2020

MATH ACROSS THE CURRICULUM

Kelsey Mason

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By
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Introduction

Some students go through the motions of learning, and may not see the things they are taught in school as being relevant to their lives. Other students experience a general lack of interest and motivation when it comes to furthering their education. This is especially prevalent in Mathematics and is sometimes due to past negative experiences and lack of confidence. The following essay contains a series of lessons intended to form a bridge between Mathematics and other disciplinary subjects offered in secondary schools. The goal of this essay is to excite students through interactions with interdisciplinary content while also touching on a wide range of student interests. The lessons in this essay are in the areas of Art, Computer Science, History, Language Arts, and Science.

Knowledge is best developed through interaction. Students will work through the following lessons independently, with partners or groups, or by teacher-lead discussion. The varied instruction is meant to provide the students with support along with the freedom to create and discover. By exploring connections to other disciplinary subjects, we hope that students who have not previously had an interest in Mathematics will discover a new appreciation for Mathematics and its applications.

Some lessons are intended to be building blocks for future learning. Both of the Science lessons are a great introduction to linear and nonlinear data analysis. Gathering data in a hands-on fashion gets students out of their desk seats and puts them in the driver seats of their learning. The Snap! lesson opens the door for students to design computer programs based on their own goals. One of the History lessons provides an introduction to Graph Theory and presents an opportunity to extend the lesson by exploring additional topics in Graph Theory.

Other lessons are intended to be self-contained. Students get to explore their creative sides by designing colorful tessellations of the plane through transformations in a lesson related to Art. In one of the Language Arts lessons, students identify patterns both in words and numbers. This process leads to their writing original song lyrics, as well as to the development of two major formulas in mathematics. In the other Language Arts lesson, students explore spatial understanding with connections to the popular book *Holes*, by Louis Sachar. The second History lesson explores the vast amount of data reported in the U.S. Census while stressing why it is so important for families to participate in the Census. In this lesson, students get to decide what state they believe will be the best place for them to live based on the data the Census provides.

We have designed each lesson to be appropriately challenging for students ranging from grades 7-12. The targeted grade levels are listed in the introduction to each lesson. We have included both student and teacher versions of each worksheet.

Tessellations

Lesson Overview

Brief Description: Students will use GeoGebra and their knowledge of transformations to tessellate the plane using both regular and irregular polygons.

Time: 150 Minutes

Grades: 7-9

Topic: Transformations

Paired with: Art

Materials:

- Computers with internet access for GeoGebra (GeoGebra can also be downloaded to the desktop)
- Graph paper
- Student handouts (guided notes, worksheet)

In this lesson, students will apply rigid transformations of the plane to form tessellations. When a figure undergoes a rigid transformation, the lengths and angles of the original figure remain unchanged. Therefore, the resulting image is congruent to the original figure. The lesson begins with a review of the four types of rigid transformations of the plane: translations, rotations, reflections, and glide reflections.

Students will begin the lesson by practicing common transformations on graph paper. Once they have completed the introductory activity, students will work in pairs to do transformations and other constructions using GeoGebra. The ultimate goal of the lesson is for students to create tessellations of the plane using an original figure and a specific type of transformation.

Early in the lesson, students are asked to create both an equilateral triangle and a regular hexagon by straightedge and compass construction. This reinforces skills that they have previously learned in a Geometry class. Teachers could omit these constructions from the lesson and instead introduce GeoGebra's Regular Polygon tool.

This lesson is designed for students who have not previously used GeoGebra. Students will begin by becoming familiar with the necessary tools in GeoGebra. The instructions in the lesson are based on the online version of GeoGebra at https://www.geogebra.org/geometry, which is freely available. Teachers may also choose to have students download GeoGebra to their devices, and then do the activity offline.

By the end of the lesson, students will create artistic tessellations that are appealing to the eye. A ***tessellation*** is a tiling of a two-dimensional surface that uses repeated copies of a finite number of geometric shapes with no overlaps or gaps. The only tessellations of the plane that consist of copies of a single regular polygon are created with equilateral triangles, squares, or regular hexagons. These tessellations are referred to as ***regular tessellations***. The reason that these three figures can tessellate a plane is that the degree measure of each interior angle is a factor of 360, so congruent copies of the shape fit together at a single vertex with no gaps or overlaps. See Figures 1 and 2.

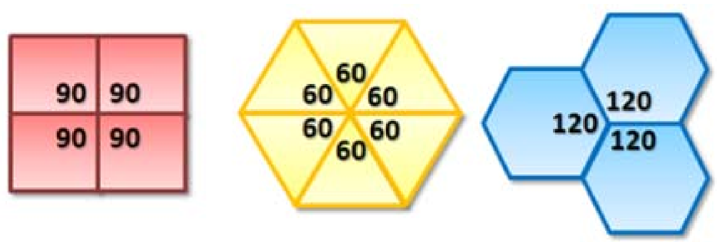


Figure 1: Squares, equilateral triangles, and regular hexagons fit together at a common vertex. [1]

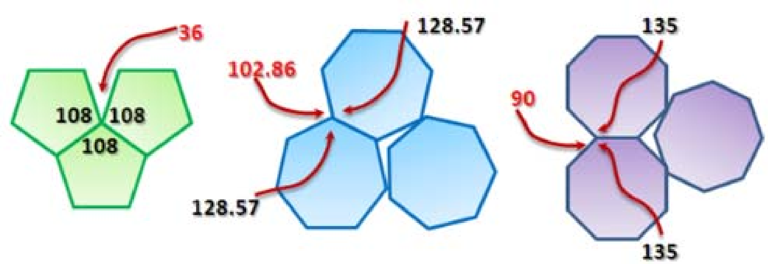


Figure 2: Regular pentagons, heptagons, and octagons do not fit together at a common vertex.[2]

[1] Regular Tessellations: Why only three of them? http://mathandmultimedia.com/2011/06/04/regular-tessellations/
[2] Ibid.

The lesson is designed to advance students through different challenges that incorporate transformations in GeoGebra while developing students' familiarity with the tessellating polygons. The difficulty and complexity increases with each challenge. By the end of the activity, students will have used translations, rotations, and reflections to successfully produce tessellations.

Resources

Bautista, Guillermo. "Regular Tessellations: Why Only Three of Them?" *Math and Multimedia*, 26 May 2016, http://mathandmultimedia.com/2011/06/04/regular-tessellations/.

"Geometry." *GeoGebra*, https://geogebra.org/geometry.

Tessellations: Guided Notes (Teacher Version)

We begin by reviewing the following transformations of shapes in the plane: translations, reflections, rotations, and glide reflections. We will then use these rigid transformations to create tessellations of the plane that are appealing to the eye. Recall that rigid transformations produce images that are congruent to the original figure.

- A translation of a figure in Euclidean geometry moves every point of the figure the same distance in the same direction.
  - This is sometimes called a slide.
  - The distance and direction are given by a translation vector.
- A rotation of a figure occurs around a fixed point referred to as the center of rotation.
  - A rotation can be in the clockwise or counterclockwise direction.
  - A rotation of the figure will alter the orientation of the figure unless the rotation is a multiple of $360°$.
- A reflection occurs when the original image is "flipped" across a line, called the line of reflection.
  - The result of a reflection is a mirror image of the original; if the plane were folded on the line of reflection, the figure and its image would match up.
  - The reflected image is the same distance from the line of reflection as the original image.
- A glide reflection is the result of a reflection followed by a translation, where the translation vector is parallel to the line of reflection. We will not use glide reflections in this lesson.

Why study transformations?

One use of transformations results in tessellations. A tessellation is a covering of a surface with a pattern of flat shapes so that there are no overlaps or gaps. A regular tessellation uses only repeated copies of a single regular polygon. The only shapes that form regular tessellations are equilateral triangles, squares, and regular hexagons. A semi-regular tessellation is composed from two or more regular polygons and the pattern at each vertex must be the same. A vertex in a tessellation is a point where vertices of the regular polygons meet. Other types of tessellations are created using irregular polygons. You will create some of these later in the lesson.

Tessellations: Worksheet (Teacher Version)

By the time you finish this worksheet, you will have created your own tessellations of the plane.

Setting the Stage

1. M.C. Escher (1898-1972) was an artist who is famous for his work with tessellations. He created amazing optical illusions in his pieces. Do a web image search for "M.C. Escher Tessellations" to look at some of Escher's pieces. See if you can recognize the use of tessellations in this artwork. With your group, spend a few minutes discussing the characteristics you see in Escher's designs. Write some characteristics below.

   Answers will vary, but students should discuss that tessellations are designs that cover the plane without any overlap or gaps.

Before you create your own tessellations electronically, do Problems 2 and 3 below to review how to perform transformations manually.

2. Plot and label the point $A=(4,6)$ on graph paper.

   a. Translate point A three units to the left and one unit down. Label the new point A'. Record the coordinates of A' here.

      $A'=(1,5)$

   b. Reflect point A through the line $y=4$. Label the new point A''. Record the coordinates of A'' here.

      $A''=(4,2)$

   c. Rotate point A $90°$ clockwise about the point $(0,0)$. Label the new point A'''. Record the coordinates of A''' here.

      $A'''=(6,-4)$

3. Plot the points $A=(2,2)$, $B=(6,2)$, and $C=(2,4)$ on a different graph than Question 1. Connect the points with straight line segments to form a triangle.
   a. Translate $\triangle ABC$ six units to the right and two units down. Label the vertices as A', B', and C' and record the coordinates here.

      $A'=(8,0)$, $B'=(12,0)$, and $C'=(8,2)$

   b. Reflect $\triangle ABC$ through the line $y=-1$. Label the vertices as A'', B'', and C'' and record the coordinates here.

      $A''=(2,-4)$, $B''=(6,-4)$, and $C''=(2,-6)$

   c. Rotate $\triangle ABC$ $90°$ clockwise about the point $(2,2)$. Label the vertices as A''', B''', and C''' and record the coordinates here.

      $A'''=(2,2)$, $B'''=(2,-2)$, and $C'''=(4,2)$

Becoming Familiar with GeoGebra

Open GeoGebra at http://www.geogebra.org/geometry. Note: When you use a tool in GeoGebra, the tool is active even after you have created what you want. If you do not want to continue using that tool, you need to switch to another tool. The Move tool is harmless, because it will allow you to click on the graph without creating anything new. If you do something you do not want in GeoGebra, you can use the undo tool in the upper left corner of the work screen. If you want to clear the workspace and start over, just reload the GeoGebra webpage.

Check off the boxes as you work through the list below.

- ☐ If you would like to see a grid, you can use the Settings button in the top right of the GeoGebra window. However, you do not need a grid for this lesson.
- ☐ Click the icons on the image to view the different work screens. The mode will provide the names of everything you have created in the workspace and the mode will

show all of the available tools. Click “More” to see the rest of the tools. If you are unsure how a tool works, you can click the tool and a description will appear at the bottom of the toolbar.

□ Create a five-sided polygon using the Polygon tool. You must start and stop at the same point to finish the polygon. When you right-click on the polygon you should see the options below. Click on the vertical dots at the far right and choose “Settings.”

Then use the “Color” tab to change the color of the polygon to red.

□ Create two line segments that intersect and put a point on that intersection by using either the Point tool or the Intersection tool. You can confirm that the point is at the intersection by adjusting the lengths of the line segments and seeing if the point moves as well.

□ Find the midpoint of a line segment using the Midpoint tool. Use the settings to change the label of this point to *M*.

□ Create a triangle with vertices *A*, *B*, and *C*. Label the line segments opposite each vertex as *a*, *b*, and *c*, respectively.

□ Create a line segment and then use the settings for the endpoints and the segment to hide all of the labels by unchecking “Show Label.” Notice that once you have the Settings box open, you can click different objects in order to change their settings.

□ Create a line segment $\overline{AB}$ and a vector $\overrightarrow{AB}$ from *A* to *B*. Translate point *B* along this vector by selecting “Translate by Vector,” clicking on point *B*, and then clicking on the vector $\overrightarrow{AB}$. The result should be a point B' a distance *AB* away from *B*, on the same line as $\overline{AB}$.

□ Create a polygon and a line segment that does not intersect the polygon. Reflect the polygon through the line by selecting “Reflect about Line,” clicking on the polygon, and then clicking on the line segment.

□ Play around with other tools in GeoGebra to familiarize yourself with the platform.

Creating Tessellations with GeoGebra

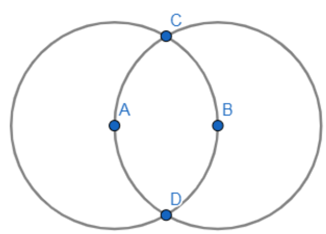
Complete the following Challenges with your partner. Your teacher must check off each Challenge before you begin the next Challenge. Find the checklist for the Challenges at the end of this worksheet. You can create a free GeoGebra account and then save your work through GeoGebra, so that if you begin a Challenge and do not finish the task you can pick up where you left off later.

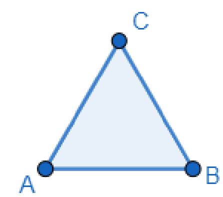
The teacher should decide whether students' finished products should have labels and/or points at the vertices. For designs without either of these, stress to the students how to hide objects and labels. A risk in hiding objects or labels early on is that students might hide an object that they need later. Objects and labels can be hidden once the design is complete, but it is moderately time consuming. Students can select multiple objects at once by holding the Ctrl button on the keyboard and then selecting the objects.

Challenge 1. Create a tessellation using equilateral triangles.

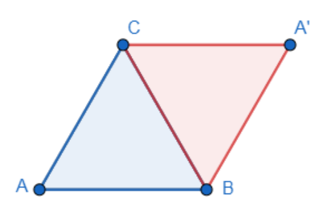
Depending on the time available, teachers may tell students to skip Step 1 and instead use the Regular Polygon tool to create the triangle.

1. Create an equilateral triangle.
   a. Construct two points A and B.
   b. Draw a circle with center A, through B, and another circle with center B, through A.
   c. Construct the intersection points C and D of these two circles.
   d. Use the Polygon tool to construct triangle ABC.
   e. Hide the circles and point D.
   f. Make all vertices the same color and size.

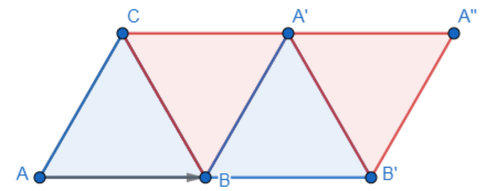




2. Reflect $\triangle ABC$ through line segment BC, and use the settings to change the color of the reflected triangle.



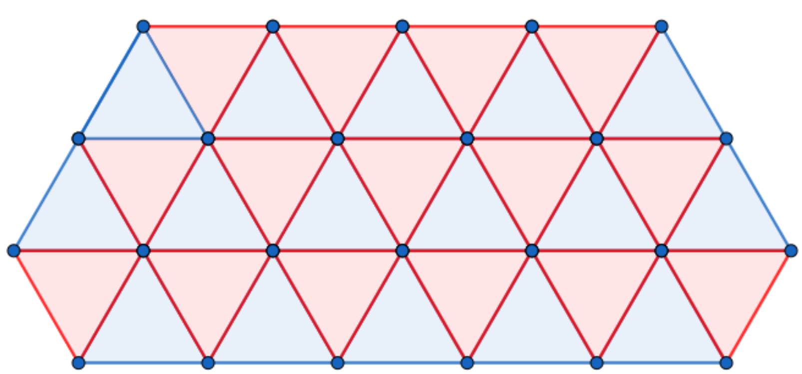
3. Translate the equilateral triangles.
   a. Create vector $\overrightarrow{AB}$.
   b. Select the entire parallelogram from Step 2 by holding the Ctrl button on the keyboard and selecting each polygon.
   c. Use the "Translate by Vector" tool to move the parallelogram a length of AB along the vector $\overrightarrow{AB}$.
   d. You should now have four equilateral triangles as shown here.



Note: The workspace gets crowded with labels pretty quickly. If you would like, you can go to the global settings and adjust the labeling setting to "No New Objects." Do this by clicking the button, then selecting "Settings," and then selecting again so you see the "Labeling" menu. If you close the web browser and return to GeoGebra later, you will have to reset these options.

4. Translate the parallelogram to create at least 8 equilateral triangles in a row. What transformation might be useful to get another row of triangles? See the example below. Create your own tessellation. Make sure you color the tessellation in a pattern, and get as creative as you would like!

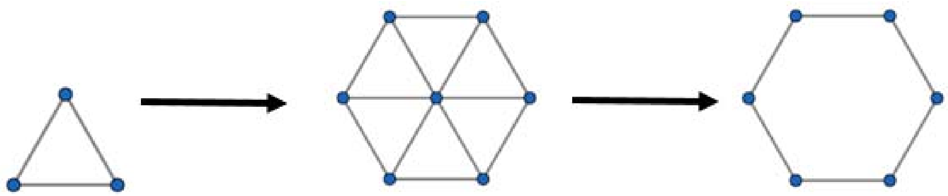
Example:



Challenge 2. Create a tessellation using a hexagon.

As in Challenge 1, students can use the Regular Polygon tool create the regular hexagon. This will cut down on time, but will not prompt students to think about the relationship between an equilateral triangle and a regular hexagon.

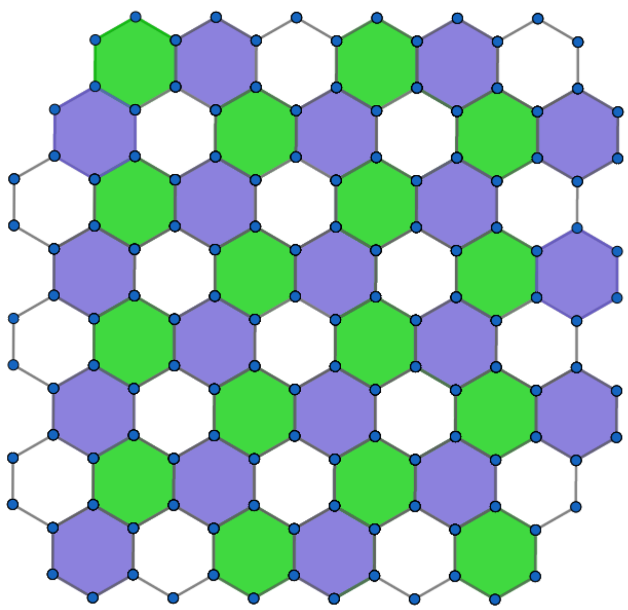
1. Start with an equilateral triangle. Do a series of reflections and/or rotations to create a regular hexagon – a six sided figure with equal side lengths. You will have to hide the interior lines of the hexagon once you have created the hexagon.



The hexagon above was created by rotating the triangle in increments of $60°$.

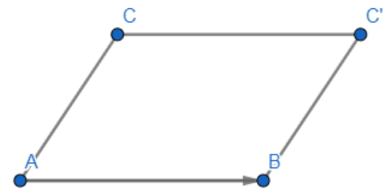
2. Hide everything but the vertices of the hexagon. Use the Polygon tool to create the interior of the regular hexagon, so you can color it. Use translations to create a tessellation with this hexagon. Hint: The translation vectors are not edges of the hexagon.

Example:

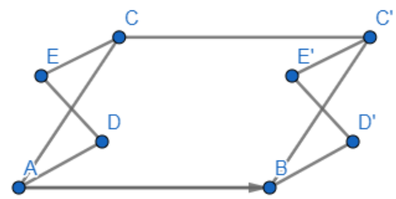


Challenge 3. Create a tessellation using only translations of an irregular polygon.

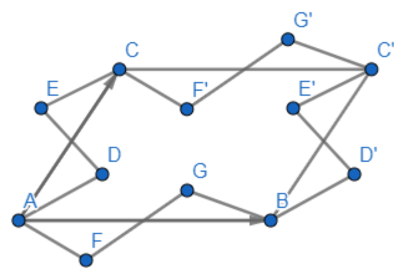
1. Create a parallelogram.
   a. Draw line segment $\overline{AB}$.
   b. Create a point C not on $\overline{AB}$.
   c. Create vector $\overrightarrow{AB}$.
   d. Use the "Translate by Vector" tool to move C a length of AB along $\overline{AB}$.
   e. Connect the vertices with line segments. (Do not use the Polygon tool.)



2. Create an irregular side.
   a. Add points E and D to opposite sides of $\overline{AC}$.
   b. Connect A, D, E, and C with line segments. Drag D and E to new locations if you want.
   c. Use the "Translate by Vector" tool to translate D and E by vector $\overrightarrow{AB}$.
   d. Connect B, D', E', and C' with line segments.

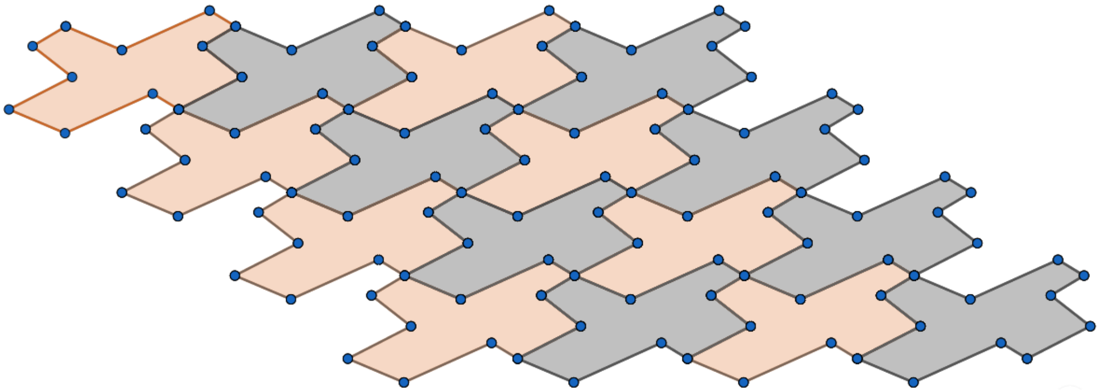


3. Create another irregular side.
   a. Create vector $\overrightarrow{AC}$.
   b. Repeat the steps in Question 2 on side $\overline{AB}$, translating by vector $\overrightarrow{AC}$.



4. Create and translate the irregular polygon.
   a. Use the Polygon tool to create the interior of the irregular polygon and hide all of the sides of the original parallelogram. This will allow you to color the polygon in the finished design. Do not hide vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   b. Translate the polygon by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ repeatedly to tessellate the plane. Color the polygons to create a pattern.

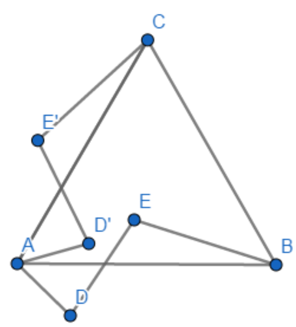
Example:

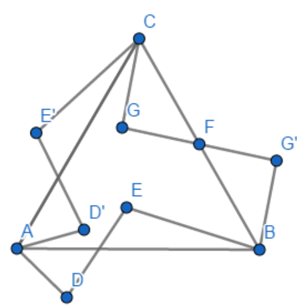


Students can move the vertices of their original polygon to alter the finished design.

Challenge 4. Create a tessellation using only rotations.

1. Build an irregular polygon based on an equilateral triangle.
   a. Construct an equilateral triangle. Refer to Challenge 1 if you need a reminder on how to do this.
   b. Create an irregular side with points *D* and *E* by putting points on opposite sides of segment $\overline{AB}$. Use line segments to connect *A*, *D*, *E*, and *B*.
   c. Select points *D* and *E* and use the "Rotate around Point" tool to rotate them $60°$ around point *A* in the direction that would send point *B* to point *C*.
   d. Connect points *A*, D', E', and *C* to create a second irregular side.
   e. Use the Midpoint tool to construct the midpoint *F* of $\overline{BC}$.
   f. Create a point *G* on one side of $\overline{CB}$.
   g. Rotate *G* 180° about *F*.
   h. Connect the points *C*, *G*, G', and *B* to create the third irregular side.





i. Hide the sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ of the equilateral triangle. If you used the polygon tool to create the triangle, hide the interior of the triangle as well.

j. Move points *D*, *E*, and *G* as needed so that no line segments are overlapping and so that the shape looks appealing to you.

k. Make *A* and *F* a different color from the other points and do not hide them, as you will need to be able to distinguish between them in the later steps.

l. Use the Polygon tool to create the interior of the polygon.

Questions:

- Why do we rotate $60°$ in part (c)?

  We rotate $60°$ because the interior angles of an equilateral triangle are each $60°$.

- When we rotate $180°$ in part (g), does it matter whether we rotate clockwise or counterclockwise?

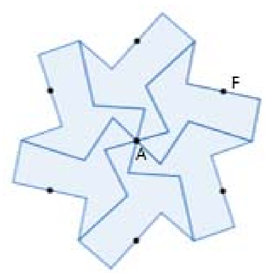
  No. Rotating by $180°$ will result in the same image whether we rotate clockwise or counterclockwise, because $180°$ is exactly half of a full rotation.

2. Rotate the polygon.

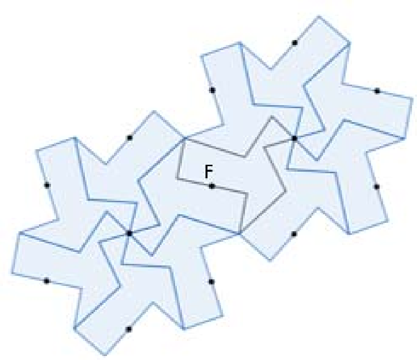
   a. Use the "Rotate around Point" tool to rotate the polygon $60°$ counterclockwise about point *A*.

   b. Continue this process, each time rotating the new image $60°$ counterclockwise about point *A*.

   c. Do this 6 times, until there is no more space around *A*.



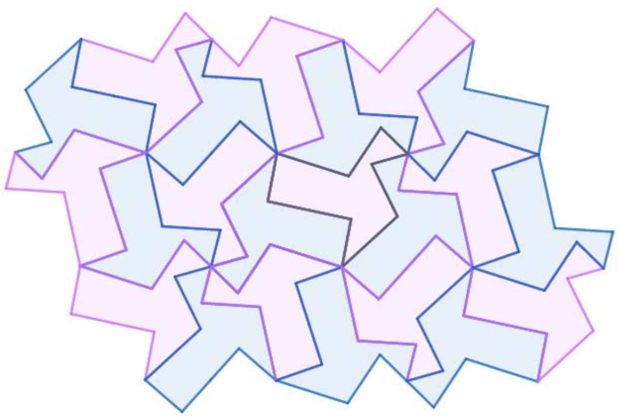
3. Rotate each of the six polygons surrounding point *A* $180°$ around point F.



4. Continue to extend the tessellation by rotating existing copies of the irregular polygon about appropriate points. When you are done, you can hide points and change the colors of the polygons to get a more appealing design.

   The centers of rotation for the additional polygons will be images of points *A* and *F* under earlier rotations.

   Example:



Challenge Checklist:

	Step 1	Step 2	Step 3	Step 4
Challenge 1				
Challenge 2			X	X
Challenge 3				
Challenge 4				

Tessellations: Guided Notes (Student Version)

We begin by reviewing the following transformations of shapes in the plane: translations, reflections, rotations, and glide reflections. We will then use these rigid transformations to create tessellations of the plane that are appealing to the eye. Recall that rigid transformations produce images that are congruent to the original figure.

- A __________________ of a figure in Euclidean geometry moves every point of the figure the same distance in the same direction.
    - This is sometimes called a __________ .
    - The distance and direction are given by a _____________ .
- A _____________ of a figure occurs around a fixed point referred to as the ____________ _______________ .
    - A rotation can be in the clockwise or counterclockwise direction.
    - A rotation of the figure will alter the orientation of the figure unless the rotation is a multiple of $360°$.
- A _______________ occurs when the original image is “flipped” across a line, called the _________________________.
    - The result of a reflection is a mirror image of the original; if the plane were folded on the line of reflection, the figure and its image would match up.
    - The reflected image is the same distance from the line of reflection as the original image.
- A ______________________ is the result of a reflection followed by a translation, where the translation vector is parallel to the line of reflection. We will not use glide reflections in this lesson.

Why study transformations?

One use of transformations results in tessellations. A __________________ is a covering of a surface with a pattern of flat shapes so that there are no overlaps or gaps. A ___________________

________________ uses only repeated copies of a single regular polygon. The only shapes that form regular tessellations are equilateral triangles, squares, and regular hexagons. A ____________________________ is composed from two or more regular polygons and the pattern at each vertex must be the same. A ______________ in a tessellation is a point where vertices of the regular polygons meet. Other types of tessellations are created using irregular polygons. You will create some of these later in the lesson.

Tessellations: Worksheet (Student Version)

By the time you finish this worksheet, you will have created your own tessellations of the plane.

Setting the Stage

1. M.C. Escher (1898-1972) was an artist who is famous for his work with tessellations. He created amazing optical illusions in his pieces. Do a web image search for "M.C. Escher Tessellations" to look at some of Escher's pieces. See if you can recognize the use of tessellations in this artwork. With your group, spend a few minutes discussing the characteristics you see in Escher's designs. Write some characteristics below.

Before you create your own tessellations electronically, do Problems 2 and 3 below to review how to perform transformations manually.

2. Plot and label the point $A = (4, 6)$ on graph paper.

   a. Translate point A three units to the left and one unit down. Label the new point A'. Record the coordinates of A' here.

   b. Reflect point A through the line $y = 4$. Label the new point A''. Record the coordinates of A'' here.

   c. Rotate point A $90°$ clockwise about the point $(0, 0)$. Label the new point A'''. Record the coordinates of A''' here.

3. Plot the points $A=(2,2)$, $B=(6,2)$, and $C=(2,4)$ on a different graph than Question 1. Connect the points with straight line segments to form a triangle.
   a. Translate $\triangle ABC$ six units to the right and two units down. Label the vertices as A', B', and C' and record the coordinates here.

   b. Reflect $\triangle ABC$ through the line $y=-1$. Label the vertices as A'', B'', and C'' and record the coordinates here.

   c. Rotate $\triangle ABC$ $90°$ clockwise about the point $(2,2)$. Label the vertices as A''', B''', and C''' and record the coordinates here.

Becoming Familiar with GeoGebra

Open GeoGebra at http://www.geogebra.org/geometry. Note: When you use a tool in GeoGebra, the tool is active even after you have created what you want. If you do not want to continue using that tool, you need to switch to another tool. The Move tool is harmless, because it will allow you to click on the graph without creating anything new. If you do something you do not want in GeoGebra, you can use the undo tool in the upper left corner of the work screen. If you want to clear the workspace and start over, just reload the GeoGebra webpage.

Check off the boxes as you work through the list below.

□ If you would like to see a grid, you can use the Settings button in the top right of the GeoGebra window. However, you do not need a grid for this lesson.

□ Click the icons on the image to view the different work screens. The mode will provide the names of everything you have created in the workspace and the mode will show all of the available tools. Click “More” to see the rest of the tools. If you are unsure how a tool works, you can click the tool and a description will appear at the bottom of the toolbar.

□ Create a five-sided polygon using the Polygon tool. You must start and stop at the same point to finish the polygon. When you right-click on the polygon you should see the options below. Click on the vertical dots at the far right and choose “Settings.”

Then use the “Color” tab to change the color of the polygon to red.

□ Create two line segments that intersect and put a point on that intersection by using either the Point tool or the Intersection tool. You can confirm that the point is at the intersection by adjusting the lengths of the line segments and seeing if the point moves as well.

□ Find the midpoint of a line segment using the Midpoint tool. Use the settings to change the label of this point to M.

□ Create a triangle with vertices A, B, and C. Label the line segments opposite each vertex as a, b, and c, respectively.

□ Create a line segment and then use the settings for the endpoints and the segment to hide all of the labels by unchecking “Show Label.” Notice that once you have the Settings box open, you can click different objects in order to change their settings.

□ Create a line segment $\overline{AB}$ and a vector $\overrightarrow{AB}$ from A to B. Translate point B along this vector by selecting “Translate by Vector,” clicking on point B, and then clicking on the vector $\overrightarrow{AB}$. The result should be a point B' a distance AB away from B, on the same line as $\overline{AB}$.

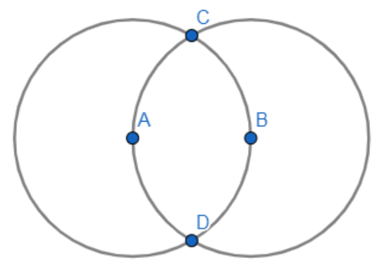
□ Create a polygon and a line segment that does not intersect the polygon. Reflect the polygon through the line by selecting “Reflect about Line,” clicking on the polygon, and then clicking on the line segment.

□ Play around with other tools in GeoGebra to familiarize yourself with the platform.

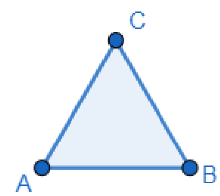
Creating Tessellations with GeoGebra

Complete the following Challenges with your partner. Your teacher must check off each Challenge before you begin the next Challenge. Find the checklist for the Challenges at the end of this worksheet. You can create a free GeoGebra account and then save your work through GeoGebra, so that if you begin a Challenge and do not finish the task you can pick up where you left off later.

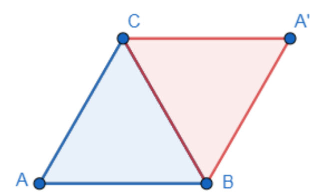
Challenge 1. Create a tessellation using equilateral triangles.



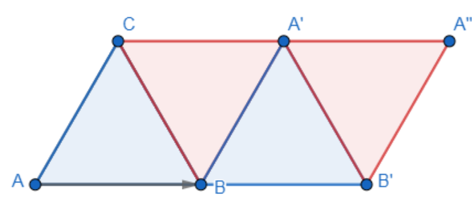
1. Create an equilateral triangle.
    a. Construct two points *A* and *B*.
    b. Draw a circle with center *A*, through *B*, and another circle with center *B*, through *A*.
    c. Construct the intersection points *C* and *D* of these two circles.
    d. Use the Polygon tool to construct triangle *ABC*.
    e. Hide the circles and point *D*.
    f. Make all vertices the same color and size.



2. Reflect $\triangle ABC$ through line segment *BC*, and use the settings to change the color of the reflected triangle.



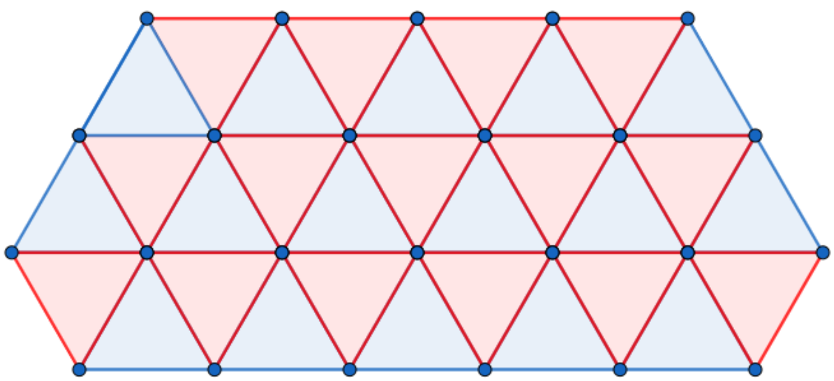
3. Translate the equilateral triangles.
    a. Create vector $\overrightarrow{AB}$.
    b. Select the entire parallelogram from Step 2 by holding the Ctrl button on the keyboard and selecting each polygon.
    c. Use the "Translate by Vector" tool to move the parallelogram a length of *AB* along the vector $\overrightarrow{AB}$.
    d. You should now have four equilateral triangles as shown here.



Note: The workspace gets crowded with labels pretty quickly. If you would like, you can go to the global settings and adjust the labeling setting to "No New Objects." Do this by clicking the button, then selecting "Settings," and then selecting again so you see the "Labeling" menu. If you close the web browser and return to GeoGebra later, you will have to reset these options.

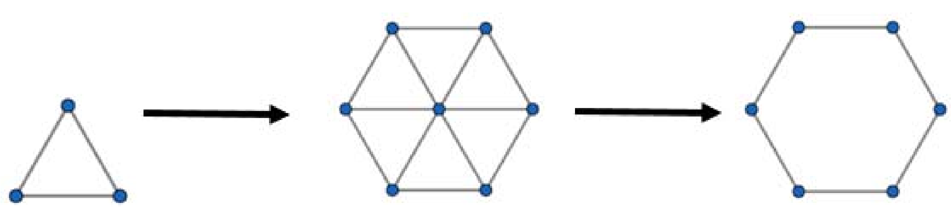
4. Translate the parallelogram to create at least 8 equilateral triangles in a row. What transformation might be useful to get another row of triangles? See the example below. Create your own tessellation. Make sure you color the tessellation in a pattern, and get as creative as you would like!

   Example:



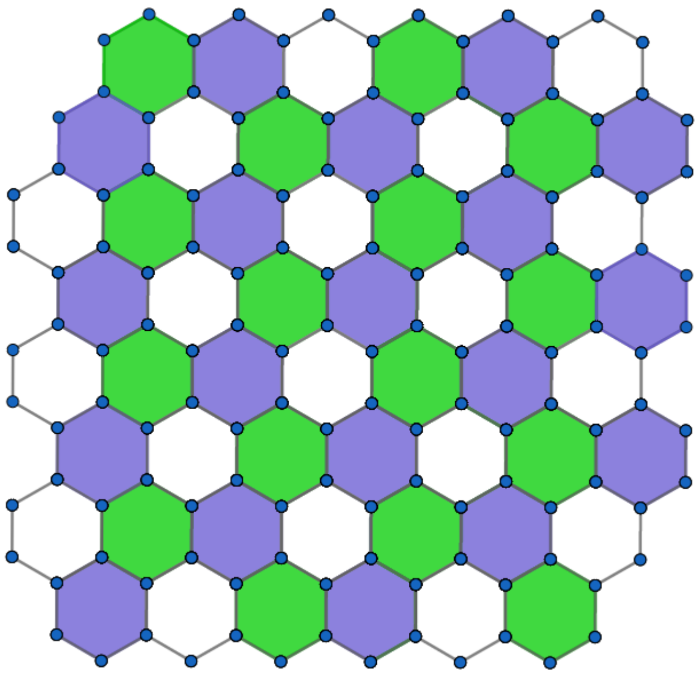
Challenge 2. Create a tessellation using a hexagon.

1. Start with an equilateral triangle. Do a series of reflections and/or rotations to create a regular hexagon – a six sided figure with equal side lengths. You will have to hide the interior lines of the hexagon once you have created the hexagon.



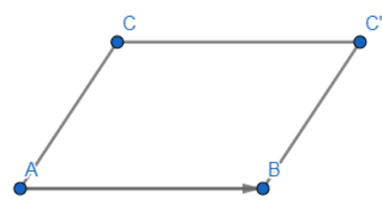
2. Hide everything but the vertices of the hexagon. Use the Polygon tool to create the interior of the regular hexagon, so you can color it. Use translations to create a tessellation with this hexagon. Hint: The translation vectors are not edges of the hexagon.

