. \tag{4.4.8.3}$$

Extend $I \sqcup \{\alpha\}$ to a basis $\mathfrak{B}$ of $[I_Q]^*_{d_i}$, then $\mathfrak{B} \setminus \{\alpha\}$ defines (up to scaling) a polynomial $\ell_{s'} \in [I_Q]_{d_i}$. Furthermore, the linear form α_{d_j} does not vanish on $\ell_{s'}$, hence $\ell_{s'} \notin [(\ell_1, .., \ell_{s'-1}) + I_Q^2]_{d_j}$. This ensures that $[(\ell_1, .., \ell_{s'-1}) + I_Q^2]_{d_j} \subsetneq [(\ell_1, .., \ell_{s'}) + I_Q^2]_{d_j}$.

Continuing we need to show that $\ell_{s'}$ satisfies (Cond. II). For each $P_i \in \operatorname{Supp}(Z)$ let r_i denote the number of copies of P_i appearing in $E_Z \setminus I$. Letting $(Z : \ell_j) := \sum_{i=0}^{s} m_{i,j} P_i$ we get for all $A \subseteq \operatorname{Supp}(Z)$ that

$$m - 1 + \sum_{P_i \in A} m_{i,j} \leq m - 1 + \sum_{P_i \in A} r_i \leq \sum_{\substack{j=1 \\ j\neq j_i}} \operatorname{rk}_{d_j}(A),$$

where the second inequality holds by eq. (4.4.8.3). Thus we have established the existence of $(\ell_1, .., \ell_s)$.

Continuing with the proof of claim 4.4.8, for each $1 \leq i \leq s$ set $W_i = (Z : \ell_i) = \sum_{j=0}^{s} m_{j,i} P_j$. By (Cond. II) and our inductive hypothesis there exists $g_{i,1}, \dots, g_{i,t} \in I_Q^{m-1}$ and $f_{i,1}, \dots, f_{i,t} \in R \setminus I_Q$ so that $g_{i,j} f_{i,j} \in [I_{W_i + (m-1)Q}]_{D - \deg \ell_i}$ and setting $J_i = (g_{i,1}, \dots, g_{i,t})$ we have $[J_i + I_Q^m]_{D - \deg \ell_i} = [I_Q^{m-1}]_{D - \deg \ell_i}$.

We see that $\ell_i g_{i,j} f_{i,j} \in I_{Z+mQ}$, $f_{i,j} \notin I_Q$ and $\ell_i g_{i,j} \in I_Q^m$, so to establish the claim and thus proposition 4.4.7 it suffices to show that

$$\left[\left(\sum_{i=1}^{s'} \ell_i J_i \right) + I_Q^{m+1} \right]_D = [I_Q^m]_D.$$

Define J as the ideal $\sum_{i=1}^{s'} \ell_i J_i$. As $\ell_i I_Q^m \subseteq I_Q^{m+1}$ we have

$$\begin{aligned} [J + I_Q^{m+1}]_D &= \sum_{i=1}^{s} \ell_i [J_i + I_Q^m]_{D - \deg \ell_i} \\ &= \sum_{i=1}^{s} \ell_i [I_Q^{m-1}]_{D - \deg(\ell_i)} \end{aligned}$$

As I_Q^{m-1} is generated in degree $m-1$ then $\ell_i [I_Q^{m-1}]_{D - \deg(\ell_i)} = [(\ell_i)]_{D-m+1} \cdot [I_Q^{m-1}]_{m-1}$. Therefore,

$$\begin{aligned}[J+I_Q^{m+1}]_D &= \sum_{i=1}^{s} \ell_i [I_Q^{m-1}]_{D-\deg(\ell_i)} \\ &= \sum_{i=1}^{s} [(\ell_i)+I_Q^2]_{D-m+1} \cdot [I_Q^{m-1}]_{m-1} \\ &= [(\ell_1,\dots,\ell_s)+I_Q^2]_{D-m+1} \cdot [I_Q^{m-1}]_{m-1} \\ &= [I_Q]_{D-m+1} \cdot [I_Q^{m-1}]_{m-1} = [I_Q^m]_D\end{aligned}$$

Thus establishing the claim and finishing the proof. □

We close this section by noting that proposition 4.4.7 together with proposition 4.4.5 implying the following theorem.

Theorem 4.4.9. *ST_2 holds. In fact let $Z = \sum_{i=0}^{s} m_i P_i \subseteq \mathbb{P}^n_{\mathbb{K}}$ be a fat point subscheme with $s \geq 2$ and $m_i \leq 2$ for each $1 < i \leq k$. For $Y \subseteq \operatorname{Supp}(Z)$ let $h_Y(d) := \dim_{\mathbb{K}}[R/I_Y]_d$. If for a given tuple of integers $(d_1, .., d_k)$ so that for every $A \subseteq \operatorname{Supp}(Z)$ with $|A| \geq 2$ we have*

$$-1 + \sum_{P_i \in A} m_i \leq \sum_{i=1}^{k} (h_A(d_i) - 1)\,.$$

Then $\operatorname{r}(Z) \leq \sum_{i=1}^{k} d_i$.

Proof. The proof follows directly from proposition 4.4.7 and proposition 4.4.5. We further note that we can take m_0 and m_1 to be arbitrary. To see this note that $\operatorname{r}(m_0P_0+m_1P_1) = m_0+m_1-1$ and so since $-1+m_0+m_1 \leq \sum_{i=1}^{k}(h_{P_0+P_1}(d_i)-1) = k$, we have $\operatorname{r}(m_0P_0+m_1P_1) \leq k \leq \sum_{i=1}^{k} d_i$.

Therefore, conjecture 4.4.1 holds for the case of two points. Since MS_1 holds by proposition 4.4.5, applying the inductive technique of claim 4.4.8 yields the result. □

Chapter 5 Very Unexpected Hypersurfaces

Given subschemes $X, Y \subseteq \mathbb{P}^n$ a natural topic of study is their intersection $X \cap Y$. Algebraically, this operation is not so well behaved, while $I(X) + I(Y) \subseteq I(X \cap Y)$ this containment is almost always strict. Restricting to the case where $X \cap Y = \emptyset$, so $I(X \cap Y) = R$ we in fact have that $I(X) + I(Y) \neq R$ unless $X = \emptyset$ or $Y = \emptyset$.

The natural followup question then becomes for which d is $[I(X)]_d + [I(Y)]_d = [R]_d$, in fact this formula holds for all $d \gg 0$. If $\dim[I(X)]_d$ and $\dim[I(Y)]_d$ are known and $\dim[I(X)]_d + \dim[I(Y)]_d \geq \dim[R]_d$, then the formula $\dim[I(X)]_d + \dim[I(Y)]_d = \dim[I(X \cup Y)]_d + \dim[I(X \cap Y)]_d$, reduces the question to the following: "For which d is

$$\dim[I(X) \cap I(Y)]_d = \max\{0, \dim[I(X)]_d - \dim[R/I(Y)]_d\}?\text{''}$$

If Y is in someway generic, we might expect the above formula to hold for all d, and refer to the number on the right as an "expected dimension". This chapter concerns itself with this problem in the case that X is a finite set of points and Y is a fat linear subspace (usually of codimension 2). Much of the content in this chapter has appeared as a preprint in [Tro20].

Continuing with this discussion, we consider the projective coordinate ring $R = \mathbb{C}[X_0, X_1, X_2, X_3]$ of $\mathbb{P}^3_{\mathbb{C}}$. For $Z \subseteq \mathbb{P}^n$ a finite set of fat points and a generic linear subspace $L \subset \mathbb{P}^3$ we expect that

$$[I(Z) \cap I(L)^m]_d = \min\{0, \dim[I(Z)]_d - \dim[R/I(L)^m]_d\}. \tag{1}$$

For instance, taking $m = 1$ we see that vanishing on L imposes $\binom{\dim L + d}{d}$ conditions on forms of degree d. Hence, we might expected that $\dim[I(Z) \cap I(L)]_d = \max\left\{0, \dim[I(Z)]_d - \binom{\dim L + d}{d}\right\}$.

Many papers have been written exploring when eq. (1) fails to give the actual dimension. If Z is a set of double points with general support and L is a general point with $m = 2$, the celebrated theorem of Alexander and Hirschowitz [AH95] gives a complete characterization of when equation eq. (1) fails to give the correct count.

More recently, a number of papers have been released (see for instance [HMT19], [Dum+19], [BMSS18] and [HMNT18]), which studied failure of expected dimension when Z is a reduced set of points under the name unexpected hypersurfaces or unexpected curves. These papers all study the above linear systems, where Z is a reduced sets of points in $\mathbb{P}^n$, and L is (possibly multiple) general linear subspace with an associated multiplicity. Many of these papers took inspiration from the paper [CHMN18]. The authors of [CHMN18] built off of earlier work in [DIV14] and introduced the concept of unexpected curves in $\mathbb{P}^2$. Namely, they said that Z admits unexpected curves in degree d if for a general point X,

$$\dim[I(Z) \cap I(X)^{d-1}]_d > \max\left\{0, \dim[I(Z)]_d - \dim[R/I(X)^{d-1}]_d\right\}.$$

The authors of [CHMN18] were able to give a full characterization of the degrees in which a set of points admits unexpected curves. Surprisingly, this characterization does not directly depend on the dimensions of either $[I(Z)]_d$ or $[I(Z) \cap I(X)^{d-1}]_d$. Namely, this information can be replaced with combinatorial information about Z, and data coming from the (reduced) module of derivations, $D_0(\mathcal{A}_Z)$, of the line arrangement, $\mathcal{A}_Z$, dual to Z. See definition 5.2.18 for the definition of splitting type.

Theorem 5.4.1 ([CHMN18])**.** *For a finite set of points $Z \subseteq \mathbb{P}^2$, let $\mathcal{A}_Z$ denote the dual line arrangement, and let (a_1, a_2) denote the splitting type of the bundle defined by $D_0(\mathcal{A}_Z)$. Then exactly one of the following statements holds:*

(i) There is some line $L \subseteq \mathbb{P}^2$ with $|L \cap Z| > a_1 + 1$, in which case $|L \cap Z| = a_2 + 1$ and Z never admits unexpected curves.

(ii) Z admits unexpected curves in degree d for precisely those d with $a_1 < d < a_2$.

This result allowed researchers to discover many new examples of unexpected curves by taking advantage of decades of prior research on line arrangements.

Given the observed connection between certain line arrangements and unexpected curves, it is natural to wonder if a similar connection exists in higher dimensions. In this chapter we show that this is true at least to a certain extent. More specifically, if $Z \subseteq \mathbb{P}^n$ is a finite set of points and L is a general codimension 2 linear subspace, we establish a general duality connecting the module of derivations $D_0(\mathcal{A}_Z)$ of the dual hyperplane arrangement to the intersection of ideals $[I(Z) \cap I(L)^d]_{d+1}$. In particular this allows us to recover $\dim[I(Z) \cap I(L)^d]_{d+1}$ from knowledge of the splitting type of $D_0(\mathcal{A}_Z)$.

In order to generalize 5.4.1, we introduce a modified definition of unexpected hypersurface which we call very unexpected hypersurfaces. Given a generic linear subspace L, we say a finite set of points $Z \subseteq \mathbb{P}^n$ admits very unexpected L-hypersurfaces if the intersection $[I(Z) \cap I(L)^{d-1}]_d$ is larger than expected, as long as this failure is not "easily explained" (see definition 5.4.7). Our definition of very unexpected hypersurfaces is more technical than that of unexpected hypersurfaces. However the two definitions agree in $\mathbb{P}^2$, in that a set of points $Z \subseteq \mathbb{P}^2$ admits very unexpected curves if and only if it admits unexpected curves.

This new definition has a few advantages compared with the definition for unexpected hypersurfaces. The first is that with the standard definition of unexpected hypersurfaces a generalization of theorem 5.4.1 to higher dimensions is impossible. The second is that, as we mentioned, in certain cases the "unexpectedness" can be relatively easily explained. For instance, if all of the points in Z lie on a proper subspace H, "unexpectedness" may simply be a consequence of the fact that $[I(H) \cap I(L)^{d-1}]_d \subseteq [I(Z) \cap I(L)^{d-1}]_d$ (for further discussion, see example 5.4.4). It is then somewhat surprising that by merely accounting for cases where "unexpectedness" is well explained, we are able to recover a generalization of theorem 5.4.1. More specifically, if L is a generic codimension 2 subspace, the degrees in which Z admits very unexpected L-hypersurfaces can again be characterized by the combinatorial data of Z in conjunction with the splitting type of the Derivation Bundle of $\mathcal{A}_Z$. We define, for every integer $d \geq 0$, a number $\mathrm{Ex.\,C}(Z, d)$ via a combinatorial optimization problem on the matroid of Z (see definition 5.4.19). Using this number, $\mathrm{Ex.\,C}(Z, d)$, we obtain the result below.

Theorem 5.0.1. *Let $Z \subseteq \mathbb{P}^n$ be a finite set of points, and suppose that $D_0(\mathcal{A}_Z)$ has splitting type $(a_1, .., a_n)$. Then for a fixed integer d,*

$$\sum_{i=1}^{n} \max\{0, d - a_i\} \leq \quad nd + 1 - \text{Ex. C}(Z, d)$$

and the inequality is strict if and only if Z admits very unexpected hypersurfaces in degree d.

In the case the points of Z are not too concentrated on some proper subspace we obtain the following result which mimics theorem 5.4.1.

Theorem 5.4.27. *Let $Z \subseteq \mathbb{P}^n$ and let $(a_1, .., a_n)$ be the splitting type of $D_0(\mathcal{A}_Z)$, where $a_i \leq a_{i+1}$. Suppose for all positive dimensional linear subspaces $H \subseteq \mathbb{P}^n$, we have that*

$$\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.$$

Then for an integer d the following are equivalent:

(a) Z admits very unexpected hypersurfaces in degree d.

(b) Z admits unexpected hypersurfaces in degree d.

(c) $a_1 < d < a_n$.

Moreover we show in proposition 5.5.4 that this condition holds if an irreducible reflection group $G \subseteq \mathbb{PGL}(\mathbb{K}, 2)$ acts on Z.

After discussing the theory of Unexpected Hypersurfaces in general, we apply this duality between $I(Z)$ and $D_0(\mathcal{A}_Z)$ to establish some structural results about both very unexpected Q-hypersurfaces and the module of derivations $D_0(\mathcal{A}_Z)$. Unlike the first part of the chapter where there are few dimension and field constraints, these results focus on unexpected curves in $\mathbb{P}^2_{\mathbb{C}}$. In particular, we establish the following bound sharp for all $d \geq 1$.

Theorem 5.7.6. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ and suppose that $|Z|$ admits an unexpected curve in degree $d \geq 1$, then $|Z| \leq 3d - 3$.*

We note that if d is the smallest such degree in which Z admits unexpected curves then it follows from theorem 5.4.1 that $2d + 1 \leq |Z|$. Consequently, no $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits unexpected curves in degree 3 or lower.

Additionally, we show the splitting type of $D_0(\mathcal{A}_Z)$ can be easily determined using only the initial degree of $D_0(\mathcal{A}_Z)$. We note that the splitting type is determined by the initial degree of the restriction of $D_0(\mathcal{A}_z)$ to a general line.

Theorem 5.6.9. *Let Z be a finite set of points in $\mathbb{P}^2_{\mathbb{C}}$ and let $\alpha(D_0(\mathcal{A}_Z))$ denote the initial degree of $D_0(\mathcal{A}_Z)$. Define $a = \min\left\{\alpha(D_0(\mathcal{A}_Z)), \left\lfloor \frac{|Z|-1}{2} \right\rfloor\right\}$ then $D_0(\mathcal{A}_Z)$ has splitting type $(a, |Z| - a - 1)$.*

The chapter proceeds as follows. After defining some notation in section 5.1, we discuss some needed background on the module of logarithmic derivations of a hyperplane arrangement in section 5.2. The reader familiar with Hyperplane Arrangements can likely skip this section with perhaps the exception of some non-standard notation found in definition 5.2.7 and definition 5.2.14.

We proceed in section 5.3, expanding on the Faenzi-Vallés duality between the module of derivations $D_0(\mathcal{A}_Z)$ of a hyperplane arrangement and certain elements of the ideal $I(Z)$ of points dual to $\mathcal{A}_Z$. The results of this section are not wholly original as much of this is implicit in the first section of [FV14]. Our approach however, is much more explicit and amenable to computation. It also has the advantage of working in arbitrary characteristic. We state two versions of this correspondence, the first (theorem 5.3.8) applies to the module, $D_0(\mathcal{A}_Z)$ itself, and we do not believe it has been stated before in this form. The second correspondence (theorem 5.3.14) applies to the restriction of $D_0(\mathcal{A}_Z)$ to a general line, generalizes the duality found in [FV14]. We note that despite the similarities in results, our method of proof and presentation is quite different from the one given in [FV14]. Additionally, the results here are not dependent on the characteristic of the ground field $\mathbb{K}$.

Section 5.4 introduces our definition of an very unexpected Q-hypersurface (see definition 5.4.7) for Q a generic subspace. We then look at the case when Q has codimension 2, establishing in theorem 5.4.20 that the degrees in which Z admits very unexpected Q-hypersurfaces depends only on the splitting type of $D_0(\mathcal{A}_Z)$ and a combinatorial optimization problem involving Z.

Section 5.6 starts by establishing a lifting criterion for the restriction of $D_0(\mathcal{A}_Z)$ to a general line (see proposition 5.6.2). We then recall some results on vector bundles on $\mathbb{P}^2_{\mathbb{C}}$ and apply these to show that proposition 5.6.2 has especially strong consequences in $\mathbb{P}^2_{\mathbb{C}}$ (see theorem 5.6.8 and corollary 5.6.10).

In Section 5.7 we give strong combinatorial constraints on the sets of points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ which can admit unexpected curves. In particular, if $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits an unexpected curve in degree d, then theorem 5.7.9 shows that no more than $d+1$ points of Z are in linearly general position and theorem 5.7.6 establishes a sharp bound on the number of points in Z showing that $|Z| \leq 3d-3$.

In section 5.8 we show that if $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits unexpected curves in degree d, then Z imposes independent condition on $(d-1)$ forms. We then briefly discuss generalizations to higher dimensions and some consequences.

We close with section 5.9, which discusses a few applications of these results to the field of Hyperplane arrangements. We focus on Terao's Freeness Conjecture mainly in $\mathbb{P}^2_{\mathbb{C}}$. In particular, we look at the conjecture for real line arrangements and connect it to the Weak Dirac Conjecture on real point configurations.

We have attempted to keep this Chapter as self contained and elementary as possible. This is largely true for the first 5 sections. However, in later sections we do apply some results from the theory of Vector Bundles and from the combinatorics of line arrangements in $\mathbb{P}^n_{\mathbb{C}}$.

5.1 Notation and Conventions

Throughout this chapter $\mathbb{K}$ will denote an algebraically closed of arbitrary characteristic, unless specified otherwise. However, most of these results hold as long as $\mathbb{K}$ is infinite. V and W will be dual $\mathbb{K}$-vector spaces. That is we suppose that there is a non-degenerate bilinear pairing $B(\, , \,) : V \times W \to \mathbb{K}$, inducing isomorphisms $V \cong W^*$ and $W \cong V^*$

If V is a $\mathbb{K}$ vector space, then V^* will denote the dual vector space of linear maps $V \to \mathbb{K}$. Our pairing gives isomorphisms $V \cong W^*$ and $W \cong V^*$, we denote these isomorphisms $v \mapsto \ell_v$ and $w \mapsto \ell_w$, respectively. Here $\ell_v(w) = \ell_w(v) = B(v, w)$. If $H \subseteq V$ is a linear subspace, then $H^{\perp} = \{w \in W \mid \ell_w(H) = \{0\}\}$. We similarly define $L^{\perp} \subseteq V$, for $L \subseteq W$.

$\mathrm{Sym}(V^*)$ will denote the graded $\mathbb{K}$-algebra of symmetric tensors. Given a choice of basis $\{Y_0, Y_1, .., Y_n\}$ of V^*, $\mathrm{Sym}(V^*)$, is naturally isomorphic to the polynomial algebra $\mathbb{K}[Y_0, .., Y_n]$.

Moreover, the graded ring $R = \mathrm{Sym}(V^*)$ is naturally identifiable with the projective coordinate ring of $\mathbb{P}(V)$. Dually, $S = \mathrm{Sym}(W^*)$ is the projective coordinate ring of $\mathbb{P}(W)$.

The goal of this chapter, is to relate properties of a finite set of points in $\mathbb{P}(V)$ to their dual hyperplanes in $\mathbb{P}(W)$.

5.2 Derivations of Hyperplane Arrangements

In this section we recall some facts about the module of logarithmic derivations $D(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$. In particular we state a few different known criteria for a general S derivation to lie in $D(\mathcal{A})$. We also give the definition (definition 5.2.18) of the splitting type of $D(\mathcal{A})$ which is used heavily in the sequel.

Definition 5.2.1. A (central) *Subspace Arrangement*, $\mathcal{A}$, is a finite collection of linear subspaces $\{H_0, .., H_s\}$ of a vector space W.

If each H_i is a hyperplane, we say that $\mathcal{A}$ is a *Hyperplane Arrangement*. We say $\mathcal{A}$ is *essential* if the only subspace contained in all the hyperplanes in $\mathcal{A}$ is the 0-subspace.

Remark 5.2.2. All subspace arrangements in this chapter will be central. We make this restriction in order to identify a subspace arrangement $\mathcal{A}$ in W with it's image in $\mathbb{P}(W)$, something we will do freely and often without comment.

A hyperplane arrangement is often defined in terms of a defining polynomial $Q_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \ell_H$. This is the product of linear forms each one defining a unique hyperplane in $\mathcal{A}$.

Definition 5.2.3. If S is our graded polynomial ring and M is a graded S-module, then a $\mathbb{K}$-derivation of S into M is a graded $\mathbb{K}$-linear map $\theta : S \to M$ which satisfies the Leibniz product rule. Namely for $f, g \in S$

$$\theta(f \cdot g) = \theta(f) \cdot g + f \cdot \theta(g)$$

.

These form a graded S-module, denoted $\mathrm{Der}(S, M)$, obtained by setting $(f \cdot \theta)(g) = f(\theta(g))$.

We grade $\mathrm{Der}(S, M)$ by the *polynomial degree*, namely, we set $\deg \theta = \deg(\theta(\ell))$, where $\ell \in [S]_1$.

In the case that $M = S$, we set $\mathrm{Der}(S) := \mathrm{Der}(S, S)$. In this chapter our module M will either be S or a quotient ring of S.

Definition 5.2.4. If $\mathcal{A} \subseteq \mathbb{P}(W)$ is a Hyperplane Arrangement, we define the *module of $\mathcal{A}$-derivations*, denoted $D(\mathcal{A})$, as submodule of $\mathrm{Der}(S)$ via

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(I(H)) \subseteq I(H) \text{ for all } H \in \mathcal{A}\}$$

Remark 5.2.5. Each element $\alpha \in W$ defines a $\mathbb{K}$-derivation, θ_α, of $S = \mathrm{Sym}(W^*)$. Namely, for $\ell \in [S]_1$ we set $\theta_\alpha(\ell) = \ell(\alpha)$ and extended to all of S via the Leibniz product rule.

Proposition 5.2.6. *Let $S = \mathrm{Sym}(W^*)$ and let M be a graded S-module, then there's an isomorphism of graded S-modules $\mathrm{Der}(S, M) \cong M \otimes_{\mathbb{K}} W$. Here the grading on $M \otimes_{\mathbb{K}} W$ is given by that of M.*

Consequently, theres an isomorphism $\mathrm{Der}(S) \otimes_S M \cong \mathrm{Der}(S, M)$.

Proof. Picking a basis $Y_0, .., Y_n$ for W^*, we have $S \cong \mathbb{K}[Y_0, .., Y_n]$. Let $\theta \in \mathrm{Der}(S, M)$, let $g_i = \theta(Y_i)$, by linearity and the Leibniz product rule we get these g_i completely determine θ. It follows that θ is equal to the derivation $\sum_{i=0}^n g_i \frac{\partial}{\partial Y_i}$.

Hence if $W_0, .., W_n$ is a basis of W dual to $Y_0, .., Y_n$, meaning $Y_i(W_j) = \delta_{i,j}$. Then $\theta = \sum_{i=0}^n g_i \otimes W_i \in M \otimes_{\mathbb{K}} W$ establishing the first result.

The second statement follows from the isomorphisms

$$M \otimes_S \mathrm{Der}(S) \cong M \otimes_S (S \otimes_{\mathbb{K}} W) \cong M \otimes_{\mathbb{K}} W$$

□

Definition 5.2.7. Let $S = \mathrm{Sym}(W^*)$, and fix a basis $Y_0, Y_1, .., Y_n$ of W^*, so $S \cong \mathbb{K}[Y_0, .., Y_n]$. Also take $W_0, .., W_n$ to be the dual basis of W. Given $\lambda = \sum_i f_i \otimes W_i \in S \otimes W$, the preceding proposition shows λ defines a derivation $\theta_\lambda \in \mathrm{Der}(S)$. Namely,

$$\theta_\lambda(g) = \sum_i f_i \frac{\partial g}{\partial Y_i}.$$

Moreover, λ defines a polynomial map $\rho_\lambda : W \to W$, or equivalently a rational map $\mathbb{P}(W) \to \mathbb{P}(W)$, via

$$\begin{aligned} \rho_\lambda(w) &= \sum_{i=0}^n f_i(w) W_i \\ &= (f_0(w) : f_1(w) : .. : f_n(w)) \end{aligned}$$

Finally, it defines a pairing $\langle\ ,\ \rangle_\lambda : W \times W^* \to \mathbb{K}$, linear only in W^*, where for $(s, \ell) \in W \times W^*$

$$\langle s, \ell\rangle_\lambda := \sum_{i=0}^{n} (f_i(s))(\ell(W_i));$$

or in coordinates

$$\langle (a_0, .., a_n), c_oY_0 + ..c_nY_n\rangle_\lambda := \sum_{i=0}^{n} f_i(a_0, .., a_n)c_i.$$

We extended this definition to a pairing $\langle\ ,\ \rangle_\lambda : W \times V \to \mathbb{K}$ via

$$\langle s, t\rangle_\lambda := \sum_i f_i(s)(B(t, u_i)).$$

The following is immediate from the definitions

Lemma 5.2.8. *For* $(s, t) \in W \times V$ *and* $\lambda \in S \otimes W$

$$[\theta_\lambda(\ell_t)]\,(s) = \langle s, t\rangle_\lambda = \ell_t(\rho_\lambda(s)).$$

This proposition is essentially due to Stanley, though the presentation is our own.

Proposition 5.2.9. *Let* $\lambda \in S \otimes W$*, and* $\mathcal{A} \subseteq W$ *a hyperplane arrangement with* $Q_\mathcal{A} = \prod_H \ell_H$*. Then the following are equivalent:*

(i) $\theta_\lambda \in D(\mathcal{A})$

(ii) $\theta_\lambda(\ell_H) \subseteq I(H)$ *for all* $H \in \mathcal{A}$

(iii) $\rho_\lambda(H) \subseteq H$ *for all* $H \in \mathcal{A}$

(iv) For all $H \in \mathcal{A}$*, the restriction of* $\langle -, -\rangle_\lambda$ *to* $H \times H^\perp \subseteq W \times V$ *is identically* 0.

