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THE UBIQUITY OF PHI IN HUMAN CULTURE & THE NATURAL WORLD

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I. INTRODUCTION

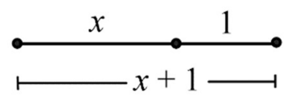
What do rabbit breeding, tornadoes, the Chambered Nautilus, a pentagram, the rhythm of a heartbeat, apple seeds, the shape of a credit card, a pinecone, the human ear, DaVinci's *Last Supper*, the structure of DNA, a light switch cover, and the structure of galaxies all have in common? Each relates to an extraordinary ratio that is highly efficient in nature, profoundly attractive to the human eye, and some claim, even divinely inspired. This special ratio is referred to as the "Golden Ratio" and is also known as the divine proportion, golden section, and golden mean. The Golden Ratio has a constant numeric value called "phi" (pronounced "FEE," or "FI") which is thought to be the most beautiful and astounding of all numbers. Phi, expressed as the Greek letter, φ, has an approximate numeric value of 1.618033988749895.

There is a generally-held consensus among classicists, and historians of mathematics, that the Golden Ratio was first understood and used by ancient Greek mathematicians during the periods known as the Classical and Hellenistic Periods of Greek Mathematics (from around 600 BCE to 600 CE). Phi was named for the Greek architect, mathematician, painter and sculptor, Pheidias, who used the divine proportion in his architecture. Because of the relationship between the Golden Ratio and the pentagon, some scholars have claimed that φ was known as early as the time of the Babylonians. Although there do exist such artifacts as cave drawings of pentagrams and pentagons as well as cuneiform tablets showing that the Babylonians had a rudimentary method for calculating the area of a pentagon, there is no conclusive evidence that they used, or were aware of, the Golden Ratio [10, p45]. While theories that the Golden Ratio was known in ancient Egypt, most specifically, that it was used to construct the Great Pyramid of Giza in Cairo (completed around 2560 BCE), are common, numerous mathematicians have presented well-researched arguments vigorously refuting these theories. Furthermore, it is important to appropriately contextualize the appearance of the Golden Ratio within the history of mathematics because the birth of Greek mathematics marked a monumental shift from the more empirical approaches of earlier civilizations to a highly sophisticated and intellectually rigorous new paradigm [1, p3]. For the purposes of this essay, the assumption is that φ was first understood during the time of the Greeks.

Since phi's digits do not terminate or repeat, φ, like π, is an irrational number meaning that it cannot be written as a ratio of two integer, or as a terminating, or repeating decimal.

The Golden Ratio is defined as the division of a line into two parts such that the following is true: the measure of the longer part ÷ the measure of the shorter part = the whole length ÷ the measure of the longer part. Consider Figure 1.1 below.

Figure 1.1



If the longer part has a length of x, and the shorter part has a length of 1, the whole length is $x + 1$. Figure 1.2 shows the solution to the quadratic equation resulting from setting up the two equal ratios and solving for x.

Figure 1.2

$$\frac{x}{1} = \frac{x+1}{x}$$

Cross-multiplication produces:

$$x^2 = 1(x+1)$$
$$x^2 = x+1$$
$$x^2 - x - 1 = 0$$

Using the quadratic formula to solve the equation:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{1 \pm \sqrt{1+4}}{2}$$
$$x_1 = \frac{1+\sqrt{5}}{2} = 1.61803...$$
$$x_2 = \frac{1-\sqrt{5}}{2} = -0.61803...$$

Disregarding the negative root, leaves 1.61803… = φ = phi!

This essay will explore various connections and coincidences related to The Golden Ratio that appear in the natural world as well as in human culture. Manifestations of φ in the structure of the universe have been observed and studied since at least the time of the ancient Greeks. The examples presented are only a very small sample of known appearances of φ and have been selected because they are expected to be of interest to the reader.

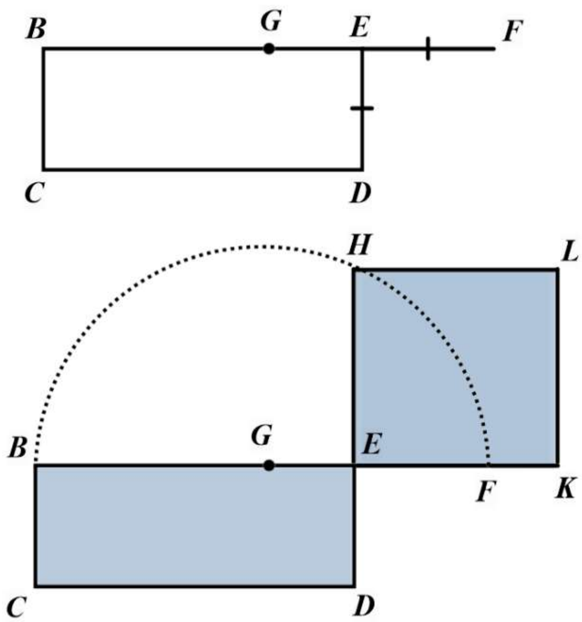
A brief history of the Golden Ratio in the ancient world will be followed by a discussion of φ in the areas of algebra, architecture, art, design, music, the natural world, human anatomy, and geometry. It will be shown that some appearances of φ, are only approximate and could be coincidental, while others are so undeniably accurate, and surprising that it is easy to understand why the Greeks would have believed they were divinely inspired.

II. THE EARLY GREEKS

The intention of Greek mathematics was not generally to solve practical problems, but rather to pursue knowledge for its own sake. Greek scholars established a contextualized intellectual model for studying abstract ideas, a model that is the very foundation for modern Western mathematical, scientific, medical, and philosophical inquiry. This involved consciously considering epistemology, understanding constraints, developing academic language, and explicitly using the concept of deductive proof. Whereas the Greeks studied broadly in many areas of what today we call philosophy, architecture, law, science, medicine, astronomy and literature, their contributions in the area of mathematics are considered to be by far the most impressive. They were committed to establishing a "conscious programme of study of abstract mathematical entities that was significantly different from empirical studies" [1, p2]. A specific indication that the Greeks pursued mathematical knowledge for its own sake is apparent in the types of problems that they posed to themselves. The nature of these problems suggests that the Greeks intended for scholars to contemplate, engage in discourse, and adhere to standards that would hold up to the highest scrutiny.

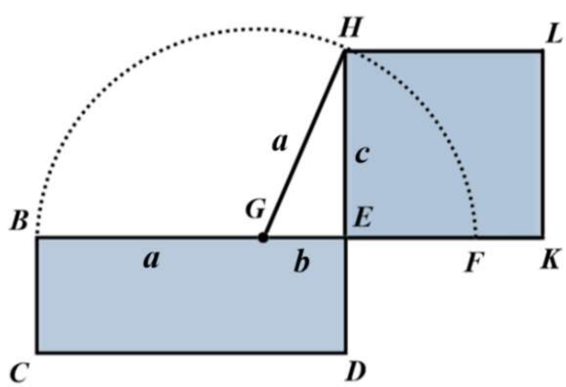
To get an idea of the types of problems the Greeks thought were important, consider the three unsolved classical Greek problems: squaring the circle, doubling the cube, and trisecting the angle. The Greeks required that these problems be solved using straightedge and compass which was the tradition established by Euclid in what is known to be the most influential textbook of all times, *The Elements*. To square a circle means to construct a square that has exactly the same area as the given circle. The problem of squaring the circle can be solved using different tools but it has never been solved using straightedge and compass (squaring the triangle, and the rectangle with straightedge and compass *can* successfully be accomplished). To get an idea of how a problem is solved using a construction with straightedge and compass, see Figure 2.1 for an example of "squaring a rectangle."

Figure 2.1



Given the rectangle *BCDE*, draw $EF = DE$. Find the midpoint of *BF* and call it *G*. Then construct a semicircle centered at *G*. Extend *DE* to intersect the semicircle at point *H*. Use length *EH* to mark off vertices at *L* and *K*. Draw *HL*, *LK* and *FK* to create square *EHLK*. The resulting square has an area equal to the area of the original rectangle *BCDE* [2, p13]. The Pythagorean Theorem can then be used to prove that the area of the rectangle is equal to the area of the square [2, p13]. See Figure 2.2.

Figure 2.2



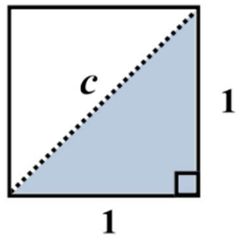
$$Area(BCDE) = BE \cdot EF = BE \cdot EF$$

$$= (a+b)(a-b) = a^2 - b^2 = c^2 .$$

Although squaring the circle with straightedge and compass is impossible, attempts to solve this problem have been made by some of the most accomplished mathematicians of all time for over 2000 years. This work has contributed significantly to human understanding in other areas of mathematics. One such example is the study of π. In the 19th century, Ferdinand von Lindemann, a German mathematician, finally proved that squaring the circle with straightedge and compass is actually impossible. His proof depends on the fact that π is a transcendental number and, therefore, cannot be a root of a polynomial equation with integer coefficients.

Though there are few surviving documents proving the contributions of any particular Greek mathematician, there are long established traditions in the history of math drawn from Euclid's *Elements* and evidence obtained before original artifacts were lost, and also from the extensive writings of philosophers such as Proclus, Plato, and Aristotle for whom some works have survived. The first two major Greek mathematicians were known to be Thales of Miletus (624–548 BCE), and Pythagoras of Samos (580–500 BCE) [10, p42]. Thales and Pythagoras were thought to have traveled to Babylon and Egypt, where they were able to obtain firsthand knowledge of ancient mathematics including geometry and astronomy. Thales is credited with being the first to make mathematics abstract, and with introducing the concept of deductive proof [10, p4], part of the reason that the excellence of Greek mathematics surpassed that of earlier civilizations. Pythagoras is known to be the originator of an astonishingly prolific intellectual movement and is a character of great intrigue as he was believed to be a mystic and a prophet. Both scholars are associated with well-known schools: Thales with the Ionian School which emphasized "rational thought over religious belief" [10, p4], and Pythagoras with the Pythagorean School, which was a secret, communal society with unusual moral requirements such as vegetarianism. For the Pythagoreans, the motto was "All is Number" [14, p45], which emphasizes the Pythagorean belief that the entirety of nature could be explained in terms of whole numbers, or ratios of whole numbers.

An essential question, key to ancient mathematics, was "is nature discrete or continuous?" [10, p2]. The early Greeks relied on The Principle of Commensurability which states that there is always some common unit that can measure any two things, i.e., that nature is discrete. In the 5th century BCE, there occurred what is referred to as "The First Great Crisis of Mathematics," the discovery, allegedly by Hippasus of Metapontum, that the Principle of Commensurability is actually false. This created a great upset because it meant that proofs, which depended on commensurability, were based on false assumptions and had to be reconsidered [10, p7]. Hippasus' discovery is easily observed in the unit square shown below. By the Pythagorean Theorem, the length of the diagonal is found to be $\sqrt{2}$.



$$1^2 + 1^2 = c^2$$
$$2 = c^2$$
$$\sqrt{2} = c$$

There is, of course, uncertainty about how exactly it was determined by Hippasus that $\sqrt{2}$ is irrational but in his writings about incommensurability, Aristotle supposedly referred to a method similar to the well-known proof, shown below, which is a proof by contradiction and depends on "the distinction between even and odd" [14, p66].

Theorem: *$\sqrt{2}$ is irrational.*

Proof: Assume that $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{p}{q}$ where p and q have no common factors,

and $2 = \frac{p^2}{q^2}$.

Then $2q^2 = p^2$ which means p^2 must be even.

Therefore, p must be even which implies that q must be odd.

Let $p = 2r$ and substitute into $2q^2 = p^2$ then

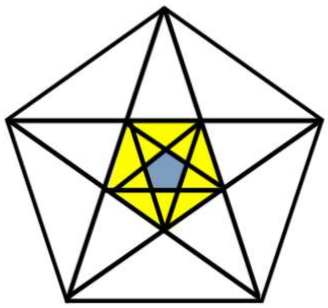
$$2q^2 = 4r^2$$
$$q^2 = 2r^2.$$

Then q^2 must be even which means q is even.

So, p and q are both even which means they do have a common factor.

By contradiction, $\sqrt{2}$ must be irrational ■

An alternative theory is that Hippasus actually drew his conclusion from making observations about pentagons. When connecting the vertices of a regular pentagon with five diagonals to form a pentagram, it can be observed that a smaller identical pentagon is created. When the process is repeated, yet another even smaller pentagon is formed. This process can be repeated indefinitely with each iteration creating a smaller and smaller pentagon. The ratio of the diagonal to the side is exactly the same for each iteration and this relationship also repeats indefinitely leading to the conclusion that the ratio of a diagonal of a pentagon to a side is not rational [14, p66]. A proof of why this is the case will be shown on the next page. If Hippasus discovered incommensurability in this way, however, then he would have proved that $\sqrt{5}$ is irrational, rather than $\sqrt{2}$, since the



ratio of the diagonal of a pentagon to the side is actually the Golden Ratio which includes $\sqrt{5}$ in its calculation. The proof that $\sqrt{5}$ is irrational is shown below. A fascinating story contradicting this hypothesis comes from Plato's dialogue, *Theaetetus* (written around 368 BCE), in which credit for the discovery of the incommensurability of the square roots of 3, 5, 6… (and all non-perfect squares up to 17) is attributed to Theaetetus but the incommensurability of $\sqrt{2}$ is explicitly not included, the theory being that it would have been common knowledge that Hippasus had already proved this fact almost 200 years earlier [13, p243].

Theorem: $\sqrt{5}$ *is irrational.*

Proof: Assume that $\sqrt{5}$ is rational.

Then $\sqrt{5} = \dfrac{p}{q}$ where p and q have no common factors,

and $5 = \dfrac{p^2}{q^2}$.

Then $5q^2 = p^2$ which means p^2 must be divisible by 5.

Therefore, p must be divisible by 5 which implies that q must not be divisible by 5.

Let $p = 5r$ and substitute into $5q^2 = p^2$ then

$$5q^2 = 25r^2$$

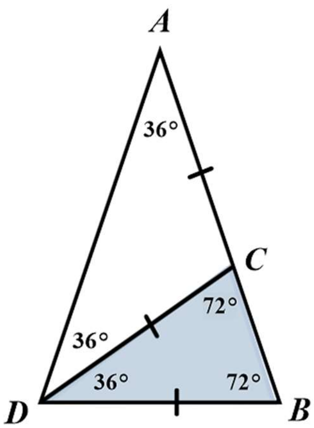
$$q^2 = 5r^2.$$

Then q^2 must be divisible by 5 which means q is divisible by 5.

So, p and q are both divisible by 5 which means they do have a common factor.

By contradiction, $\sqrt{5}$ must be irrational ■

To see precisely how the length of the diagonal of a pentagon relates to the length of its side, it is helpful to consider the following triangle similarity theorem which appears in Book VI of Euclid's *Elements* [11, p79]. ΔADB has been extracted from a regular pentagon where the side of the pentagon is length *BD,* and the diagonal is length *AB*.

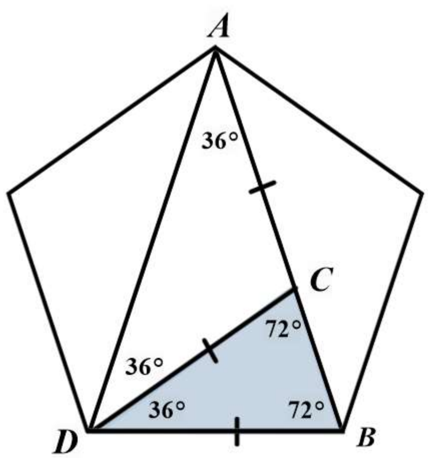


Theorem: *If two triangles have congruent angles, then the triangles are similar and corresponding sides are proportional.*

Proof:

Since ΔADB and ΔDBC have congruent angles, $\Delta ADB \sim \Delta DBC$. Then it follows that $\frac{DB}{BC} = \frac{AB}{DB}$. Since ΔADC and ΔDBC are both isosceles triangles, $DB = DC = AC$. Therefore, $\frac{AC}{BC} = \frac{AB}{AC}$ ■

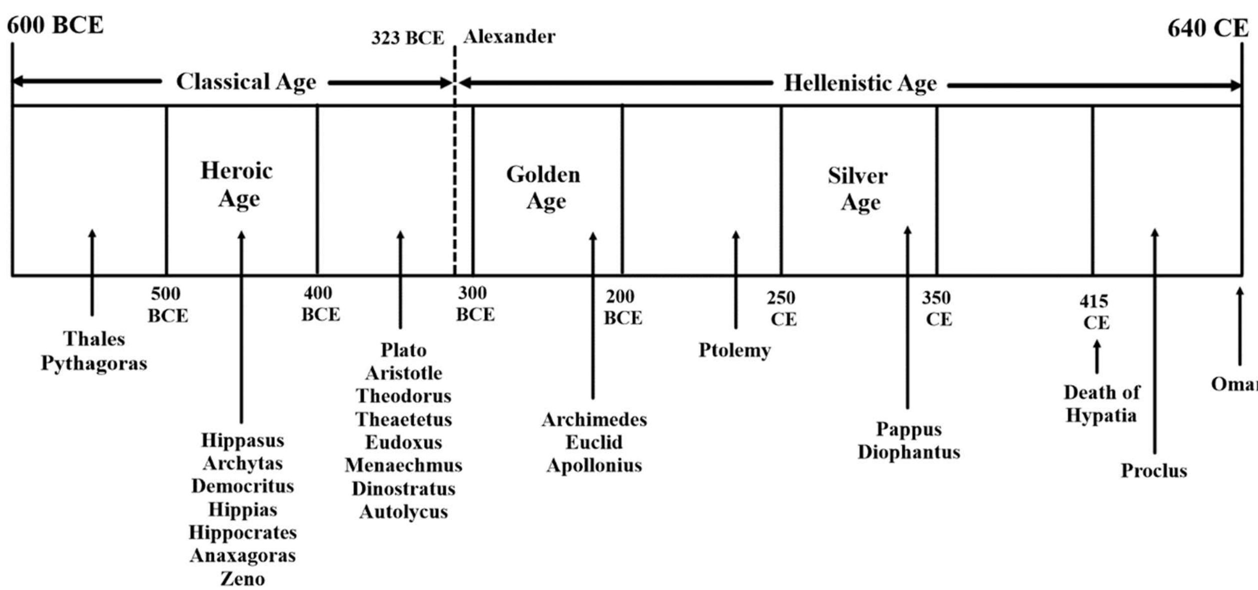
Given that $\frac{AC}{BC} = \frac{AB}{AC}$, the diagonal *AB* shown below is analogous to the length $x + 1$ in Figure 1.1. By the definition of the Golden Ratio, *AB* is divided into mean and extreme ratio. Therefore, the whole length, divided by the length of the longer section, *AC,* is equal to the Golden Ratio and then, of course, the longer section, *AC,* divided by the shorter section, *BC,* is also equal to the Golden Ratio. So, in any regular pentagon, the ratio of the length of the diagonal to the length of the side will always be equal to φ. With respect to the triangles above, since the ratios of the leg lengths to the base lengths are equal to φ, the triangles are called "Golden Triangles!"



Once Hippasus disclosed the existence of the irrational numbers to the Pythagoreans, legend has it that he met a tragic death and was possibly drowned at sea over his controversial assertion. Regardless of how historically accurate this particular story is and whether or not Hippasus came to his conclusion working with $\sqrt{2}$ or the pentagon–pentagram, the association of the Golden Ratio with the pentagram, as well as with the Platonic solids, which will be discussed later, were known to be of particular interest and curiosity to the Pythagoreans for mathematical as well as spiritual reasons. They considered the pentagram to have mystical qualities and the ubiquitous Golden Ratio, the ultimate theological and philosophical symbol, to be especially significant to human understanding of the realities of the universe.