Example:



Challenge 3. Create a tessellation using only translations of an irregular polygon.

1. Create a parallelogram.



a. Draw line segment $\overline{AB}$.

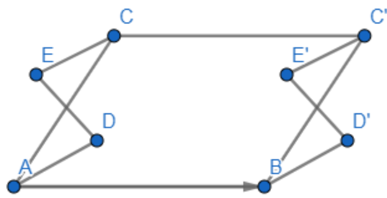
b. Create a point C not on $\overline{AB}$.

c. Create vector $\overrightarrow{AB}$.

d. Use the "Translate by Vector" tool to move C a length of AB along $\overline{AB}$.

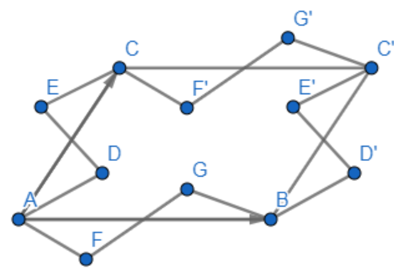
e. Connect the vertices with line segments. (Do not use the Polygon tool.)

2. Create an irregular side.



   a. Add points E and D to opposite sides of $\overline{AC}$.
   b. Connect A, D, E, and C with line segments. Drag D and E to new locations if you want.
   c. Use the "Translate by Vector" tool to translate D and E by vector $\overrightarrow{AB}$.
   d. Connect B, D', E', and C' with line segments.

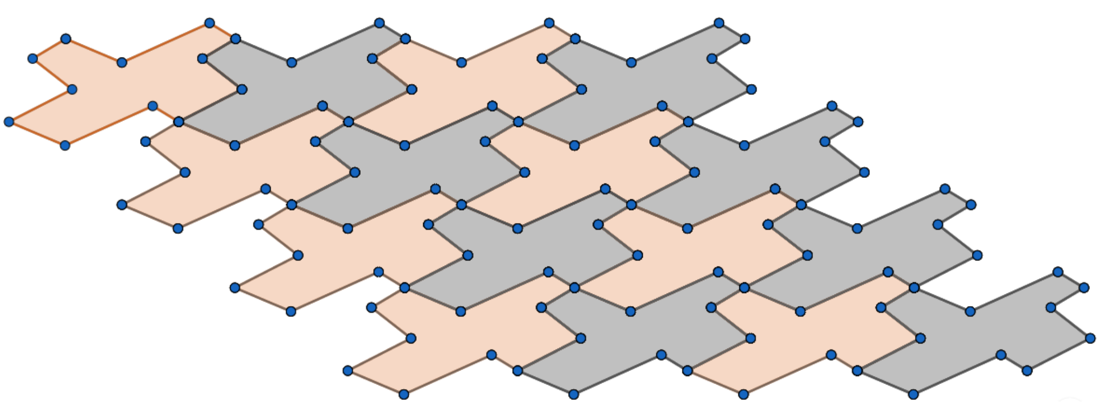
3. Create another irregular side.



   a. Create vector $\overrightarrow{AC}$.
   b. Repeat the steps in Question 2 on side $\overline{AB}$, translating by vector $\overrightarrow{AC}$.

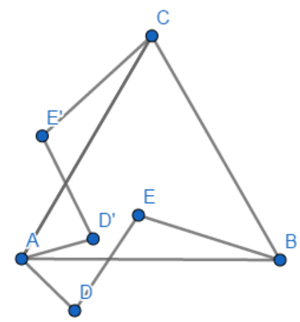
4. Create and translate the irregular polygon.
   a. Use the Polygon tool to create the interior of the irregular polygon and hide all of the sides of the original parallelogram. This will allow you to color the polygon in the finished design. Do not hide vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   b. Translate the polygon by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ repeatedly to tessellate the plane. Color the polygons to create a pattern.

Example:

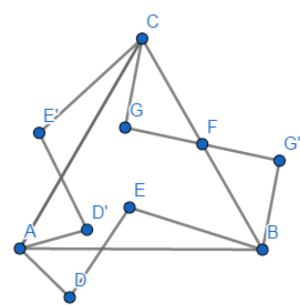


Challenge 4. Create a tessellation using only rotations.

1. Build an irregular polygon based on an equilateral triangle.
    a. Construct an equilateral triangle. Refer to Challenge 1 if you need a reminder on how to do this.
    b. Create an irregular side with points D and E by putting points on opposite sides of segment $\overline{AB}$. Use line segments to connect A, D, E, and B.



    c. Select points D and E and use the "Rotate around Point" tool to rotate them $60°$ around point A in the direction that would send point B to point C.
    d. Connect points A, D', E', and C to create a second irregular side.
    e. Use the Midpoint tool to construct the midpoint F of $\overline{BC}$.
    f. Create a point G on one side of $\overline{CB}$.
    g. Rotate G 180° about F.
    h. Connect the points C, G, G', and B to create the third irregular side.



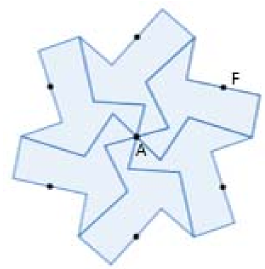
    i. Hide the sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ of the equilateral triangle. If you used the polygon tool to create the triangle, hide the interior of the triangle as well.
    j. Move points D, E, and G as needed so that no line segments are overlapping and so that the shape looks appealing to you.
    k. Make A and F a different color from the other points and do not hide them, as you will need to be able to distinguish between them in the later steps.
    l. Use the Polygon tool to create the interior of the polygon.

Questions:

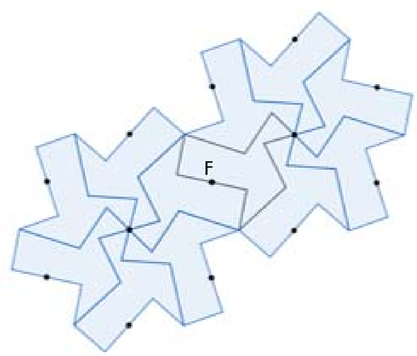
- Why do we rotate $60°$ in part (c)?

- When we rotate $180°$ in part (g), does it matter whether we rotate clockwise or counterclockwise?

2. Rotate the polygon.
   a. Use the "Rotate around Point" tool to rotate the polygon $60°$ counterclockwise about point *A*.
   b. Continue this process, each time rotating the new image $60°$ counterclockwise about point *A*.
   c. Do this 6 times, until there is no more space around *A*.

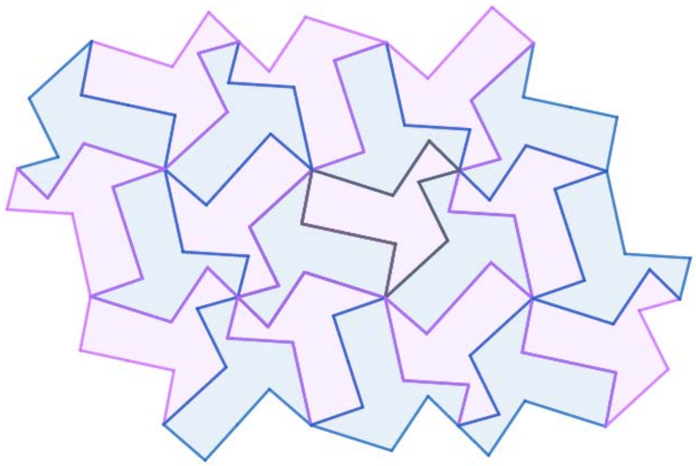


3. Rotate each of the six polygons surrounding point *A* 180˚ around point F.



4. Continue to extend the tessellation by rotating existing copies of the irregular polygon about appropriate points. When you are done, you can hide points and change the colors of the polygons to get a more appealing design.

Example:



Challenge Checklist:

	Step 1	Step 2	Step 3	Step 4
Challenge 1				
Challenge 2			X	X
Challenge 3				
Challenge 4				

Oh Snap!

Lesson Overview

Brief Description: Students will use a free online programming language called Snap! to write code, create variables, and answer challenge questions.

Time: 180 Minutes

Grades: 7-12

Topic: Computer Programming

Materials Needed:

- Computers or tablets with internet access
- Student handouts

In this lesson, students will use the programming language Snap! to learn some basics of computer programming. Snap! is an online programming environment where children and adults can become familiar with the world of Computer Science. Snap! can be used to create images, make games, prompt a user with questions, perform calculations, and much more. The freedom to create is at the fingertips of the programmer.

This lesson assumes that students have not used Snap! before. Students will start by becoming familiar with the Snap! platform. They will learn some of the commands that are available in Snap! and how to construct a sequence of commands in order to get a desired result. Specifically, students will write code to draw designs and perform calculations. They will learn how to create a variable and how to perform operations on the variables they have created. They will learn how to ask a user a question and use the answer in their code. In addition, they will learn about different types of loops and conditional statements. More information on Snap! can be found in the online reference manual located on the Snap! website's main page.

Teachers can grade student work by having students send screenshots of their code and output. We do not recommend having students submit screenshots though, because the teacher cannot then run the code in real time. Instead, students can export their Snap! projects as XML files, which they can share with their teacher. Another way of checking student work is for the teacher to check the challenges during class time, as the students complete their work.

Students will work through some examples with the teacher, and construct other examples on their own as they begin to understand Snap!. The tasks in this lesson can be completed in more than one way. As students become more familiar with Snap!, they might find more efficient ways of writing the code for a particular task. We provide one way of doing each task in the teacher notes.

Resources:

Mönig, Jens, et al. *Snap! Build Your Own Blocks*, 2020, https://snap.berkeley.edu/.

Mönig, Jens, et al. "Snap! Reference Manuel" *Snap! Build Your Own Blocks*, 2020, https://snap.berkeley.edu/snap/help/SnapManual.pdf.

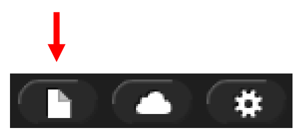
Oh Snap! Part 1 (Teacher Version)

In this lesson, we will explore Computer Science through a programing language called Snap!. Snap! is a drag-and-drop programming language that allows the user to create code to perform certain tasks. These tasks can be drawing pictures, playing games, doing calculations, and many other things. We will begin with a basic introduction to Snap!.

Logging In:

1. To launch Snap!, go to https://snap.berkeley.edu/.
2. You should create an account if you do not already have one. This will allow you to save your work and edit the code again later.
3. After you create an account, click "Run Snap!" in the toolbar on the top left of the browser window. You will be redirected to a blank canvas for coding.
4. When accessing Snap! later, just click "Log In" on the Snap! homepage.

Saving Work:

1. When you are done working on a program, click  and then select "Save As…" to save your work. Save each program separately.
2. To continue working on a program that you previously saved, click the same button and select "Open…"
3. To return Snap! to its default settings before starting to work on a new program, click the same button and select "New."

Understanding the Basics:

The Snap! workspace is separated into several regions, called the ***palette***, the ***scripting area***, the ***stage***, and the ***sprite corral***.

1. The palettes contain the elements that the programmer uses to create code.
   a. The palette area is located on the far left of the screen. Use the palette selector buttons shown below to access the eight different palettes.

   b. Click a selector button to reveal the available blocks in that specific palette. A ***block*** is a piece of code that can be dragged from the palette to the scripting area where it will be used. Several blocks pieced together form a ***script***.
   c. The blocks in a palette are color-coded, so the programmer can look at a block in the scripting area and identify what palette the block came from.

Questions:

Which palette contains the block to ask the user a question?

Sensing

Which palette contains the block to pick a random number between 1 and 10?

Operators

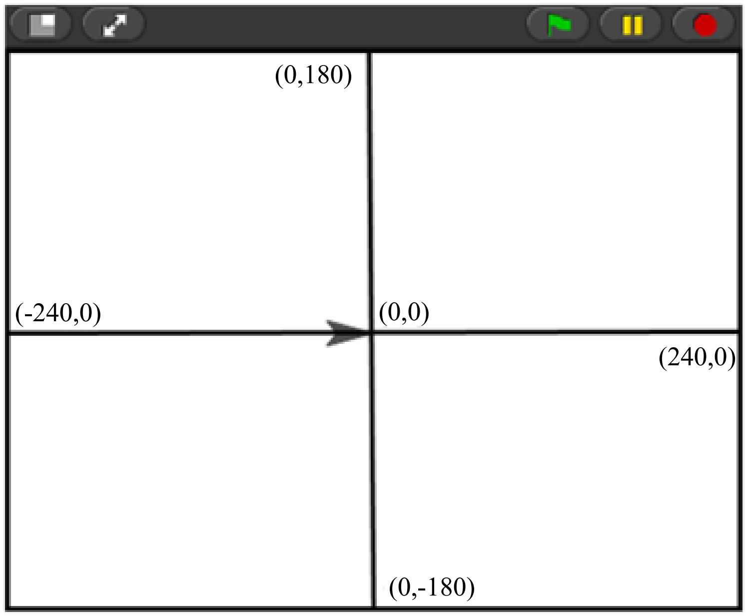
What palette contains the block to set the *x* or *y* coordinate to a certain value?

Motion

2. The ***scripting area*** is the open space in the middle of the page. This is where the programmer builds the code.

3. The ***stage*** is the white region in the Snap! workspace. This is where the output of a Snap! program will appear. To work mindfully, we must understand the format of the stage. The unit of distance in Snap! is a “step.” The default grid for the stage is 480 steps across by 360 steps

high with the point $(0,0)$ at the center. The stage is blank on a new Snap! page, but we have drawn coordinate axes in the stage shown below to demonstrate the structure of the underlying grid.



The dart shape in the stage is called a ***sprite***. The sprite must be present if we want to display anything, like a drawing or a question for the user. The default image of the sprite is  . We can click the  tool located directly under the stage in the area called the ***sprite corral*** to change the sprite image. Options include drawing one freehand or importing an image. Although it is possible to change the sprite icon we will stick with the default version for this lesson.

Directions: Read Steps 1-11 below and answer any associated questions. Then work on the program described in Step 12. You will review Steps 1-12 as a class after you have had time to work on them individually. If you finish early, start working on Challenge 1 and Challenge 2. You will turn these challenges in.

Drawing Shapes:

1. From the Control palette, drag the 

icon to the scripting area. The 

block is referred to as a ***hat block***. A hat block indicates when the script should be carried out. In this case, the code executes when the user clicks  located above the stage, and Snap! displays output (if there is any) in the stage. There are other blocks in the Control palette that we can use to initiate the code, which would be necessary if we had two (or more) different programs in the same file. For our use, we will need only one initiator.
2. Another type of block is a ***command block***. A command block corresponds to an action that Snap! knows how to carry out. Command blocks build onto one another, fitting together like puzzle pieces to create the script.
3. In order to draw anything, we need to use a pen, which is attached to the sprite. We can adjust the starting location of the sprite, the color of the pen, the size of the pen (thickness of the line it draws), and whether the pen is down (drawing) or up (not drawing). The script in Figure 1 consists of a hat block and several command blocks, and tells the computer to do the following when the user clicks :
   - Start the program (Control palette).
   - Clear anything that was previously drawn (Pen palette).
   - Lift the pen up (Pen palette).
   - Move the sprite to $(0,0)$ in the stage (Motion palette).
   - Turn the sprite so it points to the right (Motion palette).
   - Set the pen color to magenta (Pen palette).
   - Put the pen down (Pen palette).

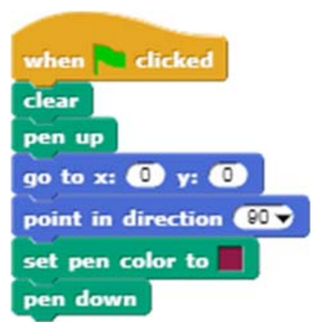


Figure 1

4. If you no longer wish to use a block that is in the scripting area, drag the block to the palette and release it. You can also remove the entire script by dragging the 

block to the palette. Build the script shown in Figure 1 in Snap!. You may set the pen color to a color of your choice. The default pen size is 1, but you may also choose to add a block from the Pen palette to resize the pen.

5. To draw a line, we need to move the sprite, and thus the pen, a certain distance. Since the grid is 480 steps by 360 steps, we need to move the pen far enough that we will actually notice the line it is drawing. Also note that the pen starts in the center of the stage. What is a reasonable distance?

   Anything between 20 and 230 steps is a reasonable distance.

6. Assume the pen moves a distance of 100 steps. Add the 

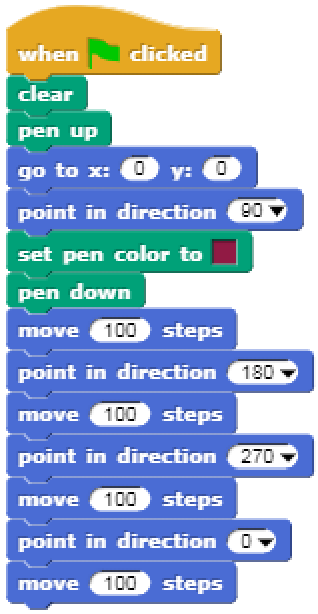
block from the Motion palette to the script. Replace the "10" in the block with 100 and run the code by clicking . What happens?

   A line that is 100 steps long appears in the stage.

7. If we want to make a square, what has to happen next?

   Turn the sprite to point either up or down so that the next pen movement will be at a right angle with the segment already created.

8. Add additional blocks as in Figure 2 to create a square that starts and ends at the initial point of $(0,0)$. Recall that the sprite points in the direction it will travel.



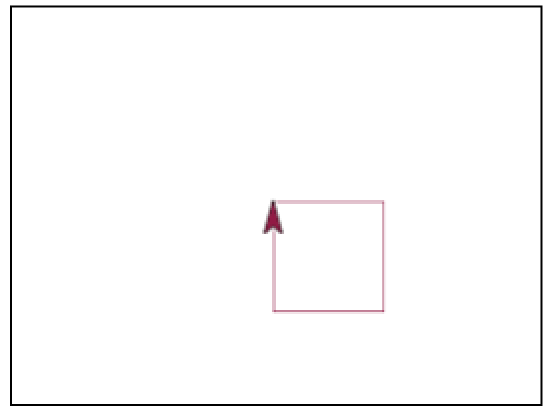


Figure 2: Script. Figure 3: Result of running the script in Figure 2.

9. In the code in Figure 2, we gave commands for four individual line segments that created the square. Is there a way to draw the same square, but with fewer lines of code? Yes! We can use what is called a ***loop***. A loop repeats a sequence of commands until a specific condition is satisfied. The loop we will use in this example is a ***repeat loop***, meaning that the code will be repeated a fixed number of times, as determined by the programmer. The block for this type of loop is shown below and can be found in the Control palette.

The default of 10 in the block means that the code will repeat 10 times. The code inside the loop we need will create one side of the square. How many times will we have to repeat the code in order to create a square?

4 times

10. We will add blocks inside the loop so that after the right number of repetitions, the program will draw a square. This means that we must specify a direction for the sprite that will work for each individual line segment. We will do this by indicating a rotation of the sprite instead of a “point in” value. After the pen draws a side of the square, what angle rotation must the sprite rotate in order to prepare for the next side?

The pen must turn 90° after it draws a side.

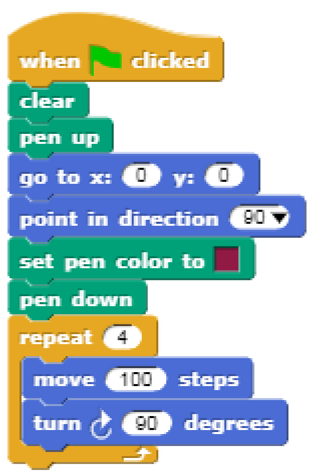
11. Use the 

or 

block from the Motion palette, depending on the desired direction of rotation. Replace the “15” with the appropriate amount of rotation.

12. The repeating part of the script goes inside the gap in the “repeat” block. This gap will expand to hold any number of blocks. Complete the script, and then run the code. If your stage does not display a square, check for errors in your script.

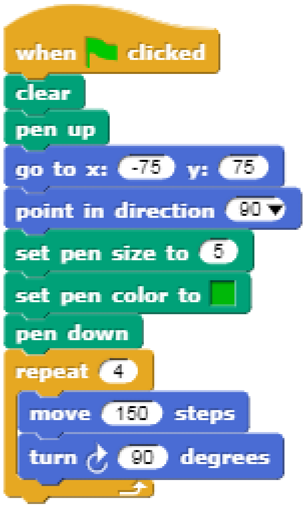
Sample script:



The output in the stage will be the same as in Figure 2.

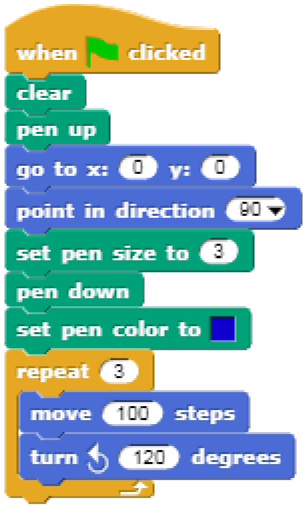
Challenge 1: Create a square that has side lengths of 150 steps with a center at $(0,0)$ and pen size of 5. You may choose the color for the pen. If the square is not centered in the stage, then adjust the script so that the center (not a corner) of the square is at $(0,0)$. Save the program when you are finished and title it "Challenge 1_*yourlastname*".

Sample script:



Challenge 2: Create an equilateral triangle with side lengths of 100 steps that is oriented so it has a horizontal bottom side, with the lower left vertex at $(0,0)$. You may choose the color and size of the pen. Save the program when you are finished and title it "Challenge 2_*yourlastname*".

Students may think that they need to rotate the pen $60°$ because the interior angles of an equilateral triangle are $60°$. The reason that the code requires a $120°$ rotation is that the sprite points in the direction of motion. After drawing an edge, it is pointing toward the outside of the triangle, and so it must rotate through the exterior angle of the triangle before drawing the next edge.



Oh Snap! Part 2 (Teacher Version)

Introducing Variables:

In Part 1, we learned how to write script in Snap! to draw shapes in the stage. Now we will learn how to use variables in a program. A variable in a computer program is a lot like a variable in a math function: both represent values that may change. In programming, we use words as variable names, rather than just using a letter like we frequently do in mathematics classes, so that we know something about the information in the variables. For example, we can use the name "TotalCost" for a variable that stores how much money we spent at the grocery store. Even moderately sized computer programs can use dozens of variables, so it is important to name variables clearly.

Directions: Work through Steps 1-16 below. You will review these steps as a class after you have had time to work on them individually. If you finish early, start working on Challenge 3 and Challenge 4. You will turn these challenges in.

1. Add variables into the program code by using blocks in the Variables palette. Before we can use a variable, we must first create it by clicking the Make a variable button. Only then can we use the variable in the script. When you click Make a variable, the default setting will create a variable for all sprites. For our use, keep that default setting.

2. Create two variables called Num1 and Num2. When you create each variable, you will see its name displayed at the top of the Variable palette, and its name and value displayed in the stage.

   If you do not want the variable displayed in the stage, you can uncheck the box next to the variable name in the palette. The variable will still be accessible, but will not be visible in the stage.

3. Using the Sensing palette, we can ask the user a question that involves the variables we created. Questions for the user will be displayed at the location of the sprite, which by default is at $(0,0)$. If during the program the sprite is sent to a different location, the text for the question will be displayed at that new location.

4. When a user answers a question, the result is stored in the answer variable, and overwrites any previous value of answer. The code is sequential, meaning that blocks are executed one at a time. If a block refers to the answer variable, it uses the value that is stored in answer at that moment. The value in answer may change later, in response to another question, so the next block that refers to answer would use the new value of the answer variable.

5. We need to store the answers to questions in variables other than answer, so that we can use the results of those questions later. To store the value of answer in another variable, drag set ▾ to 0 from the Variables palette to the scripting area. Click the arrow in the block and select the variable name. Then drag answer from the Sensing palette to the white rectangle in the block.

   Example:

   In the script above, the user is asked two questions: “What was the total cost of your grocery bill?” and “How any items did you buy?” When the user answers the first question, answer will refer to the total grocery bill. The code then stores this number in the variable “Total”. When the user answers the second question, answer will refer the number of items bought.

The code then stores this number in the variable "NumItems". If we had not put the user's first answer into the variable "Total," we would not be able to use that value later.

6. Notice that blocks for variables (, Num1, Num2) are oval and cannot be built on to the script directly. Oval blocks such as these work differently than command blocks. We call these blocks ***reporter blocks***. Reporter blocks get their name because they report values to other blocks. We use them by putting them inside command blocks.

7. Write script that asks the user to enter a number and then another number. Store the replies as Num1 and Num2. Remember that the  block is necessary in order to initiate the program to run, and that you need to store the first reply before asking the next question.

   Sample script:

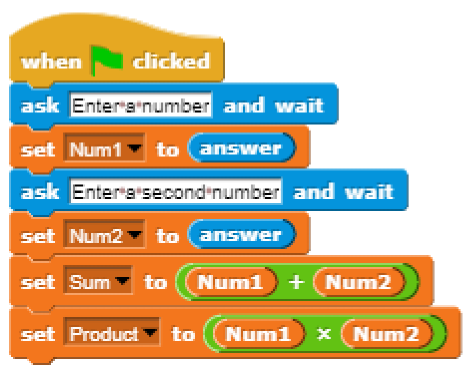
8. Now that the variables Num1 and Num2 are storing the answers that the user provided, we can use those values in mathematical operations. We will create additional variables to store the results of these operations.

9. To find the operations that we can perform in Snap!, go to the Operators palette. The most basic operations of addition, subtraction, multiplication, division, and raising something to a power are available in this palette. The corresponding blocks are shown below.

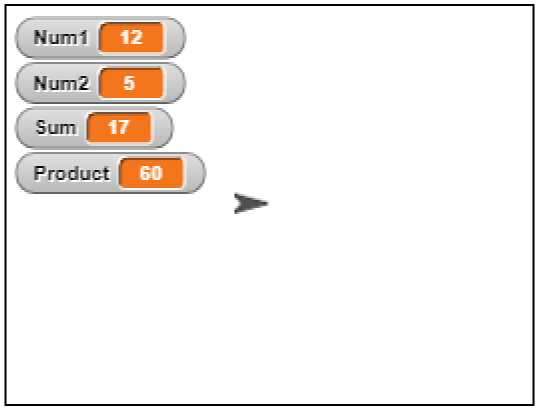
10. We can type numbers directly into the white ovals, or we can drag variable blocks into a white space. Notice that the operations are reporter blocks just like the variables. Therefore we must put these blocks into command blocks in order to use them.

11. Continue the script you started in Step 7 to find the sum and product of the two numbers the user entered. Start by creating variables named “Sum” and “Product” where you will store the answers.

    Sample script:



12. Variable values will be 0 until the first time we run the script. Run the script and answer the questions when prompted. An example output is shown in the stage below. Check the script for errors by checking the values of Sum and Product with a calculator if necessary.



In this example, the user entered 12 as
the first number and 5 as the second number.

Another useful tool in programming is a ***conditional statement***. A conditional statement is used as a way to execute a line of code only when a given condition is satisfied. Conditional statements can be placed within loops to be executed repeatedly, or they can stand alone and perform the action just once. There are two types of conditional statements in Snap!. The ***if statement*** executes when the condition statement is true, but is bypassed when the condition is false. The ***if else statement*** executes one set of instructions when the condition is satisfied and another when the condition is not satisfied. The blocks for conditional statements (shown below) are in the Control palette.

13. Do Steps 14-16 in a new browser tab so that your work in Steps 1-12 will still be accessible during the whole class discussion.

14. Write a program that asks the user to enter an even number. If the user does not enter an even number, ask them to try again. Use a variable, NumCount, to keep track of how many numbers the user enters. After the user enters an even number, the program should indicate how many times it took the user to enter an even number.

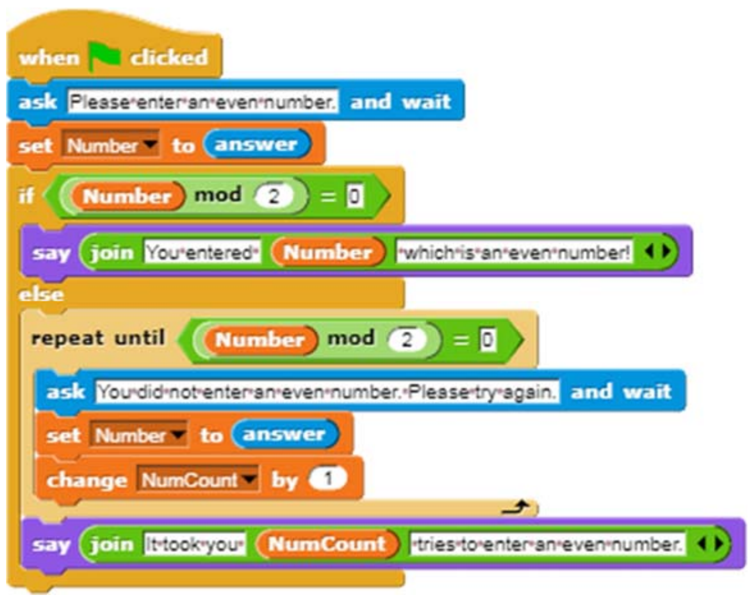
- Use a conditional statement with a "repeat until" loop, which can be found on the Control palette, to decide whether the user entered an even number. If the user entered an even number, congratulate them for following directions. If the user entered an odd number, tell them that they did not enter an even number and prompt them again to enter an even number.
- Use the 

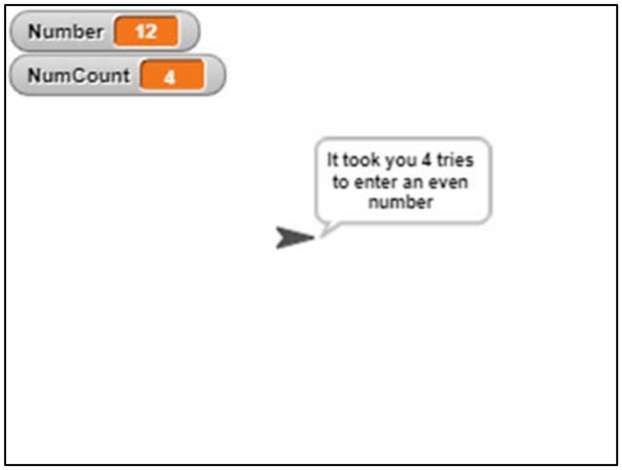
block in the Operators palette to calculate the remainder when the first number in the block is divided by the second number in the block. Recall that an even number has a remainder of 0 when divided by 2.
- Use the 

block from the Operators palette together with the 

block from the Looks palette to display responses to the user.

Sample script:





In this sample script, the user entered 13, 41, 25, and 12.

15. Run the code from Step 14 again and enter several odd numbers in a row. Is the final NumCount value correct? What is happening?

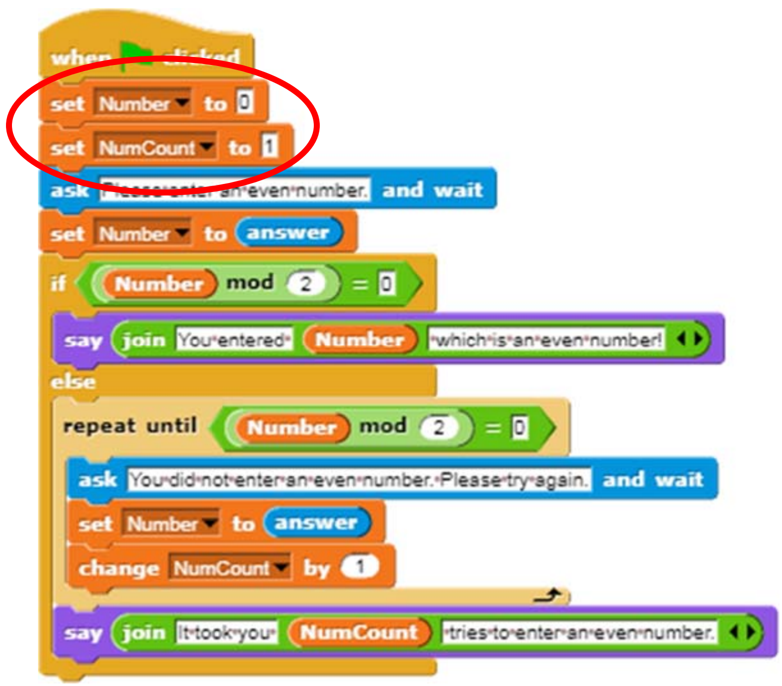
    No, the value of NumCount is not correct. At the beginning of the program, NumCount already held the result from the previous run of the code. This next run of the code added to that existing value instead of starting over.

There is an easy fix for this problem: We have to initialize the NumCount variable. This means that at the beginning of the code, we set the value of NumCount to a "starting number" that will be the same, every time we run the code. Generally, initial values of variables will be 1 or 0 because these are typical starting points for counting. In the script in Step 7, initializing the variables was not a problem because all of the variables were naturally reset by the user. To avoid overlooking a variable that should be initialized, however, good programming practice is to initialize all of the variables in a program. This also ensures that if any variables are displayed in the stage, they will not contain data from a previous run of the code. Initializing of the variables should occur at the beginning of the program.

16. Adjust the script in Step 14 so that the variables are initialized, and then run the code. Do you still notice an error in the output? If so, how can you correct the error?

In the sample script below, we need to initialize NumCount to 1, because the code adds 1 to NumCount after each odd guess. If we initialize NumCount to 0, then we are not counting the user's even guess. Students who update the value of NumCount at a different point in the script may need to initialize NumCount to 0.

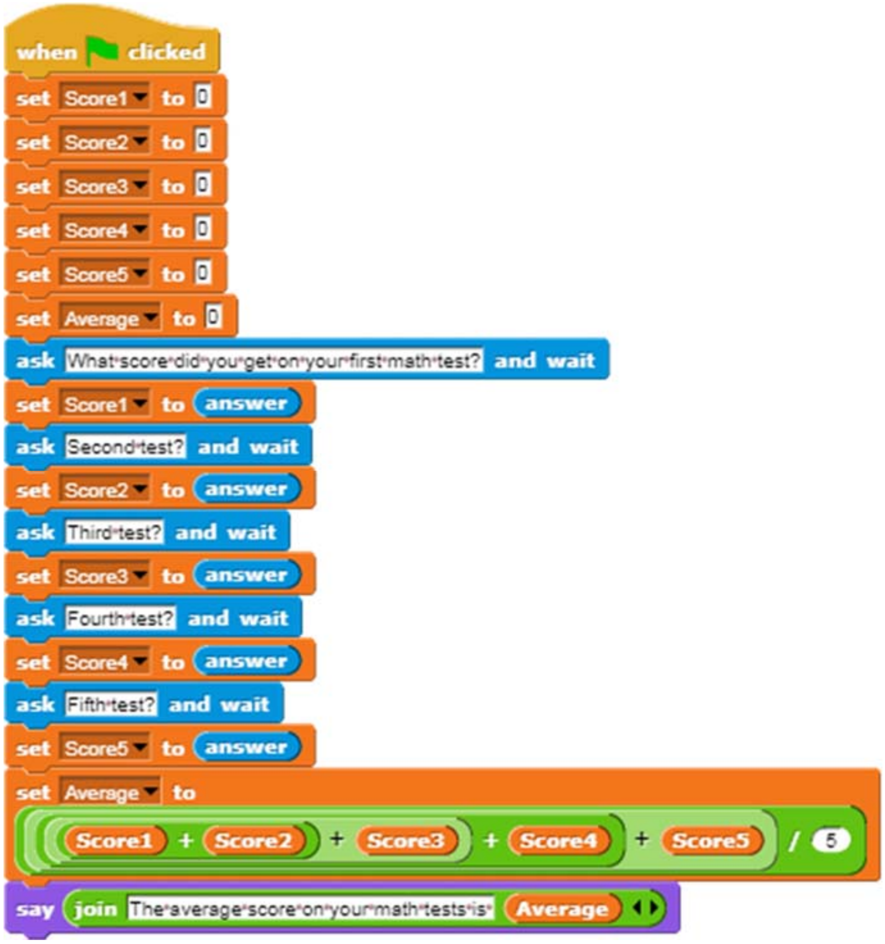
Sample script:



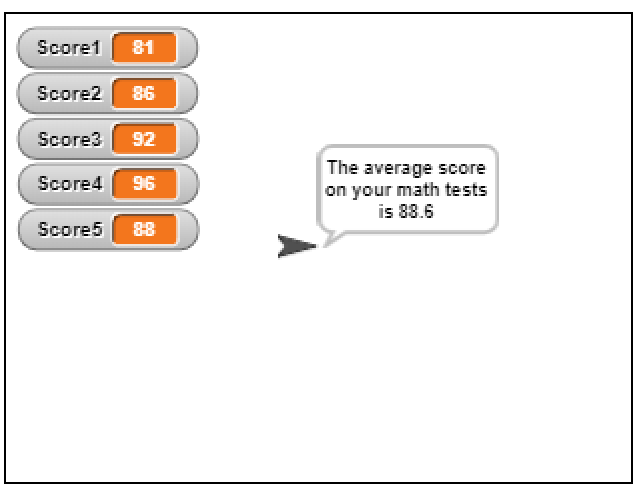
Challenge 3: Write a program that finds the average score that a user earned on their last five math tests. The code should ask the user to input their test scores and then the code will calculate the average. The code should then tell the user what their average on the math tests is. Save the program when you are finished and title it "Challenge 3_*yourlastname*".

Students must create variables for the five scores and the average. In the script below, the variables are Score1, Score2, Score3, Score4, Score5, and Average. Students may need prompting in order to realize that they can nest the operations blocks.

