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The Lexicographic Tolerable Robustness Concept for Uncertain Multi-Objective Optimization Problems: A Study on Water Resources Management

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Abstract: In this study, we introduce a robust solution concept for uncertain multi-objective optimization problems called the lexicographic tolerable robust solution. This approach is advantageous for the practical implementation of problems in which the solution should satisfy priority levels in the objective function and the worst performance vector of the solution obtained by the proposed concept is close to a reference point of the considered problem, within an acceptable tolerance threshold. Important properties of the solution sets of this introduced concept as well as an algorithm for finding such solutions are presented and discussed. We provide the implementation of the proposed lexicographic tolerable robust solution to improve understanding for practitioners by relying on the data of the water resources master plan for Serbia from Simonovic, 2009. Moreover, we are also concerned with the method of updating a desirable solution for fitting with the preferences when compromising of the multiple groups of decision makers is needed.

Keywords: uncertain multi-objective optimization; robustness concept; minmax robustness; lexicographic tolerable robust; water resources management planning

1. Introduction

Water is an essential resource for all life on the planet. Consequently, one of the biggest concerns for our water-based resources in the future is the sustainability of the current and future water resources allocation, see [1]. As water becomes more scarce in the future, the importance of how it is managed grows vastly in this research area. Finding a balance between what is needed by humans and what is needed in the environment is an important step in the sustainability of water resources. Of course, the goals of water resource planning usually involve balancing water demands and available water resources, whereas many other objectives need to be achieved, depending on the particular water system structures. Due to conflicts between multiple goal requirements, multiple criteria decision-making techniques are useful tools to explore different management options. To handle these kind of water resources management planning, many researchers adopted the techniques of multi-objective optimization to solve the problem. One may see more details in [2–6]

Actually, there is much evidence that many real-world problems are multi-faceted, occurring with multiple objectives needing to be achieved and those multiple objectives often conflict with each other, as illustrated in [7,8]. At the same time, relevant input data can sometimes be imprecise and unreliable, suffering from measurement errors or involving unknown future values, see [9,10]. These two issues have been studied in the areas of multi-objective optimization and robust optimization, respectively. Since real world problems often involve both multiple objectives and imprecise input data concurrently, there is need for the development of a concept that can address combinations of multi-objective optimization problems and robust optimization problems. Some of the prominent examples of such combinations include water resources management planning, as in [11], timetable information systems [12–14], portfolio problems [15], and flight route planning, for an overview see [16,17].

Some of the first research done in the area of uncertain multi-objective optimization was the solution concept introduced by Deb and Gupta [18] in 2006. They replaced the objective vector in a given uncertain multi-objective optimization problem with the mean effective functions computed by averaging a representative set of neighboring solutions, thereby removing the uncertainly and converting the problem to just a deterministic multi-objective optimization problem. Then, an efficient solution for that deterministic multi-objective optimization problem is considered as a robust solution for the full original uncertain multi-objective optimization problem. On the other hand, instead of using the concept of mean effective functions, Kuroiwa and Lee [19] reformulated uncertain multi-objective optimization problems by replacing the objective vector in the original problem with a vector consisting of the worst case scenario of each respective component in order to obtain a deterministic multi-objective optimization problem. Kuroiwa and Lee's approach is closely connected to the classical minmax robustness concept for single objective optimization problems, which was first introduced by Soyster [20] and subsequently extensively studied by Ben-Tal and Nemirovski [21], and Ben-Tal et al. [22]. Ehrgott et al. [23] provided another interpretation of Ben-Tal and Nemirovski's robustness concept. In [23], for each feasible solution they looked at the set of objective vectors under all scenarios and compared those sets to each other, by using the concept of set relations to define minmax robustness for uncertain multi-objective optimization problems. A similar approach was introduced by Bokrantz and Fredriksson [24], who used set relations following Ehrgott's work, but replaced the set of objective vectors of a feasible solution under all possible scenarios by its convex hull. We notice that the above three concepts are concerned with minmax robustness, since they hedge against the worst case scenarios. For more on survey and analysis of different concepts of robustness for uncertain multi-objective optimization, ones may see in [25].

On the other hand, in the single objective setting, there are various concepts which aim to overcome the limitations of strict conservativeness of the minmax concept. One such interesting concept has been introduced by Snyder [26], who investigated problems of facility location, the P-robustness solution. A solution is called P-robust if in each possible scenario, the objective value is close to the optimal value of that scenario up to P, where P is the acceptable percentage deviation from optimality. Recently, another alternative robustness concept has been proposed by Kalaï et al. [27], who came up with the concept of a lexicographic α-robust solution, which can be seen as a combination of P-robustness and minmax robustness. A lexicographic α-robust solution is a solution whose reordered cost vector with respect to scenarios is close to the ideal vector within a given threshold. We would like to notice that the concept of this solution reduces the degree of conservatism of minmax robustness by introducing the tolerance threshold and considers not only the value of the objective functions in the worst case scenario, but all scenarios, sorted from least to most wanted. For additional details the reader may see [10].