The rudimentary timeline below makes it possible to get an idea of when various ancient Greek mathematicians and philosophers likely lived and who could have been alive during the same time period. Some historians say that Thales and Pythagoras differed in age by approximately 50 years and could not have worked together. Others believe that Thales was actually Pythagoras' teacher and advisor.

Timeline of Greek Mathematicians & Philosophers [10, p51]



III. ALGEBRAIC PROPERTIES OF THE GOLDEN RATIO

The early Greeks were geometers, and though the Golden Ratio has considerable significance to geometry, part of what makes it so extraordinary is that it also has very unique and surprising algebraic properties. In Figure 1.2, the solutions to the quadratic equation obtained from the definition of the Golden Ratio were calculated to be:

$$x_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad x_2 = \frac{1-\sqrt{5}}{2}$$

The first unusual algebraic property can be observed when squaring an approximation of the irrational number, φ.

$$(1.618033989)^2 \approx 2.618033989$$

Notice the digits after the decimal point of φ^2 are the same as the digits after the decimal point of φ. The only real number whose square is equal to its base plus one is φ!

A second unusual algebraic property can be illustrated by dividing one by φ.

$$1 \div 1.618033989 \approx 0.618033989$$

Interestingly, φ is the only real number whose reciprocal is one less than the original number. In other words, φ has the following algebraic properties:

1.) $\varphi^2 = \varphi + 1$

2.) $\dfrac{1}{\varphi} = \varphi - 1$

Letting $\varphi = x$, $x^2 = x + 1$ and $\frac{1}{x} = x - 1$ can both be re-written as $x^2 - x - 1 = 0$, the original quadratic equation from which the solutions were calculated.

Additional Algebraic Properties: If the Greek letter φ designates the positive root and φ' designates the negative root, φ and φ' have the following properties [10, p75]:

3.) $\varphi\,\varphi' = -1$

4.) $\varphi + \varphi' = 1$

In this section, several unique manifestations of φ in algebra will be explored showing that the Golden Ratio turns up in many surprising places. This exploration begins with an exercise.

What is the value of the expression $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$? Evaluating a square root expression that continues forever may initially seem burdensome, perplexing, or even impossible. Using a simple but creative trick demonstrates that there is actually an efficient and beautiful method for calculating its value. First, let $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$

Then square both sides to get $x^2 = 1 + \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$

Since $x = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$, x can be substituted for $\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$ to get $x^2 = 1 + x$ which is the same equation that just appeared on the previous page. The solution is φ. So, with repeated iterations, the value of the original expression converges to the Golden Ratio! (To be rigorous, it is necessary to use the machinery of sequences and their convergences but this intuitive argument conveys the main idea.) The value of the following continued fraction can be computed in a similar way.

$$\text{Given } 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}$$

$$\text{Let } x = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}$$

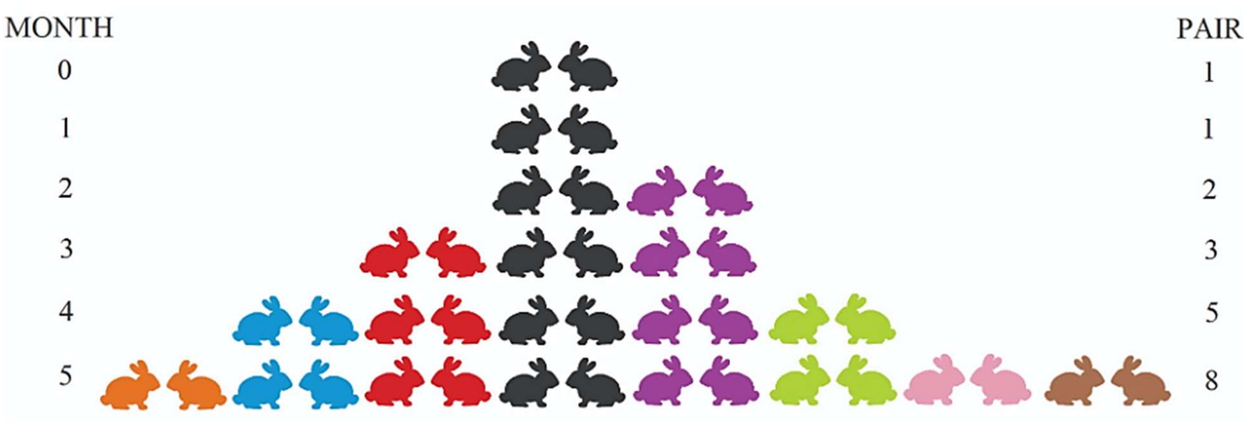
Then substitute x for the second term on the right side of the continued fraction to produce the equation, $x = 1+\frac{1}{x}$. Multiplying both sides by x will result in the equation, $x^2 = 1 + x$, which again has a solution of φ! As the fraction continues, its value converges to the Golden Ratio.

In 1202 CE, Leonardo de Pisa, better known as Leonardo Fibonacci, published his famous *Liber Abaci* (which means "book of calculation"). Fibonacci was the son of a member of the Bonacci family, Guglielmo, who traveled throughout the Mediterranean as a representative of the Italian government collecting taxes for the Republic of Pisa. Fibonacci learned a great deal about arithmetic and algebra traveling with his father in Egypt, Syria, Greece and present-day Algeria. Whereas Fibonacci acknowledges the connection between arithmetic and geometry in *Liber Abaci*, he gives the majority of his attention to arithmetic. Much of the book deals with the Hindu-Arabic numerals which Fibonacci asserts are far superior to the Roman numerals, used at the time in Europe, because they operate within a place-value system. Pisa was a busy commercial port during the 12th century and Fibonacci realized the clumsiness of using Roman numerals to conduct business and maintain trade records [11, p5]. *Liber Abaci* brought Fibonacci substantial fame and recognition, but much of the remainder of the book is considered to be tedious because it deals in great detail with cumbersome fractions used for commercial transactions. A very important exception is Fibonacci's well-known rabbit problem, which gives rise to the amazing Fibonacci sequence. The rabbit problem goes something like this:

> A farmer places one pair of baby rabbits in a fenced area. How many pairs of rabbits will be produced in one year if its assumed that every month each pair of mature rabbits produces a new pair of rabbits that is productive from its second month on? [14, p230].

After one month, the pair of rabbits is of mating age. One month after that, a second pair of rabbits is born. After the third month, the original pair gives birth to another pair. And then the fourth month, the original pair will produce a new pair and the second pair will also produce a new pair. This continues to generate the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233… which is the Fibonacci Sequence where each term (after the first two) is the sum of the two terms that came immediately before it. A visual representation of the pattern appears in Figure 3.1 on the next page. To express the property that each term is equal to the sum of the two preceding terms, the notation used to define the *n*th Fibonacci number is $F_n = F_{n-1} + F_{n-2.}$ When a term in a sequence is defined by a preceding term or terms, the sequence is "recursive." The Fibonacci sequence is thought to be the first recursive sequence known to European mathematicians.

Fibonacci's Famous Rabbit Problem



[**Figure 3.1.** Fibonacci's Rabbits. Retrieved from https://jcdr.net/articles/PDF/13317/42772_PD_(V-2-PK_PrG_OM_SL)_GC(PrG_KM_OM)_PN(SL).pdf in June, 2020]

As will be discussed later, the Fibonacci sequence appears in many other phenomena besides rabbit breeding. This is why Fibonacci is so well known and why the Fibonacci sequence is studied to such a great extent in mathematics, science, finance, and other disciplines. To understand how the Fibonacci sequence is related to the Golden Ratio, consider the values obtained when quotients of successive terms are calculated. Dividing F_n by F_{n-1} for the first fifteen pairs of consecutive terms of the Fibonacci sequence, reveals an intriguing pattern:

$$
\begin{aligned}
1/1 &= 1.000000 \\
2/1 &= 2.000000 \\
3/2 &= 1.500000 \\
5/3 &= 1.666666 \\
8/5 &= 1.600000 \\
13/8 &= 1.625000 \\
21/13 &= 1.615385 \\
34/21 &= 1.619048 \\
55/34 &= 1.617647 \\
89/55 &= 1.618182 \\
144/89 &= 1.617978 \\
233/144 &= 1.618056 \\
377/233 &= 1.618026 \\
610/377 &= 1.618037 \\
987/610 &= 1.618033
\end{aligned}
$$

As additional quotients are calculated, it can be observed that the values are alternating, slightly below and then slightly above, and gradually getting very close to 1.618033989…, the decimal approximation of the Golden Ratio! As *n* gets larger, F_n/F_{n-1} converges to φ. This type of sequence is said to be "oscillating" and "the limit of the sequence of the ratios of adjacent terms of the Fibonacci sequence is φ" [10, p76] which is expressed:

$$\lim_{n \to \infty} F_n / F_{n-1} = \varphi$$

What is truly astounding is that for *any* sequence where $F_n = F_{n-1} + F_{n-2}$, the ratios of pairs of successive terms, i.e., F_n/F_{n-1} for $n = 1, 2, 3, 4\ldots$, converge to φ. For instance, randomly choosing two and four for the first two terms, consider the resulting sequence and the associated ratios of consecutive terms:

2, 4, 6, 10, 16, 26, 42, 68, 110, 178, 288, 466, 754, 1,220, 1,974…

4/2 = 2.000000
6/4 = 1.500000
10/6 = 1.666667
16/10 = 1.600000
26/16 = 1.625000
42/26 = 1.615385
68/42 = 1.619048
110/68 = 1.617647
178/110 = 1.618182
288/178 = 1.617978
466/288 = 1.618056
754/466 = 1.618026
1,220/754 = 1.618037
1,974/1,220 = 1.618033

Just like with the Fibonacci sequence, the quotients of consecutive terms in this random sequence oscillate between values that are slightly more than phi and then slightly less. It can easily be observed that it does not take long for the values of the successive ratios to converge to φ! While this coincidence is noteworthy, most of the amazing properties of the Fibonacci sequence are unique to the sequence 1, 1, 2, 3, 5, 8...

The connection between the Fibonacci numbers and the Golden Ratio is made explicit by way of Binet's formula below which can be used to calculate the *n*th Fibonacci number:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

A simpler version of this formula can be written:

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

For example, to calculate the 15th Fibonacci number, let $n = 15$.

$$F_{15} = \frac{\varphi^{15} - (1-\varphi)^{15}}{\sqrt{5}} = 610$$

To prove Binet's formula, the quadratic equation $x^2 - x - 1 = 0$ from Figure 1.2 can be used to write expressions for all x^n where n is an integer ≥ 1.

Proof:

Since $x^2 = x + 1$,

$$\begin{aligned}
x &= x \\
x^2 &= x + 1 \\
x^3 &= x^2 \cdot x \\
&= x(x+1) \\
&= x^2 + x \\
&= (x+1) + x \\
&= 2x + 1 \\
x^4 &= x^3 \cdot x \\
&= (2x+1) \cdot x \\
&= 2x^2 + x \\
&= 2(x+1) + x \\
&= 3x + 2 \\
x^5 &= 5x + 3 \\
&\cdots \\
x^6 &= 8x + 5.
\end{aligned}$$

A pattern emerges showing that $x^n = F_n x + F_{n-1}$, i.e., the coefficient of x will be the nth Fibonacci number and the constant will be the preceding Fibonacci number. The roots of the original quadratic are $\varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi' = \frac{1-\sqrt{5}}{2}$. Since both are solutions to the quadratic, they each satisfy the equation, $x^n = F_n x + F_{n-1}$. Then, $\varphi^n = F_n \varphi + F_{n-1}$ and $(\varphi')^n = F_n \varphi' + F_{n-1}$.

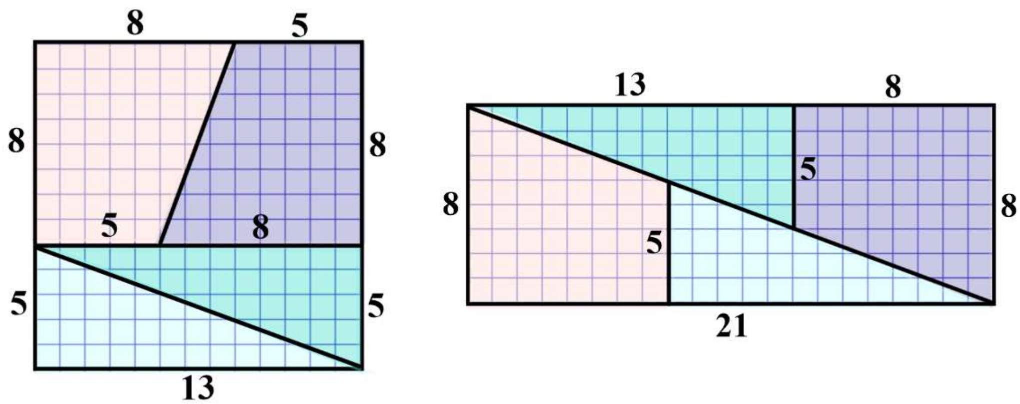
So, $\varphi^n - (\varphi')^n = F_n(\varphi - \varphi') + F_{n-1} - F_{n-1}$. Substituting the values of φ and φ' back into the equation,

then $\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n = F_n\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$.

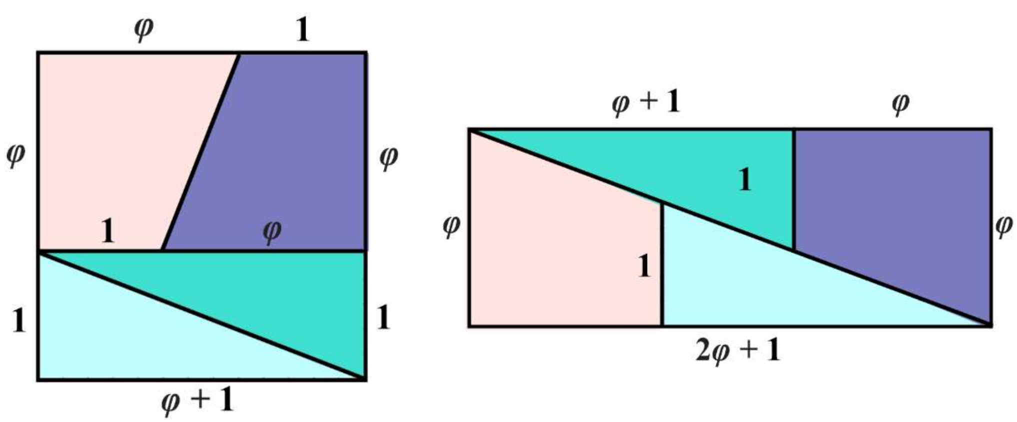
$$\text{Therefore, } F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \quad \blacksquare$$

In H.E. Huntley's book, *The Divine Proportion: A Study in Mathematical Beauty,* Huntley asserts that one of the aspects of aesthetic appeal is the feeling of "surprise at the unexpected encounter" [7, p36]. The presence of the Fibonacci sequence is not immediately perceptible in Pascal's Triangle, pictured below, but without too much work, it makes a surprising appearance. Consider not the obvious diagonals of the triangle, but rather the *shallow diagonals* which may be more easily observed in the triangle on the right. Notice that the sums (in red) of the successive shallow diagonals comprise the Fibonacci sequence!

An apparent paradox related to the Fibonacci sequence and the Golden Ratio turns up in the following problem: "Construct a square whose side has a length equal to the sum of two consecutive Fibonacci numbers…Dissect the square into the four sections indicated and fit them together to form a rectangle" [7, p48]. Choosing five and eight, it can be seen that the dimensions of the two trapezoids and two triangles are the same in each figure but when calculating the areas using the side lengths of each shape, the area of the square is $13 \cdot 13 = 169$ units^2 but the area of the rectangle is $21 \cdot 8 = 168$ units^2. No matter what two Fibonacci numbers are chosen, the areas will always differ by one unit (sometimes the square will be larger and sometimes the rectangle).



If the same problem is reconsidered with a different additive "series" though (really a finite sum), the areas will be exactly equal and the paradox disappears. Given what is known as the "golden series," *viz*., $1, \varphi, 1 + \varphi, 1 + 2\varphi, 2 + 3\varphi, 3 + 5\varphi\ldots,$ the exercise is repeated below.



The area of the square is $1 + 2\varphi + \varphi^2 = 1 + 2\varphi + \varphi + 1 = 3\varphi + 2$ and the area of the rectangle is $2\varphi^2 + \varphi = 2(\varphi + 1) + \varphi = 2\varphi + 2 + \varphi = 3\varphi + 2$. Each area is precisely $3\varphi + 2$. It turns out the golden series is the only additive series where the areas match exactly!

The series 1, φ, $1 + \varphi$, $1 + 2\varphi$, $2 + 3\varphi$, $3 + 5\varphi$... has another fascinating property when comparing its terms to corresponding terms of the sequence 1, φ, φ^2, φ^3, φ^4, φ^5…,

$$
\begin{aligned}
1 + \varphi &= \varphi^2 \\
1 + 2\varphi &= (1 + \varphi) + \varphi \\
&= \varphi^2 + \varphi \\
&= \varphi(\varphi + 1) \\
&= \varphi(\varphi^2) \\
&= \varphi^3 \\
2 + 3\varphi &= 1 + 2\varphi + 1 + \varphi \\
&= 1 + 2\varphi + \varphi^2 \\
&= (1 + \varphi)\,(1 + \varphi) \\
&= \varphi^4 \\
3 + 5\varphi &= 2\varphi + 3\varphi + 3 \\
&= 2\varphi + 3(\varphi + 1) \\
&= 2\varphi + 3\varphi^2 \\
&= \varphi(2 + 3\varphi) \\
&= \varphi^5 \\
&\dots
\end{aligned}
$$

Since each set of corresponding terms is equivalent, the series 1, φ, φ^2, φ^3, φ^4, φ^5… is equal to the golden series!

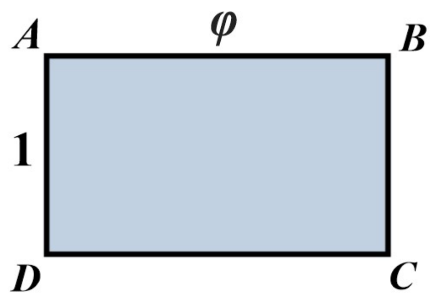
Recalling that φ' is also a solution to the original quadratic equation, $x^2 - x - 1 = 0$, an analogous relationship can be observed between the following two series:

$$1, \varphi', 1 + \varphi', 1 + 2\varphi', 2 + 3\varphi', 3 + 5\varphi' \dots \text{ and}$$
$$1, \varphi', \varphi'^2, \varphi'^3, \varphi'^4, \varphi'^5 \dots$$

Because φ' is negative, these two series have positive first terms and then alternate between negative and positive terms.

IV. THE GOLDEN RECTANGLE

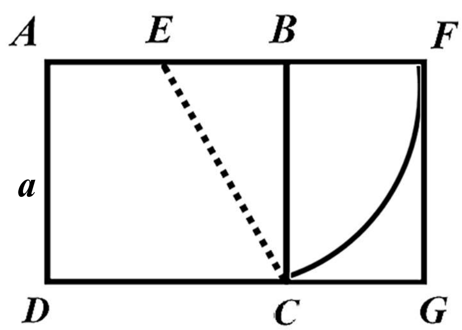
The most straightforward and frequent experiences that human beings have with the Golden Ratio involve golden rectangles. In fact, it is likely that most people on Earth, whether they are aware or not, engage with golden rectangles numerous times each day. A golden rectangle, in the very simplest sense, is a rectangle where the ratio of the length, i.e., the longer side, to the width is equal to the Golden Ratio, the most obvious example, shown below, being a rectangle with a length = φ and width = 1. Therefore, the ratio of the length to the width is $AB : AD = \varphi / 1 = \varphi$.