Sample script:

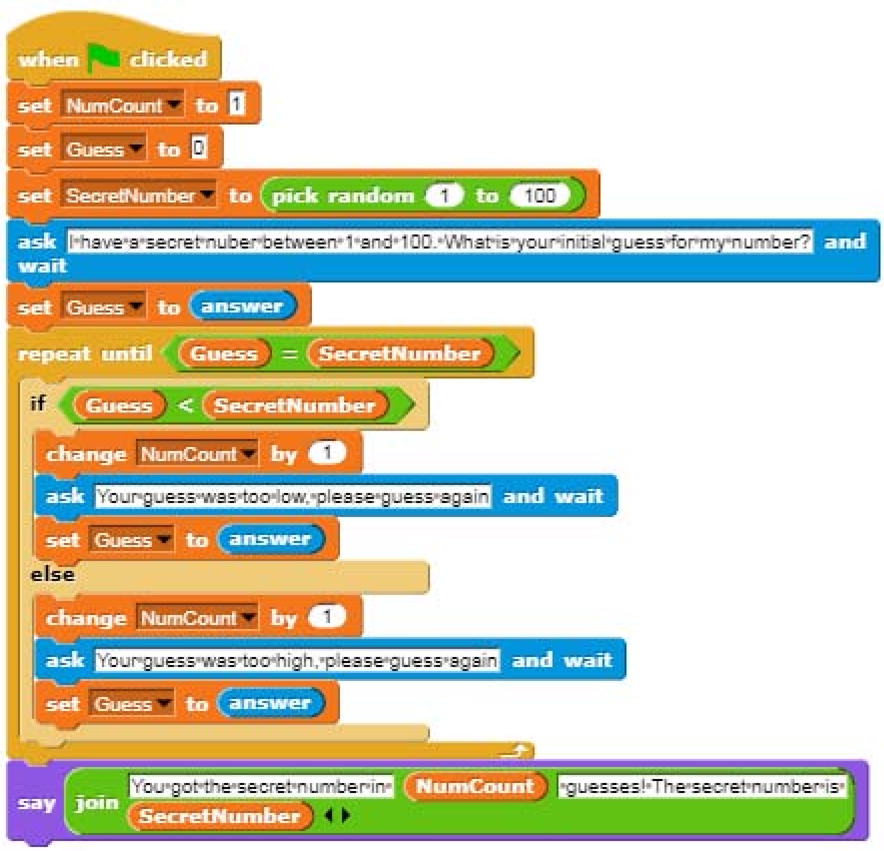


Sample output:

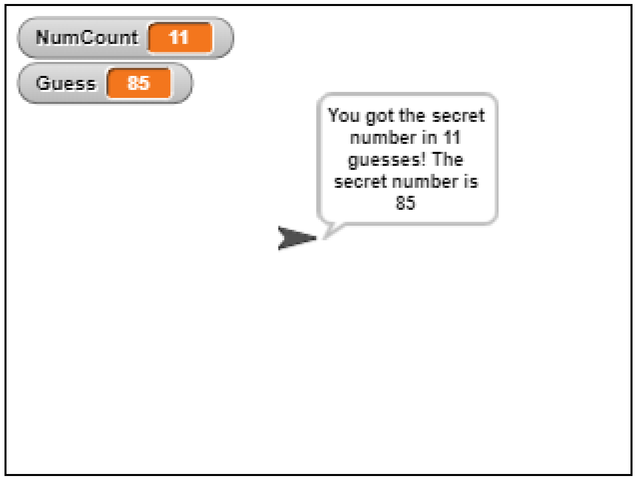


Challenge 4: Write a program that chooses a secret number between 1 and 100 and asks the user to guess the number. Create a variable and use the appropriate block from the Operators palette to generate a random value for that variable. If the user's guess is lower than the random number, they should be told that their number is too low and they need to guess again. If the user's guess is higher than the random number, they should be told that their number is too high and they need to guess again. Display the user's most recent guess in the stage. Since the user is guessing a secret number, the stage should not display the secret number. When the user gets the secret number correct, the output should display the number of guesses it took and the value of the secret number. Save the program when you are finished and title it "Challenge 4_*yourlastname*".

Sample script:



Sample output:

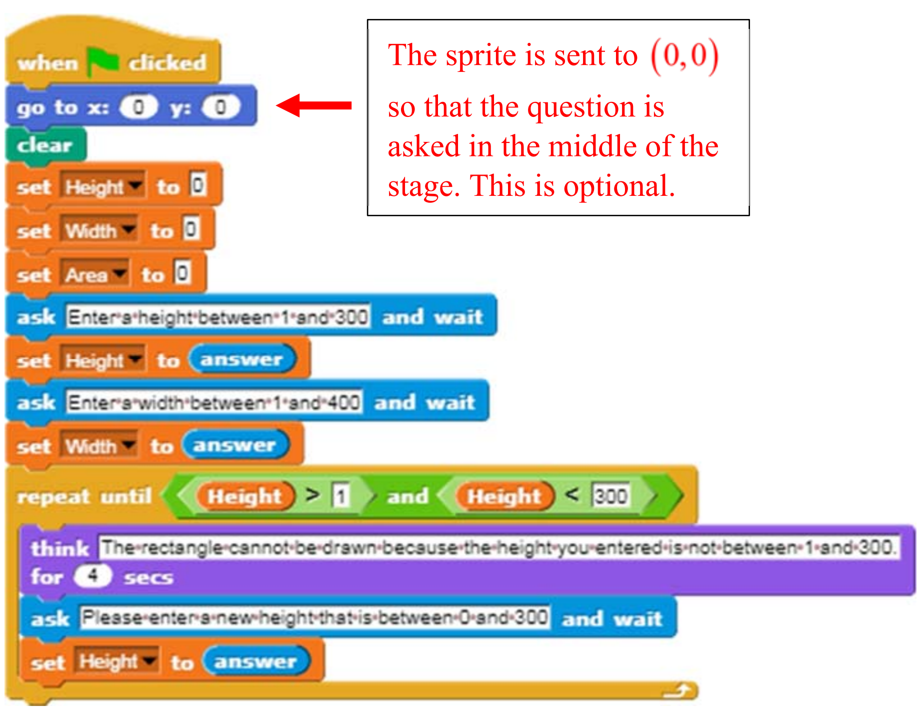


Oh Snap! Part 3 (Teacher Version)

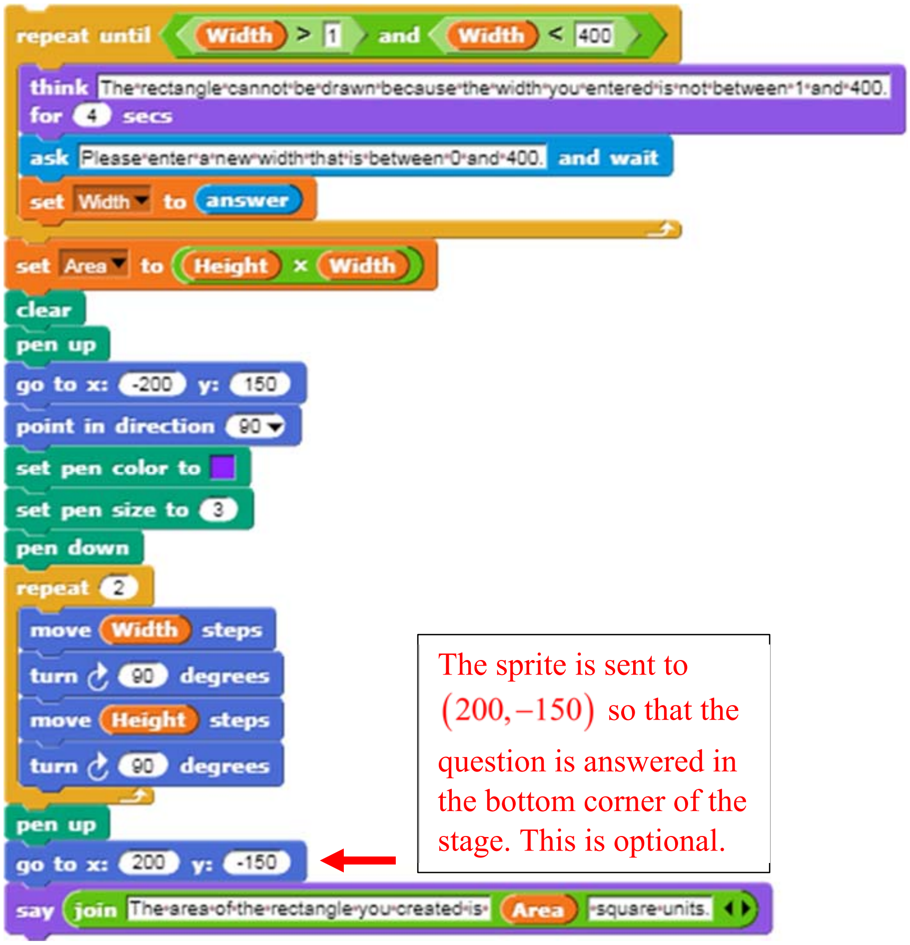
We have discussed how to draw shapes in the stage as well as how to assign variables in the code. We have looked at loops and conditional statements. Now we will combine those processes to write some more complex programs.

Challenge 5: Write a program that asks the user to enter a value for the height and width of a rectangle. The program should then draw the rectangle that the user described. Remember the size of the stage, and make sure the entire rectangle fits in the stage. State the restrictions so that the user knows what numbers they are allowed to enter for the height and width. If the user enters a number that is too large, they should be told that the rectangle cannot be drawn and to enter a new height or width (whichever applies). The program should not draw anything until the user has entered appropriate values for height and width. The program should report the area of the rectangle after drawing the rectangle. Save the program when you are finished and title it "Challenge 5_*yourlastname*".

Sample script (continued on the next page):

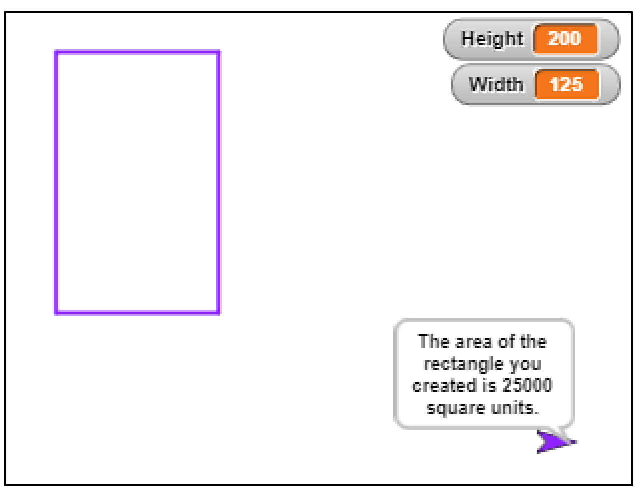


The sprite is sent to $(0,0)$ so that the question is asked in the middle of the stage. This is optional.



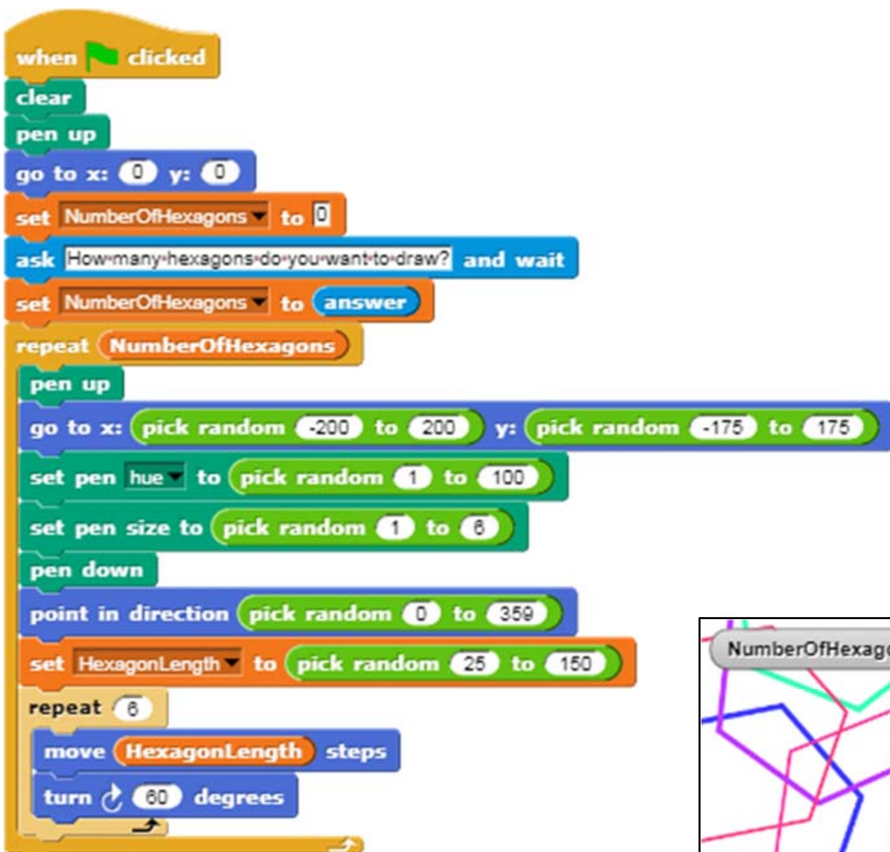
The sprite is sent to $(200, -150)$ so that the question is answered in the bottom corner of the stage. This is optional.

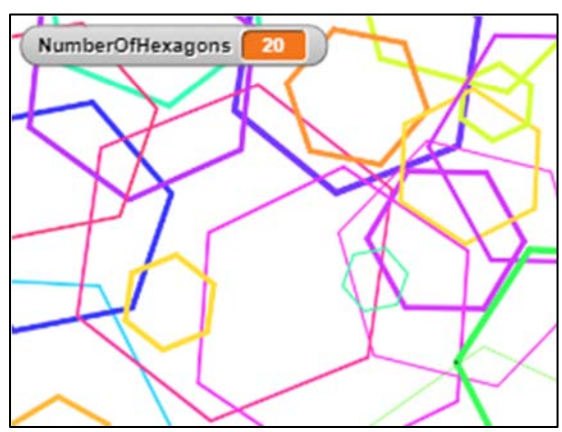
Sample output:



Challenge 6: Write a program that asks the user how many hexagons they would like to draw in the stage. Display this variable in the stage. The program should then draw that number of hexagons with random locations, sizes, orientation angles, and colors. For each hexagon set the pen hue to a random number between 1 and 100 in order to get a variety of colors. Move the sprite to a random location by setting the *x*-coordinate to a random value between -200 and 200, and setting the *y*-coordinate to a random value between -175 and 175. Note: Not all of the hexagons will be fully contained in the stage. Save the program when you are finished and title it "Challenge 6_*yourlastname*".

Sample script:





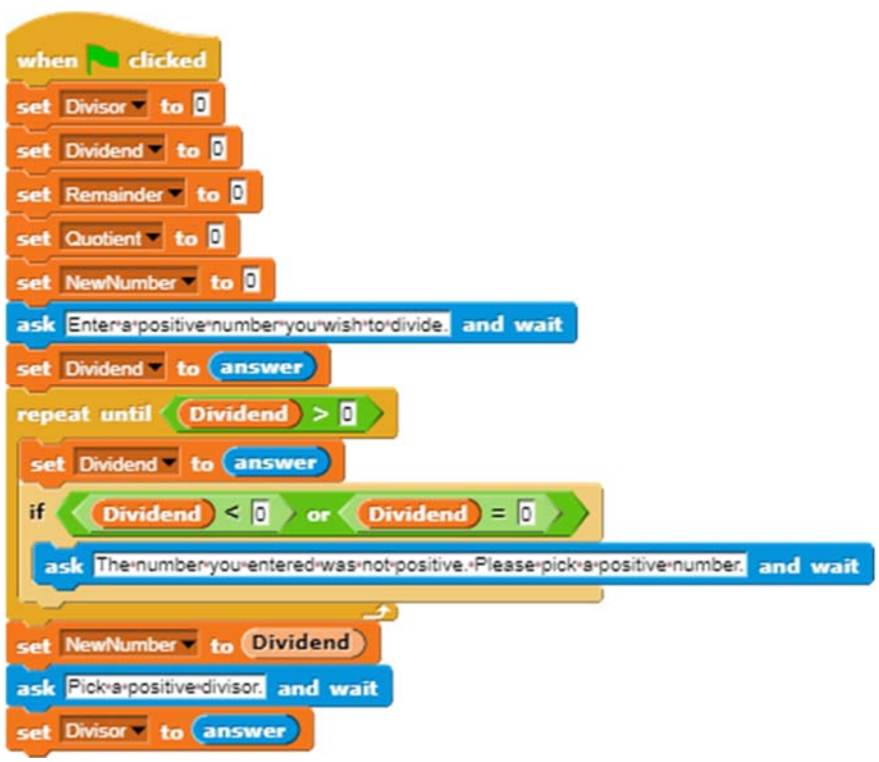
Sample Output

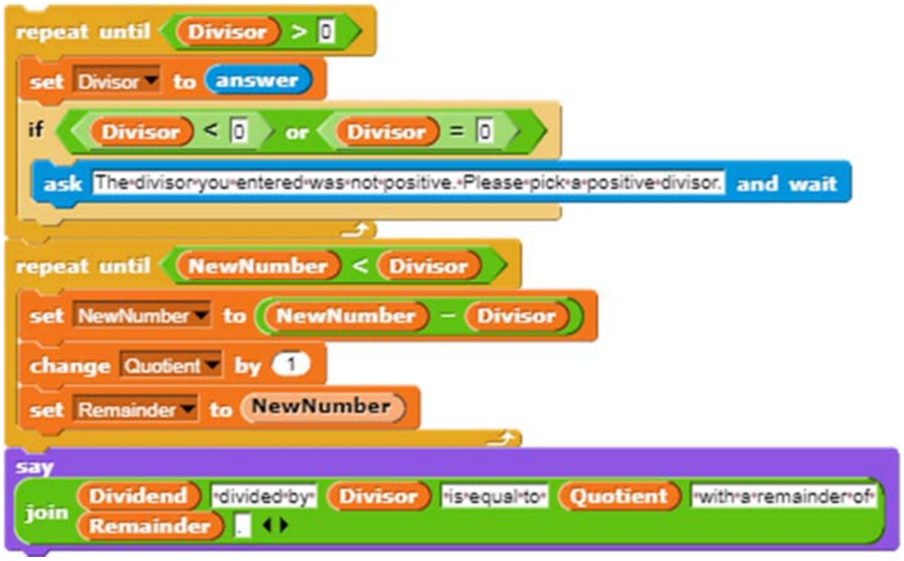
Challenge 7: Write a program that uses repeated subtraction to divide two positive numbers provided by the user. Specifically, subtract the divisor from the dividend until the result is less than the divisor. This final result is the remainder. Use conditional statements to ensure that the user enters two positive numbers. At the end of the program, remind the user of their division problem, and report the quotient and remainder.

Students who do not have prior programming experience will likely not be able to complete this without help so teachers may want to make this a bonus problem.

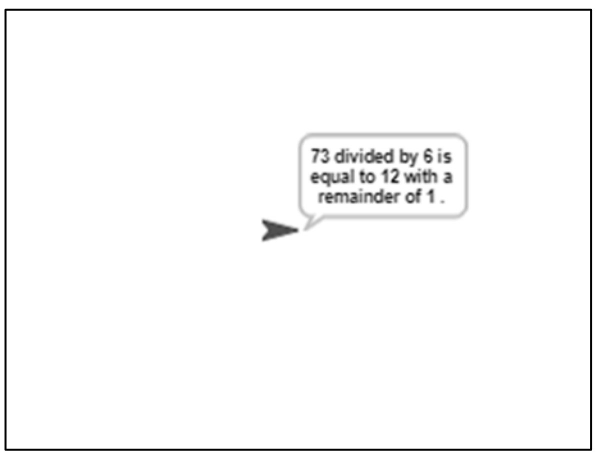
Note: Since the program will report the initial dividend at the end, we need a separate variable that changes with the repeated subtraction. In the script below the variable NewNumber initially has the same value as the user's dividend. Later in the program, the value of NewNumber decreases, until it is equal to the remainder in the division problem. Finally, the value of NewNumber is stored in a variable called Remainder.

Sample script (continued on the next page):





Sample output:



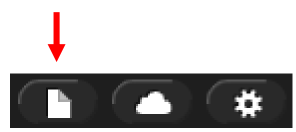
Oh Snap! Part 1 (Student Version)

In this lesson, we will explore Computer Science through a programing language called Snap!. Snap! is a drag-and-drop programming language that allows the user to create code to perform certain tasks. These tasks can be drawing pictures, playing games, doing calculations, and many other things. We will begin with a basic introduction to Snap!.

Logging In:

1. To launch Snap!, go to https://snap.berkeley.edu/.
2. You should create an account if you do not already have one. This will allow you to save your work and edit the code again later.
3. After you create an account, click "Run Snap!" in the toolbar on the top left of the browser window. You will be redirected to a blank canvas for coding.
4. When accessing Snap! later, just click "Log In" on the Snap! homepage.

Saving Work:

1. When you are done working on a program, click  and then select "Save As…" to save your work. Save each program separately.
2. To continue working on a program that you previously saved, click the same button and select "Open…"
3. To return Snap! to its default settings before starting to work on a new program, click the same button and select "New."

Understanding the Basics:

The Snap! workspace is separated into several regions, called the ***palette***, the ***scripting area***, the ***stage***, and the ***sprite corral***.

4. The palettes contain the elements that the programmer uses to create code.
    a. The palette area is located on the far left of the screen. Use the palette selector buttons shown below to access the eight different palettes.

    b. Click a selector button to reveal the available blocks in that specific palette. A ***block*** is a piece of code that can be dragged from the palette to the scripting area where it will be used. Several blocks pieced together form a ***script***.
    c. The blocks in a palette are color-coded, so the programmer can look at a block in the scripting area and identify what palette the block came from.

Questions:

Which palette contains the block to ask the user a question?

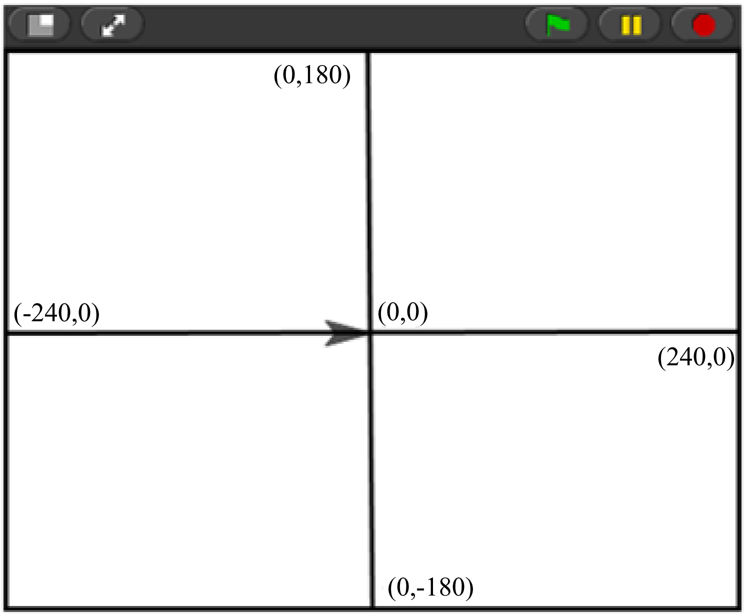
Which palette contains the block to pick a random number between 1 and 10?

What palette contains the block to set the *x* or *y* coordinate to a certain value?

5. The ***scripting area*** is the open space in the middle of the page. This is where the programmer builds the code.

6. The ***stage*** is the white region in the Snap! workspace. This is where the output of a Snap! program will appear. To work mindfully, we must understand the format of the stage. The unit of distance in Snap! is a "step." The default grid for the stage is 480 steps across by 360 steps

high with the point $(0,0)$ at the center. The stage is blank on a new Snap! page, but we have drawn coordinate axes in the stage shown below to demonstrate the structure of the underlying grid.



The dart shape in the stage is called a ***sprite***. The sprite must be present if we want to display anything, like a drawing or a question for the user. The default image of the sprite is  . We can click the  tool located directly under the stage in the area called the ***sprite corral*** to change the sprite image. Options include drawing one freehand or importing an image. Although it is possible to change the sprite icon we will stick with the default version for this lesson.

Directions: Read Steps 1-11 below and answer any associated questions. Then work on the program described in Step 12. You will review Steps 1-12 as a class after you have had time to work on them individually. If you finish early, start working on Challenge 1 and Challenge 2. You will turn these challenges in.

Drawing Shapes:

1. From the Control palette, drag the  icon to the scripting area. The  block is referred to as a ***hat block***. A hat block indicates when the script should be carried out. In this case, the code executes when the user clicks  located above the stage, and Snap! displays output (if there is any) in the stage. There are other blocks in the Control palette that we can use to initiate the code, which would be necessary if we had two (or more) different programs in the same file. For our use, we will need only one initiator.
2. Another type of block is a ***command block***. A command block corresponds to an action that Snap! knows how to carry out. Command blocks build onto one another, fitting together like puzzle pieces to create the script.
3. In order to draw anything, we need to use a pen, which is attached to the sprite. We can adjust the starting location of the sprite, the color of the pen, the size of the pen (thickness of the line it draws), and whether the pen is down (drawing) or up (not drawing). The script in Figure 1 consists of a hat block and several command blocks, and tells the computer to do the following when the user clicks :
    - Start the program (Control palette).
    - Clear anything that was previously drawn (Pen palette).
    - Lift the pen up (Pen palette).
    - Move the sprite to $(0,0)$ in the stage (Motion palette).
    - Turn the sprite so it points to the right (Motion palette).
    - Set the pen color to magenta (Pen palette).
    - Put the pen down (Pen palette).

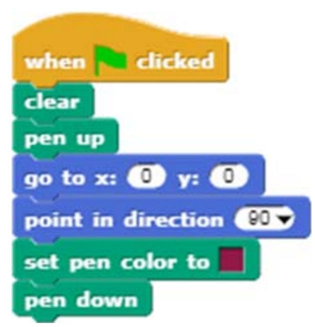


Figure 1

4. If you no longer wish to use a block that is in the scripting area, drag the block to the palette and release it. You can also remove the entire script by dragging the 

block to the palette. Build the script shown in Figure 1 in Snap!. You may set the pen color to a color of your choice. The default pen size is 1, but you may also choose to add a block from the Pen palette to resize the pen.

5. To draw a line, we need to move the sprite, and thus the pen, a certain distance. Since the grid is 480 steps by 360 steps, we need to move the pen far enough that we will actually notice the line it is drawing. Also note that the pen starts in the center of the stage. What is a reasonable distance?

6. Assume the pen moves a distance of 100 steps. Add the 

block from the Motion palette to the script. Replace the "10" in the block with 100 and run the code by clicking . What happens?

7. If we want to make a square, what has to happen next?

8. Add additional blocks as in Figure 2 to create a square that starts and ends at the initial point of $(0,0)$. Recall that the sprite points in the direction it will travel.

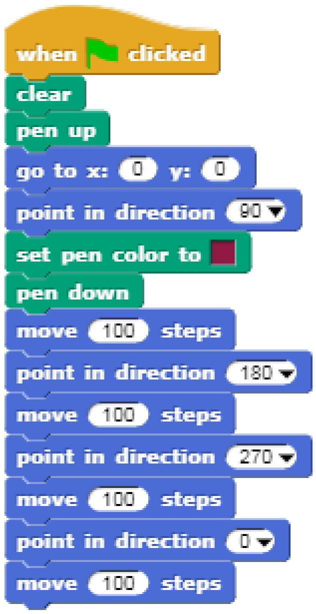


Figure 2: Script.

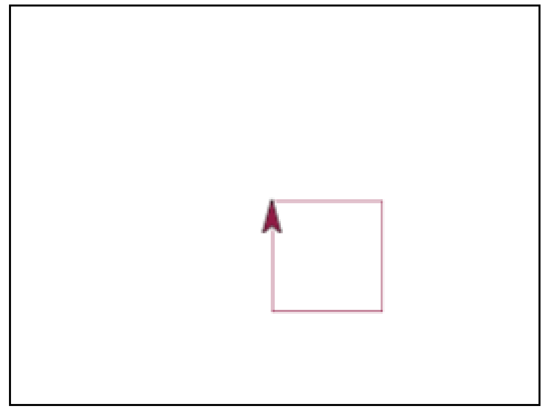


Figure 3: Result of running the script in Figure 2.

9. In the code in Figure 2, we gave commands for four individual line segments that created the square. Is there a way to draw the same square, but with fewer lines of code? Yes! We can use what is called a ***loop***. A loop repeats a sequence of commands until a specific condition is satisfied. The loop we will use in this example is a ***repeat loop***, meaning that the code will be repeated a fixed number of times, as determined by the programmer. The block for this type of loop is shown below and can be found in the Control palette.

The default of 10 in the block means that the code will repeat 10 times. The code inside the loop we need will create one side of the square. How many times will we have to repeat the code in order to create a square?

10. We will add blocks inside the loop so that after the right number of repetitions, the program will draw a square. This means that we must specify a direction for the sprite that will work for each individual line segment. We will do this by indicating a rotation of the sprite instead of a "point in" value. After the pen draws a side of the square, what angle rotation must the sprite rotate in order to prepare for the next side?

11. Use the 

or 

block from the Motion palette, depending on the desired direction of rotation. Replace the "15" with the appropriate amount of rotation.

12. The repeating part of the script goes inside the gap in the "repeat" block. This gap will expand to hold any number of blocks. Complete the script, and then run the code. If your stage does not display a square, check for errors in your script.

Challenge 1: Create a square that has side lengths of 150 steps with a center at $(0,0)$ and pen size of 5. You may choose the color for the pen. If the square is not centered in the stage, then adjust the script so that the center (not a corner) of the square is at $(0,0)$. Save the program when you are finished and title it "Challenge 1_*yourlastname*".

Challenge 2: Create an equilateral triangle with side lengths of 100 steps that is oriented so it has a horizontal bottom side, with the lower left vertex at $(0,0)$. You may choose the color and size of the pen. Save the program when you are finished and title it "Challenge 2_*yourlastname*".

Oh Snap! Part 2 (Student Version)

Introducing Variables:

In Part 1, we learned how to write script in Snap! to draw shapes in the stage. Now we will learn how to use variables in a program. A variable in a computer program is a lot like a variable in a math function: both represent values that may change. In programming, we use words as variable names, rather than just using a letter like we frequently do in mathematics classes, so that we know something about the information in the variables. For example, we can use the name "TotalCost" for a variable that stores how much money we spent at the grocery store. Even moderately sized computer programs can use dozens of variables, so it is important to name variables clearly.

Directions: Work through Steps 1-16 below. You will review these steps as a class after you have had time to work on them individually. If you finish early, start working on Challenge 3 and Challenge 4. You will turn these challenges in.

1. Add variables into the program code by using blocks in the Variables palette. Before we can use a variable, we must first create it by clicking the Make a variable button. Only then can we use the variable in the script. When you click Make a variable, the default setting will create a variable for all sprites. For our use, keep that default setting.

2. Create two variables called Num1 and Num2. When you create each variable, you will see its name displayed at the top of the Variable palette, and its name and value displayed in the stage.

   If you do not want the variable displayed in the stage, you can uncheck the box next to the variable name in the palette. The variable will still be accessible, but will not be visible in the stage.

3. Using the Sensing palette, we can ask the user a question that involves the variables we created. Questions for the user will be displayed at the location of the sprite, which by default is at $(0,0)$. If during the program the sprite is sent to a different location, the text for the question will be displayed at that new location.

4. When a user answers a question, the result is stored in the answer variable, and overwrites any previous value of answer. The code is sequential, meaning that blocks are executed one at a time. If a block refers to the answer variable, it uses the value that is stored in answer at that moment. The value in answer may change later, in response to another question, so the next block that refers to answer would use the new value of the answer variable.

5. We need to store the answers to questions in variables other than answer, so that we can use the results of those questions later. To store the value of answer in another variable, drag set [▾] to [0] from the Variables palette to the scripting area. Click the arrow in the block and select the variable name. Then drag answer from the Sensing palette to the white rectangle in the block.

Example:

In the script above, the user is asked two questions: "What was the total cost of your grocery bill?" and "How any items did you buy?" When the user answers the first question, answer will refer to the total grocery bill. The code then stores this number in the variable "Total". When the user answers the second question, answer will refer the number of items bought.

The code then stores this number in the variable “NumItems”. If we had not put the user’s first answer into the variable “Total,” we would not be able to use that value later.

6. Notice that blocks for variables (

, Num1, Num2) are oval and cannot be built on to the script directly. Oval blocks such as these work differently than command blocks. We call these blocks ***reporter blocks***. Reporter blocks get their name because they report values to other blocks. We use them by putting them inside command blocks.

7. Write script that asks the user to enter a number and then another number. Store the replies as Num1 and Num2. Remember that the 

block is necessary in order to initiate the program to run, and that you need to store the first reply before asking the next question.

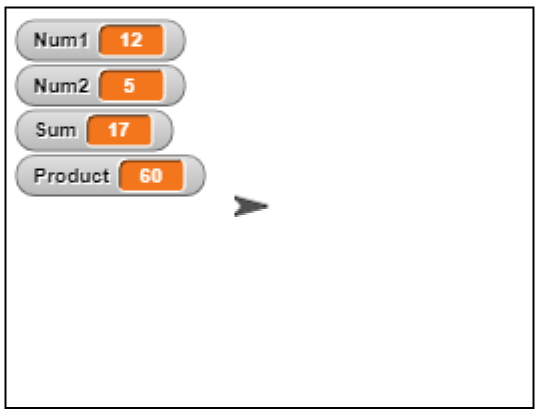
8. Now that the variables Num1 and Num2 are storing the answers that the user provided, we can use those values in mathematical operations. We will create additional variables to store the results of these operations.

9. To find the operations that we can perform in Snap!, go to the Operators palette. The most basic operations of addition, subtraction, multiplication, division, and raising something to a power are available in this palette. The corresponding blocks are shown below.

10. We can type numbers directly into the white ovals, or we can drag variable blocks into a white space. Notice that the operations are reporter blocks just like the variables. Therefore we must put these blocks into command blocks in order to use them.

11. Continue the script you started in Step 7 to find the sum and product of the two numbers the user entered. Start by creating variables named "Sum" and "Product" where you will store the answers.

12. Variable values will be 0 until the first time we run the script. Run the script and answer the questions when prompted. An example output is shown in the stage below. Check the script for errors by checking the values of Sum and Product with a calculator if necessary.



In this example, the user entered 12 as
the first number and 5 as the second number.

Another useful tool in programming is a ***conditional statement***. A conditional statement is used as a way to execute a line of code only when a given condition is satisfied. Conditional statements can be placed within loops to be executed repeatedly, or they can stand alone and perform the action just once. There are two types of conditional statements in Snap!. The ***if statement*** executes when the condition statement is true, but is bypassed when the condition is false. The ***if else statement*** executes one set of instructions when the condition is satisfied and another when the condition is not satisfied. The blocks for conditional statements (shown below) are in the Control palette.

13. Do Steps 14-16 in a new browser tab so that your work in Steps 1-12 will still be accessible during the whole class discussion.

14. Write a program that asks the user to enter an even number. If the user does not enter an even number, ask them to try again. Use a variable, NumCount, to keep track of how many numbers the user enters. After the user enters an even number, the program should indicate how many times it took the user to enter an even number.

    - Use a conditional statement with a "repeat until" loop, which can be found on the Control palette, to decide whether the user entered an even number. If the user entered an even number, congratulate them for following directions. If the user entered an odd number, tell them that they did not enter an even number and prompt them again to enter an even number.
    - Use the  block in the Operators palette to calculate the remainder when the first number in the block is divided by the second number in the block. Recall that an even number has a remainder of 0 when divided by 2.
    - Use the  block from the Operators palette together with the  block from the Looks palette to display responses to the user.

15. Run the code from Step 14 again and enter several odd numbers in a row. Is the final NumCount value correct? What is happening?

There is an easy fix for this problem: We have to initialize the NumCount variable. This means that at the beginning of the code, we set the value of NumCount to a "starting number" that will be the same, every time we run the code. Generally, initial values of variables will be 1 or 0 because these are typical starting points for counting. In the script in Step 7, initializing the variables was not a problem because all of the variables were naturally reset by the user. To avoid overlooking a variable that should be initialized, however, good programming practice is to initialize all of the variables in a program. This also ensures that if any variables are displayed in the stage, they will not contain data from a previous run of the code. Initializing of the variables should occur at the beginning of the program.

16. Adjust the script in Step 14 so that the variables are initialized, and then run the code. Do you still notice an error in the output? If so, how can you correct the error?

Challenge 3: Write a program that finds the average score that a user earned on their last five math tests. The code should ask the user to input their test scores and then the code will calculate the average. The code should then tell the user what their average on the math tests is. Save the program when you are finished and title it "Challenge 3_*yourlastname*".

Challenge 4: Write a program that chooses a secret number between 1 and 100 and asks the user to guess the number. Create a variable and use the appropriate block from the Operators palette to generate a random value for that variable. If the user's guess is lower than the random number, they should be told that their number is too low and they need to guess again. If the user's guess is higher than the random number, they should be told that their number is too high and they need to guess again. Display the user's most recent guess in the stage. Since the user is guessing a secret number, the stage should not display the secret number. When the user gets the secret number correct, the output should display the number of guesses it took and the value of the secret number. Save the program when you are finished and title it "Challenge 4_*yourlastname*".

Oh Snap! Part 3 (Student Version)

We have discussed how to draw shapes in the stage as well as how to assign variables in the code. We have looked at loops and conditional statements. Now we will combine those processes to write some more complex programs.

Challenge 5: Write a program that asks the user to enter a value for the height and width of a rectangle. The program should then draw the rectangle that the user described. Remember the size of the stage, and make sure the entire rectangle fits in the stage. State the restrictions so that the user knows what numbers they are allowed to enter for the height and width. If the user enters a number that is too large, they should be told that the rectangle cannot be drawn and to enter a new height or width (whichever applies). The program should not draw anything until the user has entered appropriate values for height and width. The program should report the area of the rectangle after drawing the rectangle. Save the program when you are finished and title it "Challenge 5_*yourlastname*".

Challenge 6: Write a program that asks the user how many hexagons they would like to draw in the stage. Display this variable in the stage. The program should then draw that number of hexagons with random locations, sizes, orientation angles, and colors. For each hexagon set the pen hue to a random number between 1 and 100 in order to get a variety of colors. Move the sprite to a random location by setting the *x*-coordinate to a random value between -200 and 200, and setting the *y*-coordinate to a random value between -175 and 175. Note: Not all of the hexagons will be fully contained in the stage. Save the program when you are finished and title it "Challenge 6_*yourlastname*".

Challenge 7: Write a program that uses repeated subtraction to divide two positive numbers provided by the user. Specifically, subtract the divisor from the dividend until the result is less than the divisor. This final result is the remainder. Use conditional statements to ensure that the user enters two positive numbers. At the end of the program, remind the user of their division problem, and report the quotient and remainder.

Informed Decisions with the Census

Lesson Overview

Brief Description: Students will analyze recent data made available by the U.S. Census Bureau to make an informed decision about what state they may want to live in.

Time: 120 Minutes

Grades: 7-9

Topic: Data analysis

Paired with: History

Materials Needed:

- Computer with internet access
- Spreadsheet application, such as Excel or Google Sheets
- Student handouts (introduction, worksheet)

In this lesson, students will obtain real-life data from the U.S. Census Bureau and then interpret and summarize their results. Students will organize data in tables, line graphs, and bar charts. They will compare demographic data from three states, and will use their analysis to decide what state they would most like to live in. They will write a summary of these results and explain how they arrived at their decision.

In order for students to accurately create the graphs necessary for the lesson, they will have to have a strong understanding of some of the graphical functions of spreadsheet software, such as Excel or Google Sheets. Teachers should present an overview of the spreadsheet application that students will use, or direct students to online tutorials. Students will need to know how to:

- Enter data into columns;
- Create line graphs and bar charts;
- Edit graph labels;
- Format the various elements of a graph;
- Copy and paste a graph into a word processing application, such as Word or Google Docs.

Students should have an opportunity to explore the Census Bureau website with their groups first. The teacher should follow up with an example showing how to filter the data to ensure that students will be able to complete the lesson.

