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PI – NOT JUST AN ORDINARY NUMBER

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By
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1. Prologue - *Pi Day*

I first witnessed how the high school where I was working as a math teacher was celebrating "Pi day" during my second year of teaching. It was an impressive school event, for which teachers and students were getting ready ahead of time. The school Math club collected and posted multiple Pi facts prior to the event; created posters, songs, and problems; and held a Pi contest on March 14. It all happened that day during lunch time. Students were asked questions on Pi, and as a reward, everybody got a generous slice of pie. Girls and boys, staff members and teachers, administration and counselors were all surrounded by Pi facts and lots of sweets. The high school even had a designated person charged with ordering pies of all kinds. The chocolate silk pies in high demand. As a new teacher, I had no idea of what this event meant for the whole school. I knew the school had a strong mathematics department, but I felt that this event was just a way to advertise the school itself.

Growing up in a different country and learning about Pi just the old-fashioned way did not allow me to explore numbers deeply. High school trigonometry was always rich in Pi encounters. Every angle we worked with was in radian measure, so Pi was always present in my schoolwork. Most of the volume formulas for solids had the number Pi incorporated, so the approximation 3.14 was in many instances my best friend. Generally, the number Pi brought joy to my math work. I knew I could simplify and reduce complex mathematics to simpler expressions. Often my calculus homework included Pi. Its presence felt like a good friend; someone you could always rely on. As a high school student, I was fascinated by calculus and algorithms were my best friends. Mathematics represented to me a fascinating language that I kept in my heart for many years. However, there were few things I did not learn in high school: the number sets, the importance of real numbers, rational and irrational numbers — the essence of algorithms. I was trained to work well with numbers, but number theory and the history of all numbers were left out. It was as if all my mathematics teachers had forgotten the parts that make mathematics beautiful. It was the reason why, when I had to deal with traditional Pi day, I felt nothing, or I felt like people got way too excited about an ordinary number.

2. An Introduction to Pi: History and Interesting Facts about Pi

My objective in this thesis is to develop a deeper understanding of Pi and some of its most beautiful aspects. Pi, denoted by the Greek letter π, is known by many as a mysterious irrational number also known as "Pee" in Europe. Most people remember Pi from school mathematics and relate Pi to formulas like the circumference of a circle, $C = 2\pi r$, and the area of a circle, $A = \pi r^2$, but many don't know what it represents. For some people, Pi is nothing more than a touch of a button on a calculator. Depending on the size of a calculator's display, the number will be:

$$3.1415927,$$
$$3.141592654,$$
$$3.14159265359,$$
$$3.1415926535897932384626433832975,$$

or even longer, when using more advanced technology [1].

The symbol π is the sixteenth letter of the Greek alphabet. In the Hebrew and the Greek languages there were no numerical symbols. The Greeks began using the letter π first associating it to the number 80.

α	Alpha	a	1	ι	Iota	i	10	ρ	Rho	r	100
β	Beta	b	2	κ	Kappa	k	20	σ (ς)	Sigma	s	200
γ	Gamma	g	3	λ	Lambda	l	30	τ	Tau	t	300
δ	Delta	d	4	μ	Mu	m	40	υ	Upsilon	u	400
ϵ	Epsilon	e	5	ν	Nu	n	50	φ	Phi	ph	500
ς	Digamma*	st	6	ξ	Xi	x	60	χ	Chi	ch	600
ζ	Zeta	z	7	ο	Omicron	o	70	ψ	Psi	ps	700
η	Eta	Ē	8	π	Pi	p	80	ω	Omega	ō	800
θ	Theta	th	9	ϙ	Koppa*	q	90	ϡ	Sampi*	sp	900

[2]

By coincidence, the Hebrew letter ף (pe) has the same value.

א	Aleph	’	1	י	Yod	y	10	ק	Qof	q	100
ב	Beth	b	2	כ	Kaph	k	20	ר	Resh	r	200
ג	Gimel	g	3	ל	Lamed	l	30	ש	Shin	sh	300
ד	Daleth	d	4	מ	Mem	m	40	ת	Tav	t	400
ה	He	h	5	נ	Nun	n	50				
ו	Vav	v	6	ס	Samek	s	60				
ז	Zayin	z	7	ע	Ayin	‘	70				
ח	Heth	kh	8	פ	Pe	p	80				
ט	Teth	t	9	צ	Tsade	ts	90				

[2]

According to history, the symbol π was first used in mathematics by William Oughtred (1575 - 1660). He chose the letter π to represent a very significant value related to circles. In 1652 he referred to the ratio $\frac{\pi}{\delta}$, where π represented the periphery (circumference , $C = 2\pi r$) of a circle and the symbol δ represented the diameter of the circle ($D = 2r$). In 1665 John Wallis (1660 - 1703) used the Hebrew letter מ (mem), to equal one-quarter of the ratio of the circumference of a circle to its diameter,

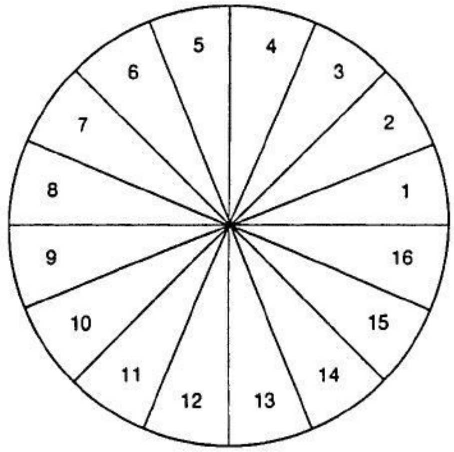
$$\text{מ} = \frac{1}{4}\cdot\frac{C}{D} = \frac{1}{4}\cdot\frac{2\pi r}{2r} = \frac{\pi}{4}.$$

In 1706, William Jones (1675 – 1749) published his book *Synopsis Palmariorium Matheseos* in which he also used the symbol π to represent the ratio of the circumference of a circle to its diameter. Later, in 1736 the Switzerland born mathematician Leonhard Euler (1707 – 1783) also used the symbol π to represent the ratio of the circumference of a circle to its diameter. However, it was after Euler’s use of the symbol π in *Introductio in Analysin Infinitorum*, that it became popular as the symbol that represents the ratio of the circumference of a circle to its diameter [1]. Symbolically this means

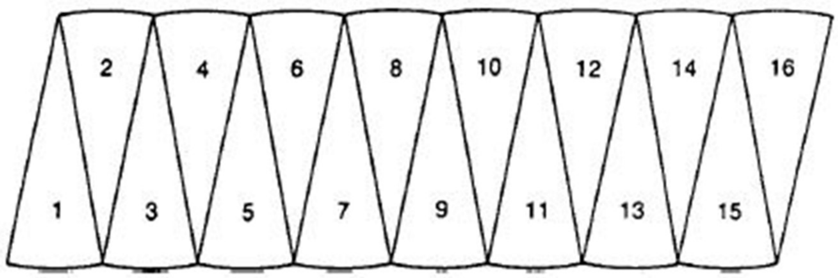
$\pi = \frac{C}{D}$, where C represents the length of the circumference and D is the length of the diameter. This ratio is a constant independent of the size of a circle. D, the diameter of a circle is twice the length of its radius or $D = 2r$, where r is the radius. If we substitute this fact, we get $\pi = \frac{C}{2r}$, which leads to the famous formula for the circumference of a circle $C = 2\pi r$. Euler was in fact the author of many symbols that are widely used in today's mathematics. Among them are: $f(x)$ — the common notation for a function, e – the base of natural logarithm, a, b, c — the sides of the lengths of a triangle, S – the semi perimeter, r – the radius of a circle inscribed in a right triangle, and $\sum$ – for the summation sign. Euler also discovered one of the most famous formulas in mathematics. His formula $e^{i\pi}+1=0$ involves five of the most important numbers in mathematics: e — the base for the natural logarithm*s* and also known as the number satisfying the equation $\ln e=\log_e e=1$, i – the imaginary number that is the solution for $x^2=-1$, π – the irrational number whose approximation is 3.14, 1 — known as the multiplicative identity, and 0 — the additive identity. Benjamin Peirce, a 19th century Harvard mathematician, claimed that this formula lacked meaning but could be proved and therefore it represented a truth.

Euler's life was as interesting as his formula. He was a Swiss mathematician who studied mathematics with his father, who himself studied with the famous mathematician Jakob Bernoulli. This connection helped Euler later when his father arranged for him to study with Johann Bernoulli, Jakob's Bernoulli's son. Thanks to Bernoulli's influence, Euler was able to work for the Russian Academy of St. Petersburg at the age of twenty and stayed there for fourteen years, eventually earning the position of Chief Mathematician. Although he spent the next twenty-five years working for the Prussian Academy, he never lost touch with the Russian Academy to which he returned to work for the last seventeen years of his life. He had an incredible memory and was known as a very productive mathematician, although he was deaf in his right ear and became blind in the last years of his lifetime. During his career he wrote 530 books and articles. Some of his manuscripts continued to appear after his death, so a total of 886 books and articles encompassed his entire life's work [1].

One familiar formula containing π is the area of a circle of radius r, $A=\pi r^2$. To derive this formula, draw a circle and divide it into sixteen equal sectors.

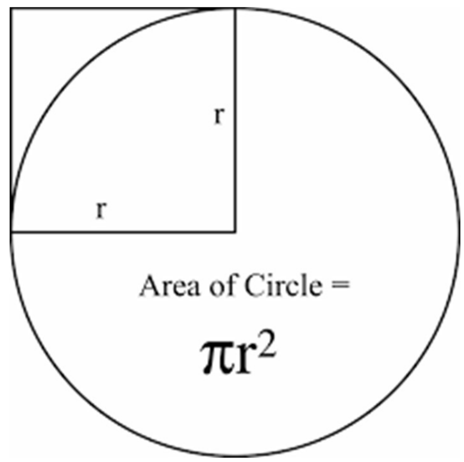


The central angle of each of the sector is $\frac{360}{16} = 22.5^{\circ}$. If we rearrange the sixteen sectors the following way:



we get what is approximately a parallelogram. The more sectors the circle is cut into, the closer to a true parallelogram the figure will get. As the number of sectors approaches infinity, the arcs will look like segments and the area of the approximating parallelogram will approach the area of the circle itself. In other words, we have something that resembles a parallelogram and in the limit its area is equal to the area of a rectangle which is equal to the area of the circle. The length of the base of the rectangle is $\frac{1}{2}C$. Since, $C = 2\pi r$, the base length is πr. The area of the rectangle equals the product of its height with its base. Since the rectangle's height is equal to the radius of the circle, the area of the rectangle is $A = \pi r^2$, which gives us the well-known formula for the area of a circle [3].

Another interesting fact about Pi is illustrated in the following diagram.



In this diagram the area of the square is r^2 but by multiplying its area by π, we convert its area to the area of the actual circle which is πr^2. Hence π can also be an interesting link for the area of a square and the area of the associated circle [1].

The number π had an interesting evolution throughout history. Archimedes of Syracuse (287 - 212 BCE) showed that the value of π lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. The Dutch mathematician Rudolph van Ceulen (1540 - 1610) calculated the value of π with thirty-five places. John Wallis (1616 - 1703), a professor of mathematics at both Cambridge and Oxford presented a formula to represent π as follows:

$$\frac{\pi}{2}=\frac{2\times 2}{1\times 3}\times\frac{4\times 4}{3\times 5}\times\frac{6\times 6}{5\times 7}\times\frac{8\times 8}{7\times 9}\times\ldots\times\frac{2n\times 2n}{(2n-1)(2n+1)}\times\ldots$$

This infinite product is equal to the value of $\frac{\pi}{2}$. This means that the finite products get arbitrarily close to the value of $\frac{\pi}{2}$ [1].

In the early 1700's Euler examined the function

$$F(x)=\frac{\sin(x)}{x}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\ .$$

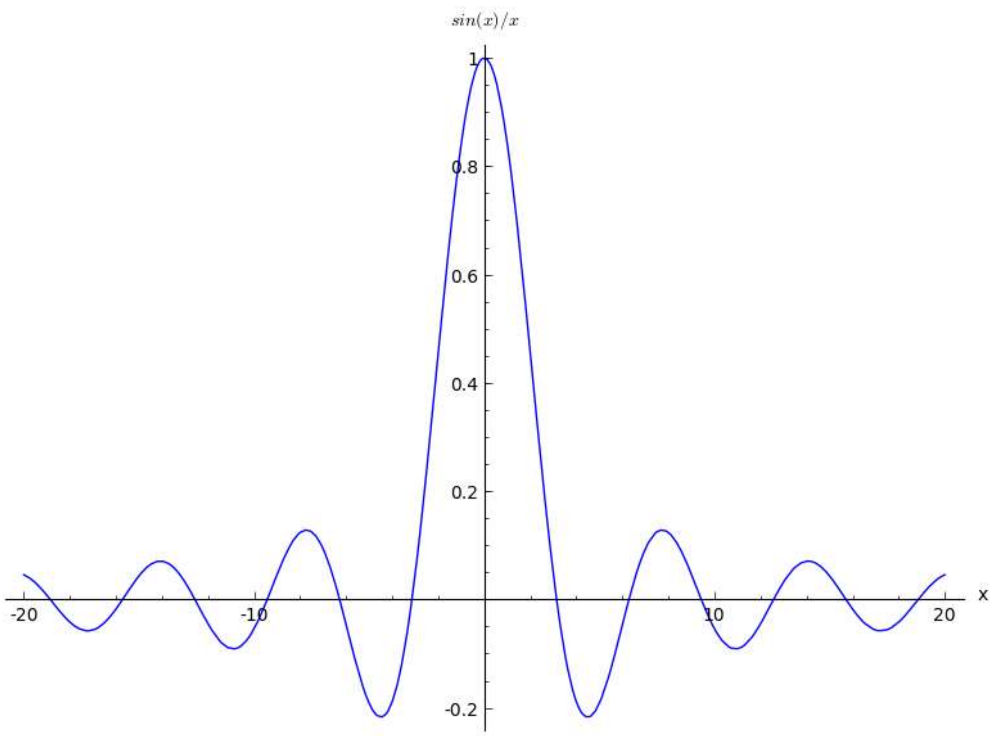
It has a value of $F(0)=1$ and an even symmetry property $F(x)=F(-x)$. Furthermore, there are an infinite number of equally spaced zeros located at $x=\pm n\pi$ with $n=1,2,3,4,...$ These facts suggest that we can try to express $F(x)=\dfrac{\sin(x)}{x}$ as a product

$$F(x)=(1-A_1x^2)(1-A_2x^2)(1-A_3x^2)...$$

which has $F(0)=1$. The constants A_n are to be adjusted to match the location of the zeros of the function. Taking $A_1=\dfrac{1}{\pi^2}$, $A_2=\dfrac{1}{4\pi^2}$, $A_3=\dfrac{1}{9\pi^2}$,… etc., the function will vanish at all its zeros and one has the symmetric product expansion

$$F(x)=\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right] \text{ [4].}$$