Two rather ordinary examples of the appearance of golden rectangles in everyday life are credit cards and light switch covers. Rounded to the nearest tenth, the ratio of the length of a credit card to its width is 4.5 inches/2.75 inches ≈ 1.6 and the ratio of the length of a standard light switch cover to its width is 3.375 inches/2.125 inches ≈ 1.6, both quite close to the value of φ which, as shown previously, is also approximately 1.6. Although this might seem like a strange coincidence, many people believe that golden rectangles are so ubiquitous in human life because their proportions just look "right." Since at least the time of the early Greeks, rectangles with these proportions have been thought to be "the most pleasing and beautiful rectangles" and have been used by humans to design everything from everyday objects that we barely think about to cultural monuments that have survived for thousands of years. While golden rectangles show up in many familiar and even mundane places, they also show up in countless extremely interesting and often surprising places. Though the rectangle above may appear unremarkable, the features of its composition and proportions alone have been a mathematician's treasure trove for millennia.

A golden rectangle can easily be constructed with straightedge and compass as shown in Figure 4.1 below. Beginning with square *ABCD*, side *AB* is bisected by point *E*. Extend side *AB* and side *DC*. Then with *E* as center, open a compass the length of *CE* and draw an arc from *C* intersecting the extension of *AB* to produce point *F*. Draw *FG* perpendicular to *DC* intersecting the extension of *DC* at point *G*. The golden rectangle is the quadrilateral *AFGD*.

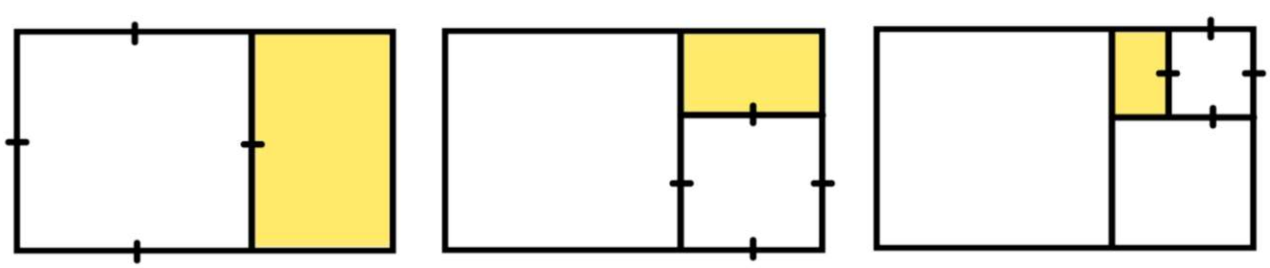
Figure 4.1



To prove that the proportions are correct, let $AD = a$. Then by the Pythagorean Theorem,

$EC = EF = \frac{\sqrt{5}}{2}a$. $AF/FG = (AE + EF)/FG = \frac{1}{2}a + \frac{\sqrt{5}}{2}a / a = \frac{1+\sqrt{5}}{2}$ ■

Since this ratio is equal to φ, *AFGD* is definitely a golden rectangle but a really incredible fact is that the smaller rectangle *CBFG* is also a golden rectangle! It turns out that no matter how many times a square is lopped off of a golden rectangle, the resulting quadrilateral will *always* be another golden rectangle. There will be more about this property in the following sections.



V. ARCHITECTURE & DESIGN

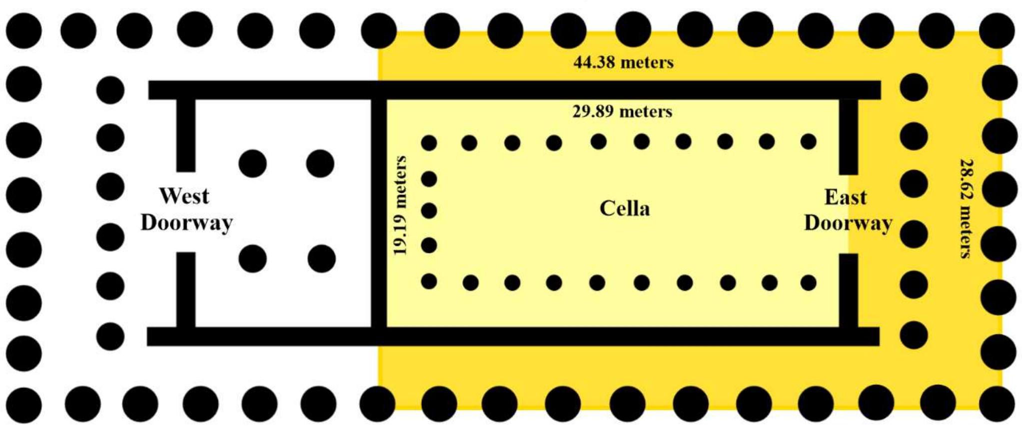
The perception of beauty is not easily apprehended or explained but since at least the time of the ancient Greeks, there has been a widely held belief that beauty relates to a particular type of symmetry and specific proportions between the part and the whole. According to Huntley, this perception is a psychological experience, part of which is primordial, and part of which is learned. He claims that “aesthetic appreciation is consummated in two stages, the first through intuition, the second through education” [4, p68]. Analyzing the intuitive aspect is challenging but Huntley has a special interest in, and particular patience for, exploring this topic. He suggests that inherent experiences of beauty could be related to certain parameters of perception, an aspect of familiarity, or maybe even the collective unconscious.

In architecture and design, the Golden Ratio and golden rectangles make quite frequent appearances. It is possible that this is not always deliberate, but the particular enthusiasm of the ancient Greeks for the Golden Ratio leads many historians to believe that φ was intentionally incorporated into the design of the Parthenon of Acropolis, pictured below. The Parthenon was built in the city-state of Athens during the Classical Age as a temple to celebrate the defeat of Persian invaders, and as a cultural symbol. Despite heavy damage due to fires, wars and earthquakes over the centuries, the Parthenon still stands today and is visited by millions of people each year. It is considered to be one of the world’s most important historic monuments.

[**Figure 5.1.** Photograph of *Parthenon of Acropolis* by Colin Dixon. Downloaded from https://explorethistown.com/what-is-the-parthenon-restoration-all-about/, June, 2020]

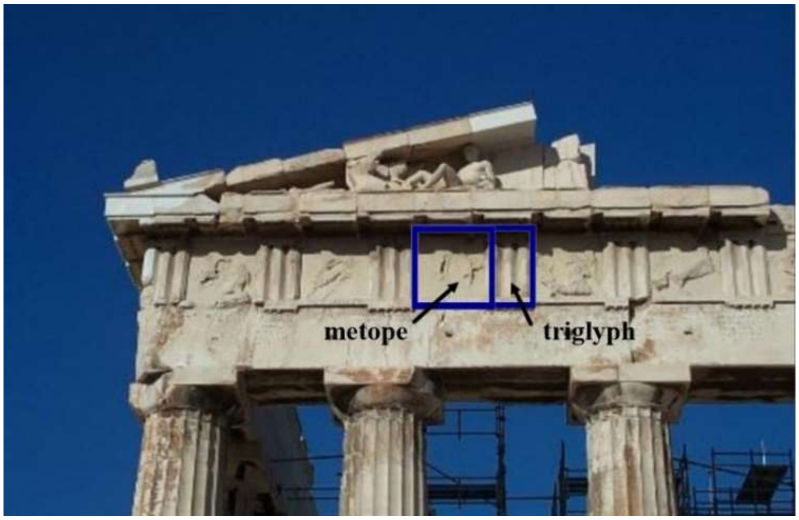
Construction of the Parthenon, ordered by the Greek statesman Pericles, began in 447 BCE and continued until 438 BCE. The Greek sculptor, Phidias, for whom φ is named, was among the architects and artists who designed the ancient temple. Depending on how measurements are taken, admittedly many of the ratios investigated appear to be only close to the Golden Ratio. From this error, sophisticated arguments have been presented, disputing the intentional use of φ in the design. While mathematical scrutiny may lead to a measure of doubt, a visual appraisal of the floor plan, reveals that the proportions of both the friezes and overall structure, are at minimum quite close to golden proportions. The arithmetic exposes errors of up to two or three percent but this discrepancy may not be detectable by the human eye. Golden rectangles, or at least approximations of, can be observed in several locations of the Parthenon's floor plan. Two obvious examples are shown in the diagram below. The ratio of the length to the width of the larger gold rectangle is 44.38 meters/28.62 meters ≈ 1.6 and the ratio of the length to the width of the interior chamber or "Cella," represented by the smaller light gold rectangle, is 29.89 meters/19.19 meters ≈ 1.6.

Parthenon Floorplan



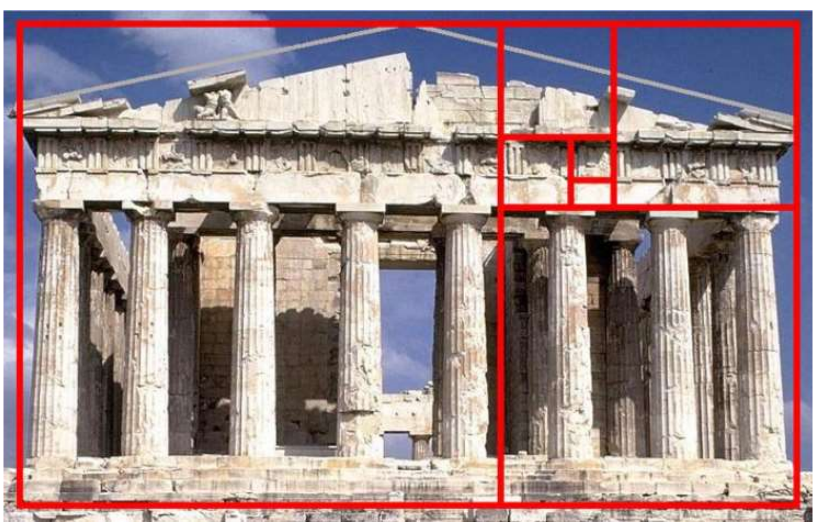
[Measurements from http://athang1504.blogspot.com/2011/01/parthenon.html, 2020]

A more compelling example of a golden rectangle might be observed in the entablature frieze on the exterior of the Parthenon. The metopes are plaques carved from marble that compose the ornamentation of the Parthenon above the Doric columns. Each side of the Parthenon has a different theme but all of the metopes depict war images. The master artist responsible for the creation of the metopes is thought to have been Phidias. The following image is of the eastern face of the Parthenon. Metopes on this side symbolize *Gigantomachy* which has to do with the battle between Olympian gods and the aggressive Giants. Each metope has a section beside it called a triglyph that looks like three miniature columns. The ratio of the length of one metope, designated in blue below, to the width is equal to approximately 1.618 and just like the golden rectangles in the previous section, once a line is draw separating the square metope and the rectangular triglyph, the resulting rectangle is also a golden rectangle.



[**Figure 5.2**. Photograph of *East Front of Parthenon* by David Gill. Downloaded from http://davidgill.co.uk/attica/parthenon/parthenoneastfro.html in June, 2020]

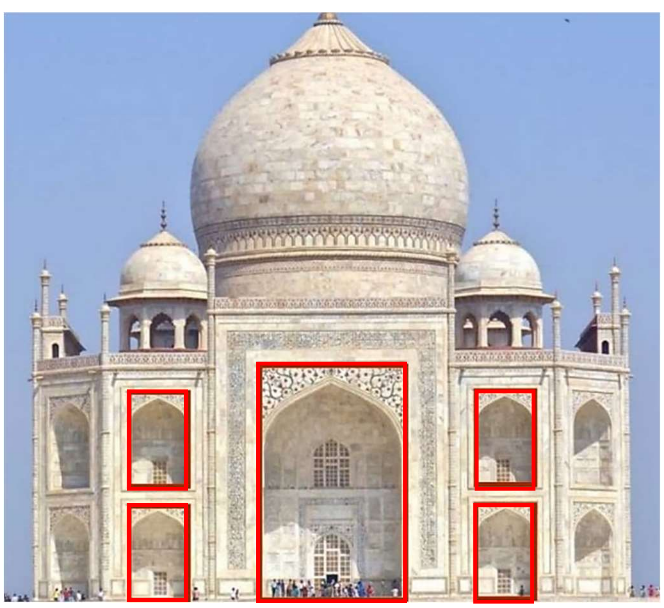
The most obvious indication of the Golden Ratio in the Parthenon's architecture is on the actual face of the structure. As can be observed in the computationally enhanced photograph below, the ratio of the overall length to the height is equal to φ!



[**Figure 5.3**. Photograph of *The Parthenon of Acropolis.* Downloaded from https://www.okeanosgroup.com/blog/ aquariums/1-618 in June, 2020]

One of the most impressive and easily recognizable buildings in the world is the Taj Mahal in Agra, India. The Taj Mahal was built by Emperor Shah Jahan after the death of his favorite wife, Mumtaz Mahal, in 1632. Construction of the main building, built as a mausoleum, was completed in 1643 while work on the ornamentation of continued for about five additional years. The Taj Mahal is constructed from ivory-white marble with beautiful domes, intricate design work, calligraphy, stone inlays, bronze detailing, carvings, splendid gardens, and many other opulent design elements consistent with Persian and Mughal architecture. Like the Parthenon, the Taj Mahal is visited by millions of people from around the world each year; in 2007, was named one of the Seven New Wonders of the World. In addition to the lavish materials and exquisite details, the design features thought to make the Taj Mahal so visually attractive are: that each element of its design can stand alone while integrating flawlessly with the whole, that self-replicating geometry contributes to a sense of visual coherence, and that its symmetry of elements has a universal aesthetic appeal.

The Golden Ratio makes countless appearances in the design of the Taj Mahal and is an important element of the self-replicating geometry mentioned above. Golden rectangles compose the face of building which can be seen in the photograph below. The main entrance and windows are all geometrically similar rectangles with the proportion of length to width approximately equal to φ. The use of golden rectangles in the design of the building is well known and is widely accepted to be one of the major reasons that the Taj Mahal is so aesthetically pleasing.



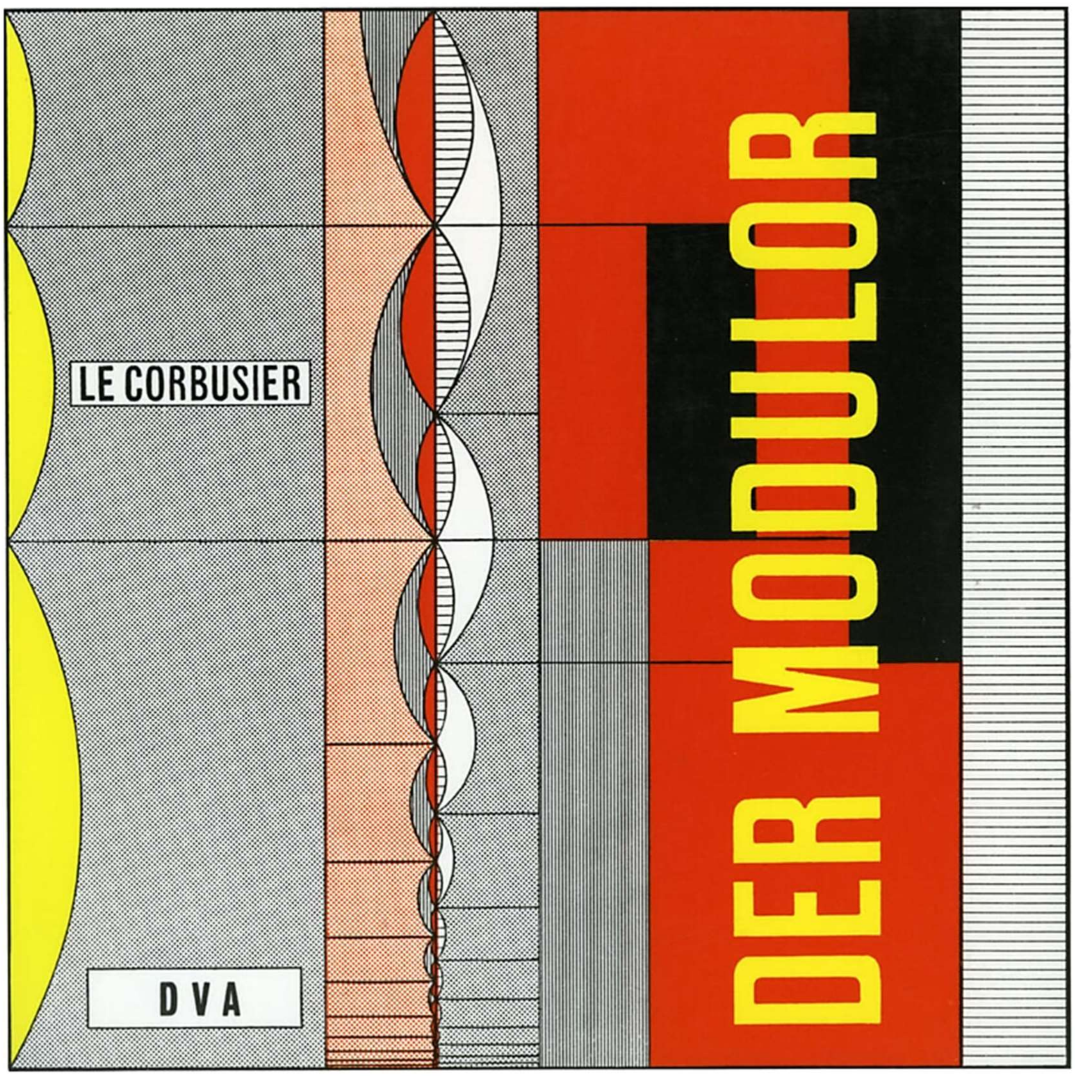
[**Figure 5.3**. Photograph of *The Taj Mahal* by Andrés Lorenzo. Downloaded from andréslorenzo-taj- mahal.com in June, 2020]

More modern examples of the Golden Ratio in architecture can be observed in the work of architects, Frank Lloyd Wright and Le Corbusier. Frank Lloyd Wright was an American architect, designer and teacher born in 1867, and Le Corbusier was a Swiss-French architect, designer and writer born in 1887. Both men had long careers. Frank Lloyd Wright designed structures that include elements of nature and are intended to be in harmony with the natural world. Le Corbusier worked in urban planning and was dedicated to improving living conditions in crowded cities. He had a particular interest in proportion as a theory of design.

Several examples of the Golden Ratio appear in the architecture of Frank Lloyd Wright. A study of Wright’s Roloson Row Houses in Chicago reveals that both the façade and the floor plan of a single row house are in the exact proportions of a *golden spiral* which because it has a growth factor of φ, is directly related to the Golden Ratio. A better-known example of a golden spiral in a Lloyd Wright building can be seen in the design of the Solomon R. Guggenheim Museum in Manhattan which is pictured below. Exactly how the golden spiral, which also appears in the natural world, is related to the Golden Ratio will be discussed in Section VIII.

[**Figure 5.4**. Photograph of *The Solomon R. Guggenheim Museum* by Andrew Pielage. Downloaded from https://ny.curbed.com/2019/10/18/20920836/nyc-museums-guggenheim-frank-lloyd-wright-architecture in June, 2020]

Though it is uncertain whether or not Frank Lloyd Wright intentionally, or only coincidentally, incorporated the Golden Ratio into his designs, Le Corbusier certainly did use it intentionally. He developed a universal measuring system which he called the "Modulor." This system is explicitly based on the Golden Ratio and the proportions of the human body. These proportions were standardized and used throughout his construction projects. In the image below, the ratio of the second vertical section to the first is φ and the ratio of the third vertical section to the second is also φ. There are many other manifestations of the Golden Ratio in this image as well.