Proof. $[(i) \iff (ii)]$ The implication $(i) \implies (ii)$ follows from the definition. For the converse note that $I(H)$ is generated by ℓ_H, so every element $f \in I(H)$ may be written $f = g\ell_H$. Applying the Leibniz product rule we get

$$\theta_\lambda(f) = \theta_\lambda(a_i)\ell_H + a_i\theta_\lambda(\ell_H).$$

The first term is necessarily in $I(H)$, and so if $\theta_\lambda(\ell_H) \in I(H)$ then we conclude that the second sum is in $I(H)$ as well, establishing the result.

$[(ii) \iff (iii) \iff (iv)]$ (ii) can be rephrased as follows: "for all $\ell \in [I(H)]_1$ and all $p \in L$, $[\theta_\lambda(\ell)](p) = 0$".

Now using the fact that $[I(H)]_1$ is naturally isomorphic to $H^\perp$ under our isomorphism $V \cong W^*$, we conclude by applying 5.2.8. □

Definition 5.2.10. Under the characterization above, the identity map on W corresponds to a derivation known as the *Euler Derivation* which we denote θ_e. In coordinates, if $S = \mathbb{K}[Y_0, .., Y_n]$, then

$$\theta_e = Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1} + \ldots + Y_n \frac{\partial}{\partial Y_n}.$$

The Euler Derivation can be alternatively characterized as the unique derivation where $\theta_e(f) = \deg(f) f$ for all homogeneous f, an identity originally due to Euler.

Definition 5.2.11. (Reduced Module of Derivations) Denoting the *Euler Derivation* by θ_e we define the *Reduced Module of Derivations*, denoted $D_0(\mathcal{A})$, as the quotient

$$D_0(\mathcal{A}) := D(\mathcal{A})/(S\theta_e).$$

By convention, we set $D(\emptyset) = \mathrm{Der}(S)$ and $D_0(\emptyset) = \mathrm{Der}(S)/(S\theta_e)$.

Definition 5.2.12. Let $\mathcal{A} \subseteq \mathbb{P}(W)$ be a hyperplane arrangement, then $D(\mathcal{A})$ defines a reflexive sheaf, $\widetilde{D(\mathcal{A})}$ on $\mathbb{P}(W)$ of rank $\dim W$.

If $L \subseteq \mathbb{P}(W)$ is a line we may tensor $\widetilde{D(\mathcal{A})}$ with the structure sheaf $\mathcal{O}_L$. This may equivalently be viewed as a sheaf of $\mathcal{O}_{\mathbb{P}(W)}$ modules, or the restriction of $\widetilde{D(\mathcal{A})}$ to L. We let $D(\mathcal{A}) \mid_L$ denote the corresponding graded module, that is

$$[D(\mathcal{A}) \mid_L]_d = H^0(\widetilde{D(\mathcal{A})} \otimes \mathcal{O}_L(-d), L)$$

We may similarly define $D_0(\mathcal{A}) \mid_L$.

If the line L is general the module $D(\mathcal{A}) \mid_L$ has an equivalent algebraic definition which we state now.

Proposition 5.2.13. *For a general line $L \subseteq \mathbb{P}(W)$, and for $\ell \in S$ let $\bar{\ell}$ denote the image of ℓ in $S/I(L)$, then*

$$D(\mathcal{A}) \mid_L = \{\theta \in \mathrm{Der}(S, S/I(L)) \mid \theta(\ell) \in (\bar{\ell}) \text{ for all } \ell \text{ dividing } Q(\mathcal{A})\}$$

and similarly for $D_0(\mathcal{A})$.

Proof. First, for any $f \in S$ we let $\bar{f}$ denote the image of f in $S/I(L)$, and similarly if $\theta = \sum_{i=0}^n f_i \frac{\partial}{\partial Y_i}$ we let $\bar{\theta} = \sum_{i=0}^n \bar{f}_i \frac{\partial}{\partial Y_i} \in \mathrm{Der}(S, S/I(L))$.

Note that $D(\emptyset) \mid_L$ is isomorphic to $\mathrm{Der}(S, S/I(L))$, and so $D(\mathcal{A}) \mid_L$ is isomorphic to a submodule of $\mathrm{Der}(S, S/I(L))$.

Now consider the case that $\mathcal{A}$ consists of a single hyperplane H. Choosing our coordinates $Y_0, .., Y_n$ so that $H = (Y_0 = 0)$, then $D(\mathcal{A})$ is free on generators $\{Y_0 \frac{\partial}{\partial Y_0}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n}\}$. Then $D(\mathcal{A}) \mid_L$ is a free $S/I(L)$ module with basis $\{\bar{Y_0} \frac{\partial}{\partial Y_0}, \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n}\}$. Yet these are also precisely the derivations $\theta \in \mathrm{Der}(S, S/I(L))$ where $\theta(X_0) \in (\bar{X_0})$, so the result follows in this case.

More generally, if $\mathcal{A} = \{H_0, H_1, \ldots, H_k\}$ and L is any line not contained in a hyperplane in $\mathcal{A}$. Then for all $i \neq j$ we have that $L \cap H_i$ and $L \cap H_j$ consist of distinct points. Consequently, letting U_i denote the complement of $\mathcal{A} \setminus \{H_i\}$, we have that $\{U_i \cap L\}_{i=0,..,k}$ is an open cover of L. Therefore, for a section $\sigma \in H^0(\widetilde{\operatorname{Der}(S)} \otimes \mathcal{O}_L(-d), L)$, we have that $\sigma \in D(\mathcal{A}) \mid_L$ if and only if

$$\sigma \mid_{U_i} \in H^0(\widetilde{D(\mathcal{A})} \otimes \mathcal{O}_L(-d), U_i \cap L) = H^0(\widetilde{D(H_i)} \otimes \mathcal{O}_L(-d), U_i \cap L)$$

for all $j = 1, .., k$.

Finally, note that for $i \neq j$ we have that $H^0(\widetilde{D(H_i)} \otimes \mathcal{O}_L(-d), U_j \cap L) = \sigma \in H^0(\widetilde{\operatorname{Der}(S)} \otimes \mathcal{O}_L(-d), U_j \cap L)$. Therefore, it follows that for $\sigma \in H^0(\widetilde{\operatorname{Der}(S)} \otimes \mathcal{O}_L(-d), L)$, that we have the following string of equivalences

$$\begin{aligned} \sigma \in H^0(\widetilde{D(\mathcal{A})} \otimes \mathcal{O}_L(-d), L) &\iff \\ \text{for all } i \in \{1, .., k\}, \sigma \mid_{U_i \cap L} \in H^0(\widetilde{D(H_i)} \otimes \mathcal{O}_L(-d), L \cap U_i) &\iff \\ \text{for all } i \in \{1, .., k\}, \sigma \in H^0(\widetilde{D(H_i)} \otimes \mathcal{O}_L(-d), L) & \end{aligned}$$

The result now follows from the previous case. □

We can emulate the constructions from 5.2.7 for the module $D_0(\mathcal{A}) \mid_L$, to achieve a characterization similar to 5.2.9.

Definition 5.2.14. Let $L \subseteq \mathbb{P}(W)$ be a line, for $\gamma = \sum_j f_j \otimes w_j \in S/I(L) \otimes W$, we obtain a pairing $\langle \, , \, \rangle_\gamma : L \times V \to \mathbb{K}$, defined by

$$\langle p, v \rangle_\gamma := \sum_j f_j(p)(\ell_v(w_j))$$

Similarly, define the polynomial map $\rho_\gamma : L \to W$ and $\theta_\gamma \in \operatorname{Der}(S, S/I(L))$.

Proposition 5.2.15. *Let $L \subseteq \mathbb{P}(W)$ be a general line, and $\mathcal{A} \subseteq \mathbb{P}^n$ a hyperplane arrangement, then for $\gamma \in S/I(L) \otimes_{\mathbb{K}} W$, the following are equivalent.*

(i) $\theta_\gamma \in D_0(\mathcal{A}) \mid_L$

(ii) $\rho(L \cap H) \subseteq H$ *for all* $H \in \mathcal{A}$

(iii) The restriction of $\langle \, , \, \rangle_\gamma$ *to* $(H \cap L) \times H^\perp$ *is identically* 0.

Proof. The proof is essentially identical to that of proposition 5.2.9 so we omit it. Note that in particular, we still have an analogue of lemma 5.2.8 for $(p, q) \in L \times V$ that

$$\langle p, q \rangle_\gamma = \theta_\gamma(\ell_q)(p) = \ell_q(\rho_\gamma(p))$$

. □

If M is a finite reflexive graded module over $\mathbb{K}[Y_0,..,Y_n]$ defining a reflexive sheaf $\tilde{M}$ on $\mathbb{P}^n_{\mathbb{K}}$. Then the restriction of $\tilde{M}$ to a general line $L \subseteq \mathbb{P}^n_{\mathbb{K}}$, defines a vector bundle over L. By the well known theorem of Birkhoff and Grothendieck, there exist integers $k_0 \leq k_1 \leq k_2 \leq .. \leq k_m$ where

$$\tilde{M} \mid_L \cong \oplus_{i=0}^m \mathcal{O}_L(-k_i).$$

If $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{K}}$ is a hyperplane arrangement, then $D(\mathcal{A})$ can be naturally identified with the first syzygy module of the ideal $J = (Q_\mathcal{A}, \frac{\partial}{\partial Y_0} Q_\mathcal{A}, \frac{\partial}{\partial Y_1} Q_\mathcal{A}, \ldots, \frac{\partial}{\partial Y_n} Q_\mathcal{A})$. This ensure that $D(\mathcal{A})$ is reflexive.

We show that $D_0(\mathcal{A})$ is reflexive for any nonempty arrangements $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{K}}$. If $|\mathcal{A}| \neq 0 \mod \operatorname{Char} \mathbb{K}$, this is well known as $J = \operatorname{Jac}(Q_\mathcal{A}) = (\frac{\partial}{\partial Y_0} Q_\mathcal{A}, \frac{\partial}{\partial Y_1} Q_\mathcal{A}, \ldots, \frac{\partial}{\partial Y_n} Q_\mathcal{A})$ and $D_0(\mathcal{A})$ can be identified with the syzygy module of $\operatorname{Jac}(Q_\mathcal{A})$.

We establish this more generally, our proof requires the following reflexive criterion. We refer to [Aut20] (see Lemma 15.23.5) for a proof.

Proposition 5.2.16 (Reflexive Criterion)**.** *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & L \arrow[r] & K \end{tikzcd}$$

is an exact sequence of finite modules, over a commutative notherian domain R. Then if L is reflexive and K is torsion free, then M is reflexive.

With this criteria we can establish our claim. This is well known for arrangements over $\mathbb{C}$ and likely in general we include it for completeness.

Proposition 5.2.17. *If $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj}(\mathbb{K}[Y_0,..,Y_n])$ is a nonempty hyperplane arrangement, then $D_0(\mathcal{A})$ is a reflexive module.*

Proof. The proof is by induction on the number of hyperplanes in $\mathcal{A}$. First we consider the case $|\mathcal{A}| = 1$ or $|\mathcal{A} = 2|$, in these cases we can choose coordinates so that $Q_\mathcal{A} = Y_0$ or $Q_\mathcal{A} = Y_0 Y_1$ respectively. It can now be checked by direct computation that $D_0(\mathcal{A})$ is free on generators $\{\frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n}\}$ and $\{Y_1 \frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_n}\}$ respectively.

For the general case if $\mathcal{A}'$ is a hyperplane arrangement with $k > 2$ hyperplanes pick two distinct hyperplane L and H in $\mathcal{A}$. Let $\mathcal{A} = \mathcal{A}' \setminus \{H\}$ and let $\mathcal{B}$ denote the hyperplane arrangement $\{L, H\}$. Then we have the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & D_0(\mathcal{A}') \arrow[r] & D_0(\mathcal{A}) \oplus D_0(\mathcal{B}) \arrow[r] & D_0(\{L\}) \end{tikzcd} .$$

As $D_0(\{L\})$ is free it is in particular torsion free. Furthermore by inductive hypothesis $D_0(\mathcal{A})$ and $D_0(\mathcal{B})$ are both reflexive so we conclude by applying the preceding proposition. □

Definition 5.2.18 (Splitting Type)**.** If $\mathcal{A} \subseteq \mathbb{P}^n$ is a hyperplane arrangement (resp. nonempty hyperplane arrangement), then there exists tuple of integers $(a_0, a_1, .., a_n)$, (resp. $(a_1, .., a_n)$) referred to as the *Splitting Type* of $D(\mathcal{A})$ (resp. $D_0(\mathcal{A})$).

This is the unique tuple satisfying $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, so that if L is a general line then there's an isomorphism

$$D(\mathcal{A}) \mid_L \cong \bigoplus_{i=0}^n S/I(L)(-a_i) \qquad \left(\text{resp. } D_0(\mathcal{A}) \mid_L \cong \bigoplus_{i=1}^n S/I(L)(-a_i)\right).$$

5.3 Derivation Bundle of Hyperplane Arrangements and the Ideals of Dual Points

In this section we introduce our duality and establish a relationship between $D_0(\mathcal{A}_Z)$ and $I(Z)$. We can summarize this relationship as follows: Given a set of points $Z \subseteq \mathbb{P}(W)$ with dual hyperplane arrangement $\mathcal{A}_Z \subseteq \mathbb{P}(V)$, we consider a ring $T = R \otimes_{\mathbb{K}} \mathbb{K}[\mathrm{Gr}(n-2, V)]$ that contains naturally isomorphic copies of $R = \mathrm{Sym}(V^*)$ and $S = \mathrm{Sym}(W^*)$. We then show in theorem 5.3.8 that $D_0(\mathcal{A})$ is isomorphic to an S-submodule of the extended ideal $I(Z)T$. This is analogous to the standard construction used in [FV14]. In theorem 5.3.10 we then give a novel interpretation of the restriction of this S-submodule to a general line.

Definition 5.3.1. Let $\bigwedge^\bullet V$, denote the exterior algebra of V. This is the graded $\mathbb{K}$-algebra generated in degree 1 by V, subject to the relation $v^2 = v \wedge v = 0$ for all $v \in V$.

Definition 5.3.2. Let $\mathrm{Gr}(k, V)$ denote the k-th grassmanian of V as a projective subvariety of $\mathbb{P}(\bigwedge^k V)$. The projective coordinate ring of $\mathrm{Gr}(k, V)$ as a quotient of the polynomial ring of the ambient space is the *Plücker Algebra*, $\mathrm{PL}(k, V)$.

Fix a set of coordinates $X_0, .., X_n$ on V, so that $\mathrm{Sym}(V^*) \cong \mathbb{K}[X_0, .., X_n]$. Extend these to coordinates on $V^{\oplus k}$ for some $1 \leq k \leq n$, by letting $A_{i,0}, .., A_{i,n}$ denote an isomorphic copy of $X_0, .., X_n$, for each $i \in \{0, .., k-1\}$. We organize these into a $k \times n+1$ matrix $\mathbf{A}$ with entries $(\mathbf{A})_{i,j} = A_{i,j}$.

Let $c(\mathrm{Gr}(k, V))$ denote the affine cone of $\mathrm{Gr}(k, V)$ as a subvariety of $\bigwedge^k V$. Then the multiplication map $\wedge : V^{\oplus k} \to c(\mathrm{Gr}(k, V)) \subseteq \bigwedge^k V$, identifies the Plücker algebra $\mathrm{PL}(k, V)$ with the $\mathbb{K}$ algebra generated by the maximal $(k \times k)$ minors of $\mathbf{A}$.

Restricting to the case where $k = n$, multiplication in $\bigwedge V$ gives a non-degenerate pairing $\wedge : V \times \bigwedge^n V \to \bigwedge^{n+1} V$. Choosing an isomorphism $\bigwedge^{n+1} V \cong \mathbb{K}$ gives an isomorphism $\bigwedge^n V \cong V^*$, natural up to a $\mathbb{K}$-scalar. We fix one of these isomorphisms and let τ denote the induced isomorphism of polynomial rings $\tau : \mathrm{Sym}(W^*) \cong \mathrm{Sym}(\bigwedge^n V^*)$. As $n = \dim V - 1$, then $\bigwedge^n V^* = \mathrm{Gr}(n, V)$, and we can identify $\mathrm{PL}(n, V)$ with $\mathrm{Sym}(\bigwedge^n V^*) \cong \mathrm{Sym}(W^*)$. We further describe τ in coordinates below.

Definition 5.3.3. Taking the definitions of X_i and $A_{j,\ell}$ from above, further require that $A_{0,i} = X_i$. Define

$$\begin{aligned}\mathbb{K}[\mathbf{A}] &:= \mathbb{K}[A_{i,j} \mid 0 \leq i \leq n-1, 0 \leq j \leq n] \\ &:= \mathbb{K}[X_0, .., X_n][A_{i,j} \mid 1 \leq i \leq n-1, 0 \leq j \leq n]\end{aligned}$$

Let $\mathrm{PL}(n)$ be the subalgebra of $\mathbb{K}[\mathbf{A}]$ generated by the determinants M_i where

$$M_i := \begin{vmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ A_{0,0} & \dots & A_{0,i-1} & A_{0,i} & A_{0,i+1} & \dots & A_{0,n} \\ A_{1,0} & \dots & & A_{1,i} & & \dots & A_{1,n} \\ \vdots & & & \vdots & & & \vdots \\ A_{n-1,0} & \dots & & A_{n-1,i} & & \dots & A_{n-1,n} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ X_0 & \dots & X_{i-1} & X_i & X_{i+1} & \dots & X_n \\ A_{1,0} & \dots & & A_{1,i} & & \dots & A_{1,n} \\ \vdots & & & \vdots & & & \vdots \\ A_{n-1,0} & \dots & & A_{n-1,i} & & \dots & A_{n-1,n} \end{vmatrix}.$$

Finally, taking Y_i to be a dual basis of X_i we define $\tau : \mathbb{K}[Y_0, .., Y_n] \to \mathrm{PL}(n)$ via $\tau(Y_i) = M_i$.

The preceding conversation shows that $\mathrm{PL}(n))$ is a polynomial algebra in the generators M_i. The lemma below shows that our definition of τ above matches the construction from the preceding remark.

Lemma 5.3.4. *Let* $v \in V = \mathrm{Spec}(\mathbb{K}[X_0, .., X_n])$ *and let* $\ell_v = \sum_{i=0}^{n} c_i Y_i \in W^*$ *be the corresponding linear form. Then as a polynomial in* $X_0, .., X_n$ *the linear form* $\tau(\ell_v) = \sum_i c_i M_i$ *vanishes on* v.

Proof. Following definition 5.3.3 we see that $\tau(\ell_v) = \sum_{i=0}^{n} c_i M_i$ is the Laplace expansion along the first row of the determinant of the matrix $\begin{bmatrix} \vec{v} \\ \mathbf{A} \end{bmatrix}$, where $\vec{v} = \begin{bmatrix} c_0 & c_1 & \dots & c_n \end{bmatrix}$. If we then evaluate $X_0, .., X_n$ at v, so that $X_i \mapsto c_i$, the matrix is singular as two rows are identical hence the determinant vanishes. $\square$

Definition 5.3.5. If $X_0, .., X_n$ form a basis of V^* and $Y_0, .., Y_n$ are dual coordinates on W^*. Then for any $\lambda = \sum_{i=0}^{n} f_i(Y_0, .., Y_n)\ell_{X_i} \in \mathrm{Sym}(W^*) \otimes W$, we define a polynomial $F_\lambda \in \mathbb{K}[\mathbf{A}]$ via

$$F_\lambda := \sum_{i=0}^{n} X_i \tau(f_i) = \sum_{i=0}^{n} X_i f_i(M_0, .., M_n)$$

Definition 5.3.6. Let $J \subseteq \mathrm{Sym}(V^*) = \mathbb{K}[X_0, \dots, X_n]$ be any homogeneous ideal. We define a graded module denoted $J^{\gg}$ over $\mathrm{PL}(n)$, thought of as a polynomial ring in the minors $M_0, .., M_n$.

First, if $\mathfrak{m} = (X_0, X_1, \dots, X_n)$ is the maximal ideal we define $\mathfrak{m}^{\gg}$ as the $\mathrm{PL}(n)$-submodule of $\mathbb{K}[\boldsymbol{A}]$ generated by $(X_0, \dots, X_n)$. We grade both $\mathrm{PL}(n)$ and $\mathfrak{m}^{\gg}$ by the X-degree, meaning $\deg(M_i) = \deg(X_i) = 1$. Equivalently, the d-th graded component of $\mathfrak{m}^{\gg}$ is generated over $\mathbb{K}$ by all terms of the form $X_i M_0^{e_0} M_1^{e_1} \dots M_n^{e_n}$ where $\sum_{i=0}^{n} e_i = d - 1$.

More generally for any homogeneous ideal J, we set $J^{\gg} = (J\mathbb{K}[\boldsymbol{A}]) \cap \mathfrak{m}^{\gg}$.

The following example and proposition that follows are the main motivation for the definition of $J^{\gg}$.

Example 5.3.7. If P is the 0-th coordinate point in $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{K}[X_0, X_1, X_2])$, then $I(P) = (X_1, X_2)$. Given $\sum_{i=0}^{2} G_i X_i \in \mathfrak{m}^{\gg}$ where G_i is a polynomial in the maximal minors M_0, M_1, M_2 of the matrix

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ A_0 & A_1 & A_2 \end{bmatrix}.$$

It's not hard to see that $\sum_{i=0}^{n} G_i X_i \in I(P)^{\gg}$ if and only if $G_0 \in I(P)\mathbb{K}[\boldsymbol{A}]$.

Treating G_0 as a polynomial in $X_0, .., X_n$ with coefficients in the ring $\mathbb{K}[A_0, A_1, A_2]$ and consider the evaluation map $e_P : \mathbb{K}[\mathbf{A}] \to \mathbb{K}[A_0, A_1, A_2]$ obtained by evaluting at P, we note that $G_0 \in I(P)\mathbb{K}[\mathbf{A}]$ if and only if $\varepsilon_P(G_0) = 0$. Furthermore, as e_P sends $M_0 \mapsto 0$, $M_1 \mapsto -A_2$ and $M_2 \mapsto A_1$ then $G_0(P) = 0$ if and only if M_0 divides G_0. From this it follows that $I(P)^{\gg}$ is generated by $\{X_0 M_0, X_1, X_2\}$.

In fact this generating set is redundant as we have the nontrivial relation $X_0 M_0 + X_1 M_1 + X_2 M_2 = 0$, and so a minimal generating set for $I(P)^{\gg}$ is given by $\{X_1, X_2\}$.

The preceding definition is motivated by the following proposition.

Theorem 5.3.8. *Let $Z \subseteq \mathbb{P}(V) = \mathrm{Proj}(\mathbb{K}[X_0, .., X_n])$ be a finite set of points, and let $\mathcal{A}_Z \subseteq \mathbb{P}(W)$ denote the dual hyperplane arrangement. Then for $\lambda \in S \otimes W$ the following are equivalent:*

- *(i)* $\theta_\lambda \in D(\mathcal{A}_Z)$
- *(ii)* $F_\lambda \in I(Z) \cdot \mathbb{K}[\mathbf{A}]$

Moreover, $F_\lambda = 0$ if and only if there exists $g \in S$ so that $\theta_\lambda = g\theta_e$, where $\theta_e = \sum_{i=0}^{n} Y_i \frac{\partial}{\partial Y_i}$ is the Euler derivation.

In essence there's an isomorphism $\eta : \mathrm{PL}(n, V) \otimes_S D_0(\mathcal{A}_Z)(-1) \to I^{\gg}(Z)$ given by

$$\eta\left(\sum_{i=0}^{n} f_i(Y_0, .., Y_n)\frac{\partial}{\partial Y_i}\right) = \sum_{i=0}^{n} f_i(M_0, .., M_n) X_i$$

The above theorem is a consequence of the following lemma which is useful in it's own right.

Lemma 5.3.9. *Fix $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_{n-1})$ a tuple of $(n-1)$ linearly independent vectors in V. Letting $\alpha_i := (\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,n})$ in our chosen set of coordinates. We define the partial evaluation map*

$$\varepsilon_{\boldsymbol{\alpha}} : \mathbb{K}[\mathbf{A}] \to \mathbb{K}[X_0, .., X_n]$$

via $\varepsilon_{\boldsymbol{\alpha}}(A_{i,j}) = \alpha_{i,j}$ for $1 \le i \le n-1$.

Let $\lambda = \sum_{i=0}^{n} f_i X_i \in [S \otimes W]_d = \mathrm{Sym}^d(W^*) \otimes W$, for any nonzero $w \in W$ where $\ell_w \in V^*$ vanishes on $\mathrm{Span}(\boldsymbol{\alpha})$*, there exists some nonzero linear form h vanishing on* $\mathrm{Span}(\boldsymbol{\alpha})$ *so that*

$$\varepsilon_{\boldsymbol{\alpha}}(F_\lambda) \equiv h^d \ell_{\rho_\lambda(w)} = h^d \left(\sum_{i=0}^{n} X_i f_i(\rho_\lambda(w))\right) \mod I(\ell_w)$$

Proof. Take $w, \boldsymbol{\alpha}$ and λ as stated above. Let $\lambda = \sum_{i=0}^{n} f_i(Y_0, .., Y_n) \otimes X_i \in [S \otimes W]_d$, then $F_\lambda = \sum_{i=0}^{n} X_i f_i(M_0, .., M_n)$.

Write $\ell_w = \sum_{i=0}^{n} c_i X_i$ and assume without loss of generality that $c_n \neq 0$. Fix some index $j \in \{0, .., n-1\}$ and let $\ell_u = c_n Y_j - c_j Y_n \in [S]_1 = W^*$. Noting that $\ell_u(w) = \ell_w(u) = 0$, we may write $\varepsilon_\alpha(\tau(\ell_u))$ as the determinant of the matrix

$$\varepsilon_\alpha(\tau(\ell_u)) = \begin{vmatrix} 0 & \dots & 0 & c_n & 0 & \dots & -c_j \\ X_0 & \dots & X_{j-1} & X_j & X_{j+1} & \dots & X_n \\ \alpha_{1,0} & \dots & & \alpha_{1,j} & & \dots & \alpha_{1,n} \\ \vdots & & & \vdots & & & \vdots \\ \alpha_{n-1,0} & \dots & & \alpha_{n-1,j} & & \dots & \alpha_{n-1,n} \end{vmatrix}$$

As $u \in \ker \ell_w$ and $\operatorname{Span}(\alpha) \subseteq \ker w$, then either $\varepsilon_{\boldsymbol{\alpha}}(\tau(\ell_u)) = 0$ and $u \in \operatorname{Span}(\alpha)$, or $\varepsilon_{\boldsymbol{\alpha}}(\tau(\ell_u)) \neq 0$ and $\operatorname{Span}(\alpha, u) = \ker \ell_w$ which implies theres a scalar $r \in \mathbb{K}$ so $\varepsilon_\alpha(\tau(\ell_u)) = r\ell_w$. In either case we have $c_n M_i - c_i M_n \equiv 0 \mod (\ell_w)$. We conclude with the equalities below where here $M_i = \varepsilon_{\boldsymbol{\alpha}}(M_i)$,

$$\begin{aligned} \varepsilon_\alpha(F_\lambda) &= \sum_{i=0}^{n} f_i(M_0, M_1, .., M_n) X_i \\ &\equiv \sum_{i=0}^{n} f_i\left(\frac{c_0}{c_n} M_n, \frac{c_1}{c_n} M_n, .., \frac{c_{n-1}}{c_n} M_n, M_n\right) X_i \qquad &\mod (\ell_w) \\ &\equiv \left(\frac{M_n}{c_n}\right)^d \sum_{i=0} f_i(c_0, .., c_n) X_i \qquad &\mod (\ell_w) \\ &\equiv \left(\frac{M_n}{c_n}\right)^d \rho_\lambda(w) \qquad &\mod (\ell_w) \end{aligned}$$

Noting that because $c_n \neq 0$, we must have that $E_n = (0 : .. : 0 : 1) \notin \operatorname{Span}(\boldsymbol{\alpha}) \subseteq \ker \ell_w$. Therefore, $\varepsilon_{\boldsymbol{\alpha}}(M_n) \neq 0$ as it is the determinant of a non-singular matrix thereby establishing the result. □

Proof of theorem 5.3.8. Let $\lambda = \sum_i f_i \otimes w_i \in S \otimes W$. We note that since $D(\mathcal{A}_Z) = \bigcap_{P \in Z} D(H_P)$ and $I(Z) = \bigcap_{P \in Z} I(P)$, it suffices to establish the equivalence in consider the case Z consists of a single point P. Furthermore, to establish the case for a single point it suffices to show that $\epsilon_\alpha(F_\lambda)$ is in $I(P)$ for every (or even for general) α. This is because $\theta \in \operatorname{Der}(S)$ is in $D(\mathcal{A}_Z)$ if and only if the restriction of θ to L is in $D(\mathcal{A}) \mid_L$ for general L, and similarly F_λ vanishes at P if and only if $\varepsilon_{\boldsymbol{\alpha}}(F_\lambda)$ vanishes on P for general $\boldsymbol{\alpha}$.