The plot of this function is:



By using Euler's infinite product of the $F(x)=\dfrac{\sin(x)}{x}$, we can easily prove John Wallis's product leading to $\dfrac{\pi}{2}$.

$$\text{In } F(x) = \frac{\sin(x)}{x} = \prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right],$$

if we let $x = \frac{\pi}{2}$ the function becomes

$$F(\frac{\pi}{2}) = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \prod_{n=1}^{\infty}(1-(\frac{\pi^2}{4n^2\pi^2}))$$

or simplified

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}(1-\frac{1}{4n^2}) = \prod_{n=1}^{\infty}(\frac{4n^2-1}{4n^2}).$$

By reversing this result

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}(\frac{4n^2}{4n^2-1}) = \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\right)\left(\frac{2n}{2n+1}\right) = \frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}...$$

we obtain Wallis's product

$$\frac{\pi}{2} = \frac{2\times 2}{1\times 3}\times\frac{4\times 4}{3\times 5}\times\frac{6\times 6}{5\times 7}\times\frac{8\times 8}{7\times 9}\times...\times\frac{2n\times 2n}{(2n-1)(2n+1)}\times...$$

such an important step in the developing π [5].

An interesting fact about π is that it is an irrational number. Aristotle (384 - 322 BCE) assumed that π is such a number, one that cannot be expressed as a fraction. However, the most fascinating fact about π is that this number cannot be calculated by a combination of the operations of addition, subtraction, multiplication, division, and root extraction, that is, it is not an algebraic number. This is the reason for which π is called a transcendental number. It is a number that is not algebraic and cannot be the root of any polynomial equation with integer coefficients. This fact was first suspected by Euler and proved in 1882 by the German mathematician Ferdinand Lindemann (1852 - 1939). In 1844, the mathematical genius Joseph Liouville (1809-1882) was the first to prove the existence of transcendental numbers. In 1851 he gave the first decimal examples such as the Liouville constant

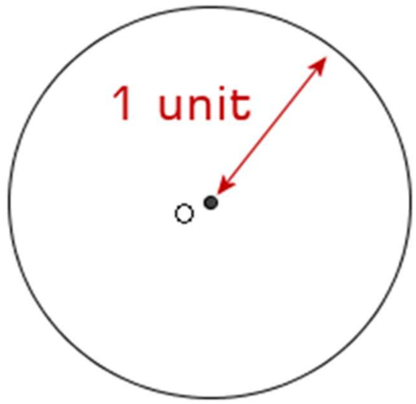
$L_b = \sum_{n=1}^{\infty} 10^{-n!} = 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + 10^{-120} + 10^{-720} + ... = 0.110001000000000000000001...$ in which the nth digit after the decimal point is 1 if n is equal to k!. Hence, the nth digit of this number is 1 only if n is one of the numbers $1! = 1, 2! = 2, 3! = 6, 4! = 24, \ldots$. Liouville showed that the number belongs to a class of transcendental numbers that can be more closely approximated by rational numbers that can any irrational algebraic number. Such numbers are called Liouville numbers, named in the honor of him. He also proved that all Liouville numbers are transcendental. Charles Hermite (1822 – 1901) proved that the number *e*, another irrational number, is transcendental in 1873. Lindemann proved that π is transcendental in 1882. The transcendence of π put an end to all hopes of those who were looking for a method to "square the circle" [1].

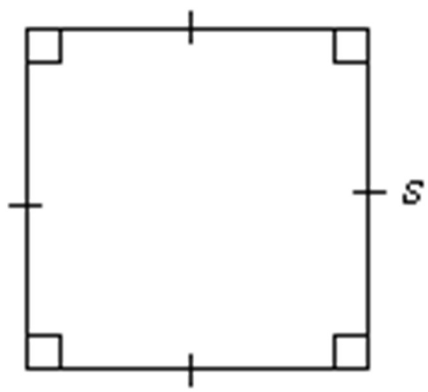
Among the impossible construction problems, squaring the circle represented a challenge for many mathematicians over time. The three actual ancient constructions using only a compass and a straight edge in Euclidean Geometry are: *trisecting an angle* (dividing a given angle into three equal angles), s*quaring a circle* (constructing a square with the same area as a given circle), and *doubling a cube* (constructing a cube with twice the given cube). People have tried for centuries to make such constructions. It was not until the development of abstract algebra in the nineteenth century that it was proven these constructions were impossible. The only shapes we can draw with a compass and straightedge are line segments and circles (or parts of circles). The only ways of constructing new points out of old points is to take the intersection of two lines, two circles, or a line and a circle. Now, if we write down the general equations for these intersections and try to solve for one of the coordinates of the intersection point, we could end up with either a linear or quadratic equation. "The coefficients of the equation involve the coordinates of the old points (or the sums, differences, products, or quotients of them). Linear equations can be solved by simple division: the equation $ax = b$ has as its solution $x = b/a$. Quadratic equations can be solved using the quadratic formula. In each case, we see that the only arithmetic operations required to calculate the new coordinate from the old coordinates are addition, subtraction, multiplication, division, and taking square roots" [6].

Therefore, if you start with some initial points whose coordinates are all rational numbers, then apply any sequence of compass-and-straightedge construction techniques, the coordinates of the points you end up with will be a very special kind of number: they will be obtainable from the rational numbers by a sequence of operations involving only addition, subtraction, multiplication, division, and the extraction of square roots. The reason the three classical constructions are impossible is the fact they all require the construction of points whose coordinates are not numbers of this type. Proving that they are not numbers of this type requires advanced mathematics from field theory. “If x is a number obtainable from the rational numbers using only addition, subtraction, multiplication, division, and the taking of square roots, then x is a solution to some polynomial equation with rational coefficients. Moreover, if one factors out irrelevant factors from this equation until one gets down to an "irreducible" polynomial equation (one that cannot be factored any further and still have rational coefficients), the degree of this polynomial will always be a power of 2. Each time you take a square root, you usually double the degree of the polynomial required to represent the number. For example, $\sqrt{2}$ can be represented as a solution to the quadratic equation $x^2 - 2 = 0$, whose degree is 2. (Of course, it's also a solution to the cubic equation $x^3 - 2x = 0$ whose degree is 3, not a power of 2; but that's because this equation can be factored, and after factoring out the x you are left with the quadratic $x^2 - 2 = 0$, which is irreducible: it can't be factored any further where the polynomial factors have rational coefficients). If we take a square root a second time, getting the number 2^4, it's a root of the fourth-degree polynomial $x^4 - 2 = 0$, which is irreducible. A number like $\sqrt{2}+\sqrt{3}$, also obtained by twice taking a square root, is also the root of an irreducible fourth-degree polynomial: $x^4 - 10x^2 + 1 = 0$. A number obtained by taking a square root three times, like $2^{\frac{1}{8}}$, is typically the root of an irreducible 8th degree polynomial, like $x^8 - 2 = 0$. This intuitive understanding is not always correct, though. For example, $\sqrt{2}+\sqrt{8}$, even though it is obtained using two square roots, is the same as $3\sqrt{2}$, which involves only one square root. The argument only shows us that the claims of the theorem are to some degree reasonable. In fact, proving the theorem (and proving, not just that there is some irreducible polynomial equation for x whose degree is a power of 2, but that every irreducible polynomial equation for x also has that same degree) involves many advanced mathematical methods” [6].

Using the theorem, it is easy to prove the impossibility of the three constructions: Doubling a cube is impossible because if you start with a cube of side length 1, you would need to construct a cube whose side length is $\sqrt[3]{2}$. But $\sqrt[3]{2}$ is a solution to the irreducible equation $x^3 - 2 = 0$ whose degree, 3, is not a power of 2 [6].

Squaring a circle is impossible because if you start with a circle of radius 1 you would need to construct a square whose side length is $\sqrt{\pi}$. But π is a transcendental number, which is not the solution to any polynomial equation with rational coefficients, let alone one whose degree is a power of 2.





Trisecting an angle is impossible because if you start with an angle of 60° (which is easily constructible, since an equilateral triangle is constructible), you would then need to be able to construct an angle of 20 degrees. This would be equivalent to constructing a point whose coordinates for the unit circle are cos(20°) and sin(20°) . This is impossible because

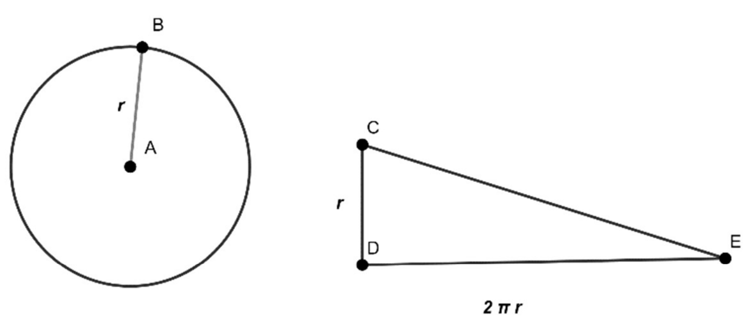
cos(20°) is a solution to the irreducible polynomial equation $8x^3 - 6x - 1 = 0$ whose degree, 3, which is not a power of 2.

3. Archimedes' Early Work

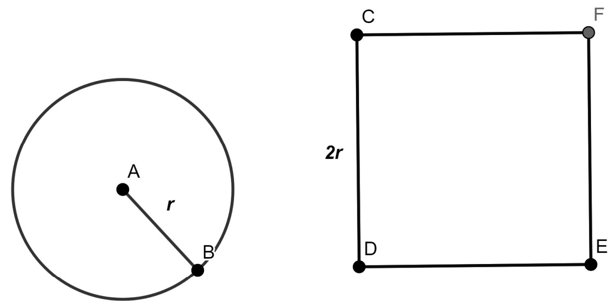
Born in Syracuse (Sicily) about 287 BCE, son of Phidias, an astronomer, Archimedes is one of the greatest mathematicians of all times. As a child he may have studied with Euclid's successors in Alexandria, Egypt. He collaborated with Conon de Samos and Eratosthenes of Cyrene and brought contributions to the world of mathematics and physics. Parts of his work are related to circles and π [1].

In Archimedes' *Measurement of the Circle*, more specifically, in the three volumes of this book, there are three propositions regarding the circle that are tied to development of π .The first proposition says that the area of a circle is equal to that of a right triangle when the legs of the right triangle are respectively equal to the radius and circumference of the circle.

$$Area = \frac{1}{2}(r)(2\pi r) = \pi r^2$$

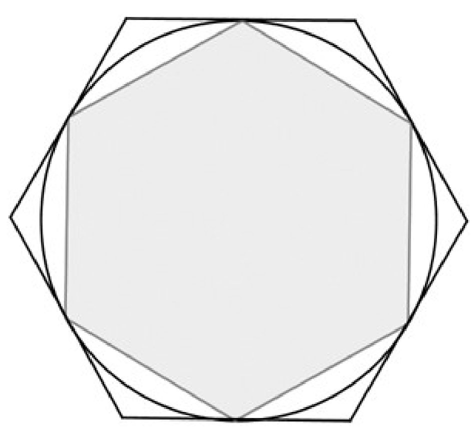


Archimedes' second proposition states that the ratio of the area of a circle to that of a square with the side equal to the circle's diameter is closed to 11:14.



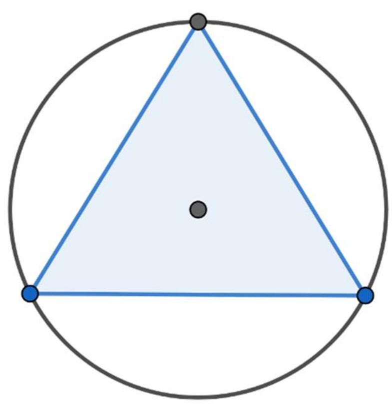
The area of the circle is πr^2 while the area of the square is $(2r)^2 = 4r^2$. The ratio of the two is $\frac{\pi r^2}{4r^2} = \frac{\pi}{4} = \frac{11}{14}$. If we simplify, we get $\pi = \frac{44}{14} = \frac{22}{7}$, which represents a very well known approximation of π [1].

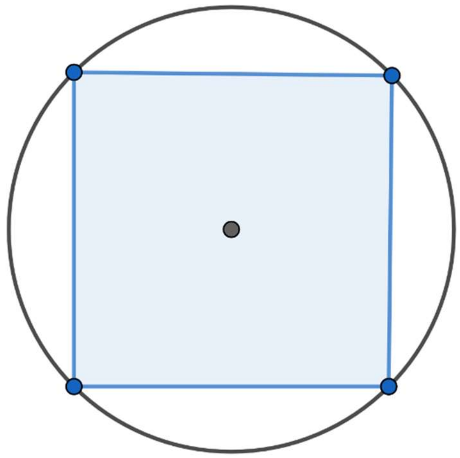
The third proposition states that the circumference of a circle is less than $3\frac{1}{7}$ times its diameter and more than $3\frac{10}{71}$ times the diameter. Archimedes got to this result by inscribing a regular hexagon into a given circle and then circumscribing a regular hexagon about the same circle.