[**Figure 5.5**. Photograph of *The Modulor* by Le Corbusier. Downloaded from https://www.library.ethz.ch/en/ms/Virtual-exhibitions/Fibonacci.-Un-ponte-sul-Mediterraneo/ Reception-of-Fibonacci-numbers-and-the-golden-ratio/Le-Corbusier-the- Modulor# in June, 2020]

It should be noted that the principle of the explicit use of the Golden Ratio and human proportion in architecture was not new but rather was explained as early as the first century CE by Roman architect and engineer, Marcus Vitruvius Pollio, in his multi-volume work, *De Architectura* [18, p508]. Le Corbusier's conceptual system, however, has an undeniable streamlined appeal, and widespread accessibility which has made him one of the most influential modern theorists on the topic.

An example of Le Corbusier's design can be seen in the photograph of his Unite d'Habitation housing complex in Marseilles, France built in 1952 and pictured below. On the façade of the building, two golden rectangles, one colored orange and the other yellow, can be easily observed side by side. This design element appears not only on the façade but throughout the design of the housing complex. Le Corbusier believed that repeated use of the golden rectangle as a design element creates harmony and unifies a structure. Because the Golden Ratio is based on the proportions of the human body, the proportions of the spaces he designed are appropriately proportioned to contain human bodies.

[**Figure 5.6**. Photograph of *Unite d'Habitation.* Downloaded from https://archi-monarch.com/theory-of-proportion/ in June, 2020]

Le Corbusier designed dozens of buildings around the world that incorporated his Modular system. One more specific example is his United Nations Secretariat Building in New York. The face of the building, which is even featured in a Donald Duck movie about the Golden Ratio, is said to be composed of three golden rectangles stacked on top of each other. When the dimensions of the building are used to confirm that these three rectangles are in fact "golden," there is an error of about two percent. Critics exploit this error to "debunk the myth" that the Golden Ratio was used in the design, however, it should be noted that there are other parameters to consider when designing a large building. That Le Corbusier developed an entire system of measurement around the Golden Ratio leaves little doubt that he used it intentionally in his designs.

These are only a very few examples of countless uses of the Golden Ratio in architecture. There are many other fascinating designs that incorporate the Golden Ratio, golden rectangles, and golden spirals that will be left to the reader to explore. One final image shown below, included for its exceptional aesthetic appeal, is the *Bramante Staircase* at The Vatican Museum. This elegant golden spiral staircase was designed by Italian architect and engineer, Giuseppe Momo in 1932. Since one side of the staircase goes up while the other goes down, the design appears to be inspired by the double helix structure of DNA, however, the structure of DNA was not discovered until 1953!

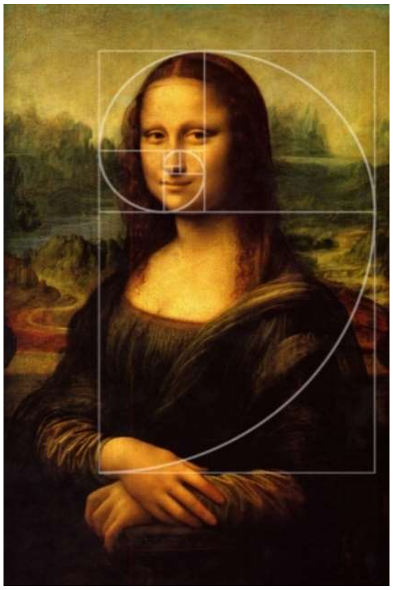
[**Figure 5.7**. Photograph of *A Staircase at the Vatican*. Downloaded from https://archiMonarch.com/theory-of-Proportion/ in June, 2020]

VI. ART

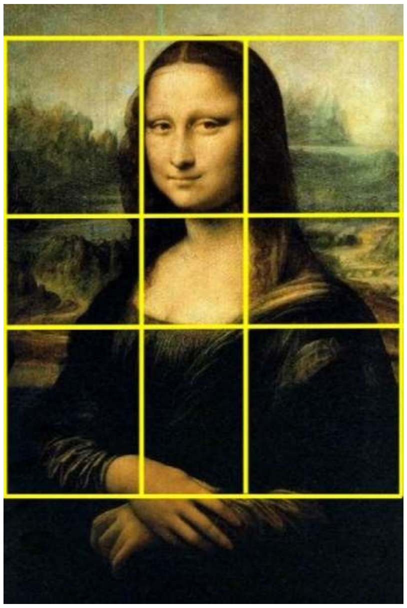
In art, like architecture, the Golden Ratio makes many appearances; these appearances occur throughout various art movements but are most prevalent during the Renaissance. The intentionality of a particular artist's use of the Golden Ratio is a topic of serious critical debate and a subject of extensive research. In the article, "On the Application of the Golden Ratio in the Visual Arts," Roger Herz-Fischler states that the only way to determine for certain that the Golden Ratio was used intentionally is with "documentary evidence that [an] artist used the Golden Ratio as a theoretical basis of his work" [6, p31]. He also points out that the Golden Ratio (in the form 1: φ) is very close to the value of the "simple proportion 5/8" or as the exact decimal, 0.625, which of course is a rational number and quite different from the irrational repeating decimal, phi. Although Herz-Fischler's point is well taken, viewing and discussing works of art considered to be, or only possibly, influenced by golden proportions seems essential to any investigation of how the Golden Ratio manifests in human culture.

The figure most prominently associated with the Golden Ratio is Leonardo da Vinci. Da Vinci is considered to be the greatest genius of the Renaissance (1300 – 1600 CE) and, some say, possibly of all time. He was an Italian polymath born in 1452 who exceled in art, science, mathematics, anatomy, architecture, music, and literature. Appearances of the Golden Ratio in da Vinci's work are thought to be numerous. A few well-known examples can be observed in his very famous paintings, the *Mona Lisa, Annunciation,* and *The Last Supper,* as well as in his *Vitruvian Man* drawings. Da Vinci was responsible for creating 60 illustrations in Luca Pacioli's book, *De Divina Proportione*, believed to have been written around 1498 but not published until 1509. As the title implies, the subject of *De Divina Proportione* is mathematical and artistic proportion, specifically the "divine" or golden proportion. While da Vinci's work with Pacioli might not stand up to the level of scrutiny required by Herz-Fischler of "documentary evidence" related to the use of the Golden Ratio in a specific work of art, da Vinci certainly would have been extremely knowledgeable about the divine proportion and likely was an expert with respect to its use in art and design. That the Golden Ratio did provide a theoretical framework for some of da Vinci's compositions is a widely held opinion that seems probable considering da Vinci's relationship to Pacioli's book.

The *Mona Lisa* is the most easily recognized portrait of all time. There are differences in opinion about when da Vinci completed the painting but it was sometime in the early 1500s during the High Renaissance. The *Mona Lisa* is on permanent display at the *Louvre* in Paris and is viewed by over 10 million people each year. The images shown below depict two different ways that the Golden Ratio expresses itself in the portrait. The first shows a golden spiral beginning at the nose, framing the subject's face and continuing to her left wrist. In the second, a golden rectangle frames the face and the grid lines emphasize the symmetry inherent in the composition.

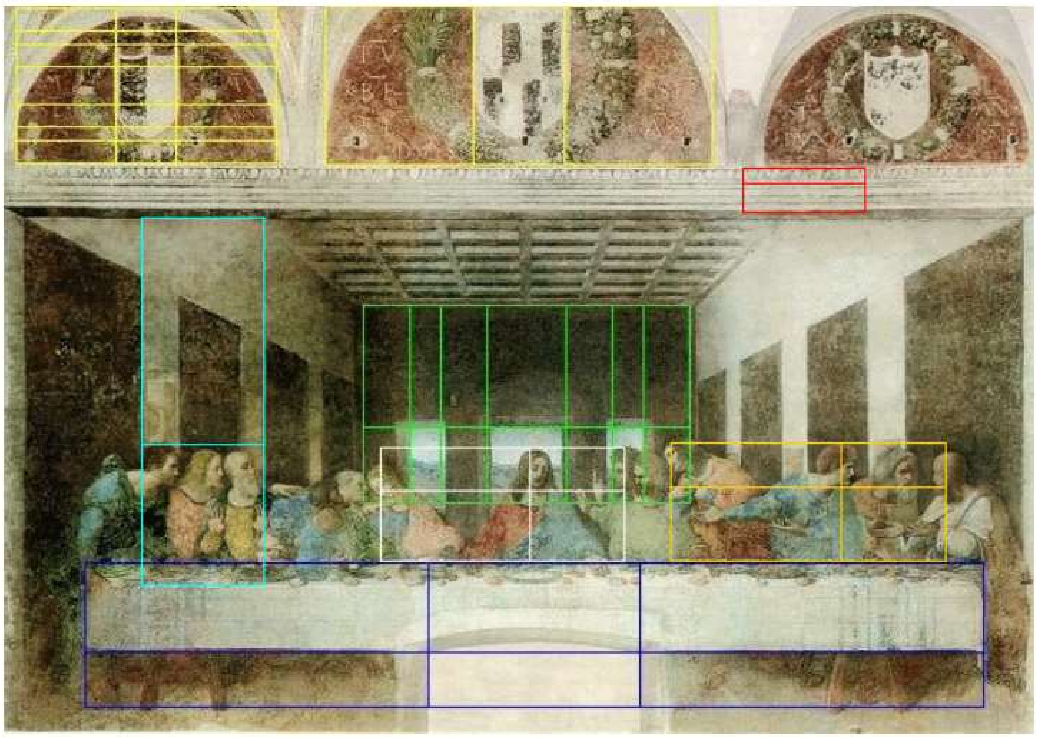


[**Figure 6.1**. Computationally enhanced image of the *Mona Lisa*. Downloaded from https://thefibonacci sequence.weebly.com/mona-lisa.html in July, 2020]



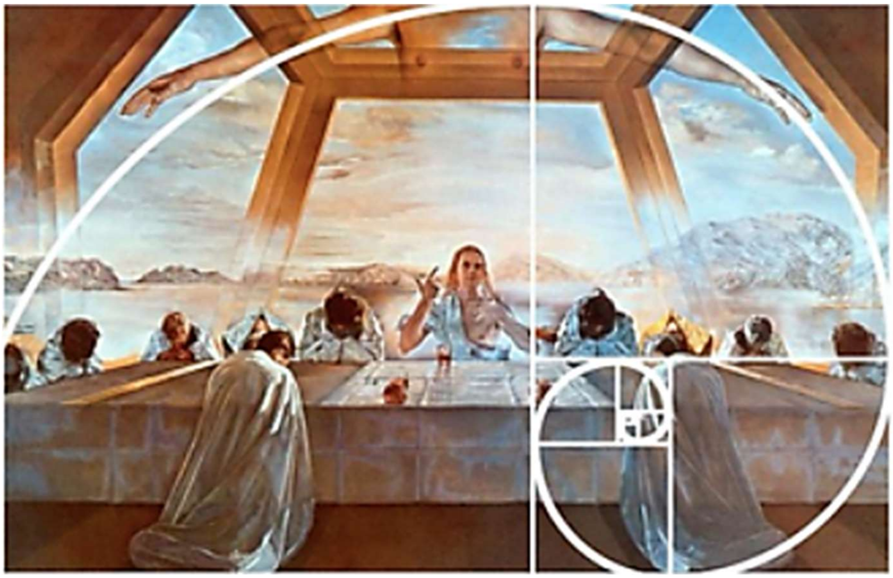
[**Figure 6.2**. Computationally enhanced image of the *Mona Lisa*. Downloaded from https://www.research- //gate.net/figure/Golden-Ratio-example-on-The-Mona-Lisa in July, 2020]

Similarly, an image of da Vinci's *The Last Supper*, completed in the 1490s, can be computationally analyzed using technology to locate the numerous golden rectangles in his design. These rectangles provide structure, harmony and consistency to the composition. This painting is also known for its exceptional use of one-point perspective which emphasizes the importance of Christ, the central figure. Despite ongoing restoration efforts, the *The Last Supper* has deteriorated significantly since it was completed, but it is still one of the best-known paintings of all time and is considered to be a masterpiece of the High Renaissance.



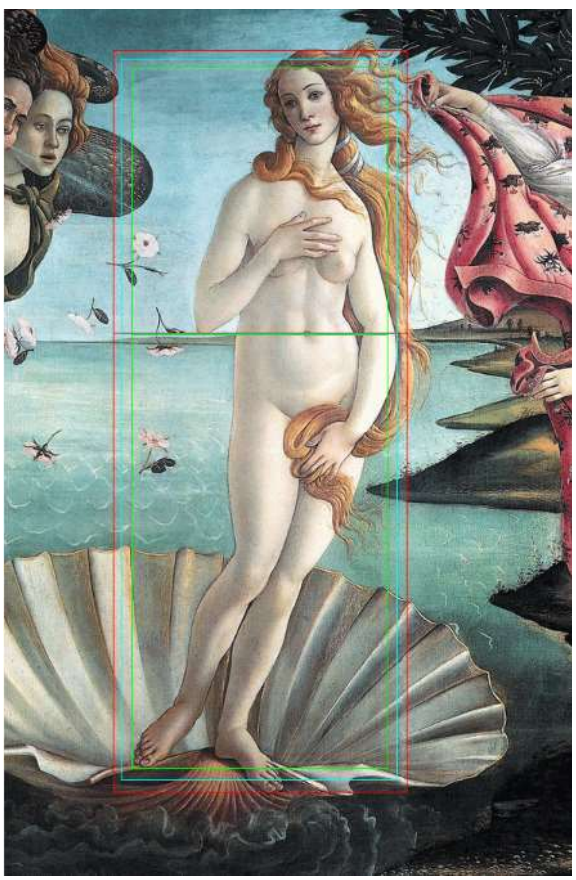
[**Figure 6.3**. Computationally enhanced image of *The Last Supper.* Downloaded from https://www.goldennumber.net/art-composition-design/ in July, 2020]

A more modern painting, completed in 1955, is Salvador Dali's *Sacrament to the Last Supper*, pictured on the next page. Not only does this painting utilize the golden spiral for its overall proportions, the background consists of an oversized dodecahedron, one of the five Platonic solids, whose 12 faces are each created by a regular pentagon. The Platonic solids, which have been known since at least the time of the Pythagoreans, have profound and important connections to the Golden Ratio and will be discussed in Section X. Dali was known to have a keen interest in mathematics and his use of the Golden Ratio is considered to be deliberate.

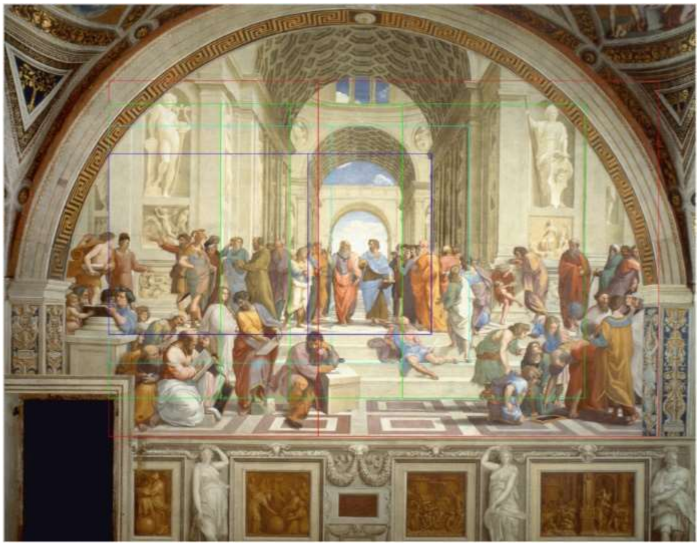


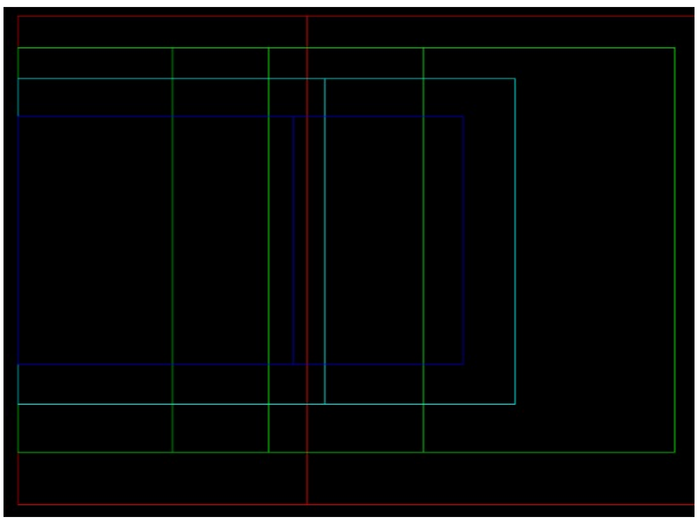
[**Figure 6.4.** Computationally enhanced image of the *Sacrament of the Last Supper*. Downloaded from http://studio.education/en/learning-community/blog/554-the-art-and-science-of-1-61803398875 in July, 2020]

Sandro Botticelli was early Renaissance painter who likely used the Golden Ratio in the composition of his work. His most famous painting, *The Birth of Venus*, was completed in 1485. The image below shows that the ratio of Venus's height to the distance to her navel is equal to φ. The three colored rectangles emphasize that these measurements can be taken at different locations however each iteration pictured produces a golden rectangle. Further evidence that Botticelli intentionally used the Golden Ratio comes from the fact that the canvas itself is a golden rectangle. The ratio of the height to the width is approximately 1.617, within one percent error of φ.



[**Figure 6.5**. Computationally enhanced image of the *The Birth of Venus*. Downloaded from http://www.goldennumber.net/art-composition-design/ in July, 2020]



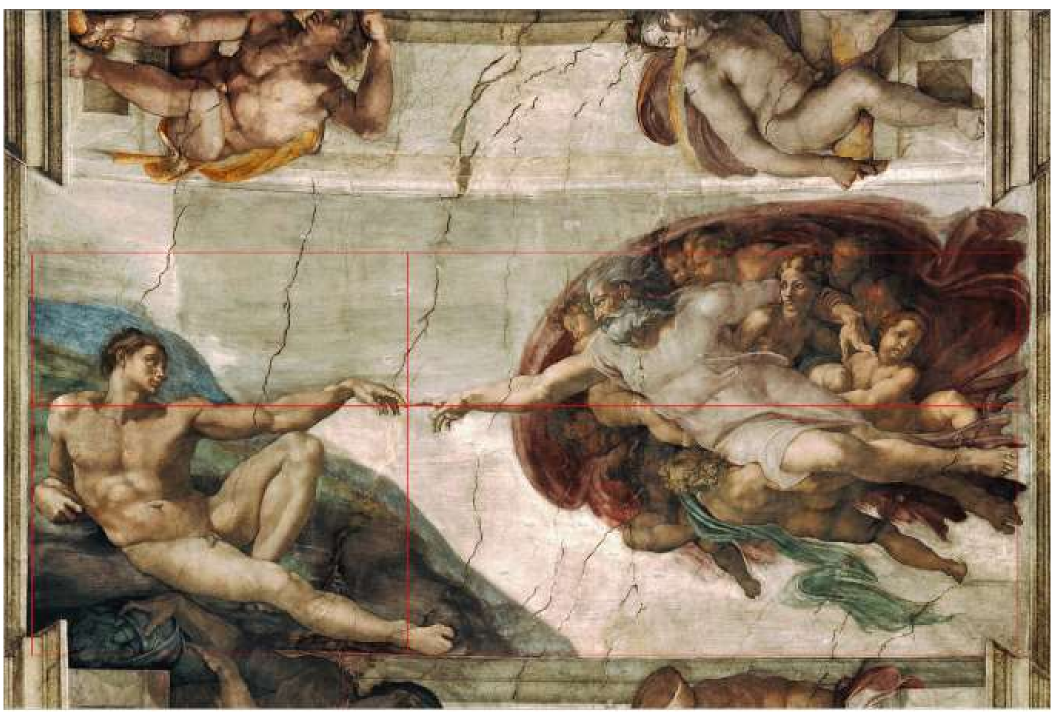


Raphael's painting, *The School of Athens,* is shown to the left. The graphic image below the painting is the result of a computer analysis revealing how the artist embeded golden rectangles into the design using a more subtle technique than with the previous examples. In viewing the painting, it is more challenging to consciously identify how the Golden Ratio is used but since from nueroscience it is known that aesthetic perception occurs in only a small fraction of a second, a positive response to the qualities of order and harmony can occur without necessarily consciously knowing that the golden rectangles are embedded in the composition.