Teachers should provide students with a hard copies of the lesson worksheets, so that they can answer the questions and complete the tables with pencil and paper. Students should also submit their graphs and final analysis electronically, as a Word document or Google Doc for example. Teachers should closely monitor students' progress through the lesson, as some students will need help to locate the correct tables on the Census website.

Resources:

"About [the 2020 Census]." The United States Census Bureau, 21 Apr. 2020, https://www.census.gov/programs-surveys/decennial-census/2020-census/about.html.

Constitution of the United States. National Archives, https://www.archives.gov/founding-docs/constitution-transcript.

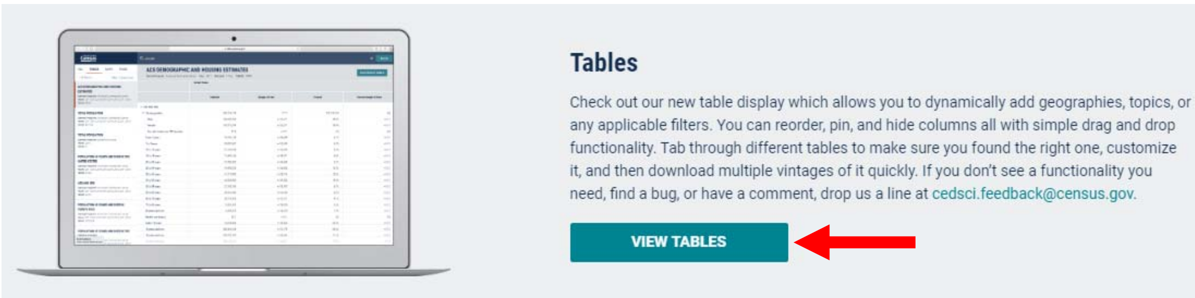
"Poverty Guidelines." U.S. Department of Health and Human Services, 17 Jan. 2020, https://aspe.hhs.gov/poverty-guidelines.

United States Census Bureau, https://data.census.gov/cedsci/.

Informed Decisions with the Census: Exploration (Teacher Version)

The paragraph that students must read first is with the student materials at the end of this section of the essay.

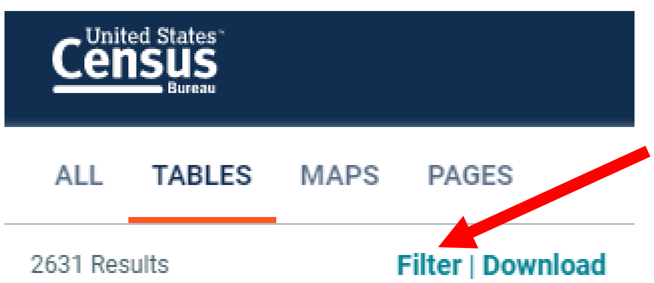
Go to https://data.census.gov/. Once on the website, click "View Tables" in order to get organized data that was collected through the Census.



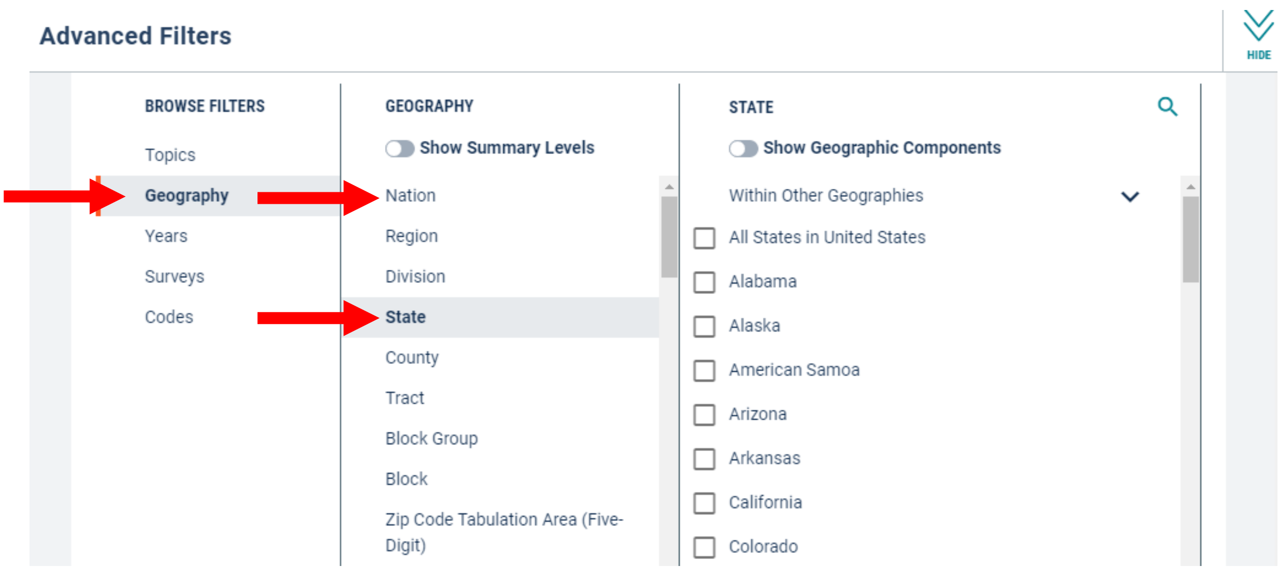
What three states will you research?

The sample work in the teacher notes is based on data from Alaska, Colorado, and Maine, and was downloaded in July 2020.

The first table you see on the Census website will most likely not relate to the lesson. **Disregard this table.** You will need to filter the data by the states you have chosen. Click "Filter" on the left side of the browser window.



Once you are in the “Filter” tab, choose “Geography,” then “State,” and finally select your three states of interest. Also, under “Nation” in the Geography category, check “United States” so you can compare each of your choices to the national data.

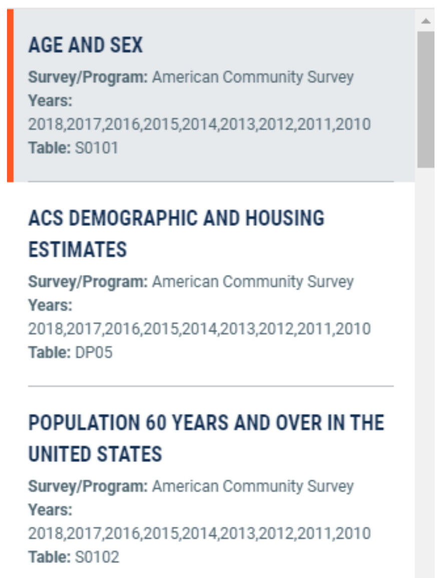


You now have experience in setting the Geography filter. To view specific data for the lesson, you will also have to filter by Topics.

In the Topics filter, there are ten categories that each have their own subgroups. Take a moment to explore these categories to see how information regarding your three states can be further broken down. To close the filter, click 

in the top right corner. The tables that pertain to the filter are listed on the left side of the window. Click on the title of a table in order to see the data in that table. Notice that we can scroll down to access more tables.

The example below shows some table options if the topics filter is "Populations and People" … "Age and Sex."



Each table provides a different breakdown of information. Take a few moments to explore the tables that are available from different filter choices. When you click on a table, you should notice a "TableID" in the title area for the table. You will need to record this ID number for the tables that you use in this lesson.

Choose categories in the Topics filter to answer the questions below. You may have to look at several tables in order to find the answers you need. If you have to scroll horizontally in order to see all of the information, you can highlight rows of the table so that you do not lose your place as you scroll.

1. Use a table to find the most recent U.S. population based on the Census. Give the name and ID number of the table you used.

   Population: 327,167,439 people   Table: S0101: Sex and Age (2018)"

2. For each of your three states, list the most recent population. Give the name and ID number of the table you used.

   a) State: Alaska   Population: 737,438 People

   b) State: Colorado   Population: 5,695,564 People

   c) State: Maine   Population: 1,338,404 People

   Table: S0101: Age and Sex (2018)

3. Navigate the Census website to find answers to the questions listed in the table on the next page. You should consider a wide variety of characteristics in making your decision, so there is space in the table for three questions based on your interests.

One of the questions on the next page is about the poverty line. For your information, the table below gives more details about the poverty line for the 48 contiguous states and the District of Columbia as of January 15th, 2020. [1]

For families/households with more than 4 persons, add $4,480 for each additional person.			
1	$12,760	3	$21,720
2	$17,240	4	$26,200

[1] U.S Federal Poverty Guidelines: https://aspe.hhs.gov/poverty-guidelines.

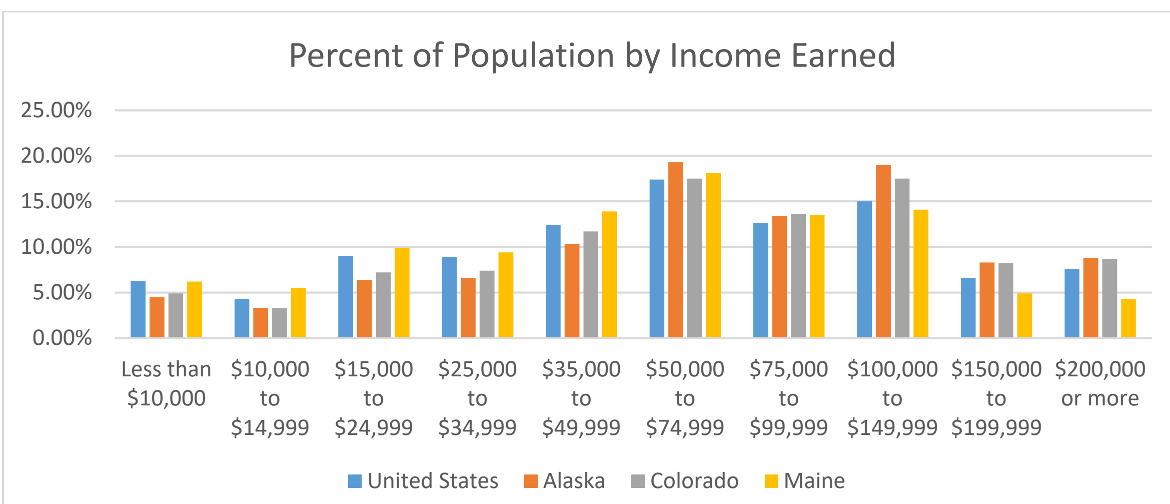
Questions	United States	State 1: Alaska	State 2: Colorado	State 3: Maine	Table Used
What percent of the population is currently unemployed?	4.9%	6.8%	3.9%	3.5%	S2301: Employment Status (2018)
What is the median household income?	$61,937	$74,346	$71,953	$55,602	S2503: Financial Characteristics (2018)
What percent of people aged 25 and older has at least completed high school?	88.3%	93.3%	93%	91.9%	S1501: Educational Attainment (2018)
What percent of people live below the poverty line?	13.1%	10.9%	9.6%	11.6%	S1701: Poverty Status in the Last 12 Months (2018)
Your Question: What is the median value of a home in your state?	$229,700	$276,100	$373,300	$197,500	DP04: Selected Housing Characteristics (2018)
Your Question: What percent of people are in my age range?	7.1%	8.5%	7.9%	6.0%	S0101: Age and Sex (2018)
Your Question: What percent of people who are enrolled in school attends public school?	83.7%	86.5%	86.6%	81.4%	S1401: School Enrollment (2018)

You will answer the next questions digitally. To easily access the tables in each part of Question 4, use the search bar at the top of the Census website with the icon and enter the ID number. You may have to reset the Geography filter after using the search feature.

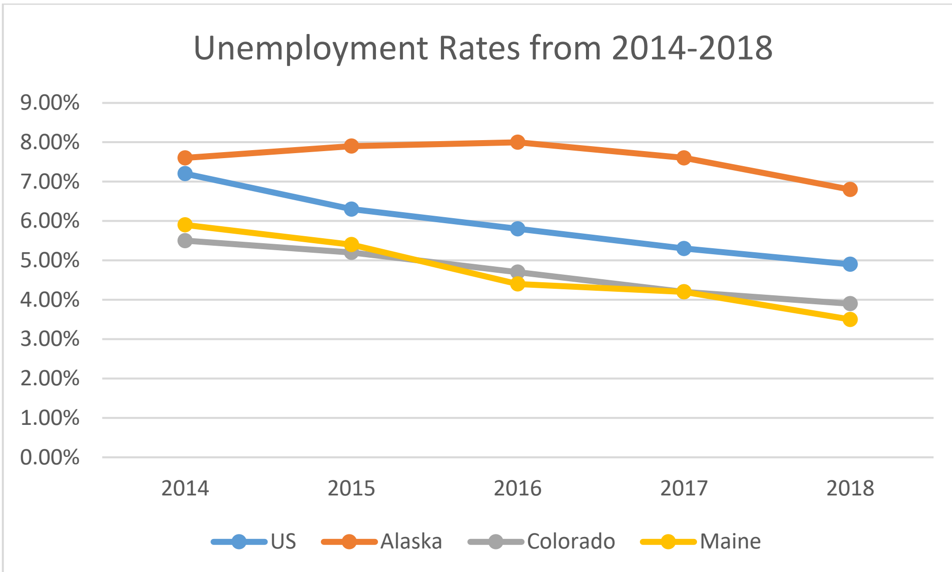
4. Collect more data from the Census website and use a spreadsheet to create graphs to help you interpret information about your states. Use the chart options in the spreadsheet application to customize the color and labels for the graphs, in a way that best fits the data (or the question you are answering). Copy and paste the graphs into an electronic document according to your teacher's instructions. Before you look at electronic graphs, read the following to remind yourself about two of the different types of graphs that can be used to display data.

    - A bar chart is a graph that presents data in vertical or horizontal rectangles representing different categories.
    - Line graphs are generally used to show how data changes over a period of time and can also be used over the same period of time to display more than one category of data.

    a) Use the table "S1901: Income in the Past 12 Months" with the most recent information to make a bar chart for the United States and each of your states, showing the percent of the population in each income bracket. Display all of the information on a single graph.



b) Use the table "CP03: Comparative Economic Characteristics" with the most recent information to make a line graph for the United States and each of your states, showing the percent of unemployment. Display all of the information on a single graph.



5. Analyze the data from the table in Question 3 and the graphs you created in Question 4. Which state would you like to live in? Why? Support your answer with specific details.

Based on all of the data I have gathered, I want to live in Alaska. There were great qualities about all of the states I researched, but Alaska stood out to me. A large portion of Alaska's population (93.3%) has at least a high school education, which is above the U.S. average. This means that the people living in Alaska value their education and I want to be surrounded by educated people. The majority of children in Alaska (86.5%) attend public school. This is good for me because I would like to be a public school teacher. It is somewhat concerning that the unemployment rate is higher than in the other states I researched, but there has been a steady decline in unemployment numbers in Alaska over the last four years. Of the entire population, the percent of people that fall in my age range is higher than any of the other states I researched. This tells me that if I were to move there, I should have an easy time meeting people close to my age. Alaska also has the highest percent of the states I researched for people who walk to work as their commute. Walking to work would be a great way to stay healthy, so if I lived in an area close to my work I would definitely join this statistic. Lastly, the income for Alaska seems to be consistent with the cost of living. Alaska has the highest percentage of people earning between $100,000 and $149,999 and the median cost of a home is just under $300,000. Because Alaska has so many people in this earnings range, they also have a higher median salary per household. Based on everything I have looked at, Alaska seems like the state for me.

6. The information you just analyzed was based on the most recent Census data results. What other information might be useful if you want to decide which state to live in? Consider factors that might not be reported in the Census data as well.

   Another thing to consider is the trends in the data over the years. By looking at how data has changed over the years, I would have a better picture of what to expect from my state in the future. Factors that would impact my decision that are not reported in the Census would be proximity to family, national park and bike path access, climate, and food culture specific to the state.

Informed Decisions with the Census: Introduction (Student Version)

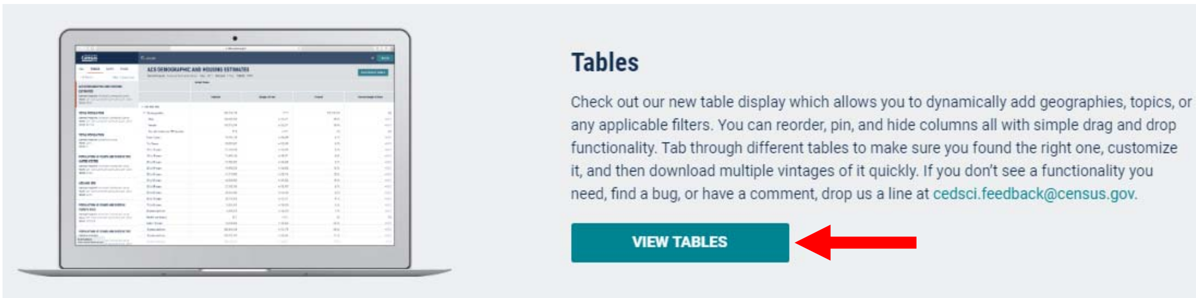
The U.S. Census is a decennial survey, meaning that it is conducted every ten years. The purpose of the Census is to update data on certain topics in the U.S. The act of completing the Census is mandated by the U.S. Constitution, which reads in Article 1, Section 2: “Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers .… The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct.”

When the Constitution was written, the founders decided that political power would be based on population rather than land or wealth. The Census creates a way to gather information on population of the entire United States. This information then determines how many seats each state has in the U.S. House of Representatives. Another major use of the Census is to allocate funding from the federal government. The funding is used for schools, hospitals, roads, public works, and other vital programs that help keep communities growing and safe. Jobs are created through responses to the Census because businesses decide where they need to build factories and offices. Home developers also use this information to see what areas can be renovated so growth can continue. The decisions that are made based on the Census impact the world around us.

In this activity, you will look at some Census data for three states and use it to make a decision about which state you may want to live in. You will also keep track of data for the United States as a whole in order to be able to make comparisons. You will look at a variety of categories such as employment, income, housing, education and more. As you work through this activity, keep in mind the types of information that would help influence your decision. You may work on this assignment in groups of up to three students. You will explore the data provided by the U.S. Census Bureau at https://data.census.gov/.

Informed Decisions with the Census: Exploration (Student Version)

Go to https://data.census.gov/. Once on the website, click “View Tables” in order to get organized data that was collected through the Census.



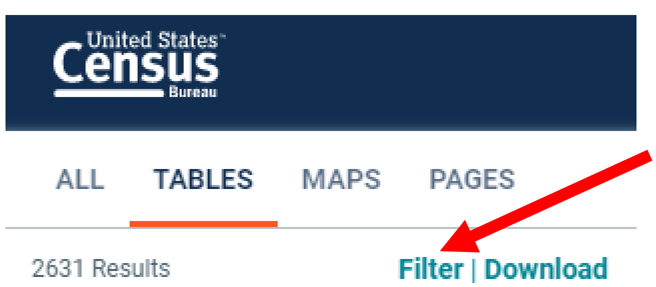
What three states will you research?

1.

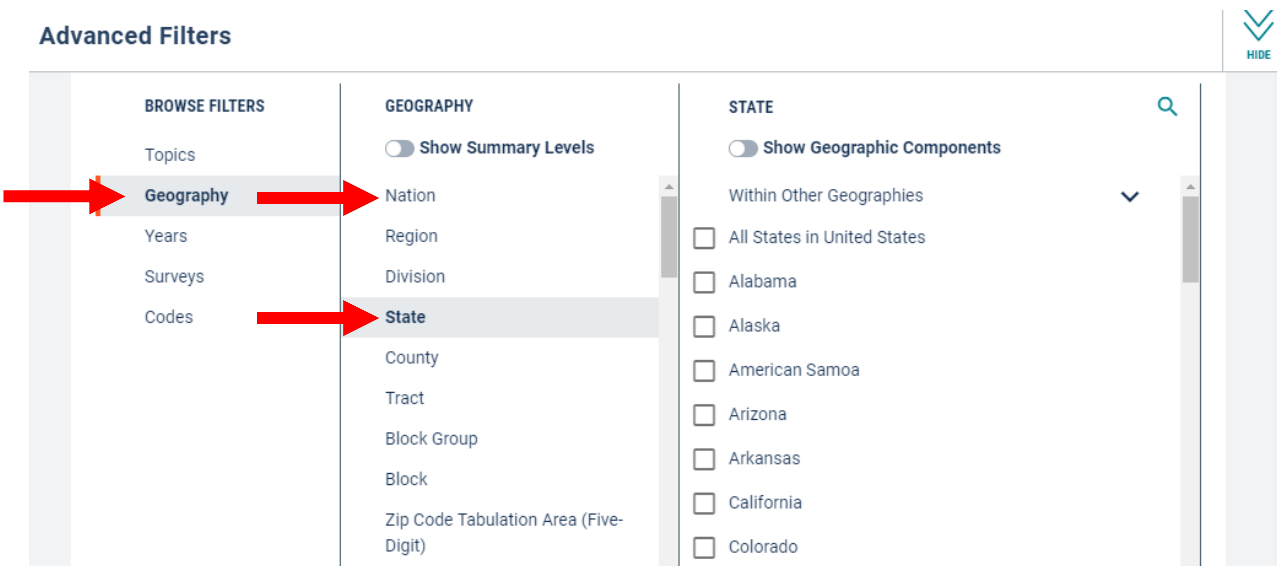
2.

3.

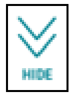
The first table you see on the Census website will most likely not relate to the lesson. **<u>Disregard this table.</u>** You will need to filter the data by the states you have chosen. Click “Filter” on the left side of the browser window.



Once you are in the "Filter" tab, choose "Geography," then "State," and finally select your three states of interest. Also, under "Nation" in the Geography category, check "United States" so you can compare each of your choices to the national data.

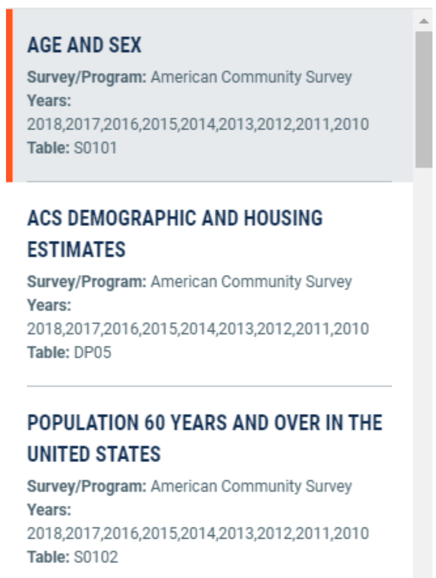


You now have experience in setting the Geography filter. To view specific data for the lesson, you will also have to filter by Topics.

In the Topics filter, there are ten categories that each have their own subgroups. Take a moment to explore these categories to see how information regarding your three states can be further broken down. To close the filter, click 

in the top right corner. The tables that pertain to the filter are listed on the left side of the window. Click on the title of a table in order to see the data in that table. Notice that we can scroll down to access more tables.

The example below shows some table options if the topics filter is “Populations and People” … “Age and Sex.”



Each table provides a different breakdown of information. Take a few moments to explore the tables that are available from different filter choices. When you click on a table, you should notice a “TableID” in the title area for the table. You will need to record this ID number for the tables that you use in this lesson.

Choose categories in the Topics filter to answer the questions below. You may have to look at several tables in order to find the answers you need. If you have to scroll horizontally in order to see all of the information, you can highlight rows of the table so that you do not lose your place as you scroll.

1. Use a table to find the most recent U.S. population based on the Census. Give the name and ID number of the table you used.

   Population: Table:

2. For each of your three states, list the most recent population. Give the name and ID number of the table you used.

   a) State: Population:

   b) State: Population:

   c) State: Population:

      Table:

3. Navigate the Census website to find answers to the questions listed in the table on the next page. You should consider a wide variety of characteristics in making your decision, so there is space in the table for three questions based on your interests.

One of the questions on the next page is about the poverty line. For your information, the table below gives more details about the poverty line for the 48 contiguous states and the District of Columbia as of January 15th, 2020. [1]

For families/households with more than 4 persons, add $4,480 for each additional person.			
1	$12,760	3	$21,720
2	$17,240	4	$26,200

[1] U.S Federal Poverty Guidelines: https://aspe.hhs.gov/poverty-guidelines.

Questions	United States	State 1:	State 2:	State 3:	Table Used
What percent of the population is currently unemployed?					
What is the median household income?					
What percent of people aged 25 and older has at least completed high school?					
What percent of people live below the poverty line?					
Your Question:					
Your Question:					
Your Question:					

You will answer the next questions digitally. To easily access the tables in each part of Question 4, use the search bar at the top of the Census website with the icon and enter the ID number. You may have to reset the Geography filter after using the search feature.

4. Collect more data from the Census website and use a spreadsheet to create graphs to help you interpret information about your states. Use the chart options in the spreadsheet application to customize the color and labels for the graphs, in a way that best fits the data (or the question you are answering). Copy and paste the graphs into an electronic document according to your teacher's instructions. Before you look at electronic graphs, read the following to remind yourself about two of the different types of graphs that can be used to display data.

   - A bar chart is a graph that presents data in vertical or horizontal rectangles representing different categories.
   - Line graphs are generally used to show how data changes over a period of time and can also be used over the same period of time to display more than one category of data.

   a) Use the table "S1901: Income in the Past 12 Months" with the most recent information to make a bar chart for the United States and each of your states, showing the percent of the population in each income bracket. Display all of the information on a single graph.

   b) Use the table "CP03: Comparative Economic Characteristics" with the most recent information to make a line graph for the United States and each of your states, showing the percent of unemployment. Display all of the information on a single graph.

5. Analyze the data from the table in Question 3 and the graphs you created in Question 4. Which state would you like to live in? Why? Support your answer with specific details.

6. The information you just analyzed was based on the most recent Census data results. What other information might be useful if you want to decide which state to live in? Consider factors that might not be reported in the Census data as well.

Westward Expansion and Graph Theory

Lesson Overview

Brief Description: Students will review westward expansion of the United States and make connections to some basics of Graph Theory.

Time: 120 Minutes

Grades: 9-12

Topic: Graph Theory

Paired with: History

Materials:

- Student handouts (westward expansion history review and map, guided notes on Graph Theory, and follow-up worksheet on Graph Theory)
- Coloring supplies
- Vocabulary treasure hunt or "I Have…, Who Has…?" game (optional)

In this lesson, students will explore some basics of Graph Theory in the context of westward expansion of the United States. They will label a map of the United States connected to westward expansion and then from that map create a dual graph, which they will use to answer questions in Graph Theory.

The first part of the lesson is intended as a review the history of westward expansion. Students will read a description of key events of westward expansion and label a corresponding map. In the second part of the lesson, the teacher will lead the class through the completion of the guided notes on Graph Theory. In the final part of the lesson, students will apply their knowledge of Graph Theory to answer follow-up questions regarding the westward expansion map. In this part, students will create a dual graph as well as color the westward expansion map based on Graph Theory concepts.

It might be helpful to add additional time to the lesson to review the Graph Theory vocabulary after completing the guided notes. One way to do this is to set up a treasure hunt around the room: Hang pieces of paper around the room; each paper should contain a word on the top half and a definition of a different word on the bottom half. Students read the word and then find the correct definition on another paper. This new paper then has a different vocabulary word for which students must find the definition. This activity is best done in pairs.

Another option for full-class review is to play "I Have…, Who Has…?." In this game, each student gets a vocabulary word and a definition of a different vocabulary word. The teacher starts by reading a definition. The student who has the word that matches the definition the teacher reads will say "I have ___________" by filling in the correct vocabulary word. Then they will verbally fill in "Who has __________?" with their definition. Students continue filling their words and definitions until all vocabulary words have been used.

The guided notes contain some Graph Theory concepts that are not included in the follow-up activity related to the westward expansion map. The additional information will give students a better sense of some aspects of Graph Theory, but can be omitted for time purposes. The topics of paths, circuits, complete graphs, bipartite graphs, and complete bipartite graphs can be removed from the lesson and students will still be able to do the rest of the activity successfully.

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"Treaty of Paris." 13 Nov. 2009, https://www.history.com/topics/american-revolution/treaty-of-paris.

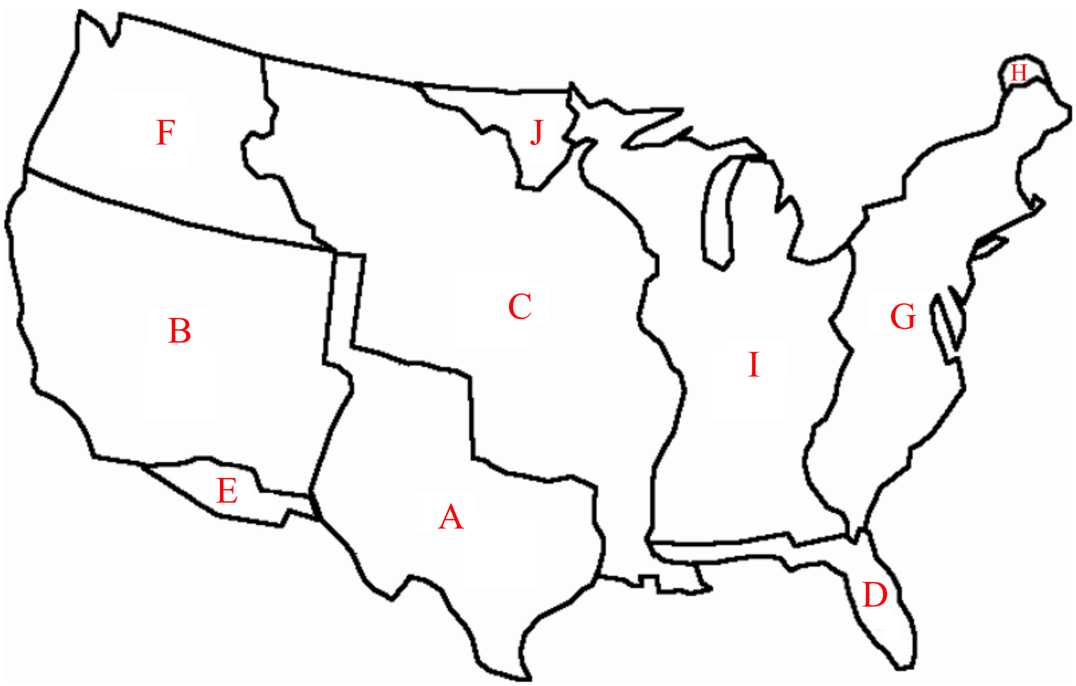
"U.S.-Canadian Border Established." 9 Feb. 2010, https://www.history.com/this-day-in-history/u-s-canadian-border-established.

Thau, Richard. "Westward Expansion," Seaford Union Free School District, 2002, https://www.seaford.k12.ny.us/Page/9701. (westward expansion map)

Westward Expansion and Graph Theory: United States Map (Teacher Version)

The paragraph that students must read is with the student materials at the end of this section of the essay.

Use the information from the paragraph on the westward expansion of the U.S. to label the map below. Put the corresponding letter in each region so it is clear and easy to read. You may work with the people in your group to make sure the labels all agree.



A: Texas Annexation (1845)

B: Mexican Cession (1848)

C: Louisiana Purchase (1803)

D: Florida Treaty (1819)

E: Gadsden Purchase (1853)

F: Oregon Treaty (1846)

G: 13 Original Colonies

H: Webster-Ashburton Treaty (1842)

I: Treaty of Paris (1783)

J: Treaty of 1818

Westward Expansion and Graph Theory: Guided Notes (Teacher Version)

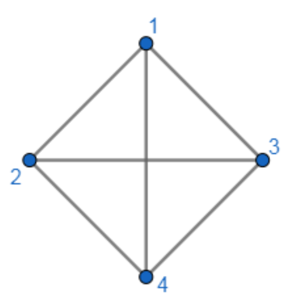
We will come back to the labeled map after introducing some topics from Graph Theory. Graph Theory is the study of ***graphs***, which are mathematical structures used to model pairwise relations between objects. This type of graph consists of vertices and edges. We will define these shortly! The graphs in Graph Theory are not the same as graphs of equations or functions that you have previously seen in math classes. Please follow along to fill in the guided notes below.

Graph: consists of a set of vertices (V) and edges (E) joining the vertices; sometimes denoted as $G=(V,E)$.

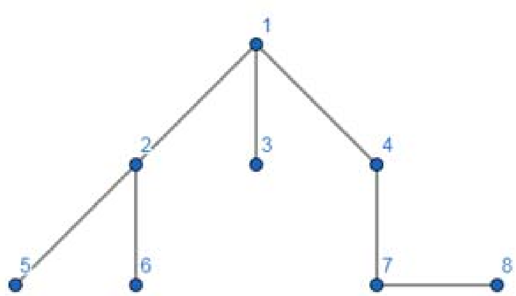
Vertex: the fundamental unit from which graphs are formed (plural ***vertices***); also called a ***node***.

Edge: gives the relationship between two vertices; usually depicted by a line segment or curve that connects two vertices.

Examples:



G_1



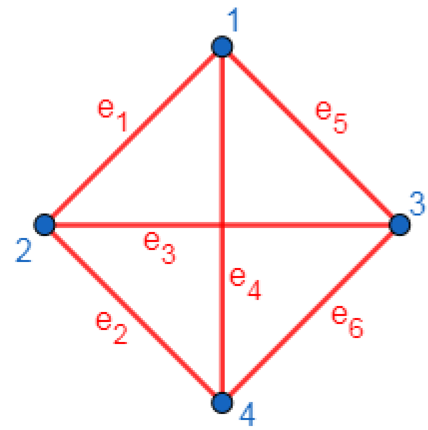
G_2

The vertex sets are $V(G_1)=\{1,2,3,4\}$ and $V(G_2)=\{1,2,3,4,5,6,7,8\}$.

The edge sets are $E(G_1)=\{(1,2),(1,4),(1,3),(2,3),(2,4),(3,4)\}$ and $E(G_2)=\{(1,2),(1,3),(1,4),(2,5),(2,6),(4,7),(7,8)\}$.

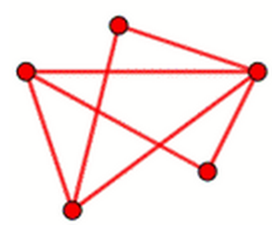
Note that edges are not ordered pairs: $(1,2)$ is the same edge as $(2,1)$.

Edges can also be labeled directly, without referring to vertices, as in the example below.



Example: Draw a graph that has 5 vertices and 7 edges.

Answers will vary.



It is possible to have a vertex without an edge, but it is not possible to have an edge without a vertex!

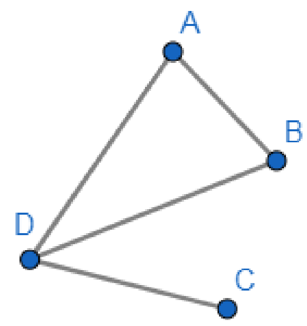
Two vertices are said to be adjacent if they have an edge between them.

An edge is said to be incident to a vertex and the vertex is incident to the edge if the vertex is one of the endpoints of the edge.

Loop: an edge that connects a vertex to itself.

Degree: number of edges incident to (touching) a node. Loops are counted twice when determining the degree of a vertex, because if we travel along a loop, we encounter the endpoint twice.

Question: For the graph below, what are $\deg(A)$, $\deg(B)$, $\deg(C)$, and $\deg(D)$?



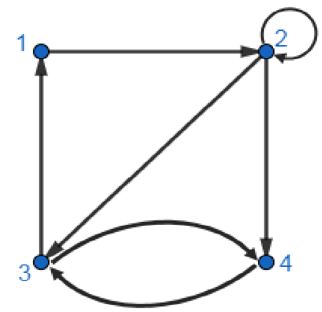
$\deg(A)=2$, $\deg(B)=2$, $\deg(C)=1$, $\deg(D)=3$

Question: Draw a loop on vertex *C* in the graph above. What is the new degree of *C*?

$\deg(C)=3$

Directed Graph: a graph in which the edges may have direction; also called a digraph.

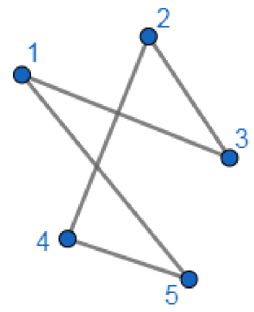
Example:



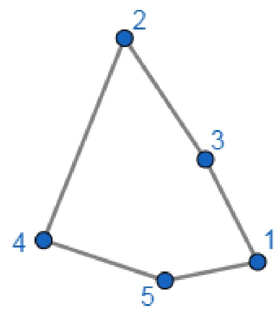
Planar Graph: a graph that can be drawn in a plane such that edges do not intersect except at the vertices.

Plane Graph: a version of a planar graph, drawn so that no two edges intersect except at the vertices.

Example:



Planar graph

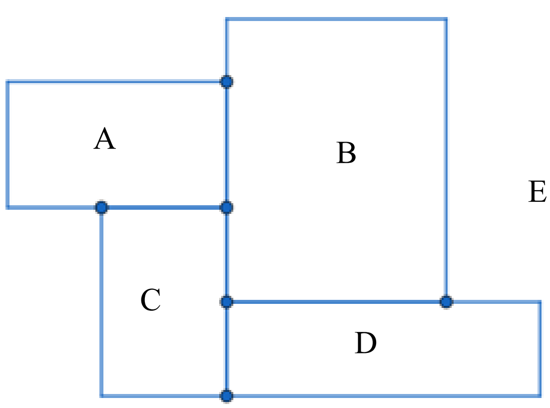


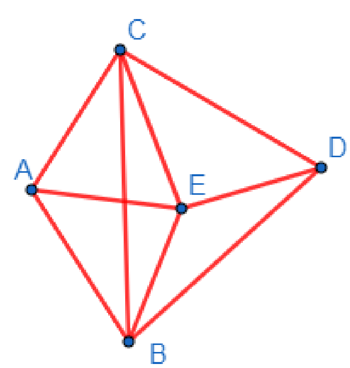
The same graph, drawn as a plane graph

Note: The depiction of the graph is generally irrelevant. The two images above represent "the same" graph because all of the relationships between the vertices are preserved despite the change in appearance.

Dual Graph: The dual graph *D* of a plane graph *G* has a vertex for each region defined by *G*, including the "outside" region. The edges in *D* represent the borders between corresponding regions in *G*.

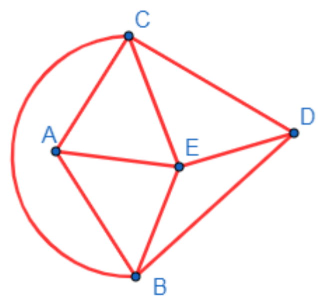
Question: In the graph on the left, the regions are *A*, *B*, *C*, *D*, and *E*. Each of these regions will become a vertex in the dual graph drawn on the right. There will be an edge between any two vertices that represent regions that share a border.





Question: Is the dual graph a planar graph? That is, can you move the vertices and/or curve the edges (if necessary) so that none of the edges cross?

Yes:



Euler's Formula: Let G be a connected plane graph, which separates the plane into r regions (including the region that surrounds the graph). Let e be the number of edges and v be the number of vertices in G. Then $r-e+v=2$.