The current paper introduces a new concept of robustness, which begins with Kalaï et al.'s [27] lexicographic α-robust solution and extends it further, abandoning the single objective setting and utilizing instead a multiobjective setting, which is suitable for the uncertainty set which is modeled as discrete set. A solution according to this concept is called "lexicographic tolerable robust solution". After introducing the fundamentals of this new concept, properties of the solution set and also an algorithm for finding this new kind of solution will be explained. The new concept will then be demonstrated on a problem of multi-objective optimization of water resources planning with an uncertainty situation. This water resources planning problem has been selected as an example because the problem's structure is such that each of the multiple objectives carry a different priority.

2. Methodology

In this section, we introduce the concept of lexicographic tolerable robust solution for an uncertain multi-objective optimization problem.

2.1. Lexicographic Robust Solutions with Respect to the Tolerance Threshold for Uncertain Multi-Objective Optimization

An uncertain multi-objective optimization problem $\mathcal{MP}(\mathcal{U})$ is given as a family of $\{\mathcal{MP}(s) | s \in \mathcal{U}\}$ of deterministic multi-objective optimization problems

$$(\mathcal{MP}(s)) \quad \min \quad f(x,s) \quad \text{subject to } x \in X \tag{1}$$

with the objective function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$, feasible set $X \subseteq \mathbb{R}^n$, and uncertainty set $\mathcal{U}$. An element $s \in \mathcal{U}$ indicates a particular value for the uncertain parameters belonging in an uncertainty set $\mathcal{U}$. That is, uncertainty is in the objective only, not in the constraints or X.

Before we are going to the approach of lexicographic tolerable robust solution, let us recall the important definition of ordering the values nonincreasingly which will be used throughout this work. We note that the specific notations which will be used throughout this work are shown in Table A1 Notations.

Definition 1. *The sort function, $sort(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$, is a function that reorders the component of each vector on $\mathbb{R}^p$ in a nonincreasing way. That is,*

$$sort(y) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p)}), \text{ for all } y \in \mathbb{R}^p \tag{2}$$

where σ is a permutation on I_p such that $y_{\sigma(1)} \geq y_{\sigma(2)} \geq \cdots \geq y_{\sigma(p)}$. In this case, we will write $sort(y) =: (sort_1(y), sort_2(y), \ldots, sort_p(y))$.

From now on, we let $\mathcal{U} = \{s_1, s_2, \ldots, s_q\}$ be the finite set of possible scenarios and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$ be the considered vector-valued function. For each $x \in \mathbb{R}^n$ and for each $i \in I_p$, we put

$$c^{(i)}(x) := (f_i(x, s_1), f_i(x, s_2), \ldots, f_i(x, s_q)), \tag{3}$$

where $f_i(x, s_j)$ is the value of the objective function for an alternative solution x under scenario s_j, for all $j \in I_q$. Subsequently, we put

$$\hat{c}^{(i)}(x) := \left(sort_1(c^{(i)}(x)), sort_2(c^{(i)}(x)), \ldots, sort_q(c^{(i)}(x))\right), \tag{4}$$

for each $i \in I_p$ and $x \in \mathbb{R}^n$. The notation $\hat{c}^{(i)}(x)$ is used to stand for the sorted vector of a vector $c^{(i)}(x)$. For the sake of simply, here we will write

$$\hat{c}^{(i)}(x) =: \left(\hat{c}_1^{(i)}(x), \hat{c}_2^{(i)}(x), \ldots, \hat{c}_q^{(i)}(x)\right), \tag{5}$$

for each $i \in I_p$. Accordingly, for each $j \in I_q$ and $x \in \mathbb{R}^n$, based on the above notations (3)–(5), the worst performance vector can be determined as follows:

$$worst_j(f(x, \mathcal{U})) := \left(\hat{c}_j^{(1)}(x), \hat{c}_j^{(2)}(x), \ldots, \hat{c}_j^{(p)}(x)\right). \tag{6}$$

Now we will introduce the concept of lexicographic robust solutions with respect to a tolerance threshold set for the considered uncertain multi-objective optimization problem. To do this, we start by introducing the notation $\inf\limits_{\text{with lex}} A$ which is used to stand for the infimum of a set A in $\mathbb{R}^p$ with respect to lexicographic order. That is, for $A \subseteq \mathbb{R}^p$, we let

$$x^{\inf} := \inf_{\text{with lex}} A \text{ if } x^{\inf} \leq_{lex} x, \text{ for all } x \in A$$

where $x^{\inf} \in \mathbb{R}^p$ and the notation $\leq_{lex}$ is defined as in Table A1.

Here, the concept of reference point is presented.

Definition 2. *The vector* $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$ *is called the reference point of the problem* $\mathcal{MP}(\mathcal{U})$ *if*

$$\hat{c}_j^* = \inf_{\textit{with lex}} \left\{ worst_j(f(x,\mathcal{U})) | x \in X \right\},$$

for each $j \in I_q$.