[**Figure 6.6.** Photograph of *The School of Athens* and digital analysis of painting. Retrieved from https://www.goldennumber.net/raphael-golden-ratio-in-renaissance-art/ July, 2020]

While the majority of the paintings already noted are from the Renaissance, golden proportions have been used in works from other time periods as well. In response to the writings of Vitruvius Pollio, the tradition of applying a theory of proportions as a basis for design "flourished in the Italian Renaissance" [4, p211]. This is why the movement provides such a multitude of expository examples, but before discussing a few modern pieces, it is essential to consider the work of just one more very important Renaissance artist.

In addition to da Vinci and Raphael, Michelangelo is considered to be one of the three masters of the High Renaissance (early 1490s – 1527 CE), and the view that he is actually the greatest artist of all time is held by many. Michelangelo was an Italian painter, sculptor, architect and poet born in 1475, a true Renaissance Man, whose genius parallels that of da Vinci. His most famous works are the Genesis frescoes on the ceiling of the Sistine Chapel and his sculpture, *David*. Michelangelo had a vast and deep knowledge of anatomy, which he obtained in part by dissecting human cadavers. *The Creation of Adam*, pictured below, is the best-known fresco of the Sistine Chapel. There is a commonly held belief that Michelangelo embedded hidden religious messages in the Sistine Chapel frescoes. One conjecture is that the *The Creation of Adam* shows that humans do not receive life from God but rather the intellect, which is the divine part. A hypothesis that the drape-like object surrounding the God image, on the right, is actually the "sagittal section of the human brain" [2, p2] supports this theory. The Golden Ratio appears in the relationship between the right edge of the fresco and the tip of God's finger and the left edge of the fresco and the tip of Adam's finger. A research project using "Image Pro Plus" software to analyze these two distances, and others in the frescoes of the Sistine Chapel, reveals that the ratio of God's distance, the longer section, to the ratio of Adam's distance is the divine proportion [2, p3]. The same project also detected over two dozen other appearances of the φ in Michelangelo's Genesis frescoes.

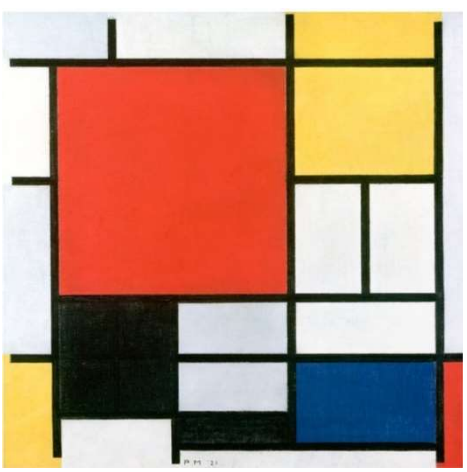


[**Figure 6.7**. Digitally analyzed photograph of *The Creation of Adam.* Downloaded from https://www.Creation-of-Adam-Campos_et_al-2015-Clinical_Anatomy, 2020]

Michelangelo's *David,* shown below, is a marble statue, depicting the figure, David, from the Biblical story of *David and Goliath*. *David* is currently on display at the Galleria dell'Accademia in Florence, Italy. Michelangelo was awarded the commission to work on the statue, which took two years of continuous work to complete, when he was only 26 years old. Because of its beauty, perfect proportions, and excellence of form, *David* was immediately recognized as a masterpiece. Like Botticelli's Venus, the ratio of David's overall height to the distance to his navel is equal to the Golden Ratio. More importantly, at least with respect to the perfect male physique, the ratio of the circumference of David's shoulders to the circumference of his waist is also equal to the Golden Ratio. A 2007 study published in the *Archives of Sexual Behavior* finds that, for women, a shoulder to waist ratio of 1.6 is a very strong predictor of the sexual attractiveness of men [20, p1]. Numerous body-building programs, including one appropriately called the "Adonis Golden Ratio" program promote working toward this ratio as a primary goal to increase attractiveness.

[**Figure 6.8**. Photograph of *David.* Retrieved from https://springsemester2015artz363.wordpress.com/proportion-scale/ in July, 2020]

Piet Modrian was a Dutch artist born in 1872. His early work is traditional, including landscapes and portraits, but he later became a pioneer of abstract art. He, along with Theo van Doesburg, founded the Di Stijl Movement (also called Neoplasticism) which they began in 1917 and has been very influential in art, design and architecture. His *Composition with Red, Yellow, Blue and Black* (Figure 6.9) was produced in 1921.



[**Figure 6.9**. *Composition with Red, Yellow, Blue and Black.* Retrieved from http://www.widewalls.ch/magazine/golden-ratio-examples-art-architecture-music/georgeus-seurat-golden-ratio in July, 2020]

Modrian limited his use of color to the primary colors, along with the values white, gray and black. He limited the forms to vertical and horizontal lines and rectangular forms. His style avoids symmetry but uses design elements to achieve balance. *Composition with Red, Yellow, Blue and Black,* like many of his abstract compositions, uses recurrent rectangles in its design, three of which have their ratios of length to width approximately equal to 1.6. Modrian was known to be interested in the connections between art, logic, and mathematics but whether or not he would have used the Golden Ratio intentionally is controversial.

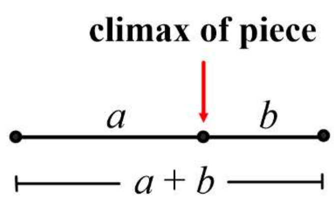
In the final examples shown below, the Golden Ratio does make intentional appearances. John Edmark is an artist, designer and mathematician who currently teaches at Stanford University. Several of his nature-influenced sculptures use the Golden Ratio as a primary source of inspiration. His work goes beyond a theory of proportions to making explicit connections between mathematics and biological phenomena. *Curling Spiral, Outer Spine,* below left, is a depiction of a golden spiral that can unwind itself. The sculptures on the right are referred to as *Blooms* sculptures. They intentionally use a golden angle (approximately 137.5 degrees) to maximize the number of leaves that can be arranged around a central stem. Edmark appropriates the golden angle from nature where it is known to optimize access to limited resources. The appearance of the Golden Ratio in biology is one of the most astounding and will be discussed in greater detail in Section VIII.

[**Figure 6.10**. *Curling Spiral, Outer Spine* and examples of *Blooms* sculptures. Retrieved from https://www.instructables.com/id/Curling-Spiral-Kinetic-Sculpture/ and from http:// www johnedmark.com/ in July, 2020]

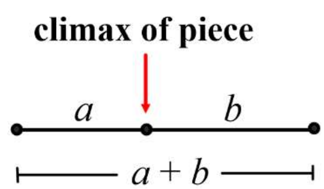
VII. MUSIC

In the areas of art and architecture, the Golden Ratio is perceived by *seeing,* but this next section will show that the Golden Ratio can also be perceived by *hearing*. Just like in the visual arts, the Golden Ratio has been used to create a kind of structure in a musical composition. Including φ as an element of an arrangement creates a limitation that helps to serve as a template. There are many examples of composers using Golden Ratio-like intervals in musical pieces. As in other disciplines, it is expected that a number of these uses are coincidental, but the possibility that some musicians make use of the Golden Ratio, and especially the related Fibonacci sequence intentionally will also be considered.

Definition 3 in Book VI of *Euclid's Elements*, presents Euclid's definition of the Golden Ratio: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser." The Golden Ratio is defined by the analogous relationship between the whole line to the longer section, and the longer to the shorter section. This relationship characterizes the most easily comprehended way that the Golden Ratio manifests in a musical composition and is illustrated by the following diagrams.

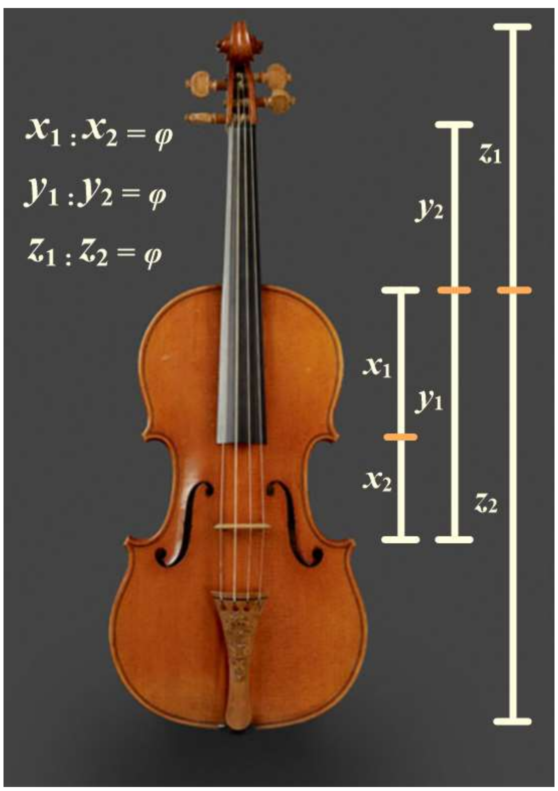


or



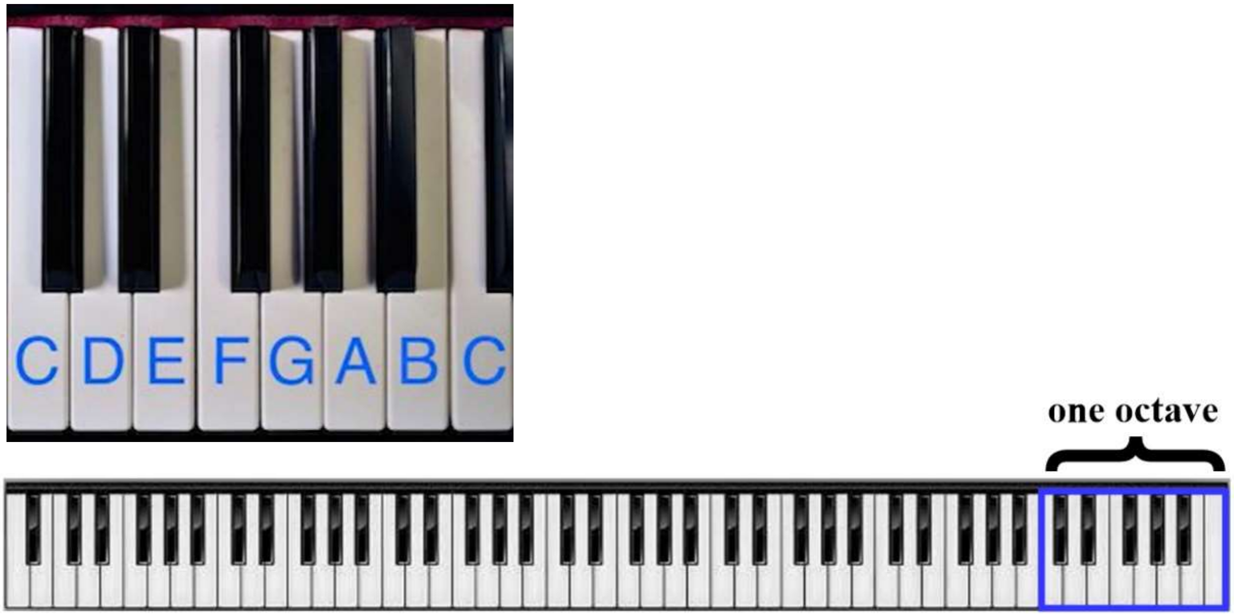
In a composition that uses the Golden Ratio as a structural element, there is a longer section, followed by the climax of the piece, and then a shorter section; the ratio of the longer to the shorter being approximately equal to φ (or vice versa). This sounds simple enough but unlike in art or design, where an entire composition can be perceived simultaneously, a musical arrangement is perceived over time. What makes the appearance of the Golden Ratio in music so fascinating is that the human brain seems to have the ability make a judgment about this climax happening at a desirable point in a song even though perception takes place over a duration and the relationship between the part of the song that occurs before the climax and the part that occurs after is not evident until the song is over. That the brain is able to perceive in this way leads to very interesting questions about the possibility of a preference for this type of proportional relationship having a biological, or evolutionary basis, and also about how aesthetic appraisals are made with respect to music.

In addition to compositions utilizing the Golden Ratio, it is also associated with musical instruments themselves. Stradivarius violins are among the most renowned instruments in the world. The superiority of a Stradivarius' tone quality is well established and has been extensively studied by physicists associated with some of the world's most prestigious universities. Theories have been proposed that the sound is enhanced by the instrument's shape, the size of its 'f' holes, a particular recipe of varnish, or even by certain imperfections. With respect to its shape, a Stradivarius violin incorporates several proportions that are equal to the Golden Ratio. These relationships are shown in the diagram below. To what degree, if any, the Golden Ratio is associated with the excellence of the violin's sound is an elusive matter that cannot easily be determined but is certainly an interesting topic of investigation.



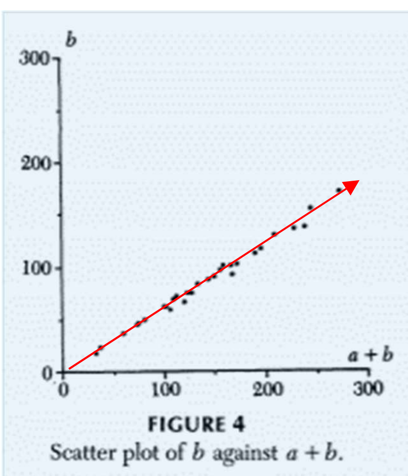
[**Figure 7.1**. Photograph of Stradivarius Violin. Retrieved from https://www.classic fm.com /lifestyle/wellbeing/what-makes-stradivarius-violin-amazing/ in July, 2020]

Recalling that the Fibonacci sequence consists of the numbers, 1, 1, 2, 3, 5, 8, 13, 22, 34…, consider as another example, an octave on a piano. An octave describes the distance between a particular note and the next time that the same note repeats. The word "octave" is derived from the Latin word for eight. A full piano has 88 keys spanning seven octaves. One octave, shown below, consists of 13 total notes with 8 white keys and 5 black keys. This, of course, may only be a coincidence but interestingly, 13, 8 and 5 are all Fibonacci numbers!



[**Figure 7.2**. Piano keyboard. Downloaded from https://www.musikalessons.com/blog/ 2016 /08/piano-keys-chart/ in July, 2020]

When Wolfgang Amadeus Mozart was only 18 years old, he began writing 19 different piano compositions which were all in sonata form. A composition of this type has two parts called the "Exposition," and the "Development & Recapitulation." Analysis of the length of the whole composition to the longer part of each sonata reveals that all of the proportions are approximately equal to 1.6. Each data point on the graph to the left represents the total length of a composition on the x–axis and the length of its Development & Capitulation on the y–axis.



[**Figure 7.3**. Graph by John Putz. https://www.yumpu com/en/document/read/40898694/the-goldensection and-the-piano-sonatas-of-mozart-john-f-putz, 2020]

The red line shown on the graph is the line, $y = \frac{1}{\varphi}x$. As can immediately be observed from the graph, all of the data points are very close to the line. While the presence of the Golden Ratio in this relationship is undeniable, that Mozart used it deliberately is uncertain. Some scholars believe that it was rather his innate sense of proportion that is responsible for the similar ratios of the whole to the part in each sonata, arguing that Mozart would not have needed to implement a methodology when he possessed such exceptional intuition [18, p280].

Frédéric Chopin, known as Poland's greatest and most popular composer, is often referenced when considering musical pieces with connections to the Golden Ratio. A frequently cited example is Chopin's Prelude in C Major, Op. 28, No. 1. The culmination of the piece occurs at the 21st measure, outlined in red below. The composition consists of a total of 34 measures with the climax taking place almost exactly at the golden mean since 34/21 ≈ 1.619. That 34 and 21 are both Fibonacci numbers again raises the interesting question of intentionality vs. coincidence. Skeptics point out that Chopin died (at the young age of 39) in 1849 but the Fibonacci numbers were not well-known until the 1870s.

Prelude in C Major, Opus 28, No.1

[**Figure 7.4**. Chopin's Prelude in C Major, Op. 28, No. 1. Downloaded from https://www.mi.sanu.ac.rs/vismath/jadrbookhtml/part42.html in July, 2020]

Other examples of musicians thought to have used the Golden Ratio in their work are French composer, Claude Debussy, and one of the most influential composers of the 20th century, Hungarian pianist, Béla Bartók. Three piano pieces by Debussy with golden proportions include *Mouvement, Hommage à Rameau*, and *Reflets dans l'eau*. Two works by Bartók where the Golden Ratio appears are the *Sonata for Two Pianos and Percussion,* and *Mikrokosmos.* Bartók who was influenced by DeBussy and also by French composer, Maurice Ravel, believed that music is deeply connected to nature and compared musical compositions to living organisms such as trees and animals. Musicologist, Ernö Lendvai, explains that *Sonata for Two Pianos and Percussion* is a particularly good example because it uses the Golden Ratio in both small components and in a broader way. As was discussed previously, a composition can be divided into two parts whose durations are proportionally related according to the Golden Ratio, but two individual notes can also be related by a golden proportion. Ratios of frequencies create notes and the key frequencies known in Western music are composed of ratios of the first seven Fibonacci numbers.

There is an old folktale that Pythagoras, upon hearing the pounding of a blacksmith's hammers, went running into his shop to discover that the sounds he was hearing were relative to the weights of the blacksmith's hammers, i.e., that there was a proportional relationship between the weight of each hammer and the musical note produced. Allegedly, there were four hammers, *A, B, C,* and *D*. Most combinations of two hammers produced consonance but the combination of hammers *B* and *C* produced dissonance. This story was first told by ancient Greek mathematician, Nicomachus, and has been repeated for centuries. It has since been determined that Nicomachus' account does not really stand up to scrutiny; however, Pythagoras is credited with being the first to perceive the intimate and important connections between music and mathematics. Huntley explains that Pythagoras "noted the curious fact that the lengths which emitted a tonic, its fifth and its octave were in the ratio 2: 3: 4" [7, p23]. Pythagoras believed that harmony in music could only come from nature and that it must be related to mathematical proportions.

Discussing music, as opposed to listening to it, certainly has its limitations, but this overview has introduced to readers how the Golden Ratio reveals itself in relationships between musical notes, in instruments themselves, and in the structures of compositions. It is well-known that some compositions are pleasant to listen to while others are not. Researchers have suggested that a piece can sound unpleasant if inconsistency is detected, e.g., if Fibonacci numbers or golden proportions are used throughout the piece and then a change in structure occurs, a part of the song might sound unpleasant, or out of place. Huntley proposes that what is pleasing to the ear is likely connected to nature. He says "intervals of music acceptable to the human ear might be the same as those first offered to ancestral man by birds" [7, p18]. Huntley's suggestion provides the perfect segue to the next section where the discussion of the manifestations of the Golden Ratio in nature will certainly be surprising and most impressive.