Question: Compute $r-e+v$ for the plane version of the dual graph in the last question. Is Euler's formula satisfied?

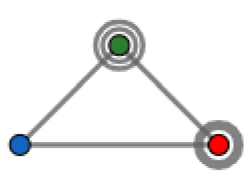
Yes, $6-9+5=2$.

Graph Coloring: a way to assign colors to the vertices of a graph so that adjacent vertices have different colors.

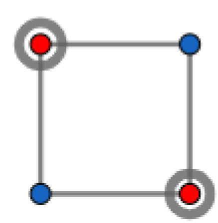
Chromatic Number: the minimum number of colors needed for a coloring of a graph.

Examples:

Chromatic Number is 2

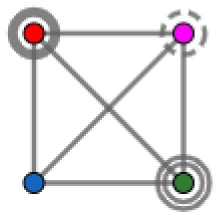


Chromatic Number is 3

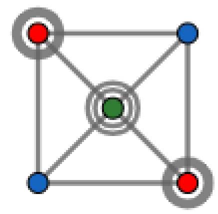


Chromatic Number is 2

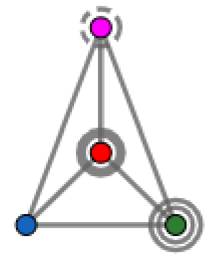
Question: What are the chromatic numbers of the following graphs?



Chromatic Number is 4



Chromatic Number is 3

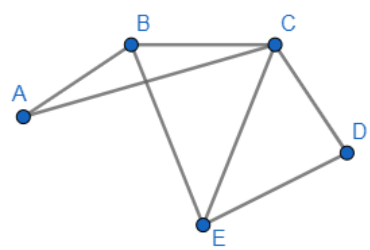


Chromatic Number is 4

Four Color Theorem: The chromatic number of a planar graph is no greater than 4.

Path: a sequence of edges such that any two consecutive edges have a common vertex.

Question: In the following graph, $\{(A,B),(B,E)\}$ is a path from A to E of length 2. Name at least two other paths from A to E.

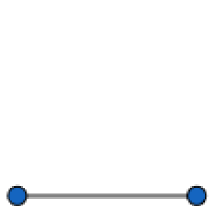


Two such paths are: $\{(A,B),(B,C),(C,E)\}$ (length 3) and $\{(A,B),(B,C),(C,D),(D,E)\}$ (length 4).

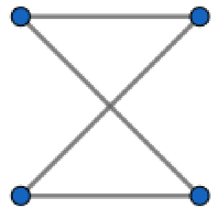
Circuit: a path that starts and ends at the same vertex; also called a ***closed path***.

Connected Graph: a graph in which any two vertices are joined by a path.

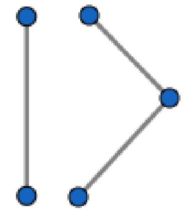
Examples:



Connected



Connected

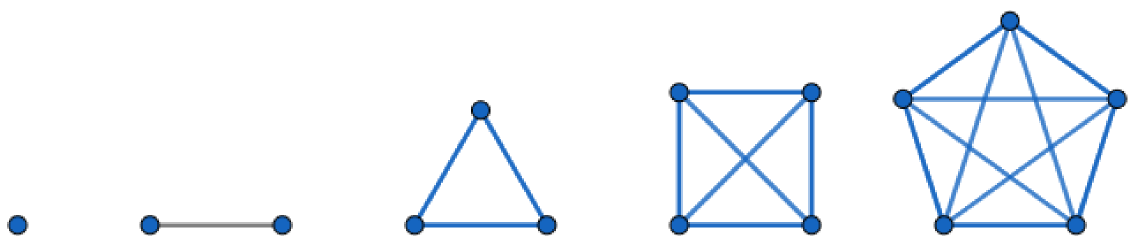


Not Connected

Complete Graph: a graph in which there exists an edge between each pair of vertices.

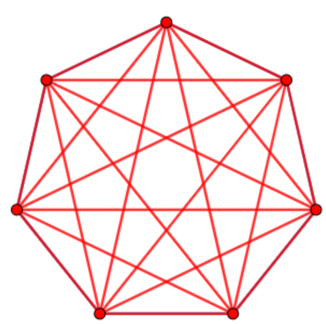
K_n **:** denotes a complete graph with n vertices.

Examples:



K_1 K_2 K_3 K_4 K_5

Question: Draw a complete graph with 7 vertices:



A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

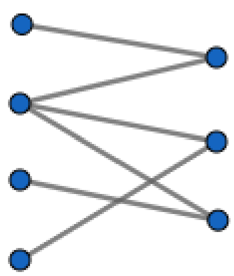
Question: Verify that the number of edges in the complete graph with 7 vertices agrees with the formula.

$$\frac{7(6)}{2} = 21$$

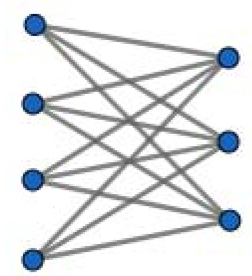
Bipartite Graph: a graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2, such that there are no edges between the vertices in V_1 and no edges between the vertices in V_2.

Complete Bipartite Graph: a graph in which each pair of vertices (v_i, v_j) in $V_1 \times V_2$ is connected by an edge.

Example: In both graphs below, V_1 consists of the vertices on the left and V_2 consists of the vertices on the right.



Bipartite Graph



Complete Bipartite Graph

Westward Expansion and Graph Theory: Worksheet (Teacher Version)

Now that we know some basics of Graph Theory, we will apply the ideas of dual graphs and graph coloring to the westward expansion map.

1. You have already labeled all but one of the regions on the map. Label the outside region *K*. How many regions does the map have? Call this number *r*.

   $r =$ 11 regions

2. Identify a vertex as a point on the map where three or more regions meet. Draw and label all of the vertices so that the first vertex you count is v_1, the second vertex you count is v_2, and so on. How many vertices are there? Call this number *v*.

   $v =$ 17 vertices

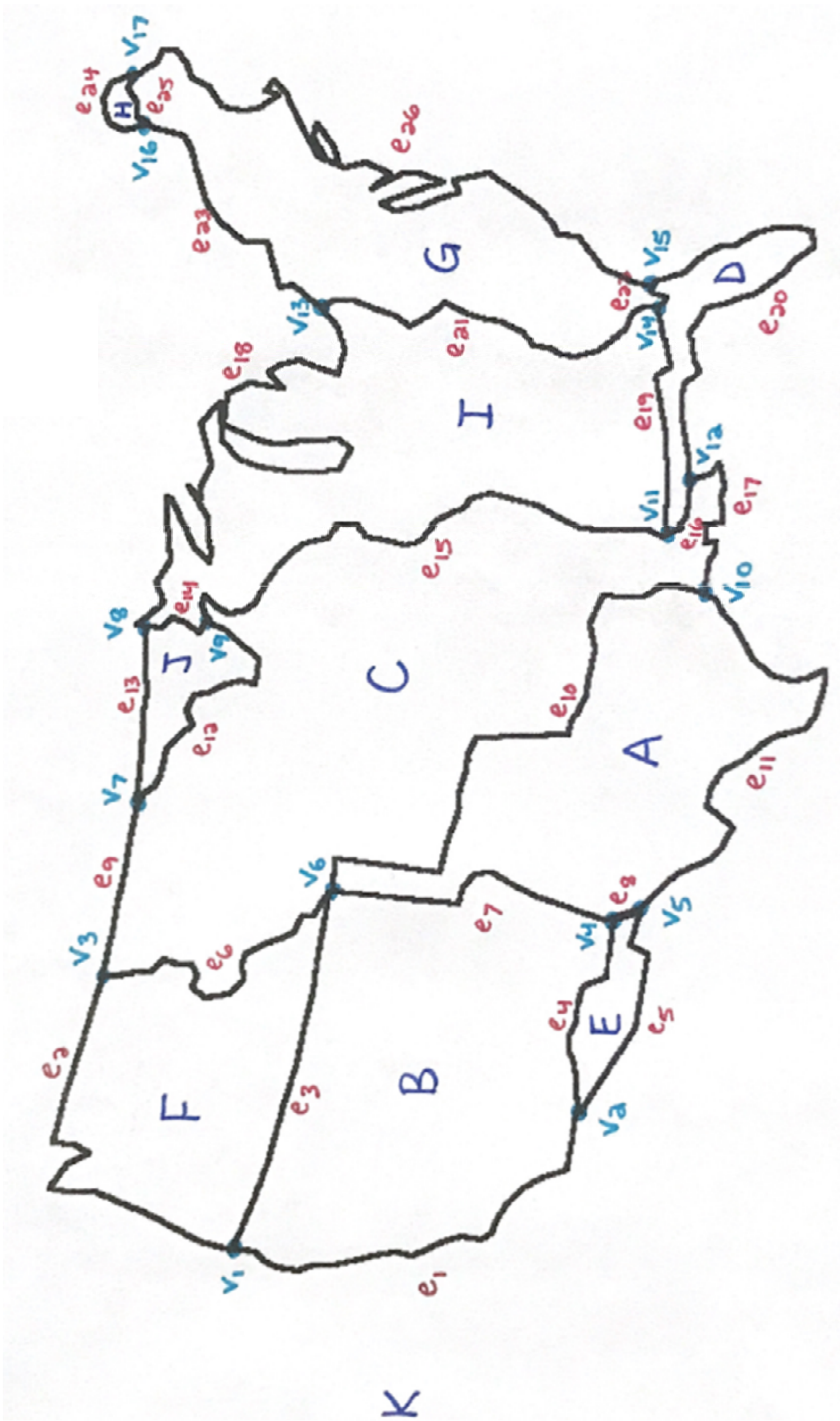
3. Identify an edge as the segment that connects two vertices on the map. Label all of the edges so that the first edge you count is e_1, the second edge you count is e_2, and so on. How many edges are there? Call this number *e*.

   $e =$ 26 edges

4. Compute the value of $r - e + v$. Is Euler's formula satisfied?

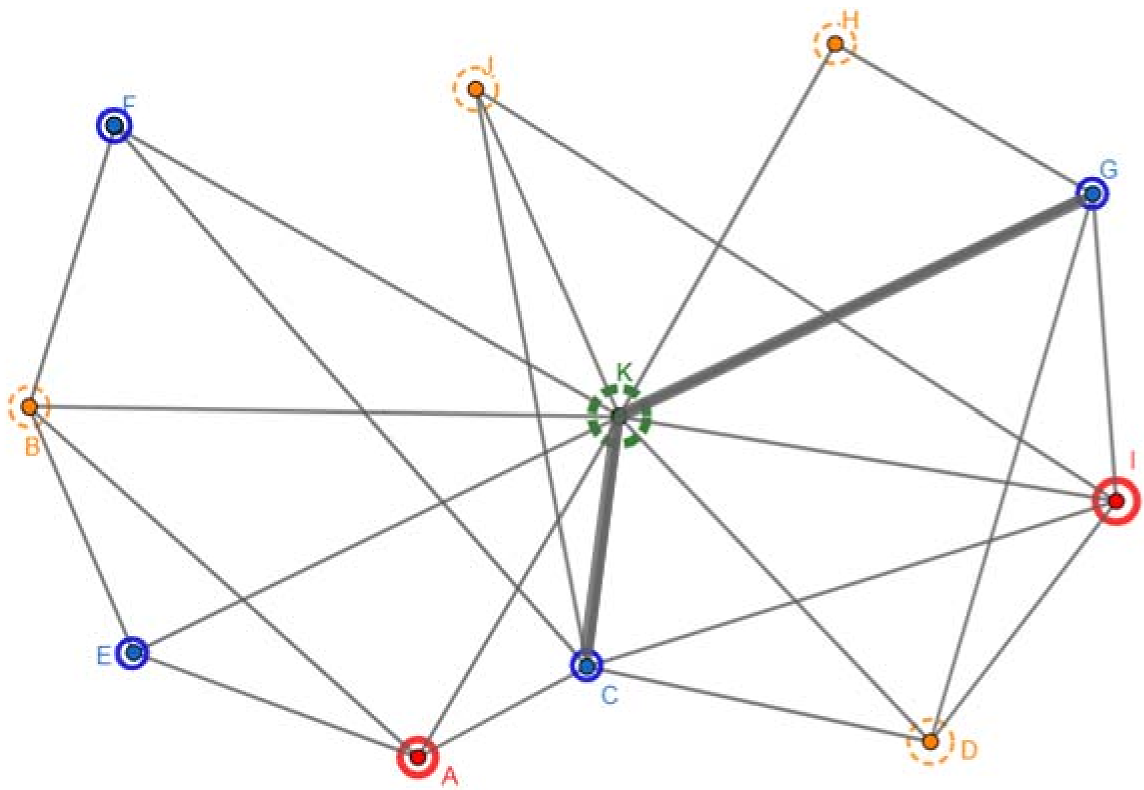
   $11 - 26 + 17 = 2$; yes, Euler's formula is satisfied.

One possible labeling of the map for Questions 2 and 3:



5. Now create a dual graph of the westward expansion map. The letters that you used to label the regions in the map will be the labels for the vertices of the dual graph. Refer to the outside region as *K*. Using the template below, draw the edges that connect the vertices in the dual graph. Since every region is touching *K*, the graph will be easier to draw if we put the vertex for *K* in the center of the diagram. Note that the location of each vertex does not matter. Recall that the important thing is the edges that connect the vertices: if two vertices are connected by an edge in one version of the graph, they must be connected by an edge in every version of the graph.

   Dual graph with coloring, for Questions 5 and 6. Note that the thick grey lines represent double edges.



The number of edges in the dual graph should match the number of edges on the map. Check now that you did not miss any edges.

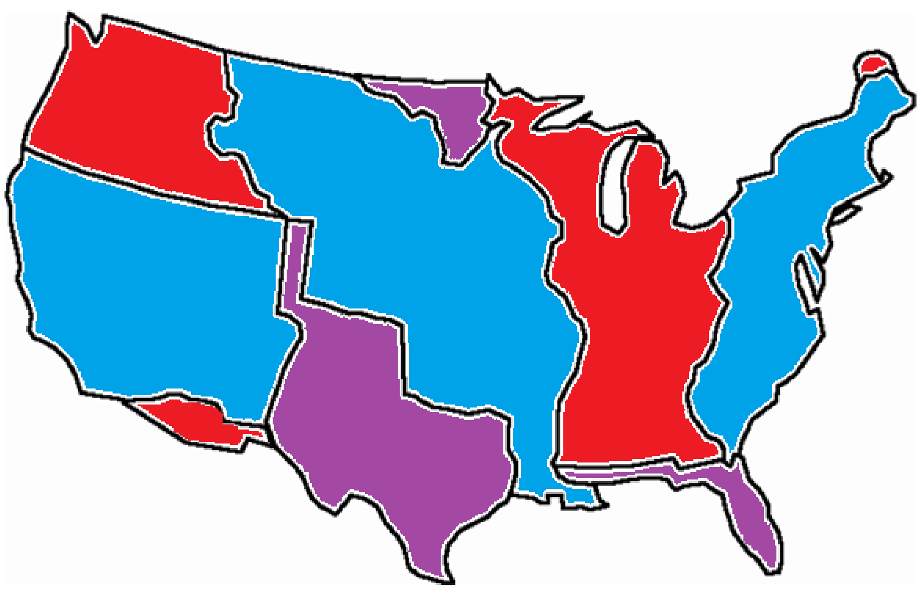
6. Color the dual graph by circling each vertex with a color. No two vertices that share an edge can be the same color. Use as few colors as possible. What is the chromatic number of the dual graph? Remember that the chromatic number is the <u>minimum</u> number of colors needed for a coloring of a graph.

   The chromatic number is 4.

7. Is the dual graph that you drew a plane graph?

   Most likely, students will connect the edges as above and therefore will not have a plane graph.

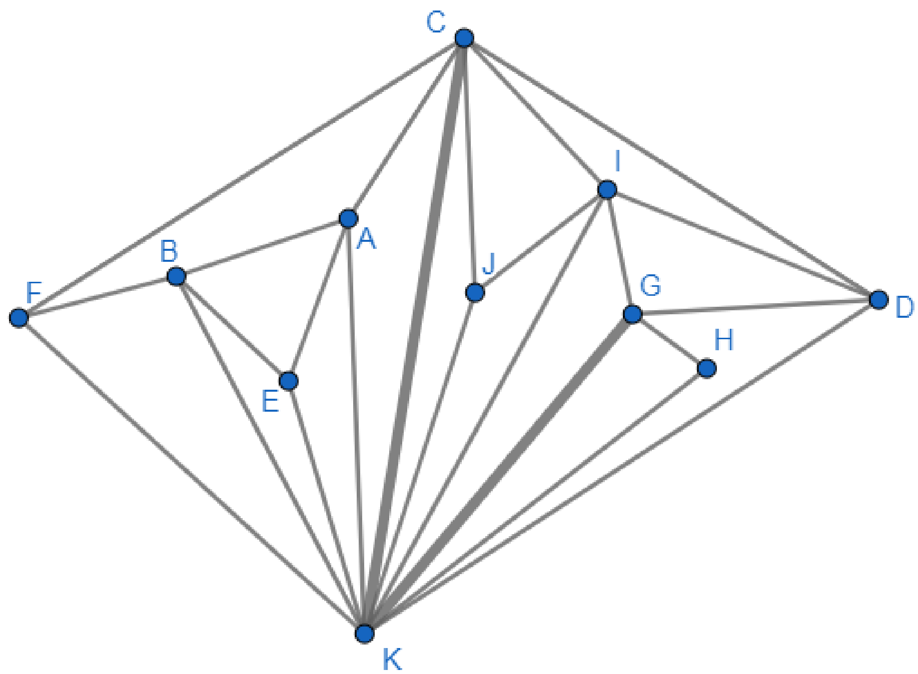
8. Color the regions of the westward expansion map in such a way that no two regions that share a border are the same color. Do this with as few colors as you can. Does the chromatic number of the map match the chromatic number of the dual graph? Explain why or why not.



   No, the map is three-colorable, but the dual graph is not. The outside region of the map is not colored, although it is used as a vertex on the dual graph. If vertex *K* were removed from the dual graph, the graph would then be three-colorable. Note: If we consider the background of the page as the color for region *K*, then the map does require four colors.

Challenge: Is the dual graph from Question 5 a planar graph? That is, can you recreate the graph by moving vertices or redrawing (possibly curved) edges so that none of the edges cross? If possible, draw this plane graph.

This is one possible solution. Note that the thick grey lines represent double edges.



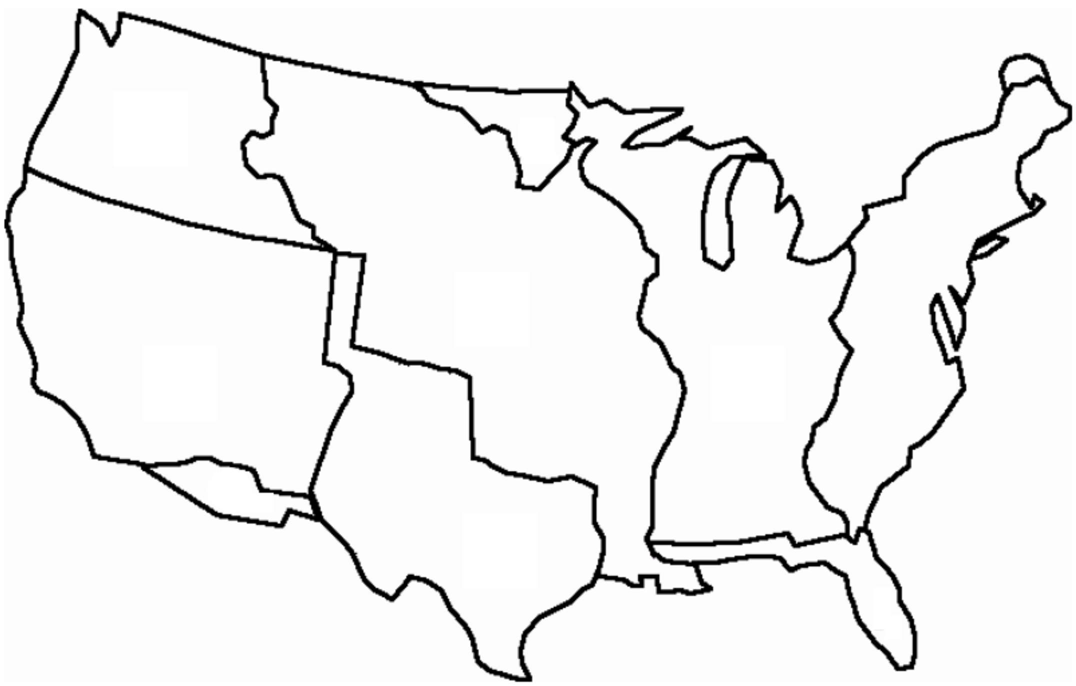
Westward Expansion and Graph Theory: History (Student Version)

In this lesson, we review the westward expansion of the U.S. and then connect it to an area of Mathematics called Graph Theory. Begin by reading the paragraph below on westward expansion.

The 13 Original Colonies were founded starting in the year 1607 with Virginia and ending with Georgia in the year 1776. During this time, the 13 Original Colonies were under the control of Great Britain. In 1783, the Treaty of Paris paved the way for westward expansion of the United States to occur. This treaty formally ended the Revolutionary War. The terms of the treaty stated that Great Britain now would recognize the 13 Original Colonies as an independent nation, the United States of America, and France would give control of land east of the Mississippi River to the U.S. With this treaty in place, America was ready for westward expansion. In 1803, the Louisiana Purchase gave full access of the Mississippi River and its tributaries to the U.S., doubling the size of land the U.S. possessed. The Treaty of 1818 created the portion of the boundary between the U.S. and what is now Canada, extending from modern-day Michigan to the Rocky Mountains. The 1819 Florida Treaty led to the U.S. gaining control of Florida from Spain. This treaty defined the boundary between the U.S. and New Spain. The Webster-Ashburton Treaty in 1842 settled a dispute between the United States and the British North American colonies, giving what is now Maine to the U.S. From there, westward expansion took off. In 1845, Texas became the 29th state through the Texas Annexation. Texas had previously been part of Mexico until it declared independence in 1836, and the annexation of Texas by the U.S. lead to the Mexican-American War from 1846-1848. The Oregon Treaty in 1846 settled the dispute between the United States and Great Britain over control of the Oregon Territory, and established the 49th Parallel as the rest of the U.S.-Canada border. Soon to follow was the 1848 Mexican Cession as a result of the Mexican-American War. The Mexican Cession gave the United States control over most of what is now the southwestern United States. This land was ceded to the U.S. by Mexico and is the third largest acquisition of territory in U.S. history. Finally, the 1853 Gadsden Purchase resulted in the addition of land from Mexico, in what is now part of Arizona and New Mexico. The Gadsden Purchase marks the end of the westward expansion of the U.S., which resulted in the area that is now the 48 contiguous states.

Westward Expansion and Graph Theory: United States Map (Student Version)

Use the information from the paragraph on the westward expansion of the U.S. to label the map below. Put the corresponding letter in each region so it is clear and easy to read. You may work with the people in your group to make sure the labels all agree.



A: Texas Annexation (1845)

B: Mexican Cession (1848)

C: Louisiana Purchase (1803)

D: Florida Treaty (1819)

E: Gadsden Purchase (1853)

F: Oregon Treaty (1846)

G: 13 Original Colonies

H: Webster-Ashburton Treaty (1842)

I: Treaty of Paris (1783)

J: Treaty of 1818

Westward Expansion and Graph Theory: Guided Notes (Student Version)

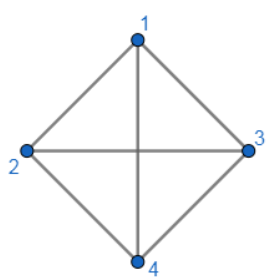
We will come back to the labeled map after introducing some topics from Graph Theory. Graph Theory is the study of ***graphs***, which are mathematical structures used to model pairwise relations between objects. This type of graph consists of vertices and edges. We will define these shortly! The graphs in Graph Theory are not the same as graphs of equations or functions that you have previously seen in math classes. Please follow along to fill in the guided notes below.

_____________: consists of a set of vertices (V) and edges (E) joining the vertices; sometimes denoted as $G=(V,E)$.

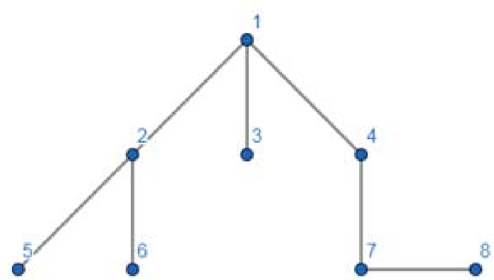
_______________: the fundamental unit from which graphs are formed (plural ***vertices***); also called a ***node***.

___________: gives the relationship between two vertices; usually depicted by a line segment or curve that connects two vertices.

Examples:



G_1



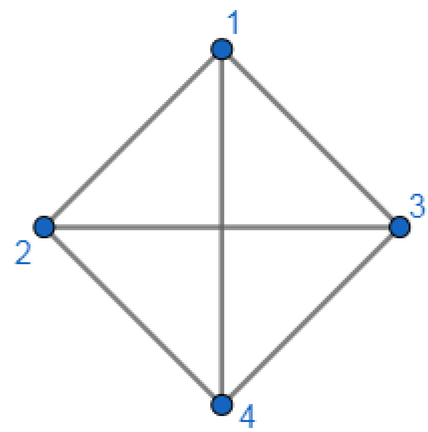
G_2

The vertex sets are:

The edge sets are:

Note that edges are not ________________ : $(1,2)$ is the same edge as $(2,1)$.

Edges can also be labeled directly, without referring to vertices, as in the example below.



Example: Draw a graph that has 5 vertices and 7 edges.

It is possible to have a vertex without an edge, but it is not possible to have an edge without a vertex!

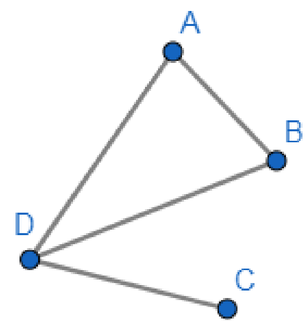
Two vertices are said to be ______________ if they have an edge between them.

An edge is said to be ____________ to a vertex and the vertex is ____________ to the edge if the vertex is one of the endpoints of the edge.

__________: an edge that connects a vertex to itself.

___________: number of edges incident to (touching) a node. _________ are counted twice when determining the degree of a vertex, because if we travel along a loop, we encounter the endpoint twice.

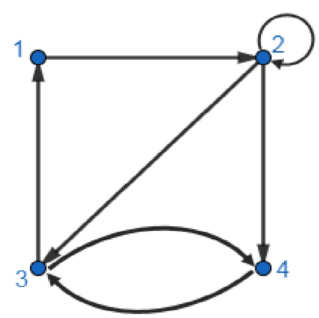
Question: For the graph below, what are $\deg(A)$, $\deg(B)$, $\deg(C)$, and $\deg(D)$?



Question: Draw a loop on vertex C in the graph above. What is the new degree of C?

_______________________: a graph in which the edges may have direction; also called a __________________.

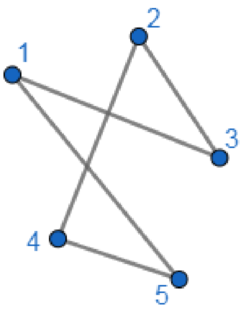
Example:



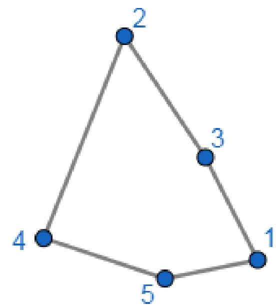
_____________________: a graph that can be drawn in a plane such that edges do not intersect except at the vertices.

____________________: a version of a planar graph, drawn so that no two edges intersect except at the vertices.

Example:



Planar graph

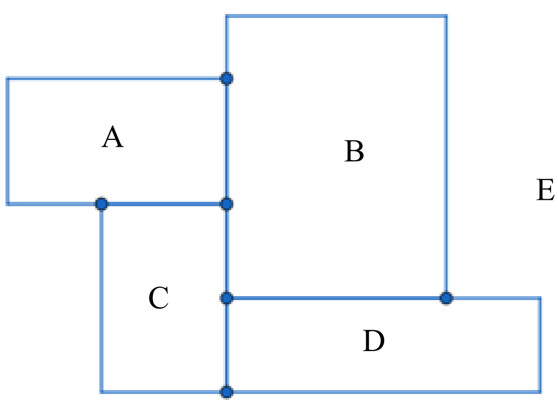


The same graph, drawn as a plane graph

Note: The depiction of the graph is generally irrelevant. The two images above represent "the same" graph because all of the relationships between the vertices are preserved despite the change in appearance.

____________________: The dual graph D of a plane graph G has a vertex for each region defined by G, including the "outside" region. The edges in D represent the borders between corresponding regions in G.

Question: In the graph on the left, the regions are A, B, C, D, and E. Each of these regions will become a vertex in the dual graph drawn on the right. There will be an edge between any two vertices that represent regions that share a border.



Question: Is the dual graph a planar graph? That is, can you move the vertices and/or curve the edges (if necessary) so that none of the edges cross?

______________________: Let G be a connected plane graph, which separates the plane into r regions (including the region that surrounds the graph). Let e be the number of edges and v be the number of vertices in G. Then ________________ .

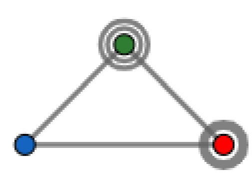
Question: Compute $r - e + v$ for the plane version of the dual graph in the last question. Is Euler's formula satisfied?

_______________________: a way to assign colors to the vertices of a graph so that adjacent vertices have different colors.

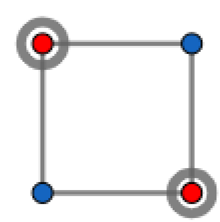
_________________________: the minimum number of colors needed for a coloring of a graph.

Examples:

Chromatic Number is 2

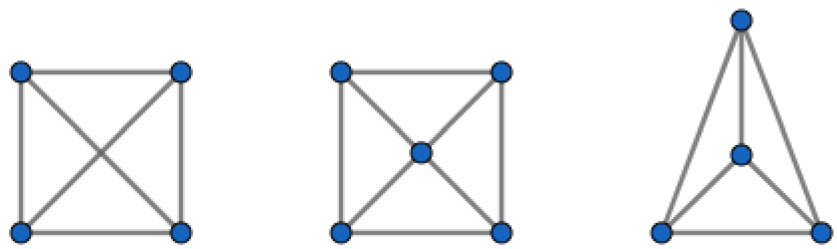


Chromatic Number is 3



Chromatic Number is 2

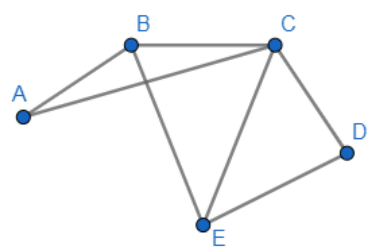
Question: What are the chromatic numbers of the following graphs?



______________________________: The chromatic number of a planar graph is no greater than 4.

_____________: a sequence of edges such that any two consecutive edges have a common vertex.

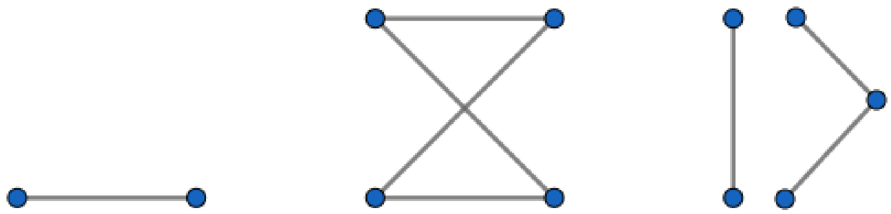
Question: In the following graph, $\{(A,B),(B,E)\}$ is a path from A to E of length 2. Name at least two other paths from A to E.



________________: a path that starts and ends at the same vertex; also called a ***closed path***.

_________________________: a graph in which any two vertices are joined by a path.

Examples:

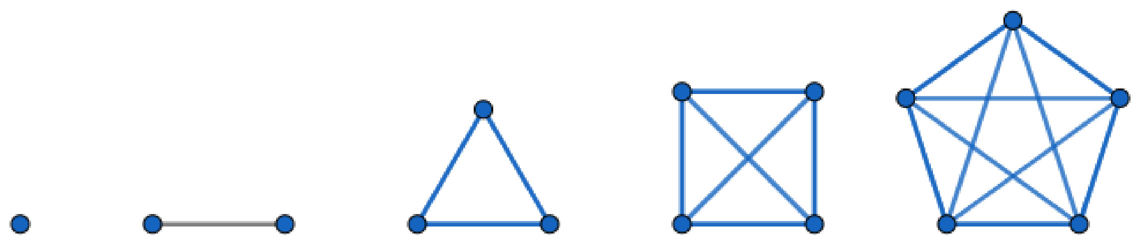


Connected Connected Not Connected

______________________: a graph in which there exists an edge between each pair of vertices.

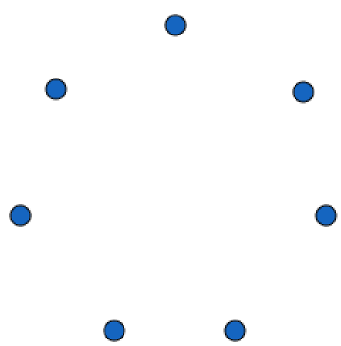
_______**:** denotes a complete graph with n vertices.

Examples:



K_1 K_2 K_3 K_4 K_5

Question: Draw a complete graph with 7 vertices:



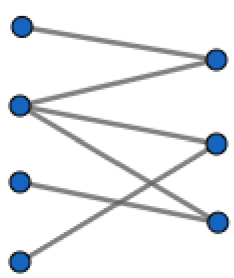
A complete graph with n vertices has _____________ edges.

Question: Verify that the number of edges in the complete graph with 7 vertices agrees with the formula.

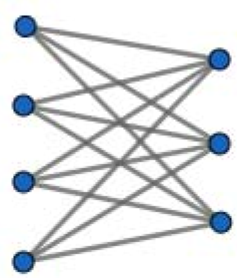
________________: a graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2, such that there are no edges between the vertices in V_1 and no edges between the vertices in V_2.

__________________________: a graph in which each pair of vertices (v_i, v_j) in $V_1 \times V_2$ is connected by an edge.

Example: In both graphs below, V_1 consists of the vertices on the left and V_2 consists of the vertices on the right.



Bipartite Graph



Complete Bipartite Graph

Westward Expansion and Graph Theory: Worksheet (Student Version)

Now that we know some basics of Graph Theory, we will apply the ideas of dual graphs and graph coloring to the westward expansion map.

1. You have already labeled all but one of the regions on the map. Label the outside region *K*. How many regions does the map have? Call this number *r*.

2. Identify a vertex as a point on the map where three or more regions meet. Draw and label all of the vertices so that the first vertex you count is v_1, the second vertex you count is v_2, and so on. How many vertices are there? Call this number *v*.

3. Identify an edge as the segment that connects two vertices on the map. Label all of the edges so that the first edge you count is e_1, the second edge you count is e_2, and so on. How many edges are there? Call this number *e*.

4. Compute the value of $r - e + v$. Is Euler's formula satisfied?

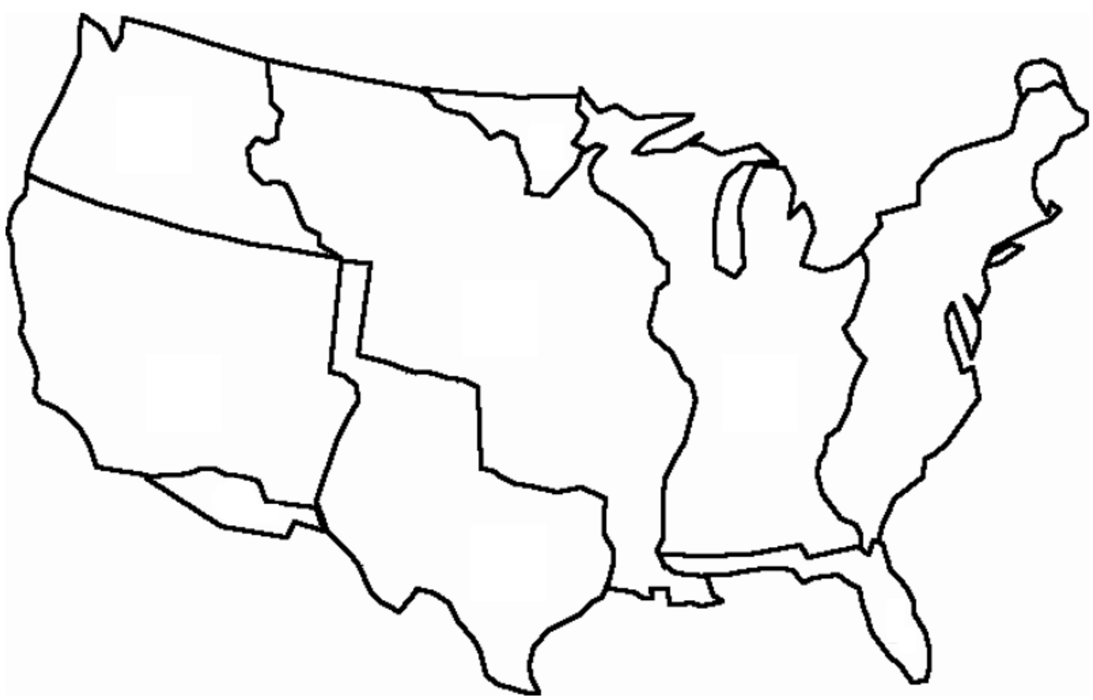
5. Now create a dual graph of the westward expansion map. The letters that you used to label the regions in the map will be the labels for the vertices of the dual graph. Refer to the outside region as *K*. Using the template below, draw the edges that connect the vertices in the dual graph. Since every region is touching *K*, the graph will be easier to draw if we put the vertex for *K* in the center of the diagram. Note that the location of each vertex does not matter. Recall that the important thing is the edges that connect the vertices: if two vertices are connected by an edge in one version of the graph, they must be connected by an edge in every version of the graph.

The number of edges in the dual graph should match the number of edges on the map. Check now that you did not miss any edges.

6. Color the dual graph by circling each vertex with a color. No two vertices that share an edge can be the same color. Use as few colors as possible. What is the chromatic number of the dual graph? Remember that the chromatic number is the <u>minimum</u> number of colors needed for a coloring of a graph.

7. Is the dual graph that you drew a plane graph?

8. Color the regions of the westward expansion map in such a way that no two regions that share a border are the same color. Do this with as few colors as you can. Does the chromatic number of the map match the chromatic number of the dual graph? Explain why or why not.



Challenge: Is the dual graph from Question 5 a planar graph? That is, can you recreate the graph by moving vertices or redrawing (possibly curved) edges so that none of the edges cross? If possible, draw this plane graph.