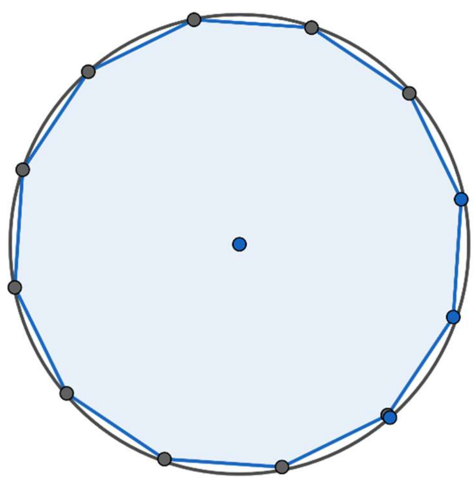
He looked for the areas of the two hexagons and then the area of the circle. He knew the area of the circle must be in between the areas of the two hexagons. For accuracy, he then repeated the calculations for regular dodecagons, then twenty-four, forty-eight, and ninety-six sided regular polygons, and each time getting closer and closer to the area of the actual circle. Finally, Archimedes concluded that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

Archimedes modified this method of estimating the value of π , by looking at the perimeters of repeated regular polygons. This resulted in a better method of calculating π. He noticed that as the number of sides of a regular polygon increases, while keeping the radius or the apothem constant, the perimeter of polygon gets closer and closer to the circumference of the circle. The circumscribed circle must contain each of the vertices of the polygon. Here is what it can look like:

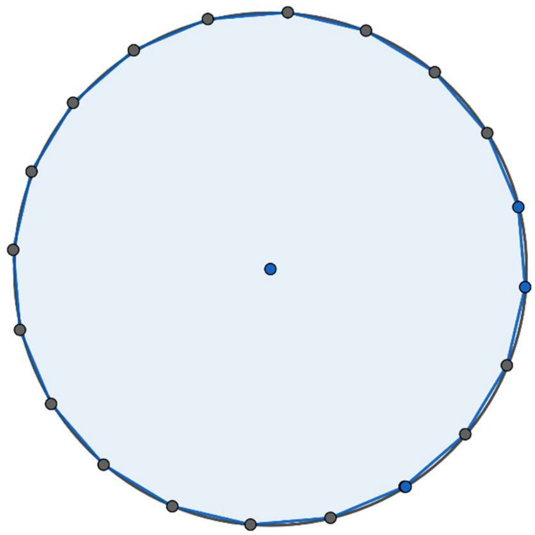




This may be much easier to see when the regular polygon's sides increase further, so it becomes a dodecagon (12 sided polygon).

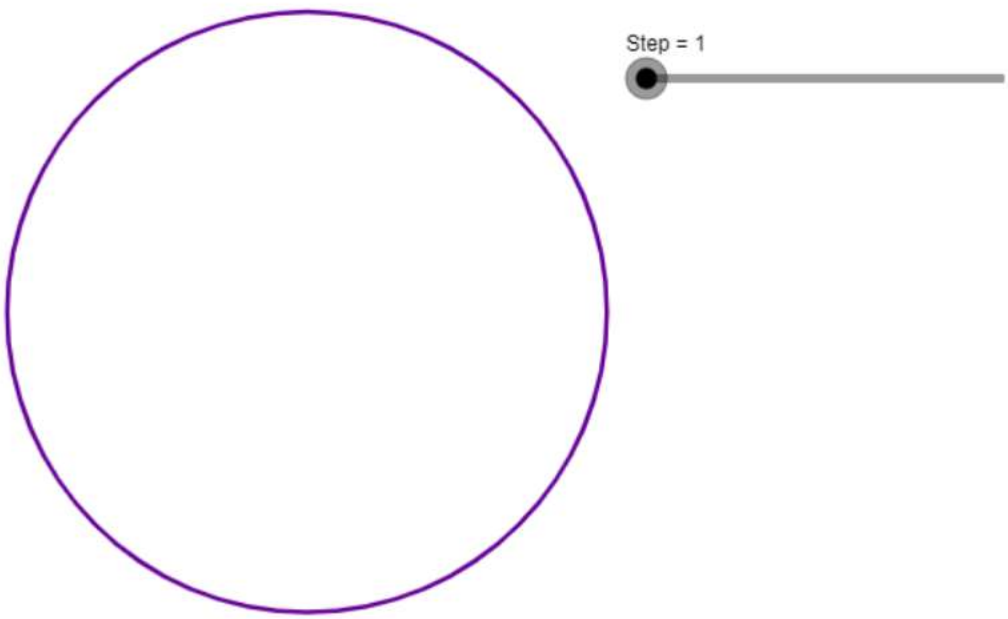


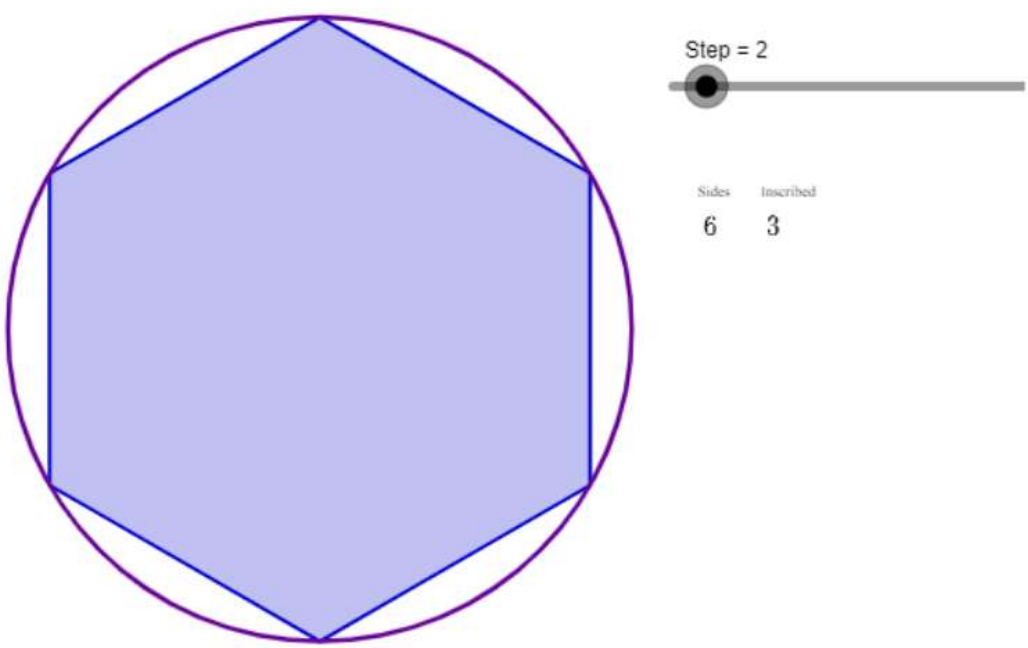
Likewise, with an icosagon (20 sided regular polygon) we can see the actual perimeter approaching the circumference of the circle.

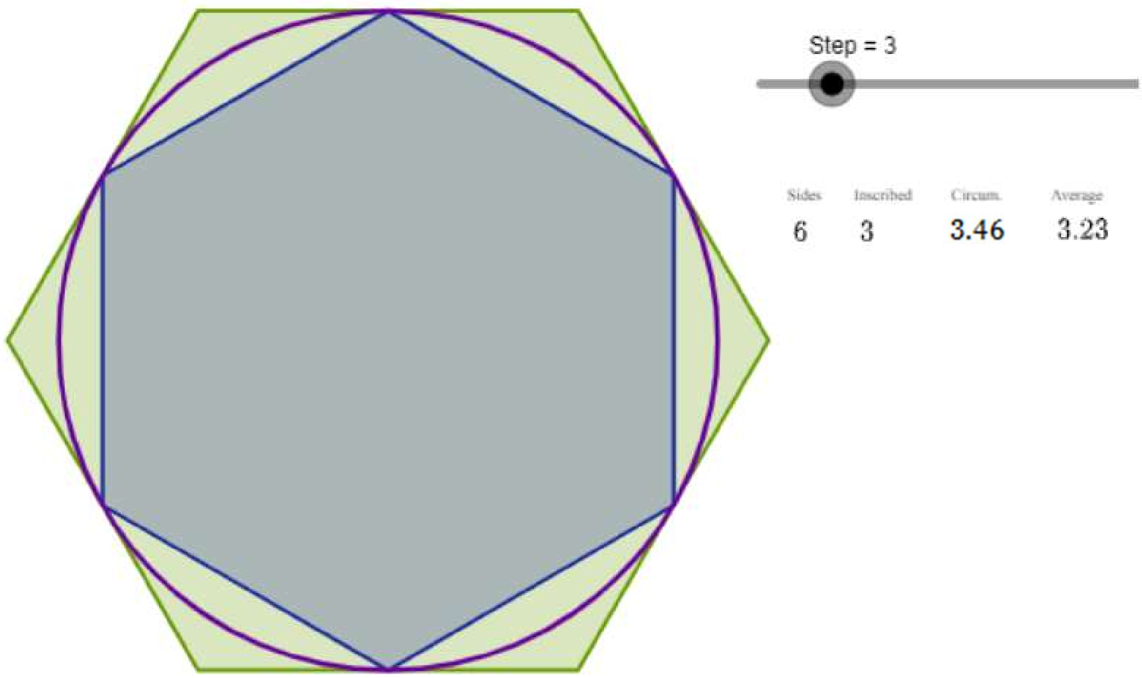


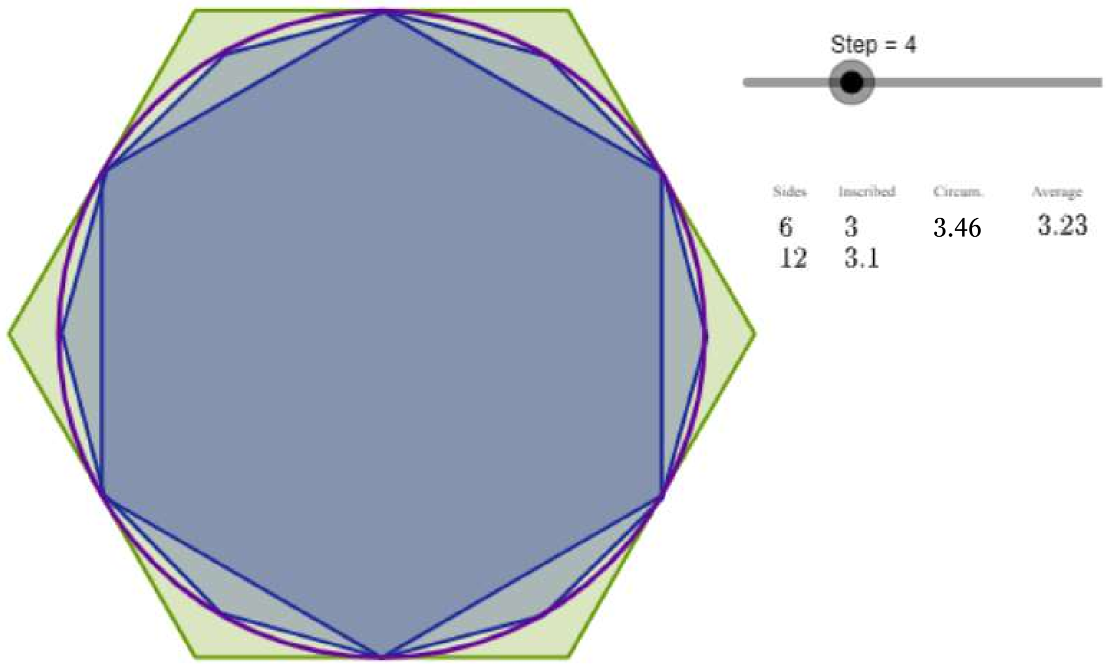
Archimedes knew that the perimeter of the n-sided regular polygon equal to $n\sin\frac{180}{n}$. With this formula he computed the perimeter of regular polygons whose circumscribed circle has a radius of $\frac{1}{2}$. Hence, for a 10,000-sided polygon and an inscribed circle with the radius of $\frac{1}{2}$, Archimedes knew that the perimeter of the polygon is approximately equal to the circumference of the circle which is $2\pi r = 2\pi(\frac{1}{2}) = \pi$. He obviously did not have the luxury of using technology. Yet, through hand calculations, he was still able to approximate the perimeter of a 96-sided regular polygon. He saw the circle as the limiting figure between the inscribed polygon and the circumscribed polygon used earlier. He also knew that the perimeter of inscribed polygon is $P_i = n\cdot\sin\frac{180^\circ}{n}$ while the perimeter of the circumscribed perimeter is $P_c = n\cdot\tan\frac{180^\circ}{n}$. Then, by taking the average of the perimeters of each pair of n-sided regular polygons, he arrived at an approximation of the circumference of the circle, which in the case of a circle with a radius of $\frac{1}{2}$ is π [1].