Continuing, assume that $\boldsymbol{\alpha}$ is sufficiently general and let ℓ_Q denote the linear form vanishing on $\boldsymbol{\alpha}$ and P. We consider $\varepsilon_{\boldsymbol{\alpha}}(F_\lambda) \mod (\ell_Q)$. By lemma 5.3.9, we get that $\varepsilon_{\boldsymbol{\alpha}}(F_\lambda) \equiv h^d \ell_{\rho_\lambda(Q)} = h^d \rho_\lambda(\ell_Q)$. Yet for general $\boldsymbol{\alpha}$, we see that $h(P) \neq 0$ so F_λ vanishes on P if and only if $\rho_\lambda(\ell_Q)$ vanishes on P.

Now for any linear form ℓ_L recall that $\ell_L(P) = 0$ if and only if the corresponding $L \in W$ lies on $P^\perp = H_P$. Hence, we apply proposition 5.2.9 and conclude the proof of the first statement with the following chain of equivalences:

$$\begin{aligned} F_\lambda \in I(P) &\iff \text{for general } \boldsymbol{\alpha},\ \varepsilon_{\boldsymbol{\alpha}}(F_\lambda) \in I(P) \\ &\iff \text{for general } \boldsymbol{\alpha},\ \rho_\lambda(\ell_Q) \in I(P) \text{ where } \ell_Q \text{ vanishes on } \operatorname{Span}(\boldsymbol{\alpha}, P) \\ &\iff \text{for general } Q \in H_p,\ \rho_\lambda(Q) \in H_P \\ &\iff \theta_\lambda \in D(H_P) \end{aligned}$$

To finish the proof, we must establish the claim about the kernel of η. We see that $F_\lambda = 0$ if and only if for general $\boldsymbol{\alpha}$ and arbitrary ℓ_H vanishing on $\boldsymbol{\alpha}$ that $\varepsilon_{\boldsymbol{\alpha}}(F_\lambda) \equiv 0 \mod I(\ell_H)$. By lemma 5.3.9 the later condition occurs precisely when $\rho_\lambda(\ell_H) \in (\ell_H)$ for every linear form ℓ_H. If this occurs we conclude for all $H \in W$ that $\rho_\lambda(H) = r_H H$ for some scalar r_H. It immediately follows that as a rational map on $\mathbb{P}(W)$, ρ_λ can be extended to the identity, allowing us to conclude that $\theta_\lambda = f\theta_e$ where θ_e is the Euler derivation. □

We note that the proof above also establishes the following.

Theorem 5.3.10. *Let $Z \subseteq \mathbb{P}(V)$ and let $\mathcal{A}_Z \subseteq \mathbb{P}(W)$ be the dual hyperplane arrangement. Let $L \subseteq \mathbb{P}(W)$ be a general line, and $Q = L^\perp \subseteq \mathbb{P}(V)$ the dual linear subspace. Then there's an isomorphism of vector spaces*

$$[I(Z) \cap I(Q)^m]_{m+1} \cong [D_0(\mathcal{A}_Z) \mid_L]_m.$$

We can in fact, prove a slightly stronger statement. Namely, the above isomorphism corresponds to an isomorphism of modules over naturally isomorphic (up to scalar) rings, we give this proof after 5.3.14. In order to make this stronger statement and to aid with the exposition for the rest of the chapter, we introduce some new notation.

Definition 5.3.11. For $Q \subseteq \mathbb{P}^n$ a codimension 2 subspace we define a ring $\mathcal{F}_Q$ via

$$\mathcal{F}_Q := \operatorname{Sym}_{\mathbb{K}}([I(Q)]_1).$$

We note that if L_0, L_1 are linear forms which generate $I(Q)$, then $\mathcal{F}_Q$ is a polynomial ring in the generators L_0, L_1.

Fix Q and let $\boldsymbol{\alpha}$ be any basis of Q. The following proposition shows that $\mathcal{F}_Q$ can be viewed yet another way, as the image of the map $\varepsilon_{\boldsymbol{\alpha}} : \operatorname{Sym}(W^*) \to \operatorname{Sym}(V^*)$.

Proposition 5.3.12. *Let $Q = \operatorname{Span}(\boldsymbol{\alpha})$, and $L = Q^\perp$, then the map $\varepsilon_{\boldsymbol{\alpha}} : S = \operatorname{Sym}(W^*) \to R = \operatorname{Sym}(V^*)$, induces an isomorphism of $\mathbb{K}$-algebras*

$$\tau_{\boldsymbol{\alpha}} : S/I(L) \to \mathcal{F}_Q.$$

Proof. First consider the restriction of $\tau_{\boldsymbol{\alpha}}$ as a map $[S]_1 \to [R]_1$. By lemma 5.3.4, $\varepsilon_{\boldsymbol{\alpha}}(\ell)$ must vanish on all points of $Q = \operatorname{Span}(\boldsymbol{\alpha})$, hence $\varepsilon_{\boldsymbol{\alpha}}(\ell) \in I(Q)$. In fact, given $P \in \mathbb{P}(V) \setminus Q$, we see again by lemma 5.3.4, that $\varepsilon_{\boldsymbol{\alpha}}(\ell_P)$ defines the hyperplane $\operatorname{Span}(Q, P)$. It follows that $\tau_{\boldsymbol{\alpha}}$ induces an isomorphism of vector spaces $[\operatorname{Sym}(W^*)/I(L)]_1 \cong [\mathcal{F}_Q]_1$.

As $\operatorname{Sym}(W^*)$ and $\mathcal{F}_Q$ are both symmetric algebras generated over $\mathbb{K}$ in degree 1. The isomorphism $\tau_{\boldsymbol{\alpha}} : \operatorname{Sym}(W^*) \to \mathcal{F}_Q$ follows. □

Definition 5.3.13. Let $Q \subseteq \mathbb{P}(V)$ be a codimension 2 subspace, and let $J \subseteq \operatorname{Sym}(V^*)$ be any homogeneous ideal. We define a graded $\mathcal{F}_Q$-module, $J_Q^{\gg}$, as the $\mathcal{F}_Q$-submodule of $\operatorname{Sym}(V^*)$ whose d-th graded component is given by

$$[I_Q^{\gg}(Z)]_d := [I(Z) \cap I(Q)^{d-1}]_d.$$

We state the full version of this duality.

Theorem 5.3.14. *Let $Z \subseteq \mathbb{P}(V) = \operatorname{Proj} R$ be a finite set of points and $\mathcal{A}_Z \subseteq \mathbb{P}(W) = \operatorname{Proj} S$ the dual hyperplane arrangement. Let $L \subseteq \mathbb{P}(W)$ be a general line, then the isomorphism of $\mathbb{K}$ algebras $\tau_Q : S/I(L) \cong \mathcal{F}_Q = \operatorname{Sym}([I(Q)]_1)$, induces an isomorphism of graded modules $I_Q^{\gg}(Z)(-1) \cong D_0(\mathcal{A}_Z) \mid_L \otimes_{S/I(L)} \mathcal{F}_Q$ via the map*

$$\eta_Q : D_0(\mathcal{A}_Z) \mid_L \otimes \mathcal{F}_Q \cong I_Q^{\gg}(Z)(-1)$$

$$\eta_Q \left(\sum_{i=0}^{n} f_i \frac{\partial}{\partial Y_i} \right) = \sum_i \tau_Q(f_i) X_i$$

Here $\{Y_i\}_{i \in [n+1]}$ and $\{X_i\}_{i \in [n+1]}$ are dual bases of W^* and V^* respectively.

Proof. This proof is very similar to the proof of theorem 5.3.8. We make note of some of the differences. Given $\gamma \in S/I(L) \otimes W$, we get both a rational map $\rho_\gamma : L \to \mathbb{P}(W)$ and a derivation θ_γ of $\operatorname{Sym}(W^*)$ into $\operatorname{Sym}(W^*)/I(L)$.

Similarly, we get a polynomial $F_\gamma \in I_Q^{\gg}(\emptyset)$ uniquely determined up to scalar. Now it again follows that $\theta_\gamma \in D_0(\mathcal{A}) \mid_L$ if and only if $\rho_\gamma(H \cap L) \subseteq H$ for all $H \in \mathcal{A}$. Additionally, we have that for any $\ell \in [I(Q)]_1$, that $F_\gamma = \ell_Q^{d-1} \rho_\gamma(\ell) \mod (\ell)$.

The proof now continues as in theorem 5.3.8. ☐

Applying this isomorphism of modules, we note that the splitting type of of $\mathcal{A}_Z$ determines the dimension of $I_Q^{\gg}(Z)$.

Corollary 5.3.15. *If $D_0(\mathcal{A}_Z)$ has splitting type $(a_1, a_2, .., a_n)$, then for a general codimension 2 linear subspace,*

$$\dim [I_Q^{\gg}(Z)]_d = \sum_{i=1}^{n} \max\{0, d - a_i\}$$

Example 5.3.16 (Ceva Configurations). Let $m \geq 2$ be an integer which is not divisible by $\operatorname{Char}(\mathbb{K})$, so that there exists a primitive m-th root of unity $\zeta \in \mathbb{K}$. Fix $n \geq 1$ and a basis $\{E_0, E_1 .., E_n\}$ (dual to $\{X_0, .., X_n\}$) of the underlying vector space of $\mathbb{P}^n_{\mathbb{K}}$. We consider the set of points $F_m \subseteq \mathbb{P}^n$ which is the projectivization of the set of vectors

$$\left\{ -E_i + \zeta^\ell E_j \in \mathbb{K}^{n+1} \mid \text{ where } 0 \leq i < j \leq n \text{ and } 0 \leq \ell < m \right\}.$$

Further define $C_m \subseteq \mathbb{P}^n$ to be the set of points which includes F_m and the $n+1$ coordinate points corresponding to our basis $\{E_0, E_1, \ldots, E_n\}$. Then C_m is a set of

$n + 1 + m\binom{n+1}{2}$ points. The corresponding ideal $I(C_m)$ has $\binom{n+1}{2} + \binom{n+1}{3} = \binom{n+2}{3}$ generators which are given by

$$\{X_iX_jX_k \mid 0 \leq i < j < k \leq n\} \cup \{X_iX_j^{m+1} + (-X_i)^{m+1}X_j \mid 0 \leq i < j \leq n\}.$$

It's dual hyperplane arrangement, $\mathcal{A}_{C_m}$, can be defined by the vanishing of the polynomial

$$Q_{C_m} = Y_0Y_1 \dots Y_n \prod_{0 \leq i < j \leq n} (Y_i^m - Y_j^m).$$

If $m \geq 3$ $\mathcal{A}_{C_m}$ is often called the *Extended Ceva Arrangement*, or complete monomial arrangement. This well studied arrangement is a reflection arrangement corresponding to the monomial group $G(m, 1, n+1)$. Then $D_0(\mathcal{A}_{C_m})$ is a free S-module and one possible choice of basis elements are the elements $\{\theta_1, \dots, \theta_n\}$ where

$$\theta_j = \sum_{i=0}^{n} Y_i^{mj+1} \frac{\partial}{\partial Y_j}.$$

Hence, we conclude that $I^{\gg}(C_m)$ is free with basis

$$F_{\theta_i} := \sum_{i=0}^{n} M_i^{mj+1} X_i,$$

where the M_i's are the minors from definition 5.3.3.

The previous results about the module $I^{\gg}(Z)$ and it's relationship with $D_0(\mathcal{A}_Z)$ can be summed up as stating that the isomorphism of projective spaces $\mathbb{P}(W) \cong \mathbb{P}(\bigwedge^n V)$ extends to an isomorphism of sheaves $\widetilde{D_0(\mathcal{A}_Z)} \cong \widetilde{I^{\gg}(Z)}(-1)$. This sheaf $\widetilde{I^{\gg}(Z)}$ and it's relationship with $\widetilde{D_0(\mathcal{A}_Z)}$ is implicit in [FV14]. The relationship of $I_Q^{\gg}(Z)$ with a general codimension 2 subspace however was only made explicit in the case $Z \subseteq \mathbb{P}^2$.

As we have committed to working algebraically we state and prove one more result which is a simple corollary of the fact that previously stated isomorphisms correspond to an underlying isomorphism of sheaves.

Proposition 5.3.17. *The following diagram commutes for all codimension* 2 *subspaces* Q*,*

$$\begin{tikzcd}
D_0(\mathcal{A}_Z) \arrow[r, "\mathrm{res}_{Q^\perp}"] \arrow[d, "\eta"] & D_0(\mathcal{A}_Z) \mid_{Q^\perp} \arrow[d, "\eta_Q"] \\
I^{\gg}(Z) \arrow[r, "\varepsilon_Q"] & I_Q^{\gg}(Z)
\end{tikzcd}$$

with the sides isomorphisms for general Q*.*

Proof. First note, that by proposition 5.3.12 we have a commuting diagram of commutative $\mathbb{K}$-algebras

$$\begin{tikzcd}
\mathrm{Sym}(W^*) \arrow[r] \arrow[d, "\tau"] & \mathrm{Sym}(W^*)/I(Q^\perp)\ , \arrow[d, "\tau_Q"] \\
\mathrm{PL}(n) \arrow[r, "\varepsilon_Q"] & \mathcal{F}_Q
\end{tikzcd}$$

where the top map sends $f \in \mathrm{Sym}(W^*)$ to its coset $\bar{f} \in \mathrm{Sym}(W^*)/I(Q^\perp)$.

Working in coordinates given $\theta = \sum_{i=0}^n F_i \frac{\partial}{\partial Y_i}$, we have that

$$\varepsilon_Q \eta(\theta) = \sum_{i=0}^n \varepsilon_Q(\tau(F_i))X_i = \sum_{i=0}^n \tau_Q(\bar{F}_i)X_i = \eta_Q(\mathrm{res}_{Q^\perp}(\theta))$$

establishing the result. □

5.4 Unexpected Hypersurfaces

In [CHMN18], the authors gave a characerization of the degrees d, in which a finite set of points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits unexpected curves in the specific case when $m = d-1$. In this section we introduce the concept of very unexpected hypersurfaces (definition 5.4.7) and study them using the duality of section 5.3. Namely in theorem 5.4.27, we achieve a higher dimensional generalization of the main result of [CHMN18] which we recall below.

Theorem 5.4.1 ([CHMN18])**.** *For a finite set of points $Z \subseteq \mathbb{P}^2$, let $\mathcal{A}_Z$ denote the dual line arrangement, and let (a_1, a_2) denote the splitting type of the bundle defined by $D_0(\mathcal{A}_Z)$. Then exactly one of the following statements holds:*

(i) There is some line $L \subseteq \mathbb{P}^2$ with $|L \cap Z| > a_1 + 1$, in which case $|L \cap Z| = a_2 + 1$ and Z never admits unexpected curves.

(ii) Z admits unexpected curves in degree d for precisely those d with $a_1 < d < a_2$.

The most striking part of this characterization is that it does not depend directly on $\dim[I(Z)]_d$ or $\dim[I(Z) \cap I(Q)]_d$, a feature also present in our generalization. As some papers have already introduced a notion of unexpected hypersurface we recall this definition below, before discussing why it is inadequate for our needs.

Definition 5.4.2. If $Z \subseteq \mathbb{P}^n = \mathrm{Proj}(R)$ is a set of points and Q is some general linear subspace, we say Z admits unexpected mQ-hypersurfaces in degree d if

$$\begin{aligned}\dim[I(Z) \cap I(Q)^m]_d &> \max\left\{0, \binom{n+d}{n} - \dim[R/I(Z)]_d - \dim[R/I(Q)^m]_d\right\} \\ &> \max\{0, \dim[I(Q)^m]_d - \dim[R/I(Z)]_d\}\end{aligned}$$

If $Z \subseteq \mathbb{P}^2$ we instead say that Z admits unexpected curves.

If Q, m or d are obvious from context, we may avoid these qualifiers and simply specify that Z admits unexpected hypersurfaces.

If we filter $[R/I(Q)^m]$ by $I(Q)$ and look at the corresponding graded module. We get that each graded component, $[I(Q)^{i-1}/I(Q)^i]$ is a free module over $[R/I(Q)]$ generated by $[I(Q)^{i-1}/I(Q)^i]_{i-1}$. It follows that

$$\dim[I(Q)^{i-1}/I(Q)^i]_d = \binom{\dim Q + d - i}{\dim Q}\binom{\mathrm{codim}\, Q - 1 + i}{\mathrm{codim}\, Q - 1}.$$

We conclude that

$$\dim[R/I(Q)^m]_d = \sum_{i=0}^{m-1} \binom{\dim Q + d - i}{\dim Q}\binom{\operatorname{codim} Q - 1 + i}{\operatorname{codim} Q - 1}.$$

This result in combination with the Chu-Vandermonde identity,

$$\sum_{j=0}^{m-b} \binom{a+j}{a}\binom{m-j}{b} = \binom{a+m+1}{a+b+1},$$

allows us to conclude the following

Proposition 5.4.3. *$Z \subseteq \mathbb{P}^n$ admits unexpected mQ-hypersurfaces in degree d if and only if letting $N_{mQ} = \sum_{i=0}^{m-1} \binom{\dim Q+d-i}{\dim Q}\binom{\operatorname{codim} Q-1+i}{\operatorname{codim} Q-1}$ and taking $M_{mQ} = \sum_{j=m}^{d} \binom{\dim Q+d-j}{\dim Q}\binom{\operatorname{codim} Q-1+j}{\operatorname{codim} Q-1}$ we have*

$$\begin{gathered} \dim[I(Z) \cap I(Q)^m]_d > \max\{0, \dim[I(Z)]_d - N_{mQ}\} \\ \textit{or equivalently} \\ \dim[I(Z) \cap I(Q)^m]_d > \max\{0, M_{mQ} - \dim[R/I(Z)]_d\} \end{gathered}$$

In particular, if $m = d-1$ and $\dim Q = n-2$ the inequality becomes

$$\dim[I(Z) \cap I(Q)^{d-1}]_d > \max\{0, nd + 1 - \dim[R/I(Z)]_d\}$$

Despite the ease in which the above definition can be stated, it has a few shortcomings. The first shortcoming is of a semantic nature, namely there are sets of points which by definition admit unexpected mQ-hypersurfaces, but where we believe the difference in dimension is unsurprising. The second issue is somewhat larger if we hope to generalize theorem 5.4.1, namely it can not be determined from $D_0(\mathcal{A}_Z)$ whether or not Z admits ostensibly unexpected $(d-1)Q$-hypersurfaces in degree d.

Both of these issues are illustrated by the following example.

Example 5.4.4. Let $H \subseteq \mathbb{P}^3$ be any plane, and let W consist of 10 of points on H. Now take two general points P_0 and P_1 not on H. Let $Z = P_0 + P_1 + W$, and $Q \subseteq \mathbb{P}^2$ a generic line, if we let ℓ_H be a linear form defining H, and ℓ_0, ℓ_1 be linear forms defining $\operatorname{Span}(Q, P_0)$ and $\operatorname{Span}(Q, P_1)$ respectively. Then taking $f = \ell_H \ell_0 \ell_1$, we get that f lies in $[I(Z + 2Q)]_3$. If the points in W are general points on H, then $h_w(3) = \min\left\{\binom{2+3}{3}, |W|\right\} = 10$, $h_Z(3) = 12$, and $h_{2Q}(3) = 4 + (2)(3) = 10$, in which case Z admits an unexpected hypersurface in degree 3.

However, taking W' to be 10 points lying on a smooth conic in H and letting $Z' = P_0 + P_1 + W'$, then $h'_W(3) = 7$ and $h'_Z(3) = 9$ so Z' does not admit unexpected hypersurfaces in degree 3.

Note though that there is an isomorphism of intersection lattices $L_{\mathcal{A}_Z} \cong L_{\mathcal{A}'_Z}$, and that both $D_0(\mathcal{A}_Z)$ and $D_0(\mathcal{A}_{Z'})$ have splitting type $(2, 4, 5)$.

In the above example, the “unexpectedness” is explained by the fact that most of the points of Z lie on the plane H. This gives us a lower bound on $\dim[I(Z+2Q)]_d$ since,

$$\dim[I(Z+2Q)]_3 > \dim[I(H+2Q) \cap I(P_0+P_1)]_3 \geq \dim[I(H+2Q)]_3 - 2.$$

Furthermore, there is no reason to expect equality in the inequality

$$\dim[I(H) \cap I(Q)^2]_3 \leq \max\left\{0, \dim[I(Q)^2]_3 - \dim[R/I(H)]_3\right\}$$

since Q and H have nonempty intersection. This situation is elaborated on further by the following proposition which computes the dimension of $[I_Q^{\gg}(H)]_d = [I(H) \cap I(Q)^{d-1}]_d$ can impose on $I(Q)^{d-1}$.

Proposition 5.4.5. *Let $H, Q \subseteq \mathbb{P}(V)$ be nonempty linear subspaces, with Q general of codimension 2. Then*

$$\dim[I_Q^{\gg}(H)]_d = \dim[I(H) \cap I(Q)^{d-1}]_d = d(\operatorname{codim} H).$$

As a consequence, if $Z \subseteq H$, then $\dim[I_Q^{\gg}(Z)]_d \geq d(\operatorname{codim} H)$.

Proof. Let $h = \dim H$. We may choose a basis $\{X_0, .., X_n\}$ of V^* so that $I(H) = (X_{h+1}, .., X_n)$. Moreover, let $\ell_i := \varepsilon_Q(M_i) \in [\mathcal{F}_Q]_1$, denote the linear form vanishing on Q and the i-th coordinate point.

We proceed by induction on h, establishing that $I^{\gg}(H)$ is a free $\mathcal{F}_Q$-module with basis $\{X_{d+1}, .., X_n\}$. First consider the case $h = 0$, so that H is the 0-th coordinate point. For each $f \in [I^{\gg}(H)_Q]_d$, we may write $f = \sum_{i=0}^n f_i X_i$ with each $f_i \in [\mathcal{F}_Q]_{d-1}$. Evaluating f at H shows that $f_0(H) = 0$. As $\mathcal{F}_Q$ is a polynomial ring in two variables, we conclude that ℓ_0 divides f_0. Using the identity $\sum_{i=0}^n X_i \ell_i = 0$, and letting $g_i = f_i - \ell_i f_0/\ell_0$ we get $f = \sum_{i=1}^n g_i X_i$. It follows that $I^{\gg}(Q)$ is a free $\mathcal{F}_Q$-module with basis $X_1, .., X_n$.

Now when $h \geq 1$, let $H_0 \subseteq H$ be the coordinate subspace, with defining ideal $I(H_0) = (X_h, .., X_n) \supset I(H)$. We get by inductive hypothesis that every element $f \in I_Q^{\gg}(H_0)$ may be written in the form $f = \sum_{i=h}^n f_i X_i$. As $X_j \in I(H)$ for $j > h$, we see that $f \in I(H)$ if and only if $f_h \in I(H) \cap \mathcal{F}_Q$. However, as $\mathbb{K}$ is infinite and $h > 0$ we have for general Q that there is no finite collection of hyperplanes through Q which vanish on H, and consequently we must have $I(H) \cap \mathcal{F}_Q = 0$. Hence $\sum_{i=h}^n f_i X_i \in I_Q^{\gg}(H)$ if and only if $f_h = 0$, and so $I_Q^{\gg}(H)$ is free with basis $X_{h+1}, .., X_n$ as claimed.

Noting that $\dim[\mathcal{F}_Q]_{t-1} = t$, we obtain the desired equality

$$\dim[I_Q^{\gg}(H)]_d = (\operatorname{codim} H)(\dim[\mathcal{F}_Q]_{d-1}) = d(\operatorname{codim} H).$$

□

Example 5.4.6. In view of the preceding lemma, we see that example 5.4.4 can be generalized. Namely, for $n > 2$ we let $H \subseteq \mathbb{P}^n$ be a proper linear subspace of dimension $d > 1$. Fix a degree $t > 1$ and let Z consist of $\binom{t+d}{t}$ general points on H, so that $\dim[R/I(Z)]_s = \min\{\binom{s+d}{s}, |Z|\}$. Then the prior lemma shows $\dim[I_Q^{\gg}(Z)]_s = \max\{(n-d)s, ns+1-|Z|\}$, and hence that Z admits unexpected Q-hypersurfaces in all degrees $2 \leq s \leq t$.

With this discussion in mind we introduce our definition of very unexpected hypersurface.

Definition 5.4.7. Let $Z \subseteq \mathbb{P}(V)$ be a finite set of points and $R = \operatorname{Sym}(V^*)$ the projective coordinate ring. For Q a generic linear subspace, we say that Z admits *very unexpected mQ-hypersurfaces* in degree d, if there is a subset $W \subseteq Z$ satisfying the following conditions:

(I) $[I(Z) \cap I(Q)^m]_d = [I(W) \cap I(Q)^m]_d$

(II) For all irreducible subvarieties $X \subseteq \mathbb{P}(V)$,

$$|W \cap X| \leq \dim[I(Q)^m/(I(X) \cap I(Q)^m)]_d$$

(III) W imposes less condition on $[I(Q)^m]_d$ than on $[R]_d$, that is

$$\dim[R/I(W)]_d > \dim[I(Q)^m/(I(W) \cap I(Q)^m)]_d$$

Remark 5.4.8. We note that condition (II) only needs to be checked on *positive dimensional* irreducible subvarieties.

Remark 5.4.9. It's possible that there are other definitions that are preferable in some ways. One change that might be useful is to require condition (ii) in the case where X is not necessarily irreducible, or if we allow Z to be nonreduced perhaps take X to be a positive dimensional subscheme. We use the above definition for now as it is strong enough for our purposes while still being relatively easy to check.

In this chapter we will be focusing on the case where $\operatorname{codim} Q = 2$ and $m = d-1$. We introduce this definition in general because we think it is a natural and potentially useful modification given our discussion in example 5.4.4.

