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Local lagged adapted generalized method of moments dynamic process

Gangaram S. Ladde

Olusegun Michael Otunuga

Nathan G. Ladde

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(54) **LOCAL LAGGED ADAPTED GENERALIZED METHOD OF MOMENTS DYNAMIC PROCESS**

(71) Applicants: **Gangaram S. Ladde**, Temple Terrace, FL (US); **Olusegun Michael Otunuga**, Huntington, WV (US); **Nathan G. Ladde**, Atlanta, GA (US)

(72) Inventors: **Gangaram S. Ladde**, Temple Terrace, FL (US); **Olusegun Michael Otunuga**, Huntington, WV (US); **Nathan G. Ladde**, Atlanta, GA (US)

(73) Assignee: **University of South Florida**, Tampa, FL (US)

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G06F 17/50	(2006.01)
G06F 17/18	(2006.01)
G06Q 10/04	(2012.01)

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CPC ***G06F 17/5009*** (2013.01); ***G06F 17/18*** (2013.01); *G06F 2217/10* (2013.01); *G06Q 10/04* (2013.01)

(58) **Field of Classification Search**
None
See application file for complete search history.

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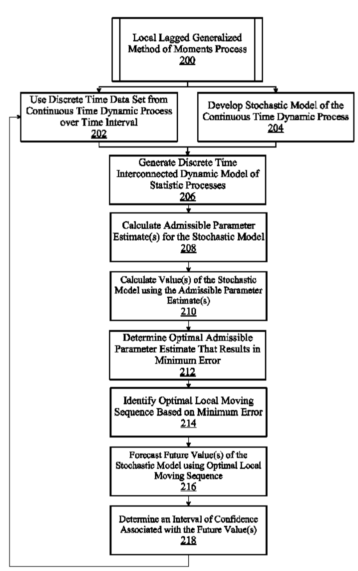
Primary Examiner — Kibrom K Gebresilassie
(74) *Attorney, Agent, or Firm* — Thomas | Horstemeyer, LLP

(57) **ABSTRACT**

Aspects of a local lagged adapted generalized method of moments (LLGMM) dynamic process are described herein. In one embodiment, the LLGMM process includes obtaining a discrete time data set as past state information of a continuous time dynamic process over a time interval, developing a stochastic model of the continuous time dynamic process, generating a discrete time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP) based on the stochastic model, and calculating a plurality of admissible parameter estimates for the stochastic model using the DTIDMLSMVSP. Further, in some embodiments, the process further includes, for at least one of the plurality of admissible parameter estimates, calculating a state value of the stochastic model to gather a plurality of state values, and determining an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values.

14 Claims, 12 Drawing Sheets



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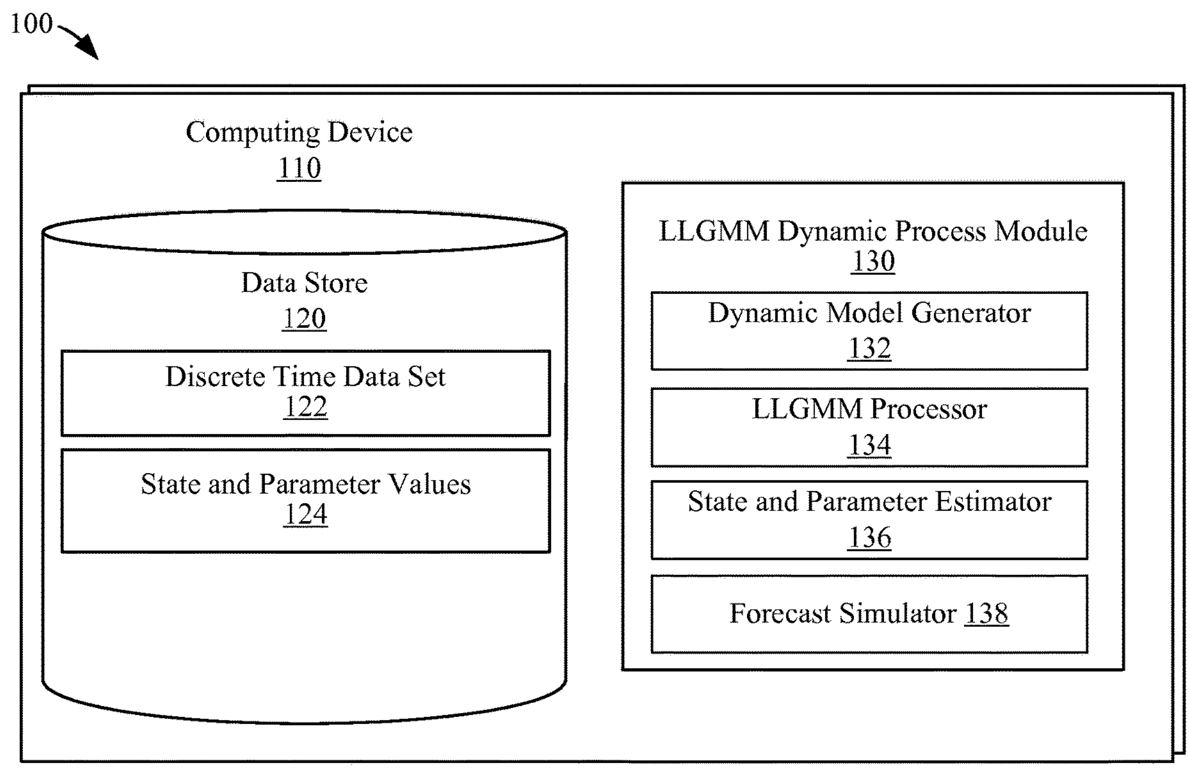
100
Computing Device
110
Data Store
120
Discrete Time Data Set
122
State and Parameter Values
124
LLGMM Dynamic Process Module
130
Dynamic Model Generator
132
LLGMM Processor
134
State and Parameter Estimator
136
Forecast Simulator 138

FIG. 1

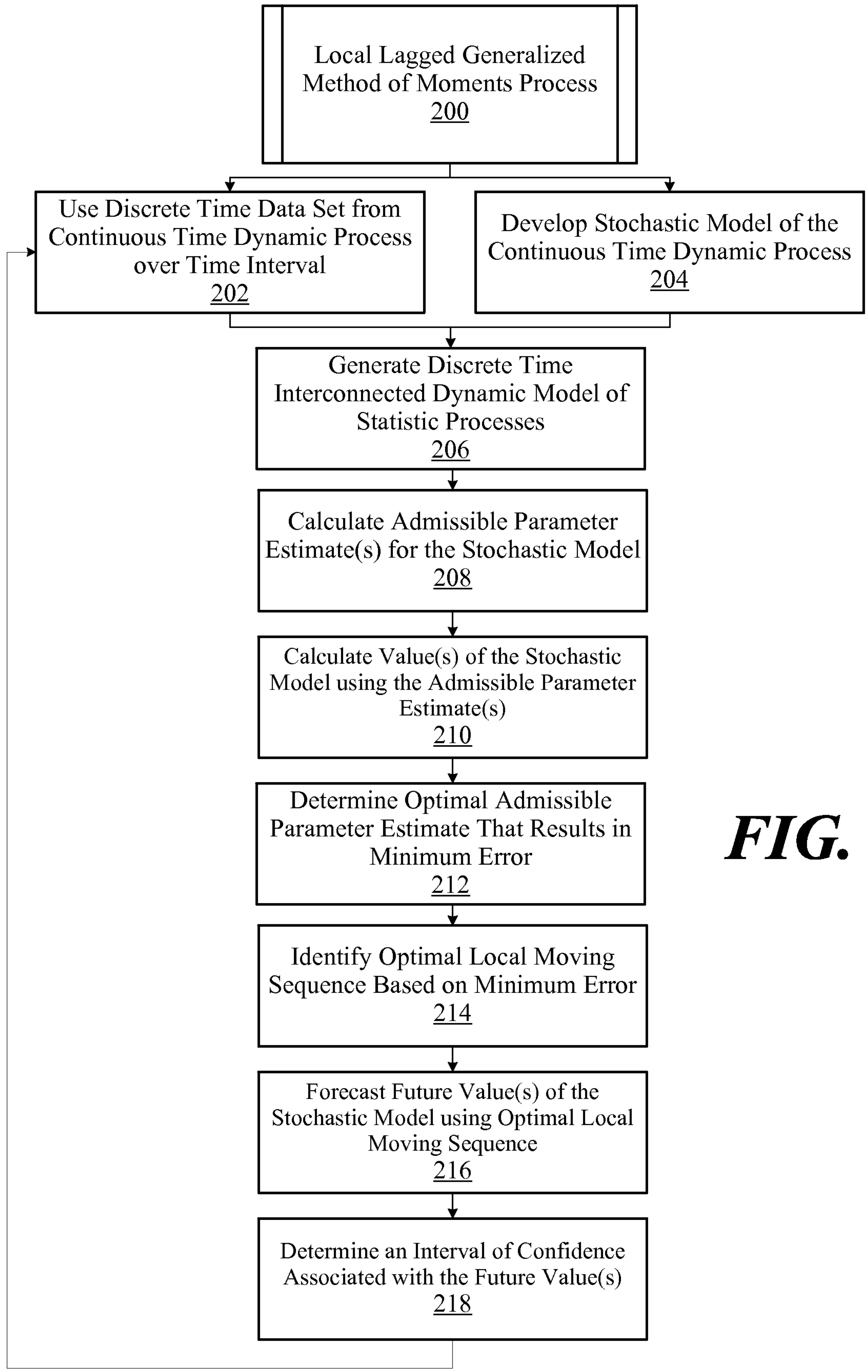
Local Lagged Generalized Method of Moments Process 200
Use Discrete Time Data Set from Continuous Time Dynamic Process over Time Interval 202
Develop Stochastic Model of the Continuous Time Dynamic Process 204
Generate Discrete Time Interconnected Dynamic Model of Statistic Processes 206
Calculate Admissible Parameter Estimate(s) for the Stochastic Model 208
Calculate Value(s) of the Stochastic Model using the Admissible Parameter Estimate(s) 210
Determine Optimal Admissible Parameter Estimate That Results in Minimum Error 212
Identify Optimal Local Moving Sequence Based on Minimum Error 214
Forecast Future Value(s) of the Stochastic Model using Optimal Local Moving Sequence 216
Determine an Interval of Confidence Associated with the Future Value(s) 218

FIG. 2

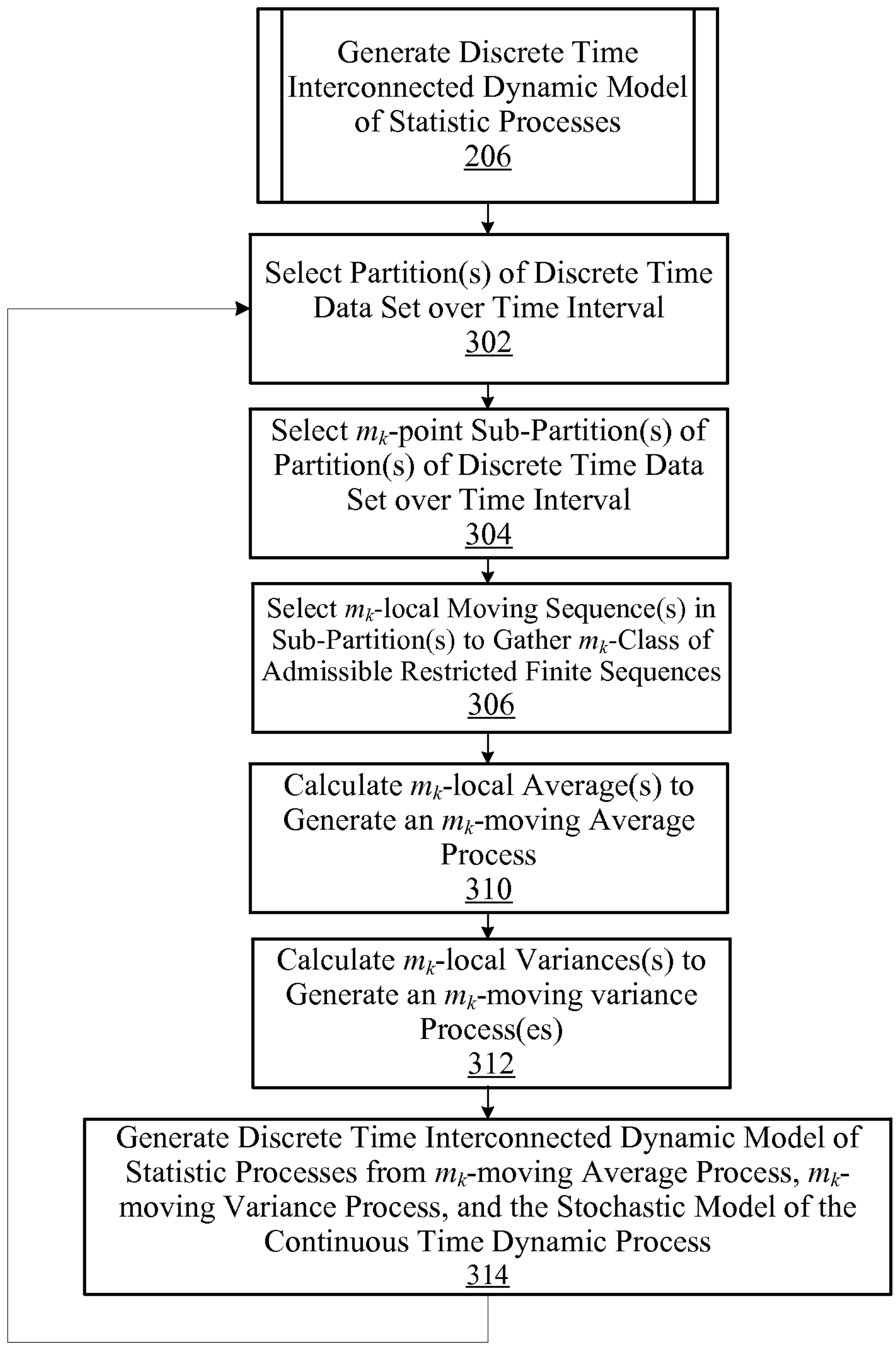
Generate Discrete Time Interconnected Dynamic Model of Statistic Processes
206
Select Partition(s) of Discrete Time Data Set over Time Interval
302
Select m_k-point Sub-Partition(s) of Partition(s) of Discrete Time Data Set over Time Interval
304
Select m_k-local Moving Sequence(s) in Sub-Partition(s) to Gather m_k-Class of Admissible Restricted Finite Sequences
306
Calculate m_k-local Average(s) to Generate an m_k-moving Average Process
310
Calculate m_k-local Variances(s) to Generate an m_k-moving variance Process(es)
312
Generate Discrete Time Interconnected Dynamic Model of Statistic Processes from m_k-moving Average Process, m_k-moving Variance Process, and the Stochastic Model of the Continuous Time Dynamic Process
314

FIG. 3

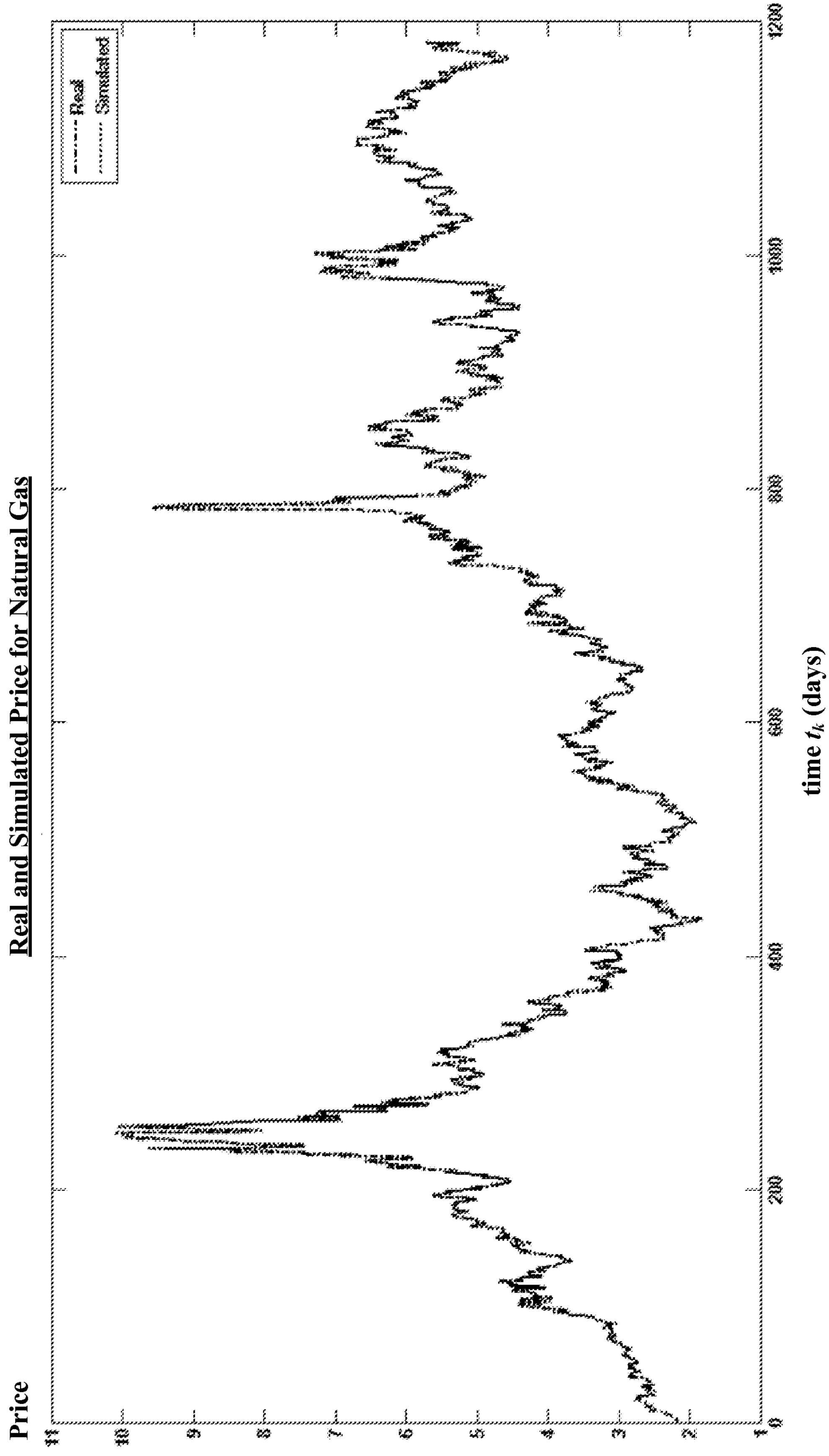
Real and Simulated Price for Natural Gas
Price
11
10
9
8
7
6
5
4
3
2
1
0
200
400
600
800
1000
1200
time t_k (days)
Real
Simulated

FIG. 4A

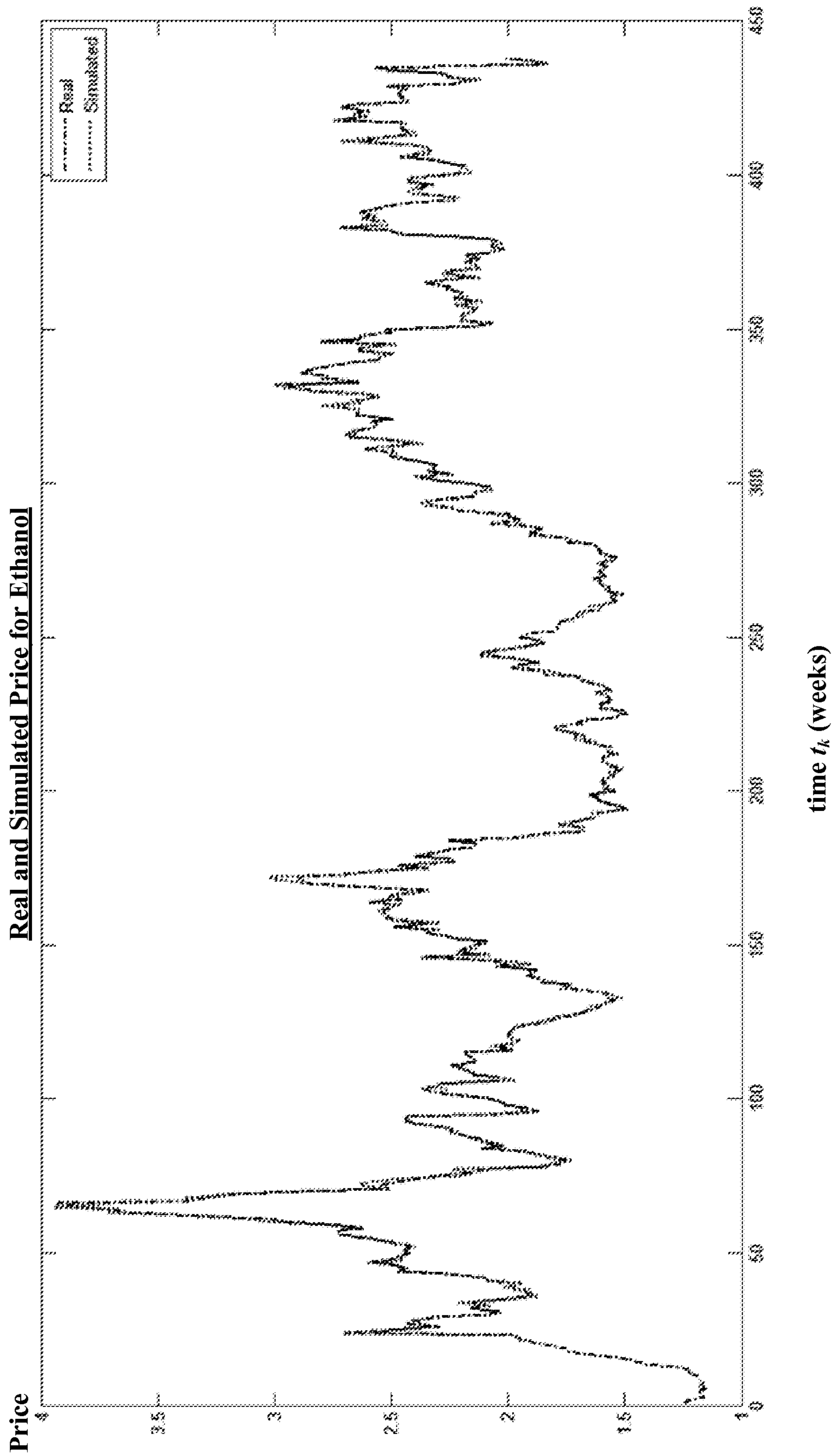
Real and Simulated Price for Ethanol
Price
4
3.5
3
2.5
2
1.5
1
Real
Simulated
0
50
100
150
200
250
300
350
400
450
time t_k (weeks)

FIG. 4B

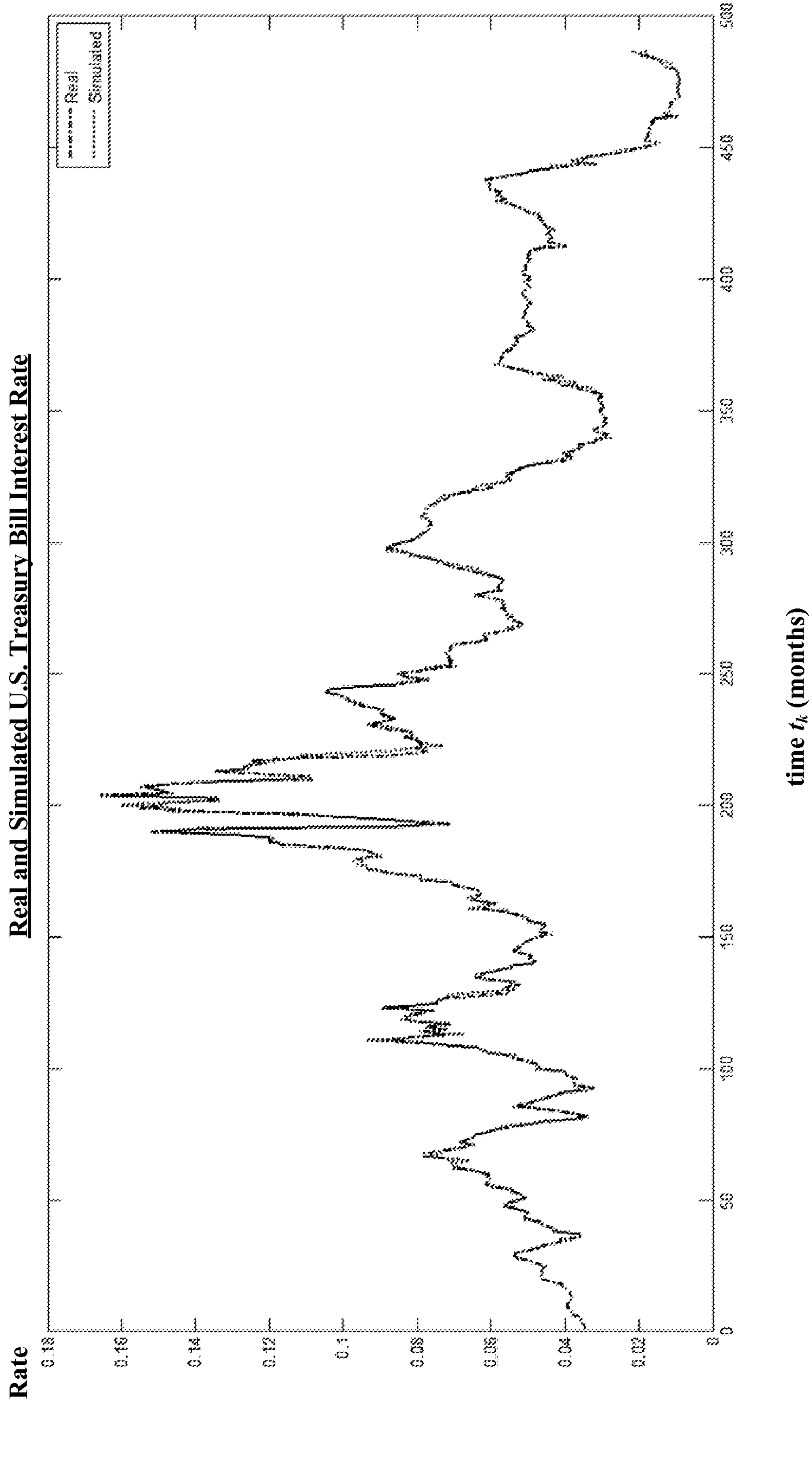
Real and Simulated U.S. Treasury Bill Interest Rate
Real
Simulated
Rate
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
50
100
150
200
250
300
350
400
450
500
time t_k (months)

FIG. 5A

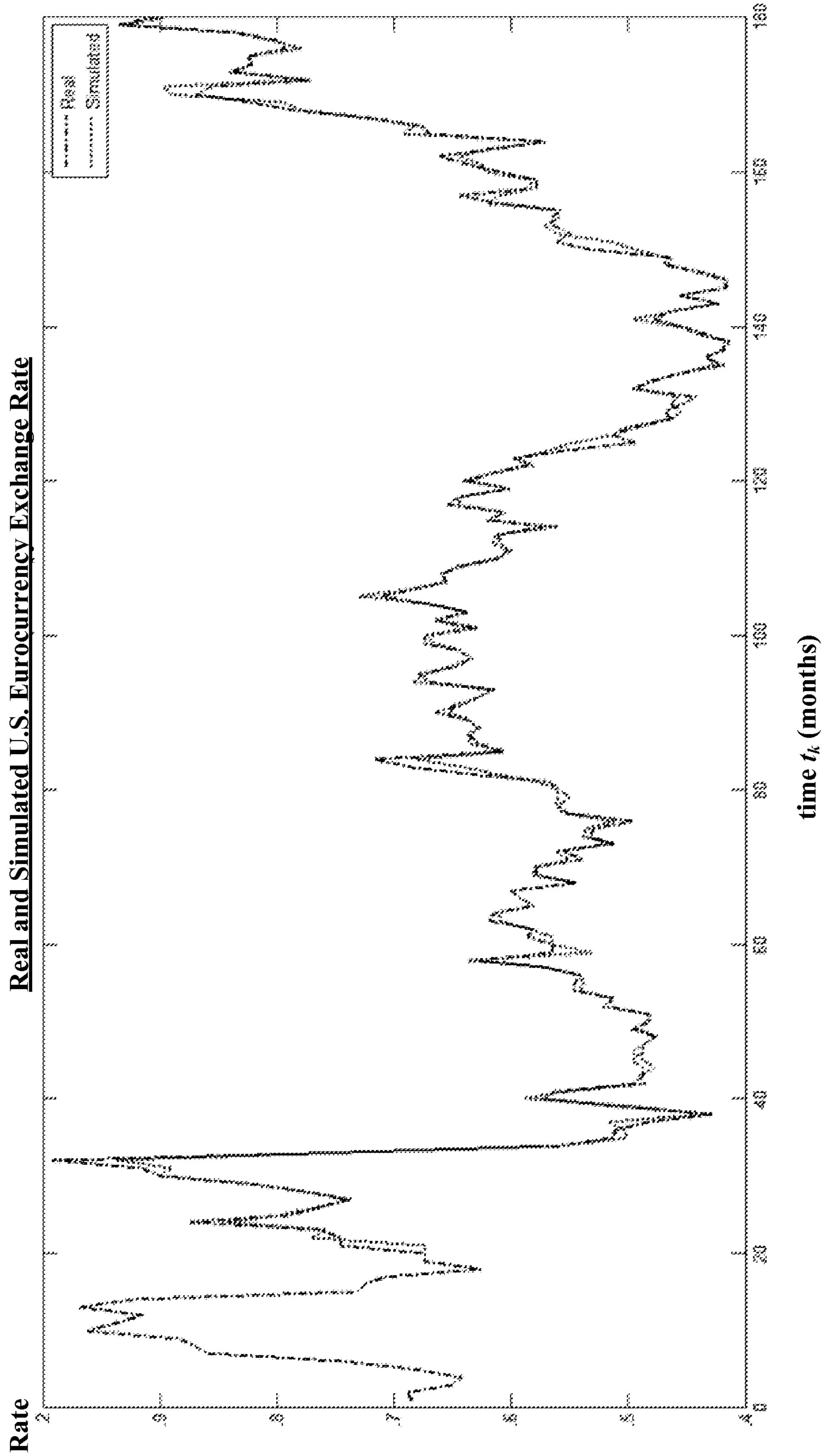
Real and Simulated U.S. Eurocurrency Exchange Rate
Rate
Real
Simulated
0
20
40
60
80
100
120
140
160
180
time t_k (months)

FIG. 5B

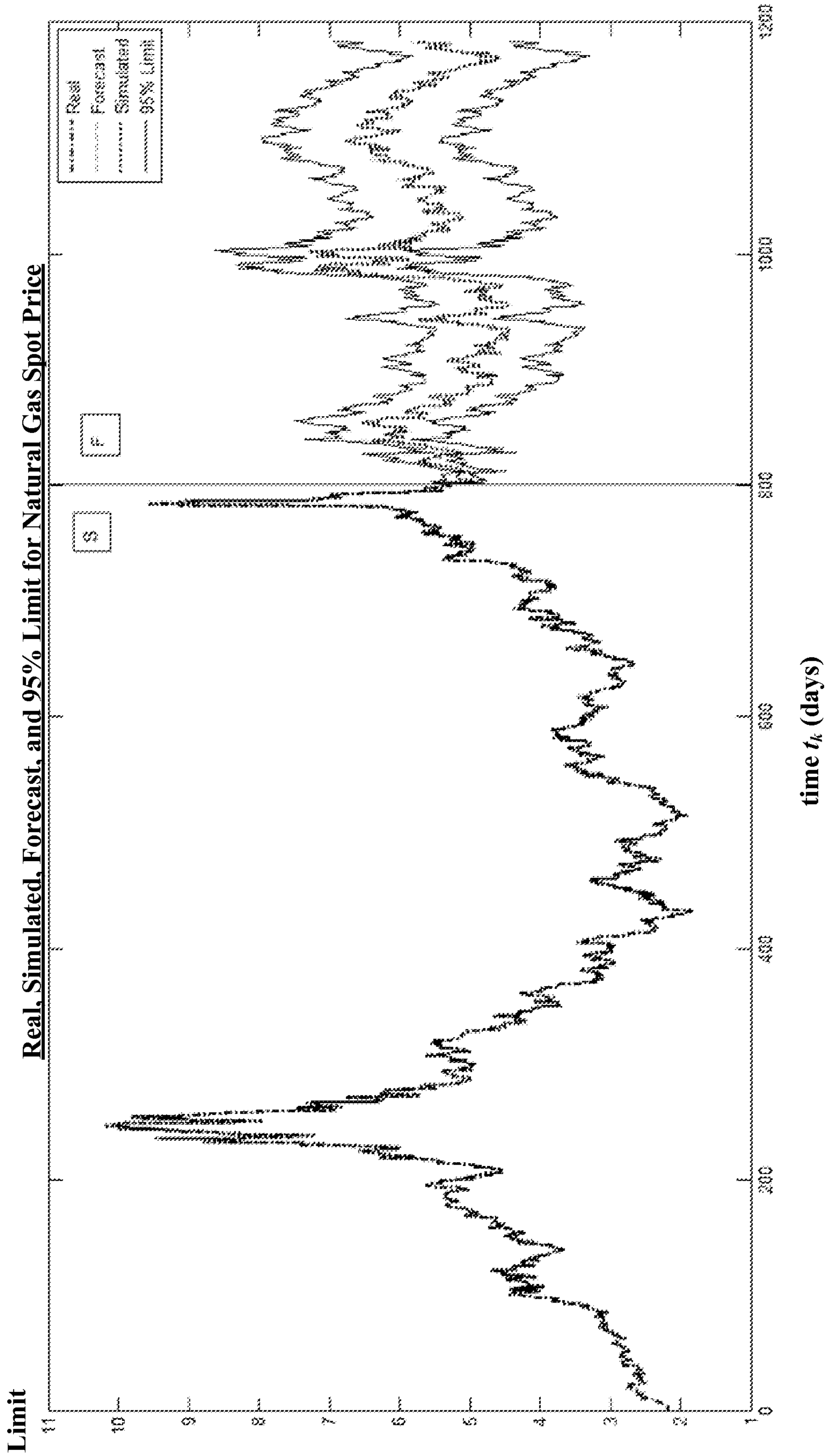
Real, Simulated, Forecast, and 95% Limit for Natural Gas Spot Price
Limit
11
10
9
8
7
6
5
4
3
2
1
0
200
400
600
800
1000
1200
time t_k (days)
S
F
Real
Forecast
Simulated
95% Limit