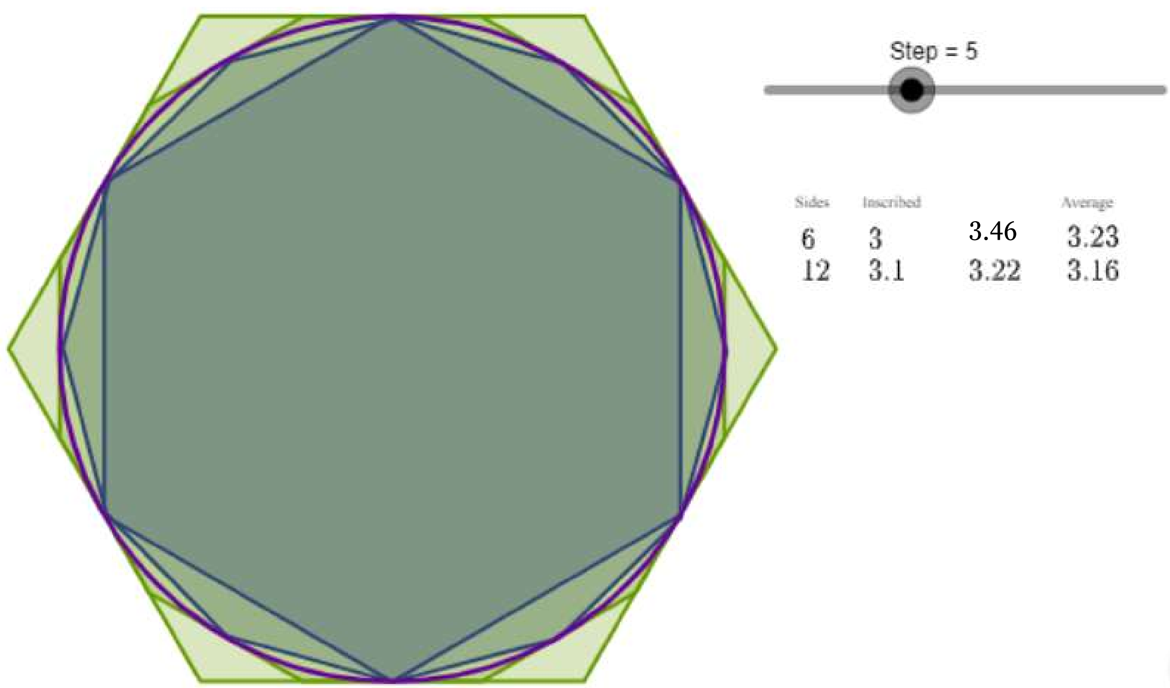
With today's technology, more specifically with GeoGebra, Paul Hartzer, a high school mathematics teacher, was able to illustrate the calculations of Archimedes' method of using the perimeters of polygons to approximate the circumference of circles. In this presentation, polygons of 6, 12, 24, 48, and 96 sides are shown, as well as a circle. Each table shows half the length of the perimeter of each inscribed and circumscribed polygon, as well as the average of those two values.

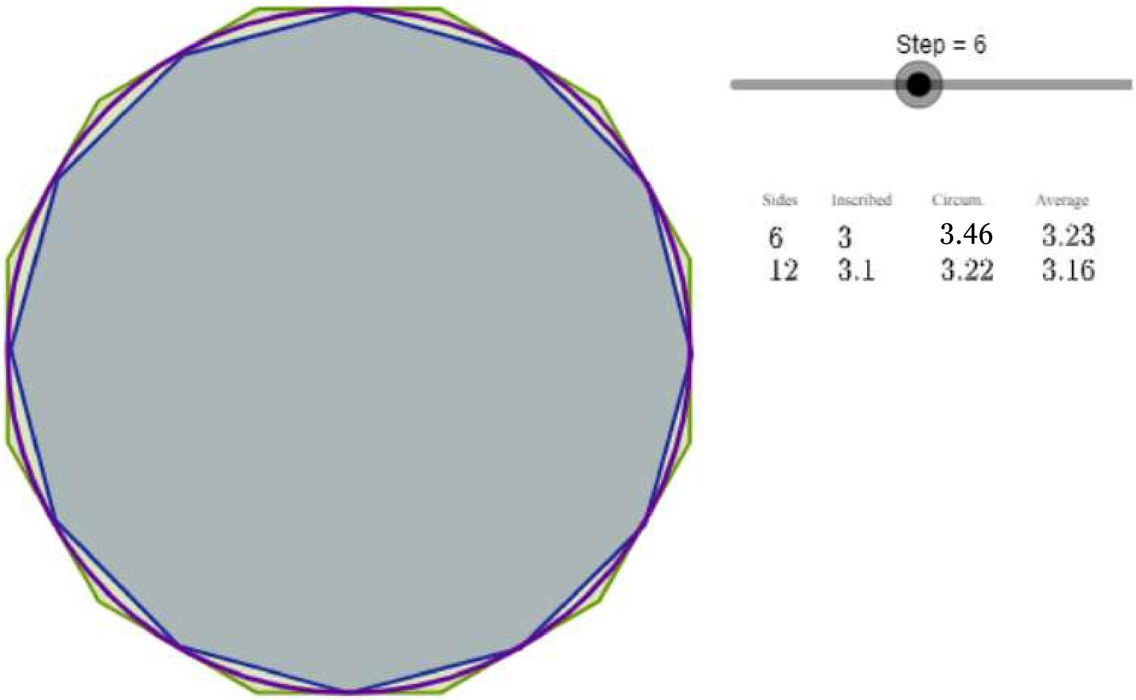




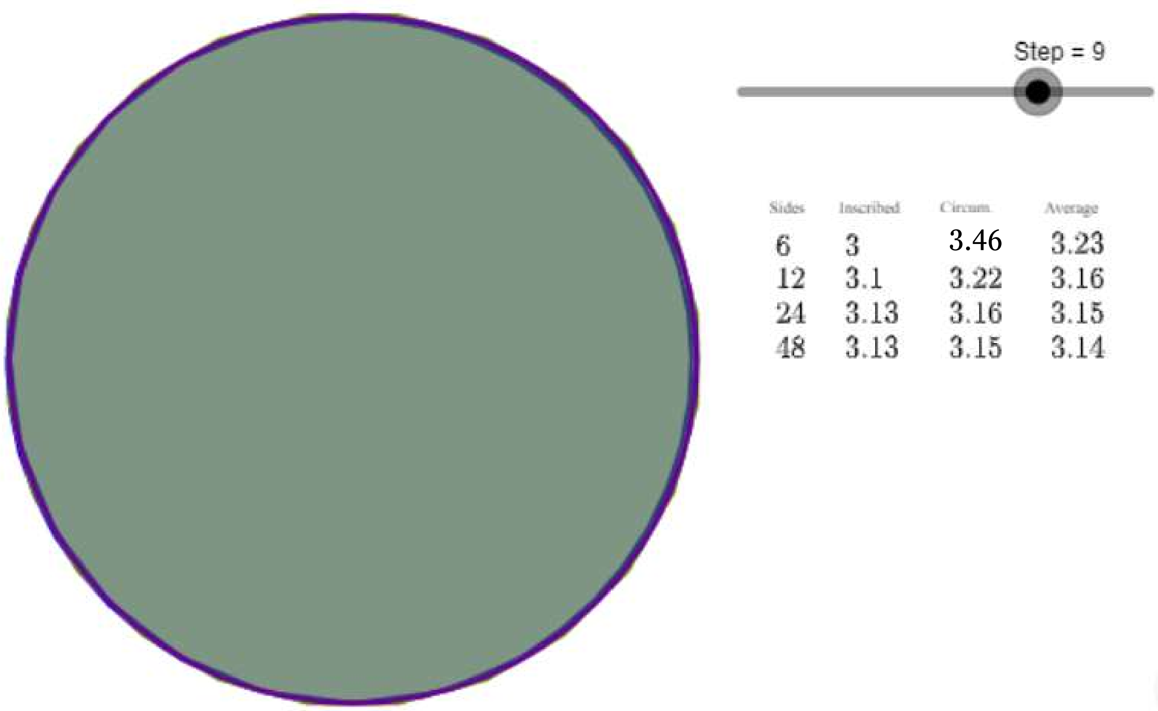

Step = 3
Sides
Inscribed
Circum.
Average
6
3
3.46
3.23


Step = 4
Sides
Inscribed
Circum.
Average
6
3
3.46
3.23
12
3.1

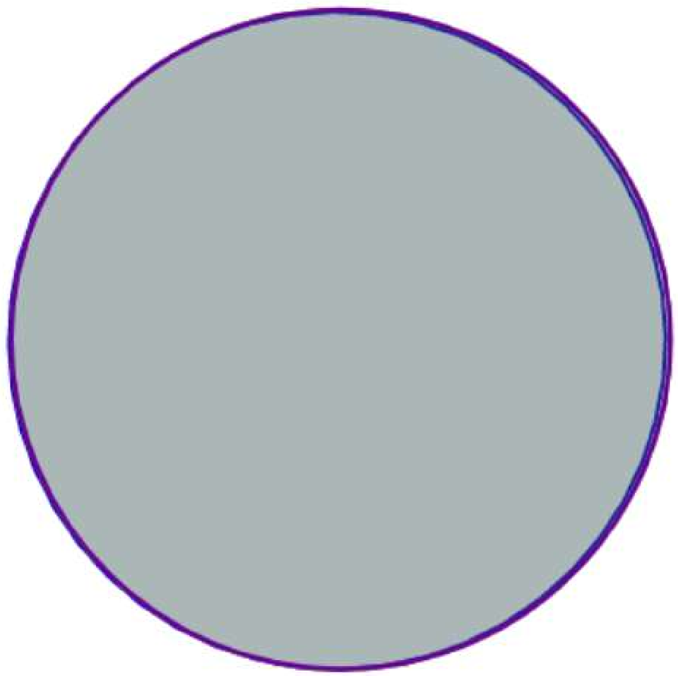

Step = 5
Sides
Inscribed
Average
6
3
3.46
3.23
12
3.1
3.22
3.16



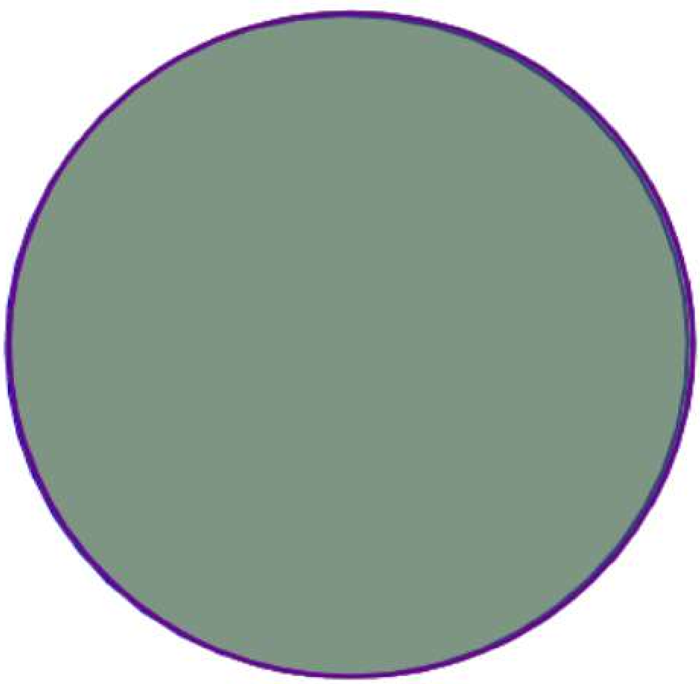
Sides	Inscribed	Circum.	Average
6	3	3.46	3.23
12	3.1	3.22	3.16



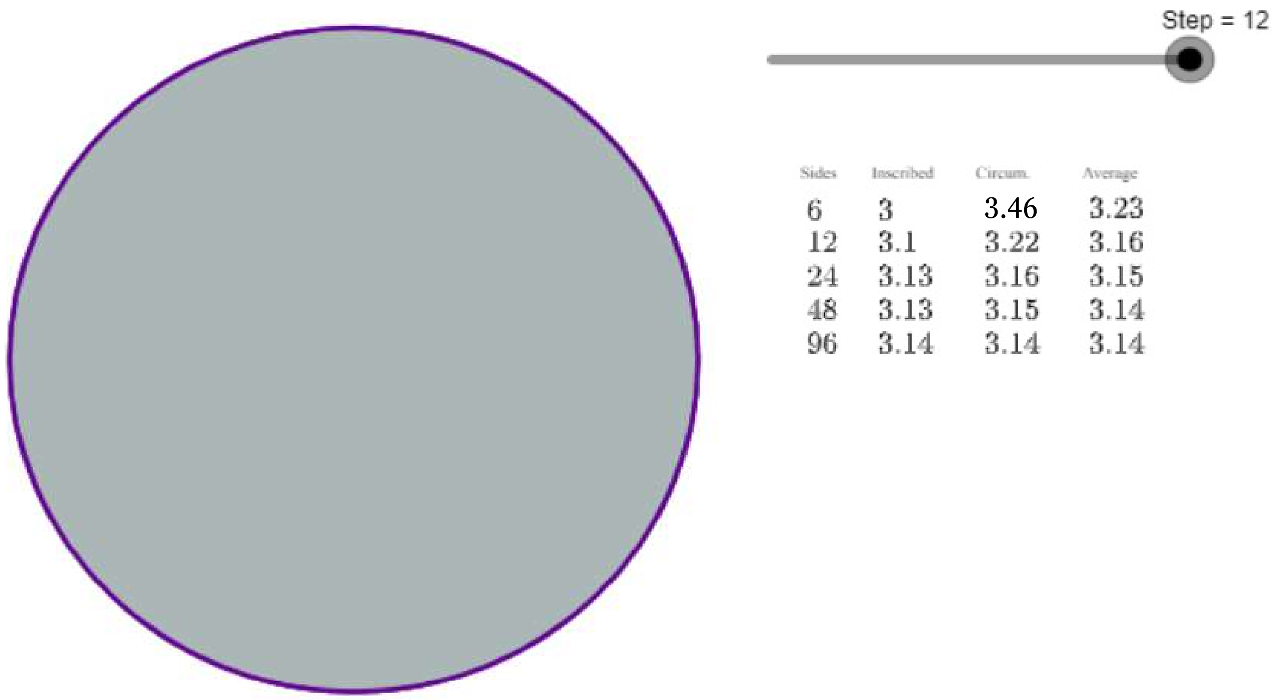
Sides	Inscribed	Circum.	Average
6	3	3.46	3.23
12	3.1	3.22	3.16
24	3.13	3.16	3.15
48	3.13	3.15	3.14



Sides	Inscribed	Circum.	Average
6	3	3.46	3.23
12	3.1	3.22	3.16
24	3.13	3.16	3.15
48	3.13	3.15	3.14



Sides	Inscribed	Circum.	Average
6	3	3.46	3.23
12	3.1	3.22	3.16
24	3.13	3.16	3.15
48	3.13	3.15	3.14
96	3.14	3.14	3.14



Sides	Inscribed	Circum.	Average
6	3	3.46	3.23
12	3.1	3.22	3.16
24	3.13	3.16	3.15
48	3.13	3.15	3.14
96	3.14	3.14	3.14

[7]

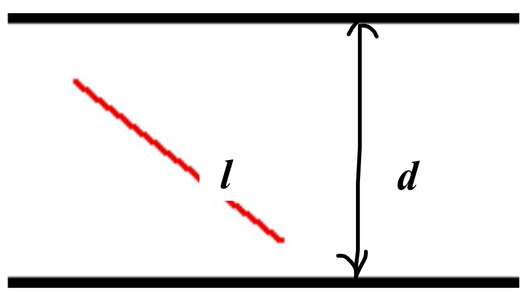
As we step through this animation, we observe what happens to the difference between the perimeter of the polygons and the circumference of the circle. When the inscribed and circumscribed polygons get to 96 sides, the perimeter is approximately 3.14. Therefore, the average is approximately 3.14, and that is close to the actual value of Pi.