VIII. THE NATURAL WORLD

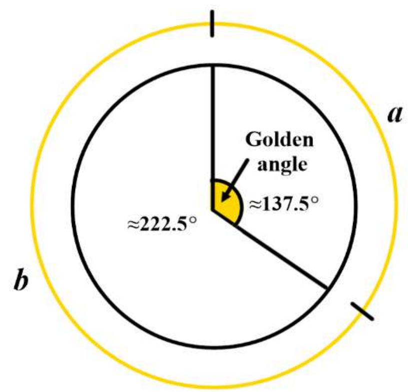
The relationship of the Golden Ratio and the Fibonacci numbers to the natural world is so absolutely stunning that natural science is really the only discipline where an investigation of φ might be as fascinating as it is in mathematics itself. The Pythagoreans began to discover surprising patterns and designs in nature and believed that understanding the meaning of symbols through mathematics is the key to comprehending the universe. 18th century Prussian geographer and naturalist Alexander von Humboldt expressed concern that the examination of nature's innate qualities would compromise the magic and dignity of the natural world, but French mathematician and philosopher of science Henri Poincaré contends that man, does not study nature "because it is something useful [but rather because it is] joyful and he finds it joyful because it is so beautiful" [12, p99]. This author's opinion is that discovering and exploring the secrets of nature leads to a deeper appreciation of its beauty and complexity and is an important impetus for the critical, often political, work that must be done to preserve its magnificence.

In studying the natural world, the ancient Greeks began to notice that the same mathematical laws applied to numerous phenomena in biology, astronomy, music, and human anatomy. The relationships between these phenomena became apparent which particularly fascinated the Pythagoreans and fueled their desire to make explicit connections between mathematics and such things as the earthly elements, celestial bodies, the Platonic solids, the human body, and the signs of the zodiac. In the 2018 paper, "Mathematical Determination in Nature–The Golden Ratio," the authors describe one such example:

> Pythagoras, together with his followers, constructed a regular pentagon based on their knowledge of the Golden intersection. They called it Health and strongly believed that it represented a pure mathematical perfection. They connected health of the human body directly with the mathematical harmony in the Golden intersection. [8, p125]

A thousand years later, in ancient Rome, a similar philosophy still prevailed. The Roman statesman, mathematician, and philosopher, Boethius, believed that the body and soul are in accordance with the very same mathematical proportions as the Cosmos. Boethius was thought to be an intermediary between the ancient and modern worlds. Interestingly, the Golden Ratio plays an important role in the training of modern-day doctors, especially cardiologists, plastic surgeons and even dentists.

Golden proportions, golden angles, and the Fibonacci sequence appear in thousands of organisms and natural phenomena as well as in the Cosmos itself. New manifestations continue to be discovered each year. In his paper, “The Golden Section and Beauty in Nature,” Ulrich Lüttege explains that the Golden section “has an important organizational role” in nature and claims that “the golden angle optimizes the packing of molecules such as seeds and fruits” [12, p98]. A golden angle is created by dividing the central 360-degree angle of a circle into two angles such that the ratio of the measure of the larger to the smaller is equal to the ratio of the measure of the whole circle to the larger angle. Solving for the smaller angle, below, the measure is determined to be about 137.5 degrees which is the golden angle!



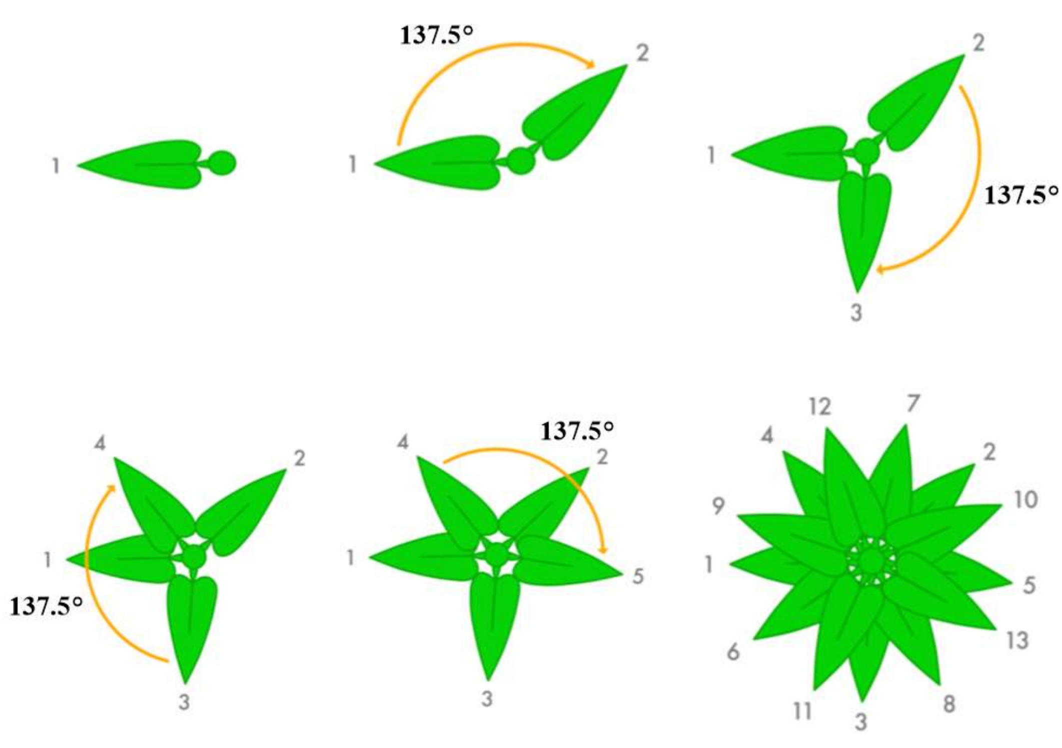
$$\frac{b}{a} = \frac{a+b}{b}$$

Letting $a = 360 - b$, then $\dfrac{b}{360-b} = \dfrac{360}{b}$

$$b^2 = 360(360-b)$$
$$b^2 = 129600 - 360b$$
$$b^2 + 360b - 1295600 = 0$$
$$b = \frac{-360 \pm \sqrt{129600 - 4(1)(-129600)}}{2(1)}$$
$$b \approx 222.5 \text{ degrees.}$$
$$\text{So, } a \approx 137.5 \text{ degrees.}$$

The diagram below shows how the development in some plants exemplifies the golden angle for leaf placement around the stem. The plant sprouts a single leaf and then when the second leaf emerges, it develops approximately 137.5 degrees away from the first leaf. The third leaf then sprouts ≈137.5 degrees from the second leaf and this pattern continues, with the new leaf always ≈137.5 degrees from the very last leaf that sprouted, until all of the leaves have emerged. By following this pattern, the plant minimizes overlap of the leaves allowing them to access as much sunlight and other resources as possible. This example of leaf phyllotaxis ensures that the arrangement will never be periodic, i.e., that any two leaves will not be in the same position around the plant's stem. This arrangement may give the plant an evolutionary advantage. Examples of species where these arrangements are easily observable include *Aeonium tabuliforme* (saucer cactus) and *Cynara cardunculus* (artichoke).

Leaf Phyllotaxis Exemplifying the Golden Ratio



[**Figure 8.1.** Diagram of the golden angle in leaf phyllotaxis. Retrieved from http://gofiguremath.org/natures-favorite-math/the-golden-ratio/the-golden-angle/#:~: text=Plants in July, 2020]

[**Figure 8.2.** Common Sunflower. Retrieved from http://gofiguremath.org/natures-favorite-math/the-golden ratio/the - golden-angle/:~:text=Plants in July, 2020]

A beautiful example of the Golden angle's organizational role can be observed in the head of a common sunflower, *Helianthus annuus*, where fruits are packed very tightly. Each fruit is produced in the center of the floral head and then must migrate outward as new fruits are created. Once the first fruit moves from the center, the second fruit automatically develops in a direction that is approximately 137.5 degrees from the path of the first fruit. This movement continues and each new fruit is positioned at an angle of ≈137.5 degrees from the very last fruit maximizing the number of fruits that can be produced by a single floral head. That this happens at all is incredibly amazing but biologists and physicists also study *how* it happens. Since a sunflower does not have a computer inside of it, complex biochemical and biomechanical processes are responsible for making sure that the fruits get dispersed in such a highly accurate arrangement. Once the fruits are in their positions, an equiangular spiral pattern becomes apparent. In a sunflower, there are 21 clockwise spirals and 34 counter-clockwise spirals that appear to emerge from the center. Noticing that 21 and 34 are both Fibonacci numbers, another connection to the Golden Ratio is revealed! 21 and 34 are in fact adjacent Fibonacci numbers. This same pattern also occurs in daisies, but it is much easier to see in sunflowers. Many other organisms also have similar arrangements with spiral patterns occurring in frequencies of adjacent Fibonacci numbers. Two well-known examples are the composite fruit structure of pineapples which have eight clockwise and 13 counterclockwise spirals, and the arrangement of the bracts on pinecones, which have five clockwise and eight counterclockwise spirals. They are easiest to see in the pinecone, pictured left.

[**Figure 8.3.** Photograph of Pinecone. Retrieved from https://www.pinterest.com /pin/11137159 59314672964/ in July, 2020.]

In Section IV, it was mentioned that given a Golden rectangle, lopping off a square produces another Golden rectangle. This process can be repeated until the resulting rectangle gets so small that it is reduced to a single point (Figure 8.4). If then a spiral is drawn such that it intersects each Golden rectangle at a vertex, as shown below, an approximation of a Golden spiral, is created (Figure 8.5). A Golden spiral is one particular case of a logarithmic, or equiangular, spiral. Huntley asserts that the Golden spiral's exceptional elegance is perceived prior to its mathematical significance being recognized, suggesting that it might be the spiral's familiarity that makes its form so appealing [7, p101].

Figure 8.4

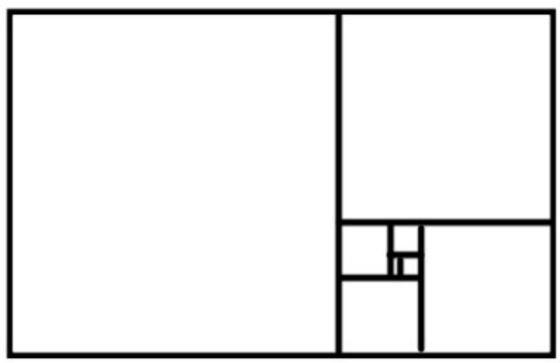
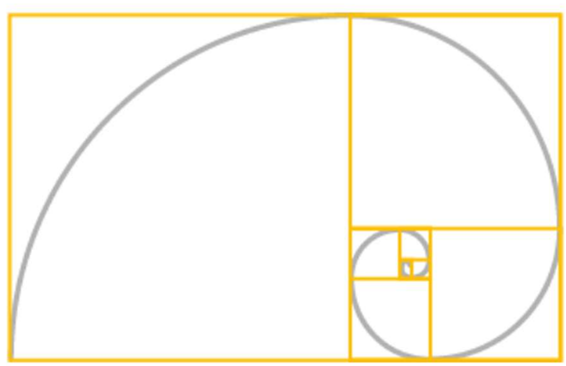


Figure 8.5



The Chambered Nautilus (pictured on the next page) is a small marine mollusk that has lived largely unchanged on Earth for over 400 million years. It is the organism most likely to be immediately associated with the Golden spiral but its familiarity does not in any way compromise the sense of awe that it inspires.

[**Figure 8.6.** Photograph by Ingo Arndt. Downloaded from https://fineartamerica.com/featured/chambered-nautilus-cross-section-ingo-arndt.html in July, 2020]

The Chambered Nautilus, which is also called a pearly Nautilus, or simply a Nautilus, is considered to be a living fossil. Its species has occupied Earth longer than there have been trees. A Nautilus' shell is created from chambers which house the Nautilus. As the Nautilus grows, it moves into larger and larger compartments that have exactly the same shape and the same proportions. Once it has moved out of a compartment, the area is walled off with mother of pearl, leaving only enough room to accommodate the Nautilus's siphuncle, the structure that enables it to adjust the gas pressure in different chambers. The empty chambers are filled with air or gas to help it stay afloat. Nautilus mollusks currently live in the Indian and Pacific Oceans but their populations are dwindling due to demand for their beautiful pearly shells.

A Golden spiral is defined by its growth factor. For every quarter turn that a Golden spiral makes, it gets larger by a factor of φ. The growth factor of a Chambered Nautilus is not typically exactly φ but like a Golden spiral, its growth is based on a geometric progression rather than a constant rate. If a Golden spiral is superimposed on an image of a Chambered Nautilus, the form of the shell will approximately follow the curve of the Golden spiral and is much closer than it would be to a Spiral of Archimedes which has a constant rate of growth. The image of the two spirals on the next page shows that the Nautilus shell is much more similar to the Golden spiral.

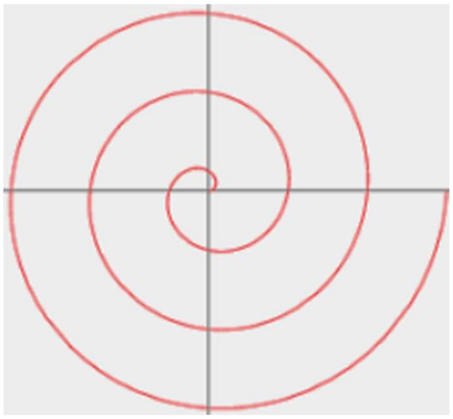


Figure 8.7 Archimedean spiral

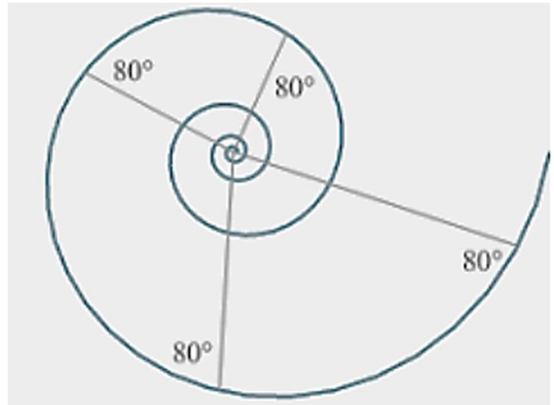


Figure 8.8 Golden spiral

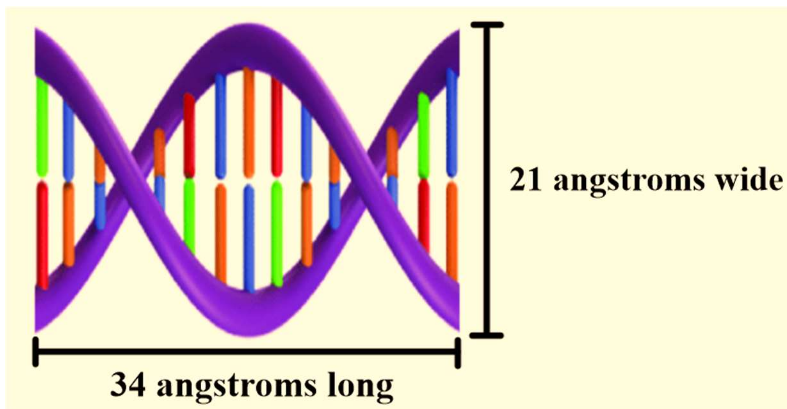
A Golden spiral is equiangular because for any radial line that is drawn from the center of the spiral to a point on the spiral, the tangent for the point and the radial line will form an angle whose measure is constant for all points on the spiral. This can be observed in Figure 8.8. A Golden spiral is a growth spiral that is also associated with other natural phenomena; a growth spiral gets larger while maintaining constant proportions and the same shape. French mathematician, philosopher, and scientist, René Descartes, was the first to describe the equiangular spiral in the year 1638. Swiss mathematician, Jakob Bernoulli, who also worked with the equiangular spiral, gave it the name *spira mirabilis,* Latin for the miraculous spiral, in 1692. Bernoulli was so fascinated by his spira mirabilis that he requested it be engraved on his tombstone but the engraver made an error and he instead ended up with an Archimedean spiral.

Messier 51a or "The Whirlpool Galaxy" (left) was discovered by Charles Messier in 1773. It is a galaxy 76,000 light years in diameter, which is about 43% the size of our own galaxy, the Milky Way.

[**Figure 8.10.** Photograph of Whirlpool Spiral. Retrieved from http://www.spacetelescope.org/images in July, 2020]

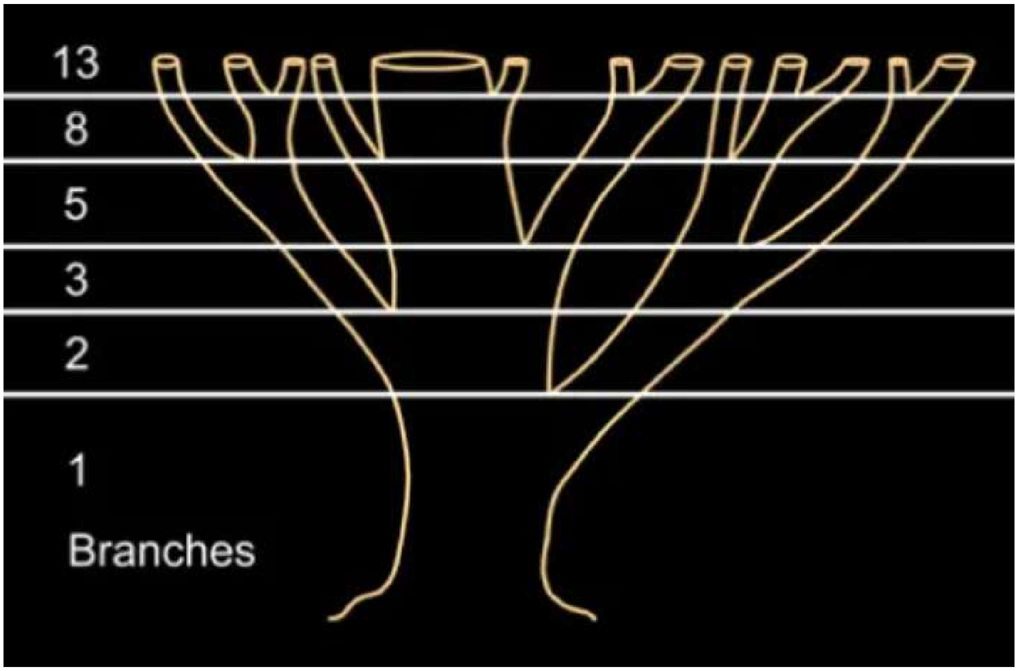
Amazingly, the Whirlpool Galaxy is 23 million light years from Earth but in a dark sky, it can be observed by amateur astronomers on Earth using only binoculars. Like the Chambered Nautilus, the Whirlpool Galaxy is thought to be about 400 million years old. It is said to replicate the Golden spiral of the Nautilus. The arms of the spiral serve an important function compressing gas so that so that new clusters of stars can be created.

DNA contains all of the genetic instructions necessary for the development and reproduction of living things (including some viruses). During the early 1950s, English chemist, Rosalind Franklin, worked on the x-ray crystallography research which led to the discovery of the structure of DNA. In 1953, Francis Crick, an American biologist, and James Watson, an English physicist, first described the 3-dimensional double helix structure that is familiar today. DNA is composed of two intertwined helixes. Each helix stores all of the information about the features of a particular organism. An angstrom is a metric unit of length, denoted by Å, that is equal to 1/10,000,000 of a millimeter. It is smaller than a nanometer and is used to measure such things as wavelengths and atoms. One complete turn of the double-helix of DNA measures 34 Å long by 21 Å wide. 34 and 21 are consecutive Fibonacci numbers whose ratio is approximately equal to 1.619!