FIG. 6A

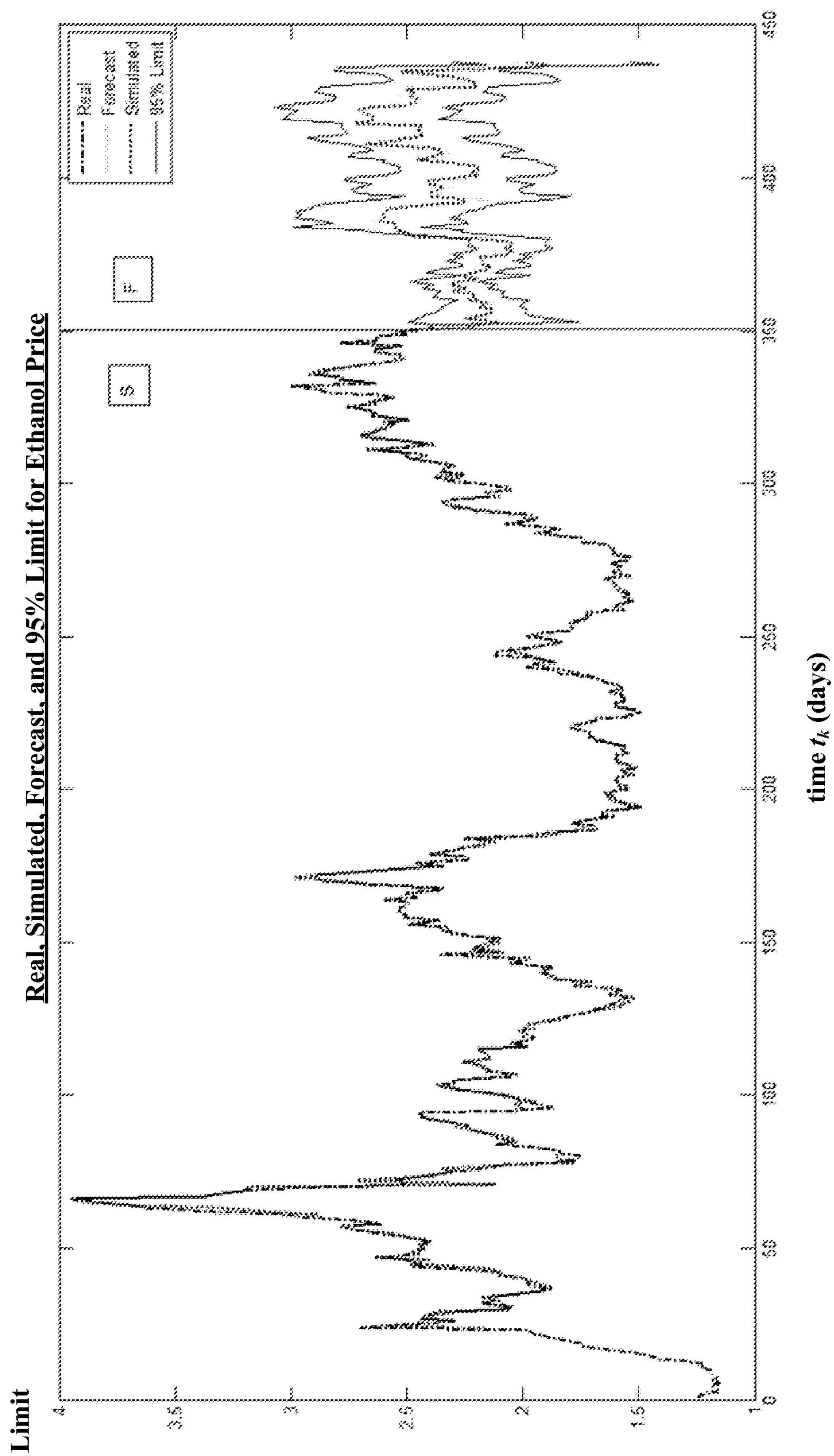
Real, Simulated, Forecast, and 95% Limit for Ethanol Price
Limit
time t_k (days)
Real
Forecast
Simulated
95% Limit
S
F
0
50
100
150
200
250
300
350
400
450
1
1.5
2
2.5
3
3.5
4

FIG. 6B

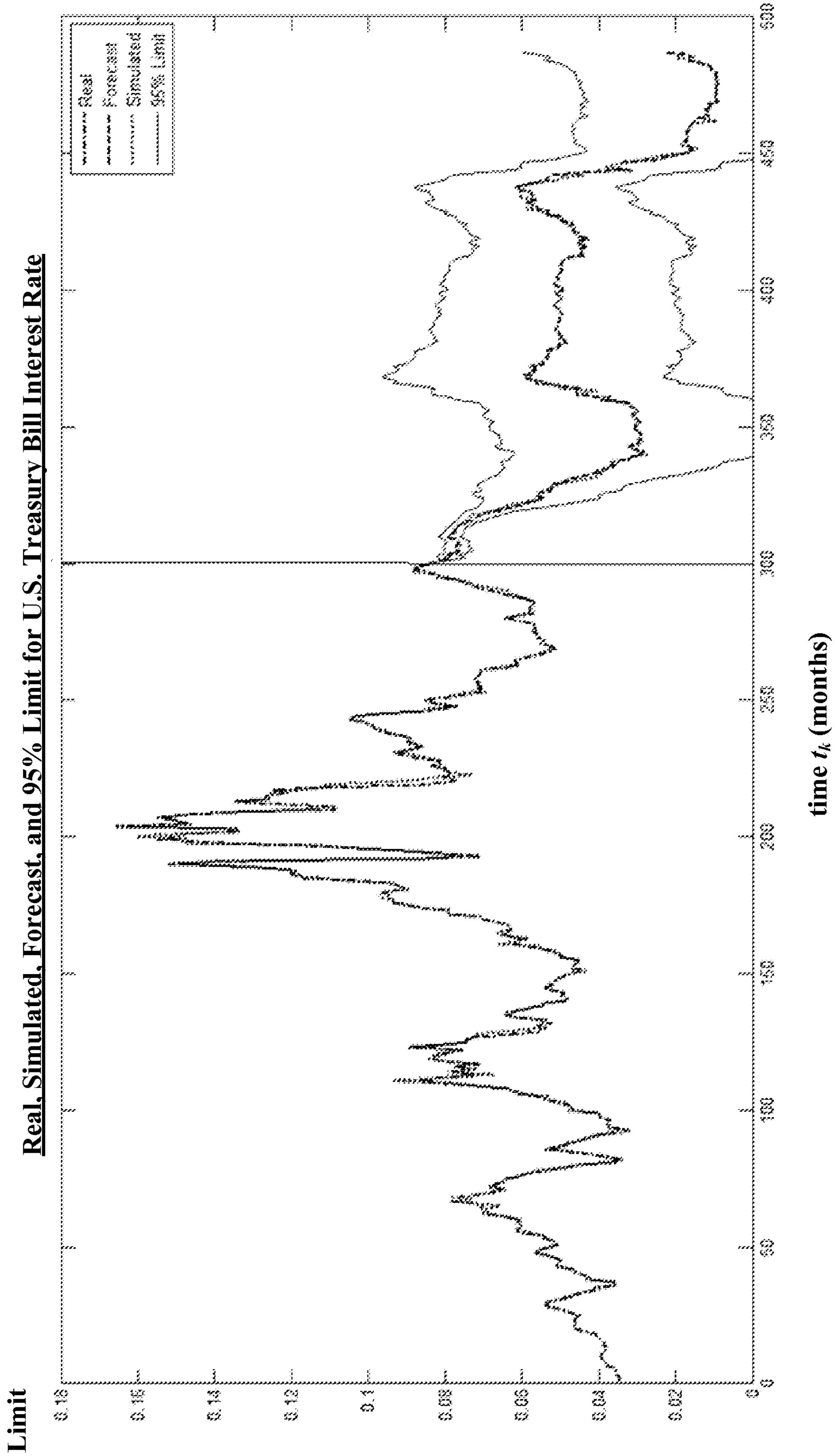
Real, Simulated, Forecast, and 95% Limit for U.S. Treasury Bill Interest Rate
Limit
time t_k (months)
Real
Forecast
Simulated
95% Limit
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0
50
100
150
200
250
300
350
400
450
500

FIG. 7A

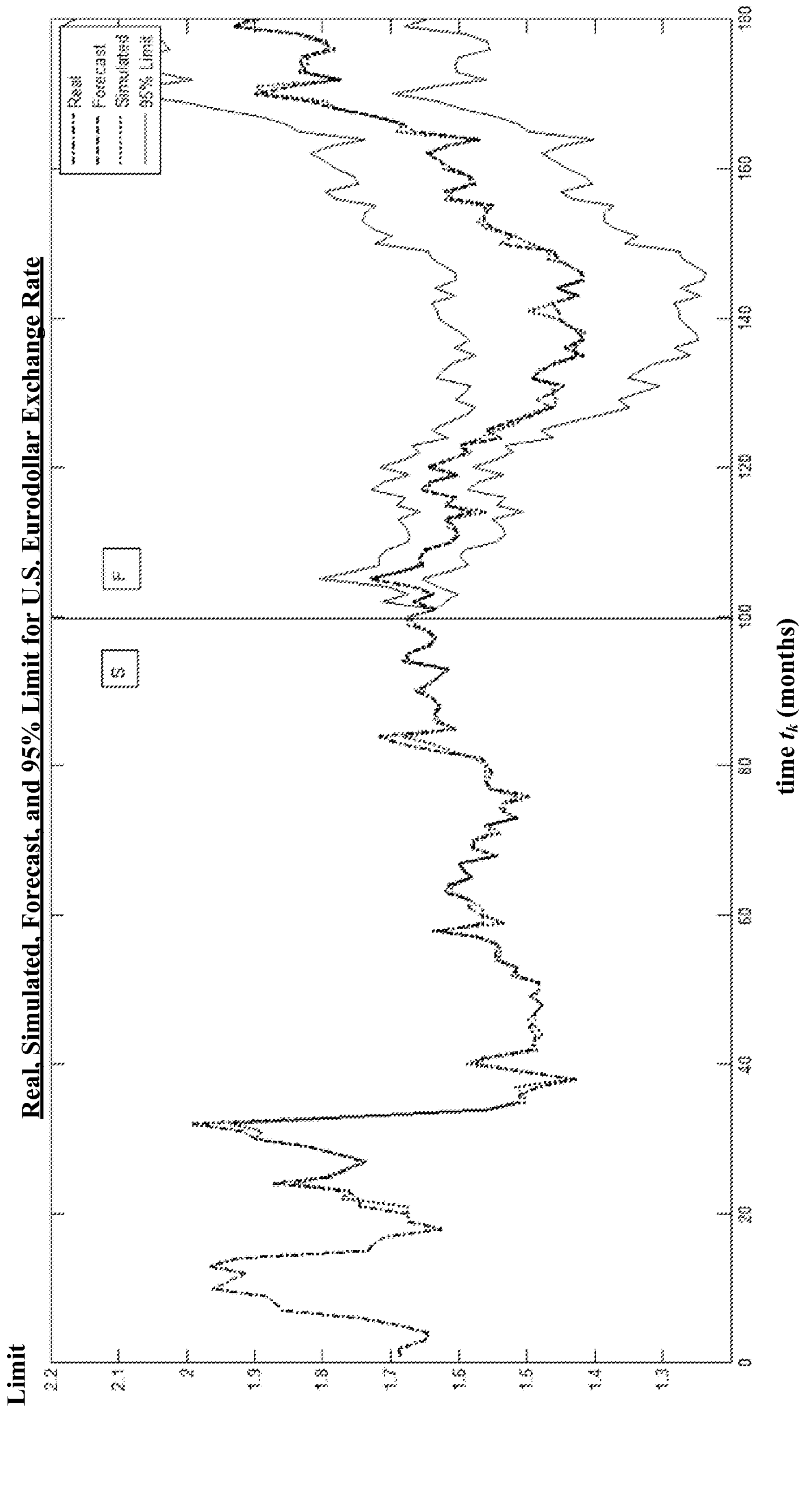
Real, Simulated, Forecast, and 95% Limit for U.S. Eurodollar Exchange Rate
Limit
2.2
2.1
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0
20
40
60
80
100
120
140
160
180
S
F
Real
Forecast
Simulated
95% Limit
time t_k (months)

FIG. 7B

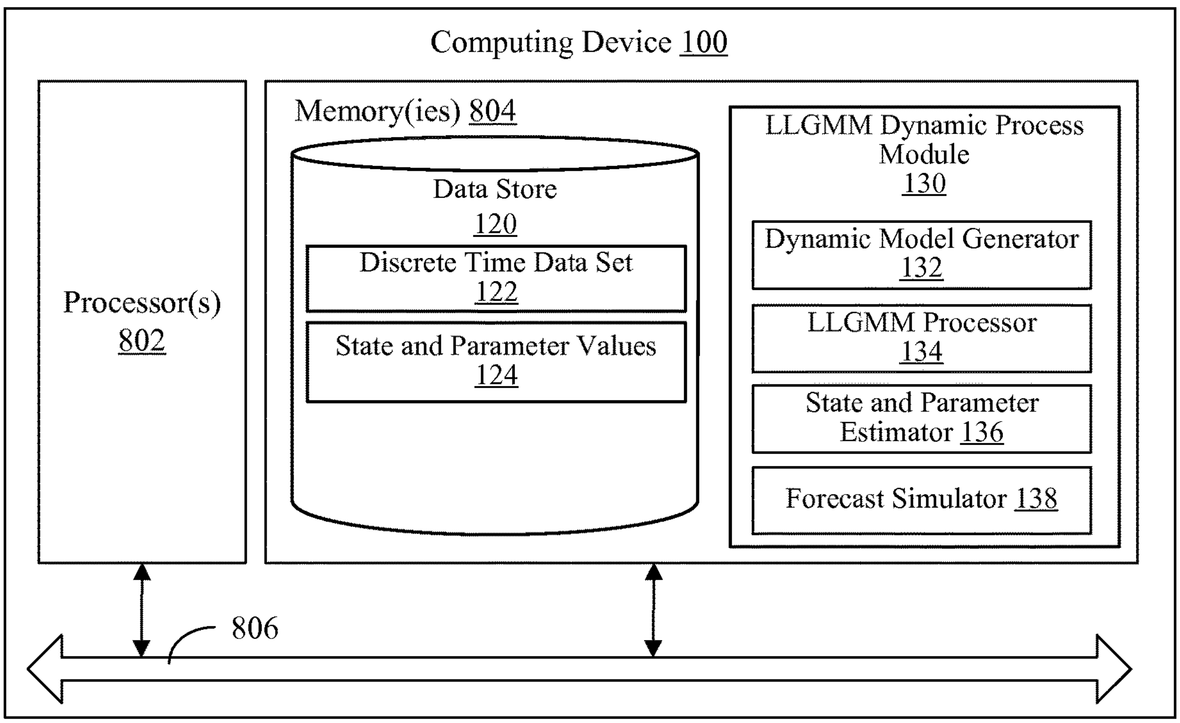
Computing Device 100
Processor(s) 802
Memory(ies) 804
Data Store 120
Discrete Time Data Set 122
State and Parameter Values 124
LLGMM Dynamic Process Module 130
Dynamic Model Generator 132
LLGMM Processor 134
State and Parameter Estimator 136
Forecast Simulator 138
806

FIG. 8

LOCAL LAGGED ADAPTED GENERALIZED METHOD OF MOMENTS DYNAMIC PROCESS

CROSS-REFERENCE TO RELATED APPLICATIONS

This application claims the benefit of U.S. Provisional Application No. 62/068,848, filed Oct. 27, 2014, the entire contents of which are hereby incorporated herein by reference. The application also claims the benefit of U.S. Provisional Application No. 62/246,189, filed Oct. 26, 2015, the entire contents of which are hereby incorporated herein by reference.

GOVERNMENT LICENSE RIGHTS

This invention was made with government support under Grant Numbers W911NF-12-1-0090 and W911NF-15-1-0182 awarded by the Army Research Office. The government has certain rights in the invention.

BACKGROUND

Tools for analyzing and managing large collections of data are becoming increasingly important. For example, data models between various commodities can be analyzed to determine whether a collaborative or competitive relationship exists between the commodities. However, traditional methods of verifying and validating nonlinear time series type data sets can encounter state and parameter estimation errors.

BRIEF DESCRIPTION OF THE DRAWINGS

Many aspects of the present disclosure can be better understood with reference to the following drawings. The components in the drawings are not necessarily to scale, emphasis instead being placed upon clearly illustrating the principles of the present disclosure. Further, in the drawings, like reference numerals designate corresponding parts throughout the several views.

FIG. **1** illustrates an example computing environment for a local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **2** illustrates a local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **3** illustrates a process of generating a discrete time interconnected dynamic model of statistic processes in the process shown in FIG. **2** according to various aspects of the embodiments described herein.

FIG. **4**A illustrates real and simulated prices for natural gas using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **4**B illustrates real and simulated prices for ethanol using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **5**A illustrates real and simulated U.S. treasury bill interest rates using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **5**B illustrates real and simulated U.S. eurocurrency exchange rates using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **6**A illustrates the real, simulated, forecast, and 95% limit natural gas spot prices using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **6**B illustrates the real, simulated, forecast, and 95% limit ethanol prices using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **7**A illustrates the real, simulated, forecast, and 95% limit U.S. treasury bill interest rates using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **7**B illustrates the real, simulated, forecast, and 95% limit U.S. Eurodollar exchange rates using the local lagged adapted generalized method of moments dynamic process according to various aspects of the embodiments described herein.

FIG. **8** illustrates an example schematic block diagram of the computing device **100** shown in FIG. **1** according to various embodiments described herein.

The drawings illustrate only example embodiments and are therefore not to be considered limiting of the scope described herein, as other equally effective embodiments are within the scope and spirit of this disclosure. The elements and features shown in the drawings are not necessarily drawn to scale, emphasis instead being placed upon clearly illustrating the principles of the embodiments. Additionally, certain dimensions may be exaggerated to help visually convey certain principles. In the drawings, similar reference numerals between figures designate like or corresponding, but not necessarily the same, elements.

DETAILED DESCRIPTION

1. Introduction

The embodiments described herein are directed to the development and application of a local lagged adapted generalized method of moments (LLGMM) dynamic process. Various embodiments of the approach can include one or more of the following components: (1) developing a stochastic model of a continuous-time dynamic process, (2) developing one or more discrete time interconnected dynamic models of statistic processes, (3) utilizing Euler-type discretized schemes for non-linear and non-stationary systems of stochastic differential equations, (4) employing one or more lagged adaptive expectation processes for developing generalized method of moment/observation equations, (5) introducing conceptual and computational parameter estimation problems, (6) formulating a conceptual and computational state estimation scheme, and (7) defining a conditional mean square ϵ-best sub-optimal procedure.

The development of the LLGMM dynamic process is motivated by and applicable to parameter and state estimation problems in continuous-time nonlinear and non-stationary stochastic dynamic models in biological, chemical, engineering, energy commodity markets, financial, medical, physical and social science, and other fields. The approach result in a balance between model specification and model prescription of continuous-time dynamic processes and the development of discrete time interconnected dynamic models of local sample mean and variance statistic processes (DTIDMLSMVSP). DTIDMLSMVSP is the generalization of statistic (sample mean and variance) for random sample drawn from the static dynamic population problems. Further, it is also an alternative approach to the generalized autoregressive conditional heteroskedasticity (GARCH) model, and it provides an iterative scheme for updating statistic coefficients in a system of generalized method of moment/observation equations. Furthermore, the application of the LLGMM to various time-series data sets demonstrates its performance in forecasting and confidence-interval problems in applied statistics.

Most existing parameter and state estimation techniques are centered around the usage of either overall data sets, batched data sets, or local data sets drawn on an interval of finite length T. This leads to an overall parameter estimate on the interval of length T. The embodiments described herein apply a new approach, the LLGMM. The LLGMM is based on a foundation of: (a) the Itô-Doob Stochastic Calculus, (b) the formation of continuous-time differential equations for suitable functions of dynamic state with respect to original SDE (using Itô-Doob differential formula), (c) constructing corresponding Euler-type discretization schemes, (d) developing general discrete time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP), (e) the fundamental properties of solution process of system of stochastic differential equations, for example: existence, uniqueness, continuous dependence of parameters.

One of the goals of the parameter and state estimation problems is for model validation rather than model misspecification. For continuous-time dynamic model validation, existing real world data sets are utilized. This real world data is time varying and sampled, drawn, or recorded at discrete times on a time interval of finite length. In view of this, instead of using an existing econometric specification/Euler-type numerical scheme, a stochastic numerical approximation scheme is constructed using continuous time stochastic differential equations for the LLGMM process described herein.

In almost all real world dynamic modeling problems, future states of continuous time dynamic processes are influenced by past state history in connection with response/reaction time delay processes influencing the present states. That is, many discrete time dynamic models depend on the past state of a system. The influence of state history, the concept of lagged adaptive expectation process, and the idea of a moving average lead to the development of the general DTIDMLSMVSP. Extensions of the discrete time sample mean and variance statistic processes are: (a) to initiate the use of a discrete time interconnected dynamic approach in parallel with the continuous-time dynamic process, (b) to shorten the computation time, and (c) to significantly reduce state error estimates.

Utilizing the Euler-type stochastic discretization, for example, of the continuous time stochastic differential equations/moment/observations and the discrete time interconnected dynamic approach in parallel with the continuous-time dynamic process (and the given real world time series data and the method of moments), systems of local moment/observation equations can be constructed. Using the DTIDMLSMVSP and the lagged adaptive expectation process for developing generalized method of moment equations, the notions of data coordination, theoretical iterative and simulation schedule processes, parameter estimation, state simulation and mean square optimal procedures are introduced. The approach described herein is more suitable and robust for forecasting problems than many existing methods. It can also provide upper and lower bounds for the forecasted state of the system. Further, it applies a nested "two scale hierarchic" quadratic mean-square optimization process, whereas existing generalized method of moments approaches and their extensions are "single-shot".

Below, using the role of time-delay processes, the concept of lagged adaptive expectation process, moving average, local finite sequences, local mean and variance, discrete time dynamic sample mean and variance statistic processes, local conditional and sequences, local sample mean and variance, the DTIDMLSMVSP is developed. A local observation system is also constructed from nonlinear stochastic functional differential equations. This can be based on the Itô-Doob stochastic differential formula and Euler-type numerical scheme in the context of the original stochastic systems of differential equations and the given data. In addition, using the method of moments in the context of lagged adaptive expectation process, a procedure is outlined to estimate state parameters. Using the local lagged adaptive process and the discrete time interconnected dynamic model for statistic process, the idea of time series data collection schedule synchronization with both numerical and simulation time schedules induces a chain of concepts further described below.

The existing GMM-based parameter and state estimation techniques for testing/selecting continuous-time dynamic models are centered around discretization and model errors in the context of the use of an entire time-series of data, algebraic manipulations, and econometric specification for formation of orthogonality condition parameter vectors (OCPV). The existing approaches lead to an overall/single-shot state and parameter estimates, and requires the ergodic stationary condition for convergence. Furthermore, the existing GMM-based single-shot approaches are not flexible to correctly validate the features of continuous-time dynamic models that are influenced by the state parameter and hereditary processes. In many real-life problems, the past and present dynamic states influence the future state dynamic. In the formulation of one of the components of the LLGMM approach, we incorporate the "past state history" via a local lagged adaptive process.

As an introduction to an LLGMM dynamic system according to various aspects of the embodiments, FIG. **1** illustrates an example computing environment **100** for LLGMM dynamic processes. The computing environment **100** includes a computing device **110**, a data store **120**, and an LLGMM dynamic process module **130**.

The computing environment **100** can be embodied as one or more computers, computing devices, or computing systems. In certain embodiments, the computing environment **100** can include one or more computing devices arranged, for example, in one or more server or computer banks. The computing device or devices can be located at a single installation site or distributed among different geographical locations. The computing environment **100** can include a plurality of computing devices that together embody a hosted computing resource, a grid computing resource, and/or other distributed computing arrangement. One example structure of the computing environment **100** is described in greater detail below with reference to FIG. **8**.

The data store **120** can be embodied as one or more memories that store (or are capable of storing) and/or embody a discrete time data set **122** and state and parameter values **124**. In addition, the data store **120** can store (or is capable of storing) computer readable instructions that, when executed, direct the computing device **110** to perform various aspects of the LLGMM dynamic processes described herein. In that context, the the data store **120** can store computer readable instructions that embody, in part, the LLGMM dynamic process module **130**. The discrete time data set **122** can include as past state information of any number of continuous time dynamic processes over any time intervals, as described in further detail below. Further the state and parameter values **124** can include both admissible parameter estimates for a stochastic model of a continuous time dynamic process and state values of the stochastic model of the continuous time dynamic process as described in further detail below.

The LLGMM dynamic process module **130** includes the dynamic model generator **132**, the LLGMM processor **134**, the state and parameter estimator **136**, and the forecast simulator **138**. Briefly, the dynamic model generator **132** can be configured to develop a one or more stochastic models of various continuous time dynamic processes. The LLGMM processor **134** can be configured to generate a DTIDMLSM-VSP based on any one of the stochastic models of the continuous time dynamic processes developed by the dynamic model generator **132**. The state and parameter estimator **136** is configured to calculate a plurality of admissible parameter estimates for the stochastic model of the continuous time dynamic process using the DTIDMLSM-VSP. The state and parameter estimator **136** can be further configured to calculate a state value of the stochastic model of the continuous time dynamic process for each of the plurality of admissible parameter estimates, to gather a plurality of state values of the stochastic model of the continuous time dynamic process. The state and parameter estimator **136** can be further configured to determine an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values. Additionally, the forecast simulator **138** can be configured to forecast at least one future state value of the stochastic model of the continuous-time dynamic process. The functional and operational aspects of the components of the LLGMM dynamic process module **130** are described in greater detail below.

This remainder of this disclosure is organized as follows: in Section 2, using the role of time-delay processes, the concept of lagged adaptive expectation process, moving average, local finite sequence, local mean and variance, discrete time dynamic sample mean and variance statistic processes, local conditional sequence, and local sample mean and variance, we develop a general DTIDMLSMVSP. DTIDMLSMVSP is the generalization of statistic of random sample drawn from the "static" population. In Section 3, a local observation system is constructed from a nonlinear stochastic functional differential equations. This is based on the Itô-Doob stochastic differential formula and Euler-type numerical scheme in the context of the original stochastic systems of differential equations and the given data. In addition, using the method of moments in the context of lagged adaptive expectation process, a procedure to estimate the state parameters is outlined.

Using the local lagged adaptive process and the discrete time interconnected dynamic model for statistic process, the idea of time series data collection schedule synchronization with both numerical and simulation time schedules induces a finite chain of concepts in Section 4, namely: (a) local admissible set of lagged sample/data/observation size, (b) local class of admissible lagged-adapted finite sequence of conditional sample/data, (c) local admissible sequence of parameter estimates and corresponding admissible sequence of simulated values, (d) ϵ-best sub-optimal admissible subset of set of m_k-size local conditional samples at time t_k in (a), (e) ϵ-sub-optimal lagged-adapted finite sequence of conditional sample/data, and (f) the ϵ-best sub optimal parameter estimates and simulated value at time t_k for k=1, 2, . . . , N in a systematic way. In addition, the local lagged adaptive process and DTIDMLSMVSP generate a finite chain of discrete time admissible sets/sub-data and corresponding chain described by simulation algorithm. The usefulness of computational algorithm is illustrated by applying the code not only to four energy commodity data sets, but also to the U.S. Treasury Bill Interest Rate data set and the USD-EUR Exchange Rate data set in finance for the state and parameter estimation problems. Further, we compare the usage of GARCH (1,1) model with the presented DTIDMLSMVSP model. We also compared the DTIDMLSMVSP based simulated volatility U.S. Treasury Bill Yield Interest rate data with the simulated work shown in Chan, K. C., Karolyi, G. Andrew, Longstaff, F. A., Sanders, Anthony B., *An Empirical Comparison of Alternative Models of the Short-Term Interest Rate*, The Journal of Finance, Vol. 47., No. 3, 1992, pp. 1209-1227 ("Chan et al").

In Section 5, the LLGMM is applied to investigate the forecasting and confidence-interval problems in applied statistics. The presented results show the long-run prediction exhibiting a degree of confidence. The use of advancements in electronic communication systems and tools exhibit that almost everything is dynamic, highly nonlinear, non-stationary and operating under endogenous and exogenous processes. Thus, a multitude of applications of the embodiments described herein exist. Some extensions include: (a) the development of the DTIDMLSMVSP and (b) the Aggregated Generalized Method of Moments AGMM of the LLGMM method are presented in Section 6. In fact, we compare the performance of DTIDMLSMVSP model with the GARCH(1,1) model and ex post volatility of Chan et al. Further, using the average of locally estimated parameters in the LLGMM, an aggregated generalized method of moment is also developed and applied to six data sets in Section 6.

In Section 7, a comparative study between the LLGMM and the existing parametric orthogonality condition vector based generalized method of moments (OCBGMM) techniques is presented. In Section 8, a comparative study between the LLGMM and some existing nonparametric methods is also presented. The LLGMM exhibits superior performance to the existing and newly developed OCBGMM. The LLGMM is problem independent and dynamic. On the other hand, the OCBGMM is problem dependent and static. In appearance, the LLGMM approach seems complicated, but it is user friendly. It can be operated by a limited theoretical knowledge of the LLGMM. Furthermore, we present several numerical results concerning both mathematical and applied statistical results showing the comparison of LLGMM with existing methods.

2. Derivation of Discrete Time Dynamic Model for Sample Mean and Variance Processes

The existing GMM-based parameter and state estimation techniques for testing/selecting continuous-time dynamic models are centered around discretization and model misspecifications errors in the context of usage of entire time-series data, algebraic manipulations, and econometric specification for formation of orthogonality condition parameter vectors (OCPV). The existing approaches lead to a single-shot for state and parameter estimates and require the ergodic stationary condition for convergence. Furthermore, the existing GMM-based single-shot approaches are not flexible to correctly validate the features of continuous-time dynamic models that are influenced by the state parameter and hereditary processes. In many real-life problems, the past and present dynamic states influence the future state dynamic. In the formulation of one of the components of the LLGMM approach, we incorporate the "past state history" via local lagged adaptive process.

Further, based on one of the goals of applied mathematical and statistical research, the embodiments described herein are applicable for various processes in biological, chemical, engineering, energy commodity markets, financial, medical, and physical and social sciences. Employing the hereditary influence of a systems, the concept of lagged adaptive expectation process, and the idea of moving average, a general DTIDMLSMVSP is developed with respect to an arbitrary continuous-time stochastic dynamic process. The development of the DTIDMLSMVSP can be motivated by the state and parameter estimation problems of any continuous time nonlinear stochastic dynamic model. Further, the idea of DTIDMLSMVSP was primarily based on the sample mean and sample variance ideas as statistic for a random sample drawn from a static population in the descriptive statistics. Using the DTIDMLSMVSP, the problems of long-term forecasting and interval estimation problems with a high degree of confidence can be addressed.

For the development of the DTIDMLSMVSP, various definitions and notations are described herein. Let τ and γ be finite constant time delays such that $0<\gamma\leq\tau$. Here, τ characterizes the influence of the past performance history of state of dynamic process, and γ describes the reaction or response time delay. In general, these time delays are unknown and random variables. These types of delay play a role in developing mathematical models of continuous time and discrete time dynamic processes. Based upon the nature of data collection, it may be necessary to either transform these time delays into positive integers or to design the data collection schedule in relations with these delays. For this purpose, the discrete version of time delays of τ and γ are defined as

$$r = \left[\left|\frac{\tau}{\Delta t_i}\right|\right] + 1, \text{ and } q = \left[\left|\frac{\gamma}{\Delta t_i}\right|\right] + 1, \tag{1}$$

respectively. For simplicity, we assume that $0<\gamma<1$ ($q=1$).

Definition 1.

Let x be a continuous time stochastic dynamic process defined on an interval $[-\tau, T]$ into $\Re$, for some $T>0$. For $t\in[-\tau, T]$, let $\mathcal{F}_t$ be an increasing sub-sigma algebra of a complete probability space for which x(t) is $\mathcal{F}_t$ measurable. Let P be a partition of $[-\tau, T]$ defined by

$$P:=\{t_i=-\tau+(r+i)\Delta t\}, \text{ for } i\in I_{-r}(N), \tag{2}$$

where

$$\Delta t = \frac{\tau+T}{N},$$

and $I_i(k)$ is defined by $I_i(k)=\{j\in\mathbb{Z} \mid i\leq j\leq k\}$.

Let $\{x(t_i)\}_{l=-r}^{N}$ be a finite sequence corresponding to the stochastic dynamic process x and partition P in Definition 1. Further, $x(t_i)$ is $\mathcal{F}_{t_i}$ measurable for $i\in I_{-r}(N)$. The definition of forward time shift operator F is given by:

$$F^i x(t_k)=x(t_{k+1}). \tag{3}$$

Additionally, $x(t_i)$ is denoted by x_i for $i\in I_{-r}(N)$.

Definition 2.

For q=1 and $r\geq1$, each $k\in I_0(N)$, and each $m_k\in I_2(r+k-1)$, a partition P_k of closed interval $[t_{k-m_k}, t_{k-1}]$ is called local at time t_k and it is defined by

$$P_k := t_{k-m_k} < t_{k-m_k+1} < \ldots < t_{k-1}. \tag{4}$$

P_k is referred as the m_k-point sub-partition of the partition P in (2) of the closed sub-interval $[t_{k-m_k}, t_{k-1}]$ of $[-\tau,T]$.

Definition 3.

For each $k\in I_0(N)$ and each $m_k\in I_2(r+k-1)$, a local finite sequence at a time t_k of the size m_k is restriction of $\{x(t_l)\}_{l=-r}^{N}$ to P_k in (4), and it is defined by

$$S_{m_k,k} := \{F^i x_{k-1}\}_{i=-m_k+1}^{0}. \tag{5}$$

As m_k varies from 2 to k+r−1, the corresponding local sequence $S_{m_k,k}$ at t_k varies from $\{x_i\}_{i=k-2}^{k-1}$ to $\{x_i\}_{i=-r+1}^{k-1}$. As a result of this, the sequence defined in (5) is also called a m_k-local moving sequence. Furthermore, the average corresponding to the local sequence $S_{m_k,k}$ in (5) is defined by

$$\bar{S}_{m_k,k} := \frac{1}{m_k}\sum_{i=-m_k+1}^{0} F^i x_{k-1}. \tag{6}$$

The average/mean defined in (6) is also called the m_k-local average/mean. Further, the m_k-local variance corresponding to the local sequence $S_{m_k,k}$ in (5) is defined by

$$s^2_{m_k,k} := \begin{cases} \dfrac{1}{m_k}\displaystyle\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1} - \frac{1}{m_k}\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2 & \text{for small } m_k \\ \dfrac{1}{m_k-1}\displaystyle\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1} - \frac{1}{m_k}\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2 & \text{for large } m_k \end{cases}. \tag{7}$$

Definition 4.

For each fixed $k\in I_0(N)$, and any $m_k\in I_2$ (k+r−1), the sequence $\{\bar{S}_{i,k}\}_{i=k-m_k}^{k-1}$ is called a m_k-local moving average/mean process at t_k. In other words, the LLGMM dynamic process includes, for each m_k-local moving sequence, calculating an m_k-local average to generate an m_k-moving average process (e.g., reference numeral **310** in FIG. **3**). Further, the sequence $\{s_{i,k}^2\}_{i=k-m_k}^{k-1}$ is called a m_k-local moving variance process at t_k. That is, for each m_k-local moving sequence, the process includes calculating an m_k-local variance to generate an m_k-local moving variance process (e.g., reference numeral **312** in FIG. **3**).

Definition 5.

Let $\{x(t_l)\}_{l=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (2). The local sample average/mean in (6) and local sample variance in (7) are called discrete time dynamic processes of sample mean and sample variance statistics.

Definition 6.

Let $\{x(t_i)\}_{i=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (2). The m_k-local moving average and variance defined in (6) and (7) are called the m_k-local moving sample average/mean and local moving sample variance at time t_k, respectively. Further, m_k-local sample average and m_k-local sample variance are referred to as local sample mean and local sample variance statistics for the local mean and variance of the continuous time stochastic dynamic process at time t_k, respectively. $\bar{S}_{m_k}$ and $s_{m_k}^{2}$ are called sample statistic time series processes.

Definition 7.

Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ-algebra $\mathcal{F}_{t_i}$, $t_i \in P$ in (2). The m_k-local conditional moving average and variance defined in the context of (6) and (7) are called the m_k-local conditional moving sample average/mean and local conditional moving sample variance, respectively.

The concept of sample statistic time-series/process extends the concept of random sample statistic for static dynamic populations in a natural and unified way. Employing Definition 7, we introduce the DTIDMLSMVSP. As described in detail below, this discrete time algorithm/model plays an important role in state and parameter estimation problems for nonlinear and non-stationary continuous-time stochastic differential and difference equations. Further, it provides feedback for both continuous-time dynamic model and corresponding discrete time statistic dynamic model for modifications and updates under the influence of exogenous and endogenous varying forces or conditions in a systematic and unified way. It is also clear that the discrete time algorithm eases the updates in the time-series statistic. Now, a change in $\overline{S}_{m_k,k}$ and $s_{m_k,k}^2$ with respect to change in time t_k can be stated.

Lemma 1. (DTIDMLSMVSP).

Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ-algebra $\mathcal{F}_{t_i}$, t_i belong to partition P in. Let $\overline{S}_{m_k,k}$ and $s_{m_k,k}^2$ be m_k-local conditional sample average and local conditional sample variance at t_k for each $k \in I_0(N)$. Using these inputs (e.g., reference numeral **314** in FIG. **3**), an example DTIDMLSMVSP can be described by

$$\begin{cases} \overline{S}_{m_{k-p+1},k-p+1} = \dfrac{m_{k-p}}{m_{k-p+1}}\overline{S}_{m_{k-p+1},k-p} + \eta_{m_{k-p},k-p}, \quad \overline{S}_{m_0,0} = \overline{S}_0 \\ S^2_{m_k,k} = \begin{cases} \dfrac{m_{k-1}}{m_k}\left[\displaystyle\sum_{i=1}^{p}\left[\dfrac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \left[\dfrac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\overline{S}^2_{m_{k-p},k-p}\right] + \varepsilon_{m_{k-1},k-1}\right], & \text{for small } m_k,\ m_{k-1} \le m_k \\ \displaystyle\sum_{i=1}^{p}\left[\dfrac{m_{k-i}-1}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \dfrac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\overline{S}^2_{m_{k-p},k-p} + \epsilon_{m_{k-1},k-1}, & \text{for large } m_k,\ m_{k-1} \le m_k \\ s^2_{m_i,i} = s_i^2,\ i \in I_{-p}(0), & \text{initial conditions} \end{cases} \end{cases} \tag{8}$$

where

$$\begin{cases} \eta_{m_{k-p},k-p} = \dfrac{1}{m_{k-p+1}}\left[\displaystyle\sum_{i=-m_{k-p+1}+1}^{-m_{k-p}+1} F^i x_{k-p} - F^{-m_{k-p}+1}x_{k-p} - F^{-m_{k-p}}x_{k-p} + F^0 x_{k-p}\right], \\ \varepsilon_{m_{k-1},k-1} = \dfrac{m_k-1}{m_k}\left[\displaystyle\sum_{i=1}^{p}\dfrac{(F^{-i+1}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\dfrac{(F^{-i+1-m_{k-i}}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\dfrac{(F^{-i+2-m_{k-i}}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \\ \quad \dfrac{m_k-1}{m_k}\left[\displaystyle\sum_{i=1}^{p}\left[\dfrac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \sum_{i=1}^{p}\left[\dfrac{\sum_{\substack{l,s=-i+2-m_{k-i+1}\\ l\ne s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right] - \\ \quad \dfrac{1}{m_k}\displaystyle\sum_{\substack{l,s=-m_k+1\\ l\ne s}}^{0} F^l x_{k-1}F^s x_{k-1}, \end{cases} \tag{9}$$

$$\begin{cases} \epsilon_{m_{k-1},k-1} = \displaystyle\sum_{i=1}^{p}\dfrac{(F^{-i+1}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\dfrac{(F^{-i+1-m_{k-i}}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} - \\ \displaystyle\sum_{i=1}^{p}\dfrac{(F^{-i+2-m_{k-i}}x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} + \sum_{i=1}^{p}\left[\dfrac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \\ \displaystyle\sum_{i=1}^{p}\left[\dfrac{\sum_{\substack{l,s=-i+2-m_{k-i+1}\\ l\ne s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right] - \dfrac{1}{m_k-1}\sum_{\substack{l,s=-m_k+1\\ l\ne s}}^{0} F^l x_{k-1}F^s x_{k-1} \end{cases}$$

Remark 1.

The interconnected dynamic statistic system (8) can be re-written as the one-step Gauss-Sidel dynamic system of iterative process described by

$$X(k;p) = A(k, X(k-1;p);p)X(k-1;p) + e(k;p), \tag{10}$$

$$\text{where } X(k;p) = \begin{pmatrix} X_1(k;p) \\ X_2(k;p) \end{pmatrix},$$

-continued

$$X_1(k;p)=\overline{S}_{m_{k-p+1}k-p+1},\ X_2(k)=\begin{pmatrix} S^2_{m_{k-p+1},k-p+1}\\ S^2_{m_{k-p+2},k-p+2}\\ \vdots\\ S^2_{m_{k-1},k-1}\\ S^2_{m_k,k}\end{pmatrix},$$

$$A(k,X(k-1;p);p)=\begin{pmatrix} A_{11}(k;p) & A_{12}(k;p)\\ A_{21}(k,X(k-1;p);p) & A_{22}(k;p)\end{pmatrix},$$

$$A_{11}(k;p)=\frac{m_{k-p}}{m_k-p+1},$$

$$A_{12}(k;p)=(0\ \ 0\ \ \dots\ \ 0),$$

$$A_{21}(k;p)=\begin{cases}\begin{pmatrix}0\\0\\\vdots\\0\\ \dfrac{(m_k-1)m_{k-p}}{m_k\prod_{j=0}^{p-1}m_{k-j}}\overline{S}_{m_{k-p},k-p}\end{pmatrix}, & \text{for small } m_k\\ \begin{pmatrix}0\\0\\\vdots\\0\\ \dfrac{m_{k-p}}{\prod_{j=0}^{p-1}m_{k-j}}\overline{S}_{m_{k-p},k-p}\end{pmatrix}, & \text{for large } m_k,\end{cases},$$

$$A_{22}(k;p)=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & 0 & 0 & 0 & \ddots & \vdots\\ 0 & \dots & 0 & 0 & 0 & 1\\ \dfrac{(m_k-1)m_{k-p}}{m_k\prod_{j=0}^{p-1}m_{k-j}} & \dfrac{(m_k-1)m_{k-p+1}}{m_k\prod_{j=0}^{p-2}m_{k-j}} & \dots & \dfrac{(m_k-1)m_{k-p+i-1}}{m_k\prod_{j=0}^{p-i}m_{k-j}} & \dots & \dfrac{(m_k-1)m_{k-1}}{m_k^2}\end{pmatrix},\ \text{for small } m_k,$$

$$\text{and}\ \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ \vdots & 0 & 0 & 0 & \ddots & \vdots\\ 0 & \dots & 0 & 0 & 0 & 1\\ \dfrac{m_{k-p}-1}{\prod_{j=0}^{p-1}m_{k-j}} & \dfrac{m_{k-p+1}-1}{\prod_{j=0}^{p-2}m_{k-j}} & \dots & \dfrac{m_{k-p+i-1}-1}{\prod_{j=0}^{p-i}m_{k-j}} & \dots & \dfrac{m_{k-1}-1}{m_k^2}\end{pmatrix},\ \text{for large } m_k$$

$$e(k;p)=\begin{pmatrix}e_1(k;p)\\ e_2(k;p)\end{pmatrix},$$

$$e_1(k;p)=\eta_{m_{k-p},k-p},$$

-continued

$$e_2(k;p)=\begin{pmatrix}0\\0\\\vdots\\ \epsilon^*_{m_{k-1},k-1}\end{pmatrix},$$

$$\epsilon^*_{m_{k-1},k-1}=\begin{cases}\epsilon_{m_{k-1},k-1}, & \text{for small } m_k\\ \epsilon_{m_{k-1},k-1}, & \text{for large } m_k\end{cases}$$

Remark 2.

For each $k\in I_0(N)$, p=2, and small m_k, the inter-connected system (8) reduces to the following special case

$$X(k;2)=A(k,\mathbb{X}(k-1;2);2)X(k-1;2)+e(k;2), \tag{11}$$

where X(k; 2), A(k; 2) and e(k; 2) are defined by

$$X(k;2)=\begin{pmatrix}X_1(k;2)\\ X_2(k;2)\end{pmatrix},$$

$$X_1(k;2)=\overline{S}_{m_{k-1},k-1},$$

$$X_2(k;2)=\begin{pmatrix}s^2_{m_{k-1},k-1}\\ s^2_{m_k,k}\end{pmatrix},\ A(k;2)=\begin{pmatrix}A_{11}(k;2) & A_{12}(k;2)\\ A_{21}(k;2) & A_{22}(k;2)\end{pmatrix},$$

$$A_{11}(k;2)=\frac{m_{k-2}}{m_k-1},\ A_{12}(k;2)=(0\ \ 0),$$

$$A_{21}(k;2)=\begin{pmatrix}0\\ \dfrac{(m_k-1)m_{k-2}}{m_k^2m_{k-1}}\overline{S}_{m_{k-2},k-2}\end{pmatrix},$$

-continued

$$A_{22}(k;2)=\begin{pmatrix}0 & 1\\ \dfrac{(m_k-1)m_{k-2}}{m_k^2m_{k-1}} & \dfrac{(m_k-1)m_{k-1}}{m_k^2}\end{pmatrix},$$

$$e(k;2)=\begin{pmatrix}e_1(k;2)\\ e_2(k;2)\end{pmatrix};\ e_1(k;2)=\eta_{m_{k-2},k-2},$$

-continued

$$e_2(k;2) = \begin{pmatrix} 0 \\ \varepsilon_{m_{k-1},k-1} \end{pmatrix}$$

$$\begin{cases} \eta_{m_{k-2},k-2} = \dfrac{1}{m_k}\left[\displaystyle\sum_{i=-m_{k-1}+1}^{-m_{k-2}+1} F^i x_{k-2} - F^{-m_{k-2}+1}x_{k-2} - F^{-m_{k-2}}x_{k-2} + F^0 x_{k-2}\right], \\ \varepsilon_{m_{k-1},k-1} = \dfrac{m_k-1}{m_k}\left[\dfrac{(F^0x_{k-1})^2 - (F^{-m_{k-1}}x_{k-1})^2 - (F^{1-m_{k-1}}x_{k-1})^2}{m_k} + \dfrac{(F^{-1}x_{k-1})^2 - (F^{-1-m_{k-2}}x_{k-1})^2 - (F^{-m_{k-2}}x_{k-1})^2}{m_k m_{k-1}}\right] + \\ \quad \dfrac{m_k-1}{m_k}\left[\dfrac{\displaystyle\sum_{i=-m_{k-1}}^{-m_{k-2}}(F^i x_{k-1})^2}{m_k m_{k-1}} + \dfrac{\displaystyle\sum_{\substack{i,j=-m_{k-1}\\ i\neq j}}^{-1} F^i x_{k-1}F^j x_{k-1}}{m_k m_{k-1}} + \dfrac{\displaystyle\sum_{i=1-m_k}^{1-m_{k-1}}(F^i x_{k-1})^2}{m_k}\right] - \dfrac{\displaystyle\sum_{\substack{i,j=1-m_k\\ i\neq j}}^{0} F^i x_{k-1}F^j x_{k-1}}{m_k^2}. \end{cases}$$

Remark 3.

Define

$$\varphi_1 = \frac{m_k-1}{m_k}\frac{m_{k-1}}{m_k},\ \varphi_2 = \frac{m_k-1}{m_k}\frac{m_{k-2}}{m_k m_{k-1}},\text{ and } \varphi_3 = \frac{m_{k-2}}{m_{k-1}}.$$

For small m_k, $m_{k-1} \leq m_k$, $\forall k$, we have $\varphi_1<1$, $\varphi_2<1$, and $\varphi_3 \leq 1$. From $0<\varphi_i$, i=1, 2, 3, and the fact that

$$\varphi_1+\varphi_2 = \frac{m_k-1}{m_k^2}\left[m_{k-1}+\frac{m_{k-2}}{m_{k-1}}\right] \leq \frac{m_k-1}{m_k^2}[m_{k-1}+1] \leq \frac{m_k^2-1}{m_k^2} < 1,$$

the stability of the trivial solution (e.g., X(k; 2)=0) of the homogeneous system corresponding to (10) follows. Further, under the above stated conditions, the convergence of solutions of (10) also follows.

Remark 4.

From Remark 2, the local sample variance statistics at time t_k depends on the state of the m_{k-1} and m_{k-2}-local sample variance statistics at time t_{k-1} and t_{k-2}, respectively, and the m_{k-2}-local sample mean statistics at time t_{k-2}.

Remark 5.

Aspects of the role and scope of the DTIDMLSMVSP can be summarized. First, the DTIDMLSMVSP is the second component of the LLGMM approach. The DTIDMLSMVSP is valid for a transformation of data. It is generalization of a "statistic" of a random sample drawn from "static" population problems. Further, Lemma 1 provides iterative scheme for updating statistic coefficients in the local systems of moment/observation equations in the LLGMM approach. This accelerates the speed of computation. The DTIDMLSMVSP does not require any type of stationary condition. The DTIDMLSMVSP plays a significant role in the local discretization and model validation errors. Finally, the approach to the DTIDMLSMVSP is more suitable for forecasting problems, as further emphasized in the subsequent sections.

Remark 6.

The usefulness of the DTIDMLSMVSP arises in estimation of volatility process of a stochastic differential or difference equations. This model provides an alternative approach to the GARCH(p,q) model. Below, the m_k-local sample variance statistics are compared with the GARCH (p,q) model to show that the m_k-local sample variance statistics give a better forecast than the GARCH(p,q) model.

3. Theoretical Parametric Estimation Procedure

In this section, a foundation based on a mathematically rigorous theoretical state and parameter estimation procedure is formulated for a very general continuous-time nonlinear and non-stationary stochastic dynamic model described by a system stochastic differential equations. This work is not only motivated by the continuous-time dynamic model validation problem in the context of real data energy commodities, but also motivated by any continuous-time nonlinear and non-stationary stochastic dynamic model validation problems in biological, chemical, engineering, financial, medical, physical and social sciences, among others. This is because of the fact that the development of the existing Orthogonality Condition Based GMM (OCBGMM) procedure is primarily composed of the following five components: (1) testing/selecting continuous-time stochastic models for a particular dynamic process described by one or more stochastic differential equations, (2) using either a Euler-type discretization scheme, a discrete time econometric specification, or other discretization scheme regarding the stochastic differential equation specified in (1), (3) forming an orthogonality condition parameter vector (OCPV) using algebraic manipulation, (4) using (2), (3) and the entire time series data set, finding a system of moment equations for the OCBGMM, and (5) single-shot parameter and state estimates using positive-definite quadratic form. The existing OCBGMM lacks the usage of Itô-Doob calculus, properties of stochastic differential equations, and a connection with econometric based discretization schemes, the orthogonality conditional vector, and the quadratic form.

In this section, an attempt is made to eliminate the drawbacks, operational limitations, and the lack of connectivity and limited scope of the OCBGMM. This is achieved by utilizing (i) historical role played by hereditary process in dynamic modeling, (ii) Itô-Doob calculus, (iii) the fundamental properties of stochastic system of differential equations, (iv) the lagged adaptive process, (v) the discrete time interconnected dynamics of local sample mean and variances statistic processes model in Section 2 (Lemma 1), (vi) the Euler-type numerical schemes for both stochastic differential equations generated from the original stochastic systems of differential equations and the original stochastic systems of differential equations, (vii) systems of moments/observation equations, and (viii) local observation/measurements systems in the context of real world data.

Starting in this section, parts of the the LLGMM dynamic process **200** shown in FIG. **2** are also described. At reference numeral **202**, the process **200** includes obtaining a discrete time data set as past state information of a continuous time dynamic process over a time interval, such as the [−τ,T] described herein. The discrete time data set can be stored in the data store **120** as the discrete time data set **122**. Further, at reference numeral **204**, the process **200** includes developing a stochastic model of a continuous time dynamic process.

As one example of a stochastic model of a continuous time dynamic process, a general system of stochastic differential equations under the influence of hereditary effects in both the drift and diffusion coefficients is described by

$$dy=f(t,y_t)dt+\sigma(t,y_t)dW(t), y_{t_0}=\varphi_0, \tag{12}$$

where, $y_t(\theta)=y(t+\theta)$, $\theta\in[-\tau,0]$, f, σ: $[0, T]\times C\rightarrow\Re^q$ are Lipschitz continuous bounded functionals, $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau,0]$ into $\Re^q$ equipped with the supremum norm, W(t) is standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$, $\varphi_0\in C$, $y_0(t_0+\theta)$ is $\mathcal{F})_{t_0}$ measurable, the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, each $\mathcal{F}_t$ with $t\geq t_0$ contains all $\mathbb{P}$-null events in F, and the solution process $y(t_0,\varphi_0)(t)$ is adapted and non-anticipating with respect to $(\mathcal{F})_{t\geq 0}$.

3.1 Transformation of System of Stochastic Differential Equations (12)

At reference numeral **206**, the process **200** includes generating a DTIDMLSMVSP based on the stochastic model of the continuous time dynamic process. As part of the conceptual aspects of generating the DTIDMLSMVSP, at reference numeral **206**, the process **200** can include transforming the stochastic model of the continuous time dynamic process into a stochastic model of a discrete time dynamic process utilizing a discretization scheme. For example, let $V\in C[[-\tau, \infty]\times\Re^q, \Re^m]$. Its partial derivatives V_t,

$$V_t, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial y^2}$$

exist and are continuous. The Itô-Doob stochastic differential formula can be applied to V to obtain

$$dV(t,y)=LV(t,y,y_t)dt+V_y(t,y)\sigma(t,y_t)dW(t), \tag{13}$$

where the L operator is defined by

$$\begin{cases} LV(t, y, y_t) &= V_t(t, y) + V_y(t, y)f(t, y_t) + \frac{1}{2}tr(V_{yy}(t, y)b(t, y_t)) \\ b(t, y_t) &= \sigma(t, y_t)\sigma^T(t, y_t). \end{cases} \tag{14}$$

3.2 Euler-Type Discretization Scheme for (12) and (13)

For (12) and (13), the Euler-type discretization scheme can be presented as

$$\begin{cases} \Delta y_i &= f(t_{i-1}, y_{t_{i-1}})\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W_{i-1},\ i\in I_1(N) \\ \Delta V(t_i, y(t_i)) &= LV(t_{i-1}, y(t_i), y_{t_{i-1}})\Delta t_i + V_y(t_{i-1}, y(t_{i-1}))\sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i) \end{cases}, \tag{15}$$

and $\mathcal{F}_{t_{i-1}}\equiv\mathcal{F}_{i-1}$ can be defined as the filtration process up to time t_{i-1}.

3.3 Formation of Generalized Moment Equations from (15)

As another part of the conceptual aspects of generating the DTIDMLSMVSP, at reference numeral **206**, the process **200** can also include developing a system of generalized method of moments equations from the stochastic model of the discrete time dynamic process. For example, with regard to the continuous time dynamic system (12) and its transformed system (13), the more general moments of $\Delta y(t_i)$ are:

$$\begin{cases} E[\Delta y(t_i) \mid \mathcal{F}_{i-1}] &= f(t_{i-1}, y_{t_{i-1}})\Delta t_i, \\ E[(\Delta y(t_i) - E[\Delta y(t_i) \mid \mathcal{F}_{i-1}]) (\Delta y(t_i) - E[\Delta y(t_i) \mid \mathcal{F}_{i-1}])^T \mid \mathcal{F}_{i-1}] &= \sigma(t_{i-1}, y_{t_{i-1}})\sigma^T(t_{i-1}, y_{t_{i-1}})\Delta t_i, \\ E[\Delta V(t_i, y(t_i)) \mid \mathcal{F}_{i-1}] &= LV(t_{i-1}, y(t_i), y_{t_{i-1}})\Delta t_i, \\ E[(\Delta V(t_i, y(t_i)) - E[\Delta V(t_i, y(t_i)) \mid \mathcal{F}_{i-1}])(\Delta V(t_i, y(t_i)) - E[\Delta V(t_i, y(t_i)) \mid \mathcal{F}_{i-1}])^T \mid \mathcal{F}_{i-1}] &= B(t_{i-1}, y(t_{i-1}), y_{t_{i-1}}) \end{cases}$$

where $B(t_{i-1},y(t_{i-1}),y_{t-1})=V_y(t_{i-1},y(t_{i-1}))b(t_{i-1},y_{t-1})V_y(t_{i-1},y(t_{i-1}))^T\Delta^t$, and T stands for the transpose of the matrix.

3.4 Basis for Local Lagged Adaptive Discrete Time Expectation Process

From (15) and (16),

$$\begin{cases} \Delta y_i &= E[\Delta y(t_i) \mid \mathcal{F}_{i-1}] + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W_{i-1},\ i\in I_1(N) \\ \Delta V(t_i, y(t_i)) &= E[\Delta V(t_i, y(t_i)) \mid \mathcal{F}_{i-1}] + V_y(t_{i-1}, y(t_{i-1}))\sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i) \end{cases}. \tag{17}$$

This provides the basis for the development of the concept of lagged adaptive expectation with respect to continuous time stochastic dynamic systems (12) and (13). This also leads to a formulation of m_k-local generalized method of moments at t_k.

Remark 7.

(Block Orthogonality Condition Vector for (12) and (13)). From (17), one can define a block vector of orthogonality condition as

$$H(t_{i-1}, y(t_i), y(t_{i-1})) = \begin{pmatrix} \Delta y(t_i) - f(t_{i-1}, y(t_{i-1}))\Delta t_i \\ \Delta V(t_{i-1}, y(t_i)) - LV(t_{i-1}, y(t_{i-1}), y_{t_{i-1}})\Delta t_i \end{pmatrix}. \tag{18}$$

Further, unlike the orthogonality condition vector defined in the literature, the definition of the block vector of orthogonality condition (18) is based on the discretization scheme associated with nonlinear and non-stationary continuous-time stochastic system of differential equations (12) and (13) and the Itô-Doob stochastic differential calculus.

Example 1

For V(t,y) in (13) defined by

$$V(t, y) = \|y\|_p^p = \sum_{j=1}^{n} |y^j|^p, \tag{19}$$

$$dV = \left[p\sum_{j=1}^{n} |y^j|^{p-1}\mathrm{sgn}(y^j)f(t, y_t^j) + \frac{p(p-1)}{2}|y^j|^{p-2}\sigma(t, y_t^j)\right]dt + p\sum_{j=1}^{n} |y^j|^{p-1}\mathrm{sgn}(y^j)\sigma(t, y_t^j)dW^j.$$

Hence, the discretized form of (19) is given by

$$\Delta V_i = \left[p\sum_{j=1}^{n} |y_{i-1}^j|^{p-1}\mathrm{sgn}(y_{i-1}^j)f(t_{i-1}, y_{t_{i-1}}^j) + \right. \tag{20}$$

-continued

$$\frac{p(p-1)}{2}|y_{i-1}^j|^{p-2}\sigma(t_{i-1}, y_{t_{i-1}}^j)\Bigg]dt +$$

$$p\sum_{j=1}^{n} |y_{i-1}^j|^{p-1}\operatorname{sgn}(y_{i-1}^j)\sigma(t_{i-1}, y_{t_{i-1}}^j)dW_i^j.$$

In this special case, (17) reduces to

$$\begin{cases} \Delta y_i & = \begin{array}{c} E[\Delta y(t_i) \mid \mathcal{F}_{i-1}] + \\ \sigma(t_{i-1}, y_{t_{i-1}})\Delta W_{i-1},\ i \in I_1(N) \end{array} \\ \Delta\left(\sum_{j=1}^{n}|y_i^j|^p\right) & = \begin{array}{c} E\left[\Delta\left(\sum_{j=1}^{n}|y_i^j|^p\right)\middle|\mathcal{F}_{i-1}\right] + \\ p\sum_{j=1}^{n}|y_{i-1}^j|^{p-1}\operatorname{sgn}(y_{i-1}^j) \\ \sigma(t_{i-1}, y_{t_{i-1}}^j)dW_i^j \end{array} \end{cases}. \tag{21}$$

Example 2

We consider a multivariate AR(1) model as another example to exhibit the parameter and state estimation problem. The AR(1) model is of the following type

$$x_t=a_{t-1}x_{t-1}+\sigma_{t-1}e_t, x(0)=x_0, \text{ for } t=0,1,2, \ldots, t, \ldots, N, \tag{22}$$

where x_t, $x_0 \in \Re^n$, $e_t \in \Re^m$ is $\mathcal{F}_t$ a measurable normalized discrete time Gaussian process, and a_{t-1} and σ_{t-1} are n×n and n×m discrete time varying matrix functions, respectively. Here

$$\begin{pmatrix} E[x_t \mid \mathcal{F}_{i-1}] \\ E[x_t x_t^T \mid \mathcal{F}_{i-1}] \end{pmatrix} = \begin{pmatrix} a_{t-1}x_{t-1} \\ a_{t-1}x_{t-1}(a_{t-1}x_{t-1})^T + \sigma_{t-1}(\sigma_{t-1})^T \end{pmatrix}. \tag{23}$$

In this case, the block orthogonality condition vector is based on a multivariate stochastic system of difference equation and difference calculus for (22) and (23), given by

$$H(t_{i-1}, x_t, x_{t-1}, a_{t-1}, \sigma_{t-1}) = \begin{pmatrix} x_t - a_{t-1}x_{t-1} \\ \Delta V(x_t) - LV(t, x_{t-1})\Delta t \end{pmatrix}, \tag{24}$$

where Δ and L are difference and L operators with respect to $V=x_t x_t^T$ for $x \in \Re^n$, and are defined by

$$\begin{cases} \Delta V(x_t) = V(x_t) - V(x_{t-1}), \text{ for } t = 1, 2, \ldots, t, \ldots, N \\ LV(t, x_{t-1}) = a_{t-1}x_{t-1}((2 + a_{t-1})x_{t-1})^T + \sigma_{t-1}\sigma_{t-1}^T \end{cases}, \tag{25}$$

and differential of V with respect to multivariate difference system (22) parallel to continuous-time version (13) is as:

$$\Delta V(x_t)=a_{t-1}x_{t-1}((2+a_{t-1})x_{t-1})^T+\sigma_{t-1},\sigma_{t-1}{}^T+2(1+a_{t-1}x_{t-1})(\sigma_{t-1},e_t)^T. \tag{26}$$

From the above, it is clear that the orthogonality condition parameter vector in (24) is constructed with respect to multivariate stochastic system of difference equations and elementary difference calculus.

Remark 8.

From the transformation of system of stochastic differential equations (13) in Sub-section 3.1, the construction of Euler-type Discretization Scheme for (12) and (13) in Sub-section 3.2, the Formation of Generalized Moment Equations from (15) in Sub-section 3.3, and the Basis for Local Lagged Adaptive Discrete time Expectation Process in Sub-section 3.4, the system is in the correct framework for mathematical reasoning, logical, and interconnected/interactive within the context of the continuous-time dynamic system (12).

Further, a continuous-time state dynamic process described by systems of stochastic differential equations (12) moves forward in time. The theoretical parameter estimation procedure in this section adapts to and incorporates the continuous-time changes in the state and parameters of the system and moves into a discrete time theoretical numerical schemes in (15) as a model validation of (12). It further successively moves in the local moment equations within the context of local lagged adaptive, local discrete time statistic and computational processes in a natural, systematic, and coherent manner. On the other hand, the existing OCBGMM approach is "single-shot" with a global approach, and it is highly dependent on the second component of the OCBGMM. That is, the use of either Euler-type discretization scheme or a discrete time econometric specification regarding the stochastic differential equation. We refer to OCBGMM as the single-shot or global approach with formation of a single moment equation in a quadratic form.

Below, a result is stated that exhibits the existence of solution of system of non linear algebraic equations. For the sake of reference, the Implicit Function Theorem is stated without proof.

Theorem 2 (Implicit Function Theorem).

Let $F=\{F_1, F_2, \ldots, F_q\}$ be a vector-valued function defined on an open set $S \in \Re^{q+k}$ with values in $\Re^q$. Suppose $F \in C'$ on S. Let $(u_0; v_0)$ be a point in S for which $F(u_0; v_0)=0$ and for which the q×q determinant $\det [D_j F_i(u_0; v_0)] \neq 0$. Then there exists a k-dimensional open set T_0 containing v_0 and unique vector-valued function g, defined on T_0 and having values in $\Re^q$, such that $g \in C'$ on T_0, $g(v_0)=u_0$, and $F(g(v); v)=0$ for every $v \in T_0$.

Illustration 1: Dynamic Model for Energy Commodity Price.

As one example, the stochastic dynamic model of energy commodities described by the following nonlinear stochastic differential equation is considered:

$$dy=ay(\mu-y)dt+\sigma(t,y_t)ydW(t), y_{t_0}=\varphi_0, \tag{27}$$

where $y_t(\theta)=y(t+\theta)$; $\theta \in [-\tau,0]$, μ, $a \in \Re$, the initial process $\varphi_0=\{y(t_0+\theta)\}_{\theta \in [-\tau,0]}$ is $\mathcal{F}_{t_0}$—measurable and independent of $\{W(t), t \in [0,T]\}$, W(t) is a standard Wiener process defined in (12), $\sigma:[0, T]\times C \rightarrow \Re^+$ is a Lipschitz continuous and bounded functional, and C is the Banach space of continuous functions defined on $[-\tau,0]$ into $\Re$ equipped with the supremum norm.

Transformation of Stochastic Differential Equation (27).

A Lyapunov function $V(t,y)=\ln(y)$ in (13) is picked for (27). Using Itô-differential formula,

$$d(\ln(y)) = \left[a(\mu - y) - \frac{1}{2}\sigma^2(t, y_t)\right]dt + \sigma(t, y_t)dW. \tag{28}$$

The Euler-Type Discretization Schemes for (27) and (28).

By setting $\Delta t_i = t_i - t_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, the combined Euler discretized scheme for (27) and (28) is

$$\begin{cases} \Delta y_i = a y_{i-1}(\mu - y_{i-1})\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}}) y_{i-1} \Delta W(t_i),\ y_{t_0} = \varphi_0, \\ \Delta(\ln(y_i)) = \left[a(\mu - y_{i-1}) - \frac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right] \Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i),\ y_{t_0} = \varphi_0. \end{cases} \tag{29}$$

where $\varphi_0 = \{y_i\}_{i=-r}^{0}$ is a given finite sequence of $\mathcal{F}_0-$ measurable random variables, and it is independent of $\{\Delta W(t_i)\}_{i=0}^{N}$.

Generalized Moment Equations.

Applying conditional expectation to (29) with respect to

$$\begin{aligned} &\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}, \\ &\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] = a y_{i-1}(\mu - y_{i-1})\Delta t \\ &\mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}] = \left[a(\mu - y_{i-1}) - \frac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right]\Delta t. \\ &\mathbb{E}[(\Delta(\ln(y_i)) - \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}] = \sigma^2(t_{i-1}, y_{t_{i-1}})\Delta t. \end{aligned} \tag{30}$$

Basis for Lagged Adaptive Discrete Time Expectation Process.

From (30), (29) reduces to

$$\begin{cases} \Delta y_i = \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] + \sigma(t_{i-1}, y_{t_{i-1}}) y_{i-1} \Delta W(t_i) \\ \Delta(\ln(y_i)) = \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}] + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i) \end{cases}. \tag{31}$$

Equation (31) provides the basis for the development of the concept of lagged adaptive expectation process with respect to continuous time stochastic dynamic systems (27) and (28).

Remark 9. Orthogonality Condition Vector for (27) and (28).

Following Remark 7 and using (29), (30), and (31), the orthogonality condition vector with respect to continuous-time stochastic dynamic model (27) is represented by

$$H(t_{i-1}, y(t_i), y(t_{i-1})) = \begin{pmatrix} \Delta y(t_i) - a y(t_{i-1})(\mu - y(t_{i-1}))\Delta t_i \\ \Delta \ln(y(t_i)) - L\ln(y(t_{i-1}), y_{t_{i-1}})\Delta t_i \\ (\Delta \ln(y(t_i)) - L\ln(y(t_{i-1}), y_{t_{i-1}})\Delta t_i)^2 - \sigma^2(t_{i-1}, y_{t_{i-1}})\Delta t_i \end{pmatrix}, \tag{32}$$

wherein L ln

$$(y(t_{i-1}), y_{t_{i-1}})\Delta t_i = \left(a(\mu - y(t_{i-1})) - \frac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right)\Delta t_i.$$

Unlike the orthogonality condition vector defined in the literature, this orthogonality condition vector is based on the discretization scheme (29) associated with nonlinear continuous-time stochastic differential equations (27) and (28) and the Itô-Doob stochastic differential calculus.

Local Observation System of Algebraic Equations.

For $k \in I_0(N)$, applying the lagged adaptive expectation process from Definitions 3-7, and using (8) and (31), a local observation/measurement process is formulated at t_k as one or more algebraic functions of m_k-local restriction sequence of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to a subpartition P_k in Definition 2 as:

$$\begin{cases} \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] = a\left[\frac{\mu}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1} - \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}] = a\left[\mu - \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t - \frac{1}{2m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[(\Delta(\ln(y_i)) - \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}], \end{cases} \tag{33}$$

$$\hat{\sigma}_{m_k,k}^2 = \begin{cases} \frac{1}{m_k \Delta t}\sum_{i=k-m_k}^{k-1} \mathbb{E}[(\Delta(\ln(y_i)) - \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}] & \text{if } m_k \text{ is small} \\ \frac{1}{(m_k - 1)\Delta t}\sum_{i=k-m_k}^{k-1} \mathbb{E}[(\Delta(\ln(y_i)) - \mathbb{E}[\Delta(\ln(y_i)) \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}] & \text{if } m_k \text{ is large.} \end{cases}$$

From the third equation in (33), it follows that the average volatility square $\hat{\sigma}_{m_k,k}^2$ is given by

$$\hat{\sigma}_{m_k,k}^2 = \frac{s_{m_k,k}^2}{\Delta t}, \tag{34}$$

where $s_{m_k,k}^2$ is the local sample variance statistics for volatility at t_k in the context of $x(t_i) = \Delta(\ln(y_i))$.

We define

$$\begin{aligned} &F_1(\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(\ln y_i) \mid \mathcal{F}_{i-1}]; a, \mu) = \frac{\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}]}{m_k} - a\left[\frac{\mu \sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k} - \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}\right]\Delta t \\ &F_2(\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(\ln y_i) \mid \mathcal{F}_{i-1}]; a, \mu) = \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta(\ln y_i) \mid \mathcal{F}_{i-1}] - a\left[\mu - \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t + \frac{s_{m_k,k}^2}{2}. \end{aligned} \tag{35}$$

Then, we have

$$\begin{cases} F_1(\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(\ln y_i) \mid \mathcal{F}_{i-1}]; a, \mu) = 0, \\ F_2(\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(\ln y_i) \mid \mathcal{F}_{i-1}]; a, \mu) = 0. \end{cases} \tag{36}$$

Let $F=\{F_1, F_2\}$. The determinant of the Jacobian matrix of F is given by

$$JF(a,\mu) = -\frac{a}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^2 - \frac{1}{m_k}\left(\sum_{i=k-m_k}^{k-1} y_{i-1}\right)^2\right](\Delta t)^2 = -a\,\mathrm{var}\left(y(t_{i-1})_{i=k-m_k}^{k-1}\right)(\Delta t)^2 \neq 0, \tag{37}$$

provided that $a\neq 0$ or the sequence $\{x(t_{i-1})\}_{i=-r+1}^{N}$ is neither zero nor a constant. This fulfils the hypothesis of Theorem 2.