Remark 5.4.10. Despite the fact the above definition is strictly stronger than definition 5.4.2, the two definitions agree in $\mathbb{P}^2$. This is a consequence of the fact that the only positive dimensional subvarieties that are needed to check in condition (ii) are hypersurfaces. More generally, if $Z \subseteq \mathbb{P}^n$ is a finite set of points contained in a hypersurface defined by $(f = 0)$ and $Q \in \mathbb{P}^n$ is the generic point. Then applying the dimension count from 5.4.3, that

$$\begin{aligned}\dim[I(Q)^m/(f) \cap I(Q)^m]_d &= \dim[I(Q)^m/(fI(Q)^m)]_d \\ &= \dim[I(Q)^m]_d - \dim[I(Q)^m]_{d-\deg f} \\ &= \max\left\{0, \binom{n+d}{n} - \binom{n+d-f}{n}\right\}\end{aligned}$$

It follows for all m and d that

$$\begin{aligned}\dim[R/I(Z)]_d &\leq \dim[R/(f)]_d \\ &\leq \binom{n+d}{n} - \binom{n+d-\deg(f)}{n} \\ &\leq \dim[I(Q)^m/((f)\cap I(Q)^m)]_d\end{aligned}$$

Establishing that condition (iii) could never be satisfied under these conditions.

A similar argument shows that if Q is a hyperplane, then no set of points Z can admit very unexpected mQ-hypersurfaces.

One potential issue with 5.4.7 is that condition (II) seems difficult to verify, given that naively there is a potentially infinite number of irreducible varieties we must check. However, we make a few observations showing that it is easier to verify than it may seem, and can be reduced to a finite number of subvarieties.

Suppose that $Z \subseteq \mathbb{P}^n$ admits unexpected mQ-hypersurfaces in degree d and furthermore, that there's no $P \in Z$ where $[I(Z)\cap I(Q)^m]_d \subsetneq [I(Z-P)\cap I(Q)^m]_d$. This is a relatively harmless assumption since if such a P does exist, then $Z \setminus P$ still admits unexpected hypersurfaces in degree d.

Now if there is some positive dimensional variety $X_1 \subseteq \mathbb{P}^n$ so that $|Z\cap X_1| > \dim[I(Q)^m/I(X_1)\cap I(Q)^m]_d$. Then $Z\cap X_1$ imposes less than $|Z\cap X_1|$ conditions on $I(Q)^m$ and so we may can find a subset $U_1 \subseteq Z\cap X_1$ with $|U_1| = \dim[I(Q)^m/(I(X_1)\cap I(Q)^m)]_d$ and $[I(Q)^m \cap I(U_1)]_d = [I(Q)^m\cap I(X\cap Z)]_d$. Setting $Z_1 = (Z\setminus X)\cup U_1$ we make two observations both of which follow readily:

(A) $[I(Z)\cap I(Q)^m]_d = [I(Z\setminus X)\cap I(X\cap Z)\cap I(Q)^m]_d = [I(Z_1)\cap I(Q)^m]_d$

(B) If there's a strict containment $[I(X_1)\cap I(Q)^m]_d \subsetneq [I(U_1)\cap I(Q)]_d$, then Z_1 admits unexpected hypersurfaces if and only if Z does.

We may continue in this way stopping when we find a subset $Z_k \subseteq Z_{k-1} \subseteq ... \subseteq Z$, where either

1. Z_k does not admit unexpected hypersurfaces; or
2. $W = Z_k$ satisfies the conditions $(I), (II)$ and (III) of definition 5.4.7.

If Z_k does not admit unexpected hypersurfaces then by observation (B), we must have $[I(X_k)\cap I(Q)^m]_d = [I(U_k)\cap I(Q)^m]_d$. Then

$$[I(Z)\cap I(Q)^m]_d \subseteq [I(U_k)\cap I(Q)^m]_d = [I(X_k)\cap I(Q)^m]_d.$$

Hence, the polynomials in $[I(Z)\cap I(Q)^m]$ vanish on the positive dimensional variety X_k.

From the preceding discussion we can conclude the following proposition.

Proposition 5.4.11. *Let $Z \subseteq \mathbb{P}^n$ be a set of points which admits unexpected mQ-hypersurfaces in degree d. Then there exists $W \subseteq Z$, so that W satisfies conditions I and II of definition 5.4.7 and Z admits very unexpected hypersurfaces if and only if W admits unexpected hypersurfaces.*

With this discussion in mind we introduce the following definition.

Definition 5.4.12. Fix positive integers m, n, c and d. If $Z \subseteq \mathbb{P}^n$ is a finite set of points we set

$$\mathrm{B.\,loc}_d(Z, m, c) := \bigcap_Q V([I(Z) \cap I(Q)^m]_d).$$

Where Q is over all linear subspaces of dimension c. Moreover, we set $\mathrm{B.\,loc}_d(Z) := \mathrm{B.\,loc}_d(Z, d-1, n-2)$ as this is the case we will focus on.

If $m = d-1$ and $c = n-2$, we also define $\mathrm{B.\,loc}_d(M)$ for a submodule $M \subseteq \mathfrak{m}^{\gg}$ via

$$\mathrm{B.\,loc}_d(M) = \bigcap_{F \in [M]_d} \bigcap_{Q \in \mathrm{Gr}(n-2,n)} V(\varepsilon_Q(F)).$$

That is $\mathrm{B.\,loc}(F_\sigma)$ is the intersection of all the hypersurfaces defined by $\varepsilon_Q(F_\delta)$ as Q varies.

From the discussion proceeding this definition, we may conclude the following

Proposition 5.4.13. *Fix m, n, c and d as above. For $Z \subseteq \mathbb{P}^n = \operatorname{Proj} R$, and Q the generic c-dimensional linear subspace, we have Z admits very unexpected mQ-hypersurfaces if and only if there's a subset $W \subseteq Z$ satisfying*

(I) $[I(Z) \cap I(Q)^m]_d = [I(W) \cap I(Q)^m]_d$

(II') For all irreducible components X of $\mathrm{B.\,loc}_d(Z, m, c)$

$$|W \cap X| \leq \dim[I(Q)^m/(I(X) \cap I(Q)^m)]_d$$

(III) W imposes less condition on $[I(Q)^m]_d$ than on $[R]_d$, that is

$$\dim[R/I(W)]_d > \dim[I(Q)^m/(I(W) \cap I(Q)^m)]_d$$

Consequently, if $\dim \mathrm{B.\,loc}_d(Z, m, c) = 0$ *then Z admits very unexpected hypersurfaces if and only if Z admits unexpected hypersurfaces.*

Remark 5.4.14. Note that $W \subseteq Z$ may satisfy (II') without satisfying (II). For instance the points $Z = C_5$ dual to the Ceva Arrangement $\mathcal{A}_{C_5} \subseteq \mathbb{P}^2_{\mathbb{C}}$ consist of 18 points which admit unexpected curves in all degrees d with $6 < d < 11$. Taking $d = 7$ we note that $W = Z$ does not satisfy condition (II) of definition 5.4.7, since taking $X = \mathbb{P}^2$ we see that $|W \cap X| = 18 > 15 = \dim[I(Q)^6/(0)]_7$.

More generally, if H is a 2-dimensional linear subspace in $\mathbb{P}^3$ then taking $Z \subseteq H$ it follows from proposition 5.5.11 that $W = Z$ does not satisfy condition (II) in degree $d = 7$. However, in either case $W = Z$ satisfies condition (II') above.

Example 5.4.15. It should be noted here that $\mathrm{B.\,loc}_d(Z)$ and $\mathrm{B.\,loc}_d(I^{\gg}(Z))$ are not necessarily the same. For instance, if $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ is 5 general points, then a computation shows that $[I^{\gg}(Z)]_3 = 0$, and so $\mathrm{B.\,loc}_3(I^{\gg}(Z)) = \mathbb{P}^2$. Yet a direct computation shows that $\dim[I(Z) \cap I(Q)^2]_3 = 2$ and $\mathrm{B.\,loc}_3(Z) = Z$. It is true, however, that $\mathrm{B.\,loc}_d(Z) \subseteq \mathrm{B.\,loc}_d(I^{\gg}(Z))$.

Remark 5.4.16. From here on we restrict the view of the chapter, to the case where $c = n - 2$ and $m = d - 1$ that is we study $[I(Z) \cap I(Q)^{d-1}]_d$.

The following proposition provides a classification of those varieties that can appear in $\mathrm{B.loc}_d(Z)$.

Proposition 5.4.17. *For any submodule $M \subseteq \mathfrak{m}^{\gg}$, (resp. $Z \subseteq \mathbb{P}^n$) the base locus $\mathrm{B.loc}_d(M)$ (resp. $\mathrm{B.loc}_d(Z)$) is a union of linear subspaces.*

Proof. We prove both statements in parallel, let $B = \mathrm{B.loc}_d(M)$ or $B = \mathrm{B.loc}_d(Z)$.

Let C be a positive dimensional irreducible subvariety which is contained in B and not a linear subspace. We establish that $\mathrm{Span}(C) \subseteq B$ from which the result follows.

First we show for a general hyperplane H, that $\mathrm{Span}(H \cap C) = H \cap \mathrm{Span}(C)$. Note that $\mathrm{Span}(H \cap C) \subseteq H \cap \mathrm{Span}(C)$ and so it suffices to show they have the same dimension. To do this take $c_1, .., c_t \in C$ to be $t = \dim \mathrm{Span}(C)$ linearly independent points, and let L be any hyperplane containing $\mathrm{Span}(c_1, .., c_t)$, but with $C \nsubseteq L$. Then

$$\dim \mathrm{Span}(L \cap C) = \dim \mathrm{Span}(C) - 1 = \dim(L \cap \mathrm{Span}(C)).$$

It now follows that $\dim \mathrm{Span}(H \cap C) \geq -1 + \dim \mathrm{Span}(C)$ for a general hyperplane $H \subseteq \mathrm{Span}(C)$, since among hyperplanes H which properly intersect C, the quantity $\dim \mathrm{Span}(H \cap C)$ is lower semi-continuous. So in particular,

$$\begin{aligned}\dim \mathrm{Span}(C) - 1 = \dim \mathrm{Span}(L \cap C) &\leq \dim \mathrm{Span}(H \cap C) \\ &\leq \dim(H \cap \mathrm{Span}(C)) = \dim \mathrm{Span}(C) - 1.\end{aligned}$$

Thus establishing the claim.

Proceeding let $Q \subseteq \mathbb{P}^n$ be a general codimension 2 subspace , and let ℓ a general linear form vanishing on Q. As Q is a hypersurface considered as a subvariety of $(\ell = 0)$, we get for any $f \in \varepsilon_Q([M]_d)$ (resp. any $f \in [I_Q^{\gg}(Z)]_d$) that there exist linear forms $r \in [R]_1$ and $\ell_Q \in I(Q)$ so that

$$f = (\ell_Q)^{d-1} r \mod (\ell).$$

Note that as ℓ is general, we may assume that $f \neq 0 \mod (\ell)$. Since Q is general we can assume that for every positive dimensional component C of B, that $C \nsubseteq Q$ and furthermore that Q contains no component of $C \cap (\ell = 0)$. As r is linear, it vanishes on $\mathrm{Span}((\ell = 0) \cap B) = \mathrm{Span}(B) \cap (\ell = 0)$. It follows that for any component C of B that f vanishes on a general hyperplane section of $\mathrm{Span}(C)$. As $\mathrm{Span}(C)$ is irreducible we conclude that if $\dim(C) > 0$ then f vanishes on $\mathrm{Span}(C)$ as desired. □

In fact more can be said about the varieties that appear as the base loci of $I^{\gg}(Z)$. The following gives a classification of such subvarieties.

Proposition 5.4.18. *Given a subvariety $B \subseteq \mathbb{P}^n$ there exists a set of points $Z \subseteq \mathbb{P}^n$ and an integer d so*

$$B = \mathrm{B.loc}_d(Z)$$

if and only if $B = \bigcup_{i=1}^k H_i$ *where* H_i *are pairwise disjoint linear subvarieties so for all* $J \subseteq \{1, .., k\}$ *with* $|J| \geq 2$ *we have*

$$\sum_{i \in J} \dim H_i < \dim \operatorname{Span}\left(\bigcup_{i \in J} H_i\right). \tag{5.4.18.1}$$

Proof. We first prove the forward implication. Note from proposition 5.4.17 we obtain that B every irreducible component of B is linear. Therefore, we can write $B = W \cup \bigcup_{i=1}^s H_i$ where W is a finite set of points, and each H_i is a positive dimensional linear subspace, so $Z \setminus W \subseteq \bigcup_{i=1}^s H_i$. We can further assume that $W \cap H_i = \emptyset$, and that $H_j \not\subseteq H_i$ for each pair of indices $i \neq j$. We observe that B satisfies the hypothesized condition if and only if $B \setminus W$ satisfies the condition, and therefore we assume that $B = \bigcup_{i=1}^s H_i$.

Now take $Q \subseteq \mathbb{P}^n$ a general codimension 2 subspace, and let L be a general hyperplane containing Q. If ℓ, ℓ_Q are linear forms so ℓ defines L and $I(Q) = (\ell, \ell_Q)$, we see by lemma 5.3.9 $f \in [I_Q^{\gg}(Z)]_d$ can be written as

$$f \equiv (\ell_Q)^{d-1} r \mod (\ell)$$

for some linear form r. Since Q and L are general we may assume that no irreducible component of $B \cap L$ is contained in Q. Therefore, $B \cap L$ is contained in the subvariety defined by the ideal (ℓ, r).

We now establish 5.4.18.1, proceeding by induction on $k = |J|$. Keeping the notation from the previous paragraph, we make two key observations before continuing with the proof. The first is that since $\operatorname{Span}\left(\bigcup_{j \in J} H_j\right)$ is not contained in $\operatorname{B.loc}_d(Z)$ we must have that $\operatorname{Span}\left(\bigcup_{j \in J} H_j\right) \cap L \not\subseteq V(\ell, r)$. Consequently the following inequality holds

$$\dim \operatorname{Span}\left(\bigcup_{j \in J} (H_j \cap L)\right) < \dim \operatorname{Span}\left(\bigcup_{j \in J} H_j\right) \cap L. \tag{5.4.18.2}$$

In particular $\dim \operatorname{Span}\left(\bigcup_{j \in J} (H_j \cap L)\right) \leq \dim \operatorname{Span}\left(\bigcup_{j \in J} H_j\right) - 2$. The second observation is that in view of the above inequality it suffices to establish that

$$\dim \operatorname{Span}\left(\bigcup_{j \in J} (H_j \cap L)\right) = -1 + \sum_{j \in J} \dim H_j. \tag{5.4.18.3}$$

We continue with the induction, establishing inductively the above equality. If $J = \{i, j\}$, note that if $H_i \cap H_j \neq \emptyset$ then $\dim(H_i \cap H_j \cap L) = \dim(H_i \cap H_j) - 1$ (recall by convention $\dim \emptyset = -1$) and so

$$\begin{aligned} \dim \operatorname{Span}((H_i \cap L) \cup (H_j \cap L)) &= \dim(H_i \cap L) + \dim(H_j \cap L) - \dim(H_i \cap H_j \cap L) \\ &= \dim(H_i) + \dim(H_j) - \dim(H_i \cap H_j) - 1 \\ &= \dim(\operatorname{Span}(H_i \cup H_j) \cap L) \end{aligned}$$

which contradicts eq. (5.4.18.2). Therefore, $H_i \cap H_j = \emptyset$ and $H_i \cap H_j \cap L = \emptyset$. Hence $\dim \operatorname{Span}((H_i \cap L) \cup (H_j \cap L)) = \dim \operatorname{Span}(H_i \cap L) + \operatorname{Span}(H_j \cap L) + 1 = \dim(H_i) + \dim(H_j) - 1$.

Now suppose that $|J| = \ell > 2$ and that for all $J' \subsetneq J$ with $|J'| \geq 2$ that eq. (5.4.18.3) holds. Furthermore assume for simplicity that $J = \{1, .., \ell\}$. We proceed by contradiction assuming that eq. (5.4.18.3) does not hold. For each $1 \leq i \leq \ell$ find an basis, B_i, of $H_i \cap L$ (aka, an affinely independent subset which spans H_i). Let $\mathfrak{B}' = \bigcup_{i=1}^{\ell-1} B_i$, then by inductive hypothesis $\mathfrak{B}'$ is affinely independent and there exists a proper subset $R \subsetneq B_\ell$ so that $\mathfrak{B}' \cup R$ is a basis of $\operatorname{Span}\left(\bigcup_{j \in J}(H_j \cap L)\right)$. Pick any $x \in H_\ell \setminus L$, we claim that $\mathfrak{B} = \mathfrak{B}' \cup R \cup \{x\}$ is a basis of $\operatorname{Span}\left(\bigcup_{j \in J} H_j\right)$. Assuming this claim we see that $\dim \operatorname{Span}\left(\bigcup_{j \in J}(H_j \cap L)\right) = |\mathfrak{B} \setminus \{x\}| = |\mathfrak{B}| - 1 = \dim \operatorname{Span}\left(\bigcup_{j \in J} H_j\right) \cap L$ contradicting eq. (5.4.18.2), and therefore establishing the result.

It suffices to show that $H_j \subseteq \operatorname{Span}(\mathfrak{B})$ for $1 \leq j \leq \ell$. If $j = \ell$, this follows as $H_\ell = \operatorname{Span}((H_\ell \cap L) \cup \{x\}) \subseteq \operatorname{Span}(\mathfrak{B})$. Otherwise if $j \neq \ell$ pick a sufficiently general hyperplane $L' \subseteq \mathbb{P}^n$ so that for all $1 \leq i \leq \ell - 1$ with $i \neq j$ we have $L' \cap H_i = L \cap H_i$. We may similarly assume we have $H_\ell \cap \operatorname{Span}(\bigcup_{i=1}^{\ell-1}(H_i \cap L')) \neq \emptyset$, and so by dimension counting we see

$$(L' \cap H_j) \subseteq (L' \cap L \cap H_j) + \sum_{i \neq j}(H_i \cap L') \subseteq L' \cap \operatorname{Span} \mathfrak{B}.$$

Now since $H_j = \operatorname{Span}((L' \cap H_j) \cup (L \cap H_j))$ we conclude that $H_j \subseteq \operatorname{Span}((L' \cap \operatorname{Span}(\mathfrak{B}) \cup (L \cap \operatorname{Span}(\mathfrak{B}))) \subseteq \operatorname{Span}(\mathfrak{B})$. Establish the result.

For the reverse direction, consider a finite set of points $Z \subseteq \mathbb{P}^n$. If $(a_1, a_2, \ldots, a_n)$ is the splitting type of $D_0(\mathcal{A}_Z)$ then by corollary 5.3.15 Z imposes independent conditions of $[I(Q)^{d-1}]_d$ for all $d > a_n$. Since $\sum_{i=1}^n a_i = |Z| - 1$ and $0 \leq a_1 \leq \ldots \leq a_n$ we conclude that for any $P \notin Z$ we have that $[I_Q^{\gg}(Z)]_{|Z|+1} \neq [I^{\gg}(Z + P)]_{|Z|+1}$, and hence that $\operatorname{B.loc}_d(Z) = Z$ for all $d > |Z|$.

Now consider $B = Z \cup \bigcup_{i=1}^k H_i$ where Z is a finite set of points and $\{H_1, .., H_k\}$ is a collection of positive dimensional linear subspace satisfying eq. (5.4.18.1), and where $\sum_{i=1}^k \dim H_i = n - 1$. We note that it suffices to show that there exists some $F \in [I^{\gg}(\emptyset)]_d$ so that $F \neq 0$ and $\operatorname{B.loc}(F) \supseteq B$. This is because by the first part of the proposition we necessarily have that $\operatorname{B.loc}(F) = B$. Furthermore taking Z_i to be $\binom{\dim H_i + d}{d}$ sufficiently general points on H_i, we have $[I(Z_i)]_d = [I(H_i)]_d$ and so setting $Z = Z \cup Z_1 \cup \ldots \cup Z_k$ it follows $[I(Z_B)]_d = [I(B)]_d$ and $\operatorname{B.loc}_d(Z_B) = B$.

We construct F as follows. Again take L to be a sufficiently general hyperplane and construct a basis $b_{i,0}, \ldots, b_{i,d_i}$ of H_i for all $0 \leq i \leq k$ where $d = \dim H_i$ and

$b_{i,1}, \dots, b_{i,d_i} \in L \cap H_i$. We now let N denote the $(n+k) \times (n+1)$ matrix

$$N = \left[\begin{array}{cccc} X_0 & X_1 & \dots & X_n \\ \multicolumn{4}{c}{b_{0,0}} \\ \multicolumn{4}{c}{\vdots} \\ \multicolumn{4}{c}{b_{0,d_0}} \\ \multicolumn{4}{c}{b_{1,0}} \\ \multicolumn{4}{c}{\vdots} \\ \multicolumn{4}{c}{b_{1,d_1}} \\ \multicolumn{4}{c}{\vdots} \\ \multicolumn{4}{c}{b_{k,d_k}} \end{array}\right]$$

where $b_{i,j}$ denotes the corresponding row vector. Let $\mathcal{C} = \{0, \dots, d_1\} \times \dots \times \{0, \dots, d_k\}$ for each $\boldsymbol{j} = (j_1, \dots, j_k) \in \mathcal{C}$ we take $N_{\boldsymbol{j}}$ to denote the determinant of the minor of N obtained by removed the rows corresponding to $b_{1,j_1}, \dots, b_{k,j_k}$.

Further let $\epsilon_{\boldsymbol{j}} = (-1)^{\sum_{i=1}^{k} j_i}$, and assuming $b_{i,j} = (b_{i,j,0} : \dots : b_{i,j,n})$ let $M_{b_{i,j}} = \sum_{k=0}^{n} b_{i,j,k} M_k$ as in definition 5.3.3. We then define

$$F = \left(\prod_{P \in Z} M_p \right) \left(\sum_{\boldsymbol{j} \in \mathcal{C}} \epsilon_{\boldsymbol{j}} N_{\boldsymbol{j}} \prod_{i=1}^{k} M_{b_{i,j_i}} \right).$$

First note that $F \neq 0$ since letting $Q = \operatorname{Span}\left(\bigcup_{i=1}^{k} L \cap H_i \right)$, it follows $\varepsilon_Q(F) = \varepsilon_Q \left(\prod_{P \in Z} M_p \right) \varepsilon_Q \left(N_{(0,\dots,0)} \prod_{i=1}^{k} M_{b_{i,0}} \right)$. Furthermore, this is nonzero since the all of the matrices involved are nonsingular.

We complete the proof in this case by showing $B \supseteq \mathrm{B.loc}(F)$. For any element v_ℓ of H_ℓ, we can write $v_\ell = \sum_{i=0}^{d_j} t_i b_{\ell,i}$ for some scalars $t_i \in \mathbb{K}$. Pick some $\boldsymbol{j} = (j_1, \dots, j_k) \in \mathcal{C}$, and for $0 \leq i \leq d_j$ define $\boldsymbol{j}_i = (j_1, \dots, j_{\ell-1}, i, j_{\ell+1}, \dots, j_k)$, we show that

$$F_{\ell,\boldsymbol{j}} = \sum_{i=0}^{d_\ell} \varepsilon_{\boldsymbol{j}_i} N_{\boldsymbol{j}_i} \left(M_{b_{\ell,i}} \prod_{\substack{m=1 \\ m \neq \ell}}^{k} M_{b_{m,j_m}} \right)$$

is 0 when evaluated at v_ℓ. Since F can be written as sums of the terms of the form above this establishes the result. Define a matrix $\hat{N}_{\boldsymbol{j},\ell}$ to be the determinant of the maximal minor of N obtained by removing the first row $\begin{bmatrix} X_0 & \dots & X_n \end{bmatrix}$ and the rows corresponding to b_{m,j_m} for all $1 \leq m \leq k$ with $m \neq \ell$. The using row operations and that some of $\{b_{\ell,0}, .., b_{\ell,d_\ell}\}$ are rows of $N_{\boldsymbol{j}_i}$, we see that $N_{\boldsymbol{j}_i}(v_\ell) = (-1)^\delta (-1)^i t_i \hat{N}_{\boldsymbol{j},\ell}(v_\ell)$ where $(-1)^\delta$ is some fixed sign. Finally we apply this to see

$$\begin{aligned} F_{\ell,\boldsymbol{j}}(v_\ell) &= \sum_{i=0}^{d_\ell} \varepsilon_{\boldsymbol{j}_i} N_{\boldsymbol{j}_i}(v_\ell) \left(M_{b_{\ell,i}}(v_\ell) \prod_{\substack{m=1 \\ m\neq\ell}}^{k} M_{b_{m,j_m}}(v_\ell) \right) \\ &= \sum_{i=0}^{d_\ell} \varepsilon_{\boldsymbol{j}_i} (-1)^{\delta} (-1)^{i} t_i \hat{N}_{\boldsymbol{j},\ell}(v_\ell) \left(M_{b_{\ell,i}}(v_\ell) \prod_{\substack{m=1 \\ m\neq\ell}}^{k} M_{b_{m,j_m}}(v_\ell) \right) \\ &= \varepsilon_{\boldsymbol{j}} (-1)^{\delta} (-1)^{j_\ell} \hat{N}_{\boldsymbol{j},\ell}(v_\ell) \left(\prod_{\substack{m=1 \\ m\neq\ell}}^{k} M_{b_{m,j_m}}(v_\ell) \right) \left(\sum_{i=0}^{d_\ell} t_i M_{b_{\ell,i}}(v_\ell) \right) \\ &= \varepsilon_{\boldsymbol{j}} (-1)^{\delta} (-1)^{j_\ell} \hat{N}_{\boldsymbol{j},\ell}(v_\ell) \left(\prod_{\substack{m=1 \\ m\neq\ell}}^{k} M_{b_{m,j_m}}(v_\ell) \right) \left(M_{v_\ell}(v_\ell) \right). \end{aligned}$$

Since $M_{v_\ell}(v_\ell) = 0$ we conclude.

Lastly we have the general case where $B = Z \cup \bigcup_{i=0}^{k} H_i$ and $n-1-\sum_{i=0}^{k} H_i = r > 0$. In this case let B' denote the union of B and a sufficiently general subspace H_{k+1} of dimension r. Then we see from the previous case that $B' = \mathrm{B.loc}_d(Z_{B'})$ for $d = |Z| + 2k$. Hence, $\mathrm{B.loc}(Z_B) \subseteq B'$, now as H_{k+1} is sufficiently general we can easily see that $\mathrm{B.loc}(Z_B) \subseteq B$. □

Combining the preceding propositions with proposition 5.4.3, it follows that conditions (I) and (II) of proposition 5.4.13 can be checked by looking at the combinatorics of linear subspaces spanned by subsets of Z. With this in mind we introduce the definition below.

Definition 5.4.19. Given a finite set of points $Z \subseteq \mathbb{P}^n$ and a real number $d \in \mathbb{R}$ we define the *modified expected number of conditions*, as the integer $\mathrm{Ex.\,C}(Z,d)$, which is the solution to optimization problem

$$\mathrm{Ex.\,C}(Z,d) = \min \left\{ \sum_{i=0}^{s} (d \dim(H_i) + 1) \;\middle|\; \begin{array}{l} \text{where } \{H_0, \dots, H_s\} \text{ are nonempty} \\ \text{linear subspaces with } Z \subseteq \bigcup_{i=0}^{s} H_i \end{array} \right\}.$$

It turns out that the linear program defined in definition 5.4.19 can be studied via the techniques defined in section 3.1.

We are now ready to state and prove the main result of this section.