4. Probability and Buffon's Needle Problem

The French naturalist, mathematician, cosmologist, and encyclopedist Georges Louis Leclerc, Comte de Buffon (1707 - 1788) is generally remembered for his *Histoire Naturelle, Générale et Particulière.* This work contained 37 volumes and was written in an ingenious style and read by every cultivated person in Europe. Originally designed to cover all the natural habitats, the *Histoire Naturelle* was referring specific to animals and minerals. The animals described by Buffon were birds and quadrupeds. Buffon's *Histoire Naturelle* was translated at that time into multiple languages, and therefore made him one of the most read authors and a true competitor to Montesquieu, Rousseau, and Voltaire [8]. In mathematics, he is known for his French translation of Newton's *Method of Fluxions*, the main framework of today's calculus and even more so for "Buffon's needle problem."

For many, the needle problem is one of the oldest geometric probability problems. In this problem he is making a fine connection among the value of π and probability.

Suppose you have a piece of paper with ruled parallel lines throughout, equally spaced — with a distance d between the lines and a needle of length $l \le d$.

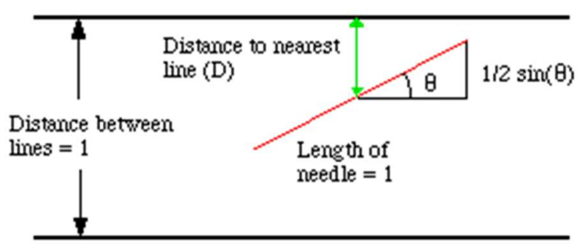


You toss the needle onto the paper many times. Buffon tossed the needle thousands of times to conclude that the probability that the needle will touch one of the ruled lines is $\frac{2l}{\pi d}$. Thirty-five years later, Pierre Simon Laplace popularized the problem. Laplace was one of the most well-known French mathematicians and showed a great interest for the field of probability. If one wants to try this experiment, he or she will begin by simplifying the problem, without loss of generality, by letting $l = d$. The probability of the needle touching one of the lines is now $\frac{2}{\pi}$. So now, $\pi = \frac{2}{P}$, where P is the probability that the needle will intersect a line, which is

$P =$ number of line touching tosses / number of all tosses

Hence by substitution $\pi \approx 2$ / (number of all tosses / number of intersection tosses) [9].

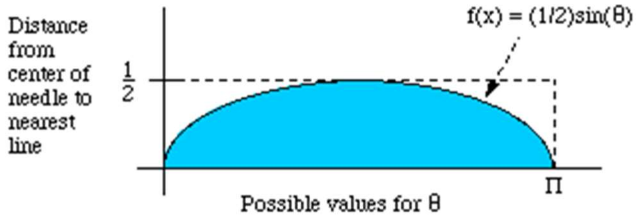
When looking at this experiment we can identify two scenarios: a base case and the other cases. In the base case, the length of the needle is one-unit and the distance between the lines are also one unit. We have two variables, the angle at which the needle falls (θ) and the distance from the center of the needle to the closest line (D). The angle can vary from 0 to 180 degrees and it is measured against a line parallel to the lines on the paper. The distance from the center to the closest line can never be more than half the distance between the lines. The diagram below illustrates the case where the needle misses the line.



[9]

The needle in the diagram misses the line. The needle will touch the line if the distance to the nearest line (D) is less than or equal to ½ times the sine function of theta.

This is, $D \le \frac{1}{2}\sin\theta$. The question is how many times will this situation occur? In the diagram below we have D along the ordinate and θ along the abscissa. The values on or below the curve represent a hit where $D \le \frac{1}{2}\sin\theta$. So, the probability of the needle hitting the line is the ratio of the shaded area over the area of the whole rectangle.



[9]

We can calculate this probability by calculating the two areas. The shaded area is given by the integral:

$$\int_0^{\pi} f(x)d\theta = \int_0^{\pi} \frac{1}{2}\sin\theta d\theta = -\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0 = 1 .$$

The area of the entire rectangle is $\frac{1}{2}\pi$. Therefore, the ratio of the two areas represents geometrically the probability: $P = 1/\frac{1}{2}\pi = \frac{2}{\pi}$, as previously stated. This is approximately .6366197. To estimate π , we can simply multiply the experimental ratio by 2. Hence, $\pi \approx 2$ / (number of total drops/ number of hits). The other cases would be those resulting from the relationship between the length of the needles and the distance between the lines. The case of the distance between the lines being larger than the needle is an extension of the simple case above. The probability of a hit in this new case is:

$\frac{2l}{\pi d}$, where l is the length of the needle and d is the distance between the lines. The situation where the needle is longer than the distance between the lines leads to a more accurate result for which we encourage the reader to pursue [9].

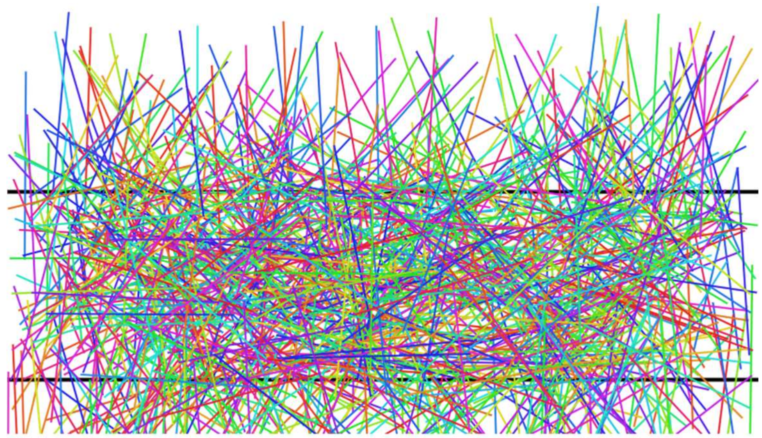
However, going back to the relationship between the number of tosses and probability of a hit, the more tosses you have, the more accurate is your estimate of the number π . In 1901 Mario Lazzarini, an Italian mathematician, tried the needle experiment with 3,408 tosses and got $\pi \approx$ 3.1415929.

With a computer simulation, one could easily get close to π . With a simulation created by the University of Illinois, we only need to press the “drop” button and drop the chosen number of needles. The computer simulation performs the calculations.

Dropping more needles allows you to approximate π more precisely.

Here are few cases:

A. Number of drops = 5002

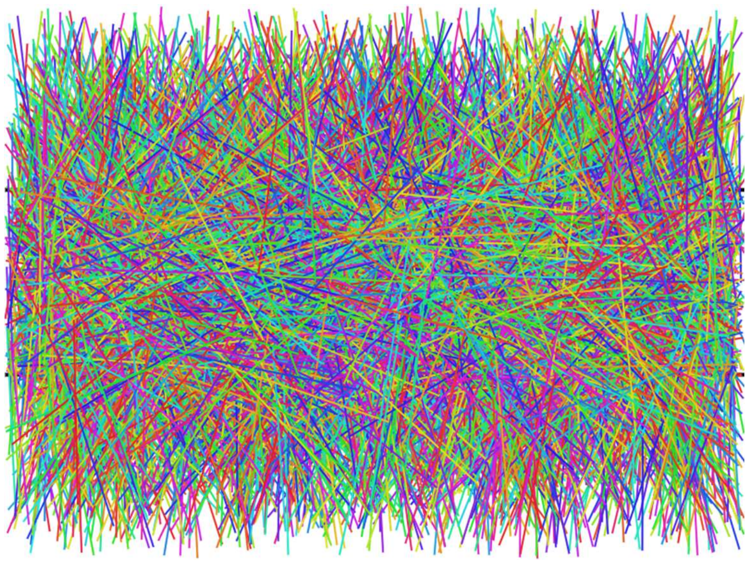


Drop Amount 5002 Drop

Measurement	Value
Needle Scale	1
Extent = Perimeter / Greatest Vertex Distance	1
Number of Drops	5002
Number of Hits	3204
Drops / Hits	1.5611735330836454
π ≈ 2 * Extent * Scale * Drops / Hits	3.1223470661672907

Needle Scale 1 Start Over

B. Number of Drops = 15,002

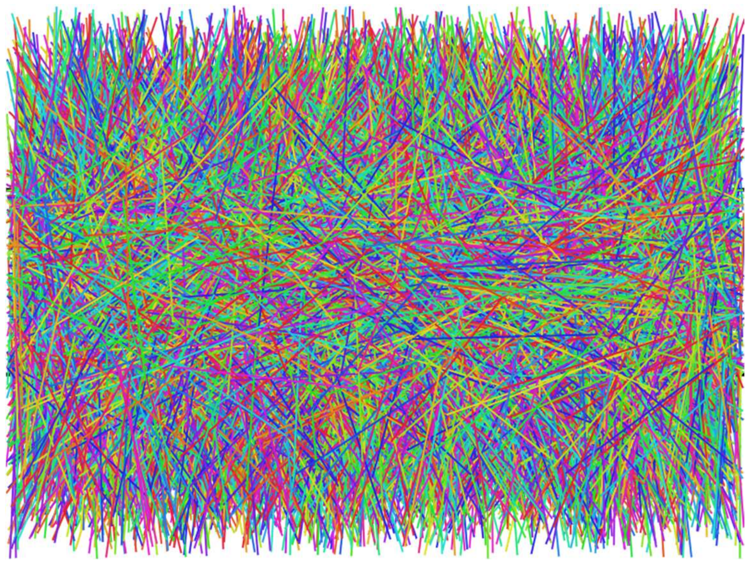


Drop Amount 15,002 Drop

Measurement	Value
Needle Scale	1
Extent = Perimeter / Greatest Vertex Distance	1
Number of Drops	15002
Number of Hits	9597
Drops / Hits	1.5631968323434406
$\pi \approx 2$ * Extent * Scale * Drops / Hits	3.1263936646868813

Needle Scale 1 Start Over

C. Number of drops = 20,000



Drop Amount 20000 Drop

Measurement	Value
Needle Scale	1
Extent = Perimeter / Greatest Vertex Distance	1
Number of Drops	20000
Number of Hits	12721
Drops / Hits	1.5722034431255405
π ≈ 2 * Extent * Scale * Drops / Hits	3.144406886251081

Needle Scale 1 Start Over

[9]

Buffon's needle method is not the most efficient way to calculate π . However, we can conclude that the probability of a tossed needle intersecting a line is related to π and therefore, is related to the ratio of the circumference of a circle of any radius to its diameter.

Due to frequent Pi encounters, mathematicians have asked themselves many questions about Pi and some have found astonishing answers. For example, *what is the probability of two random integers being coprime* [10]? I will now explain why the surprising answer is $\frac{6}{\pi^2}$. In number theory, two integers *a* and *b* are said to be coprime if the only positive integer (factor) that divides both is 1. So, in other words, the two numbers have no common factors.

A short proof for this probability result using Euler's famous equation $\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{6}$, was published by two high school students, Aaron Abrams and Matteo Paris in the *College Mathematics Journal.* It was introduced by a letter from Henry L. Adler where he tells the actual story of the proof. He originally answered the question with a traditional proof that required prior knowledge of number theory. However, the two authors, both high school students at the time, came up with a much simpler proof. Adler was the one to encourage the boys to share their proof with the readers of the *College Mathematics Journal* [10].

Assume that *p* is the probability that two random integers are coprime. They first showed that the probability of $\gcd(a,b) = k$, for $k = 1,2,3,\ldots$ is $\frac{p}{k^2}$.