Thus, by the application of Theorem 2 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{x(t_i)\}_{i=k-m_k}^{k-1}$, there exists a unique solution of system of algebraic equations (36), $\hat{a}_{m_k,k}$ and $\hat{\mu}_{m_k,k}$ as a point estimates of a and μ, respectively.

We also note that the estimated values of a and μ change at each time t_k. For instance, at time $t_0=0$ and the given $\mathcal{F}_{-1}$ measurable discrete time process $y_{-r+1}, y_{-r+2}, \ldots, y_{-1}$, (33) reduces to

$$\begin{cases} \frac{1}{m_0}\sum_{i=-m_0}^{0} \Delta y_i &= a\left[\frac{\mu}{m_0}\sum_{i=-m_0}^{0} y_{i-1} - \frac{1}{m_0}\sum_{i=-m_0}^{0} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_0}\sum_{i=-m_0}^{0} \Delta(\ln y_i) &= a\left[\mu - \frac{1}{m_0}\sum_{i=-m_0}^{0} y_{i-1}\right]\Delta t - \frac{s_{m_0,0}^2}{2}, \\ \hat{\sigma}_{m_0,0}^2 &= \frac{s_{m_0,0}^2}{\Delta t}. \end{cases} \tag{38}$$

The initial solution of algebraic equations (38) at time t_0 is given by

$$\begin{cases} \hat{a}_{m_0,0} = \dfrac{\left(\frac{1}{m_0}\sum_{i=-m_0}^{0}\Delta(\ln y_i) + \frac{s_{m_0,0}^2}{2}\right)\left(\frac{1}{m_0}\sum_{i=-m_0}^{0} y_{i-1}\right) - \frac{1}{m_0}\sum_{i=-m_0}^{0}\Delta y_i}{\frac{1}{m_0}\left[\sum_{i=-m_0}^{0} y_{i-1}^2 - \frac{1}{m_0}\left(\sum_{i=-m_0}^{0} y_{i-1}\right)^2\right]\Delta t}. \\ \hat{\mu}_{m_0,0} = \dfrac{\frac{1}{m_0\Delta t}\sum_{i=-m_0}^{0}\Delta(\ln y_i) + \frac{s_{m_0,0}^2}{2\Delta t} + \frac{\hat{a}_{m_0,0}}{m_0}\left(\sum_{i=-m_0}^{0} y_{i-1}\right)}{\hat{a}_{m_0,0}} \\ \hat{\sigma}_{m_0,0}^2 = \dfrac{s_{m_0,0}^2}{\Delta t} \end{cases} \tag{39}$$

At time $t_1=1$ and the given $\mathcal{F}_0$ measurable discrete time process $y_{-r}, y_{-r+1}, \ldots, y_{-1}, y_0$, (33) reduces to

$$\begin{cases} \frac{1}{m_1}\sum_{i=1-m_1}^{0} \Delta y_i &= a\left[\frac{\mu}{m_1}\sum_{i=1-m_1}^{0} y_{i-1} - \frac{1}{m_1}\sum_{i=1-m_1}^{0} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_1}\sum_{i=1-m_1}^{0} \Delta(\ln y_i) &= a\left[\mu - \frac{1}{m_1}\sum_{i=1-m_1}^{0} y_{i-1}\right]\Delta t - \frac{s_{m_1,1}^2}{2}, \\ \hat{\sigma}_{m_1,1}^2 &= \frac{s_{m_1,1}^2}{\Delta t}. \end{cases} \tag{40}$$

The solution of algebraic equations (40) is given by

$$\begin{cases} \hat{a}_{m_1,1} = \dfrac{\left(\frac{1}{m_1}\sum_{i=1-m_1}^{0}\Delta(\ln y_i) + \frac{s_{m_1,1}^2}{2}\right)\left(\frac{1}{m_1}\sum_{i=1-m_1}^{0} y_{i-1}\right) - \frac{1}{m_1}\sum_{i=1-m_1}^{0}\Delta y_i}{\frac{1}{m_1}\left[\sum_{i=1-m_1}^{0} y_{i-1}^2 - \frac{1}{m_1}\left(\sum_{i=1-m_1}^{0} y_{i-1}\right)^2\right]\Delta t}. \\ \hat{\mu}_{m_1,1} = \dfrac{\frac{1}{m_1\Delta t}\sum_{i=1-m_1}^{0}\Delta(\ln y_i) + \frac{s_{m_1,1}^2}{2\Delta t} + \frac{\hat{a}_{m_1,1}}{m_1}\left(\sum_{i=1-m_1}^{0} y_{i-1}\right)}{\hat{a}_{m_1,1}} \\ \hat{\sigma}_{m_1,1}^2 = \dfrac{s_{m_1,1}^2}{\Delta t} \end{cases} \tag{41}$$

Likewise, for k=2, we have

$$\begin{cases} \hat{a}_{m_2,2} = \dfrac{\left(\frac{1}{m_2}\sum_{i=2-m_2}^{1}\Delta(\ln y_i) + \frac{s_{m_k,k}^2}{2}\right)\left(\frac{1}{m_2}\sum_{i=2-m_2}^{1} y_{i-1}\right) - \frac{1}{m_2}\sum_{i=2-m_2}^{1}\Delta y_i}{\frac{1}{m_2}\left[\sum_{i=2-m_2}^{1} y_{i-1}^2 - \frac{1}{m_2}\left(\sum_{i=2-m_2}^{1} y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_2,2} = \dfrac{\frac{1}{m_2\Delta t}\sum_{i=2-m_2}^{1}\Delta(\ln y_i) + \frac{s_{m_2,2}^2}{2\Delta t} + \frac{\hat{a}_{m_2,2}}{m_2}\left(\sum_{i=2-m_2}^{1} y_{i-1}\right)}{\hat{a}_{m_2,2}}, \\ \hat{\sigma}_{m_2,2}^2 = \dfrac{s_{m_2,2}^2}{\Delta t}. \end{cases} \tag{42}$$

Hence, from (33) and applying the principle of mathematical induction, we have

$$\begin{cases} \hat{a}_{m_k,k} = \dfrac{\left(\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\Delta(\ln y_i) + \frac{s_{m_k,k}^2}{2}\right)\left(\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}\right) - \frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\Delta y_i}{\frac{1}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^2 - \frac{1}{m_k}\left(\sum_{i=k-m_k}^{k-1} y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_k,k} = \dfrac{\frac{1}{m_k\Delta t}\sum_{i=k-m_k}^{k-1}\Delta(\ln y_i) + \frac{s_{m_k,k}^2}{2\Delta t} + \frac{\hat{a}_{m_k,k}}{m_k}\left(\sum_{i=k-m_k}^{k-1} y_{i-1}\right)}{\hat{a}_{m_k,k}}, \\ \hat{\sigma}_{m_k,k}^2 = \dfrac{s_{m_k,k}^2}{\Delta t}. \end{cases} \tag{43}$$

Remark 10.

We note that without loss in generality, the discrete time data set $\{y_{-r+i}: i\in I_1(r-1)\}$ is assumed to be close to the true values of the solution process of the continuous-time dynamic process. This assumption is feasible in view of the uniqueness and continuous dependence of solution process of stochastic functional or ordinary differential equation with respect to the initial data.

Remark 11.

If the sample $\{y_i\}_{i=k-m_k-1}^{k-1}$ is a constant sequence, then it follows from (43) and the fact that $\Delta(\ln y_i)=0$ and $s_{m_k,k}^2=0$, that

$$\hat{\mu}_{m_k,k} \to \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}.$$

Hence, it follows from (33) that $\hat{a}_{m_k,k}=0$.

Remark 12.

The estimated parameters a, μ, and σ^2 depend upon the time at which data point is drawn. This is expected because of the nonlinearity of the dynamic model together with environmental stochastic perturbations generate non stationary solution process. Using this locally estimated parameters of the continuous-time dynamic system, we can find the average of these local parameters over the entire size of data set as follows:

$$\begin{cases} \overline{a} & = \frac{1}{N}\sum_{i=0}^{N} a_{\hat{m}_i,i}, \\ \overline{\mu} & = \frac{1}{N}\sum_{i=0}^{N} \mu_{\hat{m}_i,i} \\ \overline{\sigma^2} & = \frac{1}{N}\sum_{i=0}^{N} \sigma^2_{\hat{m}_i,i}. \end{cases} \tag{44}$$

Here, $\overline{a}$, $\overline{\mu}$, and $\overline{\sigma^2}$ are referred to as aggregated parameter estimates of a, μ, and σ^2 over the given entire finite interval of time, respectively.

Remark 13.

The DTIDMLSMVSP and its transformation of data are utilized in (33), (34), (35), (43), and (44) for updating statistic coefficients of equations in (30). This accelerates the computation process. Furthermore, the DTIDMLSMVSP plays a significant role in the local discretization and model validation errors.

Illustration 2: Dynamic Model for U.S. Treasury Bill Interest Rate and the USD-EUR Exchange Rate.

As noted above, at reference numeral **204** in FIG. **2**, the process **200** includes developing a stochastic model of a continuous time dynamic process. As another example of this, the scheme presented above can be applied for estimating parameters of a continuous-time model for U.S. Treasury Bill Interest Rate and USD-EUR Exchange Rate processes. By employing dynamic modeling process, a continuous time dynamic model of interest rate process under random environmental perturbations can be described by

$$dy=(\beta y+\mu y^{\delta})dt+\sigma y^{\gamma}dW(t), y(t_0)=y_0, \tag{45}$$

where $\beta, \mu, \delta, \sigma, \gamma \in \Re$; $y(t, t_0, y_0)$ is adapted, non-anticipating solution process with respect to $\mathcal{F}_t$, the initial process y_0 is $\mathcal{F}_{t_0}$ measurable and independent of $\{W(t), t\in[t_0, T]\}$, and W(t) is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$.

Transformation of Stochastic Differential Equation (45).

As part of the conceptual aspects of generating the DTIDMLSMVSP, at reference numeral **206** in FIG. **2**, the process **200** can include transforming the stochastic model of the continuous time dynamic process into a stochastic model of a discrete time dynamic process utilizing a discretization scheme. As another example of this, for (45), the Lyapunov functions $V_1(t,y)=\frac{1}{2}y^2$ and $V_2(t,y)=\frac{1}{3}y^3$ as in (13) are considered. The Itô-differentials of V_i, for i=1, 2, are given by

$$\begin{cases} dV_1 = \left[y(\beta y+\mu y^{\delta})+\frac{1}{2}\sigma^2 y^{2\gamma}\right]dt+\sigma y^{\gamma+1}dW \\ dV_2 = [y^2(\beta y+\mu y^{\delta})+\sigma^2 y^{2\gamma+1}]dt+\sigma y^{\gamma+2}dW \end{cases}. \tag{46}$$

The Euler-Type Numerical Schemes for (45) and (46)).

Following the approach in Section 3.5, the Euler discretized scheme (Delta=1) for (45) is defined by

$$\begin{cases} \Delta y_i & = (\beta y_{i-1}+\mu y_{i-1}^{\delta})+\sigma y_{i-1}^{\gamma}\Delta W(t_i) \\ \frac{1}{2}\Delta(y_i^2) & = y_{i-1}(\beta y_{i-1}+\mu y_{i-1}^{\delta})+\frac{1}{2}\sigma^2 y_{i-1}^{2\gamma}+\sigma y_{i-1}^{\gamma+1}\Delta W_i \\ \frac{1}{3}\Delta(y_i^3) & = y_{i-1}^2(\beta y_{i-1}+\mu y_{i-1}^{\delta})+\sigma^2 y_{i-1}^{2\gamma+1}+\sigma y_{i-1}^{\gamma+2}\Delta W_i \end{cases}. \tag{47}$$

Generalized Moment Equations.

As another part of the conceptual aspects of generating the DTIDMLSMVSP, at reference numeral **206** in FIG. **2**, the process **200** can also include developing a system of generalized method of moments equations from the stochastic model of the discrete time dynamic process. As another example of this, applying conditional expectation to (47) with respect to $\mathcal{F}_{i-1}$,

$$\begin{aligned} \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] &= \beta y_{i-1}+\mu y_{i-1}^{\delta} \\ \frac{1}{2}\mathbb{E}[\Delta(y_i^2)\mid \mathcal{F}_{i-1}] &= \beta y_{i-1}^2+\mu y_{i-1}^{\delta+1}+\frac{1}{2}\sigma^2 y_{i-1}^{2\gamma} \\ \frac{1}{3}\mathbb{E}[\Delta(y_i^3)\mid \mathcal{F}_{i-1}] &= \beta y_{i-1}^3+\mu y_{i-1}^{\delta+2}+\frac{1}{2}\sigma^2 y_{i-1}^{2\gamma+1}. \\ \mathbb{E}[(\Delta y_i-\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}] &= \sigma^2 y_{i-1}^{2\gamma}, \\ \frac{1}{4}\mathbb{E}\left[(\Delta(y_i^2)\mathbb{E}[\Delta(y_i^2)])^2 \,\middle|\, \mathcal{F}_{i-1}\right] &= \sigma^2 y_{i-1}^{2\gamma+2} \end{aligned} \tag{48}$$

Basis for Lagged Adaptive Discrete Time Expectation Process.

From (48), (47) reduces to

$$\begin{cases} \Delta y_i & = \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}]+\sigma y_{i-1}^{\gamma}\Delta W(t_i) \\ \frac{1}{2}\Delta(y_i^2) & = \frac{1}{2}\mathbb{E}[\Delta(y_i^2)\mid \mathcal{F}_{i-1}]+\sigma y_{i-1}^{\gamma+1}\Delta W_i \\ \frac{1}{3}\Delta(y_i^3) & = \frac{1}{3}\mathbb{E}[\Delta(y_i^3)\mid \mathcal{F}_{i-1}]+\sigma y_{i-1}^{\gamma+2}\Delta W_i \end{cases}. \tag{49}$$

Remark 14. (Orthogonality Condition Vector for (45) and (46)).

Again, imitating Remarks 7, 8 and 9 in the context of (45), (46), (47), (48), and (49), the orthogonality condition vector with respect to the continuous-time stochastic dynamic model (45) is

$$H(t_{i-1}, y(t_i), y(t_{i-1})) = \begin{pmatrix} \Delta y(t_i) - (\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1}))\Delta t_i \\ \frac{1}{2}\Delta(y^2(t_i)) - L(y^2(t_{i-1}))\Delta t_i \\ \frac{1}{3}\Delta(y^3(t_i)) - L(y^3(t_{i-1}))\Delta t_i \\ (y(t_i) - (\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1}))\Delta t_i)^2 - \sigma^2 y^{2\gamma}(t_{i-1})\Delta t_i \\ \left(\frac{1}{2}\Delta(y^2(t_i)) - L(y^2(t_{i-1}))\Delta t_i\right)^2 - \sigma^2 y^{2\gamma+2}(t_{i-1})\Delta t_i \end{pmatrix}, \tag{50}$$

where

$$L(y^2(t_{i-1}))\Delta t_i = \left(y(t_{i-1})(\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1})) + \frac{1}{2}\sigma^2 y^{2\gamma}(t_{i-1})\right)\Delta t_i$$

and

$$L(y^3(t_{i-1}))\Delta t_i = (y^2(t_{i-1})(\beta y(t_{i-1}) + \mu y^{\delta}(t_{i-1})) + \sigma^2 y^{2\gamma+1}(t_{i-1}))\Delta t_i.$$

Further, unlike other orthogonality condition vectors, this orthogonality condition vector is based on the discretization scheme (47) associated with nonlinear continuous-time stochastic differential equations (45) and (46).

Local Observation System of Algebraic Equations.

Following the argument used in (33), for $k \in I_0(N)$, applying the lagged adaptive expectation process from Definitions 3-7, and using (8) and (48), we formulate a local observation/measurement process at t_k as a algebraic functions of m_k-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to subpartition P in Definition 2, as

$$\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k} + \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k} \tag{51}$$

$$\frac{1}{2m_k}\sum_{i=k-m_k}^{k-1} [\mathbb{E}[\Delta(y_i^2) \mid \mathcal{F}_{i-1}] - \mathbb{E}[(\Delta y_i \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}]] = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k} + \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k}$$

$$\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}[\Delta(y_i^3) \mid \mathcal{F}_{i-1}] - \sigma^2 y_{i-1}^{2\gamma+1}\right] = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} + \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k}$$

$$\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[(\Delta y_i - \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}] = \sigma^2\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma}}{m_k},$$

$$\frac{1}{4m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}\left[(\Delta(y_i^2) - \mathbb{E}[\Delta(y_i^2)])^2 \mid \mathcal{F}_{i-1}\right] = \sigma^2\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma+2}}{m_k}.$$

Following the approach discussed in Section 5, the solution of $\sigma_{m_k,k}$ is given by

$$\sigma_{m_k,k} = \left[\frac{s_{m_k,k}^2}{\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}}}\right]^{1/2}, \tag{52}$$

and $\gamma_{m_k,k}$ satisfies the following nonlinear algebraic equation

$$s_{m_k,k}^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+2} - \frac{1}{4}s_{m_k,k}^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}} = 0, \tag{53}$$

where $s_{m_k,k}^2$, and $s_{m_k,k}^2$ denotes the local moving variance of Δy_i and $\Delta(y_i^2)$ respectively.

To solve for the parameters β and μ, and δ, we define the conditional moment functions

$$F_j \equiv F_j(\mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(y_i)^2 \mid \mathcal{F}_{i-1}], \mathbb{E}[\Delta(y_i)^3 \mid \mathcal{F}_{i-1}]),\ j = 1, 2, 3$$

as

$$F_1 = \frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \mathbb{E}[\Delta y_i \mid \mathcal{F}_{i-1}] - \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k} - \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k} \tag{54}$$

$$F_2 = \frac{1}{2m_k}\sum_{i=k-m_k}^{k-1} [\mathbb{E}[\Delta(y_i^2) \mid \mathcal{F}_{i-1}], \mathbb{E}[(\Delta y_i - \mathbb{E}[(\Delta y_i \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}]] - \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k} - \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k}$$

$$F_3 = \frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}[\Delta(y_i^3) \mid \mathcal{F}_{i-1}] - \sigma^2 y_{i-1}^{2\gamma+1}\right] - \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} - \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k}.$$

Using (51), we have

$$\begin{cases} F_1 = 0 \\ F_2 = 0. \\ F_3 = 0 \end{cases} \tag{55}$$

Let F={F₁, F₂, F₃}. The determinant of the Jacobian matrix of F is given by

$$JF(\beta,\mu,\delta) = -\frac{1}{m_k^3}\det\begin{pmatrix} \sum_{i=k-m_k}^{k-1} y_{i-1} & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta} & \sum_{i=k-m_k}^{k-1} (\ln y_{i-1})y_{i-1}^{\delta} \\ \sum_{i=k-m_k}^{k-1} y_{i-1}^2 & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1} & \sum_{i=k-m_k}^{k-1} (\ln y_{i-1})y_{i-1}^{\delta+1} \\ \sum_{i=k-m_k}^{k-1} y_{i-1}^3 & \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2} & \sum_{i=k-m_k}^{k-1} (\ln y_{i-1})y_{i-1}^{\delta+2} \end{pmatrix} \neq 0, \tag{56}$$

provided $\delta \neq 1$ and the sequence $\{y(t_{i-1})\}_{i=k-m_k}^{k-1}$ is neither zero nor a constant sequence. Thus, by the application of Theorem 2 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{y(t_i)\}_{i=k-m_k}^{k-1}$, $\delta \neq 1$, there exist a solution of system of algebraic equations (55) $\hat{\beta}_{m_k,k}$, $\hat{\mu}_{m_k,k-1}$, $\hat{\delta}_{m_k,k}$ as a point estimates of β and μ, and δ respectively.

The solution of system of algebraic equations (55) is given by

$$\begin{cases} \hat{\mu}_{m_k,k} = \dfrac{\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \Delta y_i \sum_{i=k-m_k}^{k-1} y_{i-1}^2 - \frac{1}{2}\left[\frac{1}{m_k}\sum_{i=k-m_k}^{k-1} \Delta(y_i^2) - s_{m_k,k}^2\right]\sum_{i=k-m_k}^{k-1} y_{i-1}}{\frac{1}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}} \sum_{i=k-m_k}^{k-1} y_{i-1}^2 - \sum_{i=k-m_k}^{k-1} y_{i-1}^{1+\delta_{m_k,k}} \sum_{i=k-m_k}^{k-1} y_{i-1}\right]}, \\ \hat{\beta}_{m_k,k} = \dfrac{\sum_{i=k-m_k}^{k-1} \Delta y_i - \hat{\mu}_{m_k,k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}}}{\sum_{i=k-m_k}^{k-1} y_{i-1}} \end{cases} \tag{57}$$

where $\delta_{m_k,k}$ satisfies the third equation in (51) described by

$$\frac{1}{3m_k}\sum_{i=k-m_k}^{k-1} \Delta(y_i^3) - \frac{\sigma_{m_k,k}^2}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+1} - \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} - \mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k} = 0 \tag{58}$$

The parameters of continuous-time dynamic process described by (45) are time-varying functions. This justifies the modifications/correctness needed for the development of continuous-time models of dynamic processes.

Remark 15.

The illustrations presented above exhibit the important features described in Remark 8 of the theoretical parameter estimation procedure. The illustrations further clearly differentiate the Itô-Doob differential formula based formation of orthogonality condition vectors in Remarks 9 and 14 and the algebraic manipulation and discretized scheme using the econometric specification based orthogonality condition vectors.

Remark 16. The DTIDMLSMVSP and its transformation of data are utilized in (51), (52), (53), (57), and (58) for updating statistic coefficient of equations in (45). Again, this accelerates the computation process. Furthermore, the DTIDMLSMVSP plays a significant role in the local discretization and model validation errors.

4. Computational Algorithm

In this section, the computational, data organizational, and simulation schemes are outlined. The ideas of iterative data process and data simulation process time schedules in relation with the real time data observation/collection schedule are also introduced. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is important to determine an admissible set of local conditional sample average and sample variance, in particular, the size of local conditional sample in the context of a partition of time interval $[-\tau, T]$. Further, the discrete time dynamic model of conditional sample mean and sample variance statistic processes in Section 2 and the theoretical parameter estimation scheme in Section 3 coupled with the lagged adaptive expectation process motivate to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

4.1. Coordination of Data Observation, Iterative Process, and Simulation Schedules

Without loss of generality, we assume that the real data observation/collection partition schedule P is defined in (2). Now, we present definitions of iterative process and simulation time schedules.

Definition 8.

The iterative process time schedule in relation with the real data collection schedule is defined by

$$IP = \{F^{-r}t_i: \text{ for } t_i \in P\}, \tag{59}$$

where $F^{-r}t_i = t_{i-r}$, and F^{-r} is a forward shift operator.

The simulation time is based on the order p of the time series model of m_k-local conditional sample mean and variance processes in Lemma 1 in Section 2.

Definition 9.

The simulation process time schedule in relation with the real data observation schedule is defined by

$$SP = \begin{cases} \{F^{r}t_i: \text{ for } t_i \in P\}, \text{ if } p \le r \\ \{F^{p}t_i: \text{ for } t_i \in P\}, \text{ if } p > r \end{cases}. \tag{60}$$

Remark 17.

The initial times of iterative and simulation processes are equal to the real data times t_r and t_p, respectively. Further, iterative and simulation process times in (59) and (60), respectively, justify Remark 10. In short, t_i is the scheduled time clock for the collection of the i th observation of the state of the system under investigation. The iterative process and simulation process times are t_{i+r} and t_{i+p}, respectively.

4.2. Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, a few concepts are introduced below, including local admissible sample/data observation size, m_k-local admissible conditional finite sequence at $t_k \in SP$, and local finite sequence of parameter estimates at t_k.

Referring back to the drawings, as part of the computational aspects of generating the DTIDMLSMVSP at reference numeral **206** (FIG. **2**), in FIG. **3** the process includes selecting at least one partition P in the time interval $[-\tau,0]$ of the discrete time data set $[-\tau,T]$ as past state information of a continuous time dynamic process at reference numeral

302. As described herein, multiple partitions P in the time interval $[-\tau,0]$ can be selected in the iterative, nested process.

Definition 10.

For each $k \in I_0(N)$, we define local admissible sample/data observation size m_k at t_k as $m_k \in OS_k$, where

$$OS_k = \begin{cases} I_2(r+k-1), & \text{if } p \le r, \\ I_2(p+k-1), & \text{if } p > r, \end{cases} . \tag{61}$$

Further, OS_k is referred as the local admissible set of lagged sample/data observation size at t_k. In other words, at reference numeral **304** in FIG. **3**, at each time point in the partition P, the process includes selecting an m_k-point sub-partition P_k of the partition P, the m_k-point sub-partition having a local admissible lagged sample observation size OS_k based on p, r, and a sub-partition time observation index size k.

Definition 11.

For each admissible $m_k \in OS_k$ in Definition 10, an m_k-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at t_k to subpartition P_k of P in Definition 3 is defined by $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$. Further, an m_k-class of admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k is defined by

$$\mathcal{AS}_k = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} : m_k \in OS_k\} = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}\}_{m_k \in OS_k}. \tag{62}$$

In other words, at reference numeral **306** in FIG. **3**, for each m_k-point in each sub-partition P_k, the process includes selecting an m_k-local moving sequence in the sub-partition to gather an m_k-class of admissible restricted finite sequences.

Without loss of generality, in the case of energy commodity model, for example, for each $m_k \in OS_k$, the corresponding m_k-local admissible adapted finite sequence of conditional sample/data observation at t_k, $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$ is found. Using this sequence and (43), $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}_{m_k,k}^2$ are computed. This leads to a local admissible finite sequence of parameter estimates at t_k defined on OS_k as follows: $\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}_{m_k,k}^2)\}_{m_k \in OS_k} = \{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}_{m_k,k}^2)\}_{m_k \in 2}^{r+k-1}$ or $\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}_{m_k,k}^2)\}_{m_k \in 2}^{p+k-1}$. It is denoted by

$$(\mathcal{A}_k, \mathbb{M}_k, \mathcal{S}_k) = \{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}_{m_k,k}^2)\}_{m_k \in OS_k}. \tag{63}$$

4.3. Conceptual Computation of State Simulation Scheme:

For the development of a conceptual computational scheme, the method of induction can be employed. The presented simulation scheme is based on the idea of lagged adaptive expectation process. An autocorrelation function (ACF) analysis performed on $s_{m_k,k}^2$ suggests that the discrete time interconnected dynamic model of local conditional sample mean and sample variance statistic in (8) is of order p=2. In view of this, the initial data is identified. Referring to FIG. **3**, at reference numeral **308**, the process includes, for each of the plurality of admissible parameter estimates, calculating a state value of the stochastic model of the continuous time dynamic process to gather a plurality of state values of the stochastic model of the continuous time dynamic process. For example, it is possible to begin with a given set of initial data y_{t_0}, $\{\hat{s}_{m_0,0}^2\}_{m_0 \in OS_0}$, $\{\hat{s}_{m_{-1},-1}^2\}_{m_{-1} \in OS_{-1}}$, and $\{\hat{s}_{m_{-1},-1}^2\}_{m_{-1} \in OS_{-1}}$. Let $y_{m_k,k}^s$ be a simulated value of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k corresponding to a local admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{m-1} \in \mathcal{AS}_k$ in (62). This simulated value is derived from the discretized Euler scheme (29) by

$$y_{m_k,k}^s = y_{m_{k-1},k-1}^s + \hat{a}_{m_{k-1},k-1}(\hat{\mu}_{m_{k-1},k-1} - y_{m_{k-1},k-1}^s) y_{m_{k-1},k-1}^s \Delta t + \sigma_{m_{k-1},k-1} y_{m_{k-1},k-1}^s \Delta W_{m_k,k}. \tag{64}$$

Further, let

$$\{y_{m_k,k}^s\}_{m_k \in OS_k} \tag{65}$$

be a m_k-local admissible sequence of simulated values corresponding to m_k-class $\mathcal{AS}_k$ of local admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k in (62). That is, at reference numeral **208** in FIG. **2**, the process **200** can include calculating a plurality of m_k-local admissible parameter estimates for the stochastic model of the continuous time dynamic process using the DTIDMLSMVSP.

4.4. Mean-Square Sub-Optimal Procedure

Using the m_k-local admissible parameter estimates, at reference numeral **210** in FIG. **2**, the process **200** can include calculating a state value of the stochastic model of the continuous time dynamic process for each of the plurality of admissible parameter estimates, to gather a plurality of state values of the stochastic model of the continuous time dynamic process. Further, at reference numeral **312** in FIG. **2**, the process **200** includes determining an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values. For example, to find the best estimate of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k from a m_k-local admissible finite sequence $\{y_{m_k,k}^s\}_{m_k \in OS_k}$ of a simulated value of $\{\mathbb{E}[y_i|\mathcal{F}_{k-1}]\}$, we need to compute a local admissible finite sequence of quadratic mean square error corresponding to $\{y_{m_k,k}^s\}_{m_k \in OS_k}$. The quadratic mean square error is defined below.

Definition 12.

The quadratic mean square error of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ relative to each member of the term of local admissible sequence $\{y_{m_k,k}^s\}_{m_k \in OS_k}$ of simulated values is defined by

$$\Xi_{m_k,k,y_k} = (\mathbb{E}[y_k|\mathcal{F}_{k-1}] - y_{m_k,k}^s)^2. \tag{66}$$

For any arbitrary small positive number ϵ and for each time t_k, to find a best estimate from the m_k-local admissible sequence $\{y_{m_k,k}^s\}_{m_k \in OS_k}$ of simulated values, the following ϵ-sub-optimal admissible subset of set of m_k-size local admissible lagged sample size m_k at t_k (OS_k) is defined as

$$\mathcal{M}_k = \{m_k : \Xi_{m_k,k,y_k} < \epsilon \text{ for } m_k \in OS_k\}. \tag{67}$$

There are three different cases that determine the ϵ-best sub-optimal sample size $\hat{m}_k$ at time t_k.

Case 1: If $m_k \in \mathcal{M}_k$ gives the minimum, then m_k is recorded as $\hat{m}_k$.

Case 2: If more than one value of $m_k \in \mathcal{M}_k$, then the largest of such m_k's is recorded as $\hat{m}_k$.

Case 3: If condition (67) is not met at time t_k, (e.g., $\mathcal{M}_k = \emptyset$), then the value of m_k where the minimum

$$\min_{m_k} \Xi_{m_k,k,y_k}$$

is attained, is recorded as $\hat{m}_k$. The ϵ-best sub-optimal estimates of the parameters $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}_{m_k,k}^2$ at the ϵ-best sub-optimal sample size $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma_{\hat{m}_k,k}^2$, respectively. It should be appreciated that the three cases described above present only one example way that a minimum error can be determined, and other ways are within the scope of the embodiments.

At reference numeral **214**, the process **200** further includes identifying an optimal m_k-local moving sequence $\hat{m}_k$ among the m_k-class of admissible restricted finite sequences based on the minimum error. For example, the simulated value $y_{m_k,k}^s$ at time t_k with $\hat{m}_k$ is now recorded as the ϵ-best sub-optimal state estimate for $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k. This ϵ-best sub-optimal simulated value of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k is denoted by $y_{\hat{m}_k,k}^s$.

In addition to comparative statements in Sections 2 together with Remarks 7, 8, 9, 13, 14, 15, and 16, the following comparisons between the LLGMM and the existing OCBGMM are noted: The LLGMM approach is focused on parameter and state estimation problems at each data collection/observation time t_k using the local lagged adaptive expectation process. LLGMM is discrete time dynamic process. On the other hand, the OCBGMM is centered on the state and parameter estimates using the entire data that is to the left of the final data collection time $T_N=T$. Implied weakness in forecasting, as seen in the next section, is explicitly shown with the OCBGMM approach and the ensuing results.

It is noted that Remark 8 exhibits the interactions/interdependence between the first three components of LLGMM, including (1) the development of the stochastic model for continuous-time dynamic process, (2) the development of the discrete time interconnected dynamic model for statistic process, and (3) using the Euler-type discretized scheme for nonlinear and non-stationary system of stochastic differential equations and their interactions. On the other hand, the OCBGMM is partially connected. From the development of the computational algorithm in Section 4, the interdependence/interconnectedness of the four remaining components of the LLGMM, including (4) employing lagged adaptive expectation process for developing generalized method of moment equations, (5) introducing conceptual computational parameter estimation problem, (6) formulating conceptual computational state estimation scheme, and (7) defining conditional mean square ϵ-sub optimal procedure are clearly demonstrated. Further, the components above and the data are directly connected with the original continuous-time SDE. On the other hand, the OCBGMM is composed of single size, single sequence, single estimates, single simulated value, and single error. Hence, the OCBGMM is the "single shot approach". Further, the OCBGMM is highly dependent on its second component rather than the first component.

As discussed above, the LLGMM is a discrete time dynamic system composed of seven interactive interdependent components. On the other hand, the OCBGMM is static dynamic process of five almost isolated components. Furthermore, the LLGMM is a "two scale hierarchic" quadratic mean-square optimization process, but the optimization process of OCBGMM is "single-shot". Further, the LLGMM performs in discrete time but operates like the original continuous-time dynamic process. As further shown below, the performance of the LLGMM approach is superior to the OCBGMM and IRGMM approaches.

The LLGMM does not require a large size data set. In addition, as k increases, it generates a larger size of lagged adapted data set, and thereby it further stabilizes the state and parameter estimation procedure with finite size data set, on the other hand the OCBGMM does not have this flexibility. The local adaptive process component of LLGMM generates conceptual finite chain of discrete time admissible sets/sub-data. The OCBGMM does not possess this feature.

The LLGMM generates a finite computational chain. The OCBGMM does not possess this feature. A further comparative summary analysis is described in Sections 6 and 7 in context of conceptual, computational, and statistical settings and exhibiting the role, scope, and performance of the LLGMM.

Remark 19.

The choice of p=2 can be determined based on the statistical procedure known as the Autocorrelation Function Analysis (AFA).

Illustration 1: Application of Conceptual Computational Algorithm to Energy Commodity Data Set.

As one example, the conceptual computational algorithm is applied to the real time daily Henry Hub Natural gas data set for the period 01/04/2000-09/30/2004, the daily crude oil data set for the period 01/07/1997-06/02/2008, the daily coal data set for the period of 01/03/2000-10/25/2013, and the weekly ethanol data set for the period of 03/24/2005-09/26/2013. The descriptive statistics of data for daily Henry Hub Natural gas data set for the period 01/04/2000-09/30/2004, the daily crude oil data set for the period 01/07/1997-06/02/2008, the daily coal data set for the period of 01/03/2000-10/25/2013, and the weekly ethanol data set for the period of 03/24/2005-09/26/2013, are recorded in the Table 1 below.

TABLE 1

Descritive Statistics			
Data Set Y	N	$\hat{Y}$	Std(Y)
Nat. Gas	1184 (days)	4.5504	1.5090)
Crude Oil	4165 (days)	54.0093	31.0248
Coal	3470 (days)	27.1441	17.8394
Ethanol	438 (weeks)	2.1391	0.4455

Sample size, mean, and standard deviation of energy commodities data are computed. N represents the sample size of corresponding data set.

Graphical, Simulation and Statistical Results—Case 1.