Theorem 5.4.20. *Let $Z \subseteq \mathbb{P}(V)$ be a finite set of points. Then Z admits unexpected $(d-1)Q$-hypersurfaces in degree d if and only if*

$$\dim[I(Q)^{d-1}]_d - \dim[I(Q)^{d-1} \cap I(Z)]_d < \mathrm{Ex.\,C}(Z,d)$$

or in the notation of section 5.3,

$$\dim[I(Q)^{d-1}]_d - \dim[I_Q^{\gg}(Z)]_d = \dim[I(Q)^{d-1}/I_Q^{\gg}(Z)]_d < \mathrm{Ex.\,C}(Z,d)$$

Proof of Theorem 5.4.20. Fix a integer d, note that it follows from the definition that

$$\operatorname{Ex.C}(Z,d) = \min \left\{ \sum_{i=0}^{s} (d \dim \operatorname{Span}(A_i) + 1) \;\middle|\; \begin{array}{l} \text{where } \{A_0, ..., A_s\} \\ \text{form a partition of } Z \end{array} \right\}.$$

Before proving either direction of the equivalence. We establish the claim below.

Claim 5.4.21. $\operatorname{Ex.C}(Z,d)$ is equal to the largest size of a subset $B \subseteq Z$ which satisfies the following 3 conditions

(C1) $[I_Q^{\gg}(B)]_d = [I_Q^{\gg}(Z)]_d$

(C2) For all linear subspaces L, $|B \cap L| \leq \dim[I(Q)^{d-1}/I_Q^{\gg}(L)]_d = d(\dim B) + 1$

(C3) B imposes independent conditions on d forms.

proof of claim. Applying results from [Edm] (see theorem (8) and comment (16)), we may define a matroid M_d on the set Z whose independent sets are precisely those $I \subseteq Z$ where $|A| \leq d \dim(\operatorname{Span} A) - 1$ for all nonempty $A \subseteq I$. The linear programming duality given in [Edm], now states that

$$\operatorname{rk}(M_d) = \operatorname{Ex.C}(Z,d).$$

From this we can conclude that $\operatorname{Ex.C}(Z,d)$ is equal to the largest size of a subset which satisfies condition $(C2)$, namely any basis of M_d works. To finish the proof of the claim we find a basis of M_d satisfying $(C1)$ and $(C3)$.

By proposition 5.4.11, there is some $W \subseteq Z$ so that W satisfies conditions $(C1)$ and $(C2)$. As W satisfies $(C2)$ it is independent in M_d and we can therefore extend it to a basis $W \subseteq B$ of M_d. Now as $W \subseteq B \subseteq Z$ we have that $[I^{\gg}(B)]_d = [I^{\gg}(Z)]_d$, and therefore B satisfies $(C1)$.

Lastly, we note that theorem 4.3.3 ensures that B since B satisfies $(C2)$ it necessarily imposes independent conditions on d forms, thereby establishing condition $(C3)$ and the claim. □

Now continuing with the proof of the equivalence. If $\dim[I(Q)^{d-1}]_d - \dim[I_Q^{\gg}(Z)]_d$ is less than $\operatorname{Ex.C}(Z,d)$, then we can find some $B \subseteq Z$ so that $|B| = \operatorname{Ex.C}(Z,d)$ and B satisfies conditions $(C1)$, $(C2)$ and $(C3)$. We then have

$$\dim[I(Q)^{d-1}]_d - \dim[I(Q)^{d-1} \cap I(Z)]_d < \operatorname{Ex.C}(Z,d) = |B| = \dim[\operatorname{Sym}(V^*)/I(B)]_d.$$

Letting $W = B$, we see that W satisfies the necessary criteria of definition 5.4.7, and so Z admits very unexpected hypersurfaces.

Conversely, suppose that Z admits very unexpected hypersurfaces. Then by definition there exists $U \subseteq Z$ so that for general Q the following conditions hold:

(I) $[I_Q^{\gg}(U)]_d = [I_Q^{\gg}(Z)]_d$.

(II) For all linear subspaces L, we have

$$|U \cap L| \leq \dim[I(Q)^{d-1}/I_Q^{\gg}(L)]_d = d(\dim L) + 1.$$

(III) $\dim[R/I(U)]_d > \dim[I(Q)^{d-1}/I_Q^{\gg}(U)]_d$.

Finding a subset $W \subseteq U$ so that $[I(U)]_d = [I(W)]_d$ and W imposes independent conditions on d forms. We get by the claim above that $|W| \leq \mathrm{Ex.\,C}(Z,d)$ and so

$$\dim[I(Q)^{d-1}/I_Q^{\gg}(U)]_d < \dim[R/I(U)]_d = |W| < \mathrm{Ex.\,C}(Z,d).$$

□

Remark 5.4.22. Let $L \subseteq \mathbb{P}^n$ be a nonempty linear subspace. We note that the above proof relies on a somewhat remarkable agreement between the dimension $\dim[I(Q)^{d-1}/(I(L) \cap I(Q)^{d-1})]_d$ and the quantity $d \dim L + 1$ appearing in the inequality from theorem 4.3.3. This is even more remarkable considering that the proof of theorem 4.3.3 is almost entirely combinatorial relying on a generalization of Edmonds Matroid Partition Theorem.

Combining the above result with theorem 5.3.14, we obtain the following as a corollary.

Theorem 5.4.23. *Let $Z \subseteq \mathbb{P}^n$ be a finite set of points, and suppose that $D_0(\mathcal{A}_Z)$ has splitting type $(a_1, .., a_n)$. Then for a fixed integer d,*

$$\sum_{i=1}^{n} \max\{0, d - a_i\} \leq \quad nd + 1 - \mathrm{Ex.\,C}(Z,d)$$

and the inequality is strict if and only if Z admits very unexpected hypersurfaces in degree d.

Remark 5.4.24. Note one consequence of this is if Z admits very unexpected hypersurfaces in degree d, then $a_1 < d < a_n$.

Proof. Let $H_1, .., H_s$ be any collection of linear subspaces covering Z. Note that $I_Q^{\gg}(Z) \supseteq \bigcap_{i=1}^{s} I_Q^{\gg}(H_i)$ and that $\bigcap_{i=1}^{s} I_Q^{\gg}(H_i)$ is the kernel of the canonical map $[I(Q)^{d-1}]_d \to \oplus_{i=1}^{s} [I(Q)^{d-1}/I_Q^{\gg}(H_i)]_d$. We have by dimension counting that for a fixed d

$$\dim[I_Q^{\gg}(Z)]_d \geq \dim \bigcap_{i=1}^{s} [I_Q^{\gg}(H_i)]_d \geq nd + 1 - \left(\sum_{i=1}^{s} d \dim(H_i) + 1 \right).$$

Taking $H_1, .., H_s$ so $\sum_{i=1}^{s} d \dim(H_i) + 1 = \mathrm{Ex.\,C}(Z,d)$, the rest follows directly from theorem 5.4.20 and corollary 5.3.15.

The final consequence follows since if $d \leq a_1$ then $I_Q^{\gg}(Z) = 0$, if $d \geq a_n$ then note that $\mathrm{Ex.\,C}(Z,d) \leq |Z|$, and so

$$nd - (|Z| - 1) = \sum_{i=1}^{n} \max\{0, d - a_i\} \geq nd + 1 - \mathrm{Ex.\,C}(Z,d) \geq nd + 1 - |Z|$$

Establishing that $\mathrm{Ex.\,C}(Z,d) = |Z|$ and that the middle inequality is an equality.

□

The following lemma, shows that the inequality in the preceding corollary above may be replaced by

$$\sum_{i=1}^{n} \max\{0, a_i - d\} \geq |Z| - \mathrm{Ex.\,C}(Z,d) \geq 0.$$

Lemma 5.4.25. *Let $Z \subseteq \mathbb{P}^n$ be a finite set of points and suppose that $(a_1, .., a_n)$ is the splitting type of $D_0(\mathcal{A}_Z)$. Then for all real numbers c and d*

$$\begin{aligned} \sum_{i=1}^{n} \max\{0, d - a_i\} \geq & \quad nd + 1 - c \\ \iff \sum_{i=1}^{n} \max\{0, a_i - d\} \geq & \qquad |Z| - c \end{aligned}$$

Proof. Using that $\sum_{i=1}^{n} a_i = |Z| - 1$ we obtain

$$\begin{aligned} \sum_{i=1}^{n} \max\{0, d - a_i\} \geq & \quad nd + 1 - c \iff \\ \left(\sum_{i=1}^{n} d - a_i\right) - \left(\sum_{j;a_j \geq d} d - a_j\right) \geq & \quad nd + 1 - c \iff \\ nd - (|Z| - 1) + \sum_{j;a_j \geq d} (a_j - d) \geq & \quad nd + 1 - c \iff \\ \sum_{i=1}^{n} \max\{0, a_i - d\} \geq & \qquad |Z| - c \end{aligned}$$

□

We now conclude this section by discussing a few conditions on Z which makes it easier to determine if Z has very unexpected hypersurfaces in some degree d. The first is a consequence of the preceding lemma and theorem 5.4.20.

Corollary 5.4.26. *Let $Z \subseteq \mathbb{P}^n$ be a finite set of points, with $(a_1, a_2, .., a_n)$ the splitting type of $D_0(\mathcal{A}_Z)$. Suppose we have for a fixed integer $d \geq 0$ that*

$$\mathrm{Ex.\,C}(Z,d) = \min\{|Z|, nd + 1\}.$$

Then the following are equivalent:

(a) Z admits very unexpected hypersurfaces in degree d

(b) Z admits unexpected hypersurfaces in degree d

(c) $a_1 < d < a_n$

Proof. Before proving any of the necessary equivalences note that since $\operatorname{Ex.C}(Z, d) \leq |Z|$ and $\operatorname{Ex.C}(Z, d) \leq d \dim(\mathbb{P}^n) + 1$, that $\operatorname{Ex.C}(Z, d)$ is at most $\min\{nd + 1, |Z|\}$.

$[(a) \iff (c)]$ First, as mentioned after theorem 5.4.23 we have that $(a) \implies (c)$. For the reverse direction assume that $a_1 < d < a_n$. First in the case that $\operatorname{Ex.C}(Z, d) = nd + 1$ we see that Z admits unexpected hypersurfaces in degree d as $d > a_1$, and so the inequality in theorem 5.4.23 is strict. For the case when $\operatorname{Ex.C}(Z, d) = |Z|$, we similarly conclude by applying lemma 5.4.25 and using that $d < a_n$.

$[(a) \iff (b)]$ The forward direction is by definition. For the reverse we use the equivalence of (a) and (c), and note it suffices to show that Z cannot admit unexpected hypersurfaces in degree d if $d \leq a_1$ or $d \geq a_n$. If $d \leq a_1$, we note this is impossible as $[I_Q^{\gg}(Z)]_d = 0$. If $d \geq a_n$, then

$$\dim[I^{\gg}(Z)]_d = \sum_{i=1}^{n} \max\{0, d - a_n\} = nd - \sum_{i=1}^{n} = nd - (|Z| - 1) = nd + 1 - |Z|.$$

As $\dim[I(Q)^{d-1}]_d = nd + 1$ we conclude that Z imposes independent conditions on $[I(Z)]_d$, and so Z cannot admit unexpected hypersurfaces. $\square$

In the case that the points of Z are not too concentrated on one or more proper subspaces, it turns out that $\operatorname{Ex.C}(Z, d) = \max\{nd + 1, |Z|\}$ holds for all d and we obtain the following result.

Theorem 5.4.27. *Let $Z \subseteq \mathbb{P}^n$ and let $(a_1, .., a_n)$ be the splitting type of $D_0(\mathcal{A}_Z)$, where $a_i \leq a_{i+1}$. Suppose for all positive dimensional linear subspaces $H \subseteq \mathbb{P}^n$, we have that*

$$\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.$$

Then for an integer d the following are equivalent:

(a) Z admits very unexpected hypersurfaces in degree d.

(b) Z admits unexpected hypersurfaces in degree d.

(c) $a_1 < d < a_n$.

Proof. By corollary 5.4.26, it suffices to show that $\operatorname{Ex.C}(Z, d) = \min\{nd + 1, |Z|\}$. Let $\mathcal{H} = \{H_1, .., H_s\}$ be a collection of positive dimensional linear subspaces, so that setting $W = Z \setminus \bigcup_{i=1}^{s} H_i$ we have

$$|W| + \sum_{i=1}^{s} d \dim(H_i) + 1 = \operatorname{Ex.C}(Z, d).$$

As $|W| + \sum_{i=1}^{s} d \dim(H_i) + 1$ is at a minimum, we make the following observations:

(Ob. 1) $d \dim(H_i) + 1 \leq |H_j \cap Z|$.

(Ob. 2) For all $J \subseteq \mathcal{H}$ we have $\sum_{H_j \in J} d \dim(H_j) + 1 \leq d \dim \operatorname{Span}\left(\bigcup_{H_j \in J} H_j\right) + 1$.

(Ob. 3) $\sum_{i=1}^{s} \dim(H_i) \leq \dim \operatorname{Span}\left(\bigcup_{i=1}^{s} H_i\right) < n$.

(Ob. 1) and (Ob. 2) must hold since otherwise we could find a set of points W' and a collection of subspaces $\mathcal{H}' = \{H_1', ..., H_k'\}$ with $Z \subseteq W' \cup \bigcup_{H_i' \in \mathcal{H}'} H_i'$ and $\sum_{H_i' \in \mathcal{H}'} d \dim H_i' + 1 < \operatorname{Ex.C}(Z, d)$. For instance, in (Ob. 1) we would consider $W' = W \cup (Z \cap H_i)$ and $\mathcal{H}' = \mathcal{H} \setminus \{H_i\}$. (Ob. 3) is a consequence of (Ob. 2).

Note that (Ob. 1) implies that $\operatorname{Ex.C}(Z, d) = |Z|$ for all $d \geq \frac{|Z|-1}{n} \geq \frac{|Z \cap H_i|-1}{\dim H_i}$, so suppose that $nd + 1 < |Z|$. Let $g_i = |Z \cap H_i| - (d \dim(H_i) + 1) \geq 0$, and note that by hypothesis $\frac{g_i}{\dim H_i} = \frac{|Z \cap H_i|-1}{\dim H_i} - d \leq \frac{|Z|-nd-1}{n}$. Combining this with our formula for $\operatorname{Ex.C}(Z, d)$, we obtain the following

$$\begin{aligned} \operatorname{Ex.C}(Z, d) &= |W| + \sum_{i=1}^{s} (d \dim(H_i) + 1) \\ &= |Z| - \sum_{i=1}^{s} g_i \geq |Z| - \sum_{i=1}^{s} (\dim H_i) \left(\frac{|Z| - nd - 1}{n} \right) \\ &\geq |Z| - (|Z| - nd - 1) \left(\frac{\sum_{i=1}^{s} \dim H_i}{n} \right) \end{aligned}$$

Now as $\sum \dim H_i \leq n$ by (Ob. 3) we obtain $\operatorname{Ex.C}(Z, d) \geq |Z| - (|Z| - nd - 1) = nd + 1$. As it's always true that $\operatorname{Ex.C}(Z, d) \leq nd + 1$, the result now follows. ☐

Remark 5.4.28. To close we spell out the connection between theorem 5.4.27 and the original theorem 5.4.1 from [CHMN18]

proof of Theorem 5.4.1. Let $Z \subseteq \mathbb{P}^2$ and let (a_1, a_2) be the splitting type of $D_0(\mathcal{A}_Z)$. First consider the case where there exists some $L \subseteq \mathbb{P}^2$ so that $|L \cap Z| > a_1 + 1$. Let $Q \in \mathbb{P}^2$ be a general point, and take $f \in [I_Q^{\gg}(Z)]_{a_1+1}$ be a minimal generator of $I^{\gg}(Z)$, then applying Bezout's Theorem we see that L must be a component of the variety $f = 0$. Hence, f factors as $f = \ell g$ where ℓ is the linear form defining L and $g \in [\mathcal{F}_Q]_{a_1} = [I(Q)^{a_1}]_{a_1}$ is a product of linear forms. As f is a minimal generator and Q is general it follows each linear form in g vanishes at precisely one point of $|Z \setminus L|$. Therefore, as $a_1 = \deg g = |Z \setminus L|$, we conclude that $|Z \cap L| = |Z| - |Z \setminus L| = a_1 + 1$.

Now noting that if $\operatorname{Ex.C}(Z, d) = \sum_{i=1}^{k} d \dim H_i + 1$ for linear subspaces, H_i that we must have $d \dim(H_1 + H_2) + 1 \geq d \dim H_1 + d \dim H_2 + 2$. It follows that $\dim(H_1 + H_2) > \dim H_1 + \dim H_2$, in the case that $Z \subseteq \mathbb{P}^2$, this implies that there is at most one line or plane among the H_i. Therefore, we conclude that $\operatorname{Ex.C}(Z, d) = \min\{2d + 1, (d \dim L + 1) + |Z \setminus L|, |Z|\}$ or equivalently

$$\operatorname{Ex.C}(Z, d) = \begin{cases} 2d + 1 & \text{If } d \leq a_1 \\ d + 1 + a_1 & \text{If } a_1 \leq d \leq a_2 \\ a_1 + a_2 + 1 & \text{If } a_2 \leq d \end{cases}$$

Applying theorem 5.3.14 and a direct comparison now shows that Z admits no unexpected curves, establishing this case.

For the other case we have $|L \cap Z| \leq a_1 + 1$ for all $L \subseteq \mathbb{P}^2$. Then for all lines $L \subseteq \mathbb{P}^2$ we have the inequality $|Z \cap L| \leq a_1 + 1 \leq \left\lfloor \frac{|Z|-1}{2} \right\rfloor + 1$. Subtracting through by 1 gives

$$|Z \cap L| - 1 \leq a_1 \leq \frac{|Z| - 1}{2}$$

allowing us to conclude by theorem 5.4.27.

□

5.5 Computations and Examples of Unexpected Hypersurfaces

Combinatorial Optimization problems similar to the linear program, $\mathrm{Ex.\,C}(Z, d)$, from in definition 5.4.19 have been studied before. One notable instance of this is in the chapter [Nar91]. In [Nar91] the author fixed a submodular function $\mu : S \to \mathbb{R}$ and a real parameter λ, and studied the optimization problem

$$\min \left\{ \sum_{i=0}^{t} \mu(S_i) - \lambda \;\middle|\; \{S_0, .., S_t\} \text{ is a partition of } S \right\}.$$

It was shown in section 3 of [Nar91] that for a fixed μ and λ that there is a unique *finest* and a unique *coarsest* partition of S achieving this minimum. Here we say a partition π if *finer* than the partition τ (or equivalently that τ is *coarser* than π) and write $\pi \leq \tau$, if every block of π is contained in a block of τ. In section 4 of [Nar91] an algorithm was given which solves this problem for a fixed μ. It was shown in particular that minimum is a piecewise linear function of λ.

We note that $\mathrm{Ex.\,C}(Z, d)$ is equivalent to

$$d \min \left\{ \sum_{i=0}^{t} \left(\mathrm{rk}_{M(Z)}(A_i) - \frac{d-1}{d} \right) \;\middle|\; \{A_0, .., A_t\} \text{ is a partition of } Z \right\}$$

and so the algorithm given in [Nar91] can be used to solve $\mathrm{Ex.\,C}(Z, d)$.

Definition 5.5.1. Let $Z \subseteq \mathbb{P}^n$ be a finite set of points, for each $d \geq 0$, we define the *modified expected base locus*, which we denote $\mathrm{Ex.\,Bl}(Z, d)$ to be the coarsest partition in the partition order which satisfies

$$\sum_{B \in \mathrm{Ex.Bl}(Z,d)} (d \dim \mathrm{Span}(B) + 1) = \mathrm{Ex.\,C}(Z, d).$$

Meaning that if Π is any other partition with $\sum_{P \in \Pi} (d \dim \mathrm{Span}(P) + 1) = \mathrm{Ex.\,C}(Z, d)$, then for every $P \in \Pi$ there is some $B \in \mathrm{Ex.\,Bl}(Z, d)$ so that $P \subseteq B$.

Section 3 of [Nar91] establishes not only that $\mathrm{Ex.\,Bl}(Z, d)$ exists, but also that in the partition order $\mathrm{Ex.\,Bl}(Z, d) \geq \mathrm{Ex.\,Bl}(Z, d + 1)$. We now make a few observations about $\mathrm{Ex.\,Bl}(Z, d)$ and $\mathrm{Ex.\,C}(Z, d)$ in order to compute $\mathrm{Ex.\,C}(Z, d)$ more easily. These results are heavily influenced by the results and techniques in [Nar91]. However, our results are stronger in some cases as we can take advantage of the fact that $\mathrm{rk}_{M(Z)}$ is the rank function of a matroid, and not merely a submodular function.

Lemma 5.5.2. *For any real number $d > 0$, and for distinct blocks $B_1, .., B_k \in \operatorname{Ex.Bl}(Z, d)$ we have*

$$\sum_{i=1}^{k} (d \dim \operatorname{Span}(B_i) + 1) < d \dim \operatorname{Span}\left(\bigcup_{i=1}^{k} B_i\right) + 1$$

In particular, for each pair of distinct blocks B_1 and B_2, $\operatorname{Span}(B_1)$ and $\operatorname{Span}(B_2)$ are disjoint subspaces.

Similarly, if $C_1 \sqcup ... \sqcup C_\ell$ is a partition of a block $B_\ell \in \operatorname{Ex.Bl}(Z, d)$ into nonempty subsets, then

$$\sum_{j=1}^{\ell} (d \dim \operatorname{Span}(C_j) + 1) \geq d \dim \operatorname{Span}(B) + 1$$

Proof. If $\operatorname{Ex.Bl}(Z, d) = \{B_1, .., B_m\}$ then let $A = \left\{\bigcup_{i=1}^{k} B_i, B_{k+1}, ..., B_m\right\}$. As A is coarser than $\operatorname{Ex.Bl}(Z, d)$, we get

$$\sum_{a \in A} d \dim \operatorname{Span}(a) + 1 > \sum_{b \in \operatorname{Ex.Bl}(Z,d)} d \dim \operatorname{Span}(b) + 1.$$

Subtracting away the shared terms now gives the desired inequality.

The proof of the second claim follows similarly, since we must have

$$\left(\sum_{j=1}^{\ell} d \dim \operatorname{Span}(C_j) + 1\right) + \left(\sum_{B \in \operatorname{Ex.Bl}(Z,d); B \neq B_\ell} d \dim \operatorname{Span}(B) + 1\right) \geq \sum_{B \in \operatorname{Ex.Bl}(Z,d)} d \dim \operatorname{Span}(B) + 1$$

□

It can be somewhat laborious to determine if a given set of points satisfies the combinatorial condition in theorem 5.4.27. Furthermore, most of the observed configurations of points Z which admit unexpected curves possess certain kinds of symmetry, namely their dual arrangements $\mathcal{A}_Z$ are reflection arrangements. We designed this next proposition with these examples in mind.

Definition 5.5.3. A *psuedoreflection* is a matrix $R \in \mathbb{GL}(n, \mathbb{K})$ so that $R^k = I_n$ for some $k > 1$ and the set of points in $\mathbb{K}^n$, which are fixed by R, denoted Fix_R, form a hyperplane. A *reflection group* is a subgroup, G, of $\mathbb{GL}(n, \mathbb{K})$, which is generated by *psuedoreflections*. G is an *irreducible reflection group* if there no nontrivial G-invariant subspace of $\mathbb{K}^n$.

Proposition 5.5.4. *If $Z \subseteq \mathbb{P}^n_{\mathbb{K}}$ is a finite set of points, and there is an irreducible reflection group $G \subseteq \mathbb{PGL}(\mathbb{K}, n)$ acting on Z. Then for all positive dimensional linear subspaces $H \subseteq \mathbb{P}^n$ we have*

$$\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.$$

Consequently by corollary 5.4.26, Z admits very unexpected hypersurfaces in degree d for precisely those d with $a_1 < d < a_n$, where $(a_1, .., a_n)$ is the splitting type of $D_0(\mathcal{A}_Z)$.

We first note a useful criterion, which is used in the proof of the above proposition.

Claim 5.5.5. Let $Z \subseteq \mathbb{P}^n$, then the following are equivalent:

1. For all positive dimensional subspaces $H \subseteq \mathbb{P}^n$,
$$\frac{|Z \cap H| - 1}{\dim H} \leq \frac{|Z| - 1}{n}.$$
2. $\operatorname{Ex.C}(Z, q) = \min\{qn + 1, |Z|\}$ for all $q \in \mathbb{Q}$.

proof of claim. The forward direction is established in theorem 5.4.27. For the reverse direction, we prove the contrapositive. Namely, suppose that there is some $H \subseteq \mathbb{P}^n$ with $\frac{|Z\cap H|-1}{\dim H} > \frac{|Z|-1}{n}$. Then choose any q with
$$\frac{|Z \cap H| - 1}{\dim H} > q > \frac{|Z| - 1}{n}.$$
Note then that $|Z \cap H| > q \dim H + 1$ and that $qn + 1 > |Z|$, hence we have that
$$\operatorname{Ex.C}(Z, d) \leq q \dim H + 1 + |Z \setminus H| < |Z| < qn + 1$$
establishing the result. ☐

Proof of proposition 5.5.4. First, note that if G is any group acting on Z then this action extends to the lattice of partitions of Z. Furthermore, if Π is any partition of Z, then for any $g \in G$ we have $\sum_{P\in\Pi} d \dim \operatorname{Span}(P) + 1 = \sum_{P\in\Pi} d \dim \operatorname{Span}(gP) + 1$. From this it follows that $\operatorname{Ex.Bl}(Z, d)$ is fixed by the G action, in the sense that blocks of $\operatorname{Ex.Bl}(Z, d)$ are taken to other blocks of $\operatorname{Ex.Bl}(Z, d)$.

Now we continue to establishing the proposition. By the preceding claim it suffices to show that for rational q, $\operatorname{Ex.Bl}(Z, q)$ is either the discrete or the indiscrete partition. Suppose that $B \in \operatorname{Ex.Bl}(Z, d)$ is a block with $|B| \geq 2$, let $r \in G$ be a psuedoreflection and $H_r = \operatorname{Fix}_r$ the hyperplane of the points fixed by r. As $\dim \operatorname{Span}(B) \geq 1$ then consequently $H_r \cap \operatorname{Span}(B)$ and hence $\operatorname{Span}(rB) \cap \operatorname{Span}(B)$ are both nonempty. Applying lemma 5.5.2, we see that we must have $rB = B$ and so $r \operatorname{Span}(B) = \operatorname{Span}(B)$. Therefore, $\operatorname{Span}(B)$ is a nonzero G-invariant subspace of $\mathbb{P}^n$. As G is an irreducible reflection group we must have that $\operatorname{Span}(B) = \mathbb{P}^n$ and so $B = Z$ by lemma 5.5.2. ☐

For a set of points $Z \subseteq \mathbb{P}^n$, if the splitting type of $D_0(\mathcal{A}_Z)$ is known, then determining when Z admits very unexpected hypersurfaces comes down to computing $\operatorname{Ex.C}(Z, d)$. The following two propositions can be useful in determining $\operatorname{Ex.C}(Z, d)$. The first places bounds on how $\operatorname{Ex.C}(Z, d)$ can change between degrees.