Since the probability that *k* divides *a* is $\frac{1}{k}$, the probability of the event that *k* divides both *a* and *b* is $\frac{1}{k^2}$. The probability of the independent event that *a* and *b* have no other factors is equal to the probability that $\gcd(\frac{a}{k},\frac{b}{k}) = 1$, which is *p* by assumption. Hence the probability such that $(a,b) = k$ is $\frac{p}{k^2}$. Since any pair of positive integers must have a greatest common divisor, the sum of the probabilities that $\gcd(a,b) = k$ for $k = 1,2,3,\ldots$ must be 1, so that

$$1=\sum_{k=1}^{\infty}\frac{p}{k^2}=p\sum_{k=1}^{\infty}\frac{1}{k^2}=p\cdot\frac{\pi^2}{6}. \text{ Thus, } p=\frac{6}{\pi^2} \text{ [11]}$$

5. Pi an Irrational Number, Number Sets and Pi as a Limit

The great astronomer, geographer, and mathematician Claudius Ptolemaeus, known as Ptolemy (83 – 161 CE), wrote in 150 CE his astronomical treatise *Almagest*. With the help of the sexagesimal system, a base 60 numerical system, Ptolemy was able to express the fractional parts of numbers. In particular, his table of chords, which was essentially the only extensive table for more than a millennium, has fractional parts of a degree in base 60 [12]. A chord of a circle is a line segment whose endpoints are on the circle. Ptolemy used a circle whose diameter is 120. He tabulated the length of a chord whose endpoints are separated by an arc of n degrees, for n ranging from 0 to 180 by increments of $\frac{1}{2}$. In modern notation, the length of the chord of circle with diameter 120 corresponding to an arc of θ degrees is

$$\text{chord}(\theta) = 120\sin(\frac{\theta}{2}) = 60(2\sin(\frac{\pi\theta}{360}\text{ radians}))$$

where θ is in between 0 to 180 degrees and the chord of θ is in between 0 to 120 degrees.
The fractional parts of chord lengths were expressed in sexagesimal (base 60) numerals. For example, the length of a chord subtended by a 112° arc has a length of approximately

$$99 + \frac{29}{60} + \frac{5}{60^2} = 99.4847\overline{2}\,.$$

Likewise, using the same system, the square root of 2, the length of the diagonal of a unit square, was approximated by the Babylonians of the Old Babylonian Period (1900 BC – 1650 BC) as

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \frac{30547}{21600} \approx 1.41421296...$$

Using this system, Ptolemy was able to approximate π as

$$3 + \frac{8}{60} + \frac{30}{60^2} = 3\frac{17}{120} = 3.141666... = 3.141\overline{6} = 3.14167\,.$$

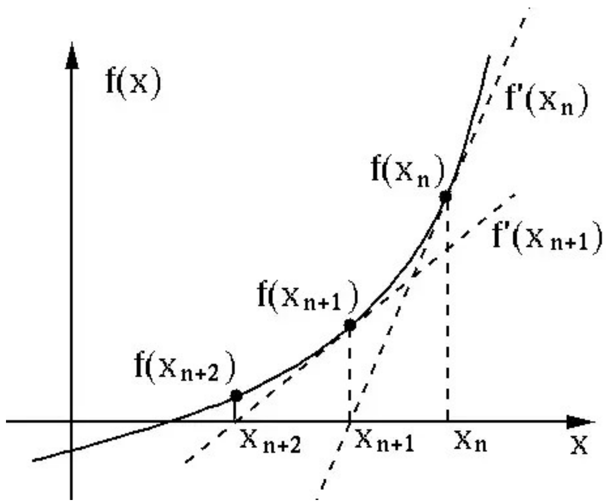
At the time, other than the approximation of π by Archimedes, this one was one of the most accurate results [12]. The existence of irrational numbers was already acknowledged at that time.

The Jewish philosopher Maimonides (1135 – 1204) first refers to irrational numbers in his comments on the Bible, when he explicitly says that the ratio of the circle's diameter to its circumference is not known and it can never be expressed precisely. He also explains that a circle, whose diameter is one handbreadth, has a circumference of approximately three and one seventh handbreadth. Therefore, by reversing the order we get the ratio of circumference and diameter of a circle to be

$$\frac{3\frac{1}{7}}{1} = 3\frac{1}{7} = 3.142857...$$

which is approximately the value of π [1].

By definition an irrational number is any real number that cannot be expressed as the quotient of two integers. For example, there is no number among integers and fractions that equals the square root of 2, which is the length of the diagonal of a square whose side is one unit long. Early in the history of mathematics it became necessary to extend the concept of number to include irrational numbers. Each irrational number can be expressed as an infinite decimal expansion with no regularly repeating digit or a finite group of digits. The natural numbers, also called the counting numbers $\mathbb{N}$, are part of the integers $\mathbb{Z}$, which are part of the rational numbers $\mathbb{Q}$, which together with the irrational numbers comprise the set of real numbers $\mathbb{R}$. The real number system together with the two operations of addition and multiplication, and the ordering property of the real numbers, $(\mathbb{R},+,\cdot,<)$, satisfies the properties of an ordered field. Under both addition and multiplication, the associative property, the commutative property, the distributive property, the identity elements, and the inverse elements constitute the properties of a field. The completeness property of the real numbers implies that $(\mathbb{R},+,\cdot,<)$ is a complete field. The completeness property of real numbers states that every nonempty set of real numbers that is bounded from above contains a least upper bound that is itself a real number. The completeness property of real numbers plays a major role in defining the irrational numbers. Completeness implies that there are no gaps or holes on the real number line. Irrational numbers are the least upper bounds of convergent increasing sequences of rational numbers. Hence, they fill in the holes of the real number line. This can be illustrated by using Newton's method. In numerical analysis, Newton's method, also known as the Newton–Raphson method, after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeroes) of a real-valued function.



The most basic version of this algorithm starts with a single-valued function f defined for a real variable x, the function's derivative f', and an initial guess x_0 for a root of f. If the function satisfies necessary assumptions and the initial guess is close to the root, then a better approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Geometrically, $(x_1, 0)$ is the intersection of the x-axis and the tangent of the graph of f at $(x_0, f(x_0))$. That is the improved number which represents the unique root of the linear approximation at the initial point. The process is repeated as iteratively

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

until an acceptable value is reached.

Now let us examine our irrational number $x = \sqrt{2}$. Consider $f(x) = x^2 - 2$.

Then, $f'(x) = 2x$ and

$$x_{n+1} = x_n - \frac{x_n^{\,2} - 2}{2x_n} = x_n - (\frac{1}{2}x_n - \frac{1}{x_n}) = \frac{1}{2}x_n + \frac{1}{x_n}.$$

Begin with:

$$x_1 = 2$$
$$x_2 = \frac{1}{2} \cdot 2 + \frac{1}{2} = \frac{3}{2} = 1.5$$
$$x_3 = \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{\frac{3}{2}} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1.41\overline{6}$$
$$x_4 = \frac{1}{2} \cdot \frac{17}{12} + \frac{1}{\frac{17}{12}} = \frac{17}{24} + \frac{12}{17} = \frac{289 + 288}{408} = \frac{577}{408} = 1.414225686...$$

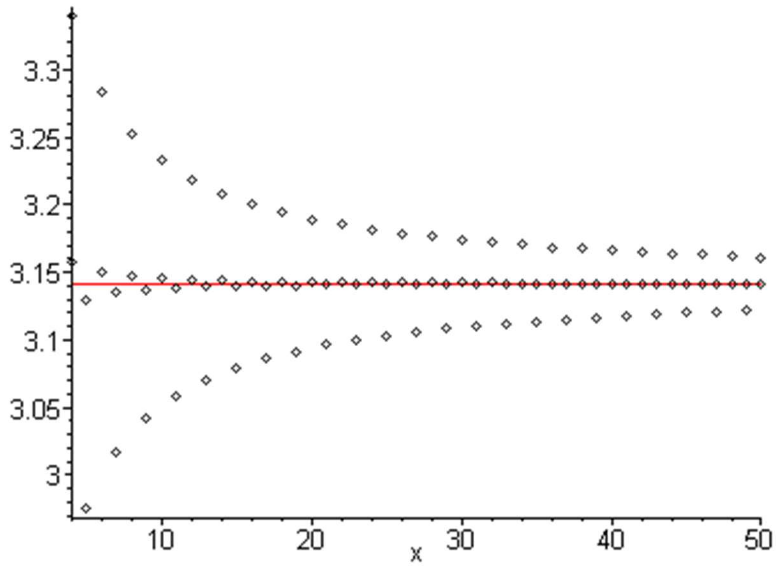
If we arrange these terms in a sequence, $2, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots$, then the terms become arbitrarily close to $\sqrt{2}$. Hence $\sqrt{2}$ is the least upper bound and is a real number due to the completeness property. Hence it fills a hole on the real number line that was not filled by any rational number.

This can be done for all irrational numbers, not just the square roots. Hence π can also be obtained this way. To prove this fact, begin with:

$$x_1 = 3$$
$$x_2 = 3.1 = \frac{31}{10}$$
$$x_3 = 3.14 = \frac{314}{100}$$
$$x_4 = 3.141 = \frac{3141}{1000}$$

These fractions become closer and closer to π but never equal π. So, the limit of this sequence of fractions is π. Hence, π is the least upper bound, or the smallest upper bound of all the upper bounds and is one of the irrational numbers that completes the real number line.

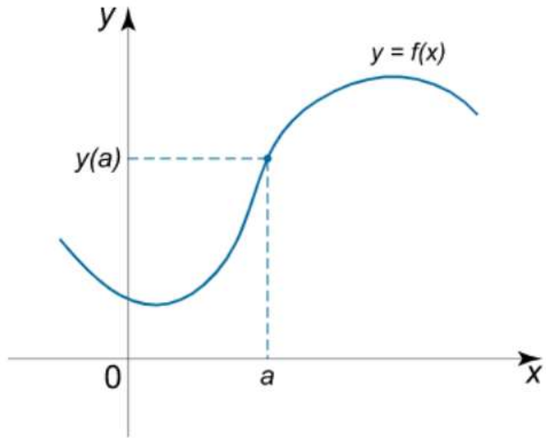
The monotone convergence theorem states that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum. In the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum. Hence a bounded monotonic sequence of real numbers will converge to a real number that is the limit. This completeness property ensures us that this real number is one that fills in the hole on the real number line.



This completeness fact relating to real and therefore, irrational numbers, allows us to see the set of all real numbers as made up of the rational numbers together with the limits of all convergent sequences of rational numbers, which in fact include the set of irrationals. Hence, $\mathbb{R} = \mathbb{Q}$ U {limits of all convergent sequences of rational numbers}.

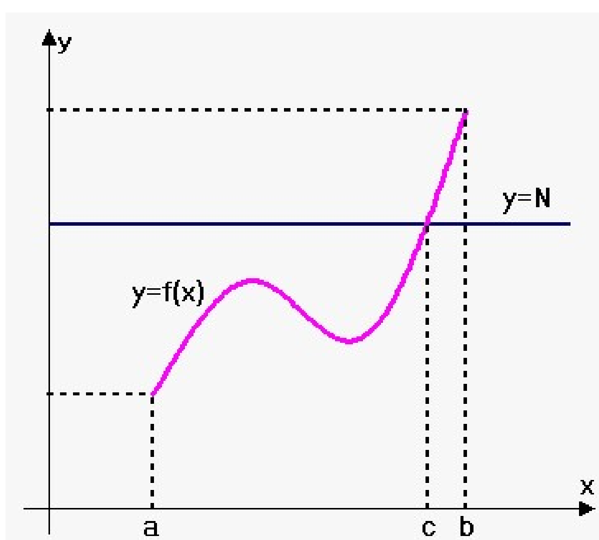
There are some other results in mathematics which illustrate the need for irrational numbers. The calculus concept of continuity, the Intermediate Value Theorem, and the Extreme Value Theorem are just a few examples where the irrational numbers are essential.

A function f is continuous at $x=a$ if $\lim_{x \to a} f(x) = y(a) = f(a)$.

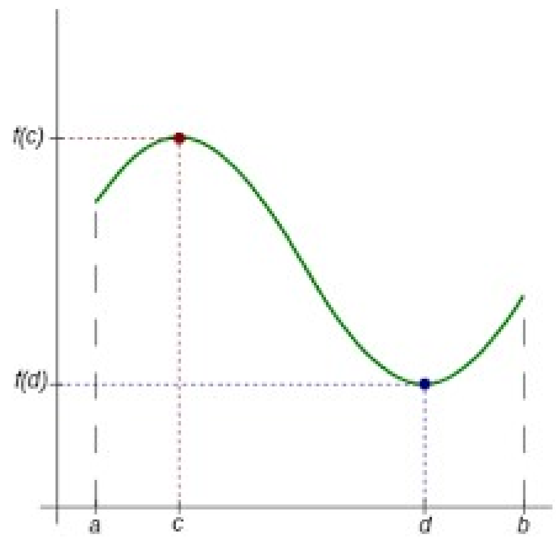


Irrational numbers are the limits and fill in the holes allowing continuity of the number line. Without the completeness property we would have holes in the curve of the function, so the function would not be continuous.

The Intermediate Value Theorem applies indirectly to irrational numbers as well. The Intermediate Value Theorem states that for any continuous function and any number N between $f(a)$ and $f(b)$, there must be a number c in between a and b for which $f(c) = N$.



Just as in the previous instances, for the Extreme Value Theorem the irrational numbers are needed.



The extreme value theorem states that if a real-valued function *f* is continuous on the closed interval [a, b], then *f* must attain a maximum and a minimum, each at least once. The function is continuous and bounded on [a, b], and it cannot have a hole at the extreme; hence, irrational numbers are needed.