Three cases are considered for the initial delay r and show that, as r increases, the root mean square error reduces significantly. Here, we pick r=5, Δt=1, ε=0.001, and p=2, the ε-best sub-optimal estimates of parameters a, μ and σ^2 at each real data times are exhibited in Table 2.

Table 2 shows the ε-best sub-optimal local admissible sample size $\hat{m}_k$ and the parameters $a_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}^2$, and $\mu_{\hat{m}_k,k}$ for four price energy commodity data at time t_k. This was based on p≤r, and the initial real data time-delay r=5. We further note that the range of the ε-best sub-optimal local admissible sample size $\hat{m}_k$ for any time $t_k\in[5,25]\cup[1145,1165]$, $t_k\in[5,25]\cup[2440,2460]$, $t_k\in[5,25]\cup[2865,2885]$, $t_k\in[5,25]\cup[375,395]$ for natural gas, crude oil, coal and ethanol data, respectively as

$$2\le\hat{m}_k\le 5. \tag{68}$$

TABLE 2

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
	Natural gas			
5	3	0.0001	2.2231	0.6011
6	3	0.0002	2.2160	0.6122
7	3	0.0002	2.2513	0.6087
8	4	0.0002	2.2494	0.1628
9	4	0.0002	2.2658	−0.1497

TABLE 2-continued

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
10	4	0.0003	2.1371	0.1968
11	4	0.0004	2.5071	−0.2781
12	4	0.0000	2.2550	0.3545
13	4	0.0005	2.5122	0.6246
14	4	0.0015	2.4850	0.5604
15	3	0.0007	2.5378	0.4846
16	3	0.0007	2.5715	0.7737
17	5	0.0011	2.5688	0.5984
18	4	0.0010	2.5831	0.5423
19	5	0.0007	2.5893	0.4256
20	5	0.0006	2.6100	0.0683
21	5	0.0007	2.3171	0.2893
22	4	0.0015	2.7043	0.6983
23	3	0.0009	2.6590	0.8316
24	3	0.0010	2.6917	0.1822
25	4	0.0017	2.5620	0.2201
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
1145	4	0.0003	5.7203	0.1225
1146	3	0.0003	5.6651	0.2031
1147	3	0.0002	5.6601	0.3133
1148	5	0.0006	5.6909	0.216
1149	3	0.0003	5.6982	0.2404
1150	5	0.0006	5.6108	0.1362
1151	5	0.0006	5.61	0.1089
1152	5	0.0006	5.4383	0.06272
1153	4	0.0003	5.4307	0.1755
1154	5	0.0005	5.4155	0.1569
1155	3	0.0004	5.3742	−2.275
1156	5	0.0006	5.4405	0.1392
1157	4	0.0003	5.4423	0.2339
1158	4	0.0008	5.4276	0.1712
1159	5	0.0006	5.3958	0.1309
1160	3	0.0002	5.3557	−0.1882
1161	3	0.0003	5.5081	−0.0696
1162	4	0.0003	4.908	0.0381
1163	4	0.0002	5.0635	0.1038
1164	3	0.0002	5.082	0
1165	4	0.0002	5.1099	−0.2756
		Crude oil		
5	3	0.0001	24.4100	0.0321
6	3	0.0002	24.7165	0.0341
7	4	0.0003	25.5946	0.0537
8	5	0.0006	25.5550	0.0467
9	4	0.0006	25.5695	0.0499
10	4	0.0004	25.4787	0.0221
11	3	0.0001	25.7742	0.0100
12	3	0.0002	26.9477	−0.0157
13	3	0.0001	25.8786	−0.0112
14	5	0.0005	22.1834	0.0049
15	5	0.0004	23.5425	0.0010
16	4	0.0002	23.8500	0.0000
17	4	0.0002	23.8486	0.0502
18	5	0.0004	23.2913	−0.0113
19	3	0.0000	24.4715	0.1282
20	3	0.0004	24.3878	0.0415
21	5	0.0003	24.3336	0.2067
22	4	0.0002	23.9993	0.0200
23	4	0.0001	24.1909	−0.0894
24	3	0.0002	25.0812	−0.0252
25	3	0.0002	22.2942	0.0064
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2440	5	0.0003	58.431	0.0141
2441	5	0.0003	57.205	0.0084
2442	4	0.0001	57.554	0.0165
2443	5	0.0003	57.871	0.0168
2444	5	0.0003	60.441	0.0023
2445	5	0.0003	38.954	−0.0006
2446	4	0.0006	59.659	0.0165
2447	4	0.001	59.548	0.016
2448	4	0.0007	58.964	0.0115
2449	4	0.0005	58.415	0.0166
2450	5	0.0003	58.61	0.0193
2451	4	0.0004	59.244	0.0091
2452	5	0.0003	58.955	0.0143
2453	4	0.0004	59.508	0.0179

TABLE 2-continued

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
2454	4	0.0003	59.978	0.0193
2455	5	0.0003	59.957	0.0199
2456	4	0.0005	59.849	0.0163
2457	5	0.0004	59.441	0.0095
2458	4	0.0003	58.479	0.0103
2459	4	0.0002	57.917	0.0158
2460	4	0.0005	56.122	0.0062
		Coal		
5	3	0.0001	11.5534	0.0142
6	3	0.0000	11.2529	0.4109
7	3	0.0001	9.9161	0.0165
8	3	0.0002	11.4663	−0.0403
9	3	0.0005	10.5922	−0.0843
10	4	0.0009	8.9379	0.0714
11	4	0.0023	8.9051	0.1784
12	3	0.0015	9.0169	0.0855
13	3	0.0020	8.6231	0.0739
14	2	0.0001	10.0100	0.0564
15	5	0.0067	9.5281	0.0741
16	4	0.0058	6.1821	0.0694
17	4	0.0015	8.8087	0.0404
18	4	0.0035	9.0681	0.0652
19	3	0.0040	9.0752	0.1527
20	3	0.0049	9.0801	0.1405
21	4	0.0043	8.9898	0.0946
22	5	0.0054	8.9148	0.0036
23	4	0.0018	8.6771	0.0884
24	5	0.0035	8.7586	0.0985
25	5	0.0006	8.4779	−0.1155
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2865	3	0.001	37.657	0.0397
2866	3	0.0006	37.73	0.0468
2867	5	0.0014	39.6	0.0087
2868	3	0.0006	38.769	0.0331
2869	5	0.0019	38.272	0.0245
2870	3	0.0014	37.627	0.0234
2871	3	0.0004	37.753	−0.243
2872	4	0.0008	36.11	0.0101
2873	5	0.0015	33.823	0.0042
2874	4	0.0009	35.221	0.0183
2875	5	0.0011	33.381	0.0084
2876	4	0.0007	34.6	0.0228
2877	3	0.001	34.463	0.0441
2878	5	0.0009	34.583	0.0334
2879	5	0.0008	34.63	0.0443
2880	4	0.0005	35.221	0.0207
2881	5	0.0007	35.249	0.0196
2882	3	0.0003	35.583	0.1566
2883	4	0.0004	36.036	0.0224
2884	3	0.0005	36.276	0.0373
2885	4	0.0004	36.195	0.0374
		Ethanol		
5	2	0.0002	1.1767	0.5831
6	5	0.0008	1.1717	0.5159
7	4	0.0007	1.1707	1.4925
8	5	0.0008	1.1713	1.4791
9	5	0.0006	1.1709	2.1406
10	4	0.0004	1.1900	0.8621
11	3	0.0025	1.1900	0.3719
12	3	0.0004	1.2188	0.5368
13	5	0.0004	1.1120	12.2917
14	5	0.0007	1.1669	−0.9289
15	5	0.0014	0.7492	−0.0879
16	5	0.0011	1.7968	0.3087
17	5	0.0002	1.8484	−0.1901
18	5	0.0003	1.1650	−0.1611
19	5	0.0022	1.8943	0.1502
20	5	0.0047	1.8144	0.2073
21	4	0.001	1.8400	0.0464
22	3	0.0020	3.7350	0.1628
23	3	0.0008	1.9905	0.1599
24	3	0.0018	1.9006	−3.4926
25	4	0.0234	2.4827	0.1837
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .

TABLE 2-continued

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
375	3	0.0008	2.1456	1.1005
376	4	0.0012	2.0689	0.2666
377	3	0.0009	2.0538	0.4339
378	3	0.0008	2.054	0.7726
379	4	0.0007	2.0551	0.7588
380	3	0.0003	2.0692	4.5252
381	5	0.0021	1.995	−0.4407
382	5	0.0025	1.3252	−0.048
383	5	0.0023	0.82891	−0.04
384	4	0.0025	2.5937	0.3073
385	3	0.0064	2.6054	0.6097
386	5	0.0044	2.5947	0.4157
387	3	0.0035	2.595	0.354
388	3	0.0018	2.6054	0.6561
389	5	0.0043	2.5992	0.3862
390	3	0.0009	2.5812	0.3334
391	4	0.0013	2.6299	−0.3594
392	4	0.0013	2.6776	−0.2827
393	4	0.0011	1.5114	0.0394
394	3	0.0006	2.2927	0.5982
395	5	0.0035	2.3275	0.3191

Remark 20.

From (68), the following conclusions can be drawn:

(a) From (61) and Definition 10 (OS_k), at teach time t_k for the four energy price data sets, the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ is attained on the subset {2, 3, 4, 5} of (OS_k). Hence, the ϵ-best sub-optimal local state and parameter estimates are obtained in at most four iterates rather than k+r−1.

(b) The basis for the conclusion (a) is due to the fact that the ϵ-best sub-optimization process described in Subsections 4.3 and 4.4 stabilize the local state and parameter estimations at each time t_k.

(c) From (a) and (b), we further remark that, in practice, the entire local lagged admissible set OS_k of size m_k at time t_k is not fully utilized. In fact, for any m_k in OS_k and $m_k > \hat{m}_k$ such that as m_k approached to k+r−1, the corresponding state and parameters relative to m_k approach to the ϵ-best sub-optimal local state and parameter estimates relative to the ϵ-best sub-optimal local admissible sample size at time t_k. This is not surprising because of the nature of the state hereditary process, that is, as the size of the time-delay m_k increases, the influence of the past state history decreases.

(d) From (c), we further conclude that the second DTIDMLSMVSP and the fourth component (local lagged adaptive process) of the LLGMM are stabilizing agents. This justifies the introduction of the term conceptual computational state and parameter estimation scheme. These components play a role of not only the local ϵ-best suboptimal quadratic error reduction, but also local error stabilization problem depending on the choice of ϵ.

(e) The conclusions (a), (b), (c) and (d) are independent of a "large" data size and stationary conditions.

Remark 21.

We remark that $\{\mu_{\hat{m}_i,i}\}_{i=0}^{N}$ and $\{a_{\hat{m}_i,i}\}_{i=0}^{N}$ are discrete time ϵ-best sub-optimal simulated random samples generated by the scheme described at the beginning of Section 4.5.

Remark 22.

We have used the estimated parameters $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, and $\sigma_{\hat{m}_k,k}^2$ in Table 2 to simulate the daily prices of natural gas, crude oil, coal, and ethanol. Using the computer readable instructions described herein and the parameters described in Table 2, we simulate the daily prices of natural gas, crude oil, coal, and ethanol. For this purpose, we pick ϵ=0.001; for each time t_k, we estimate the simulated prices $y_{\hat{m}_k,k}^s$.

Among the collected values m_k, the value that gives the minimum Ξ_{m_k,k,y_k} is recorded as $\hat{m}_k$. If condition (67) is not met at time t_k, the value of m_k where the minimum

$$\min_{m_k} \Xi_{m_k,k,y_k}$$

is attained, and is recorded as $\hat{m}_k$. The ϵ-best sub-optimal estimates of the parameters $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}_{m_k,k}^2$ at $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma_{\hat{m}_k,k}^2$, and the value of $y_{m_k,k}^s$ at time t_k corresponding to $\hat{m}_k$, $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma_{\hat{m}_k,k}^2$ is also recorded as the ϵ-best sub-optimal simulated value $y_{\hat{m}_k,k}^s$ of y_k. A detailed algorithm is given in Appendix D. In Table 3, the real and LLGMM simulated price values of the energy commodities: Natural gas, Crude oil, Coal, and Ethanol are exhibited in columns 2-3, 6-7, 10-11, and 14-15, respectively. The absolute error of each of the energy commoditys simulated value is shown in columns 4, 8, 12, and 16, respectively.

TABLE 3

Real, Simulation using LLGMM method, and absolute error of simulation with starting delay r = 5.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	$\lvert$Error$\rvert$ $\lvert y_k - y^s_{\hat{m}_k,k}\rvert$
	Natural gas		
5	2.216	2.216	0
6	2.260	2.253	0.007
7	2.244	2.241	0.003
8	2.252	2.249	0.003
9	2.322	2.329	0.007
10	2.383	2.376	0.007
11	2.417	2.417	0.000
12	2.559	2.534	0.025
13	2.485	2.554	0.069
14	2.528	2.525	0.003
15	2.616	2.615	0.001
16	2.523	2.478	0.045
17	2.610	2.638	0.028
18	2.610	2.606	0.004
19	2.610	2.614	0.004
20	2.699	2.726	0.027
21	2.759	2.748	0.011
22	2.659	2.638	0.021
23	2.742	2.737	0.005
24	2.562	2.561	0.001
25	2.495	2.487	0.008
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
1145	5.712	5.709	0.003
1146	5.588	5.592	0.004
1147	5.693	5.650	0.043
1148	5.791	5.786	0.005
1149	5.614	5.458	0.156
1150	5.442	5.460	0.018
1151	5.533	5.571	0.038
1152	5.378	5.397	0.019
1153	5.373	5.374	0.001
1154	5.382	5.420	0.038
1155	5.507	5.501	0.006
1156	5.552	5.551	0.001
1157	5.310	5.272	0.038
1158	5.338	5.348	0.010
1159	5.298	5.353	0.055
1160	5.189	5.207	0.018
1161	5.082	5.087	0.005
1162	5.082	5.207	0.125
1163	5.082	4.783	0.299

TABLE 3-continued

Real, Simulation using LLGMM method, and absolute error of simulation with starting delay r = 5.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\|y_k - y^s_{\hat{m}_k,k}\|$
1164	4.965	4.849	0.116
1165	4.767	4.733	0.034
	Crude oil		
5	25.200	25.200	0
6	25.100	25.077	0.023
7	25.950	25.606	0.344
8	25.450	25.494	0.044
9	25.400	25.411	0.011
10	25.100	24.981	0.119
11	24.800	24.763	0.037
12	24.400	24.301	0.099
13	23.850	24.862	1.012
14	23.850	23.961	0.111
15	23.850	24.010	0.160
16	23.900	24.071	0.171
17	24.500	24.554	0.054
18	24.800	24.795	0.005
19	24.150	24.165	0.015
20	24.200	23.971	0.229
21	24.000	24.028	0.028
22	23.900	23.886	0.014
23	23.050	23.253	0.203
24	22.300	22.586	0.286
25	22.450	22.418	0.032
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2440	57.350	57.298	0.052
2441	56.740	56.650	0.090
2442	57.550	57.613	0.063
2443	59.090	59.152	0.062
2444	60.270	58.926	1.344
2445	60.750	59.675	1.075
2446	58.410	59.408	0.998
2447	58.720	58.917	0.197
2448	58.640	58.502	0.138
2449	57.870	58.721	0.851
2450	59.130	58.985	0.145
2451	60.110	60.087	0.023
2452	58.940	58.858	0.082
2453	59.930	59.390	0.540
2454	61.180	60.283	0.897
2455	59.660	59.939	0.021
2456	58.590	58.49	0.100
2457	58.280	58.624	0.344
2458	58.790	59.188	0.398
2459	56.23	55.442	0.788
2460	55.900	56.055	0.155
	Coal		
5	10.560	10.560	0
6	10.240	10.436	0.196
7	10.180	10.325	0.145
8	9.560	10.072	0.512
9	8.750	8.338	0.412
10	9.060	9.072	0.012
11	8.880	9.084	0.204
12	9.440	9.581	0.141
13	10.310	9.739	0.571
14	9.810	9.633	0.177
15	9.060	9.197	0.137
16	8.750	8.806	0.056
17	8.820	8.879	0.059
18	9.560	9.326	0.234
19	8.820	8.749	0.071
20	8.820	8.774	0.046
21	8.690	8.867	0.177
22	8.630	8.519	0.111
23	8.690	8.693	0.003
24	8.940	8.952	0.012
25	9.310	9.374	0.064

TABLE 3-continued

Real, Simulation using LLGMM method, and absolute error of simulation with starting delay r = 5.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\|y_k - y^s_{\hat{m}_k,k}\|$
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2865	29.310	29.065	0.245
2866	28.680	28.619	0.061
2867	26.770	28.408	1.638
2868	27.450	27.480	0.03
2869	27.000	27.250	0.250
2870	26.670	26.544	0.126
2871	26.510	26.497	0.013
2872	26.480	26.463	0.017
2873	25.150	25.781	0.631
2874	25.570	25.615	0.045
2875	25.880	25.948	0.068
2876	25.240	25.451	0.211
2877	25.000	24.649	0.351
2878	25.080	24.984	0.096
2879	25.050	25.158	0.108
2880	25.890	25.835	0.055
2881	25.230	25.211	0.019
2882	25.940	25.727	0.213
2883	25.260	25.347	0.087
2884	25.250	25.276	0.026
2885	26.060	25.660	0.400
	Ethanol		
	1.190	1.190	0
	1.150	1.174	0.024
	1.180	1.180	0.000
	1.160	1.148	0.012
	1.190	1.196	0.006
	1.190	1.209	0.019
	1.225	1.186	0.039
	1.220	1.217	0.003
	1.290	1.250	0.040
	1.410	1.320	0.090
	1.470	1.392	0.078
	1.530	1.461	0.069
	1.630	1.545	0.085
	1.7.50	1.743	0.007
	1.750	1.858	0.108
	1.840	1.886	0.046
	1.895	1.916	0.021
	1.950	2.034	0.084
	1.974	2.033	0.059
	2.700	2.011	0.69
	2.515	2.332	0.179
	. . .	. . .	. . .
	. . .	. . .	. . .
	2.073	2.019	0.054
	2.020	2.003	0.017
	2.073	2.094	0.021
	2.065	2.076	0.011
	2.055	2.061	0.006
	2.209	2.169	0.040
	2.440	2.208	0.232
	2.517	2.220	0.297
	2.718	2.362	0.356
	2.541	1.687	0.146
	2.566	2.607	0.041
	2.626	1.549	0.077
	2.587	2.606	0.019
	2.628	2.624	0 004
	2.587	2.556	0.031
	2.536	2.546	0.010
	2.420	2.425	0.005
	2.247	2.245	0.002
	2.223	1.196	0.027

TABLE 3-continued

Real, Simulation using LLGMM method, and absolute error of simulation with starting delay r = 5.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	$\lvert$Error$\rvert$ $\lvert y_k - y^s_{\hat{m}_k,k}\rvert$
	2.390	1.381	0.009
	2.380	2.398	0.018

Graphical, Simulation and Statistical Results-Case 2.

For a better simulation result, we increase the magnitude of time delay r. We pick r=10, Δt=1, ϵ=0.001, and p=2, the ϵ-best sub-optimal estimates of parameters a, μ and σ^2 at each real data times are exhibited in Table 4.

TABLE 4

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\alpha_{\hat{m}_k,k}$ for initial delay r = 10.

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
		Natural gas		
11	8	0.0003	2.0015	0.1718
12	6	0.0003	2.1346	0.0131
13	7	0.0004	2.5701	0.0630
14	9	0.0007	2.6746	0.0461
15	7	0.0012	2.4415	0.407!
16	3	0.0013	2.5549	0.4621
17	8	0.0015	2.5576	0.1934
18	8	0.0014	2.5628	0.2495
19	7	0.0015	2.5705	0.3522
20	9	0.0011	2.5943	0.2946
21	9	0.0010	2.6947	0.0775
22	9	0.0010	2.6464	0.1883
23	3	0.0009	2.7139	0.6983
24	10	0.0013	2.6421	0.2966
25	9	0.0018	2.6387	0.2382
26	2	0.0015	2.5223	0.6595
27	4	0.0018	2.5464	0.3474
28	3	0.0008	2.5780	0.2807
29	2	0.0011	2.6588	−0.1271
30	7	0.0031	2.5610	0.3718
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
1145	4	0.0002	5.7205	0.1225
1146	4	0.0005	5.6485	0.0951
1147	4	0.0005	5.6704	0.2152
1148	7	0.0007	5.7158	0.1245
1149	4	0.0004	5.6800	0.2544
1150	6	0.0007	5.6551	0.1455
1151	4	0.0007	5.5648	0.0971
1152	10	0.0026	5.5582	0.0588
1153	5	0.0006	5.4049	0.1000
1154	5	0.0004	5.4155	0.1569
1155	8	0.0010	5.4718	0.0725
1156	7	0.0007	5.4528	0.1645
1157	8	0.0009	5.4595	0.2011
1158	5	0.0007	5.4185	0.1614
1150	7	0.0008	5.5905	0.1281
1160	9	0.0011	5.5567	0.0975
1161	8	0.0008	4.9559	0.0155
1162	8	0.0007	5.0020	0.0210
1165	7	0.0004	5.0947	0.0752
1164	5	0.0001	4.9554	0.0671
1165	9	0.0009	4.0877	0.0148
		Crude oil		
11	4	0.0003	24.3532	0.0100
12	4	0.0001	25.8537	−0.0157
13	3	0.0003	25.8786	−0.0152
14	10	0.0010	24.0633	0.0084
15	10	0.0009	22.7352	0.0025
16	4	0.0002	23.8665	0.0423
17	7	0.0005	24.0777	0.0194

TABLE 4-continued

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\alpha_{\hat{m}_k,k}$ for initial delay r = 10.

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
18	9	0.0008	24.2210	0.0138
19	7	0.0006	24.1147	0.0268
20	6	0.0004	24.2748	0.0256
21	7	0.0005	24.2175	0.0258
22	4	0.0002	23.9993	0.0317
23	10	0.0008	23.8479	0.0130
24	10	0.0009	24.7657	−0.0087
25	4	0.0001	21.8903	0.0115
26	4	0.0003	22.2871	0.0258
27	10	0.0011	35.7200	−0.0010
28	4	0.0003	22.1582	0.0391
29	6	0.0004	22.2194	0.0401
30	7	0.0005	22.296	0.0394
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2440	6	0.0004	58.4990	0.0149
2441	6	0.0004	57.7330	0.0070
2442	8	0.0006	58.1010	0.0086
2443	8	0.0006	58.2670	0.0105
2444	6	0.0004	60.6030	0.0027
2445	6	0.0003	70.6110	0.0005
2446	7	0.0003	58.6010	0.0072
2447	9	0.0009	58.7720	0.0077
2448	4	0.0006	58.9640	0.0115
2449	10	0.0011	58.4730	0.0073
2450	4	0.0003	58.5010	0.0344
2451	3	0.0003	59.6250	0.0077
2452	5	0.0003	58.9550	0.0143
2453	10	0.0014	59.3090	0.0137
2454	10	0.0013	59.4310	0.0108
2455	10	0.0012	59.2480	0.0133
2456	9	0.0010	59.3460	0.0112
2457	6	0.0005	59.2690	0.0106
2458	4	0.0002	58.4790	0.0103
2459	3	0.0004	58.4160	0.0976
2460	10	0.0014	57.0380	0.0026
		Coal		
11	6	0.0015	8.5931	0.0245
12	10	0.0011	9.1573	0.0208
13	2	0.0029	7.666.3	−0.0520
14	5	0.0053	9.7962	0.0481
15	10	0.0041	9.4047	0.0496
16	5	0.0050	9.4886	0.0694
17	10	0.0048	9.1694	0.0598
18	4	0.0016	9.0681	0.1119
19	4	0.0043	9.0152	0.1527
20	3	0.0039	9.0801	0.1613
21	3	0.0030	8.7421	0.0946
22	8	0.0085	8.8853	0.0944
23	3	0.0010	8.6669	0.1055
24	6	0.0060	8.7592	0.0967
25	7	0.0064	8.8440	0.0908
26	8	0.0067	8.8464	0.0895
27	3	0.0012	9.0667	0.1633
28	8	0.0053	8.9557	0.0539
29	4	0.0007	9.0561	0.1246
30	8	0.0041	8.9685	0.1025
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2865	4	0.0001	29.6070	0.0559
2866	6	0.0005	29.5520	0.0215
2867	7	0.0008	29.8620	−0.0251
2868	5	0.0002	27.4.500	0.0255
2869	7	0.0016	26.8240	0.0056
2870	5	0.0010	27.0540	0.0542
2871	6	0.0009	26.7590	0.0182
2872	5	0.0006	26.4540	0.0220
2875	5	0.0004	26.6850	−0.1455
2874	9	0.0025	25.9970	0.0151
2875	5	0.0014	25.5990	0.0552
2876	4	0.0010	25.5580	0.0545
2877	10	0.0027	25.2940	0.0067
2878	6	0.0012	25.5500	0.0591
2879	9	0.0019	25.2960	0.0155

TABLE 4-continued

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\alpha_{\hat{m}_k,k}$ for initial delay r = 10.				
t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
2880	9	0.0017	25.4620	0.0264
2881	7	0.0012	25.5400	0.0569
2882	9	0.0018	25.4510	0.0416
2885	7	0.0011	25.5550	0.0445
2884	9	0.0016	25.5400	0.0445
2885	4	0.0005	25.5440	0.0675
		Ethanol		
11	6	0.0009	1.1830	0.8082
12	6	0.0009	1.2087	0.3843
31	9	0.0013	4.0236	0.0040
14	2	0.0009	1.1073	0.0509
15	9	0.0024	1.0755	−0.1896
16	2	0.0025	2.8800	0.0289
17	9	0.0023	0.9139	−0.1012
18	2	0.0018	0.7387	−0.0826
19	7	0.0017	2.0655	0.0896
20	8	0.0023	2.2742	0.0690
21	7	0.0014	2.4094	0.0554
22	6	0.0029	2.0457	0.1327
23	7	0.0016	2.0441	0.1332
24	9	0.0020	1.3966	−0.2082
25	6	0.0200	2.4981	0.1465
26	7	0.0173	2.3356	0.1927
27	9	0.0143	2.3860	0.1416
28	8	0.0138	2.3919	0.2196
29	7	0.0152	2.4087	0.3983
30	10	0.0106	2.3164	0.2386
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
375	5	0.0008	2.1469	0.9842
376	4	0.0009	2.0689	0.2666
377	6	0.0011	2.0999	0.2756
378	7	0.0014	2.0924	0.2551
379	10	0.0044	2.0941	0.2867
380	5	0.0007	2.0731	0.8434
381	6	0.0017	2.0214	−0.4677
382	6	0.0024	1.4504	−0.0549
383	6	0.0017	1.6343	−0.0794
384	10	0.0057	2.7780	0.0309
385	8	0.0039	2.7055	0.0750
386	6	0.0018	2.6000	0.3021
387	8	0.0031	2.6118	0.1997
388	6	0.0027	2.6058	0.6130
389	8	0.0035	2.5973	0.4169
390	5	0.0024	2.5947	0.5364
391	5	0.0019	2.6500	−0.2801
392	5	0.0017	2.6321	−0.3394
393	6	0.0020	3.0563	−0.0442
394	9	0.0055	2.4093	0.0868
395	4	0.0027	2.3140	0.4706

Table 4 shows the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ and the parameters $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma_{\hat{m}_k,k}^2$ for four price energy commodity data at time t_k. This was based on p, r, and the initial real data time delay r=10. We further note that the range of the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ for any time $t_k \in [11, 30] U [1145,1165]$, $t_k \in [11,30] U [2440, 2460]$, $t_k \in [11,30] U [2865,2885]$, and $t_k \in [11, 30] U [375,395]$ for natural gas, crude oil, coal and ethanol data, respectively, is $2 \leq \hat{m}_k \leq 10$. Further, all comments that are made with regard to Table 2 regarding the four energy commodities remain valid with regard to Table 4.

In Table 5, the real and LLGMM simulated price values of each of the four energy commodities: Natural gas, Crude oil, Coal, and Ethanol are exhibited in columns 2-3, 6-7, 10-11, and 14-15, respectively. The absolute error of each of the energy commodities simulated value is shown in columns 4, 8, 12, 16, respectively.

TABLE 5

Real, Simulation using LLGMM method, and absolute error of simulation using starting delay r = 10.			
t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\lvert y_k - y^s_{\hat{m}_k,k}\rvert$
	Natural gas		
10	2.3830	2.3830	0.0000
11	2.4170	2.4179	0.0009
12	2.5590	2.4935	0.0655
13	2.4850	2.4949	0.0099
14	2.5280	2.5123	0.0157
15	2.6160	2.6158	0.0002
16	2.5230	2.5233	0.0003
17	2.6100	2.6314	0.0214
18	2.6100	2.5852	0.0248
19	2.6100	2.6130	0.0030
20	2.6990	2.6728	0.0262
21	2.7590	2.7601	0.0011
22	2.6590	2.6427	0.0163
23	2.7420	2.7365	0.0055
24	2.5620	2.5610	0.0010
25	2.4950	2.5455	0.0505
26	2.5400	2.5245	0.0155
27	2.5920	2.5996	0.0076
28	2.5700	2.5849	0.0149
29	2.5410	2.5403	0.0007
30	2.6180	2.6151	0.0029
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
1145	5.712	5.7533	0.0413
1146	5.588	5.5892	0.0012
1147	5.693	5.7143	0.0213
1148	5.791	5.8127	0.0217
1149	5.614	5.5940	0.0200
1150	5.442	5.6266	0.1846
1151	5.533	5.5122	0.0208
1152	5.378	5.3971	0.0191
1153	5.373	5.3496	0.0234
1154	5.382	5.3735	0.0085
1155	5.507	5.5360	0.0290
1156	5.552	5.5507	0.0013
1157	5.310	5.3019	0.0081
1158	5.338	5.3884	0.0504
1159	5.298	5.2554	0.0426
1160	5.189	5.1644	0.0146
1161	5.082	5.0874	0.0054
1162	5.082	5.0977	0.0157
1163	5.082	5.1334	0.0514
1164	4.965	5.0340	0.0690
1165	4.767	4.9143	0.1473
	Crude oil		
10	25.1000	25.1000	0.0000
11	24.8000	25.0181	0.2181
12	24.4000	24.3221	0.0779
13	23.8500	23.7260	0.1240
14	23.8500	24.4203	0.5703
15	23.8500	23.8174	0.0326
16	23.9000	23.8845	0.0155
17	24.5000	24.0924	0.4076
18	24.8000	24.3340	0.4660
19	24.1500	24.1566	0.0066
20	24.2000	24.5277	0.3277
21	24.0000	23.7803	0.2197
22	23.9000	24.1935	0.2935
23	23.0500	23.0564	0.0064
24	22.3000	23.2208	0.9208
25	22.4500	23.1610	0.7140
26	22.3500	22.7275	0.3775
27	21.7500	21.5907	0.1593
28	22.1000	22.0868	0.0132
29	22.4000	22.4301	0.0301
30	22.5000	22.6614	0.1614
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2440	57.35	57.762	0.412

TABLE 5-continued

Real, Simulation using LLGMM method, and absolute error of simulation using starting delay r = 10.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\|y_k - y^s_{\hat{m}_k,k}\|$
2441	56.74	56.743	0.0028
2442	57.55	57.739	0.189
2443	59.09	58.925	0.1646
2444	60.27	59.663	0.607
2445	60.75	61.161	0.4109
2446	58.41	58.011	0.3994
2447	58.72	58.762	0.042
2448	58.64	58.409	0.2309
2449	57.87	57.762	0.1081
2450	59.13	59.243	0.1135
2451	60.11	60.068	0.0419
2452	58.94	58.956	0.0155
2453	59.93	59.924	0.0062
2454	61.18	62.168	0.9876
2455	59.66	59.381	0.2786
2456	58.59	58.468	0.1224
2457	58.28	58.487	0.2067
2458	58.79	58.896	0.1058
2459	56.23	57.202	0.9715
2460	55.9	56.87	0.9701
		Coal	
10	9.0600	9.0600	0.0000
11	8.8800	8.8800	0.0000
12	9.4400	9.4216	0.0184
13	10.3100	10.0621	0.2479
14	9.8100	9.8058	0.0042
15	9.0600	8.8075	0.2525
16	8.7500	8.4774	0.2726
17	8.8200	8.7839	0.0361
18	9.5600	9.3610	0.1990
19	8.8200	8.6667	0.1533
20	8.8200	8.7833	0.0367
21	8.6900	8.5498	0.1402
22	8.6300	8.7065	0.0765
23	8.6900	8.7620	0.0720
24	8.9400	8.9706	0.0306
25	9.3100	8.8231	0.4869
26	8.9400	8.9945	0.0545
27	8.9400	8.9676	0.0276
28	9.1300	9.1741	0.0441
29	9.1900	9.1766	0.0134
30	8.5700	8.4567	0.1133
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2865	29.31	29.518	0.2083
2866	28.68	28.495	0.1851
2867	26.77	28.727	1.9571
2868	27.45	26.979	0.471
2869	27.00	26.879	0.121
2870	26.67	27.32	0.6499
2871	26.51	25.468	1.0415
2872	26.48	26.263	0.2174
2873	25.15	25.395	0.2445
2874	25.57	25.555	0.0153
2875	25.88	26.08	0.2003
2876	25.24	25.528	0.2879
2877	25	25.337	0.3375
2878	25.08	24.685	0.3951
2879	25.05	24.848	0.2024
2880	25.89	25.638	0.2518
2881	25.23	25.405	0.1749
2882	25.94	25.739	0.2007
2883	25.26	24.858	0.4025
2884	25.25	25.147	0.1028
2885	26.06	25.613	0.4475
		Ethanol	
10	1.1900	1.1900	0.0000
11	1.2250	1.2249	0.0001
12	1.2200	1.2425	0.0225
13	1.2900	1.2278	0.0622
14	1.4100	1.5339	0.1239
15	1.4700	1.3390	0.1310
16	1.5300	1.5745	0.0445
17	1.6300	1.5996	0.0304
18	1.7500	1.6320	0.1180
19	1.7500	1.7495	0.0005
20	1.8400	1.8586	0.0186
21	1.8950	1.8874	0.0076
22	1.9500	1.9257	0.0243
23	1.9740	1.9548	0.0192
24	2.7000	2.1431	0.5569
25	2.5150	2.6941	0.1791
26	2.2900	2.2753	0.0147
27	2.4400	2.3645	0.0755
28	2.4150	2.4019	0.0131
29	2.3000	2.2440	0.0560
30	2.1000	2.2048	0.1048
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
375	2.073	2.0662	0.0068
376	2.02	2.0267	0.0067
377	2.073	2.0731	0.0001
378	2.065	2.0709	0.0059
379	2.055	2.0232	0.0318
380	2.209	2.2109	0.0019
381	2.44	2.296	0.144
382	2.517	2.4074	0.1096
383	2.718	2.6839	0.0341
384	2.541	2.5246	0.0164
385	2.566	2.5629	0.0031
386	2.626	2.6248	0.0012
387	2.587	2.5871	0.0001
388	2.628	2.6363	0.0083
389	2.587	2.5332	0.0538
390	2.536	2.5374	0.0014
391	2.42	2.3401	0.0799
392	2.247	2.1792	0.0678
393	2.223	2.1661	0.0569
394	2.39	2.5122	0.1222
395	2.38	2.3583	0.0217

Graphical, Simulation and Statistical Results-Case 3.