Can You Dig It?

Lesson Overview

Brief Description: Students will answer questions about area and volume in the context of the book *Holes*, by Louis Sachar.

Time: 90 Minutes

Grades: 7-9

Topic: Area and volume of circles and cylinders

Paired with: Language Arts

Materials:

- Student handouts (guided notes, activity sheet)
- Manipulatives or software for modeling arrangements of circles (optional)

Students will begin the lesson by building a strong foundation for understanding area of circles and volume of cylinders. Students will develop formulas for these and make sense of why the formulas are true. Then students will use the formulas to answer questions about area and volume in the context of *Holes*, by Louis Sachar. Students who have not previously read *Holes* can probably find it at their local library if they want to read it. This lesson works well in conjunction with an English class in which the students are reading *Holes,* but the students will be able to complete the lesson even if they have not read the book.

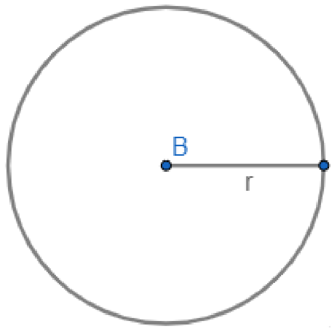
The class will develop the necessary formulas for the lesson together by following along with the guided notes. When the guided notes are finished, the students can then work in pairs to answer the follow-up questions on the activity sheet.

References:

Sachar, Louis. *Holes*. New York; Farrar, Straus and Giroux; 1998.

Can You Dig It? Guided Notes (Teacher Version)

We start with a review of information about circles and cylinders.



Here we have circle *B* with radius *r*.

What do we mean by the ***diameter*** of a circle? Find an expression for the diameter of circle *B*.

The diameter of a circle is the distance across the circle, as measured through the center of the circle. An expression for the diameter is $d = 2r$.

The constant π is defined as the circumference (*C*) of any circle divided by its diameter (*d*). Use the definition of π to develop a formula for the circumference of a circle in terms of the radius. Show your work below.

$$\pi = \frac{C}{d}$$
$$\pi = \frac{C}{2r}$$
$$C = 2\pi r$$

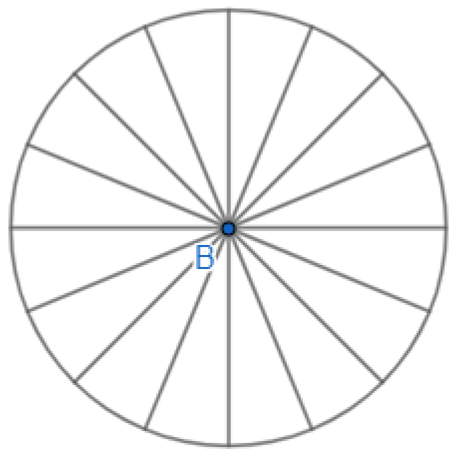
or

$$\pi = \frac{C}{d}$$
$$C = \pi d$$
$$C = 2\pi r$$

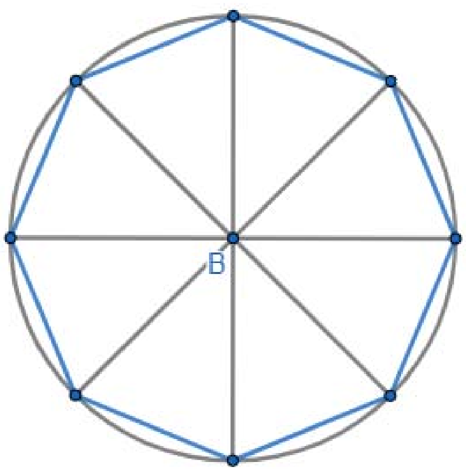
Give an expression for the area of circle *B*.

$A = \pi r^2$

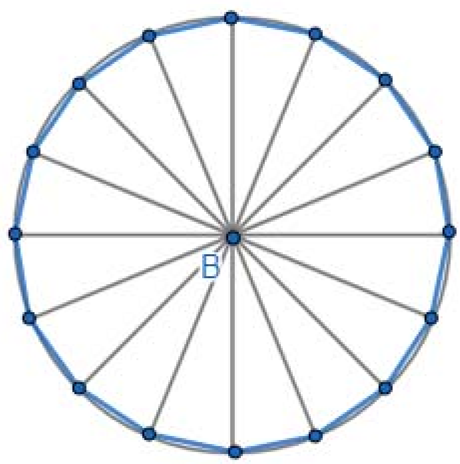
Are we able to prove the formula for area of a circle? Sort of. A formal proof requires calculus, but we can use what we know about triangles to generate a proof for our own understanding. Imagine that a circle is split into *n* congruent sectors as shown below. In this example, $n = 16$.



Join endpoints of sectors to form an *n*-sided polygon which is broken into *n* isosceles triangles.



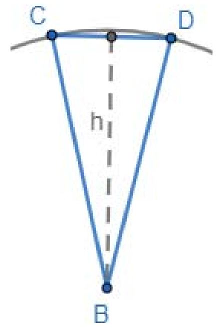
$n = 8$



$n = 16$

As *n* gets larger, the polygon gets closer to being a circle. (Note: In Calculus, we call this process a ***limit***.)

Here is one isosceles triangle, $\triangle BCD$.



We know that the area of $\triangle BCD$ is $A = \frac{1}{2}bh$ where *b* is the base of the triangle and *h* is the height. If we extend the height all the way to the edge of the circle, we see that *h* is a little less than the radius of circle *B*. Also, the base of the triangle is a little less than $m\widehat{CD} = \frac{2\pi r}{n}$. So the area of the triangle is a little less than

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(r)\left(\frac{2\pi r}{n}\right) \\ &= \frac{\pi r^2}{n}. \end{aligned}$$

Since the polygon has *n* of these triangles, the area of the whole polygon is a little less than

$$n\left(\frac{\pi r^2}{n}\right) = \pi r^2.$$

Then we let *n* get really big. The area of the circle is the limit of the area of the polygon: πr^2.

The boys dig three-dimensional holes, so we will also consider volume. To develop the formula for the volume of a cylinder we will use the formula for the area of a circle.

What shape is the base of a cylinder?

Circle

What shape do we get if we slice the cylinder parallel to the base? How does the slice compare to the base?

The slice is a circle that is congruent to the base.

For any solid where slices parallel to the base are congruent to the base, the volume is the area of the base times the height. We already know this for a rectangular prism for example: the area of the base is lw and the volume is lwh. If we apply this principle to a cylinder, what do we get for the volume?

$V = \pi r^2 h$

Can You Dig It? Activity Sheet (Teacher Version)

In the book *Holes* by Louis Sachar, the boys at a juvenile detention camp have to dig holes every day. The holes are circular and must be five feet in diameter and five feet deep. The boys use their shovels to measure the diameter and depth of the holes. The boys are told that digging holes in the desert builds character, but the real reason they are forced to dig holes is that the warden is hoping to come across a buried fortune from many years before.

Now that we have reviewed some basic information regarding circles and cylinders, we can answer math questions connected to situations in *Holes*. Your answers to the questions below will incorporate your knowledge of area and volume. In all of the questions, the holes are as described above unless otherwise noted.

1. Since the diameter of each hole is five feet, what is the radius?

   $r = 2.5$ feet

2. What is the area of the circle at the top (or bottom) of each hole? Leave the answer in terms of π. Show your work and do not forget to include units in the answer.

$$\begin{aligned} A &= \pi r^2 \\ &= \pi (2.5)^2 \\ &= 6.25\pi \text{ square feet} \end{aligned}$$

3. If we double the diameter of the hole, do we also double the area of the hole?

   Notice that $2 \cdot (2.5)^2 = 12.5$, but $5^2 = 25$, so no, we cannot just double the area.

4. The area of a circle with a diameter of 10 feet is how many times the area of a circle with a diameter of 5 feet?

The area of a 10-foot circle is 25π, which is 4 times the area of a 5-foot circle.

5. Does the comparison in Question 4 hold in general? That is, if the diameter of a circle *C* is twice as long as the diameter of circle *B,* can we multiply the area of circle *B* by the constant in Question 4 to get the area of circle *C*? Explain.

Yes, the area of circle *C* will always be 4 times the area of circle *B*. The radius of circle *B* is *r* and the radius of circle *C* is $2r$. So

$$\text{Area of Circle } C = \pi(2r)^2 = 4\pi r^2 = 4\left(\pi r^2\right) = 4(\text{Area of Circle } B).$$

6. What formula would we use to figure out how much dirt is displaced when one of the boys digs a hole? Find how much dirt is displaced from the hole. Leave the answer in terms of π. Show your work and do not forget to include units in the answer.

Formula: $V = \pi r^2 h$

Work:
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2.5)^2(5) \\ &= 31.25\pi \text{ cubic feet} \end{aligned}$$

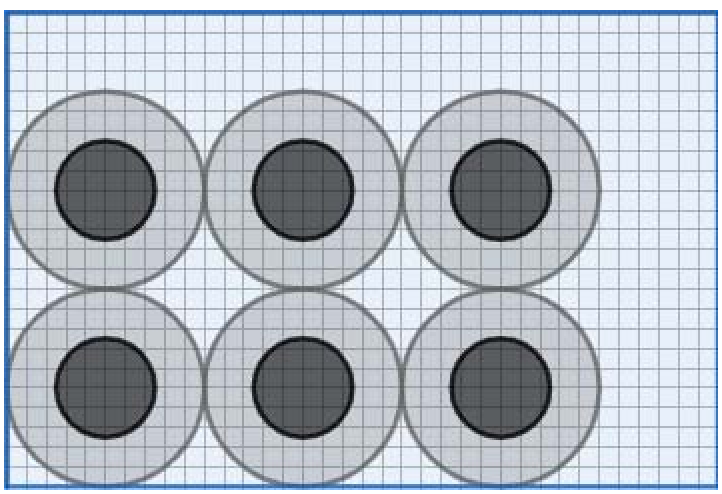
7. If the boys dig the holes in a rectangular grid, how many holes will fit in a plot of land that is 24 feet by 36 feet?

Because $\frac{24}{5} = 4.8$ and $\frac{36}{5} = 7.2$, four holes will fit across the 24-foot side of the plot and seven holes will fit across the 36-foot side. The total number of holes is thus $4 \times 7 = 28$ holes.

8. There must be sufficient space between the holes so that the walls of the holes do not collapse, though. Suppose that the holes must be at least 5 feet apart and at least 2.5 feet from the edges of the plot in Question 7.
   a. Let x be the number of holes across the length and let y be the number of holes across the width. Write an expression in terms of x for the total distance of the holes and the "buffer" around them across the length of the plot. Do the same for the width of the plot, in terms of y. Then solve for x and y.

      We can treat each hole as a 5-foot circle surrounded by a 2.5-foot "buffer," so the circles are 10 feet across and are allowed to touch. The holes and their buffers require $10x$ feet across the length of the plot, and $10y$ feet across the width. So, $10x \le 36$ and $10y \le 24$. Therefore, $x \le 3.6$ and $y \le 2.4$. Since x and y must be whole numbers, there are 3 holes across the length and 2 holes across the width, for a total of 6 holes.

   b. Draw a picture showing one way to arrange the holes in the plot.



      Each square in the grid represents one square foot. The black circles represent the holes and the grey rings represent the buffer of 2.5 feet around each hole. As long as no two grey rings overlap then the condition that the holes must have 5 feet between them is met.

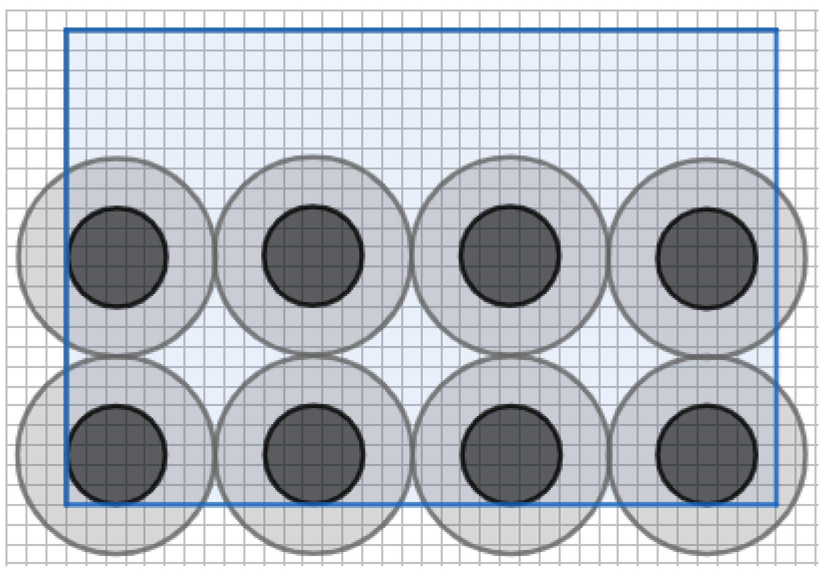
9. Now suppose that the holes must still be at least 5 feet apart in a rectangular grid, but the edges of the holes may touch the border of the plot of land.

   a. Does the answer from Question 8 change? Explain your reasoning below.

   In the rectangular grid, there are x holes and $x-1$ gaps between them. Using x and y as in Question 9, the distance for the holes are $5x$ and the total distance between the holes is $5(x-1)$. Therefore, the holes and the buffer space require $5x+5(x-1)=10x-5$ feet across the width. So $10x-5 \le 36$. For a similar reason, $10y-5 \le 24$.

   Therefore, $x \le 4.1$ and $y \le 2.9$. Since x and y must be whole numbers, there are 4 holes across the length and 2 holes across the width, for a total of 8 holes.

   b. Draw a picture showing one way to arrange the holes in the plot.



   As before, the black circles represent the holes and the grey rings represent a buffer of 2.5 feet around each hole. The blue rectangle is the plot of land.

10. How much of the land in Question 9 has not been dug up? Show your work.

Total area of rectangle:

$$\begin{aligned} A &= l \cdot w \\ &= 36 \cdot 25 \\ &= 900 \text{ square feet} \end{aligned}$$

Total area of holes:

$$\begin{aligned} A &= 8\left(\pi(2.5)^2\right) \\ &= 8(6.25\pi) \\ &= 50\pi \text{ square feet} \end{aligned}$$

Land not used:

$$\begin{aligned} L &= 900 - 50\pi \\ &\approx 742.92 \text{ square feet} \end{aligned}$$

11. What percent of the land is untouched in Question 9?

$$\text{Question 9: } \frac{742.92}{900} \approx .825$$

So roughly 82.5% of the land goes untouched.

12. Suppose that each boy throws the dirt from his hole onto a single mound. What (approximate) shape is this mound? What is the formula for the volume of this shape?

A cone.

$V = \pi r^2 \frac{h}{3}$

13. Use what you know about the volume of displaced dirt from Question 6 to approximate how tall the mound is if its base has a radius of 3.5 feet. Show your work.

$$V = \pi r^2 \frac{h}{3}$$
$$31.25\pi = \pi(3.5)^2 \frac{h}{3}$$
$$31.25\pi = \pi(12.25)\frac{h}{3}$$
$$31.25 = \frac{12.25}{3}h$$
$$h \approx 7.65 \text{ feet}$$

14. What is the approximate radius of the mound if the height is 4 feet? Show your work.

$$V = \pi r^2 \frac{h}{3}$$
$$31.25\pi = \pi r^2 \frac{4}{3}$$
$$\frac{3}{4}(31.25) = \left(r^2 \frac{4}{3}\right)\frac{3}{4}$$
$$\frac{93.75}{4} = r^2$$
$$\sqrt{\frac{93.75}{4}} = \sqrt{r^2}$$
$$r \approx 4.84 \text{ feet}$$

15. Would the discarded dirt fit exactly back into the hole? What impact does this have on the answer to Question 13?

The discarded dirt is looser now that it has been dug up than when it was in the hole, meaning that the dirt probably takes up more space than it originally did. Therefore, the answer in Question 13 is an underestimate.

16. The main character in *Holes* is Stanley Yelnats. His friend Zero looks out for Stanley and even helps dig Stanley's hole each day. Suppose Stanley takes four hours longer than Zero to dig a hole. When they dig one hole together, it takes five hours. How long does it take Stanley to dig his own hole? How long does it take Zero to dig his own hole?

Let x represent the amount of time it takes Stanley to dig his hole. Then it takes Zero $x-4$ hours to dig his hole.

Therefore, Stanley digs at a rate of $\frac{1}{x}$ hole per hour and Zero digs at a rate of $\frac{1}{x-4}$ hole per hour. Together, they dig at a rate of $\frac{1}{5}$ hole per hour. Thus, we solve:

$$\begin{aligned} \frac{1}{x-4}+\frac{1}{x}&=\frac{1}{5} \\ 5x(x-4)\left(\frac{1}{x-4}+\frac{1}{x}\right)&=5x(x-4)\left(\frac{1}{5}\right) \\ 5x+5(x-4)&=x(x-4) \\ 10x-20&=x^2-4x \\ x^2-14x+20&=0 \end{aligned}$$

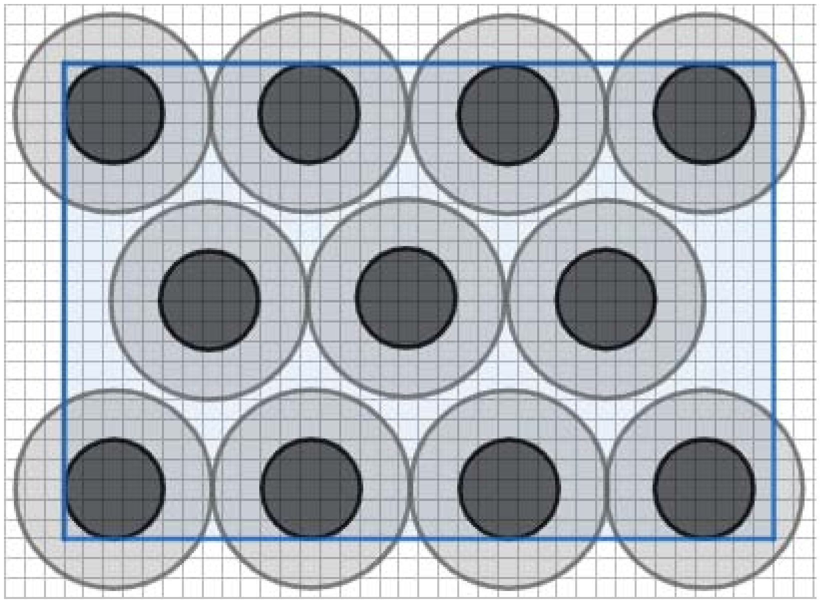
Using the quadratic formula we get:

$$\begin{aligned} x&=\frac{14\pm\sqrt{(-14)^2-4(1)(20)}}{2(1)} \\ &=\frac{14\pm\sqrt{196-80}}{2} \\ &=\frac{14\pm\sqrt{116}}{2} \end{aligned}$$

Therefore, $x\approx 12.39$ or $x\approx 1.61$.

Since it takes Zero $x-4$ hours to dig his hole, we see that the only solution is $x\approx 12.39$. Therefore it takes Stanley approximately 12.39 hours to dig his own hole, and Zero approximately 8.39 hours to dig his.

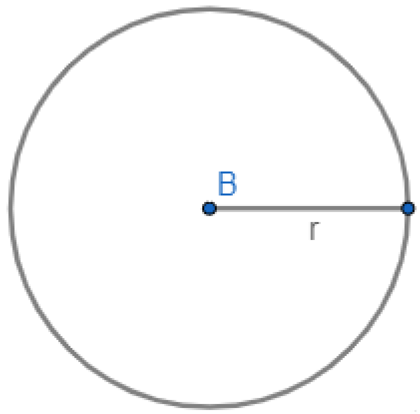
17. **Challenge:** Look back at Question 9. Suppose that the holes must still be at least 5 feet apart and the holes can touch the border of the plot of land, but the holes **do not** have to be in a rectangular grid. What is the maximum number of holes that can fit in this plot of land? Draw a picture showing your answer. Manipulatives or software like GeoGebra will be a helpful resource.



The diagram above shows that it is possible to dig 11 holes in the plot. Again, the black circles represent the holes and the grey rings represent a buffer of 2.5 feet around each hole. The blue rectangle is the plot of land.

Can You Dig It? Guided Notes (Student Version)

We start with a review of information about circles and cylinders.



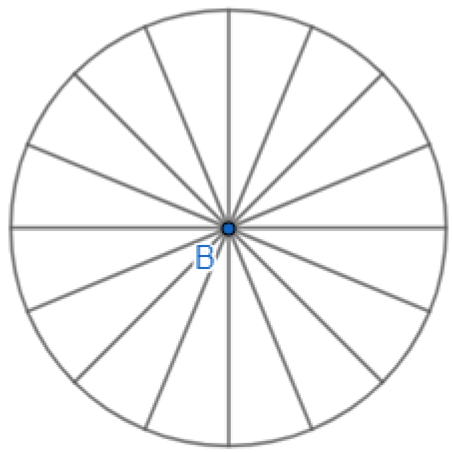
Here we have circle *B* with radius *r*.

What do we mean by the ***diameter*** of a circle? Find an expression for the diameter of circle *B*.

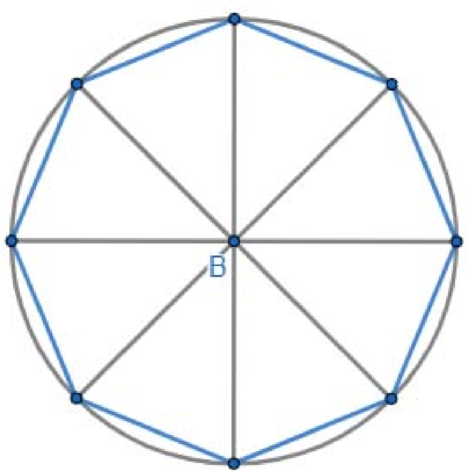
The constant π is defined as the circumference (C) of any circle divided by its diameter (d).
Use the definition of π to develop a formula for the circumference of a circle in terms of the radius. Show your work below.

Give an expression for the area of circle *B*.

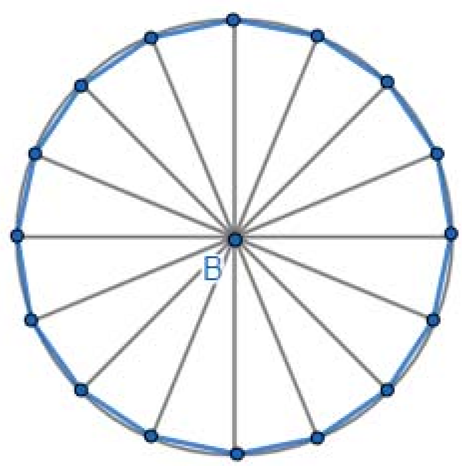
Are we able to prove the formula for area of a circle? Sort of. A formal proof requires calculus, but we can use what we know about triangles to generate a proof for our own understanding. Imagine that a circle is split into *n* congruent sectors as shown below. In this example, $n = 16$.



Join endpoints of sectors to form an *n*-sided polygon which is broken into *n* isosceles triangles.



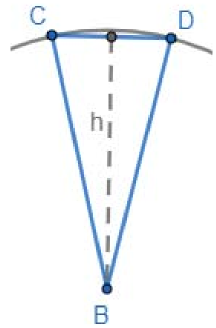
$n = 8$



$n = 16$

As *n* gets larger, the polygon gets closer to being a circle. (Note: In Calculus, we call this process a ***limit***.)

Here is one isosceles triangle, $\triangle BCD$.



We know that the area of $\triangle BCD$ is $A = \frac{1}{2}bh$ where *b* is the base of the triangle and *h* is the height. If we extend the height all the way to the edge of the circle, we see that *h* is a little less than the radius of circle *B*. Also, the base of the triangle is a little less than $m\overset{\frown}{CD} = \frac{2\pi r}{n}$. So the area of the triangle is a little less than

Since the polygon has *n* of these triangles, the area of the whole polygon is a little less than

Then we let *n* get really big. The area of the circle is the limit of the area of the polygon: _______ .

The boys dig three-dimensional holes, so we will also consider volume. To develop the formula for the volume of a cylinder we will use the formula for the area of a circle.

What shape is the base of a cylinder?

What shape do we get if we slice the cylinder parallel to the base? How does the slice compare to the base?

For any solid where slices parallel to the base are congruent to the base, the volume is the area of the base times the height. We already know this for a rectangular prism for example: the area of the base is lw and the volume is lwh. If we apply this principle to a cylinder, what do we get for the volume?

Can You Dig It? Activity Sheet (Student Version)

In the book *Holes* by Louis Sachar, the boys at a juvenile detention camp have to dig holes every day. The holes are circular and must be five feet in diameter and five feet deep. The boys use their shovels to measure the diameter and depth of the holes. The boys are told that digging holes in the desert builds character, but the real reason they are forced to dig holes is that the warden is hoping to come across a buried fortune from many years before.

Now that we have reviewed some basic information regarding circles and cylinders, we can answer math questions connected to situations in *Holes*. Your answers to the questions below will incorporate your knowledge of area and volume. In all of the questions, the holes are as described above unless otherwise noted.

1. Since the diameter of each hole is five feet, what is the radius?

2. What is the area of the circle at the top (or bottom) of each hole? Leave the answer in terms of π. Show your work and do not forget to include units in the answer.

3. If we double the diameter of the hole, do we also double the area of the hole?

4. The area of a circle with a diameter of 10 feet is how many times the area of a circle with a diameter of 5 feet?

5. Does the comparison in Question 4 hold in general? That is, if the diameter of a circle *C* is twice as long as the diameter of circle *B*, can we multiply the area of circle *B* by the constant in Question 4 to get the area of circle *C*? Explain.

6. What formula would we use to figure out how much dirt is displaced when one of the boys digs a hole? Find how much dirt is displaced from the hole. Leave the answer in terms of π. Show your work and do not forget to include units in the answer.

   Formula: Work:

7. If the boys dig the holes in a rectangular grid, how many holes will fit in a plot of land that is 24 feet by 36 feet?

8. There must be sufficient space between the holes so that the walls of the holes do not collapse, though. Suppose that the holes must be at least 5 feet apart and at least 2.5 feet from the edges of the plot in Question 7.
   a. Let x be the number of holes across the length and let y be the number of holes across the width. Write an expression in terms of x for the total distance of the holes and the "buffer" around them across the length of the plot. Do the same for the width of the plot, in terms of y. Then solve for x and y.

   b. Draw a picture showing one way to arrange the holes in the plot.

9. Now suppose that the holes must still be at least 5 feet apart in a rectangular grid, but the edges of the holes may touch the border of the plot of land.

    a. Does the answer from Question 8 change? Explain your reasoning below.

    b. Draw a picture showing one way to arrange the holes in the plot.

10. How much of the land in Question 9 has not been dug up? Show your work.

11. What percent of the land is untouched in Question 9?

12. Suppose that each boy throws the dirt from his hole onto a single mound. What (approximate) shape is this mound? What is the formula for the volume of this shape?

13. Use what you know about the volume of displaced dirt from Question 6 to approximate how tall the mound is if its base has a radius of 3.5 feet. Show your work.

14. What is the approximate radius of the mound if the height is 4 feet? Show your work.

15. Would the discarded dirt fit exactly back into the hole? What impact does this have on the answer to Question 13?

16. The main character in *Holes* is Stanley Yelnats. His friend Zero looks out for Stanley and even helps dig Stanley's hole each day. Suppose Stanley takes four hours longer than Zero to dig a hole. When they dig one hole together, it takes five hours. How long does it take Stanley to dig his own hole? How long does it take Zero to dig his own hole?

17. **Challenge:** Look back at Question 9. Suppose that the holes must still be at least 5 feet apart and the holes can touch the border of the plot of land, but the holes **do not** have to be in a rectangular grid. What is the maximum number of holes that can fit in this plot of land? Draw a picture showing your answer. Manipulatives or software like GeoGebra will be a helpful resource.

12 Days of Thanksgiving

Lesson Overview

Brief Description: Students will create a Thanksgiving version of "The 12 Days of Christmas" and then look at sums of specific sequences connected to the song.

Time: 90 Minutes

Grades: 9-12

Topic: Developing formulas for specific sums

Paired with: Language Arts

Materials: Student handouts (lyrics, worksheet, guided notes)

One focus of Language Arts is writing. The types of writing that students do often follow a specific structure. Many poems, for example, are full of patterns. Poems can rhyme every two lines, rhyme every other line, contain specific syllable counts, etc. Songs are a form of poetry sung to a beat. In this lesson, students will rewrite the lyrics to the popular Christmas song, "The 12 Days of Christmas," to make it into "The 12 Days of Thanksgiving." Students' versions must follow the same beat as the original song, so that they can sing their lyrics to the correct tune. They will then answer questions that will lead to an expression for the sum of the first n positive integers. This sum will correspond to the number of gifts listed for the n^{th} day in the song. This number refers to the number of gifts in the verse together with the number of gifts in the following refrain.

Students will have to use both recursive and explicit formulas in this lesson. For iterative processes, recursive formulas can be easy to develop, as they describe how to get from one iteration to the next. However, recursive formulas are inefficient at times because the previous term must be known in order to find the next term. If we are looking at a large n value, where n is the term number, it is possible that this task will become very time consuming. This is why an explicit formula can be more useful. Students should be able to develop a formula for the sum with the support of their groups. Once students have a formula they believe works, the teacher will work with the students to prove the result. There are several proofs of this formula. We include two of these in the teacher notes; both should be easily understood by students in grades 9-12.

Students will also develop a formula to determine the total number of gifts a person would receive through the n^{th} day in the song. This is the sum of the gifts received on Days 1, 2, 3, …, n. In order to successfully develop this formula, students will need to know both the sum of the first n positive integers as well as the sum of the squares of the first n positive integers. The formula for the sum of the squares is given to the students in the lesson. A proof of this formula is in the teacher version, for teachers to use at their discretion.

Students will need time to write their version of "The 12 Days of Thanksgiving," as either a homework assignment or during the class period. Students should work on their songs independently, so that each student writes their own unique song. The teacher will provide students with follow-up questions once all the songs are written. The students will answer these questions in groups. The teacher will then lead a discussion of the proof for the sum of the first positive n integers.

12 Days of Thanksgiving: Lyrics (Teacher Version)

Have you heard the popular song, “The 12 Days of Christmas”? The lyrics are:

On the first day of Christmas
My true love sent to me:
A partridge in a pear tree.

On the second day of Christmas
My true love sent to me:
Two turtle doves,
And a partridge in a pear tree.

The song continues in this fashion. The final verse is:

On the twelfth day of Christmas
My true love gave to me:
Twelve drummers drumming,
Eleven pipers piping,
Ten lords a-leaping,
Nine ladies dancing,
Eight maids a-milking,
Seven swans a-swimming,
Six geese a-laying,
Five gold rings,
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree.

You can do an internet search for “12 Days of Christmas song” to find various recordings if you would like to listen to the song.

In this part of the lesson, you will write a Thanksgiving version of this song. The guidelines are as follows:

- You must include 12 items that are related to Thanksgiving.
- The rhythm must match the Christmas version. You will know if the rhythm is correct if you can sing your words to the tune of the Christmas version.

Example:

Here are the first verses of "The 12 Days of Christmas" and a variation called "The 12 Days of School." The highlighting indicates the beginning of each beat. Notice that the beats in the two versions match up, so that the school version preserves the rhythm of the Christmas version.

Christmas Version: On the first day of Christ- mas, my true love sent to me _

School Version: On the first day of school _, my tea-cher gave to me _

Christmas Version: a par- tridge in a pear tree _ _.

School Version: a pen-cil in a pret-ty bag _ _.

Write the Thanksgiving verses in the space provided below. Mark the beats in your song to ensure that the rhythm is the same as in the Christmas version.

Student work will vary; here is one possible solution.

1. Christmas Version:
   On the first day of Christ-mas,
   my true love sent to me _
   A par-tridge in a pear tree_ _.

   Thanksgiving Version:
   On the first day of Thanks-gi-ving,
   my fam'-ly brought to me _
   A tur-key full of stuf-fing _ _.

2. Christmas Version:
   Two tur-tle doves

   Thanksgiving Version:
   Two dinner rolls

3. Christmas Version: Three French hens
   Thanksgiving Version: Three dogs a-bark-ing

4. Christmas Version: Four call-ing birds
   Thanksgiving Version: Four ap-ple pies

5. Christmas Version: Five gold rings_ _
   Thanksgiving Version: Five corn breads_ _

6. Christmas Version: Six geese a-lay-ing
   Thanksgiving Version: Six bowls of gra-vy

7. Christmas Version: Se-ven swans a-swim-ming
   Thanksgiving Version: Sev-en child-ren play-ing

8. Christmas Version: Eight maids a-milk-ing
   Thanksgiving Version: Eight mashed po-ta-toes

9. Christmas Version: Nine la-dies danc-ing
   Thanksgiving Version: Nine chefs a-cooking

10. Christmas Version: Ten lords a-leap-ing
    Thanksgiving Version: Ten green beans

11. Christmas Version: E-lev-en pip-ers pip-ing
    Thanksgiving Version: E-lev-en foot-balls fly-ing

12. Christmas Version: Twelve drum-mers drum-ming.
    Thanksgiving Version: Twelve hun-gry friends

12 Days of Thanksgiving: The Math (Teacher Version)

Now that you have written your own version of "The 12 days of Thanksgiving," we will investigate some math connections to this song! For all of the following questions, assume that you receive all of the items mentioned in the verse and the refrain for that specific day.

Example: On the second day you receive three gifts because the verse for the second day is:

On the second day of Christmas
My true love sent to me:
Two turtle doves,
And a partridge in a pear tree.

Work with your group to answer the following questions.

1. Let G_3, G_4, and G_5 represent the number of gifts you receive on Day 3, Day 4 and Day 5, respectively. (Note that $G_1 = 1$ and $G_2 = 3$.)
   a. How many gifts do you receive on Day 3?
      $G_3 =$ 6 Gifts
   b. How many gifts do you receive on Day 4?
      $G_4 =$ 10 Gifts
   c. How many gifts do you receive on Day 5?
      $G_5 =$ 15 Gifts

2. Do you notice a pattern in how the number of gifts changes from one day to the next?

   To get the number of gifts for a specific day, take the number of gifts on the previous day and add the day number.

3. If you receive x gifts on Day 10, how many would you receive on Day 11?

   You would receive $x+11$ gifts.

4. Suppose G_{n-1} represents the number of gifts you receive on Day $n-1$. What does G_n represent? What is the value of G_0? Give a formula for G_n that uses G_{n-1}.

   G_n represents: the number of gifts received on the n^{th} day.

   $G_0 = 0$

   $G_n = G_{n-1} + n$

5. The formula in Question 4 is called a ***recursive formula***. Can you imagine any difficulties in using this recursive formula to find the number of gifts on a particular day?

   One disadvantage is that recursive equations are inefficient for large n values. For example, in order to find G_{40} we would have to know the value of G_{39}, and in order to know the value of G_{39} we need to know the value of G_{38}, and so on.

   Another problem with the recursive equation is that it does not tell us how to find the value of G_1 (or G_0 if that is where you are starting).

Since recursive formulas are not always practical, a more efficient formula for G_n is $1+2+3+\cdots+n$. This works well until the value of n gets too large. If we want to find the sum of the integers from 1 to 67 for example, we would have to add $1+2+3+\cdots+65+66+67$, and that might take a while.

6. Complete the table and then develop a more efficient formula for $G_n = 1+2+3+\cdots+n$, the number of gifts you receive on the n^{th} day.

n	1	2	3	4	5	6
G_n	1	3	6	10	15	21

$$G_n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}$$

7. Use the formula you developed in Question 6 to determine how many gifts you would receive on Day 12. Show your work.

$$\frac{12(12+1)}{2} = \frac{12(13)}{2} = 78$$

12 Days of Thanksgiving: Guided Notes (Teacher Version)

We now prove that the formula you found in Question 6 of the worksheet is correct.

We provide two different proofs; teachers may present one of these, or another proof of their choosing. The two proofs apply the same technique, but the first is purely symbolic, and the second is more visual.

Proof 1: Let G_n represent the sum of the first n positive integers. Notice that we can add those integers in increasing order and we will get the same sum as when we add them in decreasing order, as shown below.

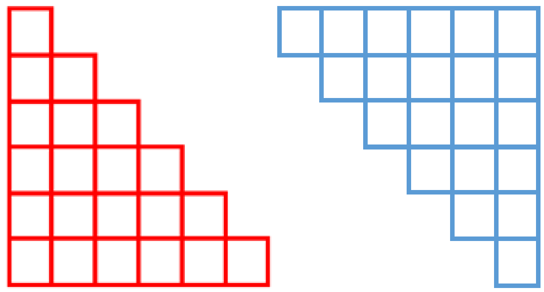
$$\begin{array}{ccccccccccc} G_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\ G_n & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

If we add these vertically we get:

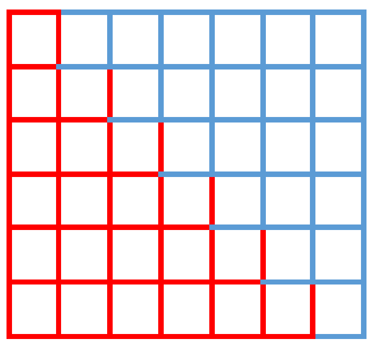
$$\begin{aligned} 2G_n &= (1+n)+(2+(n-1))+(3+(n-2))+\cdots+(n+1) \\ &= (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1). \end{aligned}$$

Notice that G_n is the sum of n numbers. So for $2G_n$, there are n copies of "$n+1$". Therefore $2G_n = n(n+1)$, and thus $G_n = \frac{n(n+1)}{2}$.

Proof 2: Consider a "triangle" with n rows of blocks where the first row contains one block, the second row contains two blocks, and so on until the last row contains n blocks. Create a second triangle by rotating the first triangle $180°$. The picture below shows both triangles for $n = 6$. The number of blocks in each triangle is $1+2+3+\cdots+n$.



Fit the two triangles together to create a rectangle as shown below.



We can see that the height of the rectangle is n blocks, and the base is $n+1$ blocks. Therefore, the total number of blocks in the rectangle is $n(n+1)$.

Since each triangle has $G_n = 1+2+3+\cdots+n$ blocks, then the rectangle has $2G_n$ blocks because it consists of two triangles. So $2G_n = n(n+1)$, and thus $G_n = \dfrac{n(n+1)}{2}$.

Now that we have an efficient formula for finding the number of gifts on the n^{th} day of Thanksgiving, another interesting thing to consider is the total number of gifts through Day *n*.