Lemma 5.5.6. *For $Z \subseteq \mathbb{P}(V)$, the sequence of forward differences*
$$\delta_d = \operatorname{Ex.C}(Z, d + 1) - \operatorname{Ex.C}(Z, d)$$
is nonincreasing. Furthermore, we have
$$\sum_{A\in\operatorname{Ex.Bl}(Z,d)} \dim(A) \geq \delta_d \geq \sum_{B\in\operatorname{Ex.Bl}(Z,d+1)} \dim(B).$$

Proof. Consider the following inequalities

$$
\begin{aligned}
&\sum_{A\in \mathrm{Ex.Bl}(Z,d)} \dim \mathrm{Span}(A) = \\
&\sum_{A\in \mathrm{Ex.Bl}(Z,d)} [(d+1)\dim \mathrm{Span}(A)+1] - \sum_{A\in \mathrm{Ex.Bl}(Z,d)} [d\dim \mathrm{Span}(A)+1] \\
&\geq \sum_{A\in \mathrm{Ex.Bl}(Z,d)} [(d+1)\dim \mathrm{Span}(A)+1] - \sum_{B\in \mathrm{Ex.Bl}(Z,d+1)} [d\dim \mathrm{Span}(B)+1] \\
&\geq \sum_{B\in \mathrm{Ex.Bl}(Z,d+1)} [(d+1)\dim \mathrm{Span}(B)+1] - \sum_{B\in \mathrm{Ex.Bl}(Z,d+1)} [d\dim \mathrm{Span}(B)+1] \\
&= \sum_{B\in \mathrm{Ex.Bl}(Z,d+1)} \dim \mathrm{Span}(B).
\end{aligned}
$$

Now noting that

$$\delta_d = \sum_{A\in \mathrm{Ex.Bl}(Z,d+2)} [(d+1)\dim \mathrm{Span}(A)+1] - \sum_{B\in \mathrm{Ex.Bl}(Z,d)} [d\dim \mathrm{Span}(B)+1]$$

establishes the result. ☐

The following proposition shows that if the splitting type $(a_1, .., a_n)$ is known it suffices to check if Z admits very unexpected hypersurfaces by only looking around the degrees in the splitting type.

Proposition 5.5.7. *Let $Z \subseteq \mathbb{P}^n$ and let $(a_1, a_2, .., a_n)$ denote the splitting type of $D_0(\mathcal{A}_Z)$. If Z does not admit very unexpected hypersurfaces in degree d, but does admit them in either degree $d-1$ or degree $d+1$, then $d = a_i$ for some i.*

Proof. Let d be an index satisfying the hypothesis. Define indexes j and ℓ so that $a_k < d$ for all $k \leq j$, and $a_k < d+1$ for all $k \leq \ell$. The proposition is established if we show $\ell > j$.

Applying the inequality from theorem 5.4.23 in degrees $d-1, d$ and $d+1$ we obtain the following three equations

(Eq. 1) $\mathrm{Ex.\,C}(Z, d-1) + \sum_{k=1}^{j}(d-1-a_k) \geq n(d-1)+1$

(Eq. 2) $\mathrm{Ex.\,C}(Z, d) + \sum_{k=1}^{\ell}(d-a_k) = nd+1$; and

(Eq. 3) $\mathrm{Ex.\,C}(Z, d+1) + \sum_{k=1}^{\ell}(d+1-a_k) \geq n(d+1)+1$.

Subtracting (Eq. 1) from (Eq. 2) and (Eq. 2) from (Eq. 3) gives (Eq. 4) and (Eq. 5) below.

(Eq. 4) $\delta_{d-1} + j = \mathrm{Ex.\,C}(Z, d) - \mathrm{Ex.\,C}(Z, d-1) + j \leq n$

(Eq. 5) $\delta_d + \ell = \mathrm{Ex.\,C}(Z, d+1) - \mathrm{Ex.\,C}(Z, d) + \ell \geq n$

By the preceding lemma $\delta_d \leq \delta_{d-1}$, with this and (Eqs. 4 & 5) we have

$$\delta_d + j \leq \delta_{d-1} + j \leq n \leq \delta_d + \ell.$$

We may conclude that $j < \ell$, if either (Eq. 4) or (Eq. 5) is strict. Yet this happens precisely when Z admits very unexpected hypersurfaces in degree $d-1$ or $d+1$. $\square$

Example 5.5.8. Let $\mathbb{F}_q$ be the finite field with $q = p^e$ elements, and $\mathbb{K}$ an infinite field containing $\mathbb{F}_q$. Let $\mathbb{P}^n_{\mathbb{F}_q} \subseteq \mathbb{P}^n_{\mathbb{K}}$ consist of those points which in homogeneous coordinates can be written as $(\alpha_0 : \alpha_1 : .. : \alpha_n)$ with $\alpha_i \in \mathbb{F}_q$. It is well known that $|\mathbb{P}^n_{\mathbb{F}_q}| = \frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + \ldots + q + 1$, and that $\mathcal{A}_{\mathbb{P}^n_{\mathbb{F}_q}}$ is free with exponents $(1, q, q^2, ..., q^n)$. The generator in degree q^i is of the form

$$\sum_{j=0}^{n} Y_j^{q^i} \frac{\partial}{\partial Y_i}$$

and so the corresponding generator of $I^{\gg}(\mathbb{P}^n_{\mathbb{F}_q})$ is

$$\sum_{j=0}^{n} M_j^q X_i = \begin{vmatrix} X_0 & X_1 & \ldots & X_n \\ X_0^{q^i} & X_1^{q^i} & \ldots & X_n^{q^i} \\ A_{1,0}^{q^i} & A_{1,1}^{q^i} & \ldots & A_{1,n}^{q^i} \\ \vdots & & \ddots & \vdots \\ A_{n-1,0}^{q^i} & \ldots & \ldots & A_{n-1,n}^{q^i} \end{vmatrix}$$

Furthermore, note that $\mathbb{P}^n_{\mathbb{F}_q}$ is acted on by the group $\mathbb{GL}(n, \mathbb{F}_q)$. This in particular contains the irreducible reflection group consisting of the permutation matrices, so by proposition 5.5.4 $\mathbb{P}^n_{\mathbb{F}_q}$ admits very unexpected hypersurfaces in all degrees d with $q < d < q^n$.

Example 5.5.9. Fix some primitive m-th root of unity $\zeta \in \mathbb{C}$, for $m \geq 2$. Define a configuration of points $F_m \subseteq \mathbb{P}^n_{\mathbb{C}}$ as consisting of the $m\binom{n+1}{2}$ points whose i-th coordinate is -1 and j-th coordinate is ζ^k for all $0 \leq k \leq d-1$ and all pairs $0 \leq i < j \leq n$. Let $C_m = F_m \cup \{E_0, E_1, .., E_m\}$ here E_i is the i-th coordinate point.

Then $\mathcal{A}_{C_m}$ is an Extended Ceva Arrangement, it is a reflection arrangement corresponding to the reflection group $G(m, 1, n+1) \subseteq \mathbb{PGL}(\mathbb{C}, n)$. The splitting type of $D_0(\mathcal{A}_{C_m})$ is $(m+1, 2m+1, ..., nm+1)$ (see [OT92] for details). As $\mathcal{A}_{C_m}$ is a reflection arrangement, we again apply proposition 5.5.4 to conclude that $\mathcal{A}_{C_m}$ admits very unexpected hypersurfaces in all degrees d with $m+1 < d < nm+1$.

Both of our classes of examples come from reflection arrangements, more generally proposition 5.5.4 gives a good criterion for determining if the points dual to a given reflection arrangement admit unexpected hypersurfaces. We note that reflection arrangements have been classified and that their exponents and hence their splitting type can be found in the appendix of [OT92].

Our final example shows that the degrees in which a set of points Z admits very unexpected hypersurfaces do not need to be consecutive. This is in contrast with the situation in the plane as shown in theorem 5.4.1. Before outlining the example we state a useful proposition and definition.

Definition 5.5.10. Let V_1 and V_2 be finite dimensional $\mathbb{K}$-vector spaces, and suppose we have finite sets of points $Z_1 \subseteq \mathbb{P}(V_1)$, $Z_2 \subseteq \mathbb{P}(V_2)$. There are inclusion maps $\iota_i : \mathbb{P}(V_i) \to \mathbb{P}(V_1 \oplus V_2)$ for $i = 1, 2$. We then define $Z_1 \oplus Z_2 \subseteq \mathbb{P}(V_1 \oplus V_2)$ as the set of points

$$Z_1 \oplus Z_2 := \iota_1(Z_1) \cup \iota_2(Z_2).$$

Proposition 5.5.11. *$Z_1 \oplus Z_2$ admits very unexpected curves in degree $d \geq 1$ if and only if Z_1 or Z_2 admits unexpected curves in degree d.*

Proof. First, note that for hyperplane arrangements $\mathcal{A}_1 \subseteq \mathbb{P}(W_1)$ and $\mathcal{A}_2 \subseteq \mathbb{P}(W_2)$ there is an arrangement $\mathcal{A}_1 \times \mathcal{A}_2 \subseteq \mathbb{P}(W_1 \oplus W_2)$ induced by the projections $p_i : \mathbb{P}(W_1 \oplus W_2) \to \mathbb{P}(W_i)$. Namely, $\mathcal{A}_1 \times \mathcal{A}_2$ is formed by taking all hyperplanes of the form $\pi_i^{-1}(H)$ for $H \in \mathcal{A}_i$.

We now note two facts:

(Fact 1) $\mathcal{A}_{Z_1 \oplus Z_2} = \mathcal{A}_{Z_1} \times \mathcal{A}_{Z_2}$,

(Fact 2) If S is the projective coordinate ring of $\mathbb{P}(W_1 \oplus W_2)$ there is an isomorphism of S-modules, $D(\mathcal{A}_{Z_1} \times \mathcal{A}_{Z_2}) \cong (S \otimes D(\mathcal{A}_{Z_2})) \oplus (S \otimes D(\mathcal{A}_{Z_1}))$.

The first can be seen by following each the constructions through the duality. We omit a proof of the second referring to [OT92] for details.

One consequence of fact 2 is that if $D_0(\mathcal{A}_1)$ has splitting type $(a_1, .., a_n)$ and $D_0(\mathcal{A}_2)$ has splitting type $(b_1, .., b_m)$, then $D_0(\mathcal{A}_1 \times \mathcal{A}_2)$ has a splitting type (up to reordering) of $(1, a_1, ..., a_n, b_1, .., b_m)$. Applying theorem 5.4.23, now yields the inequalities valid for any $d \geq 1$. Each inequality strict if and only if the corresponding set of points admits very unexpected hypersurfaces

(Ineq. 1) $\sum_{i=1}^{n} \max\{0, d - a_i\} \geq nd + 1 - \operatorname{Ex.C}(Z_1, d)$

(Ineq. 2) $\sum_{j=1}^{m} \max\{0, d - b_j\} \geq md + 1 - \operatorname{Ex.C}(Z_2, d)$

(Ineq. 3) $d - 1 + \left(\sum_{i=1}^{n} \max\{0, d - a_i\}\right) + \left(\sum_{j=1}^{m} \max\{0, d - b_j\}\right) \geq (n + m + 1)d + 1 - \operatorname{Ex.C}(Z_1 \oplus Z_2, d)$

We now claim that $\operatorname{Ex.C}(Z_1 \oplus Z_2, d) = \operatorname{Ex.C}(Z_1, d) + \operatorname{Ex.C}(Z_2, d)$. First note that if we assume this claim and subtract $d - 1$ from both sides of (Ineq. 3), then the resulting inequality may be written as the sum of (Ineq. 1) and (Ineq. 1). From this it follows that (Ineq. 3) is strict if and only if either (Ineq. 3) or (Ineq. 3) is strict and the proposition follows.

Continuing to the proof of our claim, we first note that if $d = 1$ then for any set of points $\operatorname{Ex.Bl}(Z, 1) = \{Z\}$. A direct computation establishes the claim in this case.

Now we may assume $d \geq 2$. Take a block $B \in \operatorname{Ex.Bl}(Z_1 \oplus Z_2, d)$ and define $B_1 = B \cap Z_1$ and $B_2 = B \cap Z_2$. We note that if B_1 and B_2 are nonempty, then lemma 5.5.2 states

$$d(\dim \operatorname{Span} B - \dim \operatorname{Span} B_1 - \dim \operatorname{Span} B_2) \leq 1.$$

Yet as B_1 and B_2 are contained in disjoint subspaces, $\dim \mathrm{Span}(B) = \dim \mathrm{Span}(B_1) + \dim \mathrm{Span}(B_2) + 1$ and the inequality becomes $d \leq 1$ giving a contradiction. Therefore, for each block B we have $B \subseteq Z_1$ or $B \subseteq Z_2$, and consequently $\mathrm{Ex.\,Bl}(Z_1 \oplus Z_2, d) = \Pi_1 \cup \Pi_2$ for some partitions Π_1 and Π_2 of Z_1 and Z_2 respectively. From this it readily follows from the definition that $\mathrm{Ex.\,Bl}(Z_1 \oplus Z_2, d) = \mathrm{Ex.\,Bl}(Z_1, d) \cup \mathrm{Ex.\,Bl}(Z_2, d)$. This establishes the claim that $\mathrm{Ex.\,C}(Z_1 \oplus Z_2, d) = \mathrm{Ex.\,C}(Z_1, d) + \mathrm{Ex.\,C}(Z_2, d)$ and completes our proof. □

Example 5.5.12. If $C_2, C_7 \subseteq \mathbb{P}^2_{\mathbb{C}}$ are the configurations of points described in example 5.5.9, then $C_2 \oplus C_7$ is a configuration of 33 points in $\mathbb{P}^5_{\mathbb{C}}$. The module of derivations, $D_0(\mathcal{A}_{C_2} \times \mathcal{A}_{C_7})$, has splitting type $(1, 3, 5, 8, 15)$. Using the computation from example 5.5.9 along with proposition 5.5.11, it follows that $C_2 \oplus C_7$ admits very unexpected hypersurfaces in degree d if and only if $d = 4$ or $8 < d < 15$.

5.6 A Lifting Criterion and the Structure of Unexpected Curves in $\mathbb{P}^2_{\mathbb{C}}$

One feature of the theorems 5.3.8 and 5.3.14, is they allow us to view elements of reduced Module of Derivations as explicit polynomials. This permits us to use techniques such as unique factorization and polynomial division that are not as well developed for general modules. In this section we give a few applications of this view point. First, we state a lifting criterion in proposition 5.6.2, this allows us under certain conditions to lift an element of the restricted module $D_0(\mathcal{A}_Z) \mid_L$ to the module $D_0(\mathcal{A}_Z)$. This criterion has especially strong implications in $\mathbb{P}^2_{\mathbb{C}}$, such as in theorem 5.6.8, where we show that for $Z \subseteq \mathbb{P}^2$ every polynomial defining an unexpected curve in $I_Q^{\gg}(Z)$ can be lifted to an element of $I^{\gg}(Z)$.

This result ends up putting very strong conditions on the combinatorics of sets of points Z which admit unexpected curves, which we explore in the next section.

Proposition 5.6.1. *Let $Z \subseteq \mathbb{P}(V) \cong \mathbb{P}^n$. Consider $G \in I^{\gg}(Z) \subseteq \mathbb{K}[\mathbf{A}]$. If there is some $F \in \mathbb{K}[\mathbf{A}]$ so that for general $\boldsymbol{\alpha} \in Gr(n-1, V)$ we have that $\varepsilon_{\boldsymbol{\alpha}}(F) \in I^{\gg}_{\boldsymbol{\alpha}}(Z)$ and $\varepsilon_{\boldsymbol{\alpha}}(F) \mid \varepsilon_{\boldsymbol{\alpha}}(G)$. Then F and G have a common divisor $H \in I^{\gg}(Z)$.*

Proof. For any prime ideal, I, we set $\nu_I(F)$ as the valuation $\nu_I(F) := \sup\{m \geq 0 \mid F \in I^{(m)}\}$. Now we define two ideals of $\mathbb{K}[\mathbf{A}]$, X is the ideal $(X_0, .., X_n)$ and we let M denote the ideal generated by the maximal minors of the matrix $\mathbf{A}$. Lastly, for $\boldsymbol{\alpha} \in \mathrm{Gr}(n-1, V)$, $I(\boldsymbol{\alpha})$ is the ideal of $\mathbb{K}[X_0, .., X_n] \subseteq \mathbb{K}[\mathbf{A}]$ defined by the subspace $\boldsymbol{\alpha}$.

Before continuing we note a few facts:

Fact 1 For each of the 3 ideals, X, $I(\alpha)$ and M, that we have defined we have $I^k = I^{(k)}$.

Fact 2 For any $f \in \mathbb{K}[\mathbf{A}]$ we have $\nu_X(f) \geq \nu_M(f)$ and $\nu_{I(\boldsymbol{\alpha})}(\varepsilon_{\boldsymbol{\alpha}}(f)) = \nu_M(f)$ for general $\boldsymbol{\alpha}$.

Fact 3 For any $f \in \mathbb{K}[\mathbf{A}]$, we have the inequality

$$(\nu_X(f) - \nu_M(f)) + n^2 \nu_M(f) \leq \deg(f).$$

Fact 4 If $\nu_X(f) = \nu_M(f) + 1$, then equality occurs in Fact 3 if and only if $f \in \mathfrak{m}^{\gg}$.

The first fact follows for X and $I(\boldsymbol{\alpha})$ since both are complete intersections, for M we refer to section 2.2 of [Hoc73]. The second fact follows since M is essentially $I(\alpha)$ for α the generic point. The third is a consequence of the first and that $\deg(M_i) = n^2$. Lastly, the fourth fact follows since if $\nu_X(f) = \nu_M(f)$ then setting $d = \nu_X(f)$ we have $f \in [(X_0, .., X_n)M^{d-1}]_{n^2(d-1)+1}$, but this is precisely $[\mathfrak{m}^{\gg}]_d$.

First we claim for general $\boldsymbol{\alpha}$, that any $f \in I^{\gg}_{\boldsymbol{\alpha}}(Z)$ factors into irreducible components as $f = f_0 \prod_{i=0}^{k} \ell_i$ where each ℓ_i is an element of the special fibre ring $\mathcal{F}_{\boldsymbol{\alpha}} = \mathrm{Sym}([I(\boldsymbol{\alpha})]_1)$. It suffices to show that if $f = pq$, then either p or q is in $\mathcal{F}_{\boldsymbol{\alpha}}$. Noting that $\nu_X(f) = 1 + \nu_{I(\boldsymbol{\alpha})}(f)$, and using the additive property of valuations, we obtain

$$\nu_X(p) + \nu_X(q) = 1 + \nu_{I(\boldsymbol{\alpha})}(p) + \nu_{I(\boldsymbol{\alpha})}(q) \leq 1 + \nu_X(p) + \nu_X(q).$$

Since all numbers above are integers, and $\nu_{I(\boldsymbol{\alpha})}(h) \leq \nu_X(h)$ for every polynomial h in $\mathbb{K}[X_0, .., X_n]$, we may assume without loss of generality that $\nu_{I(\boldsymbol{\alpha})}(p) = \nu_X(p)$ and $\nu_{I(\boldsymbol{\alpha})}(q) = \nu_X(q) + 1$. It now follows that $p \in [I(\boldsymbol{\alpha})^{\nu_X(p)}]_{\nu_X(p)} \subseteq \mathcal{F}_{\boldsymbol{\alpha}}$, which establishes our claim.

Continuing with the proof of the proposition, we let $\varepsilon_{\mathfrak{g}}$ denote the generic evaluation. In other words $\varepsilon_{\mathfrak{g}}$ is the inclusion $\varepsilon_{\mathfrak{g}} : \mathbb{K}[\mathbf{A}] \to \mathbb{F}[X_0, .., X_n]$, for $\mathbb{F}$ the function field $\mathbb{F} := \mathbb{K}(A_{i,j} \mid (1,0) \leq (i,j) \leq (n, n+1))$. By assumption $\varepsilon_{\mathfrak{g}}(F)$ divides $\varepsilon_{\mathfrak{g}}(G)$, so there exists $h \in \mathbb{K}[\mathbf{A}]$ and $k \in \mathbb{K}[A_{i,j} \mid (1,0) \leq (i,j) \leq (n, n+1)]$ with h and k coprime so that $\frac{h}{k}\varepsilon_{\mathfrak{g}}(F) = \varepsilon_{\mathfrak{g}}(G) \iff hF = kG$ where this last equality is in $\mathbb{K}[\mathbf{A}]$. Now by unique factorization in the polynomial ring $\mathbb{K}[\mathbf{A}]$, we get $k \mid F$. Setting $\tilde{F} = \frac{F}{k} \in \mathbb{K}[\mathbf{A}]$, we have $h\tilde{F} = G$ and $k\tilde{F} = F$. Moreover, since $\varepsilon_{\mathfrak{q}}(F) \notin \mathcal{F}_{\mathfrak{q}}$ it follows that $\tilde{F} \notin \mathcal{F}_{\mathfrak{q}}$ and so $h \in \mathcal{F}_{\mathfrak{q}}$. We finish the proof by establishing that $\tilde{F} \in I^{\gg}(Z)$.

Since F differs from $\tilde{F}$ only by $\mathbb{F}$ scalar, and $F \in I(Z)$ we have $\tilde{F} \in I(Z)$ and so it suffices to show that $\tilde{F} \in \mathfrak{m}^{\gg}$. Since $\varepsilon_{\mathfrak{q}}(\tilde{F}) \in I^{\gg}_{\mathfrak{g}}(Z)$ and $\varepsilon_{\mathfrak{q}}(h) \in \mathcal{F}_{\mathfrak{q}}$ we have the inequalities

$$\nu_M(h) \leq \nu_{I(\mathfrak{q})}(h) = \nu_X(\varepsilon_{\mathfrak{q}}(h)) = \nu_X(h); \text{ and}$$
$$\nu_M(\tilde{F}) \leq \nu_{I(\mathfrak{q})}(\tilde{F}) = \nu_X(\varepsilon_{\mathfrak{q}}(\tilde{F})) - 1 = \nu_X(\tilde{F}) - 1.$$

As $h\tilde{F} \in I^{\gg}(Z)$, we have that $\nu_M(h) + \nu_M(F) = \nu_X(h) + \nu_X(F) - 1$ and so the above inequalities must be equality. Similarly, using the inequalities $1 + n^2\nu_M(\tilde{F}) \leq \deg(\tilde{F})$, $n^2\nu_M(h) \leq \deg(h)$ and $\tilde{F}h = G \in I^{\gg}(Z)$ we have that

$$1 + n^2\nu_m(\tilde{F}) + n^2\nu_M(h) \leq \deg\tilde{F} + \deg h = \deg G = 1 + n^2\nu_M(\tilde{F}h).$$

Allowing us to conclude that $1 + n^2\nu_M(\tilde{F}) = \deg(\tilde{F})$ and $n^2\nu_M(h) = \deg(h)$ which completes the proof. □

The preceding lemma when combined with the results of section 5.3 allows us under certain circumstances to lift elements of $D_0(\mathcal{A}) \mid_L$ to elements of $D_0(\mathcal{A})$. One example of this is illustrated in the following proposition.

Proposition 5.6.2. *Let $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{K}}$ and let $(a_1, \ldots, a_n)$ denote the splitting type of $D_0(\mathcal{A})$. If $a_1 < a_2 \leq a_3 \leq \ldots \leq a_n$ and $\theta_\lambda \in D_0(\mathcal{A})$ is a nonzero element of degree $< a_2$, then $D_0(\mathcal{A})$ has a minimal generator in degree a_1.*

Proof. Using the translation given by theorem 5.3.8, there's a nonzero $F_\lambda \in [I^{\gg}(Z)]_d$ where $d < a_2 + 1$. If Q is the generic codimension 2 linear subspace, then by theorem 5.3.14 $I^{\gg}_Q(Z)$ is free on generators $f_1, .., f_n$ with $\deg f_i = a_i + 1$. Hence, $\varepsilon_Q(F_\lambda) = \sum_{i=1}^n g_i f_i$. Yet as $\deg f_j > \deg \varepsilon_Q(F_\lambda)$ for all $j \geq 2$, we must have $\varepsilon_Q(F_\lambda) = g_1 f_1$. After clearing denominators we may lift f_1 to an element $\tilde{f}_1$ of $\mathbb{K}[\mathbf{A}]$.

Now as $\varepsilon_Q(\tilde{f}_1)$ divides $\varepsilon_Q(F_\lambda)$ we see by the previous lemma that there exists $F_1 \in I^{\gg}_Q(Z)$ which divides both $\tilde{f}_1$ and F_λ. As F_1 divides $\tilde{f}_1$ we must have $F_1 \in [I^{\gg}(Z)]_{a_1+1}$ and so by theorem 5.3.8 there's a nonzero $\theta_1 \in [D_0(\mathcal{A}_Z)]_{a_1}$. ☐

The previous two propositions will prove to be especially useful when our points (or line arrangements) are in the plane $\mathbb{P}^2$. We will establish this using some results on vector bundles on $\mathbb{P}^2_{\mathbb{C}}$ which we recall now.

Definition 5.6.3. We say a vector bundle M on $\mathbb{P}^n_{\mathbb{C}}$ is semistable, if for all proper subbundles $N \subsetneq M$, we have

$$\frac{c_1(N)}{\operatorname{rank} N} \leq \frac{c_1(M)}{\operatorname{rank} M}$$

where here c_1 is the first Chern class, and $\operatorname{rank} M$ is the dimension of a fibre.

If $\operatorname{rk}(\mathcal{M}) = 2$, semistability has a simpler characterization originally due to Hartshorne (see lemma 3.1 of [Har80]).

Lemma 5.6.4. *Let $\mathcal{M}$ be a rank 2 bundle on $\mathbb{P}^n_{\mathbb{C}}$, then if $\mathcal{M}$ is semistable if and only if letting $c_1 = c_1(\mathcal{M})$*

$$H^0\left(\mathcal{M}\left(\left\lfloor \frac{-c_1 - 1}{2} \right\rfloor\right)\right) = 0.$$

In the case that our bundle $\mathcal{M}$ is the Derivation Bundle, $\widetilde{D_0(\mathcal{A})}$ of a hyperplane arrangement, it was shown by Terao that $c_1(M) = 1 - |\mathcal{A}|$, where here $|\mathcal{A}|$ is the number of hyperplanes in $\mathcal{A}$. Using this together with the previous lemma now allows us to characterize semistability of $\widetilde{D_0(\mathcal{A})}$ for $\mathcal{A}$ a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$.

Proposition 5.6.5. *For $\mathcal{A} \subseteq \mathbb{P}^2_{\mathbb{C}}$ define $d = \left\lfloor \frac{|A|-2}{2} \right\rfloor$. Then the derivation bundle $\mathcal{D}_0(\mathcal{A})$ is semistable if and only if $[D_0(\mathcal{A})]_d = 0$. In particular, if $\mathcal{D}_0(\mathcal{A})$ is not semistable, then $D_0(\mathcal{A})$ contains a nonzero derivation in degree $\left\lfloor \frac{|A|}{2} \right\rfloor - 1$.*

One property of semistable bundles is the celebrated theorem of Grauert and Mülich, which characterizes the splitting type of semistable bundles. In light of theorem 5.4.1, we see that if $D_0(\mathcal{A}_Z)$ is semistable then $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits no unexpected curves.

Theorem 5.6.6 (Grauert-Mülich)**.** *If $\mathcal{B}$ is a semistable bundle on $\mathbb{P}^n_{\mathbb{C}}$, with splitting type $a_1 \leq a_2 \leq ... \leq a_k$, then for all $1 \leq i < k$, we have $0 \leq a_{i+1} - a_i \leq 1$.*

Theorem 5.6.7. *[CHMN18] For $\mathbb{Z} \subseteq \mathbb{P}(V) \cong \mathbb{P}^2_{\mathbb{C}}$, if $\mathcal{D}_0(\mathcal{A}_Z)$ is semistable then Z admits no unexpected curves.*

This theorem in conjunction with 5.6.2 allows us to say that every unexpected curve in $\mathbb{P}^2_{\mathbb{C}}$ comes from a global section of the derivation bundle. More precisely a polynomial defining a degree d unexpected curve corresponds via the duality of theorem 5.3.8 to an element of $[D_0(\mathcal{A})]_{d-1}$.