Again, we pick r=20, Δt=1, ϵ=0.001, and p=2, the ϵ-best sub-optimal estimates of parameters a, μ and σ^2 at each real data times are exhibited in Table 6.

TABLE 6

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $_{\hat{m}_k,k}$ for initial delay r = 20.

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
		Natural gas		
21	13	0.0011	2.7056	0.0816
22	5	0.0009	2.6748	0.233
23	3	0.0013	2.7139	0.6983
24	12	0.0021	2.6197	0.2119
25	10	0.0022	2.6201	0.2199
26	5	0.0015	2.567	0.2063
27	9	0.0021	2.6295	0.1919
28	17	0.0031	2.6074	0.2204
29	11	0.0022	2.6099	0.1688
30	8	0.0014	2.5821	0.2593
31	7	0.0013	2.5605	0.3999
32	9	0.0016	2.5738	0.3887
33	16	0.0035	2.6195	0.2084
34	20	0.0041	2.6078	0.2483
35	16	0.0033	2.6031	0.2024
36	5	0.0007	2.579	0.2816

TABLE 6-continued

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $_{\hat{m}_k,k}$ for initial delay r = 20.

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
37	9	0.0013	2.5814	0.3453
38	10	0.0014	2.5836	0.3371
39	3	0.0015	2.603	0.3923
40	18	0.0048	2.6026	0.2551
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
1145	3	0.0001	5.7243	0.1464
1146	17	0.0033	5.7831	0.0272
1147	15	0.0025	5.8662	0.0337
1148	8	0.0006	5.7271	0.0741
1149	5	0.0004	5.6834	0.2598
1150	18	0.0034	5.6161	0.0138
1151	16	0.0026	5.6048	0.0268
1152	18	0.0031	5.3059	0.0099
1153	9	0.0008	5.4937	0.0517
1154	7	0.0006	5.4044	0.0549
1155	5	0.0003	5.4342	0.2005
1156	7	0.0006	5.4528	0.1(46
1157	8	0.0006	5.4395	0.2012
1158	14	0.002	5.4704	0.0583
1159	10	0.0009	5.4035	0.1412
1160	14	0.0018	5.3501	0.0373
1161	11	0.001	5.174	0.0277
1162	18	0.0029	5.1069	0.016
1163	18	0.0027	5.1426	0.0213
1164	16	0.002	5.0554	0.0297
1165	15	0.0016	5.7431	−0.0195
	Crude oil			
21	11	0.0003	24.115	0.0204
22	7	0.0003	24.215	0.0278
23	2	0.0006	24.013	−0.314
24	15	0.0007	14.246	0.0009
25	19	0.0011	18.542	0.001
26	19	0.001	21.738	0.0031
27	4	0.0001	22.135	0.0355
28	14	0.0007	20.045	0.0015
29	14	0.0007	22.096	0.0034
30	9	0.0004	22.249	0.0154
31	3	0.0002	22.739	0.0203
32	6	0.0004	22.226	0.0427
33	7	0.0005	22.084	0.0296
34	11	0.001	21.683	0.0138
35	10	0.0009	20.446	0.0041
36	3	0	21.027	0.0489
37	4	0.0002	20.962	0.0465
38	3	0.0002	21.267	−0.0327
39	13	0.0014	15.485	0.0012
40	5	0.0004	20.617	0.028
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2440	8	0.0007	58.338	0.0143
2441	20	0.0033	58.546	0.0028
2442	10	0.0008	58.056	0.0098
2443	8	0.0006	58.267	0.0106
2444	7	0.0005	58.414	0.0079
2445	7	0.0005	65.583	0.001
2446	8	0.0005	58.733	0.0078
2447	9	0.0007	58.772	0.0078
2448	20	0.0033	58.727	0.0079
2449	13	0.0013	58.371	0.0087
2450	3	0.0001	58.48	0.0345
2451	9	0.0008	59.324	0.013
2452	5	0.0005	58.955	0.0144
2453	9	0.001	59.171	0.0135
2454	15	0.002	59.298	0.0063
2455	13	0.0015	59.512	0.0126
2456	11	0.0011	59.169	0.0137
2457	12	0.0012	59.072	0.0128
2458	8	0.0006	59.427	0.0112
2459	15	0.0018	58.808	0.0092
2460	14	0.0015	58.187	0.0042

TABLE 6-continued

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $_{\hat{m}_k,k}$ for initial delay r = 20.

t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
		Coal		
21	19	0.0042	9.1915	0.0255
22	15	0.0044	9.0773	0.0601
23	19	0.0038	9.1073	0.0319
24	10	0.0035	8.8762	0.0924
25	14	0.0049	9.1783	0.0517
26	9	0.003	8.9447	0.1
27	10	0.0031	8.9442	0.1
28	6	0.0013	9.0358	0.0767
29	3	0.0006	9.4379	0.0213
30	8	0.0019	8.9685	0.1025
31	4	0.0014	8.8837	0.0869
32	15	0.0096	8.9287	0.0972
33	5	0.0013	8.7634	0.0932
34	7	0.0018	8.8238	0.0869
35	8	0.0021	8.7923	0.0823
36	9	0.0023	8.7282	0.0671
37	13	0.0062	8.7653	0.0502
38	7	0.001	8.6612	0.1378
39	20	0.0151	8.8225	0.0644
40	17	0.0101	8.8585	0.0667
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
2865	4	0.0002	29.607	0.034
2866	12	0.0023	29.257	0.0209
2867	20	0.0054	26.256	0.0021
2868	14	0.0028	28.678	0.009
2869	11	0.0019	27.482	0.0052
2870	14	0.0026	26.136	0.0023
2871	12	0.0019	25.376	0.0021
2872	9	0.0011	26.067	0.0064
2871	4	0.0003	27.22	−0.0313
2874	10	0.0016	25.744	0.0095
2875	3	0.0012	25.599	0.0532
2876	3	0.0008	25.559	0.0541
2877	5	0.0006	25.415	0.0446
2878	4	0.0005	25.193	0.0206
2879	3	0.0002	25.059	0.0528
2880	5	0.0004	25.256	0.0431
2881	5	0.0005	25.254	0.0435
2882	9	0.002	25.431	0.0417
2883	13	0.0033	25.507	0.0243
2884	20	0.006	25.52	0.0094
2885	5	0.0007	25.538	0.069
		Ethanol		
21	18	0.0024	0.7591	0.0467
22	4	0.0015	0.7929	−0.0272
23	8	0.0004	2.1528	0.0888
24	15	0.0025	1.0048	−0.1078
25	20	0.0094	−0.4372	−0.0208
26	19	0.0094	3.1726	0.0251
27	7	0.0205	2.3915	0.2198
28	17	0.0087	2.6208	0.0553
29	3	0.0218	2.3857	0.634
30	19	0.0161	2.3086	0.0752
31	18	0.0162	2.2442	0.1049
32	9	0.0279	2.3519	0.4089
33	12	0.0193	2.2912	0.2631
34	6	0.0186	2.1259	0.2733
35	20	0.0218	2.2078	0.1261
36	10	0.0199	1.9158	0.0549
37	7	0.0146	1.9215	0.088
38	7	0.0127	2.0226	0.1587
39	19	0.0413	2.1885	0.1729
40	8	0.0112	1.9751	0.1655
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
375	6	0.0013	2.1486	0.7096
376	3	0.0009	2.0699	0.2808
377	5	0.0011	2.0858	0.3308
378	11	0.007	2.1286	0.2103
379	3	0.0007	2.0623	0.6096
380	16	0.0137	2.1586	0.1983

TABLE 6-continued

Estmates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $_{\hat{m}_k,k}$ for initial delay r = 20.				
t_k	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\alpha_{\hat{m}_k,k}$
381	19	0.0185	2.2115	0.1503
382	11	0.0066	1.7644	−0.0401
383	3	0.0025	2.9233	0.1347
384	4	0.0025	2.5937	0.3073
385	5	0.0039	2.5887	0.3099
386	3	0.006	2.5861	0.4792
387	4	0.0039	2.5882	0.4761
388	11	0.0087	2.6964	0.077
389	6	0.0038	2.5952	0.4921
390	10	0.0075	2.5899	0.3122
391	9	0.0062	2.5817	0.4568
392	7	0.0038	2.6222	−0.3162
393	15	0.0142	2.5051	0.1102
394	12	0.01	2.4881	0.1156
395	3	0.0036	2.355	0.2939

Table 6 shows the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ and the parameters $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma_{\hat{m}_k,k}^2$ for four price energy commodity data at time t_k. This was based on p, r, and the initial real data time delay r=20. We further note that the range of the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ for any time $t_k \in [21, 40] \cup [1145, 1165]$, $t_k \in [21, 40] \cup [2440, 2460]$, $t_k \in [21, 40] \cup [2865, 2885]$, and $t_k \in [21, 40] \cup [375, 395]$ for natural gas, crude oil, coal and ethanol data, respectively, is $3 \leq \hat{m}_k \leq 20$. Further, all comments that are made with regard to Table 2 regarding the four energy commodities remain valid with regard to Table 6.

In Table 7, the real and LLGMM simulated price values of each of the four energy commodities, including natural gas, crude oil, coal, and ethanol, are shown, respectively. The absolute error of each of energy commodity simulated value is also shown.

TABLE 7

Real, Simulation using LLGMM method, and absolute error of simulation using starting delay r = 20.			
t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\|y_k - y^s_{\hat{m}_k,k}$
	Natural gas		
21	2.759	2.7718	0.0128
22	2.659	2.6566	0.0024
23	2.742	2.7353	0.0067
24	2.562	2.5757	0.0137
25	2.495	2.5332	0.0382
26	2.54	2.5336	0.0064
27	2.592	2.5631	0.0289
28	2.57	2.5797	0.0097
29	2.541	2.4846	0.0564
30	2.618	2.6245	0.0065
31	2.564	2.5469	0.0171
32	2.667	2.6763	0.0093
33	2.633	2.6308	0.0022
34	2.515	2.5021	0.0129
35	2.53	2.5136	0.0164
36	2.549	2.5458	0.0032
37	2.603	2.5835	0.0195
38	2.603	2.5822	0.0208
39	2.603	2.6075	0.0045
40	2.815	2.8728	0.0578
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
1145	5.712	5.7577	0.0457
1146	5.588	5.6488	0.0608
1147	5.693	5.7062	0.0132
1148	5.791	5.7917	0.0007

TABLE 7-continued

Real, Simulation using LLGMM method, and absolute error of simulation using starting delay r = 20.			
t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\|y_k - y^s_{\hat{m}_k,k}$
1149	5.614	5.5799	0.0341
1150	5.442	5.4099	0.0321
1151	5.533	5.5035	0.0295
1152	5.378	5.407	0.029
1153	5.373	5.3682	0.0048
1154	5.382	5.3827	0.0007
1155	5.507	5.4896	0.0174
1156	5.552	5.5423	0.0097
1157	5.31	5.318	0.008
1158	5.338	5.3794	0.0414
1159	5.298	5.3541	0.0561
1160	5.189	5.1838	0.0052
1161	5.082	5.3804	0.2984
1162	5.082	4.9802	0.1018
1163	5.082	5.1933	0.1113
1164	4.965	5.1925	0.2275
1165	4.767	4.7917	0.0247
	Crude oil		
21	24	24.025	0.025
22	23.9	24.093	0.193
23	23.05	23.051	0.001
24	22.3	22.887	0.587
25	22.45	22.126	0.324
26	22.35	22.409	0.059
27	21.75	22.12	0.37
28	22.1	22.137	0.037
29	22.4	22.315	0.085
30	22.5	22.531	0.031
31	22.65	22.712	0.062
32	21.95	22.003	0.053
33	21.6	21.853	0.253
34	21	21.099	0.099
35	20.95	21.012	0.062
36	21.1	20.971	0.129
37	20.8	20.786	0.014
38	20.3	20.048	0.252
39	20.25	20.244	0.006
40	20.75	20.734	0.016
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2440	57.35	57.376	0.026
2441	56.74	56.447	0.293
2442	57.55	57.523	0.027
2443	59.09	58.968	0.122
2444	60.27	60.278	0.008
2445	60.75	60.737	0.013
2446	58.41	58.494	0.084
2447	58.72	58.614	0.106
2448	58.64	58.95	0.31
2449	57.87	57.865	0.005
2450	59.13	58.967	0.163
2451	60.11	59.937	0.173
2452	58.94	59.068	0.128
2453	59.93	60.141	0.211
2454	61.18	61.53	0.35
2455	59.66	59.792	0.132
2456	58.59	58.481	0.109
2457	58.28	58.224	0.056
2458	58.79	58.928	0.138
2459	56.23	56.329	0.099
2460	55.9	54.676	1.224
	Coal		
21	8.69	8.6747	0.0153
22	8.63	8.6175	0.0125
23	8.69	8.6862	0.0038
24	8.94	8.9184	0.0216
25	9.31	9.3069	0.0031
26	8.94	8.8992	0.0408
27	8.94	8.8745	0.0655

TABLE 7-continued

Real, Simulation using LLGMM method, and absolute error of simulation using starting delay r = 20.

t_k	Real y_k	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	|Error| $\lvert y_k - y^s_{\hat{m}_k,k}\rvert$
28	9.13	9.1162	0.0138
29	9.19	9.234	0.044
30	8.57	8.5495	0.0205
31	8.69	8.7241	0.0341
32	8.88	8.8866	0.0066
33	8.57	8.5084	0.0616
34	8.75	8.7447	0.0053
35	8.63	8.6003	0.0297
36	8.44	8.412	0.028
37	8.44	8.4465	0.0065
38	8.94	8.9538	0.0138
39	9	9.0064	0.0064
40	8.94	8.8655	0.0745
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
2865	29.31	29.291	0.019
2866	28.68	28.8	0.12
2867	26.77	26.891	0.121
2868	27.45	27.316	0.134
2869	27	27.189	0.189
2870	26.67	26.812	0.142
2871	26.51	26.709	0.199
2872	26.48	26.54	0.06
2873	25.15	25.313	0.163
2874	25.57	25.47	0.1
2875	25.88	26.078	0.198
2876	25.24	25.208	0.032
2877	25	25.138	0.138
2878	25.08	25.306	0.226
2879	25.05	25.16	0.11
2880	25.89	25.509	0.381
2881	25.23	25.278	0.048
2882	25.94	25.961	0.021
2883	25.26	25.255	0.005
2884	25.25	25.298	0.048
2885	26.06	25.882	0.178
		Ethanol	
21	1.895	1.9024	0.0074
22	1.95	1.9315	0.0185
23	1.974	1.9788	0.0048
24	2.7	2.5529	0.1471
25	2.515	2.5134	0.0016
26	2.29	2.3306	0.0406
27	2.44	2.3718	0.0682
28	2.415	2.3927	0.0223
29	2.3	2.3311	0.0311
30	2.1	2.072	0.028
31	2.04	2.0323	0.0077
32	2.16	2.1561	0.0039
33	2.13	2.0796	0.0504
34	2.155	2.2141	0.0591
35	2.01	1.9687	0.0413
36	1.93	1.8762	0.0538
37	1.9	1.9186	0.0186
38	1.975	1.9052	0.0698
39	1.98	2.019	0.039
40	2	1.9385	0.0615
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
375	2.073	2.09	0.017
376	2.02	2.0589	0.0389
377	2.073	2.0601	0.0129
378	2.065	2.0312	0.0338
379	2.055	2.0725	0.0175
380	2.209	2.2254	0.0164
381	2.44	2.462	0.022
382	2.517	2.51	0.007
383	2.718	2.6979	0.0201
384	2.541	2.5164	0.0246
385	2.566	2.5328	0.0332
386	2.626	2.5831	0.0429
387	2.587	2.5606	0.0264
388	2.628	2.6322	0.0042
389	2.587	2.5651	0.0219
390	2.536	2.53	0.006
391	2.42	2.4268	0.0068
392	2.247	2.2228	0.0242
393	2.223	2.2072	0.0158
394	2.39	2.4141	0.0241
395	2.38	2.4265	0.0465

FIGS. **4**A and **4**B illustrate real and simulated prices for natural gas and ethanol using the local lagged adapted generalized method of moments dynamic process, respectively, for r=20.

Goodness-of-Ft Measures.

The goodness-of-fit measures are found for four energy commodities, natural gas, crude oil, coal and ethanol. This is achieved by using the following goodness-of-fit measures:

$$\begin{cases} \widehat{RAMSE} = \left[\frac{1}{N}\sum_{t=1}^{N}\frac{1}{S}\sum_{s=1}^{S}\left(y_i^{(s)} - y_i\right)^2\right]^{\frac{1}{2}}, \\ \widehat{AMAD} = \frac{1}{N}\sum_{t=1}^{N}\operatorname{median}_s\left(\left|y_i^s - \operatorname{median}_l(y_i^l)\right|\right), \\ \widehat{AMB} = \frac{1}{N}\sum_{i=1}^{N}\left(\left|\operatorname{median}_s(y_i^s) - y_i\right|\right), \end{cases} \tag{69}$$

where $\{y_t^s\}_{t=1, 2, \ldots, N}^{s=1, 2, \ldots, S}$ is a double sequence of simulated values at the data collected/observed time t=1, 2, . . . , N, RAMSE is the root mean square error of the simulated path, AMAD is the average median absolute deviation, and AMB is the the average median bias. The goodness-of-fit measures are computed using S=100 pseudo-data sets. The comparison of the goodness-of-fit measures RAMSE, AMAD, and AMB for the four energy commodities: natural gas, crude oil, coal, and ethanol data are recorded in Table 8.

Remark 23.

As the RAMSE decreases, then the state estimates approach to the true value of the state. As the value of AMAD increases, the influence of the random environmental fluctuations on the state dynamic process increases. In addition, if the value of RAMSE decreases and the value of AMAD increases, then the method of study possesses a greater degree of ability for state and parameter estimation accuracy and greater degree of ability to measure the variability of random environmental perturbations on the state dynamic of system. Further, as RAMSE decreases, AMAD increases, and AMB decreases, the method of study increases its performance under the three goodness of fit measures in a coherent way. On other hand, as the RAMSE increases, the state estimates tend to move away from the true value of the state. As the value of AMAD decreases, the influence of the random environmental fluctuations on state dynamic process decreases.

In addition, if the value of RAMSE increases and the value of AMAD decreases, then the method of study possesses a lesser degree of ability for state and parameter estimation accuracy and lesser degree of ability to measure the variability of random environmental perturbations on the state dynamic of system. Further, as the RAMSE increases, AMAD decreases and the AMB increases, the method of study decreases its performance under the three goodness-of-fit measures in a coherent manner.

The Comparison of Goodness-of-Fit Measures for r=5, r=10, and r=20.

The following table exhibits the goodness-of-fit measures for the energy commodities natural data, crude oil, coal, and ethanol data using the initial delays r=5, r=10, and r=20.

TABLE 8

Goodness-of-fit Measures for r = 5, r = 10 and r = 20

Goodness of-fit Measure	Natural gas	Crude oil	Coal	Ethanol
	r = 5			
$\widehat{RAMSE}$	0.1801	1.1122	1.2235	0.1001
	1.1521	24.6476	9.4160	0.3409
	1.1372	27.2707	12.8370	0.3566
	r = 10			
$\widehat{AMAD}$	0.1004	0.5401	0.8879	0.0618
	1.1330	24.5376	9.4011	0.3233
	1.1371	27.2708	12.8369	0.3566
	r = 20			
$\widehat{AMB}$	0.0674	0.4625	0.4794	0.0375
	1.1318	24.5010	9.4009	0.3213
	1.1374	27.2707	12.8370	0.3566

Remark 24.

From Tables 3, 5, and 7 it is clear that as r increases the absolute error decreases. Furthermore, the comparison of the goodness-of-fit measures in Table 8 for the natural gas, crude oil, coal, and ethanol data the energy commodities using the initial delays r=5, r=10, and r=20 shows that as the delay r increases, the root mean square error decreases significantly, AMAD decreases very slowly, and AMB remains unchanged.

Remark 25.

Computer readable instructions can be designed to exhibit the flowchart shown in FIGS. **2** and **3**. For example, computer readable instructions for parameter estimation, simulations, and forecasting can be written and tested using MATLAB®. Due to the online control nature of m_k in our model, it is worth mentioning that the execution times for each of the four commodities: Natural gas, Crude oil, Coal and Ethanol depend on the robustness of the data.

Illustration 2: Application of Presented Approach to U. S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate Data Set.

Here, the conceptual computational algorithm discussed in Section 4 is applied to estimate the parameters in equation (45) using the real time U.S. Treasury Bill Yield Interest Rate (U.S. TBYIR) and the U.S. Eurocurrency Exchange Rate (U.S. EER) data collected on Forex database.

Graphical, Simulation and Statistical Results.

Using ϵ=0.001, r=20, and p=2, the ϵ-best sub-optimal estimates of parameters β, μ, δ, σ and γ for each Treasury bill Yield and U.S. Eurocurrency rate data sets are exhibited in Tables 9 and 10, respectively.

TABLE 9

Estimates for $\hat{m}_k$, $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U.S. Treasury Bill Yield Interest Rate data.

		interest rate				
t_k	$\hat{m}_k$	$\beta_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\delta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
21	2	1.5199	−7.0332	1.46	0.0446	0.9078
22	2	1.2748	−5.919	1.46	0.0941	1.5
23	10	2.9904	−13.928	1.46	0.0576	1.5
24	12	1.8604	−8.6515	1.46	0.0895	1.5
25	6	2.1606	−10.076	1.46	0.1064	1.5
26	20	0.0199	−0.0372	1.46	0.1097	1.3872
27	16	−0.0274	0.1991	1.46	0.1066	1.4348
28	4	−0.1841	0.9753	1.46	0.1345	1.2081
29	19	0.3261	−1.3952	1.46	0.1855	0.7006
30	12	0.2707	−1.1525	1.46	0.1624	1.4187
31	13	0.543	−2.4097	1.46	0.2571	1.4986
32	11	0.5357	−2.4098	1.46	0.1962	−0.0695
33	11	0.4723	−2.1258	1.46	0.3494	0.097
34	11	−0.3697	1.4705	1.46	1.4014	1.4983
35	4	−0.7862	3.3703	1.46	0.3488	1.4993
36	6	−0.3375	1.3041	1.46	0.2914	1.4711
37	5	0.3541	−2.0609	1.46	0.2676	1.4972
38	14	0.2368	−1.1239	1.46	0.3201	1.4961
39	8	1.1109	−5.5453	1.46	0.6811	−0.7462
40	4	1.9032	−9.1187	1.46	1.1055	0.1008
41	11	0.4364	−2.1327	1.46	0.3532	1.4994
42	4	0.2942	−1.3004	1.46	0.4885	1.4975
43	5	0.4012	−1.9198	1.46	0.3418	1.5
44	3	0.2605	−1.2108	1.46	0.4133	1.4705
45	5	0.4213	−2.0086	1.46	0.3324	1.4992
. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .
420	12	3.8416	−18.331	1.46	0.1187	1.4906
421	12	2.8918	−13.821	1.46	0.6961	1.386
422	12	0.5602	−2.6281	1.46	0.3759	1.1741
423	7	0.5825	−2.7201	1.46	0.1753	1.4935
424	7	0.7397	−3.4486	1.46	0.1687	0.5396
425	7	0.2488	−1.1148	1.46	0.1819	0.6161
426	7	0.8447	−3.9535	1.46	0.4182	0.7124
427	11	−0.2202	1.098	1.46	0.2013	0.6577
428	12	−0.1169	0.6256	1.46	0.1779	0.6063
429	9	0.1464	−0.6472	1.46	0.3672	1.2589
430	9	0.0343	−0.117	1.46	0.3637	0.7374
431	9	0.1785	−0.6832	1.46	0.1395	0.5804
432	19	−0.0031	0.1015	1.46	0.1932	1.1832
433	8	0.1651	−0.6463	1.46	0.1745	0.5374
434	19	0.4102	−1.6622	1.46	0.121	0.3774
435	8	0.2941	−1.1608	1.46	0.1085	1.0262
436	19	0.3694	−1.4911	1.46	0.1547	1.4945
437	14	1.6473	−6.6877	1.46	0.2198	−0.0071
438	5	1.417	−5.7323	1.46	0.1406	−0.1462
439	17	1.3024	−5.3352	1.46	0.133	0.2225
440	9	0.2839	−1.191	1.46	0.1929	0.0883
441	17	0.2053	−0.8785	1.46	0.2007	−0.1338
442	17	−0.4585	1.6754	1.46	0.4803	0.944
443	7	−0.2917	0.8858	1.46	0.5227	−0.236
444	9	−0.023	−0.2999	1.46	0.5836	−0.2083
445	13	−0.3263	1.2217	1.46	0.2632	−0.1684

TABLE 10

Estimates for $\hat{m}_k$, $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U.S. Eurocurrency Exchange Rate.

		US Eurocurrency Exchange Rate				
t_k	$\hat{m}_k$	$\beta_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\delta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
21	2	−0.1282	0.1406	1.4892	0.0235	−1.4529
22	3	8.3385	−7.7988	1.4892	0.0256	1.4954
23	2	3.1279	−2.9205	1.4892	0.0286	1.4995
24	20	0.22	−0.1976	1.4892	0.0298	1.4948
25	18	3.0772	−2.8778	1.4892	0.016	1.4741
26	4	3.8605	−3.6034	1.4892	0.0147	1.3925
27	13	3.7355	−3.4973	1.4892	0.0395	1.4959

TABLE 10-continued

Estimates for $\hat{m}_k$, $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U.S. Eurocurrency Exchange Rate.

	US Eurocurrency Exchange Rate					
t_k	$\hat{m}_k$	$\beta_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\delta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
28	16	2.436	−2.2773	1.4892	0.0315	−0.7142
29	17	1.8545	−1.7299	1.4892	0.0159	−1.4613
30	3	6.4061	−5.9636	1.4892	0.0324	−2.4907
31	12	1.0648	−0.9689	1.4892	0.0242	1.47
32	15	0.4861	−0.4244	1.4892	0.0285	1.5
33	18	2.9505	−2.7502	1.4892	0.0267	1.4943
34	5	3.8981	−3.635	1.4892	0.0984	1.4807
35	4	0.4644	−0.4841	1.4892	0.1052	1.4884
36	3	0.753	−0.7159	1.4892	0.0474	1.4954
37	3	0.719	−0.682	1.4892	0.0472	1.4995
38	3	−0.7094	0.6544	1.4892	0.0482	1.4948
39	5	1.221	−1.1708	1.4892	0.0649	1.4741
40	9	6.7537	−6.4315	1.4892	0.0395	1.4959
41	9	1.0019	−0.9439	1.4892	0.0566	1.4962
42	11	5.5279	−5.2617	1.4892	0.0309	1.499
43	5	5.3829	−5.1253	1.4892	0.0529	0.1514
44	10	5.2433	−4.9934	1.4892	0.0483	0.8817
45	10	5.2445	−4.9945	1.4892	0.0305	1.3425
. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .
155	2	10.779	−10.219	1.4892	0.0167	0.8188
156	14	2.4641	−2.3297	1.4892	0.0227	0.8437
157	4	3.2423	−3.0622	1.4892	0.0184	1.4906
158	6	3.1716	−3.0016	1.4892	0.0204	0.4736
159	7	6.2013	−5.8656	1.4892	0.0163	0.6027
160	8	9.3459	−8.8311	1.4892	0.0207	0.6834
161	4	5.3512	−5.0566	1.4892	0.027	0.4978
162	16	−1.3298	1.2689	1.4892	0.0289	0.3431
163	12	4.7287	−4.4662	1.4892	0.0206	1.2122
164	18	6.22	−5.8772	1.4892	0.0184	1.0666
165	19	13.13	−12.394	1.4892	0.021	1.4906
166	18	7.1076	−6.6994	1.4892	0.0211	1.386
167	5	3.2762	−3.0824	1.4892	0.0255	1.1741
168	11	3.0507	−2.8403	1.4892	0.0296	1.4935
169	10	0.9617	−0.8742	1.4892	0.0234	0.5396
170	19	2.0934	−1.9275	1.4892	0.027	0.6161
171	5	0.0174	−0.0078	1.4892	0.0275	0.7124
172	7	3.2551	−3.0304	1.4892	0.0244	0.6577
173	19	0.909	−0.8452	1.4892	0.0258	0.6063
174	19	0.8669	−0.807	1.4892	0.0219	1.2589
175	10	1.9332	−1.7976	1.4892	0.0189	0.7374
176	10	13.928	−12.966	1.4892	0.0235	0.5804
177	6	8.7675	−8.1583	1.4892	0.0232	1.1832
178	9	1.3481	−1.2544	1.4892	0.0198	0.5374
179	14	0.9565	−0.8852	1.4892	0.0232	0.3774
180	8	0.7656	−0.5372	1.4892	0.0132	0.2771

Tables 9 and 10 show the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ and the corresponding parameter estimates $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$ and $\gamma_{\hat{m}_k,k}$ for the U. S. Treasury Bill Yield Interest Rate (US-TBYIR) and U. S. Eurocurrency Exchange Rate (US-EER) data at each time t_k, respectively. This is based on p≤r, and the initial real data time-delay r=20. That is, the data schedule time $t_r=t_{20}$. Furthermore, note that the range of the ϵ-best sub-optimal local admissible sample size for the U. S. TBYIR and U. S. EER data for time $t_k\in[21,45]\cup[420,445]$ and $t_k\in[21,45]\cup[155,180]$, respectively, is $2\leq\hat{m}_k\leq 20$. All comments made with regard to Table 2 remain valid with regard to Tables 9 and 10 in the context of the the U. S. treasury bill Yield Interest Rate and the U. S. Eurocurrency Exchange Rate data at time t_k and the LLGMM approach.

FIGS. 5A and 5B illustrate real and simulated U.S. treasury bill interest rates and U.S. eurocurrency exchange rates using the local lagged adapted generalized method of moments dynamic process, respectively, with r=20.

Comparison of Goodness-of-Fit Measures for U. S. TBYIR and U. S. EER Using r=20.

Table 11 compares the Goodness-of-fit Measures for the U. S. TBYIR and U. S. EER data using r=20.

TABLE 11

Goodness-of-fit Measures for the U. S. TBYIR and U. S. EER data using r = 20.

	r = 20	
Goodness of-fit Meaure	U. S. TBYIR	U. S. EER
$\widehat{\text{RAMS}}$ E	0.0024	0.0137
$\widehat{\text{AMAD}}$	0.0148	0.0718
$\widehat{\text{AMB}}$	0.0165	0.1033

5. Forecasting

Referring back to FIG. **2**, at reference numeral **216**, the process **200** further includes forecasting at least one future state value of the stochastic model of the continuous-time dynamic process using the optimal m_k-local moving sequence. Further, at reference numeral **218**, the process **200** includes determining an interval of confidence associated with the at least one future state value. In those contexts, the application of the LLGMM approach to robust forecasting and the confidence interval problems is outlined in this section. It does not require a large data size or any type of stationary conditions. First, an outline about forecasting problems is outlined. The ϵ-best sub-optimal simulated value $y_{\hat{m}_k,k}^s$ at time t_k is used to define a forecast $y_{\hat{m}_k,k}^f$ for y_k at the time t_k for each of the Energy commodity model, and the U. S. TBYIR and U.S. EER.

5.1. Forecasting for Energy Commodity Model

In the context of the illustration in Section 3.5, we begin forecasting from time t_k. Using the data up to time t_{k-1}, we compute $\hat{m}_i$, $\sigma_{\hat{m}_i,i}^2$, $a_{\hat{m}_i,i}$ and $\mu_{\hat{m}_i,i}$ for $i\in I_0(k-1)$. We assume that we have no information about the real data $\{y_i\}_{i=k}^N$. Under these considerations, imitating the computational procedure outlined in Section 4 and using equation (43), we find the estimate of the forecast $y_{\hat{m}_k,k}^f$ at time t_k by employing the following discrete time iterative process:

$$y_{\hat{m}_k,k}^f=y_{\hat{m}_{k-1},k-1}^s+a_{\hat{m}_{k-1},k-1}y_{\hat{m}_{k-1},k-1}^s(\mu_{\hat{m}_{k-1},k-1}-y_{\hat{m}_{k-1},k-1}^s)\Delta t+\sigma_{\hat{m}_{k-1},k-1}y_{\hat{m}_{k-1},k-1}^s\Delta W_k, \quad (70)$$

where the estimates $\sigma_{\hat{m}_{k-1},k-1}^2$, $a_{\hat{m}_{k-1},k-1}$ and $\mu_{\hat{m}_{k-1},k-1}$ are defined in (43) with respect to the known past data up to the time t_{k-1}. We note that $y_{\hat{m}_k,k}^f$ is the ϵ-sub-optimal estimate for y_k at time t_k.

To determine $y_{\hat{m}_{k+1},k+1}^f$, we need $\sigma_{\hat{m}_k,k}^2$, $a_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$. Since we only have information of real data up to time t_{k-1}, we use the forecasted estimate $y_{\hat{m}_k,k}^f$ as the estimate of y_k at time t_k, and to estimate $\sigma_{\hat{m}_k,k}^2$, $a_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$. Hence, we can write $a_{\hat{m}_k,k}$ as

$$a_{\hat{m}_k,k}\equiv a_{\hat{m}_k,y_{k-\hat{m}_k+1},y_{k-\hat{m}_k+2},\ldots,y_{k-1},y_{m_k,k}^f}.$$

We can also re-write

$$\mu_{\hat{m}_k,k}\equiv \mu_{\hat{m}_k,y_{k-\hat{m}_k+1},y_{k-\hat{m}_k+2},\ldots,y_{k-1},y_{m_k,k}^f}.$$

To find $y_{\hat{m}_{k+2},k-2}^{f}$, we use the estimates

$$a_{\hat{m}_{k+1},k+1} \equiv a_{\hat{m}_{k+1},y_{k-\hat{m}_k+2},y_{k-\hat{m}_k+3},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1}}$$

and

$$\mu_{\hat{m}_{k+1},k+1} \equiv \mu_{\hat{m}_{k+1},y_{k-\hat{m}_k+2},y_{k-\hat{m}_k+3},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1}}.$$

Continuing this process in this manner, we use the estimates

$$a_{\hat{m}_{k+i-1},k+i-1} \equiv a_{\hat{m}_{k+i-1},y_{k-\hat{m}_k+i},y_{k-\hat{m}_k+i+1},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1},\ldots,y^f_{\hat{m}_{k+1},k+i-1}}$$

and

$$\mu_{\hat{m}_{k+i-1},k+i-1} \equiv \mu_{\hat{m}_{k+i-1},y_{k-\hat{m}_k+i},y_{k-\hat{m}_k+i+1},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1},\ldots,y^f_{\hat{m}_{k+1},k+i-1}}$$

to estimate $y_{\hat{m}_{k+i},k+i}^{f}$.

5.1.1. Prediction/Confidence Interval for Energy Commodities

In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the $100(1-\alpha)\%$ confidence interval for the forecast of the state $y_{\hat{m}_i,i}^{f}$ at time t_i, $i \geq k$, as $y_{\hat{m}_i,i}^{f} \pm z_{1-\alpha/2}(s_{\hat{m}_{i-1},i-1}^{2})^{1/2}\, y_{\hat{m}_{i-1},i-1}^{f}$ where $(s_{\hat{m}_{i-1},i-1}^{2})^{1/2}\, y_{\hat{m}_{i-1},i-1}^{f}$ is the estimate for the sample standard deviation for the forecasted state derived from the following iterative process

$$y_{\hat{m}_k,k}^{f} = y_{\hat{m}_{k-1},k-1}^{f} + a_{\hat{m}_{k-1},k-1} y_{\hat{m}_{k-1},k-1}^{f}(\mu_{\hat{m}_{k-1},k-1} - y_{\hat{m}_{k-1},k-1}^{f})\Delta t + \sigma_{\hat{m}_{k-1},k-1} y_{\hat{m}_{k-1},k-1}^{f}\Delta W_k. \quad (71)$$

It is clear that the 95% confidence interval for the forecast at time t_i is

$$\left(y^f_{\hat{m}_i,i} - 1.96\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1},\ y^f_{\hat{m}_i,i} + 1.96\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1}\right),$$

where the lower end denotes the lower bound of the state estimate and the upper end denotes the upper bound of the state estimate.