These theorems assure us that π is in fact not just an ordinary number, but an irrational number that fills a hole in the number line and completes the real number system in a perfect way. During the Renaissance Fibonacci approximated π to be

$$\frac{1440}{458\frac{1}{3}} = 3.14181818181818181818181818181818 \cdot$$

In 1223, in *Practica Geometria,* he obtained the value from the average of

$$\frac{1440}{458\frac{1}{5}} \text{ and } \frac{1440}{458\frac{4}{9}} \text{ [1].}$$

Although his approximation was not as close as later ones, Fibonacci's work represented a major development of the number system and mathematics generally. Throughout the 16th century, Francois Viète, considered a regular polygon with $6 \cdot 2^{16} = 393{,}216$ sides and estimated π by using a method developed by the Greeks. He estimated π correct to nine decimal places, by using an infinite product. He used

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots,$$

to calculate the value of π to be in between 3.1415926535 and 3.1415926537. During the 17th century, John Wallis, a professor of mathematics derived the formula

$$\frac{\pi}{2} = \frac{2\times2}{1\times3} \times \frac{4\times4}{3\times5} \times \frac{6\times6}{5\times7} \times \frac{8\times8}{7\times9} \times \ldots \times \frac{2n\times2n}{(2n-1)(2n+1)} \times \ldots .$$

Wallis's result was then transformed into a continued fraction by William Brouncker (1620 – 1684), an English mathematician, as:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \ldots}}}}}.$$

To analyze the continued fraction, we can look at each increasing piece of the fraction, each time cutting off the rest of the fraction at the plus sign. These parts are called convergents.

The first convergent is 1. The second convergent is $1 + \frac{1^2}{2} = \frac{3}{2}$. The third convergent is

$$1 + \cfrac{1^2}{2 + \cfrac{3^2}{2}} = 1 + \cfrac{1}{2 + \cfrac{9}{2}} = 1 + \frac{2}{13} = \frac{15}{13}.$$

The fourth convergent is $\frac{105}{76}$. The fifth convergent is $\frac{945}{789}$. To get the related approximation of π, we need to multiply the reciprocal of each convergent by 4.

Hence, we get:

$$1 \cdot 4 = 4$$
$$\frac{2}{3} \cdot 4 = \frac{8}{3} \approx 2.6667$$
$$\frac{13}{15} \cdot 4 = \frac{52}{15} \approx 3.46667$$
$$\frac{76}{105} \cdot 4 = \frac{304}{105} \approx 2.8952380$$
$$\frac{789}{945} \cdot 4 = \frac{3{,}156}{945} = \frac{1{,}052}{315} \approx 3.3396825$$

The fractions get closer to the true value of $\pi = 3.14159265358979\ldots$ [1].

As I mentioned earlier in this essay, it took centuries to obtain more accurate approximations of π. In 1668 the Scottish mathematician James Gregory anticipated a result of the famous German mathematician Gottfried Wilhelm Leibniz, when he derived the following approximation:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

Using power series, Newton and Leibnitz, the founders of calculus, were able to prove

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$
$$\tan^{-1}(x) = \int (1 - x^2 + x^4 - x^6 + \ldots)dx$$

Hence,

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

and therefore, when substituting $x = 1$, they obtained

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} \cdot 1^3 + \frac{1}{5} \cdot 1^5 - \frac{1}{7} \cdot 1^7 + \ldots \text{ and } \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots).$$

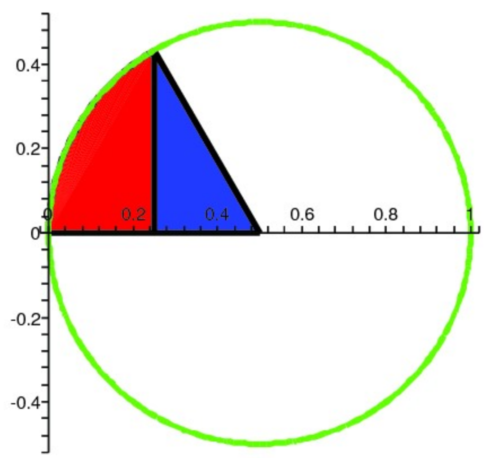
Today this method is considered not very useful because the series converges to π very slowly. It takes 500,000 terms to estimate π correctly to five decimal places and about five billion terms to estimate it to ten correct decimal places. However, one can distinguish the partial sums as follows

$$x_1 = 4$$
$$x_2 = 4(1-\frac{1}{3}) = \frac{8}{3}$$
$$x_3 = 4(1-\frac{1}{3}+\frac{1}{5}) = \frac{52}{15}$$
$$\ldots$$

which becomes Brouncker's method of estimating π. The fractions make up a sequence of rational numbers that converges in the end to our special irrational number π.

Isaac Newton's arcsine computation provided 15 digits of π. Born in 1642, Isaac Newton, was an English mathematician, physicist, and astronomer recognized as one of the most influential scientists of all times [13].

To develop π, he looked at the area A of the left-most region shown as



The area expresses as an integral is $A = \int_0^{\frac{1}{4}} \sqrt{x - x^2 dx}$. The same area refers to the circular sector and is $\frac{\pi}{24}$ minus the area of the triangle which is $\frac{\sqrt{3}}{32}$. Newton used his binomial theorem in this integral and developed

$$A = \int_0^{\frac{1}{4}} x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} dx = \int_0^{\frac{1}{4}} x^{\frac{1}{2}} (1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \ldots) dx = \int_0^{\frac{1}{4}} (x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{8} - \frac{x^{\frac{7}{2}}}{16} - \frac{5x^{\frac{9}{2}}}{128} - \ldots) dx.$$

By integrating term-by-term and combining the like terms, Newton got to:

$$\pi = \frac{3\sqrt{3}}{4} + 24[\frac{1}{12} - \frac{1}{(5\cdot 2^5)} - \frac{1}{(28\cdot 2^7)} - \frac{1}{(72\cdot 2^9)} - ...]\ [14].$$

Euler calculated π to 126 place accuracy. He created a series by taking the squares of the terms in a harmonic series. A harmonic series is created by taking the reciprocal of the terms of an arithmetic sequence, one with a common difference between terms. The simplest arithmetic series is 1, 2, 3, 4, 5… The related harmonic series is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}....$$

Euler created the series $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ...$ to estimate π . Moreover, Euler's trigonometric identity $\tan^{-1}(1) = \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3})$ produces the convergent rational series

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + ... + \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + ...\ [1].$$

He also developed the formula $e^{i\pi} + 1 = 0$ which includes the five famous numbers: $0, 1, e, i,$ and π . The number π like e is not only irrational but is also transcendental.

6. Is Pi Algebraic or Transcendental?

A transcendental number is a number that cannot be the root of any polynomial with integer coefficients. Therefore, a transcendental number is not an algebraic number and therefore must also be irrational. Transcendental numbers represent an important part of mathematics because for instance, they helped mathematicians understand that squaring a circle was not possible. On the other hand, many algebraic numbers, known also as Euclidean numbers, can be produced by a geometric construction using the Greek rules. The question of the rationality of π came to an end in the late 1700s, when Lambert and Legendre proved, by using continuous fractions, that the constant is irrational. The question of whether π was algebraic was settled in 1882, when Lindemann proved that π is transcendental. Lindemann's proof settled once and for all the impossibility of "squaring the circle"— that is constructing with straightedge and compass alone a square whose area equals that of a given circle. This was one of the great straight edge and compass problems of classical geometry, along with doubling the cube and trisecting an angle [14]. Ancient Greek geometers studying the circle had proven that the circumference, or "periphery", is proportional to the diameter, and that the area is proportional to the square of the radius. Carl Louis Ferdinand Von Lindemann (1852 – 1939) was a professor of mathematics at the University of Freiburg and a specialist in geometry and analysis and had completed his doctoral thesis on a topic in non-Euclidean geometry [14]. As a necessary introduction to von Lindemann's proof, we must recall some familiar properties of polynomial equations, and establish some fewer familiar ones. The Fundamental Theorem of Algebra tells us that any complex polynomial of degree n, with complex coefficients has exactly n complex roots if duplicate roots are counted [14].

The following proof is a variation of Lindemann's original proof by Michael Filaseta, a number theorist in the Department of Mathematics at the University of South Carolina.
To prove that π is transcendental we first need to make use of

$$I(t) = \int_0^t e^{t-u} f(u)du,$$

where t is a complex number and $f(x)$ is a polynomial with complex coefficients to be specified later. Integration by parts gives us

$$(1)\quad I(t)=e^t\sum_{j=0}^{\infty}f^{(j)}(0)-\sum_{j=0}^{\infty}f^{(j)}(t)=e^t\sum_{j=0}^{n}f^{(j)}(0)-\sum_{j=0}^{n}f^{(j)}(t),$$

where n is the degree of $f(x)$.

If $f(x)=\sum_{j=0}^{n}a_j x^j$, we set

$$\overline{f}(x)=\sum_{j=0}^{n}\left|a_j\right|x^j .$$

Then

$$\left|I(t)\right|\le\left|\int_0^t\left|e^{t-u}f(u)\right|du\right|\le|t|\max\{\left|e^{t-u}\right|\}\max\{\left|f(u)\right|\}\le|t|e^{|t|}\overline{f}(|t|).$$

To continue the proof, we must review few definitions.

An algebraic number is defined to be any complex number that is a root of a non-zero polynomial in one variable with rational coefficients. Some examples of algebraic numbers include all integers, rational numbers, and some of the irrational numbers. However, numbers like e and π are not algebraic numbers. While the set of complex numbers is uncountable, the set of algebraic numbers is countable.

An algebraic integer is a complex number that is a root of some monic polynomial, a polynomial whose leading coefficient of 1, with integer coefficients. The set of algebraic integers is closed under addition, subtraction, and multiplication. It is also a commutative subring of the complex numbers.

Let K be a number field. A rational integer of K is an integer of K that is included in the set of integers ($\mathbb{Z}$). Generally, every integer n is a rational number since each integer can be written as a fraction $n/1$. The terminology “rational integer” is used to emphasize that the number is simply an integer in the usual sense and is not an algebraic number.

A minimum polynomial of a value α is the polynomial of lowest degree having coefficients of a given type, such that α is a root of the polynomial. If the minimum polynomial exists, it is unique. The highest degree coefficient is 1 and the rest of the coefficients could be integers, rational numbers, real numbers, or others. For example, suppose $\alpha=\sqrt{2+\sqrt{3}}$. The minimum polynomial of α in $\mathbb{Q}$ is x^4-4x+1.

Minimal polynomials are useful for constructing and analyzing field extensions. The field of complex numbers $\mathbb{C}$ is an extension field of the field of real numbers $\mathbb{R}$ and $\mathbb{R}$ in turn is an extension field of the field of rational numbers $\mathbb{Q}$. Clearly then, $\mathbb{C}/\mathbb{Q}$ is also a field extension. Hence, more formally, let *L/K* be a field extension. Let $\alpha \in L$ be algebraic over *K*. The minimum polynomial of α over *K* is the unique irreducible, monic polynomial $f \in K(x)$ such that $f(\alpha)=0$.

To prove that π is transcendental we also need to consider the following lemmas:

Lemma 1

If α and β are algebraic numbers, then so are their sum, difference, product, and quotient (when β is not 0) . If α and β are algebraic integers, then so are their sum, difference, and product.

Lemma 2

If α is an algebraic number with minimum polynomial $g(x) \in \mathbb{Z}[x]$ and if b is the leading coefficient of $g(x)$, then their product $b\alpha$ is an algebraic integer.

Lemma 3

If α is an algebraic integer and is rational, then α is a rational integer.

In addition, we need to be familiar with the fundamental theorem of elementary symmetric functions. For example, consider the real function in three variables: $f(x_1,x_2,x_3)=(x-x_1)(x-x_2)(x-x_3)$. By definition, a symmetric function has the property that $f(x_1,x_2,x_3)=f(x_2,x_1,x_3)=f(x_3,x_1,x_2)$, etc. Hence the function remains the same for every permutation of its variables. This means for this function

$$(x-x_1)(x-x_2)(x-x_3)=(x-x_2)(x-x_1)(x-x_3)=(x-x_3)(x-x_1)(x-x_2)$$

and so on for all permutations of x_1,x_2,x_3.

Generally, any symmetric polynomial (respectively, symmetric rational function) can be expressed as a polynomial (respectively, rational function) in the elementary symmetric functions on those variables.

The elementary symmetric function $\prod_n$ on n variables $\{x_1, x_2, ..., x_n\}$ are defined by

$$\prod_1 = \sum_{1 \le i \le n} x_i$$
$$\prod_2 = \sum_{1 \le i < j \le n} x_i x_j$$
$$\prod_3 = \sum_{1 \le i < j < k \le n} x_i x_j x_k$$
$$\prod_4 = \sum_{1 \le i < j < k < l \le n} x_i x_j x_k x_l$$
$$...$$
$$\prod_n = \sum_{1 \le i \le n} x_i.$$

Alternatively, $\prod_j$ can be defined as the coefficients of x^{n-j} in the generating functions $\prod_n = \sum_{1 \le i \le n} (x + x_i)$.

The Fundamental Theorem of Elementary Symmetric Functions states that if R be a commutative ring and $R[x_1, x_2, ..., x_n]$ the polynomial ring in n indeterminates over R then a symmetric polynomial $f(x_1, x_2, ...x_n)$ of degree k can be expressed as a polynomial of the elementary symmetric polynomials. This representation is unique.

Let's begin by saying that if π were algebraic, then $i\pi$ would be as well (by Lemma 1). Hence, it is enough to show that $\theta = i\pi$ is transcendental. Assume otherwise. Let r be the degree of the minimal polynomial $g(x)$ for θ , and let $\theta_1, \theta_2, ..., \theta_r$ denote the conjugates for θ, where $\theta_i = \theta$ Let b denote the leading coefficient of $g(x)$.