Theorem 5.6.8. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a finite set of points. If $D_0(\mathcal{A}_Z)$ has splitting type (a_1, a_2) with $a_2 - a_1 \geq 2$ (in particular if Z admits unexpected curves), then $D_0(\mathcal{A}_Z)$ has a generator in degree a_1; Equivalently, $I^{\gg}(Z)$ has a generator in degree $a_1 + 1$.*

Proof. As $a_2 - a_1 > 1$, $\widetilde{D_0(\mathcal{A}_Z)}$ is not semistable by the Grauert-Mülich theorem. Hence by proposition 5.6.5, there's a nonzero $\theta \in D_0(\mathcal{A})$ with $\deg \theta \leq \left\lfloor \frac{|Z|-2}{2} \right\rfloor < a_2$. Applying proposition 5.6.2 now yields the required generator of degree a_1. □

Combining this with the Grauert-Mülich Theorem, we obtain the following result.

Theorem 5.6.9. *Let Z be a finite set of points in $\mathbb{P}^2_{\mathbb{C}}$ and let $\alpha(D_0(\mathcal{A}_Z))$ denote the initial degree of $D_0(\mathcal{A}_Z)$. Define $a = \min\left\{\alpha(D_0(\mathcal{A}_Z)), \left\lfloor \frac{|Z|-1}{2} \right\rfloor\right\}$ then $D_0(\mathcal{A}_Z)$ has splitting type $(a, |Z| - a - 1)$.*

Translating the above statement via the duality of theorem 5.3.8, we obtain the corollary below.

Corollary 5.6.10. *For a finite set of points $Z \subseteq \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2])$, suppose Z admits unexpected curves. If $Q = (A_0 : A_1 : A_2)$ is the generic point, then $I^{\gg}_Q(Z)$ is a free $\mathcal{F}_Q$-module on generators f and g with $\deg f < \deg g - 1$ and f can be lifted to an element F of $I^{\gg}(Z)$ with $\epsilon_Q(F) = f$.*

In particular, f can be written as

$$f = X_1 f_1 + X_2 f_2 + X_3 f_3,$$

where f_i is a polynomial of degree $(\deg f) - 1$ in the maximal minors of $\begin{bmatrix} A_0 & A_1 & A_2 \\ X_0 & X_1 & X_2 \end{bmatrix}$

The above corollary states that the polynomials defining unexpected curves are "as simple as possible", in the sense that they have the minimal possible degree as a polynomial in the coordinates of our general point Q. This stands in stark contrast to most other sets of points where this is not the case. As an illustration, taking 8 randomly chosen points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$, a computation with Macaulay2 showed that $I^{\gg}_Q(Z)$ has generators of X-degree 4 and 5. The first generator had an A-degree of 12 giving a total degree of 16, showing that the above result is far from expected. Similar computations with 6 points and 10 points gave minimal polynomials with A-degrees of 6 and 20, respectively.

Below we present a simpler example illustrating a similar point.

Example 5.6.11. If $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ consists of the 3 coordinate points and $(1 : 1 : 1)$. Then for generic Q, $I^{\gg}_Q(Z)$ has generators f_1 and f_2 of degrees 2 and 3 as polynomials in X. Many different f_2 are possible. On the other hand, if we require f_1 to be a polynomial of minimal degree in $\mathbb{K}[\mathbf{A}]$, it is unique up to $\mathbb{C}$ scalar. The corresponding polynomial formula is

$$\begin{aligned} f_1 &= (A_0 - A_1)M_1X_1 + (A_0 - A_2)M_2X_2 \\ &= (A_0 - A_1)A_2X_0X_1 - (A_0 - A_2)A_1X_0X_2 + (A_1 - A_2)A_0X_1X_2 \end{aligned}$$

where here M_i is the minor of the 2×3 matrix from the matrix above and proposition 5.3.12. f_1 is irreducible, and defines the unique smooth conic through Z and Q. As the A degree and X degree of f_1 are the same, we can see f_1 cannot be written in the form from corollary 5.6.10.

5.7 Combinatorial Constraints on Points Admitting Unexpected Curves

In this section we explore combinatorial constraints necessarily satisfied by sets of points admitting unexpected curves. Most of these constraints apply only when this unexpected curve is irreducible. Yet this turns out to be a fairly weak assumption, since if Z admits a unique unexpected curve in degree d there is always a subset $W \subseteq Z$ so $|Z \setminus W| = k$ and W admits a unique irreducible unexpected curve in degree $d - k$.

We start by exploring the consequences of corollary 5.6.10. In the case the curve of degree d is irreducible we show in lemma 5.7.5 that corollary 5.6.10 gives a bound on the number of distinct lines through a point $P \in Z$ and the remaining points of Z, showing that there are at most d lines. As $|Z| \geq 2d + 1$ this is a very strong combinatorial condition which states that on average each line through a fixed point P contains 3 or more points of Z. We are able to use this in theorem 5.7.6 to give a sharp bound on the number of points in Z, this bound is achieved by the Ceva type point configurations C_d from example 5.5.9.

Furthermore, in proposition 5.7.13 we also give and upper bound on the number of lines spanned by points of Z. We then close the section by applying a theorem of Terao to state a combinatorial condition that guarantees that Z will admit an unexpected curve.

We note that throughout this section, we often state theorems with the assumption that *"there's some nonzero (possibly irreducible)* $f \in [I^{\gg}(Z)]_d$*"*. By theorem 5.6.8 perhaps the prototypical example for us are points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admitting unexpected curves in degree d. However, this also holds in other contexts for instance if $\mathcal{A}_Z \subseteq \mathbb{P}^2_{\mathbb{K}}$ is free.

Lemma 5.7.1. *For* $Z \subseteq \mathbb{P}(V) \cong \mathbb{P}^2_{\mathbb{K}}$*, consider* $F(X_0, X_1, X_2; A_0, A_1, A_2) \in [I^{\gg}(Z)]_d$ *then for every* $P = (P_0 : P_1 : P_2) \in Z$*,*

$$\varepsilon_P(F) = F(X_0, X_1, X_2; P_0, P_1, P_2) \in I(P)^d.$$

Moreover, $\varepsilon_P(F) = 0$ *if and only if the linear form* $\ell_P = P_0M_0 + P_1M_1 + P_2M_2$ *divides* F*.*

Proof. We may choose coordinates so $P = (1:0:0)$ and $\ell_P = M_0$. Writing F as $F = F_0X_0 + F_1X_1 + F_2X_2$, then as $F \in I(P) = (X_1, X_2)$ we have

$$F(P_0, P_1, P_2, A_0, A_1, A_2) = F_0(1, 0, 0, A_0, A_1, A_2) = 0.$$

As each M_i is antisymmetric in A and X, it follows that $\varepsilon_P(F_0) = 0$ and so $F_0 \in I(P^\perp) = M_0$ by proposition 5.3.12. Then applying the identity $M_0X_0 + M_1X_1 + M_2X_2 = 0$, we may write $F = f_1X_1 + f_2X_2$ where $f_i = \left(F_i - \frac{M_i}{M_0}F_0\right)$. Noting for arbitrary $Q \in \mathbb{P}^2$, that $\varepsilon_Q(f_i) \in I(Q)^{d-1}$. It follows that

$$\varepsilon_P(F) = \varepsilon_P(f_1)X_1 + \varepsilon_P(f_2)X_2 \in (X_1, X_2)I(P)^{d-1} = I(P)^d,$$

which establishes the first statement.

Continuing with the proof of the second statement, we assume that $\varepsilon_P(F) = 0$. Noting $\varepsilon_P(M_1) = X_2$ and $\varepsilon_P(M_2) = -X_1$, it follows that $\varepsilon_P(f_1M_2 - f_2M_1) = -\varepsilon_P(f_1)X_1 - \varepsilon_P(f_2)X_2 = 0$, so $f_1M_2 - f_2M_1 \in (M_0) = \ker \varepsilon_P$. Let $\tilde{f}_1, \tilde{f}_2 \in \mathbb{K}[M_1, M_2]$ so that $f_i = \tilde{f}_i \mod (M_0)$ for each $i \in \{1, 2\}$. Then $\tilde{f}_1M_2 - \tilde{f}_2M_1 \in (M_0) \cap \mathbb{K}[M_1, M_2] = 0$, so $\tilde{f}_1M_2 = \tilde{f}_2M_1$ and we get by unique factorization that there exists some $g \in \mathbb{K}[M_1, M_2]$ with $g = \dfrac{\tilde{f}_2}{M_2} = \dfrac{\tilde{f}_1}{M_1}$.

Finally, applying the identity $X_0M_0 + X_1M_1 + X_2M_2 = 0$ again we can write

$$\begin{aligned} F &= f_1X_1 + f_2X_2 - g(M_0X_0 + M_1X_1 + M_2X_2) \\ &= (-gM_0)X_0 + (f_1 - gM_1)X_1 + (f_2 - gM_2)X_2. \end{aligned}$$

Noting that $f_i - gM_i = f_i - \tilde{f}_i \equiv 0 \mod (M_0)$, we conclude that M_0 divides F. This establishes the forward direction, the reverse direction follows as $\varepsilon_P(\ell_P) = 0$. $\square$

As we will see, the preceeding lemmas imposes a very strong combinatorial condition on the configurations of points which can admit unexpected curves. Before we state the first of these conditions we introduce a new piece of notation.

Definition 5.7.2. Let $Z \subseteq \mathbb{P}^2_{\mathbb{K}}$ be a finite configuration of points, with $|Z| \geq 2$. For each $P \in \mathbb{P}^2_{\mathbb{K}}$, define a set of lines, $L_P(Z)$, as follows

$$L_p(Z) := \{\mathrm{Span}(Q_i, P) \mid Q_i \in Z \setminus \{P\}\}.$$

Remark 5.7.3. Note $|L_P(Z)| \leq |Z \setminus \{P\}|$ with equality if and only if for distinct $Q, Q' \in |Z \setminus \{P\}|$ we have $\mathrm{Span}(Q, P) \neq \mathrm{Span}(Q', P)$.

The number $|L_p(Z)|$ defined above has an equivalent purely algebraic definition.

Lemma 5.7.4. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{K}}$ be a finite set of at least 2 points, then for any $P \in \mathbb{P}^2_{\mathbb{K}}$ we have*

$$|L_P(Z)| = \min\{d \mid [I(Z) \cap I(P)^d]_d \neq 0\}$$

Proof. Let $m = \min\{d \mid [I(Z) \cap I(P)^d]_d \neq 0\}$. For any $Q \in Z \setminus \{P\}$, we get by Bezout's Theorem that the line $\operatorname{Span}(P, Q)$ must be a component of the base locus of $[I(Z) \cap I(P)^d]_d$. Hence, letting G_p denote the product of the linear forms defining the elements of $L_p(Z)$, we have $\deg G_p = |L_p(Z)|$ and $[I(Z) \cap I(P)^d]_d = [I(L_p(Z))]_d = [(G_p)]_d$ which completes the proof. □

We introduce the first combinatorial constraint below, it occurs whenever $I^{\gg}(Z)$ contains an irreducible element. As we will see this simple constraint ends up having a number of strong consequences.

Lemma 5.7.5. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{K}}$, with $|Z| \geq 2$ and suppose $F \in [I^{\gg}(Z)]_d$ is an irreducible polynomial. Then for all $P \in Z$ we have $|L_P(Z)| \leq d$.*

Consequently, if $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits an irreducible unexpected curve in degree d, then $|L_p(Z)| \leq d$ for all $P \in Z$.

Proof. As F is irreducible, we know by lemma 5.7.1 that $\varepsilon_P(F) \neq 0$ and $\varepsilon_P(F) \in [I(P)^d \cap I(Z)]_d$ for all $P \in Z$. Hence, by lemma 5.7.4 we get $|L_P(Z)| \leq d$.

The second statement follows from the first in light of corollary 5.6.10. □

G. Dirac conjectured (see [Dir51]) that for any set Z of noncollinear points in $\mathbb{R}^2$, there always exists some $P \in Z$ with $|L_P(Z)| \geq \lfloor \frac{|Z|}{2} \rfloor$. This turned out to be false. However, since then alternative conjectures have been proposed, one version of the conjecture was established in [Han17] for points in $\mathbb{C}^2$. This result allows us theorem 5.7.6 below. We explore possible further consequences of the conjectures in section 5.9.

Theorem 5.7.6. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ and suppose that $|Z|$ admits an unexpected curve in degree $d \geq 1$, then $|Z| \leq 3d - 3$.*

Proof. This follows from 5.7.5 and Han's improvement of the Dirac Conjecture [Han17], which states for a finite set of points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ which span $\mathbb{P}^2$ there always exists some $P \in Z$ so $|L_p(Z)| \geq \frac{|Z|}{3} + 1$.

Namely, suppose Z admits an irreducible curve in degree d, then for all $P \in Z$, $|L_P(Z)| \leq d$. Now applying Han's result, there exists $P \in Z$ so

$$d \geq |L_p(Z)| \geq \frac{|Z|}{3} + 1.$$

Solving for $|Z|$ now yields $3(d-1) \geq |Z|$, the desired inequality.

□

Remark 5.7.7. It should be noted that the paper [Han17], is rather vague and states the result only for points in "the plane". However, the proof works for complex line arrangements, as the main nonelementary tool is a Hirzeburch type inequality for complex line arrangements first proved in [Boj03]

Equivalently, Langer's Inequality [Lan03], could replace and or rederive [Han17]'s result. Langer's Inequality states that letting $\ell_r = \left|\{L \subseteq \mathbb{P}^2_{\mathbb{C}} \mid |L \cap Z| = r\}\right|$ we have that if $\ell_r = 0$ for $r > \frac{2}{3}|Z|$ that

$$\sum_{P \in Z} |L_P(Z)| = \sum_{r \geq 2} r\ell_r \geq \left\lceil \frac{|Z|^2 + 3|Z|}{3} \right\rceil .$$

For further discussion see the survey article [Pok18], where the author first learned of these results.

In [CHMN18] it was shown that any set of points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ in linearly general position can never admit unexpected curves. This proposition provides a strengthening of that result, and extends it to an arbitrary field.

Proposition 5.7.8. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{K}}$ suppose there's a nonzero $F \in [I^{\gg}(Z)]_d$ for $1 \leq d \leq |Z| - 2$. Then no subset $W \subseteq Z$ with $|W| > d + 1$ is in linearly general position. If F is irreducible and d is even this can be improved to say no subset $W \subseteq Z$ with $|W| > d$ is in linearly general position.*

Proof. We first proceed in the special case that $|L_p(Z)| \leq d$ for all $P \in Z$, note that by lemma 5.7.5 this includes the case that F is irreducible. If $W \subseteq Z$ is in linearly general position, then for all $P \in W$ and all $L \in L_P(W)$, we have that $|L \cap (W \setminus \{P\})| \leq 1$. Therefore $|W| - 1 = |L_P(W)| \leq |L_p(Z)| \leq d$ implying

$$|W| \leq |L_P(W)| + 1 \leq d + 1.$$

If furthermore d is even, then suppose by contradiction that $W \subseteq Z$ is in linearly general position with $|W| = d+1$. As $|W| = d+1$ we get that $|L_P(W)| = |L_p(Z)| = d$ for all $P \in W$. Now fix some $Q \in Z \setminus W$ and define a partition Π_Q of W, where $P \in W$ is contained in the block $\operatorname{Span}(Q, P) \cap W$. Now as $\operatorname{Span}(Q, P) \in L_P(Z) = L_P(W)$, we get $|\operatorname{Span}(Q, P) \cap W| = 2$, therefore Π_Q is a partition where each block has size 2 contradicting the fact that $|W| = d + 1$ is odd.

Now continuing with the general case, let F be a nonzero possibly reducible polynomial. Let $Z' \subseteq Z$ be the subset $Z' = \{P \in Z \mid \varepsilon_P(F) \neq 0\}$, and let $T = Z \setminus Z'$. Then by lemma 5.7.1, we see that F factors as $F = G \prod_{P \in T} \ell_P$. Furthermore, $G \in I^{\gg}(Z')$ and $\varepsilon_P(G) \neq 0$ for all $P \in Z'$, so by the proof of lemma 5.7.5 we have $|L_P(Z')| \leq \deg(G) = d - |T|$ for all $P \in Z'$. If $W \subseteq Z$ is in linearly general position, then so is $W' = W \cap Z'$. Applying the result from our first case we see

$$|W| \leq |W'| + |T| \leq d + 1$$

establishing the result. □

We immediately obtain the following corollary by applying corollary 5.6.10.

Theorem 5.7.9. *If $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits an unexpected curve in degree d, then every subset $W \subseteq Z$ of points in linearly general position has*

$$|W| \leq d + 1.$$

Furthermore, we have $|W| \leq d$ if d is even and the unexpected curve is irreducible.

Remark 5.7.10. The author suspects the bound of theorem 5.7.9 can be somewhat improved over $\mathbb{C}$. Namely, given $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$, which admits an unexpected curve in degree d, then every subset $W \subseteq Z$ in linearly general position must have $|W| \leq d$. It should be noted, however, that the bound given in proposition 5.7.8 is sharp in positive characteristic at least if $d-1$ is a prime power.

Namely, let $\mathbb{K}$ be a field of characteristic $p > 0$. Let $q = p^e$ and take $Z = \mathbb{P}^2_{\mathbb{F}_q} \subseteq \mathbb{P}^2_{\mathbb{K}}$. Then as shown in example 5.5.8 Z will have an unexpected curve in degree $q+1$. A smooth conic such as $X_1^2 = X_0X_2$ will contain exactly $q+1$ points of Z which form a subset in linearly general position. This achieves the bound from proposition 5.7.8 if $q+1$ is even.

If $q+1$ is odd, then $\mathrm{Char}(\mathbb{K}) = 2$ and for each smooth conic $C \subseteq \mathbb{P}^2_{\mathbb{K}}$ there is a point $N_C \in \mathbb{P}^2_{\mathbb{K}} \setminus C$ which is contained in every tangent line of C. This is often referred to as the *nucleus* of C. As an example, we can verify that $C : X_1^2 = X_0X_2$ has nucleus $N = (0:1:0)$. In this case taking W to be $(Z \cap C) \cup \{N\}$, gives a linearly general subset of size $q+2$.

Remark 5.7.11. Proposition 5.7.5 can be applied to generalize an inductive technique, stated as Lemma 6.5 in [CHMN18], restricted to the case the bundle $\mathcal{D}_0(\mathcal{A}_Z)$ is semistable.

Proposition 5.7.12. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ and $P \in \mathbb{P}^2$ and suppose $I^{\gg}(Z)$ has splitting type (a,b) with $a \leq b$. If $L_P(Z) > a$, then $I^{\gg}(Z+P)$ has splitting type $(a+1,b)$ (or $(a,a+1)$ if $a = b$).*

In particular, if Z does not admit unexpected curves, and $L_P(Z) > \left\lceil \frac{|Z|}{2} \right\rceil$, then $Z+P$ does not admit unexpected curves.

Proof. First suppose that $[I^{\gg}(Z)]_a = 0$, then by theorem 5.6.9 we have $(a,b) = \left(\left\lfloor \frac{|Z|+1}{2} \right\rfloor, \left\lceil \frac{|Z|+1}{2} \right\rceil\right)$, and can conclude that $|Z|$ admits no unexpected curves. Now for every $P \in \mathbb{P}^2 \setminus Z$ we have that $Z+P$ does not admit unexpected curves. Since if it did we would have by theorem 5.6.8, that $[I^{\gg}_Q(Z+P)]_d \neq 0$ for some $d \leq a = \left\lfloor \frac{|Z|+1}{2} \right\rfloor$.

Now suppose that $[I^{\gg}(Z)]_a \neq 0$ and that $|L_P(Z)| > a$. Note that $\ell_p F \in [I^{\gg}(Z+P)]_{a+1}$ and it suffices to show that $[I^{\gg}(Z+P)]_a = 0$, since then by theorem 5.6.9 $I^{\gg}(Z+P)$ has splitting type $(\alpha, |Z|-\alpha)$ where $\alpha = \min\left\{a+1, \left\lfloor \frac{|Z|+2}{2} \right\rfloor\right\}$.

For any $F \in [I^{\gg}(Z+P)]_d$, we have that $\varepsilon_P(F) \in [I(P)^d \cap I(Z)]_d$. Yet as $|L_P(Z)| > a$ we have by applying lemma 5.7.4 that $\varepsilon_P(F)$ must be 0. By lemma 5.7.1 this means that ℓ_p divides F, however then F/ℓ_P is a nonzero element of $[I^{\gg}(Z)]_{a-1}$ giving a contradiction. ☐

Proposition 5.7.13. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$, define*

$$\mathcal{L} = \{\mathrm{Span}(P,Q) \mid P, Q \in Z \textit{ are distinct points } \}$$

and suppose that Z admits an irreducible unexpected curve in degree d, then

$$|\mathcal{L}| \leq d^2 - d + 1$$

.

Proof. We prove the theorem under the slightly weaker assumption that there is some irreducible $F_\lambda \in [I^{\gg}(Z)]_d$. Without loss of generality assume that $E_0 = (1:0:0) \in Z$, so we may write $F_\lambda = f_1 X_1 + f_2 X_2$ with $f_i(M_0, M_1, M_2) \in \mathbb{K}[M_0, M_1, M_2]$. Recall the map $\rho_\lambda : \mathbb{P}^2 \to \mathbb{P}^2$ from definition 5.2.7, in coordinates

$$\rho_\lambda(a_0, a_1, a_2) = (0 : f_1(a_0, a_1, a_2) : f_2(a_0, a_1, a_2)).$$

By theorem 5.3.8 we get that $\rho_\lambda(H) \subseteq H$ for all $H \in \mathcal{A}_Z$, namely all H of the form $H = P^\perp$ for $P \in Z$. Define $\mathcal{P} = \{L \cap H \mid L \text{ and } H \text{ are distinct lines in } \mathcal{A}_Z\}$. By projective duality we have that $\mathcal{L}^\perp = \mathcal{P}$ and so in particular $|\mathcal{L}| = |\mathcal{P}|$. Now as $\rho_\lambda(H) \subseteq H$ for all $H \in \mathcal{A}_Z$, we get for any $H \cap L = Q \in \mathcal{P}$ that

$$\rho_\lambda(Q) = \rho_\lambda(H \cap L) \subseteq \rho_\lambda(H) \cap \rho_\lambda(L) \subseteq H \cap L = Q.$$

Hence $\mathcal{P}$ is contained in the vanishing locus of the minors of

$$\begin{bmatrix} Y_0 & Y_1 & Y_2 \\ 0 & F_1 & F_2 \end{bmatrix}.$$

So $\mathcal{P}$ is contained in the solutions of the polynomial system

$$\begin{aligned} Y_1 F_2 - Y_2 F_1 &= 0 \\ Y_0 F_2 &= 0 \\ Y_0 F_1 &= 0 \end{aligned} \tag{5.7.13.1}$$

To count solutions, let V denote the variety defined by this system, we look at solutions on the line $Y_0 = 0$ and solutions on the subset $Y_0 \neq 0$. On $Y_0 = 0$ we get that the system (1), reduces to

$$\begin{aligned} Y_1 F_2 - Y_2 F_1 &= 0 \\ Y_0 &= 0 \end{aligned}$$

from which we get by Bezout's theorem that the number of solutions is at most $\deg(Y_0)\deg(Y_1F_2 - Y_2F_1) = d$. On the subset $Y_0 \neq 0$, the first equation in the system is redundant and the system reduces to

$$\begin{aligned} F_1 &= 0 \\ F_2 &= 0. \end{aligned}$$

As F is irreducible F_1 and F_2 have no shared component so by Bezout's Theorem this system has at most $\deg(F_1)\deg(F_2) = (d-1)^2$ solutions. Combining both results, we can conclude that

$$|\mathcal{L}| = |\mathcal{P}| \le |V \cap (Y_0 = 0)| + |V \cap (Y_0 \neq 0)| \le d + (d-1)^2 = d^2 - d + 1.$$

□

Example 5.7.14. We note that the above bound is sharp in every degree. Namely, the point configuration $C_m \subseteq \mathbb{P}^2_{\mathbb{C}}$ of example 5.5.9 achieves the bound. To see this write $C_m = \{E_0, E_1, E_2\} \cup F_m$, where $F_m = \{-E_i + \zeta^k E_j \mid 0 \leq i < j \leq 2\}$. Then the points in F_m generate $m^2 + 3$ lines, the 3 coordinate lines $(X_i = 0)$, and all lines of the form $\operatorname{Span}(-E_0 + \zeta^j E_1, -E_1 + \zeta^k E_2, -E_0 + \zeta^{j+k} E_2)$ with defining equation $\zeta^{j+k} X_1 + \zeta^k X_1 + X_2 = 0$. The only lines unaccounted for in $\mathcal{L}_{C_m}$ are those of the form $\operatorname{Span}(E_i, -E_j + \zeta^t E_k)$ with $\{i, j, k\} = \{0, 1, 2\}$ of which there are $3m$.

Then $|\mathcal{L}_{C_m}| = m^2 + 3m + 3$, and C_m admits a unique unexpected curve in degree $m + 2$. Noting that $m^2 + 3m + 3 = (m + 2)^2 - (m + 2) + 1$ we conclude that the above bound is sharp for all $d \geq 4$.

We close this section with a previously unnoticed combinatorial condition which guarantees the existence of unexpected curves.

Proposition 5.7.15. *Let $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a finite set of points. Further suppose that no line $L \subseteq \mathbb{P}^2_{\mathbb{C}}$ has $|Z \cap L| \geq \frac{|Z|-1}{2}$. Define $\mathcal{L}$ as in proposition 5.7.13. Then for any integer $d \leq \frac{|Z|+1}{2}$, if*

$$\left(\sum_{p \in Z} |L_P(Z)|\right) - |Z| - |\mathcal{L}| + 1 < (d - 1)(|Z| - d)$$

then $|Z|$ admits unexpected curves in degree d.

Proof. Let $c_t(D_0(\mathcal{A}_Z))$ denote the Chern polynomial of $D_0(\mathcal{A}_Z)$. Then by theorem 2.5 of [Sch03],

$$c_t(D_0(\mathcal{A}_Z)) = 1 - (|Z| - 1)t + \left(\sum_{L \in \mathcal{L}} (|L \cap Z| - 1) - |Z| + 1\right) t^2.$$

Noting that

$$\sum_{L \in Z} |L \cap Z| = \sum_{L \in Z} \sum_{P \in (L \cap Z)} 1 = \sum_{P \in Z} \sum_{L \in L_p(Z)} 1 = \sum_{P \in Z} |L_P(Z)|.$$

We get the following formula for $c_2(D_0(\mathcal{A}_Z))$,

$$\begin{aligned} c_2(D_0(\mathcal{A}_Z)) &= \sum_{L \in \mathcal{L}} (|L \cap Z| - 1) - |Z| + 1 \\ &= \left(\sum_{P \in Z} |L_P(Z)|\right) - |\mathcal{L}| - |Z| + 1. \end{aligned}$$

In particular, we see that our hypothesized inequality is equivalent to $c_2(D_0(\mathcal{A}_Z)) < (d - 1)(|Z| - d)$.