FIGS. **6**A and **6**B show the graphs of the forecast and 95 percent confidence limit for the daily Henry Hub Natural gas and weekly Ethanol data, respectively. Further, **6**A and **6**B show two regions: the simulation region S and the forecast region F. For the simulation region S, we plot the real data together with the simulated data. For the forecast region F, we plot the estimate of the forecast as explained in Section 5. The upper and the lower simulated sketches in FIGS. **6**A and **6**B are corresponding to the upper and lower ends of the 95% confidence interval. Next, we show graphs which exhibit the bounds of the estimates of the forecast for the four energy commodity.

5.2. Prediction/Confidence Interval for U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Rate

Following the same procedure explained in Section 5.1, we show the graph of the real, simulated, forecast and 95% confidence limit for the U. S. TBYIR and U.S. EER for the initial delay r=20. FIG. **7**A shows the real, simulated, forecast, and 95 percent confidence limit for the Interest rate data, and FIG. **7**B shows the real, simulated, forecast, and 95% confidence level for the U. S. EER.

6. The Byproduct of the Llgmm Approach

The DTIDMLSMVSP not only plays role (a) to initiate ideas for the usage of discrete time interconnected dynamic approach parallel to the continuous-time dynamic process, (b) to speed-up the computation time, and (c) to significantly reduce the state error estimates, but it also provides an alternative approach to the GARCH(1,1) model and comparable results with ex post volatility results of Chan et al. Furthermore, the LLGMM directly generates a GMM based method (e.g., Remark 12, Section 3). In this section, we briefly discuss these comparisons in the context of four energy commodity and U.S. TBYIR and EER data.

6.1 Comparison Between DTIDMLSMVSP and GARCH Model

In this subsection, we briefly compare the applications of DTIDMLSMVSP and GARCH in the context of four energy commodities. In reference to Remark 6, we compare the estimates $s_{\hat{m}_k,k}^{2}$ with the estimate derived from the usage of a GARCH(1,1) model described defined by

$$z_t \mid \mathcal{F}_{t\,;} \sim \mathcal{N}(0, h_t), \quad (72)$$
$$h_t = \alpha_0 + \alpha_1 h_{t-1} + \beta_1 z_{t-1}^2,\ \alpha_0 > 0,\ \alpha_1,\ \beta_1 \geq 0.$$

The parameters α_0, α_1, and β_1 of the GARCH(1,1) conditional variance model (72) for the four commodities natural gas, crude oil, coal, and ethanol are estimated. The estimates of the parameters are given in Table 12.

TABLE 12

Parameter estimates for Garch(1,1) Model (72).

Data Set	α_0	α_1	β_1
Natural Gas	6.863×10^{-5}	0.853	0.112
Crude Oil	9.622×10^{-5}	0.917	0.069
Coal	3.023×10^{-5}	0.903	0.081
Ethanol	4.152×10^{-4}	0.815	0.019

We later show a side by side comparison of $s_{\hat{m}_k,k}^{2}$ and the volatility described by GARCH(1,1) model described in (72) with coefficients in Table 12. The GARCH model does not estimate volatility but instead demonstrated insensitivity.

6.2 Comparison of DTIDMLSMVSP with Chan et al

In this subsection, using the U.S. TBYIR and U.S. EER data, the comparison between the DTIDMLSMVSP and ex post volatility of Chan et al is made. According to the work of Chan et al, we define the ex post volatility by the absolute value of the change in U.S. TBYIR data. Likewise, we define simulated volatility by the square root of the conditional variance implied by the estimates of the model (45). Using (45), we calculate our simulated volatility as

$$\sigma_{\hat{m}_k,k}\left(y^s_{\hat{m}_k,k}\right)^{\delta_{\hat{m}_k,k}}.$$

We compare our work (DTIDMLSMVSP) with FIG. 1 of Chan et al. Their model does not clearly estimate the volatility. It demonstrated insensitivity in the sense that it was unable to capture most of the spikes in the interest rate ex post volatility data.

6.3 Formulation of Aggregated Generalized Method of Moment (AGMM)

In this subsection, using the theoretical basis of the LLGMM and Remark 12 (Section 3), we develop a GMM based method for state and parameter estimation problems.

6.3.1. AGMM Method Applied to Energy Commodities

Using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ described by the mean value of the estimated samples $\{a_{\hat{m}_i,i}\}_{i=0}^N$, $\{\mu_{\hat{m}_i,i}\}_{i=0}^N$ and $\{\sigma_{\hat{m}_i,i}^2\}_{i=0}^N$, respectively, we discuss the simulated price values for the four energy commodities. We define

$$\bar{a} = \frac{1}{N}\sum_{i=0}^{N} a_{\hat{m}_i,i},\ \bar{\mu} = \frac{1}{N}\sum_{i=0}^{N} \mu_{\hat{m}_i,i},\ \text{and}\ \overline{\sigma^2} = \frac{1}{N}\sum_{i=0}^{N} \sigma^2_{\hat{m}_i,i},$$

respectively. Further, $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ are referred to as aggregated parameter estimates of a, μ, and σ^2 over the given entire finite interval of time, respectively.

These estimates are derived using the following discretized system:

$$y_i^{ag} = y_{i-1}^{ag} + \bar{a}(\bar{\mu} - y_{i-1}^{ag})y_{i-1}^{ag}\Delta t + \overline{\sigma^2}^{1/2} y_{i-1}^{ag}\Delta W_i, \tag{73}$$

where y_k^{ag} denotes the simulated value for y_k at time t_k at time. The overall descriptive data statistic regarding the four energy commodity prices and estimated parameters are recorded in the Table 13.

TABLE 13

Descriptive statistics for a, μ and σ^2 with time delay r = 20.

Data Set Y	$\bar{Y}$	Std (Y)	$\overline{\Delta\ln(Y)}$	var (Δln(Y))	$\bar{a}$	Std (a)	$\bar{\mu}$	Std (μ)	$\overline{\sigma^2}$	std (σ^2)	95% C.I. $\bar{\mu}$
Nat. Gas	4.5504	1.5090	0.0008	0.0015	0.1867	0.3013	4.5538	2.3565	0.0013	0.0017	(4.4196, 4.6880)
Crude Oil	54.0093	31.0248	0.0003	0.0006	0.0215	0.0517	54.0307	37.4455	0.0005	0.0008	(51.8978, 56.1636)
Coal	27.1441	17.8394	0.0003	0.0015	0.0464	0.0879	27.0567	21.3506	0.0014	0.0022	(25.8405, 28.2729)
Ethanol	2.1391	0.4455	0.0011	0.0020	0.3167	0.8745	2.1666	0.7972	0.0018	0.0030	(2.0919, 2.2414)

Table 13 shows the descriptive statistics for a, μ and σ^2 with time delay r=20. Further, $\bar{\mu}$ is approximately close to the overall descriptive statistics of the mean $\bar{Y}$ of the real data for each of the energy commodities shown in column 2. Also, $\overline{\sigma^2}$ is approximately close to the overall descriptive statistics of the variance of $\Delta\ \ln(Y)=\ln(Y_i)-\ln(Y_{i-1})$ in column 5. Further, column 12 shows that the mean of the actual data set in Column 2 falls within the 95% confidence interval of $\bar{\mu}$. This exhibits that the parameter $\mu_{\hat{m}_k,k}$ is the mean level of y_k at time t_k.

Using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ in Table 13 (columns 6, 8, and 10), the simulated price values for the four energy commodities are shown in columns 3, 6, 9 and 12 of Table 14.

TABLE 14

Real, Simulation using AGMM with r = 20.

	Natural gas			Crude oil			Coal			Ethanol	
t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)
21	2.759	2.649	21	24.00	23.974	21	8.690	9.111	21	1.895	1.834
22	2.659	2.651	22	23.900	24.204	22	8.630	9.028	22	1.950	1.854
23	2.742	2.636	23	23.050	25.229	23	8.690	9.192	23	1.974	1.798
24	2.562	2.625	24	22.300	25.586	24	8.940	9.032	24	2.700	1.858
25	2.495	2.593	25	22.450	26.470	25	9.310	8.938	25	2.515	1.830
26	2.54	2.525	26	22.350	25.953	26	8.940	8.792	26	2.290	1.954
27	2.592	2.513	27	21.750	26.229	27	8.940	9.035	27	2.440	1.926
28	2.57	2.399	28	22.100	26.555	28	9.130	9.255	28	2.415	1.939
29	2.541	2.485	29	22.400	26.402	29	9.190	9.018	29	2.300	1.883
30	2.618	2.506	30	22.500	27.34	30	8.570	8.687	30	2.100	1.880
31	2.564	2.460	31	22.650	26.24	31	8.690	8.985	31	2.040	1.817
32	2.667	2.295	32	21.950	26.765	32	8.880	9.339	32	2.160	1.810
33	2.633	2.534	33	21.600	26.358	33	8.570	9.359	33	2.130	1.774
34	2.515	2.514	34	21.000	26.87	34	8.750	9.310	34	2.155	1.717
35	2.53	2.573	35	20.950	26.835	35	8.630	9.302	35	2.010	1.658
36	2.549	2.592	36	21.100	26.725	36	8.440	9.543	36	1.930	1.607
37	2.603	2.456	37	20.800	26.439	37	8.440	9.288	37	1.900	1.645
38	2.603	2.428	38	20.300	26.916	38	8.940	9.155	38	1.975	1.635

TABLE 14-continued

Real, Simulation using AGMM with r = 20.

Natural gas			Crude oil			Coal			Ethanol		
t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)	t_k	Real	Simulated y_k^{ag} (AGMM)
39	2.603	2.505	39	20.250	26.989	39	9.000	8.469	39	1.980	1.629
40	2.815	2.526	40	20.750	26.759	40	8.940	8.899	40	2.00	1.745
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
1145	5.712	5.218	2440	57.350	48.179	2865	29.310	17.839	375	2.073	2.625
1146	5.588	5.414	2441	56.740	48.239	2866	28.680	18.563	376	2.02	2.784
1147	5.693	5.460	2442	57.550	46.984	2867	26.770	19.577	377	2.073	2.558
1148	5.791	5.464	2443	59.090	47.418	2868	27.450	19.841	378	2.065	2.670
1149	5.614	5.544	2444	60.270	48.137	2869	27.000	18.876	379	2.055	2.565
1150	5.442	5.700	2445	60.750	49.185	2870	26.670	18.465	380	2.209	2.796
1151	5.533	5.710	2446	58.410	48.271	2871	26.510	18.139	381	2.44	2.783
1152	5.378	5.936	2447	58.720	48.384	2872	26.480	17.963	382	2.517	2.659
1153	5.373	5.869	2448	58.640	47.509	2873	25.150	18.151	383	2.718	2.739
1154	5.382	5.778	2449	57.870	48.654	2874	25.570	17.987	384	2.541	2.681
1155	5.507	5.732	2450	59.130	46.883	2875	25.880	18.393	385	2.566	2.631
1156	5.552	5.816	2451	60.110	46.403	2876	25.240	18.492	386	2.626	2.638
1157	5.31	6.000	2452	58.940	45.564	2877	25.000	18.621	387	2.587	2.542
1158	5.338	6.162	2453	59.930	44.177	2878	25.080	18.806	388	2.628	2.491
1159	5.298	5.899	2454	61.180	43.112	2879	25.050	19.384	389	2.587	2.392
1160	5.189	6.008	2455	59.660	43.47	2880	25.890	20.131	390	2.536	2.393
1161	5.082	6.175	2456	58.590	41.531	2881	25.230	21.099	391	2.42	2.534
1162	5.082	6.191	2457	58.280	40.452	2882	25.940	21.499	392	2.247	2.687
1163	5.082	5.814	2458	58.790	41.968	2883	25.260	21.38	393	2.223	2.701
1164	4.965	5.701	2459	56.230	44.359	2884	25.250	20.786	394	2.39	2.703
1165	4.767	5.871	2460	55.90	44.679	2885	26.060	20.892	395	2.38	2.655
1166	4.675	5.998	2461	56.420	43.081	2886	26.030	21.269	396	2.366	2.559
1167	4.79	5.952	2462	58.010	44.235	2887	26.660	20.371	397	2.335	2.575
1168	4.631	5.782	2463	57.280	43.199	2888	27.120	19.822	398	2.428	2.466
1169	4.658	5.673	2464	60.30	42.655	2889	26.400	19.644	399	2.409	2.369
1170	4.57	5.936	2465	60.970	43.498	2890	26.940	20.602	400	2.29	2.222

TABLE 15

Comparison of Goodness-of-fit Measures for the LLGMM and AGMM method using initial delay r = 20.

Goodness	LLGMM				AGMM			
of-fit Measure	Natural Gas	Crude Oil	Coal	Ethanol	Natural gas	Crude oil	Coal	Ethanol
$\widehat{RAMSE}$	0.0674	0.4625	0.4794	0.0375	1.4968	30.7760	17.7620	0.4356
$\widehat{AMAD}$	1.1318	24.5010	9.4009	0.3213	0.0068	0.0857	0.0833	0.0035
$\widehat{AMB}$	1.1371	27.2707	12.8370	0.3566	1.2267	27.3050	13.1060	0.3579

6.3.2. Formulation of Aggregated Generalized Method of Moment (AGMM) for U.S. Treasury Bill and U.S. Eurocurrency Rate

The overall descriptive statistics of data sets regarding U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate are recorded in the following table for initial delay r=20.

TABLE 17

$\bar{Y}$	Std (Y)	$\bar{\beta}$	Std (β)	$\bar{\mu}$	Std (μ)	$\bar{\delta}$	Sts (δ)	$\bar{\sigma}$	Std (σ)	$\bar{\gamma}$	Std (γ)
Descriptive statistics for β, $\bar{\mu}$, δ, σ, and γ for the U.S. TBYIR data with initial delay r = 20											
0.05667	0.0268	0.8739	1.8129	−3.8555	8.7608	1.4600	0.00	0.3753	0.5197	1.4877	0.1357
Descriptive statistics for β, $\bar{\mu}$, δ, σ, and γ for the U.S. Eurocurrency Exchange Rate data with initial delay r = 20											
1.6249	0.1337	1.5120	2.1259	−1.1973	1.6811	1.4892	0.00	0.0243	0.0180	1.08476	1.0050

In Tables 16 and 17, the real and the LLGMM simulated rates of the U.S. TBYIR and the U. S. Eurocurrency Exchange Rate (US-EER) are exhibited in the first and second columns, respectively. Using the aggregated parameter estimates $\bar{\beta}$, $\bar{\mu}$, $\bar{\delta}$, $\bar{\sigma}$ and $\hat{\gamma}$ in the respective Tables 16 (columns: 3, 5, 7, 9 and 11) and Table 17 (columns: 3, 5, 7, 9, and 11), the simulated rates for the U.S. TYBIR and the U.S. EER are shown in the column 3 of Table 18. These estimates are derived using the following discretized system:

$$y_i^{ag}=y_{i-1}^{ag}+(\bar{\beta}y_{i-1}^{ag}+\bar{\mu}(y_{i-1}^{ag})^{\bar{\delta}})+\bar{\sigma}(y_{i-1}^{af})^{\bar{\gamma}}\Delta W_i, \quad (74)$$

where AGMM, y_k^{ag}, y_k at time t_k are defined in (73).

TABLE 18

Estimates for Real, Simulated value using LLGMM and AGMM methods for U.S. TYBIR and the U.S. EER, respectively for initial delay r = 20.

	Interest Rate Data				Eurocurrency Rate		
t_k	Real	Simulated LLGMM	Simulated AGMM	t_k	Real Real	Simulated LLGMM	Simulated AGMM
21	0.0465	0.0459	0.0326	21	1.7448	1.6732	1.655
22	0.0459	0.0467	0.0299	22	1.7465	1.7711	1.6588
23	0.0462	0.0463	0.0342	23	1.7638	1.7588	1.6096
24	0.0464	0.0463	0.034	24	1.874	1.8423	1.6251
25	0.045	0.0457	0.0365	25	1.7902	1.7971	1.6221
26	0.048	0.048	0.0447	26	1.7635	1.7668	1.5984
27	0.0496	0.0496	0.0449	27	1.74	1.7362	1.6368
28	0.0537	0.053	0.0538	28	1.7763	1.7755	1.5795
29	0.0535	0.0529	0.0535	29	1.8219	1.8224	1.5708
30	0.0532	0.0536	0.0489	30	1.8985	1.9002	1.6174
31	0.0496	0.0495	0.0575	31	1.9166	1.8897	1.6403
32	0.047	0.0479	0.0548	32	1.992	1.9361	1.6425
33	0.0456	0.0453	0.0385	33	1.7741	1.7738	1.6409
34	0.0426	0.0423	0.042	34	1.5579	1.5601	1.6759
35	0.0384	0.0413	0.0339	35	1.5138	1.5017	1.5287
36	0.036	0.0363	0.0384	36	1.5102	1.5028	1.5445
37	0.0354	0.0358	0.0457	37	1.4832	1.5171	1.6334
38	0.0421	0.0434	0.0321	38	1.4276	1.4353	1.6666
39	0.0427	0.043	0.023	39	1.51	1.4972	1.606
40	0.0442	0.044	0.0299	40	1.5734	1.588	1.662
41	0.0456	0.0463	0.0301	41	1.5633	1.5556	1.6305
42	0.0473	0.0462	0.0365	42	1.4966	1.4856	1.5987
43	0.0497	0.0512	0.0341	43	1.4868	1.4914	1.5832
44	0.05	0.0505	0.042	44	1.4864	1.4785	1.621
45	0.0498	0.0497	0.0451	45	1.4965	1.4854	1.6208
. . .	. . .	. . .	. . .		. . .	. . .	. . .
. . .	. . .	. . .	. . .		. . .	. . .	. . .
420	0.045	0.0449	0.0337	155	1.5581	1.5635	1.6326
421	0.0457	0.045	0.0309	156	1.6097	1.6195	1.574
422	0.0455	0.0459	0.0389	157	1.6435	1.6089	1.6232
423	0.0472	0.047	0.0306	158	1.5793	1.5817	1.6669
424	0.0468	0.0464	0.0385	159	1.5782	1.5826	1.649
425	0.0486	0.0481	0.0179	160	1.6108	1.6206	1.5725
426	0.0507	0.0499	0.0191	161	1.6368	1.6256	1.6879
427	0.052	0.0514	0.0257	162	1.662	1.644	1.6681
428	0.0532	0.0539	0.029	163	1.6115	1.6156	1.6534
429	0.0555	0.0546	0.0379	164	1.571	1.5708	1.6387
430	0.0569	0.0588	0.0404	165	1.6692	1.6912	1.6243
431	0.0566	0.056	0.0487	166	1.6766	1.6832	1.5822
432	0.0579	0.0587	0.0432	167	1.7188	1.7224	1.5764
433	0.0569	0.0571	0.0436	168	1.7856	1.7285	1.6206
434	0.0596	0.0602	0.0393	169	1.8225	1.7952	1.6044
435	0.0609	0.0601	0.04	170	1.8699	1.8896	1.6792
436	0.06	0.0601	0.0483	171	1.8562	1.8964	1.5417
437	0.0611	0.0604	0.0292	172	1.772	1.7717	1.6087
438	0.0617	0.0617	0.031	173	1.8398	1.8372	1.5426
439	0.0577	0.0583	0.0379	174	1.8207	1.8214	1.6147
440	0.0515	0.0509	0.0464	175	1.8248	1.8242	1.6544
441	0.0488	0.05	0.0476	176	1.7934	1.7795	1.5929
442	0.0442	0.0441	0.0516	177	1.7982	1.8056	1.5845
443	0.0387	0.0445	0.0675	178	1.8335	1.835	1.6625
444	0.0362	0.0313	0.0484	179	1.934	1.9301	1.5832
445	0.0349	0.0386	0.0484	180	1.9054	1.8939	1.5472

In Table 18, we show a side by side comparison of the estimates for the simulated value using LLGMM and AGMM methods for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate, respectively: initial delay r=20.

7. Comparisons of LLGMM with OCBGMM

In this section, we briefly compare LLGMM and OCBGMM in the frame-work of the conceptual, computational, mathematical, and statistical results coupled with role, scope and applications. For this purpose, to better appreciate and understand the comparative work, we utilize the state and parameter estimation problems for the stochastic dynamic model of interest rate that has been studied extensively in the frame-work of orthogonality condition vector based generalized method of moments (OCBGMM). Recall that the LLGMM approach is based on seven interactive components (Section 1). On the other hand, the existing OCBGMM (GMM and IRGMM) approach and its extensions are based on five components (Section 3). The basis for the formation of orthogonality condition parameter vectors (OCPV) in the LLGMM (Section 3) and OCBGMM (GMM/IRGMM) are different. In the existing OCBGMM (GMM/IRGMM), the orthogonality condition vectors are formed on the basis of algebraic manipulation coupled with econometric specification-based discretization scheme (OCPV-Algebraic) rather than stochastic calculus and a continuous-time stochastic dynamic model based OCPV-Analytic. This motivates to extend a couple of OCBGMM-based state and parameter estimation methods.

Using the stochastic calculus based formation of the OCPV-Analytic in the context of the continuous-time stochastic dynamic model (Section 3), two new OCBGMM based methods are developed for the state and parameter estimation problems. The proposed OCBGMM methods are direct extensions of the existing OCBGMM method and its extension IRGMM in the context of the OCPV. In view of this difference and for the sake of comparison, the newly developed OCBGMM and the existing OCBGMM methods are referred to as the OCBGMM-Analytic and OCBGMM-Algebraic, respectively. In particular, the GMM and IRGMM with OCPV-algebraic are denoted as GMM-Algebraic and IRGMM-Algebraic and corresponding extensions under the OCPV-Analytic as GMM-Analytic and IRGMM-Analytic, respectively.

Furthermore, using LLGMM based method, the aggregated generalized method of moments (AGMM) introduced in Subsection 3.5 and described in Subsection 6.3 is also compared along with the above stated methods, namely GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, and IRGMM-Analytic. A comparative analysis of the results of GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic and AGMM methods with the LLGMM for the state and parameter estimation problems of the interest rate and energy commodities stochastic dynamic models are briefly outlined in the subsequent subsections. First, based on Sections 1, 2, 3 and 4, we briefly summarize the comparison between the LLGMM and OCBGMM methods.

7.1 Theoretical Comparison Between LLGMM and OCBGMM

Based on the foundations of the analytical, conceptual, computational, mathematical, practical, statistical, and theoretical motivations and developments outlined in Sections 2, 3, 4 and 5, we summarize the comparison between the innovative approach LLGMM with the existing and newly developed OCBGMM methods in separate tables in a systematic manner.

Table 19 outlines the differences between the LLGMM method and existing orthogonality condition based GMM/IRGMM-Algebraic and the newly formulated GMM/IRGMM-Analytic methods together with the AGMM.

TABLE 19

Mathematical Comparison Between the LLGMM and OCBGMM

Feature	LLGMM	OCBGMM-Algebraic	OCGMM-Analytic	Justifications
Composition:	Seven components	Five components	Five components	Sections 1. 3
Model:	Development	Selection	Development/Selection	Sections 1, 3
Goal:	Validation	Specification/Testing	Validation/Testing	Sections 1, 3
Discrete-Time Scheme:	Constructed from SDE	Using Econometric specification	Constructed from SDE	Remarks 8,15
Formation of Orthogonality Vector:	Using stochastic calculus	Formed using algebraic manipulation	Using Stochastic calculus	Remarks 7, 8, 9, 14, 15

TABLE 20

Intercomponent Interaction Comparison Between LLGMM and OCBGMM

Feasture	LLGMM	OCBGMM-Algebraic	OCGMM-Analytic	Justifications
Moment Equations:	Local Lagged adaptive process	Single/global system	Single/global system	Remarks 5, 8, 18a, and 18b
Type of Moment Equations:	Local lagged adaptive process	Single-shot	Single-shot	Remarks 5, 8, 13, 15, and 16
Component Interconnections:	Strongly connected	Weakly connected	Weakly connected	Remarks 8, 13, 14, 15, 16, and 18
Dynamic and Static:	Discrete-time Dynamic	Static	Static	Remarks 5, 8, 18 and Lemma 1 (Section 2)

TABLE 21

Conceptual Computational Comparison Between LLGMM and OCBGMM

Feature	LLGMM	OCBGMM-Algebraic	OCGMM-Analytic	Justifications
Local admissible Lagged Data Size:	Multi-choice	Single-choice/data size	Singe-choice/data size	Definition 10, Remark 18, Subsection 4.2
Local admissible class of lagged finite restriction sequences	Multi-choice	Single-choice/data sequence	Single-choice/data sequence	Adapted finite restricted sample data: Definition 11, Remark 18, Subsection 4.2
Local admissible finite sequence parameter estimates:	Multi-choice	Single-shot estimate	Single-shot estimates	Subsection 4.2
Local admissible sequence of finite state simulation values:	Multi-choice	Single-choice	Single-choice	Remark 18, Subsection 4.3
Quadratic Mean Square ϵ-sub-optimal errors:	Multi-choice	Single-error	Single-error	Remark 18, Subsection 4.3
ϵ-sub-optimal local lagged sample size:	Multi-choice	Single-choice	Single-choice	Definition 12, Remark 18, Subsection 4.3
ϵ-best sub optimal sample size:	ϵ-best sub optimal choice	No-choice	No-choice	Remark 18, Subsection 4.3
ϵ-best sub optimal parameter estimated:	ϵ-best estimators	No-choice	No-choice	Remark 18, Subsection 4.3
ϵ-best sub optimal state estimate	ϵ-best sub optimal choice	No-choice	No-choice	Remark 18, Subsection 4.3

TABLE 22

Theoretical Performance Comparison Between LLGMM and OCBGMM

Feature	LLGMM	OCBGMM-Algebraic	OCGMM-Analytic	Justifications
Data Size:	Reasonable Size	Large Data Size	Large Data Size	For Respectable results
Stationary Condition:	Not required	Need Ergotic/ Asympotic stationary	Need Ergodic/ Asymptotic	For Reasonable results
Multi-level optimization:	At least 2 level hierarchical optimization	Single-shot	Single-shot	Not comparable
Admissible Strategies:	Multi-choices	Single-shot	Single-shot	Not comparable
Computational Stability:	Algorithm Converges in a single/double digit trials	Single-choice	Single-choice	Simulation results
Significance of lagged adaptive process:	Stabilizing agent	Non-existence of the feature	Non-existence	Not comparable
Operation:	Operates like Discrete time Dynamic Process	Operates like a static dynamic process	Operates like static process	Obvious, details see Sections 4, 5, 6 and 7

7.2 Comparisons of LLGMM Method with Existing Methods Using Interest Rate Stochastic Model

The continuous-time interest rate process is described by a nonlinear Itô-Doob-type stochastic differential equation:

$$dy=(\alpha+\beta y)dt+\sigma y^{\gamma}dW(t). \quad (75)$$

The energy commodities stochastic dynamic model is described in (27), in Subsection 3.5. These models would be utilized to further compare the role, scope and merit of the LLGMM and OCBGMM methods in the frame-work of the graphical, computational and statistical results and applications to forecasting and prediction with certain degree of confidence.

Remark 26.

The continuous-time interest rate model (75) was chosen so that we can compare our LLGMM method with the OCBGMM method. Our proposed model for the continuous-time interest rate model is described in (45). We will later compare the results derived using model (75) with the results using (45) from Subsections 3.6 and 4.6.

Descriptive Statistic for Time-Series Data Set.

For this purpose, first, we consider one month risk free rates from the Monthly Interest rate data sets for the period Jun. 30, 1964 to Dec. 31, 2004. Table 23 below shows some statistics of the data set shown in FIG. **19**.

TABLE 23

Statistics for the Interest Rate data for Jun. 30, 1964 to Dec. 31, 2004.

Variable	N	Mean	Std dev	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
y_t	487	0.0592	0.0276	0.9809	0.9508	0.9234	0.8994	0.8764	0.8519
Δy_t	486	−0.00003	0.0050	0.3305	−0.0919	−0.1048	−0.0351	0.0403	−0.1877

Mean, standard deviations, and autocorrelations of monthly Treasury bill yields (US TBYIR) and yield changes ρ_j denotes the autocorrelation coefficient of order j, N represents the total number of observations used.

The Orthogonality Condition Vector for (75).

First, we present the orthogonality condition parameter vectors (OCPV) for the GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, and IRGMM-Analytic methods. These orthogonality vectors are then used for the state and parameter estimation problems. For this, we need to follow the procedure (Section 3) for obtaining the analytic orthogonality condition parameter vector (OCPV-Analytic). We consider the Lyapunov functions (

$$V_1(t, y) = \frac{1}{2}y^2$$

and

$$V_2(t, y) = \frac{1}{3}y^3.$$

The Itô-differential of V_1 and V_2 with respect to (75) are:

$$\begin{cases} d\left(\frac{1}{2}y^2\right) = \left[\alpha y + \beta y^2 + \frac{1}{2}\sigma^2 y^{2\gamma}\right]dt + \sigma y^{\gamma+1}dW(t) \\ d\left(\frac{1}{3}y^3\right) = [\alpha y^2 + \beta y^3) + \sigma^2 y^{2\gamma+1}\Big]dt + \sigma y^{\gamma+2}dW(t) \end{cases}. \tag{76}$$

The component of orthogonality condition vector (OCPV-Analytic) is described by:

$$\begin{cases} \Delta y_t - (\mathbb{E}[y_t \mid \mathcal{F}_{t-1}] - y_{t-1}) \\ \frac{1}{2}\Delta(y_t^2) - \frac{1}{2}(\mathbb{E}[y_t^2 \mid \mathcal{F}_{t-1}] - y_{t-1}^2) \\ \frac{1}{3}\Delta(y_t^3) - \frac{1}{3}(\mathbb{E}[y_t^3 \mid \mathcal{F}_{t-1}] - y_{t-1}^3) \\ \mathbb{E}[(\Delta y_t - \mathbb{E}[\Delta y_t \mid \mathcal{F}_{t-1}])^2 \mid \mathcal{F}_{t-1}] - \sigma^2 y_{t-1}^{2\gamma}\Delta t. \end{cases} \tag{77}$$

where

$$\begin{cases} \mathbb{E}[y_t \mid \mathcal{F}_{t-1}] - y_{t-1} & = (\alpha + \beta y)\Delta t \\ \frac{1}{2}(\mathbb{E}[y_t^2 \mid \mathcal{F}_{t-1}] - y_{t-1}^2) & = \left[\alpha y_{t-1} + \beta y_{t-1}^2 + \frac{1}{2}\sigma^2 y_{t-1}^{2\gamma}\right]\Delta t \\ \frac{1}{3}(\mathbb{E}[y_t^3 \mid \mathcal{F}_{t-1}] - y_{t-1}^3) & = \left[\alpha y_{t-1}^2 + \beta y_{t-1}^3 + \sigma^2 y_{t-1}^{2\gamma+1}\right]\Delta t \\ \mathbb{E}[(\Delta y_t - \mathbb{E}[\Delta y_t \mid \mathcal{F}_{t-1}])^2 \mid \mathcal{F}_{t-1}] & = \sigma^2 y_{t-1}^{2\gamma}\Delta t \end{cases}. \tag{78}$$

On the other hand, using discrete time econometric specification coupled with algebraic manipulations, the components of orthogonality condition parameter vector (OCPV-Algebraic) are as follows:

$$\begin{cases} y_t - y_{t-1} - (\alpha + \beta y)\Delta t \\ y_{t-1}(y_t - y_{t-1} - (\alpha + \beta y)\Delta t) \\ (y_t - y_{t-1} - (\alpha + \beta y)\Delta t)^2 - \sigma^2 y_{t-1}^{2\gamma} \\ y_{t-1}\left[(y_t - y_{t-1} - (\alpha + \beta y)\Delta t)^2 - \sigma^2 y_{t-1}^{2\gamma}\right] \end{cases} \tag{79}$$

We apply the GMM-Algebraic, IRGMM-Algebraic, GMM-Analytic, and IRGMM-Analytic methods.

Parameter Estimates of (75) Using LLGMM Method.

Using the LLGMM method, the parameter estimates $\alpha_{\hat{m}_k,k}$, $\beta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ are shown in Table 24. Here, we use ϵ=0.001, p=2, and initial delay r=20.

TABLE 24

Estimates for $\hat{m}_k$, $\alpha_{\hat{m}_k,k}$, $\beta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U. S. Treasury Bill Yield Interest Rate data using LLGMM. Interest Rate

t_k	$\hat{m}_k$	$\alpha_{\hat{m}_k,k}$	$\beta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
21	2	0.0334	−0.7143	0.0446	1.5
22	3	0.0427	−0.9254	0.0766	1.5
23	4	0.0425	−0.9198	0.0914	1.5
24	5	0.0413	−0.8937	0.09	1.5
25	4	0.1042	−2.2619	0.1003	1.5
26	19	0.0002	0.0083	0.1043	1.5
27	14	0.0024	−0.0359	0.1281	1.5
28	5	−0.023	0.5207	0.3501	1.5
29	13	0.0037	−0.0573	0.1652	1.5
30	18	0.0008	0.001	0.1447	1.5
31	3	−0.3827	7.1316	0.26	1.5
32	19	0.006	−0.1213	0.1828	1.5
33	6	0.0063	−0.1359	0.343	1.5
34	19	0.0081	−0.1705	0.1993	1.5
35	4	−0.0166	0.2984	0.3509	1.5
36	4	−0.0059	0.0721	0.2318	1.5
37	9	−0.0035	0.0324	0.3114	1.5
38	14	0.0051	−0.1186	0.3385	1.5

TABLE 24-continued

Estimates for $\hat{m}_k$, $\alpha_{\hat{m}_k,k}$, $\beta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U. S. Treasury Bill Yield Interest Rate data using LLGMM. Interest Rate

t_k	$\hat{m}_k$	$\alpha_{\hat{m}_k,k}$	$\beta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
39	20	0.0059	−0.1294	0.282	1.5
40	12	0.0075	−0.185	0.3447	1.5
41	12	0.0099	−0.2379	0.3579	1.5
42	4	−0.0089	0.2335	0.3562	1.5
43	7	0.0074	−0.1289	0.4654	1.5
44	7	0.0182	−0.3677	0.4206	1.5
45	6	0.0106	−0.2031	0.2356	1.5
...	...	...	...	...	...
...	...	...	...	...	...
420	3	0.0836	−1.9	0.1006	1.5
421	8	0.0428	−0.9671	0.783	1.5
422	3	0.0359	−0.7857	0.1702	1.5
423	8	0.0127	−0.2766	0.1719	1.5
424	6	0.0178	−0.3857	0.1636	1.5
425	6	0.0177	−0.3685	0.1829	1.5
426	18	0.0146	−0.3172	0.3871	1.5
427	8	0.0017	−0.012	0.1788	1.5
428	4	0.009	−0.1489	0.1341	1.5
429	9	−0.0059	0.1469	0.1616	1.5
430	13	−0.0046	0.116	0.191	1.5
431	9	0.0039	−0.0532	0.1369	1.5
432	9	0.0027	−0.0287	0.1109	1.5
433	3	0.0857	−1.5	0.0952	1.5
434	9	0.0102	−0.1661	0.1197	1.5
435	9	0.0075	−0.114	0.107	1.5
436	5	0.029	−0.485	0.1446	1.5
437	4	0.0476	−0.784	0.2163	1.5
438	9	0.0122	−0.1966	0.1054	1.5
439	4	0.1626	−2.6824	0.1248	1.5
440	20	0.0072	−0.1278	0.1916	1.5
441	19	0.0084	−0.1502	0.2016	1.5
442	17	0.0024	−0.0479	0.2369	1.5
443	7	−0.0153	0.2236	0.2687	1.5
444	3	0.0054	−0.2188	0.3887	1.5
445	16	−0.0076	0.1177	0.2528	1.5

Table 24 shows the parameter estimates of $\hat{m}_k$, $\alpha_{\hat{m}_k,k}$, $\beta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ in the model (75) for U.S. Treasury Bill Yield Interest Rate data. As noted before, the range of the ϵ-best sub-optimal local admissible sample size $\hat{m}_k$ for any time $t_k \in [21,45] \cup [420,445]$ is $2 \leq \hat{m}_k \leq 20$. We also draw the similar conclusions (a) to (e) as outlined in Remark 20.