Not that $b \cdot \theta_i$ is an algebraic integer (see Lemma 2). Since $e^{i\pi} = -1$, we deduce that

$$(1 + e^{\theta_1})(1 + e^{\theta_2})...(1 + e^{\theta_n}) = 0.$$

Multiplying this expression on the left of the equation , we obtain a sum of 2^r term of the form e^{ϕ} where $\phi = \varepsilon_1\theta_1 + \varepsilon_2\theta_2 + ... + \varepsilon_r\theta_r$ with $\varepsilon_j \in \{0,1\}$, for all j. Let $\phi_1, \phi_2, ..., \phi_n$ denote the nonzero expressions of this form so that (since the remaining $2^r - n$ values of ϕ are 0)

$q + e^{\phi_1} + e^{\phi_2} \ldots + e^{\phi_n} = 0$, where $q = 2^r - n$. Let p be a large prime, and let

$$f(x) = b^{np} x^{p-1} (x-\phi_1)^p (x-\phi_2)^p ... (x-\phi_n)^p.$$

By the fundamental theorem of elementary symmetric functions and Lemma 2 and Lemma 3, $f(x) \in \mathbb{Z}[x]$. To see this more clearly, consider $\phi_1, \phi_2, ..., \phi_{2^r}$ as the complete set of ϕ's as above (so the first n are still the non-zero ones) and use that

$$\prod_{j=1}^{2^r} (x - \phi_j) = x^{2^r - n} \prod_{j=1}^{n} (x - \phi_j)$$

is symmetric in $\theta_1, \theta_2, ..., \theta_r$. Define

$$J = I(\phi_1) + I(\phi_2) + ... + I(\phi_n)$$

From (1), we deduce that

$$J = -q \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} \sum_{k=1}^{n} f^{(j)}(\phi_k)$$

where $m = (n+1)p - 1$. Observe that the sum over *k* is a symmetric polynomial in $b\phi_1, b\phi_2, ..., b\phi_n$ with integer coefficients and thus a symmetric polynomial with integers in the 2^r numbers $b\phi = b(\varepsilon_1\theta_1 + \varepsilon_2\theta_2 + ... + \varepsilon_r\theta_r)$. Hence, by the Fundamental Theorem of Elementary Symmetric Functions we obtain that this sum is a rational number. Observe that Lemma 2 and Lemma 3 imply that the sum is furthermore a rational integer. Since $f^{(j)}(\phi_k) = 0$ for $j < p$, we deduce that the double sum in the expression for J above is a rational integer divisible by $p!$. Observe that $f^{(j)}(0) = 0$ is divisible by $p!$ for $j \geq p$. Also,

$$f^{(p-1)}(0) = b^{np} (-1)^{np} (p-1)! (\phi_1 ... \phi_n)^p.$$

From the Fundamental Theorem of Elementary Symmetric Functions and Lemma 2 and Lemma 3, we deduce that $f^{(p-1)}(0)$ is a rational integer divisible by $(p-1)!$. Furthermore, if p is sufficiently large, then $f^{(p-1)}(0)$ is not divisible by p. If $p > q$, we deduce that

$$|J| \geq (p-1)!.$$

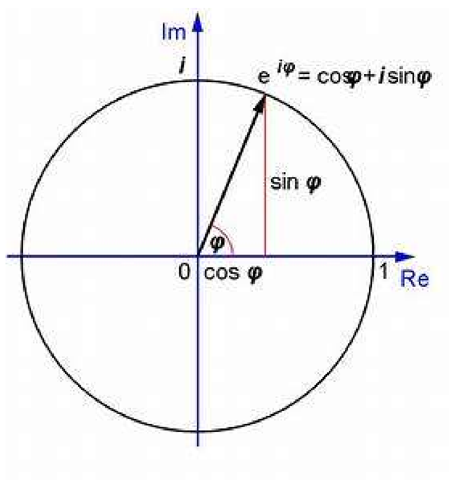
On the other hand, using the upper bound we obtained for $|I(t)|$, we have

$$|J| \leq \sum_{k=1}^{n} |\phi_k| e^{|\phi_k|} f(|\phi(k)|) \leq c_1 c_2{}^p$$

for some constants c_1 and c_2. We get a contradiction, completing the proof.

7. Euler's Identity

In complex analysis Euler's formula shows the relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number x, $e^{ix} = \cos x + i \sin x$. In this formula, where *e* is the base of the natural logarithm function, *i* is the imaginary unit, $i = \sqrt{-1}$, and cos and sin are the trigonometric functions cosine and sine respectively, with the argument *x* given in radians [15].



When $x = \pi$, Euler's formula becomes

$$e^{i\pi} = \cos\pi - i\sin\pi = -1 - i(0) = -1,$$ which is equivalent to

$$e^{i\pi} + 1 = 0,$$

which is known as Euler's Identity. Euler's Identity is an expression that lies at the heart of complex number theory. Here is a proof of Euler's formula using Taylor series expansions as well as basic facts about the powers of *i*. We know:

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i,$$
$$i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i,$$

and so on. The functions e^x, $\cos x$ and $\sin x$ of the real variable *x* can be expressed using their Taylor expansions centered at zero as:

$$
\begin{aligned}
e^{x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
\end{aligned}
$$

For complex z we define each of these functions by the above series, replacing the real variable x with the complex expression iz. This is possible because the radius of convergence of each series is infinite. We then find that

$$
\begin{aligned}
e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \frac{(iz)^7}{7!} + \frac{(iz)^8}{8!} + \dots \\
&= 1 + iz - \frac{z^2}{2!} + \frac{iz^3}{3!} - \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} + \frac{iz^7}{7!} - \frac{z^8}{8!} + \dots \\
&= (1 - \frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \frac{z^8}{8!} + \dots) + i(z + \frac{iz^3}{3!} + \frac{iz^5}{5!} + \frac{iz^7}{7!} + \dots) \\
&= \cos z + i \sin z
\end{aligned}
$$

The rearrangement of terms is justified because each series is absolutely convergent [16].

Euler's highly valued expression is known as the "gold standard of mathematical beauty" because it links seemingly different branches of mathematics in an exquisitely simple manner. Its ability to represent a deep fundamental mathematical truth with an equation is what delights mathematicians all around the world. Its aesthetic comes from Euler's ability to connect the five royal constants [15].

Euler's identity contains Euler's number e, the base of the natural logarithm that is extensively used in calculus. It is a transcendental number whose value is 2.71828…. It also has i, the complex number or imaginary unit, which is the square root of -1 or the solution of the equation $x^2+1=0$**.** It is important in electrical engineering and has provided great insights in quantum mechanics. This expression also includes π, the transcendental number obtained by the ratio of a circle's circumference to its diameter. Its value is 3.14159… This constant does not need any further introduction, as it is the most popular mathematical constant, ubiquitous in fields from Euclidean geometry to General Relativity. It also includes the first natural number 1, the multiplicative identity — any number multiplied by this identity results in the same number itself. It possesses the first whole number 0, the additive identity – any number added to this identity results in the same ordinary number itself. Therefore, Euler's Formula combines five of the most famous mathematical constants. These significant constants have many practical applications, including communication, navigation, energy, manufacturing, finance, meteorology and medicine. In this formula π, like e, is not just some ordinary number but as we have seen it is a transcendental and therefore irrational number that also has numerous applications in real life. The importance of π goes beyond mathematics and has many applications in the natural world as well. Since π is related to circles, it is used to describe the disk of the sun, the spiral of the DNA double helix, the pupil of the eye, and the concentric rings from splashes in ponds. These are just a few of the examples where we encounter π .There are of course many more. The number π also appears in the physics that describes waves, such as ripples of light and sound; π describes a river's windiness and it can be used to identify its meandering ratio and the ratio of the river's actual length to its distance from its source to its mouth measured in a straight line. Rivers that flow straight from their source to their mouth have small meandering ratios, while those that meander along the way have higher ones. The average meandering ratio of rivers gets close to π . Albert Einstein used fluid dynamics to show that rivers tend to bend into multiple curves. He was the first to explain this fascinating fact. The slightest curve in a river will produce large currents on the outer side of the curve, which will cause extreme erosion and a deeper bend. This process will gradually tighten the curve, until it causes the river to change its direction and it begins to form a new curve. The curve looks like the circumference of a circle and the distance from one bend to another looks like a diameter. More so, the ratio of the two approximates π [17].

For all these reasons, I understand today why a whole school got excited about what I thought was merely an ordinary number. An educated group of teachers transformed an entire school into a Pi celebration space. The teachers quickly engaged all students in many celebratory activities and the excitement of knowing facts about Pi was highly rewarded. It took me few years of teaching mathematics and some wonderful encounters within a mathematics masters degree program to fill in the void that I had as a student when it came to what I used to call ordinary numbers. Today, I also know the reasons for which I felt that Pi was my best friend while completing my school assignments. Pi is not just a symbol for an ordinary number. It is a remarkable friend that always brings light, peace, and structure into my meandering work; its presence gives me joy because I know that no matter how chaotic my work is, it will always help me reach my destination successfully.

8. Epilogue

The presence of π can be thought of as the result of a battle between order and chaos. In her wonderful ode to π, the Nobel-winning Polish poet Wislawa Szymborska describes the number as "the admirable number… nudging, always nudging a sluggish eternity to continue." In 1996 the Cambridge scientist Hans-Henrik Stolum published a paper in which he also concluded that π is always nudging the paths of rivers that go into a predictable mathematical pattern. By using empirical data and fluid dynamics modeling he calculated the ratio of the river's actual meandering length by the length of the direct line traced from source to sea and estimated the average is approximately 3.14 [18].

PI

by Wisława Szymborska

" The admirable number pi:
three point one four one.
All the following digits are also initial,
five nine two because it never ends.
It can't be comprehended *six five three five* at a glance,
eight nine by calculation,
seven nine or imagination,
not even *three two three eight* by wit, that is, by comparison
four six to anything else
two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi
doesn't stop at the page's edge.
It goes on across the table, through the air,
over a wall, a leaf, a bird's nest, clouds, straight into the sky,
through all the bottomless, bloated heavens.
Oh how brief — a mouse tail, a pigtail — is the tail of a comet!
How feeble the star's ray, bent by bumping up against space!

While here we have *two three fifteen three hundred nineteen*
my phone *number your shirt size the year*
nineteen hundred and seventy-three the sixth floor
the number of inhabitants sixty-five cents
hip measurement two fingers a charade, a code,
in which we find *hail to thee, blithe spirit, bird thou never wert*
alongside *ladies and gentlemen, no cause for alarm*,
as well as *heaven and earth shall pass away*,
but not the number pi, oh no, nothing doing,
it keeps right on with its rather remarkable *five*,
its uncommonly fine *eight*,
it's far from final *seven*,
nudging, always nudging a sluggish eternity
to continue." [18]

REFERENCES

1. Posamentier, S., Alfred, Lehmann Ingmar. (2004). *A Biography of the World's Most Mysterious Number. New York, Prometheus Group.*
2. Gray, Jason. (2019). Retrieved July 17, 2019, from JGray.org:

   http://www.jgray.org/revelation/numbers.html
3. Salman Khan (2017). Retrieved July 18, 2019, from Shiska:

   https://www.ekshiksha.org.in/chapter/160/areas_related_to_circles.html
4. *Euler and the Function sinx/x*. (2012, July 2). Retrieved from Department of Mechanical & Airspace Engineering: The University of Florida:

   http://www2.mae.ufl.edu/~uhk/EULER-SINE.pdf
5. *Wallis' Formula.* (2011, February 2). Retrieved from Encyclopedia of Mathematics:

   http://www.encyclopediaofmath.org/index.php?title=Wallis_formula&oldid=13195
6. *The Three Classical Impossible Constructions of Geometry.* (1999, April 19). Retrieved July 17, 2019 from Mathematics Network: University of Toronto:

   https://www.math.toronto.edu/mathnet/questionCorner/impossconstruct.html
7. Hartzer, Paul (n.d.). *The Archimedes Method of Estimating Pi.* Retrieved January 20, 2019, from GeoGebra:

   https://www.geogebra.org/m/xqXNW5TE
8. Buffon, Georges Louis Leclerc, Comte de". *Encyclopedia of Life Sciences.* Biographies Plus Illustrated, H.W. Wilson Company (2001). vnweb.hwwilsonweb.com (Accessed December 26, 2005)
9. *Buffon's Needle an Analysis and Simulation*. (2019, March), Retrieved March 25, 2019 from the MSTE — Office for Mathematics, Science, and Technology Education College of Education: The University of Illinois:

   https://mste.illinois.edu/activity/buffon/

10. *Probability of Two Integers of Being Coprime.* (July 19, 2019). Retrieved from *Cut the Knot:*

https://www.cut-the-knot.org/m/Probability/TwoCoprime.shtml

11. Alsina, Claudi, Nelsen, B., Roger. (2010). *Charming Proofs. A Journey in Elegant Mathematics.* The Mathematical Association of America.

12. Toomer, G. J. (1998). *Ptolemy's Almagest*, Princeton University Press.

13. Jonathan M. Borwein. (18 September 2013). *The Life of Pi: From Archimedes to*

14. *Eniac and Beyond. Retrieved from Springer Link.* (July 2019).

https://link.springer.com/chapter/10.1007%2F978-3-642-36736-6_24

15. Schwartz, K., Randy. (2006, February 2). *Pi is Transcendental: Von Lindemann's Proof Made Accessible to Today's Undergraduates*. Retrieved from Research Gate: https://www.researchgate.net/publication/272181195_Pi_is_Transcendental_Von_Lindemann's_Proof_Made_Accessible_to_Today's_Undergraduates.

16. *Euler's Identity. (n.d.). Retrieved from Wikipedia: The Free Encyclopedia:* https://en.wikipedia.org/wiki/Euler%27s_identity

17. Hansha,, Ozahner. (2017 November 27). *Understanding Euler's Formula.* Retrieved from Medium Daily Digest: https://medium.com/@ozanerhansha/understanding-eulers-formula-888e5f58f559

18. Wolchover Natalia. (2009 August 9*) What Makes Pi So Special.* Retrieved from Live Science: https://www.livescience.com/34132-what-makes-pi-special.html

19. Popova, Maria. (n.d.) *Pi and Meandering Paths of Rivers.* Retrieved from Brain Pickings: https://www.brainpickings.org/2018/06/21/pi-rivers/