1. For example, what is the number of gifts through Day 3? (Add the number of gifts on each of Days 1, 2, and 3.)

   $(1)+(1+2)+(1+2+3)=10\ \text{gifts}$

2. How many gifts have been received through the fourth day? The fifth day? Show your work below.

   Fourth Day: $(1)+(1+2)+(1+2+3)+(1+2+3+4)=20\ \text{gifts}$

   Fifth Day: $(1)+(1+2)+(1+2+3)+(1+2+3+4)+(1+2+3+4+5)=35\,\text{gifts}$

3. It is clear that finding the total number of gifts through the n^{th} day will become time-consuming as the value of *n* increases. As before, we can develop a more efficient formula for finding these totals. Let TG_n be the total number of gifts received through Day *n*. We will answer the questions below in order to develop a formula for TG_n.

   a. Recall that the formula for the number of gifts received on the n^{th} day is $G_n=\dfrac{n(n+1)}{2}$.

      Explain why $G_n=\dfrac{n(n+1)}{2}=\dfrac{1}{2}(n^2+n)$.

      We know that $G_n=\dfrac{n(n+1)}{2}=\dfrac{1}{2}(n^2+n)$ by distributing *n* and changing division by 2 to multiplication by $\dfrac{1}{2}$.

b. If you are asked to find the total number of gifts received through the fourth day you must add the number of gifts received on which days? Write the answer in words.

We must add the number of gifts from the first day, second day, third day, and fourth day.

c. Express the answer to part (b) using symbols. Recall that the total number of gifts received through Day 4 is represented by TG_4 and the number of gifts received on Day n is represented by G_n

$$TG_4 = G_1 + G_2 + G_3 + G_4$$

d. Write the sum for the total number of gifts received through Day n using symbols as in part (c). Show at least the first three terms and the last term. Use "$\cdots$" to represent any "hidden" terms.

$$TG_n = G_1 + G_2 + G_3 + \cdots + G_n$$

e. Replace G_1, G_2, etc. in part (d) by using the formula from part (a). Do not simplify.

$$TG_n = \frac{1}{2}\left(1^2 + 1\right) + \frac{1}{2}\left(2^2 + 2\right) + \frac{1}{2}\left(3^2 + 3\right) + \cdots + \frac{1}{2}\left(n^2 + n\right)$$

f. You should be able to factor a constant out of the sum in (e). Do so, and write the result here.

$$TG_n = \frac{1}{2}\left(\left(1^2 + 1\right) + \left(2^2 + 2\right) + \left(3^2 + 3\right) + \cdots + \left(n^2 + n\right)\right)$$

g. Recall that addition is commutative and associative. So we can change both the order and the grouping of the terms in the sum in part (f). Rewrite the sum from part (f), so that it includes a group that consists of something we saw earlier in this lesson.

$$TG_n = \frac{1}{2}\left(\left(1+2+3+\cdots+n\right)+\left(1^2+2^2+3^2+\cdots+n^2\right)\right)$$

h. We have not yet seen a formula for $1^2+2^2+3^2+\cdots+n^2$. That formula is $SS_n = \frac{n(n+1)(2n+1)}{6}$. Using this information and what we know about $1+2+3+\cdots+n$, replace parts of the equation in part (g) with more concise formulas.

$$TG_n = \frac{1}{2}\left(\left(\frac{n(n+1)}{2}\right)+\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$$

i. Now simplify further to get a "nice" formula for the total number of gifts received through the n^{th} day. There is more space at the top of the next page in case you need it.

Most students will start by expanding the products in the sum to get:

$$\begin{aligned} TG_n &= \frac{1}{2}\left(\left(\frac{n^2+n}{2}\right)+\left(\frac{\left(n^2+n\right)(2n+1)}{6}\right)\right) \\ &= \frac{1}{2}\left(\left(\frac{n^2+n}{2}\right)+\left(\frac{2n^3+3n^2+n}{6}\right)\right). \end{aligned}$$

Some students may notice that it is more efficient to instead factor out $n(n+1)$:

$$
\begin{aligned}
TG_n &= \frac{1}{2}\left(\left(\frac{n(n+1)}{2}\right)+\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\
&= \frac{n(n+1)}{2}\left(\frac{1}{2}+\frac{2n+1}{6}\right)\\
&= \frac{n(n+1)}{2}\left(\frac{3}{6}+\frac{2n+1}{6}\right)\\
&= \frac{n(n+1)}{2}\left(\frac{2n+4}{6}\right)\\
&= \frac{2n(n+1)}{2}\left(\frac{n+2}{6}\right)\\
&= n(n+1)\left(\frac{n+2}{6}\right)\\
&= \frac{n(n+1)(n+2)}{6}.
\end{aligned}
$$

4. Use the formula from Question 3(i) to determine how many gifts you would have by the end of the 12 Days of Thanksgiving. Show your work.

$$\frac{12(12+1)(12+2)}{6} = 364 \text{ gifts}$$

We include a proof that $SS_n = \frac{n(n+1)(2n+1)}{6}$ for teachers who want to develop it with their students.

Proof: Notice that for any number k, $(k-1)^3 = k^3 - 3k^2 + 3k - 1$. Solving for k^2 gives

$$k^2 = \frac{k^3 - (k-1)^3 + 3k - 1}{3}\ .$$

So

$$1^2=\frac{(1)^3-(1-1)^3+3(1)-1}{3},\quad 2^2=\frac{(2)^3-(2-1)^3+3(2)-1}{3},\quad 3^2=\frac{(3)^3-(3-1)^3+3(3)-1}{3},$$

and so on.

Thus

$$\begin{aligned}1^2+2^2+3^2+\cdots+n^2&=\frac{(1)^3-(1-1)^3+3(1)-1}{3}+\frac{(2)^3-(2-1)^3+3(2)-1}{3}+\\&\quad\frac{(3)^3-(3-1)^3+3(3)-1}{3}+\cdots+\frac{(n)^3-(n-1)^3+3(n)-1}{3}\\&=\frac{1^3-0^3+3(1)-1}{3}+\frac{2^3-1^3+3(2)-1}{3}+\\&\quad\frac{3^3-2^3+3(3)-1}{3}+\cdots+\frac{n^3-(n-1)^3+3(n)-1}{3}\\&=\frac{1}{3}\left(\begin{array}{l}\left(1^3-0^3+3(1)-1\right)+\left(2^3-1^3+3(2)-1\right)+\\\left(3^3-2^3+3(3)-1\right)+\cdots+\left(n^3-(n-1)^3+3(n)-1\right)\end{array}\right)\\&=\frac{1}{3}\left(\begin{array}{l}\left(1^3-0^3+2^3-1^3+3^3-2^3+\cdots+n^3-(n-1)^3\right)+\\\left(3(1)+3(2)+3(3)+\cdots+3(n)\right)+(-1-1-1-\cdots-1)\end{array}\right)\\&=\frac{1}{3}\left(n^3+3(1+2+3+\cdots+n)+n(-1)\right)\\&=\frac{1}{3}\left(n^3+3\left(\frac{n(n+1)}{2}\right)-n\right)\\&=\frac{1}{3}n\left(n^2+\frac{3(n+1)}{2}-1\right)\\&=\frac{1}{3}n\left(\frac{2n^2}{2}+\frac{3n+3}{2}-\frac{2}{2}\right)\\&=\frac{n\left(2n^2+3n+1\right)}{6}\\&=\frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

12 Days of Thanksgiving: Lyrics (Student Version)

Have you heard the popular song, “The 12 Days of Christmas”? The lyrics are:

On the first day of Christmas
My true love sent to me:
A partridge in a pear tree.

On the second day of Christmas
My true love sent to me:
Two turtle doves,
And a partridge in a pear tree.

The song continues in this fashion. The final verse is:

On the twelfth day of Christmas
My true love gave to me:
Twelve drummers drumming,
Eleven pipers piping,
Ten lords a-leaping,
Nine ladies dancing,
Eight maids a-milking,
Seven swans a-swimming,
Six geese a-laying,
Five gold rings,
Four calling birds,
Three French hens,
Two turtle doves,
And a partridge in a pear tree.

You can do an internet search for “12 Days of Christmas song” to find various recordings if you would like to listen to the song.

In this part of the lesson, you will write a Thanksgiving version of this song. The guidelines are as follows:

- You must include 12 items that are related to Thanksgiving.
- The rhythm must match the Christmas version. You will know if the rhythm is correct if you can sing your words to the tune of the Christmas version.

Example:

Here are the first verses of "The 12 Days of Christmas" and a variation called "The 12 Days of School." The highlighting indicates the beginning of each beat. Notice that the beats in the two versions match up, so that the school version preserves the rhythm of the Christmas version.

Christmas Version:	On the	first	day of	Christ-	mas,	my true love	sent to	me	_
School Version:	On the	first	day of	school	_,	my tea-cher	gave to	me	_

Christmas Version:	a	par-	tridge	in a	pear	tree	_ _.
School Version:	a	pen-	cil	in a	pret-ty	bag	_ _.

Write the Thanksgiving verses in the space provided below. Mark the beats in your song to ensure that the rhythm is the same as in the Christmas version.

1. Christmas Version: Thanksgiving Version:
   On the first day of Christ-mas,
   my true love sent to me _
   A par-tridge in a pear tree_ _.

2. Christmas Version: Thanksgiving Version:
   Two tur-tle doves

3. Christmas Version: Thanksgiving Version:
   Three French hens

4. Christmas Version: Thanksgiving Version:
   Four call-ing birds

5. Christmas Version: Thanksgiving Version:
   Five gold rings_ _

6. Christmas Version: Thanksgiving Version:
   Six geese a-lay-ing

7. Christmas Version: Thanksgiving Version:
   Se-ven swans a-swim-ming

8. Christmas Version: Thanksgiving Version:
   Eight maids a-milk-ing

9. Christmas Version: Thanksgiving Version:
   Nine la-dies danc-ing

10. Christmas Version: Thanksgiving Version:
    Ten lords a-leap-ing

11. Christmas Version: Thanksgiving Version:
    E-lev-en pip-ers pip-ing

12. Christmas Version: Thanksgiving Version:
    Twelve drum-mers drum-ming.

12 Days of Thanksgiving: The Math (Student Version)

Now that you have written your own version of "The 12 days of Thanksgiving," we will investigate some math connections to this song! For all of the following questions, assume that you receive all of the items mentioned in the verse and the refrain for that specific day.

Example: On the second day you receive three gifts because the verse for the second day is:

On the second day of Christmas
My true love sent to me:
Two turtle doves,
And a partridge in a pear tree.

Work with your group to answer the following questions.

1. Let G_3, G_4, and G_5 represent the number of gifts you receive on Day 3, Day 4 and Day 5, respectively. (Note that $G_1 = 1$ and $G_2 = 3$.)
    a. How many gifts do you receive on Day 3?
       $G_3 =$
    b. How many gifts do you receive on Day 4?
       $G_4 =$
    c. How many gifts do you receive on Day 5?
       $G_5 =$

2. Do you notice a pattern in how the number of gifts changes from one day to the next?

3. If you receive x gifts on Day 10, how many would you receive on Day 11?

4. Suppose G_{n-1} represents the number of gifts you receive on Day $n-1$. What does G_n represent? What is the value of G_0? Give a formula for G_n that uses G_{n-1}.

   G_n represents:

   $G_0 =$

   $G_n =$

5. The formula in Question 4 is called a ***recursive formula***. Can you imagine any difficulties in using this recursive formula to find the number of gifts on a particular day?

Since recursive formulas are not always practical, a more efficient formula for G_n is $1+2+3+\cdots+n$. This works well until the value of n gets too large. If we want to find the sum of the integers from 1 to 67 for example, we would have to add $1+2+3+\cdots+65+66+67$, and that might take a while.

6. Complete the table and then develop a more efficient formula for $G_n = 1+2+3+\cdots+n$, the number of gifts you receive on the n^{th} day.

n	1	2	3	4	5	6
G_n						

7. Use the formula you developed in Question 6 to determine how many gifts you would receive on Day 12. Show your work.

12 Days of Thanksgiving: Guided Notes (Student Version)

We now prove that the formula you found in Question 6 of the worksheet is correct.

Now that we have an efficient formula for finding the number of gifts on the n^{th} day of Thanksgiving, another interesting thing to consider is the total number of gifts through Day *n*.

1. For example, what is the number of gifts through Day 3? (Add the number of gifts on each of Days 1, 2, and 3.)

2. How many gifts have been received through the fourth day? The fifth day? Show your work below.

   Fourth Day:

   Fifth Day:

3. It is clear that finding the total number of gifts through the n^{th} day will become time-consuming as the value of *n* increases. As before, we can develop a more efficient formula for finding these totals. Let TG_n be the total number of gifts received through Day *n*. We will answer the questions below in order to develop a formula for TG_n.

   a. Recall that the formula for the number of gifts received on the n^{th} day is $G_n = \dfrac{n(n+1)}{2}$.

      Explain why $G_n = \dfrac{n(n+1)}{2} = \dfrac{1}{2}\left(n^2 + n\right)$.

b. If you are asked to find the total number of gifts received through the fourth day you must add the number of gifts received on which days? Write the answer in words.

c. Express the answer to part (b) using symbols. Recall that the total number of gifts received through Day 4 is represented by TG_4 and the number of gifts received on Day n is represented by G_n

d. Write the sum for the total number of gifts received through Day n using symbols as in part (c). Show at least the first three terms and the last term. Use "$\cdots$" to represent any "hidden" terms.

e. Replace G_1, G_2, etc. in part (d) by using the formula from part (a). Do not simplify.

f. You should be able to factor a constant out of the sum in (e). Do so, and write the result here.

g. Recall that addition is commutative and associative. So we can change both the order and the grouping of the terms in the sum in part (f). Rewrite the sum from part (f), so that it includes a group that consists of something we saw earlier in this lesson.

h. We have not yet seen a formula for $1^2+2^2+3^2+\cdots+n^2$. That formula is $SS_n=\dfrac{n(n+1)(2n+1)}{6}$. Using this information and what we know about $1+2+3+\cdots+n$, replace parts of the equation in part (g) with more concise formulas.

i. Now simplify further to get a "nice" formula for the total number of gifts received through the n^{th} day. There is more space at the top of the next page in case you need it.

4. Use the formula from Question 3(i) to determine how many gifts you would have by the end of the 12 Days of Thanksgiving. Show your work.

To Infinity and Beyond! Or Not?

Lesson Overview

Brief Description: Students will gather data by launching a rubber band to see if the relationship between distance traveled and the extension (amount of stretching) of the rubber band has a linear or nonlinear relationship.

Time: 120 Minutes

Grades: 9-12

Topic: Scatter plots and least squares regression line

Paired with: Science

Materials for each group:

- Rubber band
- Object of known mass to use to determine the spring constant
- Protractor
- Ruler and measuring tape
- Paperclip
- Graphing technology (e.g., TI-84 Plus, Desmos, Excel)
- Student handouts (guided notes, worksheet)

In this lesson, students will complete a three-part activity that involves both gathering data and then interpreting the results. In the first part, students will use guided notes to review science vocabulary related to elastic potential energy and kinetic energy. They will gather data in the second part and analyze the data in the final part.

For the activity itself, each group of students will stretch and then launch a rubber band. As they do so, they will record data to determine whether there is a correlation between the distance the rubber band is pulled back and the distance the rubber band travels when it is released. They will need to be consistent in how they measure distances. Each group should use enough different stretching distances so that they can notice a relationship between the variables. Performing three trials for each distance will lead to more reliable conclusions. After the students have gathered all of their data for the activity, they will complete the analysis of the data. Students will use some sort of technology to get an accurate display of their results and to calculate a least squares regression line for their data. Technology options include, but are not limited to: TI-84 graphing calculator, Desmos, Google Sheets, and Excel.

Groups who gather data correctly should notice a linear relationship between the distance the rubber band is stretched and the distance the launched rubber band travels. The results will be affected by launch angle as well as consistency in data collection. The teacher should have a conversation with students about extrapolating data. Even though the data should show a linear relationship, there is a limit to how far the rubber band can be stretched before the rubber band will no longer return to its original shape. At this point, any further data collected will not be consistent with previous data. This deformation occurs because the molecules in the rubber band will not go back to their original location if they are stretched beyond capacity. This breaking of molecules usually occurs when the rubber band is stretched to more than double its original length.

Students will create their own launch pads using materials on hand, such as books and binders. Each group will need their own space that is approximately five feet in width and thirty feet in length. If this space is not available in the classroom, the activity may have to be completed in the hallway, gym, cafeteria, or other open space. Roles can be assigned to each member of the group to make the data collection more efficient. Roles can include: launcher, measurer, recorder, and retriever of the rubber band. It is critical that students gather all of their data within the same class period in order to keep collection as consistent as possible. If a rubber band breaks, students will have to start over, as different rubber bands may produce different data. Informing students of this ahead of time will hopefully cut down on carelessness in using the rubber band.

This lesson assumes that students are familiar with elastic potential energy and kinetic energy, presumably from a past or current science class. They should be aware of Hooke's Law and what a spring constant is. From a math perspective, we assume the students are able to use technology to input data and calculate a least squares regression line, scatter plot, and residual plot.

Resources:

Khan Academy, 2020.

"What Is Elastic Potential Energy?." https://www.khanacademy.org/science/physics/work-and-energy/hookes-law/a/what-is-elastic-potential-energy.

"What Is Kinetic Energy?." https://www.khanacademy.org/science/ap-physics-1/ap-work-and-energy/kinetic-energy-ap/a/what-is-kinetic-energy.

Stein, Ben P. "Rubber Theory Fits without a Stretch." *Physics*, American Physical Society, 20 Feb. 2007, https://physics.aps.org/story/v19/st5.

Williams, Matt. "What Is Hooke's Law?." Phys.org, 16 Feb. 2015, https://phys.org/news/2015-02-law.html.

To Infinity and Beyond! Or Not? Guided Notes (Teacher Version)

Later in this lesson, your group will launch rubber bands to discover, based on the data you collect, a relationship between the distance a rubber band is pulled back and the distance the rubber band travels when released. You will work in groups to gather data, analyze the data, and then form a conclusion based on the results. We will start by reviewing some vocabulary that you have used in your science classes. Please fill in the guided notes below during the class discussion.

Elastic Potential Energy is the energy that is stored as a result of the deformation of an elastic object, for example the stretching of a spring. We use the formula below to calculate elastic potential energy:

$$EPE = \frac{1}{2}kx^2 .$$

In this formula,

- k is the spring constant (N/m);
- x is the extension (m); that is, the change in length of the rubber band;
- EPE is elastic potential energy (J).

Kinetic Energy is the energy that objects possess due to their motion. We use the formula below to calculate kinetic energy:

$$KE = \frac{1}{2}mv^2 .$$

In this formula,

- m is mass (kg);
- v is velocity (m/s);
- KE is kinetic energy (J).

One of the questions later in the lesson will require that you know the spring constant of the rubber band you will be using. In order to calculate the spring constant (k), we need to use a result known as Hooke's Law.

Hooke's law states that $F = kx$, where F is the force needed to stretch the rubber band a distance x beyond its "natural" length. We will determine the value of k by suspending an object from the rubber band. So F will be the weight of the object: $F = mg$, where g is 9.8 m/s^2 and m is the mass of the object in kilograms. Replace F with mg and solve for k.

$$
\begin{aligned}
F &= kx \\
mg &= kx \\
k &= \frac{mg}{x}
\end{aligned}
$$

Then $k = \frac{mg}{x}$, where x is the extension (difference in length between the non-stretched rubber band and the stretched rubber band).

To Infinity and Beyond! Or Not? Data Collection (Teacher Version)

1. Make sure you have the necessary materials for the activity: rubber band, weight, protractor, ruler, measuring tape, and paperclip.

2. Hang a weight from the rubber band and measure the resulting extension of the rubber band. To do this, hang the rubber band from a paperclip, and measure its length before and after you attach a weight to the rubber band.

   Mass of weight (kg): ____________

   Length of rubber band (m): ____________ (unstretched)   ____________ (stretched)

   Extension (m): _____________

   Calculate the spring constant. Show your work.

3. With your group, decide what launch angle you want to use for the rubber band. Use the same angle throughout the entire experiment.

4. Build a launch pad using objects like books and binders. Use a protractor to record the angle of the launch pad as measured from the horizontal.

   Angle: Angles should be greater than $10°$ and less than $60°$.

Prediction Question: Do you believe that the extension of the rubber band and the distance the rubber band travels will have a linear or nonlinear relationship? Explain your reasoning.

Answers will vary.

5. Fix one end of the rubber band and pull the other end back different distances past the initial tautness of the band and then launch the rubber band. Do not stretch the band to more than twice its unstretched length because that will ruin the data. Before launching, use the table below to record the distance that the rubber band is extended. (Remember that the extension is not the total length of the stretched rubber band.)

6. After the launch, measure the distance between the launch point and the part of the rubber band that is closest to the launch point.

7. Repeat Steps 5 and 6 until you have a full table of data. You will complete three trials for each extension distance.

The answers in the teacher notes are an example, based on one instance of the experiment, and students' results will vary.

Sample data:

Extension (cm)	**Trial 1** **Distance Traveled (cm)**	**Trial 2** **Distance Traveled (cm)**	**Trial 3** **Distance Traveled (cm)**
1	95	74	110
2	156	148	132
3	197	184	212
4	246	272	261
5	312	320	305
6	360	336	330
7	390	405	412

To Infinity and Beyond! Or Not? Data Analysis (Teacher Version)

Now that your group has gathered data on rubber band launches, you will graph and interpret the results. To do this accurately, use technology as determined by your teacher.

Start by entering the data into a calculator list or spreadsheet.

1. Create a scatter plot of the data with the extension of the rubber band as the independent variable, and distance the rubber band travels as the dependent variable.

2. Calculate the least squares regression line. Record the least squares regression line here, rounding coefficients to the nearest hundredth.

   $y = 51.29x + 45.19$

3. Calculate the correlation coefficient and r^2 value, rounding to the nearest ten thousandths.

   $r = 0.9920$ $r^2 = 0.9840$

4. Answer the questions below regarding the regression.

   a. Can we extrapolate the data to predict the distance traveled for an extension of 1 meter?

      No. There is an upper limit on the values of x for which the line is a good predictor of distance traveled. If the value of x exceeds the length of the original rubber band, the rubber band becomes deformed, and the data is no longer valid.

b. Interpret the slope and y-intercept of the regression line in the context of the experiment. Does the y-intercept value make sense?

The slope of 51.29 means that for every 1 cm of extension the rubber band is expected to travel approximately 51.29 cm.

The y-intercept of 45.19 means that if the rubber band had an extension of 0 cm it is expected to travel approximately 45.19 cm.

The fact that the y-intercept is not close to 0 suggests that there is a minimal extension length for which the regression line is a valid predictor of distance traveled.

c. Interpret the correlation coefficient and r^2 value of the data in context.

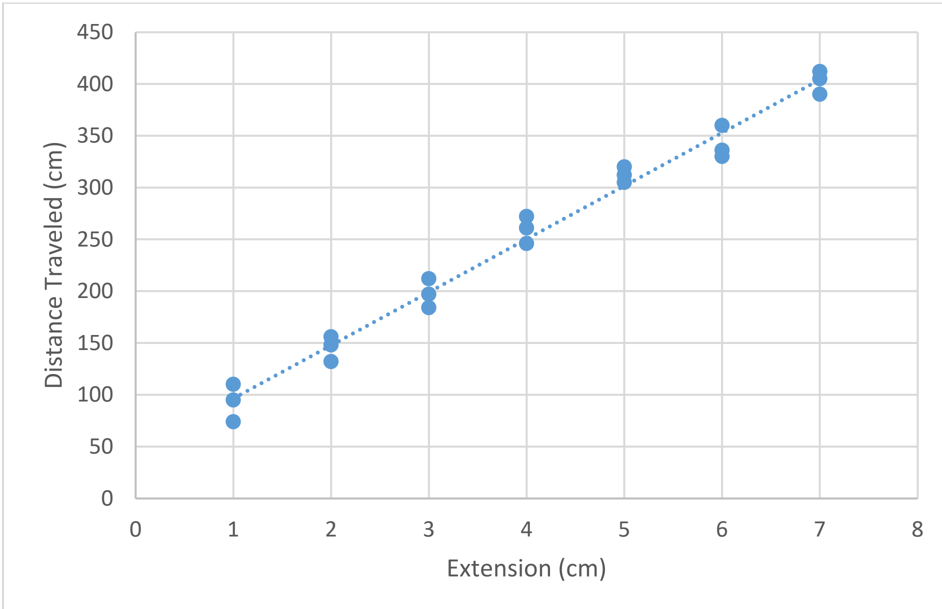
The correlation coefficient tells us that the data has a linear association with a strong positive correlation. The r^2 value indicates that 98.4% of the variation in the distance the rubber band travels can be explained by a linear relationship with the extension.

d. Are there any lurking variables in this experiment?

Lurking variables may include the effect of the wind if the experiment is conducted outside, the slope of the ground, stretchiness of the rubber band, technique of measuring the launch location, and bouncing of the rubber band after it lands.

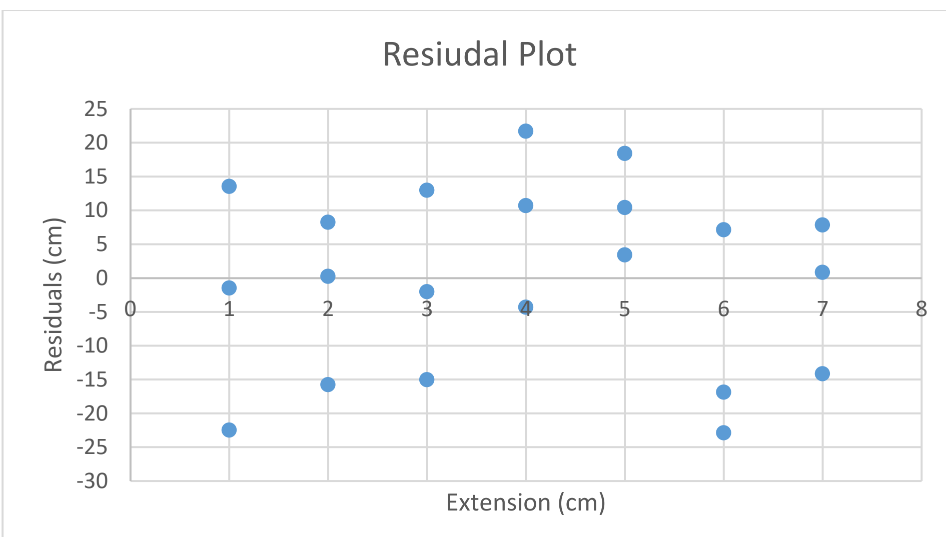
e. Sketch a scatter plot of the data that shows the least squares regression line. Be sure to label the scale on the axes.

Sample data:



f. Sketch a residual plot. Be sure to label the scale on the axes.

Sample data:



g. Is a linear function a good fit for this data set? Explain your answer.

A linear function should be a good fit for the data because the residual plot shows that there is no apparent pattern to the data and the correlation coefficient is close to 1.

5. Draw a diagram representing at what time(s) the rubber band has elastic potential energy and at what time(s) the rubber band has kinetic energy.

Drawings will vary, but students should recognize that the rubber band has elastic potential energy while it is stretched before the launch, and it has kinetic energy after it is released and before it lands.

6. What does it mean if the spring constant is large?

A large spring constant means that the rubber band is stiffer and requires more force to stretch.

7. Where you surprised by the results of the experiment? Was your earlier prediction correct?

Answers will vary.

8. We can use the information about the spring constant to calculate the velocity of the rubber band at the moment of the launch.

   a. Since elastic potential energy converts to kinetic energy, we know that $EPE = KE$. Use the equations for elastic potential energy and kinetic energy to determine v (velocity) in terms of k (spring constant), x (extension), and m (mass).

$$EPE = KE$$
$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{kx^2}{m}} = x\sqrt{\frac{k}{m}}$$

   b. You found a formula for the velocity of the rubber band at the moment it is launched. Is this velocity the same at every launch? Why or why not?

   When you pull the rubber band back further, the launch velocity is higher because the increased extension provides more elastic potential energy.

9. Use the equation from Question 8 to find the velocity of the rubber band at the moment it is launched, with an extension of 3 centimeters. Show your work and include units.

   Note: Remember that in computing *k*, we measure *x* in meters (not centimeters), in order to be consistent with the other units in the Hooke's Law equation. So in the computation below, be sure to convert centimeters to meters.

   Answers will vary. The units are m/sec.

10. Use the regression line to predict how far the rubber band will travel if the extension is 3 cm.

    Answers will vary.

11. Launch the rubber band using an extension of 3 centimeters. Was the answer from Question 10 a good prediction for the actual distance the rubber band traveled?

    Teachers may opt to skip this question if the original data collection conditions cannot be replicated.

    Answers will vary. To have the most accurate results, students need to use the same rubber band from their data collection and use the same launch angle.

To Infinity and Beyond! Or Not? Guided Notes (Student Version)

Later in this lesson, your group will launch rubber bands to discover, based on the data you collect, a relationship between the distance a rubber band is pulled back and the distance the rubber band travels when released. You will work in groups to gather data, analyze the data, and then form a conclusion based on the results. We will start by reviewing some vocabulary that you have used in your science classes. Please fill in the guided notes below during the class discussion.

___________________________ is the energy that is stored as a result of the deformation of an elastic object, for example the stretching of a spring. We use the formula below to calculate elastic potential energy:

In this formula,

- ___ is the spring constant (N/m);
- ___ is the extension (m); that is, the change in length of the rubber band;
- ______ is elastic potential energy (J).

_____________________ is the energy that objects possess due to their motion. We use the formula below to calculate kinetic energy:

In this formula,

- ___ is mass (kg);
- ___ is velocity (m/s);
- ____ is kinetic energy (J).

One of the questions later in the lesson will require that you know the spring constant of the rubber band you will be using. In order to calculate the spring constant (*k*), we need to use a result known as __________________ .

Hooke's law states that __________ , where F is the force needed to stretch the rubber band a distance x beyond its "natural" length. We will determine the value of k by suspending an object from the rubber band. So F will be the weight of the object: $F = mg$, where g is 9.8 m/s^2 and m is the mass of the object in kilograms. Replace F with mg and solve for k.

Then __________ , where x is the __________________ (difference in length between the non-stretched rubber band and the stretched rubber band).

To Infinity and Beyond! Or Not? Data Collection (Student Version)

1. Make sure you have the necessary materials for the activity: rubber band, weight, protractor, ruler, measuring tape, and paperclip.

2. Hang a weight from the rubber band and measure the resulting extension of the rubber band. To do this, hang the rubber band from a paperclip, and measure its length before and after you attach a weight to the rubber band.

   Mass of weight (kg): ____________

   Length of rubber band (m): ____________ (unstretched)   ____________ (stretched)

   Extension (m): _____________

   Calculate the spring constant. Show your work.

3. With your group, decide what launch angle you want to use for the rubber band. Use the same angle throughout the entire experiment.

4. Build a launch pad using objects like books and binders. Use a protractor to record the angle of the launch pad as measured from the horizontal.

   Angle:

Prediction Question: Do you believe that the extension of the rubber band and the distance the rubber band travels will have a linear or nonlinear relationship? Explain your reasoning.

5. Fix one end of the rubber band and pull the other end back different distances past the initial tautness of the band and then launch the rubber band. Do not stretch the band to more than twice its unstretched length because that will ruin the data. Before launching, use the table below to record the distance that the rubber band is extended. (Remember that the extension is not the total length of the stretched rubber band.)

6. After the launch, measure the distance between the launch point and the part of the rubber band that is closest to the launch point.

7. Repeat Steps 5 and 6 until you have a full table of data. You will complete three trials for each extension distance.

Extension (cm)	**<u>Trial 1</u> Distance Traveled (cm)**	**<u>Trial 2</u> Distance Traveled (cm)**	**<u>Trial 3</u> Distance Traveled (cm)**

To Infinity and Beyond! Or Not? Data Analysis (Student Version)

Now that your group has gathered data on rubber band launches, you will graph and interpret the results. To do this accurately, use technology as determined by your teacher.

Start by entering the data into a calculator list or spreadsheet.

1. Create a scatter plot of the data with the extension of the rubber band as the independent variable, and distance the rubber band travels as the dependent variable.

2. Calculate the least squares regression line. Record the least squares regression line here, rounding coefficients to the nearest hundredth.

3. Calculate the correlation coefficient and r^2 value, rounding to the nearest ten thousandths.

   $r =$ $r^2 =$

4. Answer the questions below regarding the regression.

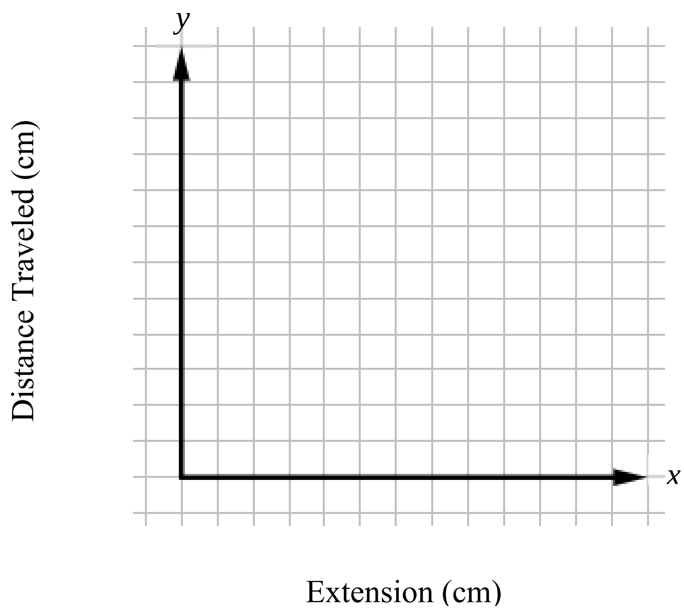
   a. Can we extrapolate the data to predict the distance traveled for an extension of 1 meter?

b. Interpret the slope and y-intercept of the regression line in the context of the experiment. Does the y-intercept value make sense?

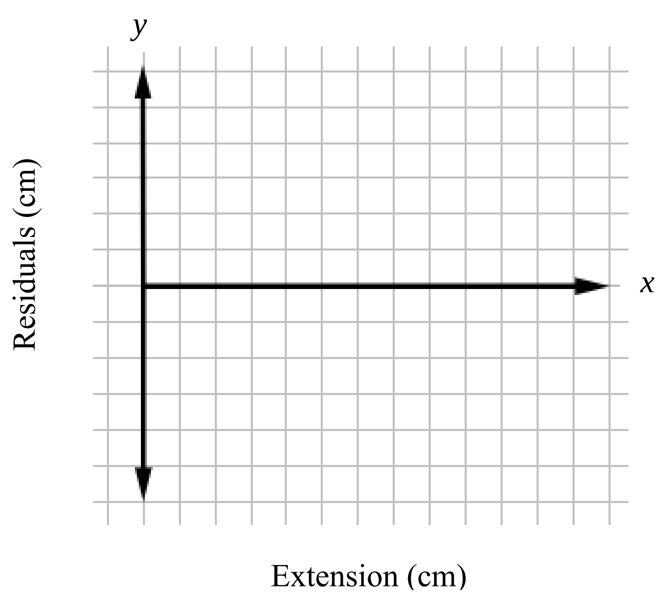
c. Interpret the correlation coefficient and r^2 value of the data in context.

d. Are there any lurking variables in this experiment?

e. Sketch a scatter plot of the data that shows the least squares regression line. Be sure to label the scale on the axes.



f. Sketch a residual plot. Be sure to label the scale on the axes.



g. Is a linear function a good fit for this data set? Explain your answer.

5. Draw a diagram representing at what time(s) the rubber band has elastic potential energy and at what time(s) the rubber band has kinetic energy.

6. What does it mean if the spring constant is large?

7. Where you surprised by the results of the experiment? Was your earlier prediction correct?

8. We can use the information about the spring constant to calculate the velocity of the rubber band at the moment of the launch.

   a. Since elastic potential energy converts to kinetic energy, we know that $EPE = KE$. Use the equations for elastic potential energy and kinetic energy to determine v (velocity) in terms of k (spring constant), x (extension), and m (mass).

   b. You found a formula for the velocity of the rubber band at the moment it is launched. Is this velocity the same at every launch? Why or why not?

9. Use the equation from Question 8 to find the velocity of the rubber band at the moment it is launched, with an extension of 3 centimeters. Show your work and include units.

   Note: Remember that in computing k, we measure x in meters (not centimeters), in order to be consistent with the other units in the Hooke's Law equation. So in the computation below, be sure to convert centimeters to meters.

10. Use the regression line to predict how far the rubber band will travel if the extension is 3 cm.

11. Launch the rubber band using an extension of 3 centimeters. Was the answer from Question 10 a good prediction for the actual distance the rubber band traveled?

Skittle Explosion

Lesson Overview

Brief Description: Students will gather data to explore exponential growth and decay.

Time: 120 Minutes

Grades: 9-12

Topic: Introduction to exponential functions

Paired with: Science

Materials for each group of students:

- Plate or tray
- Plastic or paper cup
- Roughly 75 Skittles (any item with a distinct different on the top and bottom will work)
- Graphing technology (e.g., TI-84 Plus, Desmos, Excel)
- Student worksheets (data collection and analysis, exponential functions, optional extension)

The activity in this lesson has been modified from a "Math = Love" blog post by Sarah Carter. [1]

In this lesson, students will explore exponential growth and decay by gathering their own data. This activity is intended as an introduction to exponential functions with the hope that students discover the relationship between the behavior of the function and the parameters in the symbolic form of the function. Students will work in groups of three or four.

After collecting some data, students will see if a linear model is the best fit for the data by considering the residual plot. They will realize that an exponential function better fits their data and will be able to make connections between parameters in the equation and the function behavior. Specifically, students should come to realize that the initial value of an exponential function is the same as the y-intercept of the graph of the function. They will also discover the fact

[1] "Math = Love" blog post: https://mathequalslove.blogspot.com/2014/05/modeling-exponential-growth-and-decay.html

that a base greater than 1 will result in exponential growth, while exponential decay will occur when the base is between 0 and 1. Students will use graphing technology in this lesson to graph exponential functions both efficiently and correctly as well as to compute regression equations.

Resources:

Carter, Sarah. "Modeling Exponential Growth and Decay with Skittles." *Math = Love*, 2 May 2014, https://mathequalslove.blogspot.com/2014/05/modeling-exponential-growth-and-decay.html.

Skittle Explosion: Data Collection and Analysis (Teacher Version)

In this lesson, you will gather data from an experiment using Skittles. You will analyze the data to determine if there are linear or nonlinear relationships. The equations and corresponding graphs you will discover are widely used in both math and science.

Part 1

1. Get Skittles, a cup, and a plate from your teacher. Check that all of the Skittles have an 'S' label on one and only one side. If any Skittles are not labeled correctly, please remove those Skittles from the supply.

2. Start with one Skittle. Record a 1 in the column for Trial 0 in the data collection table.

3. Place the Skittle in the cup, shake the cup, and then empty the Skittle onto the plate. If the letter S is showing, you will return the Skittle to the cup along with another Skittle. If the letter S is not showing, return the Skittle to the cup, but do not add any new Skittles. Record the number of Skittles in the cup in the column for Trial 1 in the table.