Now theorem B of [BR10] states that if (a, b) denotes the splitting type of $D_0(\mathcal{A}_Z)$ then $ab \leq c_2(D_0(\mathcal{A}_Z))$. So if we satisfy $c_2(D_0(\mathcal{A}_Z)) < (d - 1)(|Z| - d)$, then $ab <$

$(d-1)(|Z|-d)$. Letting $k = \frac{|Z|-1}{2}$, $g_1 = k-(d-1)$ and $g_2 = k-a$, then this inequality becomes

$$k^2 - g_2^2 = (k-g_2)(k+g_2) = ab < (d-1)(|Z|-d) = (k-g_1)(k+g_1) = k^2 - g_1^2.$$

Therefore, we may conclude that $g_1 < g_2$ and so $a < d-1$. Applying theorem 5.4.1 now establishes the result. $\square$

5.8 Regularity Bounds

In remark 3.8 of [CHMN18], it is claimed that the definition of unexpected curves

> " .. *leaves open the possibility that the points of* Z *do not impose independent conditions on curves of some degree* $j+1$*, and ... a general fat point* jP *fails to impose the expected number of conditions on the linear system defined by* $[I_Z]_{j+1}$*. Theorem 3.7 gives the surprising result that this is impossible.*"

However, it appears to the author that theorem 3.7 of [CHMN18] is a weaker statement than the above quotation claims. Rather it establishes that Z imposes independent conditions on a specific degree $t_z \geq j+1$, if Z admits an unexpected curve in degree $j+1$.

In this section, we establish the full claim for points $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$. In fact we prove a stronger claim. Namely if $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ admits unexpected curves in degree $d+1$, then Z imposes independent conditions on forms in degree d. This claim is false in general in positive characteristic, though it does hold for certain values of d (see proposition 5.8.2).

Before proceeding we recall the definition of Castelnuovo-Mumford Regularity, this number determines when Z imposes independent conditions on d forms.

Definition 5.8.1. Given a finite set of points $Z \subseteq \mathbb{P}(V)$ the Castelnuovo-Mumford Regularity of Z, denoted $\operatorname{reg}(Z)$, is the integer

$$\operatorname{reg}(Z) := 1 + \min\{r \mid \dim_{\mathbb{K}}[\operatorname{Sym}(V^*)/I(Z)]_r = |Z|\}.$$

It should be noted that the above definition is highly nonstandard, and applies only to this specific situation. We refer to Exercise $4E.3$ and theorem 4.2 of [Eis05], for proofs that the definition given is equivalent the standard definitions for graded modules.

This result has some applications to Terao's conjecture as well, which we explore in the last section.

Proposition 5.8.2. *Let* $\mathbb{K}$ *be an infinite field, and let* $A = \mathbb{K}[s,t]$ *be a standard graded polynomial ring on* 2 *variables, let* $\operatorname{Pow}_d : [A]_1 \to [A]_d$ *be the* d*-th power map, that is the map* $\ell \mapsto \ell^d$.

Then the image of Pow_d *spans* $[A]_d$ *over* $\mathbb{K}$ *if and only if the pair* $(\operatorname{Char}\mathbb{K}, d)$ *satisfies one of the following*

Characteristic Hypothesis

1. $(\operatorname{Char}\mathbb{K}, d) = (0, d)$; *or*
2. $(\operatorname{Char}\mathbb{K}, d) = (p, q(p^e) - 1)$ *for some* $e \geq 0$ *and* q*, with* $1 \leq q \leq p$*.*

Proof. This result is likely well known and consists of standard techniques so we only give a brief sketch.

Let L be the $(d+1) \times (d+1)$ matrix whose i-th row is $(s + a_i t)^d$ in the standard monomial basis of $[A]_d$, also suppose that $a_i \neq a_j$ for $i \neq j$. Then L can be seen to be a Vandermonde Matrix whose j column has been scaled by $\binom{d}{j}$. Using the well known Vandermonde Determinant formula the matrix is nonsingular and hence the rows span $[A]_d$ if and only if $\prod_{i=0}^{d} \binom{d}{i}$ is nonzero as an element of $\mathbb{K}$. In particular, we may conclude if $\operatorname{Char}(\mathbb{K}) = 0$.

If $\operatorname{Char}(\mathbb{K}) = p > 0$, we recall Lucas's theorem on Binomial coefficients which states $\left(\begin{matrix} d \\ i \end{matrix}\right) \not\equiv 0 \mod p$ if and only if each digit of i written in base p does not exceed the corresponding digit of d. In base p, the only numbers d where this criterion holds for all $0 \leq i \leq d$ are those d where the non-leading digits are all $p - 1$. This happens precisely when $d = qp^e - 1$ for $1 \leq q \leq p$. ☐

Proposition 5.8.3. *Let* $Z \subseteq \mathbb{P}^2_{\mathbb{K}}$ *be a finite set of points, suppose that the base locus* $\operatorname{B.loc}_{d+1}(I^{\gg}(Z))$ *is zero dimensional and that the pair* $(\operatorname{Char}\mathbb{K}, d)$ *satisfies the characteristic hypothesis of proposition 5.8.2. Then* $\operatorname{reg}(Z) \leq d + 1$.

Proof. Let $\mathbb{P}^2 = \operatorname{Proj}(R)$, where $R = \mathbb{K}[X_0, X_1, X_2]$. Take $\ell \in [R]_1$ to be a general linear form and consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & {[R/I(Z)]_{t-1}} \arrow[r, "\ell"] & {[R/I(Z)]_t} \arrow[r] & {[R/(I(Z) + (\ell))]_t} \arrow[r] & 0 \end{tikzcd}$$

Letting $h_Z(t) := \dim[R/I(Z)]_t$ we conclude that for integers t that

$$h_z(t) - h_z(t-1) = \dim[R/(I(Z) + \ell)]_t.$$

Furthermore, as $R/(I(Z) + \ell)$ is principally generated we can conclude that $h_z(t) = h_z(t-1)$ if and only if $h_z(t-1) = |Z|$. From this it follows from definition 5.8.1 that $\operatorname{reg}(Z) = \min\{r \mid [R/(I(Z) + (\ell))]_r = 0\}$ and it suffices to prove that $[R/(I(Z) + (\ell))]_{d+1} = 0$.

Fix $F_\lambda \in [I^{\gg}(Z)]_{d+1}$ with $\dim \operatorname{B.loc}(F_\lambda) = 0$. For all points Q on the line $\ell = 0$, we have by lemma 5.3.9 that

$$\varepsilon_Q(F_\lambda) = h_Q^d \rho_\lambda(\ell) \mod (\ell)$$

for some linear form h_Q vanishing on Q. Noting $\varepsilon_Q(F_\lambda) \in I(Z)$, we get an inclusion of $\mathbb{K}$-vector spaces

$$[(I(Z) + (\ell))/(\ell)]_{d+1} \supseteq \operatorname{Span}\{h_Q^d \rho_\lambda(\ell) + (\ell) \mid Q \in \mathbb{P}^2 \text{ and } \ell(Q) = 0\}.$$

By proposition 5.8.2 the set $\{h_Q^d \mid Q \in (\ell = 0)\}$ spans $[R/(\ell)]_d$ and so

$$[I(Z) + (\ell)/(\ell)]_{d+1} \supseteq \rho_\lambda(\ell)[R/(\ell)]_d.$$

Hence, $[I(Z)+(\ell)]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell))]_{d+1}$. Let $P \in \mathbb{P}^2$ denote the point defined by the ideal $(\ell, \rho_\lambda(\ell))$, as $\operatorname{B.loc}(F_\lambda) \cap (\ell = 0) = \emptyset$ we can find some $H \in \mathbb{P}^2$ so that $\varepsilon_H(F_\lambda) \in I(Z)$ and $\varepsilon_H(F_\lambda) \notin I(P) = (\ell, \rho_\lambda(\ell))$. Hence,

$$[I(Z)+(\ell)]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell))]_{d+1} + [\varepsilon_H(F_\lambda)]_{d+1} = [R]_{d+1}$$

allowing us to conclude that $[R/(I(Z)+(\ell))]_{d+1} = 0$ as desired. $\square$

Remark 5.8.4. It should be noted that the assumptions of the above theorem, may be relaxed in various ways to give slightly different bounds, which also require different proofs, and possibly stronger assumptions. We have choosen to give only the proof above for the sake of brevity, but will briefly comment on two of the possible changes now.

1. The condition $0 = \dim \operatorname{B.loc}_{d+1}(I^{\gg}(Z))$ may be replaced with the weaker condition that $0 = \dim \operatorname{B.loc}_{d+1}(Z)$. However the bound then becomes $\operatorname{reg}(Z) \leq d+2$. The proof is similar to above, but ℓ is replaced with a line through a general point Q, which also vanishes on a point in Z. It can then only be concluded that $\dim[R/(I(Z)+(\ell))]_{d+1} \leq 1$. This worse bound of $\operatorname{reg}(Z) \leq d+2$ is in fact sharp, as can be illustrated by taking Z to be $2d+3$ points on a smooth conic.

   Interestingly, this technique can be used to give a completely geometric proof of theorem 4.3.3 for points in $\mathbb{P}^2_{\mathbb{C}}$, in contrast to the combinatorial proof given in chapter 4.

2. A generalization to $\mathbb{P}^n$, at least in characteristic 0, is possible however the proof becomes more involved and/or additional assumptions on $[I^{\gg}(Z)]_{d+1}$ are necessary. The main technique is still roughly the same except now induction is needed. After showing $[I(Z)+(\ell)]_{d+1} \supseteq (\ell, \rho_\lambda(\ell))$ one proceeds as before showing

   $$[I(Z)+(\ell)]_{d+1} = [I(Z)+(\ell, \rho_\lambda(\ell))]_{d+1} \supseteq [(\ell, \rho_\lambda(\ell), \rho_\lambda^2(\ell))]_{d+1} \supseteq \ldots$$

   If $(\ell, \rho_\lambda(\ell), \ldots, \rho_\lambda^{n-1}(\ell))$ is the ideal of a point we then proceed as in the proposition.

Combining the above from some results from earlier sections, we may obtain the following result

Theorem 5.8.5. *If $Z \subseteq \mathbb{P}^2_{\mathbb{C}}$ has an unexpected curve in degree d, then* $\operatorname{reg}(Z) \leq d$. *In particular, Z imposes independent conditions on forms of degree $d-1$.*

Proof. Suppose that Z admits an unexpected curve in degree d, without loss of generality we assume that Z does not admit an unexpected curve in degree $d-1$. Then by theorem 5.4.1, we note that $|Z| \geq 2d+1$ and that no line L contains more that $d+1$ points of Z. Additionally by theorem 5.6.8, there exists $F \in [I^{\gg}(Z)]_d$

defining the curve. We claim that $\dim \mathrm{B.loc}(F) = 0$, which in light of proposition 5.8.3 establishes the claim.

Proceeding by contradiction assume that $\dim \mathrm{B.loc}(F) = 1$, applying proposition 5.4.17, we get $\mathrm{B.loc}(F)$ has a component which is a line. If $\ell \in \mathbb{C}[X_0, X_1, X_2]$ is the linear form defining this line, L, then viewing it as a polynomial in the ring $\mathbb{C}[X_0, X_1, X_2, A_0, A_1, A_2]$ and applying lemma 5.6.1, we get $F = \ell h$ with $h \in \mathbb{C}[M_0, M_1, M_2]$. As $\varepsilon_Q(h) \in [I(Q)^{\deg h}]_h$ for all $Q \in \mathbb{P}^2$ we get that the variety $V(\varepsilon_Q(F))$ is a union of L and at most $\deg h$ lines through Q. For a general $Q \in \mathbb{P}^2_{\mathbb{C}}$, each line in $V(\varepsilon_Q(h))$ contains at most one point of Z. This forces us to conclude that $|L \cap Z| \geq |Z| - \deg h \geq d + 2$ giving us our desired contradiction. □

We note that the theorem 5.8.5, gives a decent criteria purely algebraic criteria for establishing that a set of points Z does not admit unexpected curves in a given degree. We explore this a bit in the next section in the context of Terao's Conjecture.

5.9 Applications to Terao's Conjecture in $\mathbb{P}^2$

A much studied problem in the theory of line arrangements is the freeness of the module of derivations $D(\mathcal{A}_Z)$. One reason for this in particular is that if $D(\mathcal{A}_Z)$ is free then many of the invariants of $D(\mathcal{A}_Z)$ can be determined from combinatorics of the intersection lattice $L(\mathcal{A}_Z)$ (or equivalently the matroid $M(Z)$). A major open problem in the study of Hyperplane arrangements is Terao's Freeness Conjecture

Conjecture 5.9.1 (Terao's Freeness Conjecture). *Over $\mathbb{C}$ freeness of $D_0(\mathcal{A})$ can be determined by the intersection lattice $L(\mathcal{A}_Z)$.*

Remark 5.9.2. The above conjecture is usually stated for $D(\mathcal{A})$. However the two versions are equivalent because over $\mathbb{C}$, $D(\mathcal{A})$ splits as $(\mathrm{Sym}(V^*))\theta_e \oplus D_0(\mathcal{A})$.

One natural question to ask given theorem 5.3.8, is what freeness of $D_0(\mathcal{A}_Z)$ says about $I(Z)$. Namely, can $D_0(\mathcal{A}_Z)$ be characterized in terms of Z? The following proposition (which is well known to experts) is helpful in addressing this question.

Proposition 5.9.3. *$D_0(\mathcal{A})$ is free if and only if for a general line $L \subseteq \mathbb{P}(W)$, the restriction map $D_0(\mathcal{A}) \to D_0(\mathcal{A}) \mid_L$ is surjective.*

Proof. The forward implication is clear. For the reverse implication, we apply a corollary of Saito's Criterion which can be found as theorem 4.23 of [OT92]. Namely $D_0(\mathcal{A})$ is free if there exists $\theta_1, .., \theta_n \in D_0(\mathcal{A}_Z)$ which are linearly independent over the projective coordinate ring, S, of $\mathbb{P}(W)$, and where $\sum_{i=1}^n \deg(\theta_i) = |\mathcal{A}| - 1$.

So suppose that $\mathrm{res}_L : D_0(\mathcal{A}) \to D_0(\mathcal{A}_Z) \mid_L$ is surjective for a general line L. Let $\bar{\theta}_1, ..., \bar{\theta}_n$ be a $S/I(L)$-basis of $D_0(\mathcal{A}_Z) \mid_L$, then for each i we can find $\theta_i \in D_0(\mathcal{A})$ so that $\mathrm{res}_L(\theta_i) = \bar{\theta}_i$. As $\sum_{i=1}^n \deg(\theta_i) = \sum_{i=1}^n \deg(\bar{\theta}_i) = |\mathcal{A}| - 1$ it suffices to show that the θ_i are linearly independent over S. Yet if $\sum_{i=1}^n s_i \theta_i = 0$ for some $s_i \in S$ and some index j, then $\sum_{i=1}^n \bar{s}_i \bar{\theta}_i = 0$ in $D_0(\mathcal{A}_z) \mid_L$. As L is general if $s_j \neq 0$ for some index j then we can assume that $s_j \notin I(L)$ which gives a non-trivial relation among $\bar{\theta}_1, .., \bar{\theta}_n$ and a contradiction. □

Corollary 5.9.4. *$\mathcal{A}_Z$ is a free arrangement if and only if the evaluation map $\varepsilon_Q : I^{\gg}(Z) \to I_Q^{\gg}(Z)$ is surjective for general Q.*

Theorem 5.6.8 has some applications to Terao's conjecture, namely we give a new criterion for determining Freeness.

Proposition 5.9.5. *Let $\mathcal{A} \subseteq \mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}(S)$ with splitting type (a_1, a_2). If $a_2 - a_1 \geq 2$, then $D_0(\mathcal{A})$ is free if and only if it has a minimal generator in degree a_2.*

Proof. We use the criterion from proposition 5.9.3. As $a_2 - a_1 > 2$, then we may apply theorem 5.6.8 to see there nonzero $\theta_1 \in [D_0(\mathcal{A})]_{a_1}$ and so the image of the restriction map contains the generator of $D_0(\mathcal{A}) \mid_L$ in degree a_1. If $D_0(\mathcal{A})$ has a minimal generator θ_2 in degree a_2, then $\theta_2 \neq f\theta_1$ for any $f \in S$. For a general line L, we still have $\mathrm{res}_L(\theta_2) \notin S/I(L)\,\mathrm{res}_L(\theta_1)$ and conclude that $\{\mathrm{res}_L(\theta_1), \mathrm{res}_L(\theta_2)\}$ is a generating set for $D_0(\mathcal{A}) \mid_L$. ☐

Additionally, it is well known that freeness can be determined from combinatorics and the splitting type. More precisely,

Proposition 5.9.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be hyperplane arrangements in $\mathbb{P}^n$, and suppose $\mathcal{A}$ and $\mathcal{B}$ have isomorphic intersection lattices. Suppose that $D_0(\mathcal{A})$ is free, then $D_0(\mathcal{B})$ is free if and only if it has the same splitting type as $\mathcal{A}$.*

Proof. By a theorem of Terao $c_2(D_0(\mathcal{A}))$ is determined solely by $L_{\mathcal{A}}$. The result is now a consequence of the criterion that $D_0(\mathcal{A})$ is free if and only if $c_2(D_0(\mathcal{A})) = a_1 a_2$ where (a_1, a_2) is the splitting type, see for instance [BR10]. ☐

This characterization allows us to generalize a theorem of [Sch03] which was stated only for balanced free arrangements in $\mathbb{P}^2_{\mathbb{C}}$. Here balanced means free arrangements with splitting type (a, a) or $(a, a+1)$.

Theorem 5.9.7. *For a finitely generated graded module M, let $\alpha(M)$ denote the initial degree of M, that is*

$$\alpha(M) := \inf\{d \in \mathbb{Z} \mid [M]_d \neq 0\}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorially equivalent line arrangements $\mathbb{P}^2_{\mathbb{K}}$. If $D_0(\mathcal{A})$ is free, then

$$\alpha(D_0(\mathcal{B})) \leq \alpha(D_0(\mathcal{A}))$$

with equality if and only if $\mathcal{B}$ is free.

In particular, if $\mathcal{A}$ is free with exponents $(1, a, b)$ and $\mathcal{B}$ is not free, then $D_0(\mathcal{B})$ has a generator in degree $< a$ and all other minimal generators are in degree $> b$.

Remark 5.9.8. Note if $\mathbb{K} \subseteq \mathbb{C}$, the following argument can be slightly simplified by applying theorem 5.6.8.

Proof. The reverse direction is immediate as in that case $D_0(\mathcal{A})$ and $D_0(\mathcal{B})$ are isomorphic.

To prove the forward implication, we apply the characterization given in 5.9.6. Hence, assume that $D_0(\mathcal{A})$ has splitting type (a_1, a_2), and $D_0(\mathcal{B})$ has splitting type (b_0, b_1), with $(b_0, b_1) \neq (a_0, a_1)$. It suffices to show that $\alpha(D_0(\mathcal{B})) < \alpha(D_0(\mathcal{A}))$.

By [Yuz93] freeness is an open property. Hence, if $\mathcal{B}$ is not free it lies on closed subvariety of $V_{L(\mathcal{A})}$ the variety parameterizing arrangements with intersection lattice isomorphic to $L(\mathcal{A})$. We can view $D(\mathcal{B})$ as the kernel of the linear map $\operatorname{Der}(S) \to \prod_{H \in \mathcal{B}} S/(I(H))$ which maps $\theta \mapsto (\theta(\ell_H))_{\{H \in \mathcal{B}\}}$, so by lower semicontinuity of rank we may conclude that $\dim[D_0(\mathcal{B})]_d \geq \dim[D_0(\mathcal{A})]_d$ for all d. Applying the same argument to the restriction of $D_0(\mathcal{B})_d$ to a general line, we see that $b_0 < a_0 \leq a_1 < b_1$. As $\dim[D_0(\mathcal{B})]_{a_0} \geq \dim[D_0(\mathcal{A})]_{a_0} > 0$, we can apply proposition 5.6.2 to get

$$\alpha(D_0(\mathcal{B})) = b_0 < a_0 = \alpha(D_0(\mathcal{A})).$$

The final sentence follows from this and proposition 5.9.5. □

One corollary of the above theorem is an extension of a theorem of [FV14] over $\mathbb{C}$, to positive characteristic.

Corollary 5.9.9. *If $\mathcal{A} \subseteq \mathbb{P}^2_{\mathbb{K}}$ is a free arrangement with splitting type (a_1, a_2) and some point $P \in \mathbb{P}^2_{\mathbb{K}}$ is incident to at least a_1 lines of $\mathcal{A}$. Then any arrangement over $\mathbb{P}^2_{\mathbb{K}}$ combinatorially equivalent to $\mathcal{A}$ is also free.*

Proof. Let $\mathcal{B}$ be an arrangement that is combinatorially equivalent to $\mathcal{A}$, and let (b_1, b_2) denote it's splitting type. Dualizing the problem statement, we see that there is a line H containing at least a_1 points of $\mathcal{B}^\perp$ so by theorem 5.4.1 and the prior theorem we have $a_1 - 1 \leq b_1 \leq a_1$.

Let h denote the linear form defining this line. Furthermore for a general Q, let g denote the product of linear forms through Q and each point not on H, then $hg \in [I^{\gg}(\mathcal{B}^\perp)]_{a_2+2}$. If $b_1 = a_1 - 1$, then hg would correspond by theorem 5.3.8 to a minimal generator of $D_0(\mathcal{B})$ in degree $b_2 = a_2 + 1$. The prior theorem together with proposition 5.9.6 now gives a contradiction. □

We now close by discussing connections between Terao's conjecture and a conjecture due to Dirac. It was conjectured in [Dir51] that for every finite set of points non collinear points $Z \subseteq \mathbb{P}^2_{\mathbb{R}}$, that there is always some $Q \in Z$ so that

$$|L_Q(Z)| = |\{\operatorname{Span}(Q, P) \mid P \in Z \setminus Q\}| \geq \frac{|Z|}{2}.$$

However, some counterexamples have been found to the original formulation (see [Grü72]). This has lead to two reformulations of the original conjecture which we reprint below.

Conjecture 5.9.10 (Weak Dirac Problem)**.** *Determine the smallest constant C, so that for every finite set of noncollinear points $Z \subseteq \mathbb{P}^2_{\mathbb{R}}$, there exists some $Q \in Z$ where*

$$|L_Q(Z)| \geq \frac{|Z|}{C}$$

Conjecture 5.9.11 (Strong Dirac Conjecture)**.** *There exists some constant $c_0 > 0$ so that for every set of finite noncollinear points $Z \subseteq \mathbb{R}^2$, there exists some $Q \in Z$ so that*

$$|L_Q(Z)| \geq \frac{Z}{2} - c_0$$

Counterexamples have been found to Dirac's Original Conjecture for every odd $n = |Z|$ with the exception of those n of the form $n = 12k + 11$ with $k \geq 4$ (see [AIKN11]). Despite that the known counterexamples only barely break the original conjecture bound. Most satisfy the Strong Dirac Conjecture with $c_0 = 1/2$ and all but finitely many satisfy the conjecture with $c_0 = 3/2$.

We now show that any minimal counterexample to Terao's Conjecture for real line arrangements must itself be a counterexample to the original Conjecture of G.Dirac, and must be extremal in the regards to the other two cases.

Theorem 5.9.12. *Let $\mathcal{A}$ and $\mathcal{B} \subseteq \mathbb{P}^2$ be real (or complex) line arrangements, which form a counter example to Terao's conjecture. Meaning $L_{\mathcal{A}} \cong L_{\mathcal{B}}$, but $D_0(\mathcal{A})$ is free with splitting type (a_1, a_2) where as $D_0(\mathcal{B})$ is not free. Furthermore, suppose there is no pair of lines $(L, L') \in \mathcal{A} \times \mathcal{B}$ we can remove to get subarrangements $\mathcal{A}' = \mathcal{A} \setminus \{L\}$ and $\mathcal{B}' = \mathcal{B} \setminus \{L'\}$ forming a smaller counterexample.*

Then letting $\mathcal{A}^\perp$ be the set of points dual to $\mathcal{A}$ we have

$$|L_P(Z)| \leq a_1 \leq \left\lfloor \frac{|Z| - 1}{2} \right\rfloor .$$

Our proof of the above theorem relies on the following proposition which seems useful in it's own right. It is related to Terao's well known Addition-Deletion Theorem

Proposition 5.9.13. *Let $\mathcal{A}_z \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a free line arrangement with splitting type (a_1, a_2) and Z the dual set of points. If there is some $P \in Z$ with $|L_P(Z)| > a_1 + 1$, then $|L_P(Z)| = a_2 + 1$ and $\mathcal{A}_W$ is free where $W = Z \setminus \{P\}$.*

Proof. By theorem B of [BR10], $c_2(D_0(\mathcal{A}_Z)) \geq a_1 a_2$, and $\mathcal{A}$ is free if and only if equality holds. Furthermore, if $L_P(Z) > a_1 + 1$, then letting $F \in [I^{\gg}(Z)]$ it follows by lemma 5.7.5 that $\varepsilon_P(F) = 0$. Then by lemma 5.7.1 the linear form, ℓ_p, defining the line dual to P must divide F. However, then we necessarily have $F/\ell_P \in [I^{\gg}(W)]_{a_1} \cong [D_0(\mathcal{A})]_{a_1 - 1}$ so $\mathcal{A}_W$ must have splitting type $(a_1 - 1, a_2)$.

We note that it suffices to show that $D_0(\mathcal{A}_W)$ is free, since Terao's Famous Addition-Deletion Formula then ensures that $L_P(W) = a_2 + 1$. Yet this follows since if F and G freely generate $D_0(\mathcal{A}_Z)$, then F/ℓ_P and G must generate $D_0(\mathcal{A}_W)$. □

Proof of theorem 5.9.12. By the preceding proposition there exists no $P \in Z$ with $|L_P(Z)| > a_1 + 1$. Furthermore, by Terao's Addition-Deletion formula there is no $P \in L_P(Z)$ with $|L_p(Z)| = a_1 + 1$, since then letting $\mathcal{A}' = \mathcal{A} \setminus \{\ell_0 = 0\}$ we would get a smaller counterexample. Hence, for all $P \in Z$ we have

$$|L_p(Z)| \leq a_1 \leq \frac{|Z| - 1}{2}$$

establishing the result. □

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Vita

William Trok

Education

University of Kentucky Lexington, KY, USA.

- Ph.D., Mathematics August 2020
  - *Advisor:* Uwe Nagel

University of the Pacific Stockton, CA, USA.

- B.S., Mathematics May 2014
  - Minor in Computer Science

Awards and Honors

University of Kentucky

- Max Steckler Fellowship (2015-2017)
- Summer Research Fellowship (Summer 2017)

Publications

- U. Nagel and B. Trok, *Segre's Regularity Bound for Fat Point Schemes.* Annali della Scuola Normale Superiore di Pisa.
- L. Farnik, F. Gallupi, L. Sodomaco, and B. Trok, *On the unique unexpected quartic in* $\mathbb{P}^2$. (To appear in Journal of Algebraic Combinatorics)
- U. Nagel and B. Trok, *Interpolation and the Weak Lefschetz Property.* Transactions of the American Mathematical Society.
- B. Trok, *Projective Duality, Unexpected Hypersurfaces and Logarithmic Derivations of Hyperplane Arrangements.* Preprint at ArXiv:2003.02397.

Professional Teaching Experience

University of Kentucky 2014-2020.