Parameter Estimates of (75) Using OCBGMM Methods.

Following Remark 12, we define the average $\overline{\alpha}$, $\overline{\beta}$, $\overline{\sigma}$, and $\overline{\gamma}$ by

$$\begin{cases} \alpha = \frac{1}{N}\sum_{k=1}^{N} \alpha_{\hat{m}_k,k}, \\ \overline{\beta} = \frac{1}{N}\sum_{k=1}^{N} \beta_{\hat{m}_k,k} \\ \overline{\sigma} = \frac{1}{N}\sum_{k=1}^{N} \sigma_{\hat{m}_k,k}, \\ \overline{\gamma} = \frac{1}{N}\sum_{k=1}^{N} \gamma_{\hat{m}_k,k}, \end{cases} \tag{80}$$

where the parameters $\alpha_{\hat{m}_k,k}$, $\beta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ are each estimated in Table 25 at time t_k using LLGMM method.

Imitating the argument used in Subsection 6.3, the parameters and state are also estimated. These parameter estimates are shown in the row of AGMM approach in Table 25. We also estimate the parameters in (75) by following both the GMM-algebraic and GMM-analytic frame-work. Similarly, the parameter estimates (75) are determined under the IRGMM-algebraic and IRGMM-analytic approaches. These parameter estimates are recorded in rows of GMM-algebraic, GMM-analytic, IRGMM-algebraic, and IRGMM-analytic approaches, respectively, in Table 25.

Comparison of Goodness-of-Fit Measures.

In order to statistically compare the different estimation techniques we estimate the statistics RAMSE, AMAD, and AMB defined in (69). The goodness-of-fit measures are computed using S=100 pseudo-data sets of the same sample size, and the real data set, N=487 months. The t-statistics of each parameter estimate is in parenthesis, the smallest value of RAMSE for all method is italicized. The goodness-of-fit measures RAMSE, AMAD and AMB are recorded under the columns 6, 7, and 8 respectively.

TABLE 25

Comparison of parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB under the usage of GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM, and LLGMM methods.

Method	α	β	σ	γ	$\widehat{\text{RAMSE}}$	$\widehat{\text{AMAD}}$	$\widehat{\text{AMB}}$
GMM-Algebraic	0.0017 (1.53)	−0.0308 (−1.33)	0.4032 (1.55)	1.5309 (3.21)	0.0424	0.0098	0.0195
GMM-Analytic	0.0009 (1.06)	−0.0153 (−0.90)	0.0184 (1.25)	0.4981 (1.73)	0.0315	0.0161	0.0190
IRGMM-Algebraic	0.0020 (0.32)	−0.0410 (−0.21)	0.207 (0.25)	1.3031 (1.02)	0.03186	0.00843	0.01972
IRGMM-Analytic	0.0084 (0.44)	−0.1436 (−0.40)	0.1075 (0.22)	1.3592 (1.01)	0.0278	0.0028	0.01968
AGMM	0.0084 (0.41)	−0.1436 (−0.33)	0.1075 (0.25)	1.3592 (0.98)	0.0288	0.0047	0.0207
LLGMM					0.0027*	0.0146	0.0178

The LLGMM estimates are derived using initial delay r=20, p=2 and ϵ=0.001. Among these stated methods, the LLGMM method generates the smallest RAMSE value. In fact, the RAMSE value is smaller than the one tenth of any other RAMSE values. Further, second, third and fourth smaller RAMSE values are due to the IRGMM-Analytic, AGMM and GMM-Analytic methods, respectively. This exhibits the superiority of the LLGMM method over all other methods. We further observe that the LLGMM approach yields the smallest AMB in comparison with the OCBGMM approaches. The GMM-Analytic, IRGMM-Analytic and IRGMM-Algebraic rank the second, third and fourth smaller values, respectively. The high value of AMAD for the LLGMM method signifies that the LLGMM captures the influence of random environmental fluctuations on the dynamic of interest rate process. We further note that the first, second, third, and fourth smaller AMB values are due to the GMM-Analytic, LLGMM, IRGMM-Algebraic, and GMM-Algebraic methods, respectively. Again, from Remark 23, the smallest RAMSE, higher AMAD, and smallest AMB value under the LLGMM method exhibit the superior performance under the three goodness-of-fit measures. We also notice that the performance of stochastic calculus based-OCPV-Analytic methods, namely, GMM-Analytic, IRGMM-Analytic and AGMM is better than the performance of OCPV-Algebraic based, GMM-Algebraic, and IRGMM-Algebraic approaches. In short, this suggests that the OCPV-Analytic based GMM methods are more superior than the OCPV-Algebraic based GMM methods.

TABLE 26

Parameter estimates and goodness of fit tests for one month risk free rates for periods June 1964-December 1981 and January 1982-December 2004.

Orthogonality Condition	June 1964- December 1981 $\widehat{RAMSE}$	January 1982- December 2004 $\widehat{RAMSE}$
GMM-Algebraic	0.0468	0.0377
GMM-Analytic	0.0315	0.0347
IRGMM-Algebraic	0.0307	0.0326
IRGMM-Analytic	0.0200	0.0215
LLCIMM	0.0030*	0.0017*

Table 26 shows the goodness-of-fit measures RAMSE using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, and LLGMM method for two separate sub-periods: 06/1964-12/1981 and 01/1982-12/2004. Among all methods, the LLGMM method generates the smallest RAMSE value for each sub-period. Further, the goodness-of-fit measure RAMSE regarding the LLGMM method is less than the one sixth, and one twelfth of any other RAMSE value, respectively. The IRGMM-Analytic, IRGMM-Algebraic, GMM-Analytic, and GMM-Algebraic methods are in second, third, fourth and fifth place.

Comparative Analysis of Forecasting with 95% Confidence Intervals.

Using data set June 1964 to December 1989, the parameters of model (75) are estimated. Using these parameter estimates, we forecasted the monthly interest rate for Jan. 1, 1990 to Dec. 31, 2004.

TABLE 27

Parameter estimates in (75) in the context of the data from June 1964 to December 1989.

Method	α	β	σ	γ
GMM-Algebraic	0.0033	−0.051	0.4121	1.5311
GMM-Analytic	0.0009	−0.0155	0.0197	0.4854
IRGMM-Algebraic	0.0023	−0.0421	0.3230	1.3112
IRGMM-Analytic	0.0084	−0.1436	0.1073	1.3641
AGMM	0.01.54	−0.2497	0.2949	1.4414

7.3 Comparisons of LLGMM Method with Existing and Newly Introduced OCBGMM Methods Using Energy Commodity Stochastic Model

Using the stochastic dynamic model in (27) of energy commodity represented by the stochastic differential equation

$$dy=ay(\mu-y)dt+\sigma(t,y_t)ydW(t),y(t_0)=y_0, \tag{81}$$

the orthogonality condition parameter vector (OCPV) is described in (30) in Remark 9.

Based on a discretized scheme using the econometric specification, the orthogonality condition parameter vector in the context of algebraic manipulation is as:

$$\begin{cases} y_t - y_{t-1} - ay_{t-1}(\mu - y_{t-1})\Delta t \\ y_{t-1}(y_t - y_{t-1} - ay_{t-1}(\mu - y_{t-1})\Delta t) \\ (y_t - y_{t-1} - ay_{t-1}(\mu - y_{t-1})\Delta t)^2 - \sigma^2 y_{t-1}^2] \end{cases}. \tag{82}$$

The goodness-of-fit measures are computed using pseudo-data sets of the same sample size as the real data set: (i) N=1184 days for natural gas data, (ii) N=4165 days for crude oil data, (iii) N=3470 for coal data, and (iv) N=438 weeks for ethanol data. The smallest value of RAMSE for all method is italicized.

TABLE 28

Parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM and LLGMM methods for natural gas data

Method	a	μ	σ^2	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
GMM-Algebraic	0.0023	5.3312	0.0019	1.5119	0.0663	1.1488
GMM-Analytic	0.0018	5.4106	0.0015	1.5014	0.0538	1.1677
IRGMM-Algebraic	0.2000	4.4996	0.0010	1.4985	0.0050	1.2299
IRGMM-Analytic	0.1998	4.4917	0.0011	1.4901	0.0044	1.2329
AGMM	0.1867	4.5538	0.0013	1.4968	0.0068	1.2267
LLGMM				0.0674*	1.1318	1.1371

TABLE 29

Parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM and LLGMM methods for crude oil data

Method	a	μ	σ^2	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
GMM-Algebraic	0.0023	54.4847	0.0005	39.2853	0.3577	29.1587
GMM-Analytic	0.0021	51.2145	0.0006	38.8007	0.5181	28.7414
IRGMM-Algebraic	0.0000	88.5951	0.0005	30.7511	0.0920	27.5791
IRGMM-Analytic	0.0021	51.2195	0.0005	28.9172	0.2496	27.3564
AGMM	0.0215	54.0307	0.0005	30.776	0.0857	27.3050
LLGMM				0.4625*	24.501	27.2707

TABLE 30

Parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM and LLGMM methods for coal data

Method	a	μ	σ^2	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
GMM-Algebraic	0.0000	94.4847	0.0006	22.6866	0.2015	16.3444
GMM-Analytic	0.0000	94.4446	0.0006	21.6564	0.2121	16.3264
IRGMM-Algebraic	0.0027	34.4838	0.0013	17.6894	0.3438	13.4981
IRGMM-Analytic	0.0021	23.1151	0.0005	17.6869	0.3448	13.4989
AGMM	0.0464	27.0567	0.0014	17.7620	0.0833	13.106
LLGMM				0.4794*	9.4009	12.8370

respectively. The LLGMM estimates are derived using initial delay r=20, p=2 and ϵ=0.001. Among all methods under study, the LLGMM method generates the smallest RAMSE value. In fact, the RAMSE value is smaller than the $\frac{1}{22}$, $\frac{1}{62}$, $\frac{1}{36}$, and $\frac{1}{10}$ of any other RAMSE values regarding the natural gas, crude oil, coal and ethanol, respectively. This exhibits the superiority of the LLGMM method over all other methods. We further observe that the LLGMM approach yields the smallest AMB and highest AMAD value regarding the natural gas, crude oil, coal and ethanol. The high value of AMAD for the LLGMM method signifies that the LLGMM captures the influence of random environmental fluctuations on the dynamic of energy commodity process. From Remark 23, the smallest RAMSE, highest AMAD, and smallest AMB value under the LLGMM method exhibit the superior performance under the three goodness-of-fit measures.

Ranking of Methods Under Goodness of Fit Measure.

TABLE 32

Ranking of natural gas, crude oil, coal, and ethanol under three statistical measures
RANK OF METHODS UNDER GOODNESS OF FIT MEASURE

	Natural gas			Crude oil			Coal			Ethanol		
Method	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
GMM-Algebraic	6	2	2	6	3	6	6	5	6	6	3	6
GMM-Analytic	5	3	3	5	2	5	5	4	5	5	2	5
IRGMM-Algebraic	4	5	5	3	5	4	4	3	3	4	5	4
IRGMM-Analytic	2	6	6	2	4	3	3	2	4	3	4	3
AGMM	3	4	4	4	6	2	2	6	2	2	6	2
LLGMM	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 31

Parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM and LLGMM methods for ethanol

Method	a	μ	σ^2	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
GMM-Algebraic	0.0000	94.4847	0.0006	22.6866	0.2015	16.3444
GMM-Analytic	0.0000	94.4446	0.0006	21.6564	0.2121	16.3264
IRGMM-Algebraic	0.0014	3.4506	0.0026	0.5844	0.0322	0.4346
IRGMM-Analytic	0.0015	3.4506	0.0026	0.5813	0.0336	0.4303
AGMM	0.3167	2.166	0.0018	0.4356	0.0035	0.3579
LLGMM				0.0375*	0.3213	0.3566

Tables 28, 29, 30, and 31 show a comparison parameter estimates of model (75) and the goodness-of-fit measures RAMSE, AMAD, and AMB using GMM-Algebraic, GMM-Analytic, IRGMM-Algebraic, IRGMM-Analytic, AGMM and LLGMM methods for the daily natural gas data, daily crude oil data, daily coal data, and weekly ethanol data,

Remark 27.

The ranking of LLGMM is top one in all three goodness-of-fit statistical measures for all four energy commodity data sets. Further, one of the IRGMM-Analytic and AGMM is ranked either as top 2nd or 3rd under RAMSE measure. This exhibits the influence of the usage of stochastic calculus based orthogonality condition parameter vectors (OCPV-Analytic).

7.4 Comparison of Goodness of Fit Measures of Model (45) with (75) Using LLGMM Method

As stated in Remark 26, we compare the Goodness of fit Measures RAMSE, AMAD, and AMB using the U.S. Treasury Bill Interest Rate data and the LLGMM applied to the model validation problems of two proposed continuous-time dynamic models of U.S. Treasury Bill Interest Rate process described by (45) and (75). The LLGMM state estimates of (45) and (75) are computed under the same initial delay r=20, p=2, and ϵ=0.001. The results are recorded in the following table.

TABLE 33

Comparison of goodness of fit measure of model (45) with model (75)

LLGMM	$\widehat{RAMSE}$	$\widehat{AMAD}$	$\widehat{AMB}$
Model (45)	0.0024*	0.0145	0.0178
Model (75)	0.0027	0.0146	(10178

Table 33 shows that the goodness-of-fit measures RAMSE, AMAD, and AMB of the LLGMM method using both models (75) and (45) are very close. Model (45) appears to have the least RAMSE value. This shows that the LLGMM result performs better using model (45) than using model (75) since it has a lower root mean square error. The

AMAD value using (75) is larger than the value using (45). This suggests that the influence of the random environmental fluctuations on state dynamic model (75) is higher than using the model (45). The AMB value derived using both models appeared to be the same, indicating that both model give the same average median bias estimates. Based on this statistical analysis, we conclude that (45) is most appropriate continuous-time stochastic dynamic model for the short-term riskless rate model which includes many well-known interest rate models.

8. Comparison of LLGMM with Existing Nonparametric Statistical Methods

In this section, we compare our LLGMM method with existing nonparametric methods. We consider the following existing nonparametric methods.

8.1 Nonparametric Estimation of Nonlinear Dynamics by Metric-Based Local Linear Approximation (LLA)

The LLA method assumes no functional form of a given model but estimates from experimental data by approximating the curve implied by the function by the tangent plane around the neighborhood of a tangent point. Suppose the state of interest x_t at time t is differentiable with respect to t and satisfies $dx_t=f(x_t)dt$, where $f:\Re^k \to \Re$ is a smooth map, $x_t \in \Re^k$. . The approximation of the curve $f(x_t)$ in a neighbourhood $U_\epsilon(x_0)=\{x: d(x,x_0)<\epsilon\}$ is defined by a tangent plane at x_0

$$y_r = f(x_0) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x_0)(x_i - x_0),$$

where d is a metric on $\Re^k$. Allowing error in the equation and assigning a weight $w(x_t)$ to each error terms ϵ_t, the method reduces to estimating parameters

$$\beta_i = \frac{\partial f}{\partial x_i}(x_0),\ 1 = 1, 2, \ldots, k$$

in the equation

$$w(t)y_t = \beta_0 \cdot w(x_t) + \sum_{i=1}^{k} \beta_i \cdot w(x_t)(x_{t,i} - x_{0,i}).$$

Applying the standard linear regression approach, the least square estimate $\hat{\beta}$ is given by

$$\hat{\beta}=(\tilde{X}^T\tilde{X})_{-1}\tilde{X}^T\tilde{Y}, \tag{83}$$

where

$$\tilde{x}_i = (w(x_{t_1})(x_{t_1,1} - x_{0,i}), \ldots, w(x_{t_n})(x_{t_n,i} - x_{0,i}))^T,\ i = 1, \ldots, k.$$
$$\tilde{w} = (w(x_{t_1}), \ldots, w(x_{t_n}))^T$$
$$\tilde{Y} = (w(x_{t_1})y_{t_1}, \ldots, w(x_{t_n})y_{t_n})^T$$
$$\tilde{X} = (\tilde{w}, \tilde{x}_1, \ldots, \tilde{x}_k).$$

Particularly, the trajectory $f(x_{t_i})$ is estimated by choosing $x_0=x_{t_i}$, for each i=1, 2, . . . , n, respectively. We use $d(x, x_0)=|x-x_0|$, where |.| is the standard Euclidean metric on $\Re^k$, and $w(x)=\phi(d(x, x_0))$, where $\phi(u)=K(u/\epsilon)$ and K is the Epanechnikov Kernel $K(x)=0.75(1-x^2)_+$.

8.2 Risk Estimation and Adaptation after Coordinate Transformation (REACT) Method

Given n pairs of observations $(x_1, Y_1), \ldots, (x_n, Y_n)$, the REACT method, the response variable Y is related to the covariate x (called a feature) by the equation

$$Y_i=r(x_i)+\sigma\epsilon_i, \tag{84}$$

where $\epsilon_i \sim N(0, 1)$ are IID, and $x_i=i/n$, i=1, 2, . . . , n. The function r(x) is approximated using orthogonal cosine basis ϕ_i, i=1, 2, 3, . . . of [0,1] described by

$$\phi_1(x)\equiv 1,\ \phi_j(x)=\sqrt{2}\cos((j-1)\pi x), j\geq 2. \tag{85}$$

The function r(x), expanded as

$$r(x) = \sum_{j=1}^{\infty} \theta_j \phi_j(x) \tag{86}$$

where

$$\theta_j = \int_0^1 \phi_j(x) r(x)\, dx$$

is approximated. The function estimator

$$\hat{r}(x) = \sum_{j=1}^{\hat{J}} Z_j \phi_j(x)$$

where

$$Z_j = \frac{1}{n}\sum_{i=1}^{n} Y_i \phi_j(x_i),\ j = 1, 2, \ldots, n$$

and $\hat{J}$ is found so that the risk estimator

$$\hat{R}(J) = \frac{J\hat{\sigma}^2}{n} + \sum_{j=J+1}^{n}\left(Z_j^2 - \frac{\hat{\sigma}^2}{n}\right)$$

is minimized, $\hat{\sigma}^2$ is the estimator of variance of Z_j.

8.3 Exponential Moving Average Method (EMA)

The EMA for an observation y_t at time t may be calculated recursively as

$$S_t=\alpha y_t+(1-\alpha)S_{t-1},\ t=1,2,3, \ldots ,n, \tag{87}$$

where $0<\alpha\leq 1$ is a constant that determines the depth of memory of S_t.

8.4 Goodness-of-Fit Measures for the LLA, REACT, and EMA Methods

In this subsection, we show the goodness-of-fit measures for the LLA, REACT, and EMA methods. We use $\hat{J}=183$ for the REACT method and $\alpha=0.5$ for the EMA method.

TABLE 34

Goodness-of-fit measures for the LLA, REACT, and EMA methods.

Goodness of-fit Measure	Natural gas	Crude oil	Coal	Ethanol
		LLGMM method		
$\widehat{RAMSE}$	0.0674	0.4625	0.4794	0.0375
$\widehat{AMAD}$	1.1318	24.5010	9.4009	0.3213
$\widehat{AMB}$	1.1371	27.2707	12.8370	0.3566
		LLA Method		
$\widehat{RAMSE}$	0.3114	1.9163	2.1645	0.2082
$\widehat{AMAD}$	1.1406	24.3266	9.4511	0.3290
$\widehat{AMB}$	1.2375	27.2713	12.8388	0.3677
		REACT method		
$\widehat{RAMSE}$	0.1895	2.0377	2.0162	0.0775
$\widehat{AMAD}$	1.1779	24.6967	9.3791	0.3291
$\widehat{AMB}$	1.12352	27.2711	12.8369	0.3566
		EMA method		
$\widehat{RAMSE}$	0.1222	0.7845	0.8233	0.0682
$\widehat{AMAD}$	1.1336	24.5858	9.4183	0.3159
$\widehat{AMB}$	1.2352	27.2710	12.8370	0.3567

Comparison of the results derived using these non-parametric methods with the LLGMM method show that the results derived using the LLGMM method is far better than results of the nonparametric methods.

FIG. **8** illustrates an example schematic block diagram of the computing device **100** shown in FIG. **1** according to various embodiments described herein. The computing device **100** includes at least one processing system, for example, having a processor **802** and a memory **804**, both of which are electrically and communicatively coupled to a local interface **806**. The local interface **806** can be embodied as a data bus with an accompanying address/control bus or other addressing, control, and/or command lines.

In various embodiments, the memory **804** stores data and software or executable-code components executable by the processor **802**. For example, the memory **804** can store executable-code components associated with the visualization engine **130** for execution by the processor **802**. The memory **804** can also store data such as that stored in the device data store **120**, among other data.

It is noted that the memory **804** can store other executable-code components for execution by the processor **802**. For example, an operating system can be stored in the memory **804** for execution by the processor **802**. Where any component discussed herein is implemented in the form of software, any one of a number of programming languages can be employed such as, for example, C, C++, C #, Objective C, JAVA®, JAVASCRIPT®, Perl, PHP, VISUAL BASIC®, PYTHON®, RUBY, FLASH®, or other programming languages.

As discussed above, in various embodiments, the memory **804** stores software for execution by the processor **802**. In this respect, the terms "executable" or "for execution" refer to software forms that can ultimately be run or executed by the processor **802**, whether in source, object, machine, or other form. Examples of executable programs include, for example, a compiled program that can be translated into a machine code format and loaded into a random access portion of the memory **804** and executed by the processor **802**, source code that can be expressed in an object code format and loaded into a random access portion of the memory **804** and executed by the processor **802**, or source code that can be interpreted by another executable program to generate instructions in a random access portion of the memory **804** and executed by the processor **802**, etc.

An executable program can be stored in any portion or component of the memory **804** including, for example, a random access memory (RAM), read-only memory (ROM), magnetic or other hard disk drive, solid-state, semiconductor, or similar drive, universal serial bus (USB) flash drive, memory card, optical disc (e.g., compact disc (CD) or digital versatile disc (DVD)), floppy disk, magnetic tape, or other memory component.

In various embodiments, the memory **804** can include both volatile and nonvolatile memory and data storage components. Volatile components are those that do not retain data values upon loss of power. Nonvolatile components are those that retain data upon a loss of power. Thus, the memory **804** can include, for example, a RAM, ROM, magnetic or other hard disk drive, solid-state, semiconductor, or similar drive, USB flash drive, memory card accessed via a memory card reader, floppy disk accessed via an associated floppy disk drive, optical disc accessed via an optical disc drive, magnetic tape accessed via an appropriate tape drive, and/or other memory component, or any combination thereof. In addition, the RAM can include, for example, a static random access memory (SRAM), dynamic random access memory (DRAM), or magnetic random access memory (MRAM), and/or other similar memory device. The ROM can include, for example, a programmable read-only memory (PROM), erasable programmable read-only memory (EPROM), electrically erasable programmable read-only memory (EEPROM), or other similar memory device.

The processor **802** can be embodied as one or more processors **802** and the memory **804** can be embodied as one or more memories **804** that operate in parallel, respectively, or in combination. Thus, the local interface **806** facilitates communication between any two of the multiple processors **802**, between any processor **802** and any of the memories **804**, or between any two of the memories **804**, etc. The local interface **806** can include additional systems designed to coordinate this communication, including, for example, a load balancer that performs load balancing.

As discussed above, the LLGMM dynamic process module **130** can be embodied, at least in part, by software or executable-code components for execution by general purpose hardware. Alternatively the same can be embodied in dedicated hardware or a combination of software, general, specific, and/or dedicated purpose hardware. If embodied in such hardware, each can be implemented as a circuit or state machine, for example, that employs any one of or a combination of a number of technologies. These technologies can include, but are not limited to, discrete logic circuits having logic gates for implementing various logic functions upon an application of one or more data signals, application specific integrated circuits (ASICs) having appropriate logic gates, field-programmable gate arrays (FPGAs), or other components, etc.

The flowchart or process diagrams in FIGS. **2** and **3** are representative of certain processes, functionality, and operations of the embodiments discussed herein. Each block can represent one or a combination of steps or executions in a process. Alternatively or additionally, each block can represent a module, segment, or portion of code that includes program instructions to implement the specified logical function(s). The program instructions can be embodied in the form of source code that includes human-readable statements written in a programming language or machine code that includes numerical instructions recognizable by a suitable execution system such as the processor **802**. The machine code can be converted from the source code, etc. Further, each block can represent, or be connected with, a circuit or a number of interconnected circuits to implement a certain logical function or process step.

Although the flowchart or process diagrams in FIGS. **2** and **3** illustrate a specific order, it is understood that the order can differ from that which is depicted. For example, an order of execution of two or more blocks can be scrambled relative to the order shown. Also, two or more blocks shown in succession in FIGS. **2** and **3** can be executed concurrently or with partial concurrence. Further, in some embodiments, one or more of the blocks shown in FIGS. **2** and **3** can be skipped or omitted. In addition, any number of counters, state variables, warning semaphores, or messages might be added to the logical flow described herein, for purposes of enhanced utility, accounting, performance measurement, or providing troubleshooting aids, etc. It is understood that all such variations are within the scope of the present disclosure.

Also, any logic or application described herein, including the LLGMM dynamic process module **130** that are embodied, at least in part, by software or executable-code components, can be embodied or stored in any tangible or non-transitory computer-readable medium or device for execution by an instruction execution system such as a general purpose processor. In this sense, the logic can be embodied as, for example, software or executable-code components that can be fetched from the computer-readable medium and executed by the instruction execution system. Thus, the instruction execution system can be directed by execution of the instructions to perform certain processes such as those illustrated in FIGS. **2** and **3**. In the context of the present disclosure, a "non-transitory computer-readable medium" can be any tangible medium that can contain, store, or maintain any logic, application, software, or executable-code component described herein for use by or in connection with an instruction execution system.

The computer-readable medium can include any physical media such as, for example, magnetic, optical, or semiconductor media. More specific examples of suitable computer-readable media include, but are not limited to, magnetic tapes, magnetic floppy diskettes, magnetic hard drives, memory cards, solid-state drives, USB flash drives, or optical discs. Also, the computer-readable medium can include a RAM including, for example, an SRAM, DRAM, or MRAM. In addition, the computer-readable medium can include a ROM, a PROM, an EPROM, an EEPROM, or other similar memory device.

A phrase, such as "at least one of X, Y, or Z," unless specifically stated otherwise, is to be understood with the context as used in general to present that an item, term, etc., can be either X, Y, or Z, or any combination thereof (e.g., X, Y, and/or Z). Similarly, "at least one of X, Y, and Z," unless specifically stated otherwise, is to be understood to present that an item, term, etc., can be either X, Y, and Z, or any combination thereof (e.g., X, Y, and/or Z). Thus, as used herein, such phases are not generally intended to, and should not, imply that certain embodiments require at least one of either X, Y, or Z to be present, but not, for example, one X and one Y. Further, such phases should not imply that certain embodiments require each of at least one of X, at least one of Y, and at least one of Z to be present.

Although embodiments have been described herein in detail, the descriptions are by way of example. The features of the embodiments described herein are representative and, in alternative embodiments, certain features and elements may be added or omitted. Additionally, modifications to aspects of the embodiments described herein may be made by those skilled in the art without departing from the spirit and scope of the present invention defined in the following claims, the scope of which are to be accorded the broadest interpretation so as to encompass modifications and equivalent structures.

Therefore, at least the following is claimed:

1. A local lagged adapted generalized method of moments (LLGMM) process to simulate a forecast using measured data, the process to simulate comprising:

developing a stochastic model of a continuous time dynamic process;

obtaining a discrete time data set measured for at least one commodity as past state information of the continuous time dynamic process over a time interval;

generating a discrete time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP) based on the stochastic model of the continuous time dynamic process and the discrete time data set measured for at least one commodity;

calculating, by at least one computer, a plurality of admissible parameter estimates for the stochastic model of the continuous time dynamic process, to forecast a price of the at least one commodity, using the DTIDMLSMVSP;

for each of the plurality of admissible parameter estimates, calculating, by the at least one computer, a state value of the stochastic model of the continuous time dynamic process to gather a plurality of state values of the stochastic model of the continuous time dynamic process; and

determining an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values, wherein generating the DTIDMLSMVSP further comprises:

at each time point in a partition of the time interval, selecting, by the at least one computer, an m_k-point sub-partition of the partition, the m_k-point sub-partition having a local admissible lagged sample observation size based on an order of a model, a response delay associated with the continuous time dynamic process, and a sub-partition time observation index size; and

for each m_k-point in each sub-partition, selecting, by the at least one computer, an m_k-local moving sequence in the sub-partition to gather an m_k-class of admissible restricted finite sequences.

2. The LLGMM process according to claim **1**, wherein generating the DTIDMLSMVSP further comprises:

for each m_k-local moving sequence, calculating, by the at least one computer, an m_k-local average to generate an m_k-moving average process; and

for each m_k-local moving sequence, calculating, by the at least one computer, an m_k-local variance to generate an m_k-local moving variance process.

3. The LLGMM process according to claim **2**, wherein generating the DTIDMLSMVSP further comprises:

transforming the stochastic model of the continuous time dynamic process into a stochastic model of a discrete time dynamic process utilizing a discretization scheme; and developing a system of generalized method of moments equations from the stochastic model of the discrete time dynamic process.

4. The LLGMM process according to claim 2, further comprising identifying an optimal m_k-local moving sequence among the m_k-class of admissible restricted finite sequences based on the minimum error.

5. The LLGMM process according to claim 4, wherein determining the optimal admissible parameter estimate comprises:

identifying one m_k-local moving sequence among the m-class of admissible restricted finite sequences as the optimal m_k-local moving sequence when the one m_k-local moving sequence is associated with the minimum error; and

selecting a largest m_k-local moving sequence among the m_k-class of admissible restricted finite sequences as the optimal m_k-local moving sequence when more than one m_k-local moving sequence in the m_k-class of admissible restricted finite sequences is associated with the minimum error.

6. The LLGMM process according to claim 4, further comprising forecasting at least one future state value of the stochastic model of the continuous-time dynamic process using the optimal m_k-local moving sequence.

7. The LLGMM process according to claim 6, further comprising determining an interval of confidence associated with the at least one future state value.

8. A local lagged adapted generalized method of moments (LLGMM) system to simulate a forecast using measured data, comprising:

a memory that stores a discrete time data set measured for at least one commodity as past state information of a continuous time dynamic process over a time interval and computer readable instructions for an LLGMM process; and

at least one computing device coupled to the memory and configured, through the execution of the computer readable instructions for the LLGMM process, to:

generate a discrete time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP) based on a stochastic model of a continuous time dynamic process and the discrete time data set measured for at least one commodity;

calculate a plurality of admissible parameter estimates for the stochastic model of the continuous time dynamic process, to forecast a price of the at least one commodity, using the DTIDMLSMVSP;

for each of the plurality of admissible parameter estimates, calculate a state value of the stochastic model of the continuous time dynamic process to gather a plurality of state values of the stochastic model of the continuous time dynamic process;

determine an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values;

at each time point in a partition of the time interval, select an m_k-point sub-partition of the partition, the m_k-point sub-partition having a local admissible lagged sample observation size based on an order of a model, a response delay associated with the continuous time dynamic process, and a sub-partition time observation index size; and

for each m_k-point in each sub-partition, select an m_k-local moving sequence in the sub-partition to gather an m_k-class of admissible restricted finite sequences.

9. The LLGMM system according to claim 8, wherein the at least one computing device is further configured to:

for each m_k-local moving sequence, calculate an mA-local average to generate an m_k-moving average process; and

for each m_k-local moving sequence, calculate an mA-local variance to generate an m_k-local moving variance process.

10. The LLGMM system according to claim 9, wherein the at least one computing device is further configured to:

transform the stochastic model of the continuous time dynamic process into a stochastic model of a discrete time dynamic process utilizing a discretization scheme; and

develop a system of generalized method of moments equations from the stochastic model of a discrete time dynamic process.

11. The LLGMM system according to claim 9, wherein the at least one computing device is further configured to identify an optimal m_k-local moving sequence among the m_k-class of admissible restricted finite sequences based on the minimum error.

12. The LLGMM system according to claim 11, wherein the at least one computing device is further configured to:

identify one m_k-local moving sequence among the m_k-class of admissible restricted finite sequences as the optimal m_k-local moving sequence when the one m_k-local moving sequence is associated with the minimum error; and

select a largest m_k-local moving sequence among the m-class of admissible restricted finite sequences as the optimal m_k-local moving sequence when more than one m_k-local moving sequence in the m_k-class of admissible restricted finite sequences is associated with the minimum error.

13. The LLGMM process according to claim 11, wherein the at least one computing device is further configured to forecast at least one future state value of the stochastic model of the continuous-time dynamic process using the optimal m_k-local moving sequence.

14. A non-transitory computer readable medium including computer readable instructions stored thereon that, when executed by at least one computing device, direct the at least one computing device to perform a local lagged adapted generalized method of moments (LLGMM) process to simulate a forecast using measured data, the process to simulate comprising:

obtaining a discrete time data set measured for at least one commodity as past state information of a continuous time dynamic process over a time interval;

generating a discrete time interconnected dynamic model of local sample mean and variance statistic processes (DTIDMLSMVSP) based on a stochastic model of a continuous time dynamic process and the discrete time data set measured for at least one commodity;

calculating, by the at least one computing device, a plurality of admissible parameter estimates for the stochastic model of the continuous time dynamic process, to forecast a price of the at least one commodity, using the DTIDMLSMVSP;

for each of the plurality of admissible parameter estimates, calculating, by the at least one computer, a state value of the stochastic model of the continuous time dynamic process to gather a plurality of state values of the stochastic model of the continuous time dynamic process; and determining an optimal admissible parameter estimate among the plurality of admissible parameter estimates that results in a minimum error among the plurality of state values, wherein generating the DTIDMLSMVSP further comprises:

at each time point in a partition of the time interval, selecting, by at least one computer, an m_k-point sub-partition of the partition, the m_k-point sub-partition having a local admissible lagged sample observation size based on an order of a model, a response delay associated with the continuous time dynamic process, and a sub-partition time observation index size;

for each m_k-point in each sub-partition, selecting, by the at least one computer, an m_k-local moving sequence in the sub-partition to gather an m_k-class of admissible restricted finite sequences;

for each m_k-local moving sequence, calculating, by the at least one computer, an m_k-local average to generate an m_k-moving average process; and

for each m_k-local moving sequence, calculating, by the at least one computer, an m_k-local variance to generate an m_k-local moving variance process.

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