[**Figure 8.11.** Fibonacci Numbers in DNA. Retrieved from https://www.researchgate.net/figure/A-The-double-stranded structure-of-DNA in July, 2020]

Through an examination of the Golden ratio and the Fibonacci sequence in the natural world, designs and patterns from nature can be more acutely perceived and better appreciated. From Ilić again, “there are clear marks and signs of mathematical regularity which appear in nature showing fascinating accuracy” [8, p124]. DNA is only one such example of this splendid accuracy. The Fibonacci sequence also shows up in the numbers of petals on flowers such as lilies, asters, and cosmos, in reproductive dynamics, in the number of seeds in fruits like apples and bananas, and in the optimal growth patterns of trees (shown on the next page) which maximizes the ability of a tree to grow more leaves.



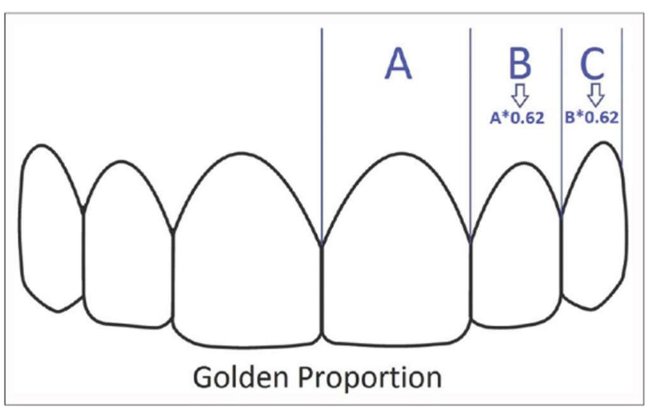
[**Figure 8.11**. Optimal growth pattern in trees. https://io9.gizmodo.com/15-uncanny-examples-of-the-golden-ratio-in-nature-5985588 in July, 2020]

The Golden spiral can be observed in hurricanes, animal flight patterns, seed heads, and spiral galaxies. The Golden section appears in the proportions of animals such as dolphins, tigers, starfish and ants as well as in the anatomy of humans. Ilić's paper points out that "the intellectual elite of wide domains have been deeply fascinated by the Golden Ratio for more than 24 centuries" [8, p125]. Human beings' modern fascination is the result of the same deep curiosity of the Pythagoreans; what do these coincidences mean, and does the universe actually have a predetermined mathematical construction?

IX. HUMAN ANATOMY

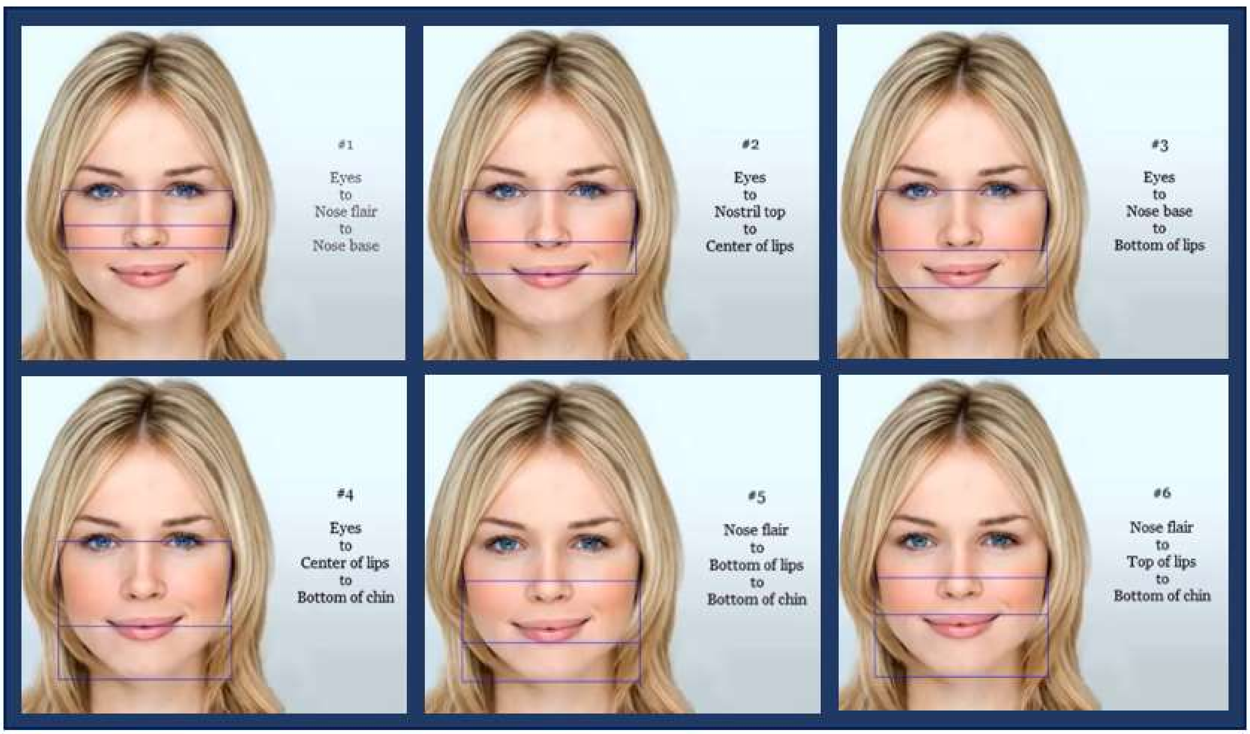
It should be no surprise that the Golden Ratio's organizing role in nature also extends to human beings. Golden proportions, Fibonacci numbers, and the Golden spiral all reveal themselves in the bodies of human beings. Though φ is generally the irrational constant much less well known than π, its connection to human anatomy, which has been highlighted since the time of the ancient Greeks, is well accepted in Western medicine. Modern doctors and dentists are taught that the golden proportion is associated with health and that ideal proportions should be based on the Golden Ratio [22, p713]. This is of particular interest to plastic surgeons and dentists who are often employed to help a patient modify their appearance to more closely align with idealized Western beauty standards which, like in art and architecture, often have as their foundation a theory of proportions.

The shape of the mouth and teeth, along with facial symmetry which depends on the structure of the mouth and jaw, are accepted to be a primary determinant of facial beauty. Dental professionals understand this and have tried to establish parameters to help people achieve beautiful smiles. In 1978, Dr. Edwin Levin, who practiced dentistry in London until his retirement in 2004, observed that in the most aesthetically pleasing smiles, "the width of the maxillary incisor [to] the width of the lateral incisor is in the golden proportion" [9, p9]. Similarly, the width of the lateral incisor (tooth B in Figure 9.1 below) is in the golden proportion to the width of the canine (tooth C). This observation led to the development of a widely-used diagnostic grid, called the "Phi Dental Grid," which is available in different sizes to help dentists improve the aesthetics of the anterior teeth.



[**Figure 9.1**. Ideal Proportions for Anterior Teeth. Retrieved from http://www.jidonline.com/viewimage.asp?img=JInterdiscipDentistry_2018_8_2_62_233620 in July, 2020]

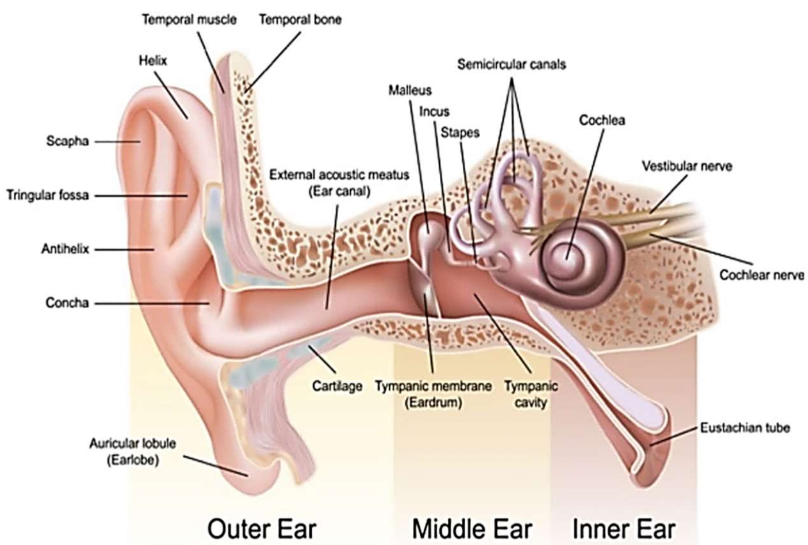
In her paper, “A Compendium of the Fibonacci Ratio,” Priyanka Katyal asserts that the Golden Ratio “epitomizes beauty, congruence, and balance in physical form” [9, p4]. Plastic surgeons are trained to consider golden proportions when performing aesthetic procedures on people’s faces. Figure 9.2 shows photographs of the woman voted Britain’s “Most Beautiful Face of 2012.” Measurements taken at numerous places on her face show that the vertical proportions consistently adhere to the Golden Ratio. The Golden Ratio can also be applied to the horizonal proportions of the face, e.g., the ideal ratio of the distance between the eyes to the length of one eye is thought to be φ. A face where the ratio of the overall width to the width of the eyes is equal to φ is also said to be more beautiful.



[**Figure 9.2.** Images of golden proportions in vertical facial proportions. Retrieved from https://www.medisculpt.co.za/golden-ratio-beautiful-face/ in July, 2020]

It should be noted that both dentistry (particularly orthodontics) and plastic surgery also address functional problems such as breathing difficulties, migraines, hearing deficiencies, sleep apnea, and obviously injuries and accidents. Subtle adjustments to the structure of a patient’s face can greatly increase functioning. Parameters for proportions established by the Golden Ratio help to ensure that changes are not arbitrary.

An interesting example of the Golden spiral in the human body appears in the inner ear. As can be seen in Figure 9.3, the cochlea replicates the spiral of the Nautilus. The cochlea is a complex organ whose function is to translate sound vibrations into nerve impulses that get sent to the brain. Depending on the frequency and intensity, different parts of the cochlea move in different ways. Research shows that the spiral shape of the cochlea makes it possible for humans to hear low-frequency sounds.



[**Figure 9.3.** Anatomy of the human ear. Retrieved from ///https://drjillgordon.com/how-hearing-works in July, 2020]

Lüttge argues that "developmental stability is based on [an] organism's ability to minimize random perturbations during [its] development" [12, p102]. Modern researchers are finding that healthy vascular systems depend on minimal deviations, or perturbations, from ideal proportions over a patient's lifespan. The Golden Ratio makes some surprising appearances in the area of cardiology. With respect to the measurement of the heart, based on research, the ideal ratio between the overall vertical measure of the heart and the left ventricle transverse is thought to be φ. Studies of cardiac patients conducted in China and Sweden have found that patients with heart failure are very unlikely to have measurements whose ratio is close to φ [8, p127]. This leads to the conclusion that the Golden Ratio can be used to detect subtle deviations from the norm, which should really be thought of as deviations from stability, earlier in the disease process to help improve outcomes for patients.

A 2019 paper reporting research on blood pressure and mortality finds that ratios of systolic to diastolic blood pressure values that deviate from the Golden Ratio are associated with greater "all-cause mortality." The research, which was conducted in the United States, includes a total of 31,622 participants, 2,820 of whom died during the course of the study. Through an analysis of the patients who died, the researchers reached the following conclusion:

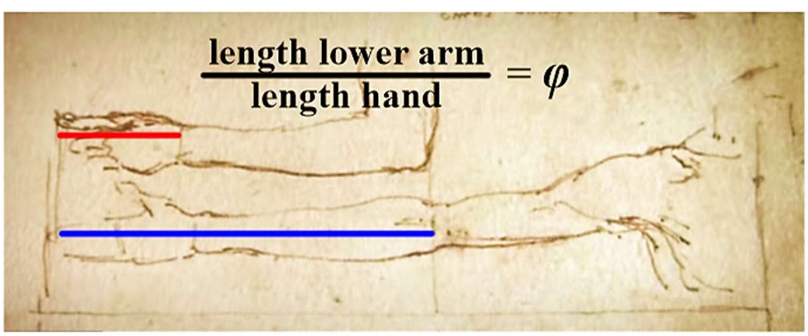
> By this preliminary analysis, it was found that participants with SBP/DBP values that deviate from the golden ratio have a significantly higher risk of death, regardless from other established risk factors in comparison to individuals whose BP values fulfill the golden proportion. [17, p56]

The research team suggests that rather than using absolute parameters for each blood pressure value (that frequently change due to conflicting research) to assess risk, it might be more helpful to calculate ratios of systolic to diastolic blood pressure to look for deviations from the most stable value, φ. Lüttge explains that "fluctuating asymmetry is a widely used measure of developmental instability" and describes a stability continuum (shown below) ranging from perfect symmetry to death [12, p102].

Perfect Symmetry **Deviations** **Collapse** **Death**

Integrating this type of subtle change into Western medicine makes it more similar to Chinese medicine where health is believed to depend on balance and harmony.

There are many other examples of relationships between the Golden Ratio and human anatomy. The lengths of the bones in hand are said to be Fibonacci numbers. The proportion of arm length (from the elbow to the fingertips) to hand length is ideally equal to φ. Similar relationships are apparent in the sections of the fingers and in fact, since these measurements are considered to be so reliable that, "a stable use of measurement units taken from human body parts" has been utilized since the time of the Greeks [21, p509]. The proportion of the length to the width of a woman's uterus, which fluctuates between 2 and 1.4 throughout her life, is approximately equal to φ during her most fertile years, and women whose uteri proportions are closest to φ are found to be the most fertile [23, p716]. Again, considering cardiology, even the Golden angle makes an appearance in human anatomy: "angles between the outflow tract



[**Figure 9.5**. Arm proportions by da Vinci. Retrieved from https://www.youtube.com/watch?v=GGUOtwDhyzc, 2020] axis and inflow tract axis and angle between pulmonary trunk and ascending aorta approximate the golden angle" [8, p6].

By now the case should be made that the Golden Ratio "is a systemic property of many systems in nature" [12, p102]. As in art, it seems to provide an underpinning of order, balance, and harmony. Where the Golden Ratio presents itself as an ideal in human health, deviation from it is associated with disease and possibly even death. Lüttge quotes researcher Shu-Kun Lin in his paper who claims that "symmetry is beautiful because it renders stability" [12, p103]. Human health depends on stability and what is healthy is perceived as beautiful. Investigating how the Golden Ratio manifests itself with regard to human anatomy is fascinating in itself but also helps to challenge the traditional Western disease fighting paradigm which might rather be thought of as restoration of internal stability and order. In addition, it emphasizes how intimately connected φ is to the human body giving it a special status among other mathematical constants such as π and e.

X. GEOMETRY

There is no greater expression of the Golden Ratio's magnificence than in geometry. The Greeks' first encounter with the Golden Ratio would have likely occurred while working with the geometry of the pentagram–pentagon. A pentagram which is inscribed inside of a regular pentagon was adopted as the secret symbol of Pythagoras and his followers so that they could recognize, and be recognized by, fellow members. Huntley asserts that "all the great knowledge of the Pythagoreans meets in their symbol, a pentagram, which is called *pentagramma*, inscribed into a pentagon" [7, p38]. The pentagramma and the Golden Ratio are inexorably linked and the symbolism of the figure is directly related to the regenerative properties of the Golden Ratio which are perceptible in its construction. The Greeks believed that each of the five points on the pentagram were associated with one of the earthly elements, earth, air, fire, water and idea. It has also been suggested that the pentagram is the symbol of life itself connecting it to the five fingers and five toes of humans. Some even believe that it is a symbol of power and immunity that is used to ward off evil.

According to Roger Herz-Fischler in his book, *A Mathematical History of the Golden Number*, "pentalpha" may have been another name for the pentagrammma. The Pythagoreans were thought to have used the symbol in letters that they sent to each other. They began their correspondences with the greeting, "Health to you" [5, p65], rather than more customary salutations, highlighting the importance that they placed on health.

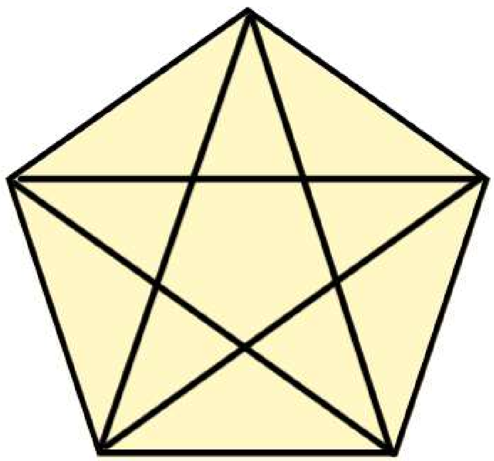
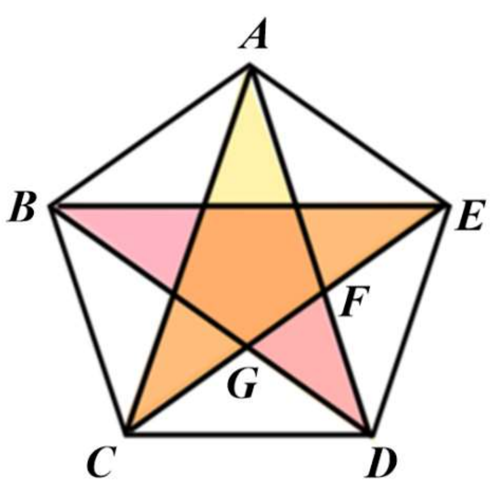


Figure 10.1 *Pentagramma*

The fundamental characteristics related to the proportions of the side lengths of the pentagram and pentagon, including the presence of the Golden Ratio in these measures, was previously discussed in Section II but other φ relationships were not emphasized. Given the ubiquity of the golden mean in the proportions of their symbol, the pentagramma, it is no wonder that the Pythagoreans would have assumed an exalted status for the Golden Ratio. Figure 10.2 below enumerates these relationships.