4. Again, shake the cup, and then empty the cup onto the plate. Count the number of Skittles that have the letter S showing. Put all of the Skittles from the plate back into the cup and add an additional Skittle for every Skittle that was showing an S. Record the total number of Skittles that are now in the cup in the column for Trial 2. Repeat for up to 10 trials or until you run out of candy. If you do not have enough Skittles left to complete a trial, record the number of Skittles that should be in the cup, and then end the experiment.

Example: Suppose you have 12 candies in Trial 5. When you empty the candies on the plate, 5 have an S showing, and 7 do not. You would then put 17 candies into the cup for the next trial (12 for the number of Skittles you emptied onto the plate and 5 additional Skittles because that is how many had an S showing.) You would record a total of 17 Skittles for Trial 6.

5. Each trial will consist of the following; in this order:
   - Shaking the Skittles in the cup;
   - Emptying the cup onto the plate;
   - Counting the number of Skittles that have the letter S showing and putting this number of additional Skittles into the cup;
   - Putting all of the Skittles from the plate back into the cup;
   - Recording the total number of Skittles that are in the cup.

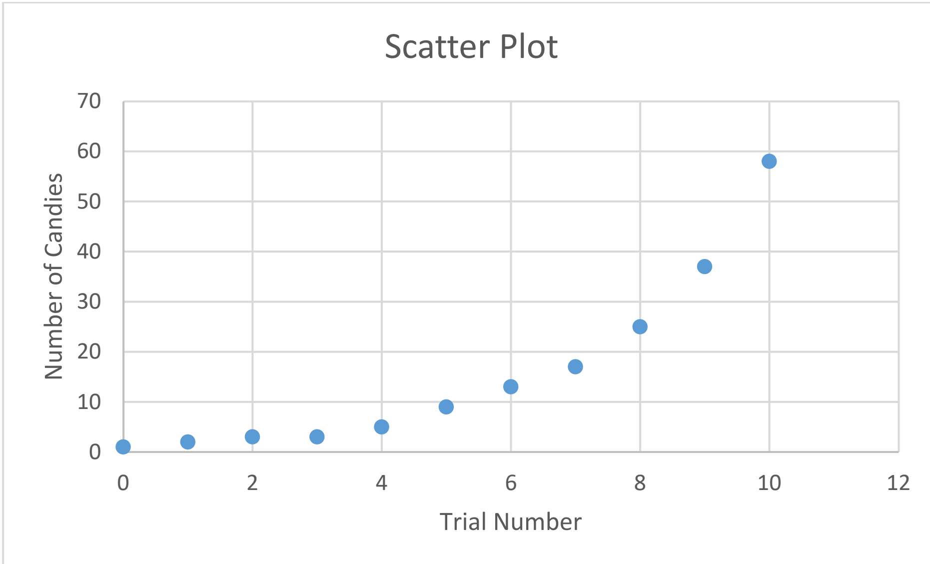
Record the data in the table below.

The answers in the teacher notes are an example, based on one instance of the experiment, and students' results will vary.

Trial Number	0	1	2	3	4	5	6	7	8	9	10
Number of Candies	1	2	3	3	5	9	13	17	25	37	58

If students have trouble getting good data results, they can start with five Skittles in their cup instead of just one. This adjustment will cause the growth of the Skittles to start more quickly. However, that means that many more Skittles will be needed for later trials. In this case, eight trials may be sufficient in order to see the trend. Collecting the data this way will be dependent on access to Skittles.

6. Sketch a graph of the data you collected. Be sure to label the scale on the axes.



7. Does there appear to be any correlation between the number of trials and the number of candies? If so describe the form, direction, strength, and any outliers of the correlation.

As the number of trials increases, so does the number of candies. The form seems to be nonlinear with a strong positive correlation and no apparent outliers.

8. Use a calculator or spreadsheet to calculate a least squares regression line for the data. Write the equation of the line below.

$y = 4.79x - 8.23$

9. Calculate the correlation coefficient and r^2 value, rounding to the nearest ten thousandths.

$r = 0.8849$ $r^2 = 0.7831$

10. Explain the meaning of each of the following as it relates to the experiment.

   a. Slope of the regression line:

      For each trial, the number of Skittles in the cup is expected to be approximately four more than in the previous trial.

   b. y-intercept of the regression line:

      We would expect to have approximately -8 Skittles in Trial 0. (So we would start the first trial with -8 Skittles.)

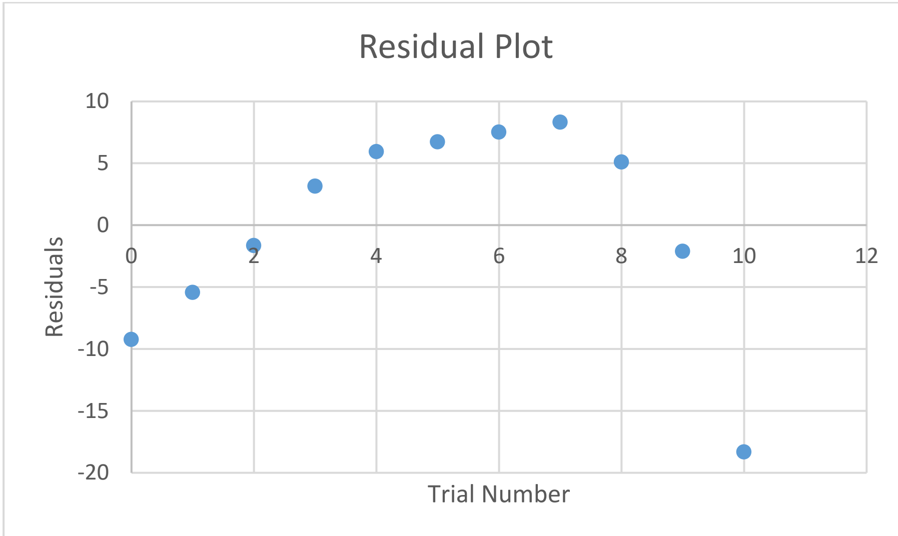
   c. r^2 value:

      The r^2 values tells us that 78% of the variation in the number of Skittles can be explained by a linear relationship with the number of trials.

11. Use a graphing calculator or spreadsheet to add the least squares line to the scatter plot of the data. Does the linear equation appear to fit the data? Explain why or why not. (If the equation "fits" the data, it will allow you to predict how many Skittles (y) would be left after (x) trials.)

    No, the linear equation does not appear to fit the data. The data appears to follow a curved trajectory.

12. Draw a sketch of the residual plot. Be sure to label the scale on the axes.



13. Based on the residual plot, is a linear model appropriate for this data? Explain why or why not.

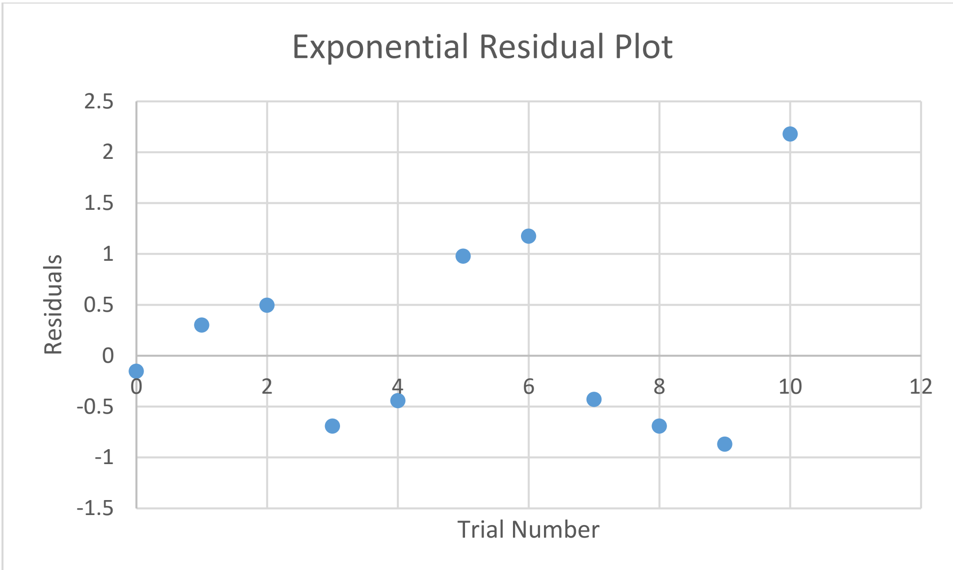
No, a linear model is not appropriate because there is a clear pattern in the residual plot.

14. A linear model is not a good fit for the data, but we can try an exponential model instead. Calculate the exponential regression equation from the data and write the result below.

$y = 1.16(1.48)^x$

Students may see the equation in the form $y = Ae^{kx}$, depending on the graphing technology they are using. In this case, help students rewrite the equation in the form $y = a \cdot b^x$ using the fact that $e^{kx} = \left(e^k\right)^x$. (So $b = e^k$.)

15. Graph the new residual plot, and use it to determine if the regression equation in Question 14 is a better fit for the data. Be sure to label the scale on the axes. In your reasoning, include the newly calculated correlation coefficient and r^2 value.



The residual plot shows that an exponential equation better fits the data. In addition, $r = 0.9954$ and $r^2 = 0.9908$.

16. Use the regression equation from Question 14 to determine the number of candies you would expect to have in the 20th trial. Show your work.

$$
\begin{aligned}
y &= 1.16(1.48)^x \\
&= 1.16(1.48)^{20} \\
&= 249.119
\end{aligned}
$$

We would expect about 249 Skittles in Trial 20.

Part 2

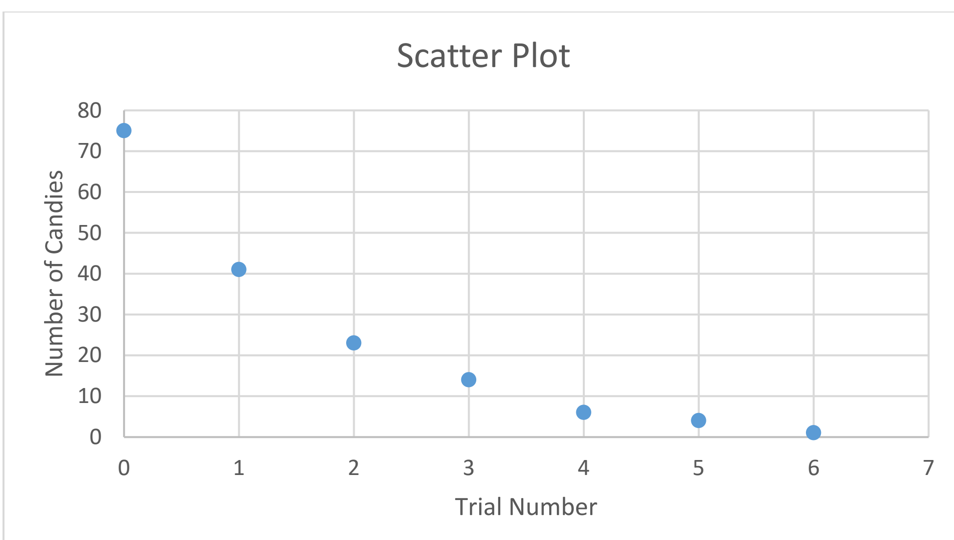
1. Start with all of the candies. Count them and record this number for Trial 0. Place them in the cup, shake the cup, and empty the cup. Place each candy that is showing an S back in the cup, and discard the others. Record the number of Skittles that you return to the cup in the table below for Trial 1. Repeat this process until you have fewer than 3 Skittles but more than 0 Skittles.

   The answers in the teacher notes are an example, based on one instance of the experiment, and students' results will vary.

Trial Number	0	1	2	3	4	5	6	7	8	9	10
Number of Candies	75	41	23	14	6	4	1	-	-	-	-

2. Sketch a graph of the data you collected. Be sure to label the scale on the axes.

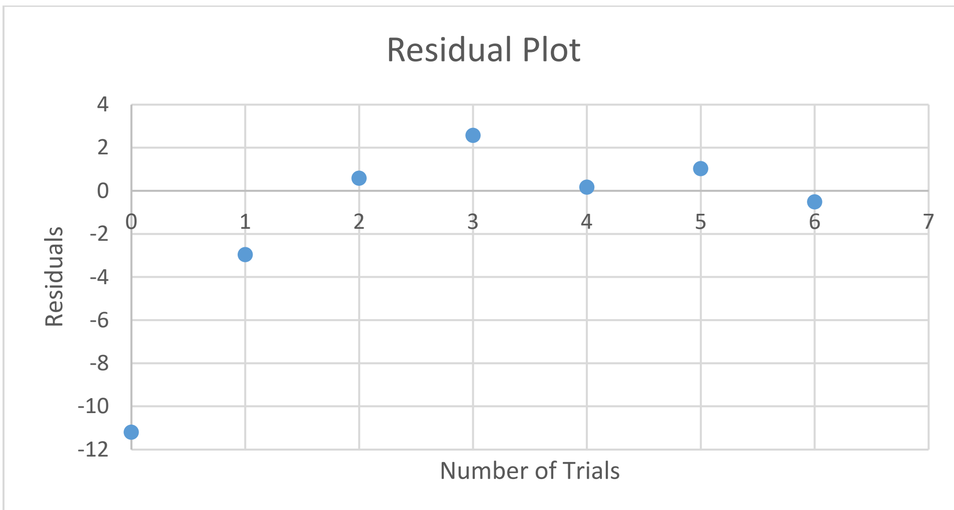
   The answers in the teacher notes are an example, based on one instance of the experiment, and students' results will vary.



3. In Part 1, an exponential function was a good fit for the data. Calculate the exponential regression equation for the newly gathered data and write the result below.

$y = 86.20(0.51)^x$ (Students may need to convert from the $y = Ae^{kx}$ form of the equation.)

4. Graph the residual plot, and use it to determine if the regression equation in Question 3 is a good fit for the data. Be sure to label the scale on the axes. In your reasoning, include the correlation coefficient and r^2 value.



The residual plot indicates that an exponential function is a good representation of the data. In addition, $r = -0.9875$ and $r^2 = 0.9751$.

5. Based on the regression equation, how many candies would you expect to have after 20 trials? Show your work. Does this answer make sense in the context of the problem?

$$\begin{aligned} y &= 86.2(.51)^x \\ &= 86.2(.51)^{20} \\ &= 0.000122 \end{aligned}$$

No, there will always be a whole number of Skittles left after each toss. So an answer of 0.000122 Skittles does not make sense.

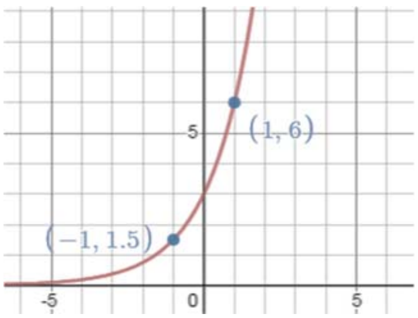
Further explanation for teachers: The original activity should result in a base of roughly 1.5 for exponential growth. This is because roughly half of the Skittles will land with an S facing up when the cup is emptied. This means that half of the original number plus the original number of Skittles are returned to the cup. The base for the decay function should be roughly 0.5, because about half of the original number of Skittles are returned to the cup for the next trial.

Skittle Explosion: Exponential Functions (Teacher Version)

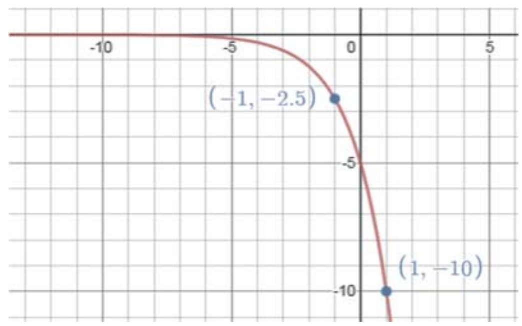
You developed two different exponential equations as a result of the Skittle experiment. The first equation represents ***exponential growth*** and the second equation represents ***exponential decay***. We now look further into exponential functions to investigate the relationship between the constants in the equation and the behavior of the function.

1. Use technology to graph the given functions. Copy a sketch of each graph in the empty space below the equation. You may have to adjust the display window in order to see the graph.

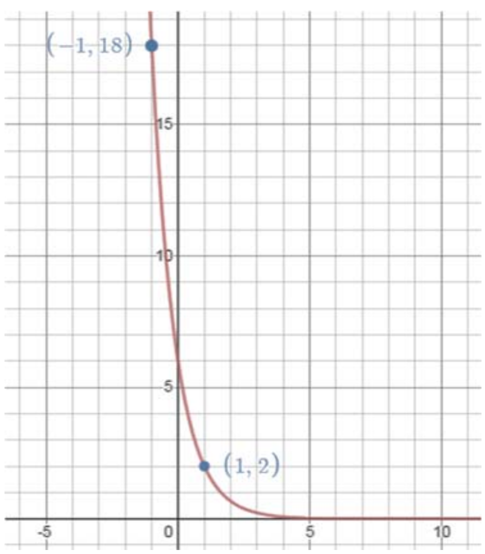
a. $y = 3(2)^x$



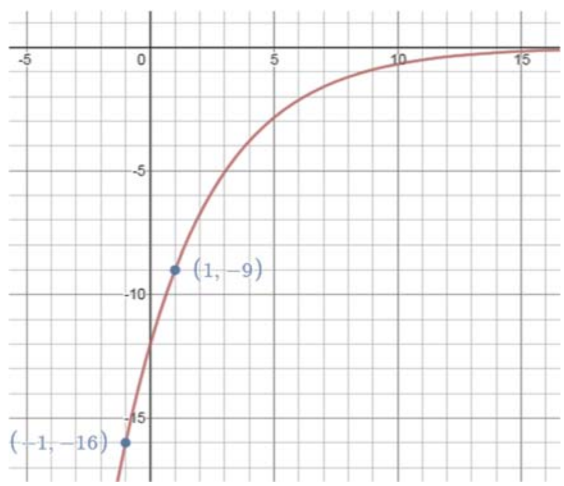
b. $y = -5(2)^x$



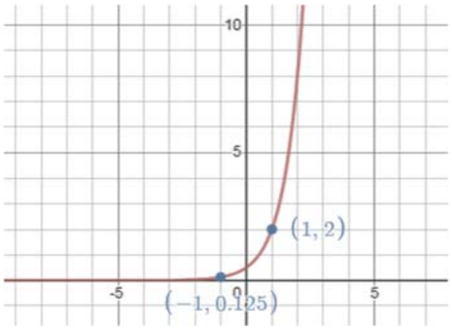
c. $y = 6\left(\frac{1}{3}\right)^x$



d. $y = -12\left(\frac{3}{4}\right)^x$



e. $y = 4(25)^x$



f. $y = \frac{1}{2}(-3)^x$

No graph exists.

2. Exponential equations can be expressed in the form $y = a \cdot b^x$ where b is a positive real number. Use the examples above to make a conjecture about how the values of a and b affect the graph of the equations.

   The value of a is the y-coordinate of the y-intercept of the graph. If $b > 1$, the graph increases and if $0 < b < 1$, then the graph decreases. The value of b also impacts the steepness of the graph. The graph increases more quickly for larger b values when $b > 1$, and decreases more quickly for smaller b values when $0 < b < 1$. The graphs always go through the points $(1, b)$ and $\left(-1, \frac{1}{b}\right)$.

3. Can you find equations for the exponential functions represented by these tables? Do this without using graphing technology or computing regression equations.

a.

x	0	1	2	3	4
y	15	30	60	120	240

$y = 15(2)^x$

b.

x	0	1	2	3	4
y	729	243	81	27	9

$y = 729\left(\frac{1}{3}\right)^x$

c.

x	0	1	2	3	4
y	100	50	25	12.5	6.25

$y = 200\left(\frac{1}{2}\right)^x$

4. The table below shows some partial data for an exponential function.

x	0	1	2	3	4
y		12			96

a. Find an equation for the function.

Methods for finding the correct equation will vary. One method is guess-and-check and will probably be the approach must students will take to discover that $y = 6(2)^x$.

Another approach is to start with $a \cdot b^1 = 12$ and $a \cdot b^4 = 96$. So, $\frac{a \cdot b^4}{a \cdot b^1} = \frac{96}{12}$, and thus $b^3 = 8$ so $b = 2$.

b. Can we use the data in the table to deduce that the function is exponential?

No. With only two data points the data could have any relationship.

5. In exponential growth functions, what happens to the y-values as x increases? What about exponential decay functions?

   Growth: As x increases, the y-value goes towards infinity.

   Decay: As x increases, the y-value is gets closer and closer to 0 without ever reaching it.

6. What are the domain and range of exponential functions? As mathematical functions (unrelated to Skittles), consider whether the x-values could be negative.

   Domain: $\mathbb{R}$
   Since the base is positive, the exponent can be any value.

   Range: $(0, \infty)$
   Since the exponent can be arbitrarily large (positive or negative), we can get any positive number as a result of the function.

   Note: The line $y = 0$ is a horizontal asymptote for both exponential growth and decay.

Skittle Explosion: Optional Extension (Teacher Version)

Create an angle with the vertex at the center of the plate. Decide if you would like to use the interior or exterior angle for the experiment. To model exponential growth, start with 5 Skittles and empty them onto the plate so that they are roughly evenly distributed across the plate. If the Skittles land in the interior (exterior) of the angle, they mutate and double. Put the Skittles (including the newly "created" Skittles) back in the cup and repeat until you have gathered enough data to see a trend. To model exponential decay, start with 75 Skittles and empty them on the plate so that they are roughly evenly distributed across the plate. Any that land in the interior (exterior) of the angle become contaminated. Count the safe Skittles and put them back into the cup. Discard the contaminated Skittles. Repeat the process with the safe Skittles until you have gathered enough data to see a trend. Record the data and regression equation for both exponential growth and decay on a separate piece of paper. Be sure to make note of the measure of the angle that you used for the data collection (the measure of the exterior angle is greater than $180°$). Do you notice any connections between the angle and the regression equation?

In order for this experiment to work, the plate must be circular. For the decay version, we recommend using the interior angle, or starting with more than 75 Skittles.

For exponential growth and decay, the angle will determine the approximate value of the base in the exponential function. Let θ be the angle measurement in degrees. Then the fraction of Skittles that typically land in the target region is $\frac{\theta}{360}$. So the base for the exponential growth function will be approximately $1+\frac{\theta}{360}$, and the base for the exponential decay function will be approximately $\frac{\theta}{360}$.

Students can do a gallery walk around the room to see their classmates' equations and central angles to help make the connection more clear.

Skittle Explosion: Data Collection and Analysis (Student Version)

In this lesson, you will gather data from an experiment using Skittles. You will analyze the data to determine if there are linear or nonlinear relationships. The equations and corresponding graphs you will discover are widely used in both math and science.

Part 1

1. Get Skittles, a cup, and a plate from your teacher. Check that all of the Skittles have an 'S' label on one and only one side. If any Skittles are not labeled correctly, please remove those Skittles from the supply.

2. Start with one Skittle. Record a 1 in the column for Trial 0 in the data collection table.

3. Place the Skittle in the cup, shake the cup, and then empty the Skittle onto the plate. If the letter S is showing, you will return the Skittle to the cup along with another Skittle. If the letter S is not showing, return the Skittle to the cup, but do not add any new Skittles. Record the number of Skittles in the cup in the column for Trial 1 in the table.

4. Again, shake the cup, and then empty the cup onto the plate. Count the number of Skittles that have the letter S showing. Put all of the Skittles from the plate back into the cup and add an additional Skittle for every Skittle that was showing an S. Record the total number of Skittles that are now in the cup in the column for Trial 2. Repeat for up to 10 trials or until you run out of candy. If you do not have enough Skittles left to complete a trial, record the number of Skittles that should be in the cup, and then end the experiment.

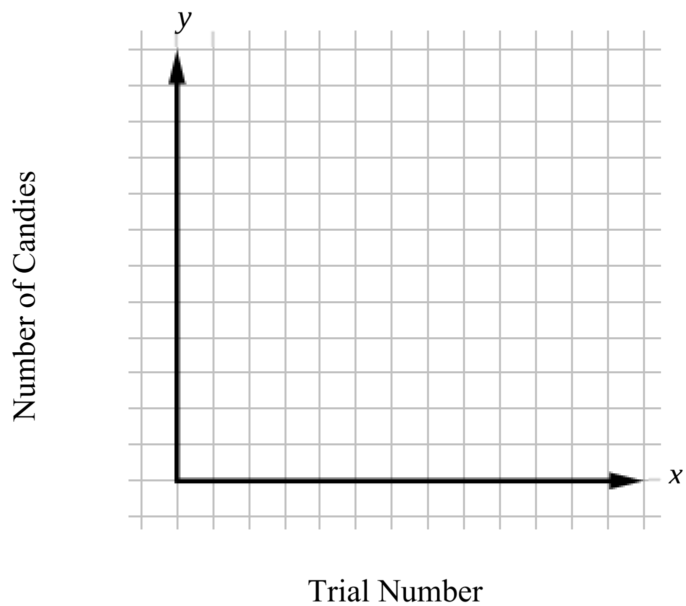
Example: Suppose you have 12 candies in Trial 5. When you empty the candies on the plate, 5 have an S showing, and 7 do not. You would then put 17 candies into the cup for the next trial (12 for the number of Skittles you emptied onto the plate and 5 additional Skittles because that is how many had an S showing.) You would record a total of 17 Skittles for Trial 6.

5. Each trial will consist of the following; in this order:
   - Shaking the Skittles in the cup;
   - Emptying the cup onto the plate;
   - Counting the number of Skittles that have the letter S showing and putting this number of additional Skittles into the cup;
   - Putting all of the Skittles from the plate back into the cup;
   - Recording the total number of Skittles that are in the cup.

Record the data in the table below.

Trial Number	**Number of Candies**
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

6. Sketch a graph of the data you collected. Be sure to label the scale on the axes.



7. Does there appear to be any correlation between the number of trials and the number of candies? If so describe the form, direction, strength, and any outliers of the correlation.

8. Use a calculator or spreadsheet to calculate a least squares regression line for the data. Write the equation of the line below.

9. Calculate the correlation coefficient and r^2 value, rounding to the nearest ten thousandths.

$r =$ $\qquad\qquad\qquad$ $r^2 =$

10. Explain the meaning of each of the following as it relates to the experiment.

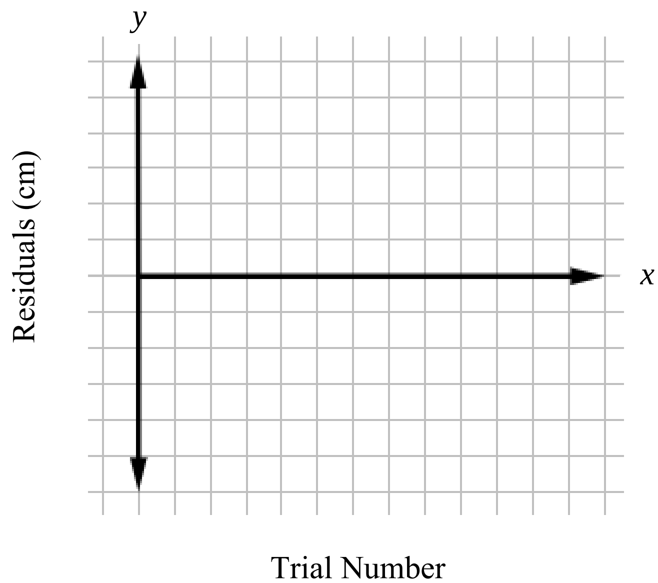
    a. Slope of the regression line:

    b. y-intercept of the regression line:

    c. r^2 value:

11. Use a graphing calculator or spreadsheet to add the least squares line to the scatter plot of the data. Does the linear equation appear to fit the data? Explain why or why not. (If the equation "fits" the data, it will allow you to predict how many Skittles (y) would be left after (x) trials.)

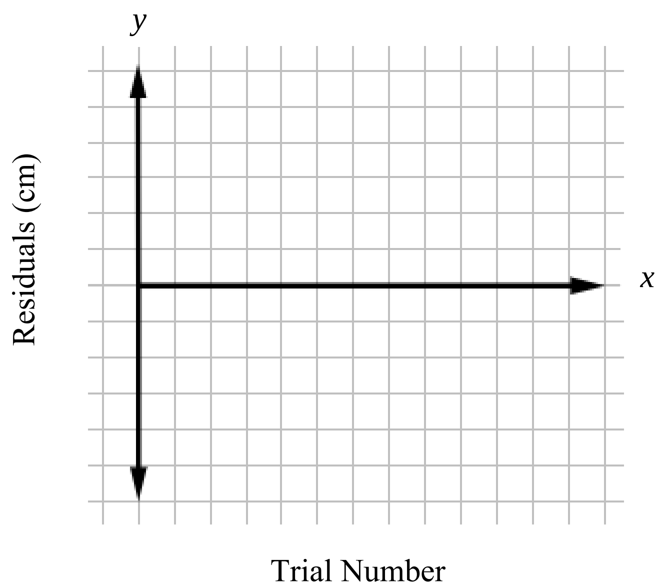
12. Draw a sketch of the residual plot. Be sure to label the scale on the axes.



13. Based on the residual plot, is a linear model appropriate for this data? Explain why or why not.

14. A linear model is not a good fit for the data, but we can try an exponential model instead. Calculate the exponential regression equation from the data and write the result below.

15. Graph the new residual plot, and use it to determine if the regression equation in Question 14 is a better fit for the data. Be sure to label the scale on the axes. In your reasoning, include the newly calculated correlation coefficient and r^2 value.



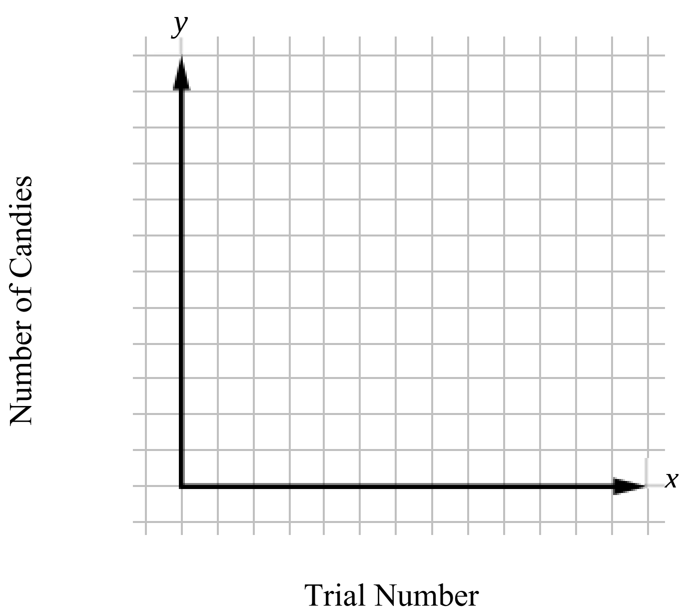
16. Use the regression equation from Question 14 to determine the number of candies you would expect to have in the 20^{th} trial. Show your work.

Part 2

1. Start with all of the candies. Count them and record this number for Trial 0. Place them in the cup, shake the cup, and empty the cup. Place each candy that is showing an S back in the cup, and discard the others. Record the number of Skittles that you return to the cup in the table below for Trial 1. Repeat this process until you have fewer than 3 Skittles but more than 0 Skittles.

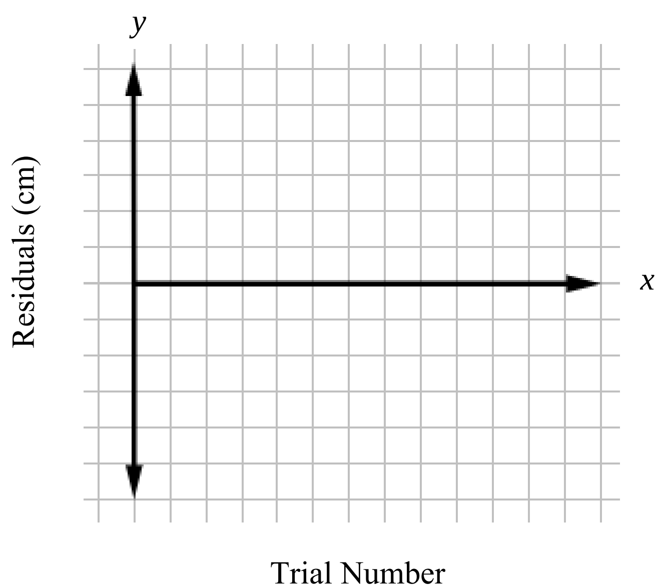
Trial Number	0	1	2	3	4	5	6	7	8	9	10
Number of Candies											

2. Sketch a graph of the data you collected. Be sure to label the scale on the axes.



3. In Part 1, an exponential function was a good fit for the data. Calculate the exponential regression equation for the newly gathered data and write the result below.

4. Graph the residual plot, and use it to determine if the regression equation in Question 3 is a good fit for the data. Be sure to label the scale on the axes. In your reasoning, include the correlation coefficient and r^2 value.



5. Based on the regression equation, how many candies would you expect to have after 20 trials? Show your work. Does this answer make sense in the context of the problem?

Skittle Explosion: Exponential Functions (Student Version)

You developed two different exponential equations as a result of the Skittle experiment. The first equation represents ***exponential growth*** and the second equation represents ***exponential decay***. We now look further into exponential functions to investigate the relationship between the constants in the equation and the behavior of the function.

7. Use technology to graph the given functions. Copy a sketch of each graph in the empty space below the equation. You may have to adjust the display window in order to see the graph.

   a. $y = 3(2)^x$

   b. $y = -5(2)^x$

   c. $y = 6\left(\frac{1}{3}\right)^x$

   d. $y = -12\left(\frac{3}{4}\right)^x$

e. $y = 4(25)^x$

f. $y = \frac{1}{2}(-3)^x$

8. Exponential equations can be expressed in the form $y = a \cdot b^x$ where b is a positive real number. Use the examples above to make a conjecture about how the values of a and b affect the graph of the equations.

9. Can you find equations for the exponential functions represented by these tables? Do this without using graphing technology or computing regression equations.

a.

x	0	1	2	3	4
y	15	30	60	120	240

b.

x	0	1	2	3	4
y	729	243	81	27	9

c.

x	0	1	2	3	4
y	100	50	25	12.5	6.25

10. The table below shows some partial data for an exponential function.

x	0	1	2	3	4
y		12			96

a. Find an equation for the function.

b. Can we use the data in the table to deduce that the function is exponential?

11. In exponential growth functions, what happens to the y-values as x increases? What about exponential decay functions?

Growth:

Decay:

12. What are the domain and range of exponential functions? As mathematical functions (unrelated to Skittles), consider whether the x-values could be negative.

Skittle Explosion: Optional Extension (Student Version)

Create an angle with the vertex at the center of the plate. Decide if you would like to use the interior or exterior angle for the experiment. To model exponential growth, start with 5 Skittles and empty them onto the plate so that they are roughly evenly distributed across the plate. If the Skittles land in the interior (exterior) of the angle, they mutate and double. Put the Skittles (including the newly "created" Skittles) back in the cup and repeat until you have gathered enough data to see a trend. To model exponential decay, start with 75 Skittles and empty them on the plate so that they are roughly evenly distributed across the plate. Any that land in the interior (exterior) of the angle become contaminated. Count the safe Skittles and put them back into the cup. Discard the contaminated Skittles. Repeat the process with the safe Skittles until you have gathered enough data to see a trend. Record the data and regression equation for both exponential growth and decay on a separate piece of paper. Be sure to make note of the measure of the angle that you used for the data collection (the measure of the exterior angle is greater than 180°). Do you notice any connections between the angle and the regression equation?

Conclusion

The topics in these interdisciplinary Mathematics lessons address a wide variety of student interests. The creation and discovery students will experience through the lessons will help to make their learning relevant. In addition to increased student engagement, students will also benefit from seeing a discussion of topics from multiple viewpoints, including methods of presentation that may differ from their mathematics textbooks. This will help with both their understanding and retention of the topics.

Interdisciplinarity has benefits for teachers as well. Collaborating with teachers from other disciplines leads to increased creativity in the planning of lessons. It ensures that lessons contain factual information from all of the content areas covered and leads to increased knowledge for both teachers. Mathematics topics can get dense quickly, so collaborating with a non-Mathematician can force the Math teacher to develop descriptions that are more easily understood by students. In many schools, teachers socialize primarily with teachers of their own content area, due to common academic backgrounds and the structure of the school. Collaborating with teachers from other disciplines allows social connections to form that may not have previously existed.

Although we have provided only eight lessons in this essay, some possibilities for other interdisciplinary lessons include:

- Mathematics of voting, paired with Social Studies
- Mathematics in music (theory), paired with Music
- Mathematics in music (sound), paired with Physics
- Mathematics in text mining, paired with Language Arts
- Mathematics in perspective drawing, paired with Art
- Mathematics in performance statistics, paired with Physical Education

There are endless options for what hardworking teachers can develop together. We hope that these lessons will help foster an interest in mathematics across the curriculum.

Appendix: TI-84 Plus Graphing Calculator Instructions

The instructions below discuss linear regression on a TI-84 Plus graphing calculator, and can be easily modified for exponential regression.

1. Enter data into lists L_1 and L_2:
   a. Press [STAT].
   b. Select 1: Edit…
   c. Clear out any existing data by highlighting L_1, pressing [CLEAR] and then [ENTER]. Repeat for L_2 if necessary.
   d. Type x values in L_1 and y values in L_2.

2. Calculate the least squares regression line:
   a. Press [STAT].
   b. Select CALC (use the right arrow).
   c. Select 4:LinReg(ax+b).
      Settings:
         Xlist: L_1
         Ylist: L_2
         FreqList: (leave empty)
         Store RegEQ: Y_1 (Press [VARS], Y-VARS (use the right arrow), select 1: Function…, select 1: Y_1)
         Calculate (enter)

3. Turn Diagnostics On (if the correlation coefficient was not displayed as a result of Step 2):
   a. Press [2nd] [0] (Catalog) and then i.
   b. Scroll down and select DiagnositicOn.
   c. Press [ENTER] on the home screen.
   d. Repeat the linear regression from Step 2.

4. View scatter plot with least squares regression line:
   a. Equation should already be in Y_1 from Step 2.
   b. Press [2nd] [Y=] (Stat Plot) and select Plot1.
   c. Highlight "On."
   d. Make sure the Xlist is L_1 and the Ylist is L_2.
   e. Press [ZOOM], and then select ZoomStat to get an appropriate viewing window.

5. Entering Residuals in L_3:
   a. Press [STAT].
   b. Select 1: Edit…
   c. Highlight L_3.
   d. Press 2nd [STAT].
   e. Select RESID.
   f. Press [ENTER] so that L_3 fills up with calculated residuals.

6. Graph Residuals
   a. Press [2nd] [Y=] and turn Plot1 off and turn Plot2 on.
   b. Make sure the Xlist is L_1 and the Ylist is L_3 in Plot2.
   c. Press [ZOOM], and then select ZoomStat to display the residual plot.