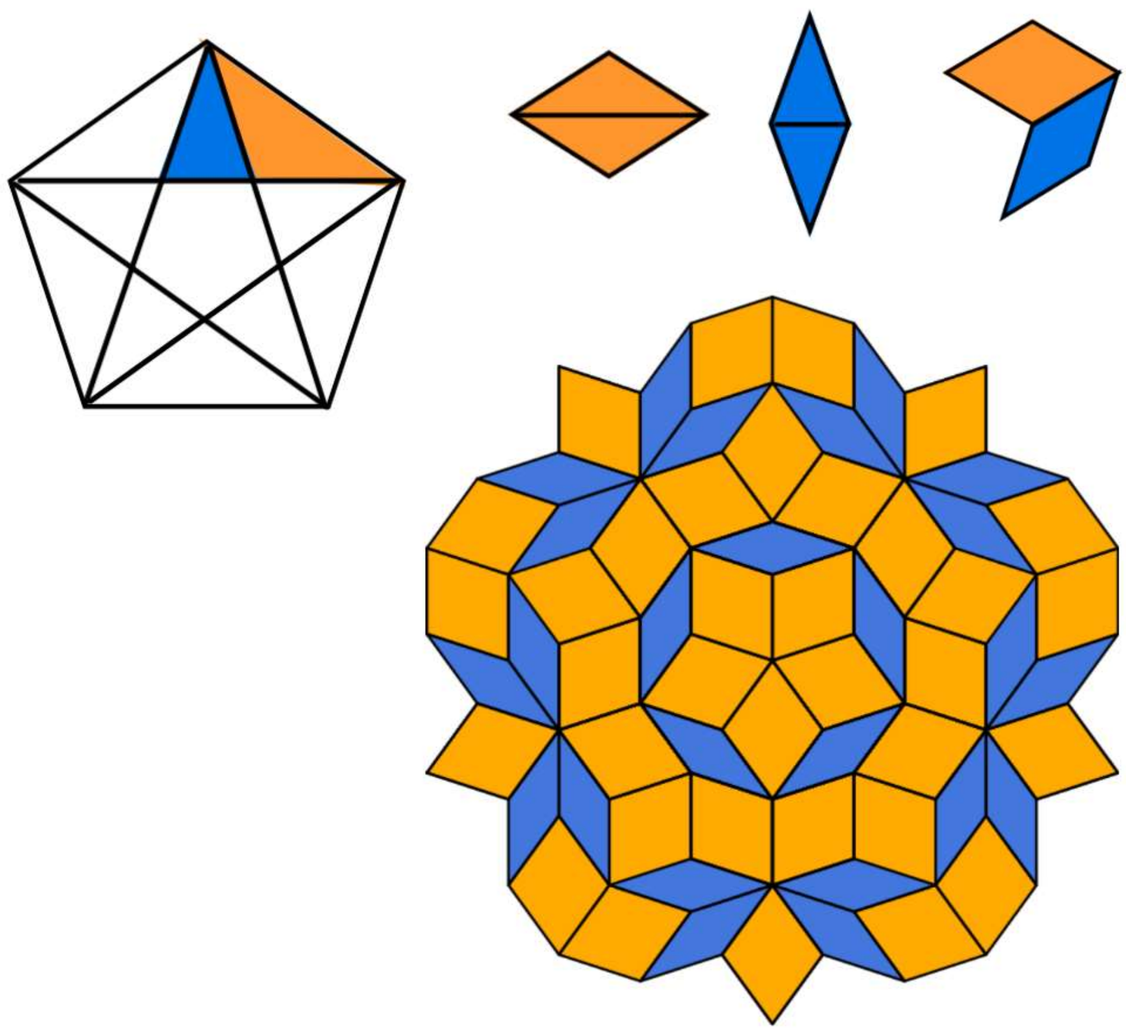
Figure 10.2 Numerous Golden Relationships in the Pentagramma



$$\frac{AC}{CD} = \varphi \qquad \frac{CD}{CG} = \varphi$$

$$\frac{BG}{GD} = \varphi \qquad \frac{FD}{FG} = \varphi$$

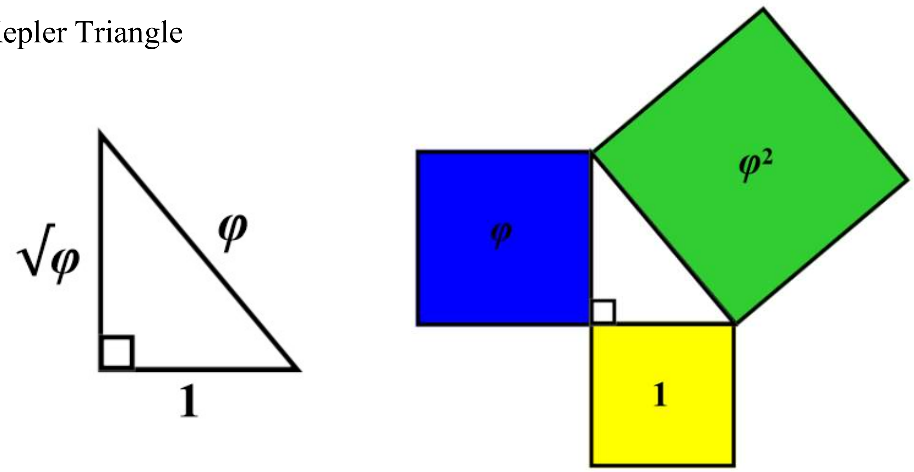
In the 1970s Sir Roger Penrose, a British mathematical physicist, used these relationships, specifically the proportions between the two types of triangles that compose the pentagram–pentagon (shown in blue and orange below), to create the beautiful Penrose tiling. Since pentagons cannot tesselate the plane like triangles, squares, and hexagons, he instead took advantage of the triangles of the pentagramma combining them in such a way to make tiling the plane possible. The image below shows the beautiful, sophisticated pattern resulting from the careful placement of two types of quadrilaterals generated by the golden triangles.



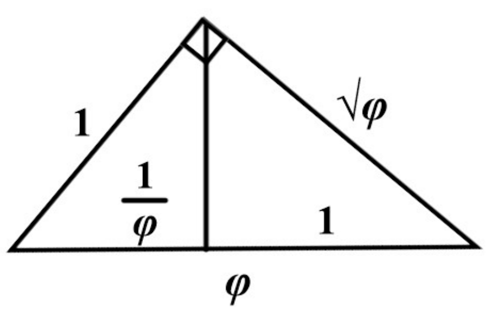
[**Figure 10.3.** Penrose Tiling. Retrieved from https://www.nist. gov/image/penrose-tiling in 2020]

Johannes Kepler, the German mathematician, astronomer, and astrologer was an important figure during the scientific revolution of the 17th century. He is said to have been the person responsible for giving the Golden Ratio the name, "divine proportion." Kepler made an interesting observation between φ and the right triangle. Since φ has the unusual algebraic property that $\varphi + 1 = \varphi^2$, he drew an analogy between this property and the Pythagorean Theorem, $a^2 + b^2 = c^2$, leading to the discovery of the "Kepler Triangle" which has side lengths 1, $\sqrt{\varphi}$, and φ.

Figure 10.4 Kepler Triangle



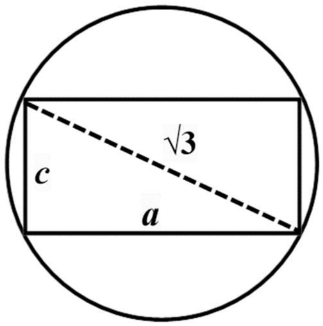
Kepler then realized this interesting fact, "if on a line which is divided in extreme and mean ratio, one constructs a right-angled triangle such that the right angle is on the perpendicular put at the section point, then the smaller leg will equal the larger segment of the divided line" [13, p28].



The Golden Ratio shows up in many other such places. Following is a clever problem explained by Huntley involving a 3-dimensional figure that he calls "the golden cuboid." He introduces it by saying "here is another example of *Phi* appearing out of the blue!" [7, p99].

The problem presented by Huntley is this: find the dimensions of a "rectangular parallelpiped of unit volume which has a diagonal two units in length" [7, p98]. He begins with the base of the figure, shown below, inscribed in the unit sphere.

Figure 10.5 Base of Golden Cuboid



Let the lengths of the edges be *a, b, c*. Then

$$a \cdot b \cdot c = 1 \qquad \text{(i)}$$

$$\sqrt{a^2 + b^2 + c^2} = 2 \qquad \text{(ii)}$$

Without loss of generality, let $b = 1$. Then $a^2 + c^2 = 3$. From Figure 10.5, $a \cdot c$ has a maximum value when $a = c = \sqrt{3/2}$ so $a \cdot c$ may have any value from zero to 3/2.
Since from (i), $c = 1/a$ and substituting into (ii) results in the equation,

$$a^2 + 1/a^2 = 3 \ , \quad \text{i.e.,} \quad a^4 - 3a^2 + 1 = 0 \, .$$

Then,

$$a^2 = \frac{3+\sqrt{5}}{2} = 1 + \varphi = \varphi^2$$

$$a = \varphi.$$

Since from (i),

$$c = 1/\varphi,$$

The required ratios must be

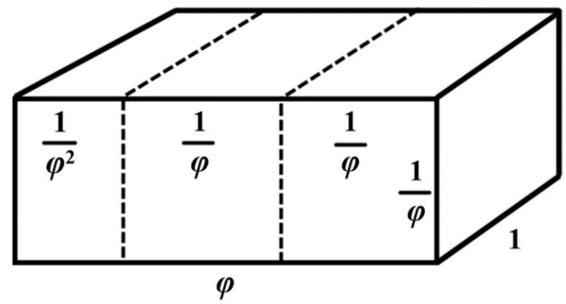
$$a : b : c = \varphi : 1 : 1/\varphi.$$

It is then verified that

$\varphi \cdot 1 \cdot \varphi^{-1} = 1$, the volume from (i), and

$\sqrt{\varphi^2 + 1 + \varphi^{-2}} = 2$, the diagonal from (ii).

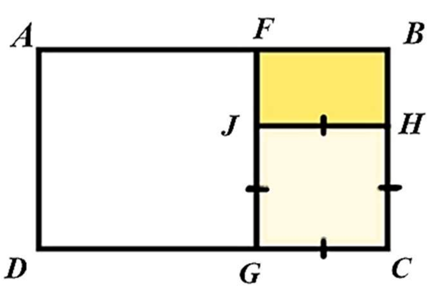
Figure 10.6 The Golden Cuboid



Huntley then enumerates four impressive properties of the Golden Cuboid explaining that with diligence, more properties might be revealed:

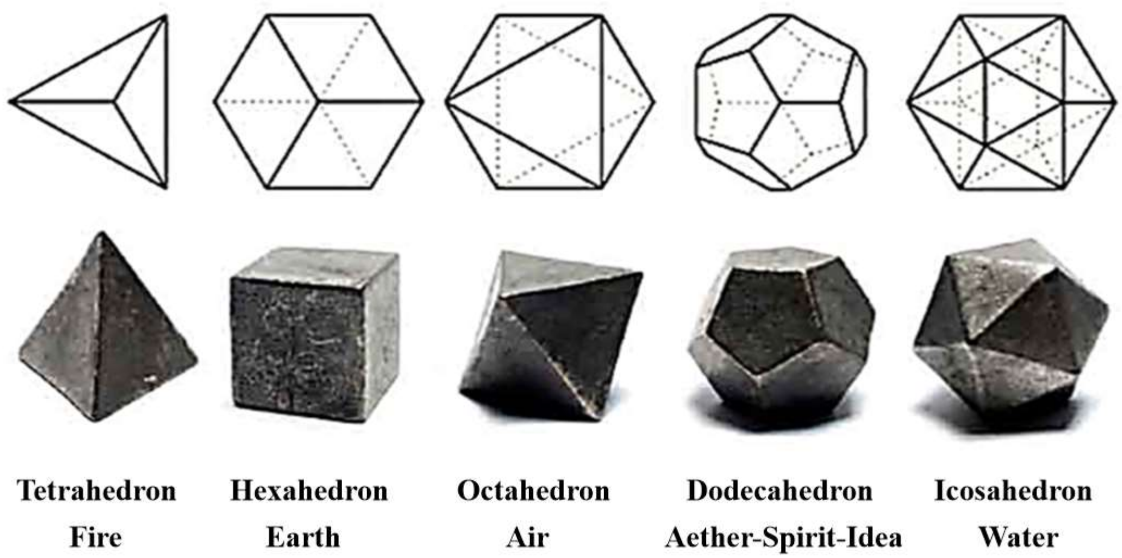
1. The lengths of the edges and the areas of the faces are in geometric progression: $\varphi^{-1}:1:\varphi = 1:\varphi:\varphi^2$.
2. Four of the six rectangles are golden rectangles.
3. While its volume is that of a unit cube, the total surface area of the golden cuboid is $2(\varphi + 1 + \varphi^{-2}) = 4$.
4. The ratio of the area of a sphere circumscribing it to that of the cuboid is π–an interesting result. [7, p99]

Perhaps the *most* fascinating property of the Golden Cuboid relates back to the Golden Rectangle. Recall that when a square is lopped off of a Golden Rectangle, another Golden Rectangle is created (and this this process repeated indefinitely until the rectangle becomes a point will always yield the same result). It turns out that the area of the resulting smaller rectangle will always be φ^{-2} times the area of the original, e.g., the area of rectangle *ABCD* multiplied by φ^{-2} will produce the area of rectangle *FBCG.* If *two* cuboids of square cross-section are cut from the Golden Cuboid (see dashed lines in Figure 10.6), the edges of the remaining cuboid will have measures in the same ratio as the original! Again, this process can be repeated with the same result until the cuboid encloses a "limiting point" [7, p100].



Propositions 13–17 in Euclid's *Elements* Book XIII present descriptions of the five Platonic solids which are the tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron. The Platonic solids "are associated with the name of Plato because of his efforts to relate them to the important entities of which the world is made" [7, p31]. In his dialogue, *Timaeus*, Plato elucidates his theory that everything in the universe is composed of some combination of these elements. Since at least the time of Euclid, it has been known that there are five, and no more than five, Platonic solids.

Platonic Solids



[**Figure 10.7**. Image downloaded from https://aeqai.com/main/2013/12/geometrically-ordered-design-the-solids-of-plato/ in July, 2020]

The Platonic solids are defined to have the following three properties:

1. All faces are congruent polygons.
2. The same number of faces meet at each vertex.
3. The same number of edges meet at each vertex.

Leonhard Euler was an 18th century Swiss mathematician, and physicist who made many important contributions in various areas of mathematics and is considered to be one of the founders of pure mathematics. Euler's topological formula, $F + V = E + 2$, relates the number of faces, vertices, and edges of the various polyhedra. This formula is used to prove that there are only five Platonic solids.

Polyhedron	Faces	Edges	Vertices	Verify Euler's Formula
Tetrahedron	4 faces	6 edges	4 vertices	4 + 4 = 6 + 2
Hexahedron	6 faces	12 edges	8 vertices	6 + 8 = 12 + 2
Octahedron	8 faces	12 edges	6 vertices	8 + 6 = 12 + 2
Dodecahedron	12 faces	30 edges	20 vertices	12 + 20 = 30 + 2
Icosahedron	20 faces	30 edges	12 vertices	20 + 12 = 30 + 2

Theorem: *There are exactly five Platonic solids.*

Proof:

Observe that $\chi(S^2) = 2 = V - E + F$ regardless of triangulation. If q = the number of edges that meet at each vertex and p = the number of edges in each face, then for any regular/Platonic tiling/triangulation of S^2, V is related to E: $qV = 2E$, F is related to E: $pF = 2E$.

$$\chi(S^2) = 2 = V - E + F$$

$$\frac{2E}{q} - E + \frac{2E}{p} = 2 \qquad p \geq 3, q \geq 3,$$

$$\frac{2E}{2E \cdot q} - \frac{E}{2E} + \frac{2E}{2E \cdot p} = \frac{2}{2E}$$

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{1}{e}$$

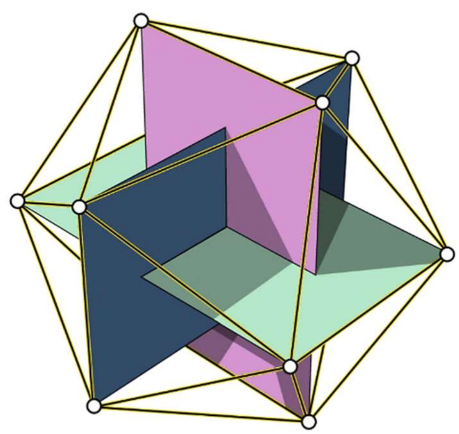
$$\frac{1}{q} - \frac{1}{e} + \frac{1}{p} = \frac{1}{2}$$

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{e} > \frac{1}{2}$$

Checking the p and q values for each solid verifies that these are the only combinations of values that yield results less than 1/2, therefore there can only be five Platonic solids ■ [19, p62]

That the Platonic solids could possibly represent all of the entities that everything in the universe is made of meant that they were held in the highest esteem by the Greeks. The characteristics of the polyhedra emphasize their significance as foundational elements of the natural world. Five being the number of total polyhedra is significant in itself. The Greeks associated the even number, two, with masculinity and the odd number, three, with femininity. These two numbers "taken together comprise the principle and sources of generation" [15, p23].

The pentagonal faces of the dodecahedron have an obvious connection to the Golden Ratio which also makes several other appearances in the features of the polyhedra. An icosahedron can be inscribed inside an octahedron in such a way that each vertex of the icosahedron divides the edge of the octahedron into golden proportions. Golden rectangles can be observed in the icosahedron when three coplanar groups of four vertices are connected by perpendicular quadrilaterals (see Figure 10.8 below). Similarly, three perpendicular golden rectangles are formed when the "centroids of the twelve pentagonal faces of a dodecagon, divisible into three coplanar groups of four," are connected by quadrilaterals [7, p33]. These unexpected appearances of the Golden Ratio contribute to the "aura of mystery" surrounding the Platonic solids. [7, p31].



[**Figure 10.8**. The Icosahedron. Retrieved from archimedes-lab.com/wp/2020/03/03/ icosahedron -with-golden-ratio-cross-sections/ in July, 2020]

These are just a very few examples of how the Golden Ratio expresses itself in Geometry. Discovering additional appearances is a rich topic for further inquiry.

XI. CONCLUSION

Manifestations of the Golden Ratio in nature, culture, and the universe itself have been explored through numerous and varied examples. There are, of course, many more but these provide a glimpse into the types of coincidences and connections that have intrigued humans for 25 centuries. Huntley asserts that the Golden Ratio, or divine proportion, was an idea "that appealed strongly to the aesthetic sensibilities of the ancient Greeks from the time of Pythagoras" [7, p2]. The Golden Ratio's aesthetic appeal is just as overwhelming today. It is as mysterious and charming as it has ever been, and its beauty persists undiminished.

Huntley also claims that the perception of beauty is a psychological experience, dependent on two factors, one biologically inherited and instinctual, the other acquired through education. He quantifies this experience arguing that more learning leads to an enhanced ability for aesthetic appreciation. Education accounts for the feeling of satisfaction and awe gleaned from working through a challenging proof, or suddenly realizing that a problem has a deft solution. In his paper, "The Golden Section in Beauty and Nature," Ulrich Lüttge probes the other side of the perception of beauty, the part that is inborn. Through research in neuroscience, it is known that humans are more likely to perceive beauty in symmetry [16, p103]. Though the Golden Ratio makes manifest balance and order in a physical form, perhaps this is an oversimplification. Lüttge asks "do we find [the Golden Ratio] beautiful because our brains are adapted to it, or is it a link between science and a transcendental dimension?" [12, p99]. He argues that beauty cannot be assessed through reason; it can only be a "transcendental category," and believes that "optimization is the cutting edge between science and transcendence" [9, p101].

This is not so different from the perspective of the Pythagoreans who were also particularly interested in the mysterious and spiritual qualities of the Golden Ratio. Today, as in eras since the time of the Greeks, defining beauty is an elusive matter but mathematics undeniably enhances the ability of human beings to perceive structure and harmony in the universe, and to more fully experience aesthetic pleasure related to both nature and culture. In 1509, Luca Pacioli's three-volume *De Divina Proportione* referred to itself as:

> A work necessary for all the clear-sighted and inquiring human minds, in which everyone who loves to study philosophy, perspective, painting, sculpture, architecture, music and other mathematical disciplines will find a

> very delicate, subtle and admirable teaching and will delight in diverse questions touching on a very secret science. [13, p60]

Pacioli makes clear what has just been shown, that a theory of proportions permeates various academic disciplines such as art, architecture, music, biology and anatomy. He also acknowledges that the Golden Ratio is an exemplary topic of study for "inquiring human minds." As Huntley suggests, it offers the possibility of attaining a deeper level of aesthetic appreciation through greater knowledge. That the Golden Ratio has a plethora of unique and amazing properties, and manifests itself in so many interesting and surprising places, including those most intimately linked to the universe and humanity itself, ensures that it will continue to endure through the ages as a worthwhile subject of research for the most curious minds.

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