We now present the solution concept of this paper.

Definition 3. *Let* $\mathcal{MP}(\mathcal{U})$ *be an uncertain multi-objective optimization problem with the reference point* $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$*. For each* $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0,\infty)^{p\times q}$*, the set of lexicographic tolerable robust solutions with respect to the tolerance threshold* α*, which will be denoted by* $LRS(\alpha)$*, is*

$$LRS(\alpha) := \bigcap_{j=1}^{q} \left\{ x \in X | worst_j(f(x,\mathcal{U})) \in \left(\hat{c}_j^* + \alpha_j\right) - \mathbb{R}^p_{\succsim} \right\}.$$

2.2. Algorithm for $LRS(\alpha)$

We begin this section with results on the tolerance threshold, results have been obtained to help decision makers in finding acceptable threshold to guarantee the nonemptyness of the solution set. In doing so, for the sake of simplicity, we use the following notations:

$$\max(x) := \max\{x_1, x_2, \ldots, x_n\},$$

and

$$x + \varepsilon := (x_1 + \varepsilon, x_2 + \varepsilon, \ldots, x_n + \varepsilon),$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $\varepsilon \in \mathbb{R}$.

Proposition 1. *Let* X *be a feasible set and* $\mathcal{MP}(\mathcal{U})$ *an uncertain multi-objective optimization problem with the corresponding reference point* $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$*. Let* $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$ *where* $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$*, for all* $j \in I_q$*, such that*

$$\alpha^{\inf} := \inf_{x\in X} \max\left(\Delta_x\right), \tag{7}$$

and $\Delta_x = \begin{bmatrix} worst_1(f(x,\mathcal{U})) - \hat{c}_1^* \\ worst_2(f(x,\mathcal{U})) - \hat{c}_2^* \\ \cdot \\ \cdot \\ \cdot \\ worst_q(f(x,\mathcal{U})) - \hat{c}_q^* \end{bmatrix} \in \mathbb{R}^{pq}.$

Then, for each $\varepsilon > 0$*, we have*

(i) $LRS(\alpha + \varepsilon) \neq \emptyset$, *and*
(ii) $LRS(\alpha - \varepsilon) = \emptyset$.

Proof. See in Appendix B.1 □

The next theorem provides a threshold vector α such that the solution set $LRS(\alpha)$ is nonempty.

Theorem 1. *(Nonemptyness) Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multi-objective optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p\times q}$, and $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ be continuous functions, for all $i \in I_p$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$ where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$, such that a threshold value $\alpha^{\inf}$ is defined as (7). If X is a compact set then $LRS(\alpha)$ is nonempty.*

Proof. See in Appendix B.2 □

By considering the Proposition 1 and Theorem 1, one can see that the tolerance threshold which is defined by (7) will be used to compute the best choice among the alternative solutions for the solution concept in Definition 3. In other words, the solution set due to Definition 3, for the considered uncertain multi-objective optimization problem $\mathcal{MP}(\mathcal{U})$, is the set $LRS(\alpha)$ when α is computed by (7). The following Theorem 2 will lead to a method for computing an element in the such set $LRS(\alpha)$.

Theorem 2. *Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multi-objective optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p\times q}$, where $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right) \in \mathbb{R}^p$, for all $j \in I_q$. Let $\alpha := (\alpha_1, \ldots, \alpha_q) \in [0, \infty)^{p\times q}$ be such that $\alpha_j := \left(\alpha_j^{(1)}, \alpha_j^{(2)}, \ldots, \alpha_j^{(p)}\right) \in \mathbb{R}^p$ for all $j \in I_q$. Then, we have*

$$\bigcap_{(i,j)\in I_p\times I_q} L_{(i,j)} \subseteq LRS(\alpha),$$

where $L_{(i,j)} = \left\{x \in X | \hat{c}_j^{(i)}(x) \leqslant \hat{c}_j^{*(i)} + \alpha_j^{(i)}\right\}$ for all $i \in I_p$ and $j \in I_q$.

Proof. See in Appendix B.3 □

Based on Theorem 2, we now suggest a method for finding a solution to the problem $\mathcal{MP}(\mathcal{U})$ in the set $LRS(\alpha)$.

Algorithm 1: Finding a solution of $\mathcal{MP}(\mathcal{U})$.

Input: Uncertain multi-objective optimization problem $\mathcal{MP}(\mathcal{U})$.
Step.1: For each fixed $j \in I_q$, find the reference point $\hat{c}_j^*$.
Step.2: Compute a tolerance threshold valued $\alpha^{\inf}$ of the problem $\mathcal{MP}(\mathcal{U})$ as defined in the Equation (7) of Proposition 1.
Step.3: For each fixed $i \in I_p$ and $j \in I_q$, compute the level set $L_{(i,j)}$ by

$$L_{(i,j)} = \left\{x \in X | \hat{c}_j^{(i)}(x) \leqslant \hat{c}_j^{*(i)} + \alpha^{\inf}\right\},$$

where $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$.
Step.4: Find an element x^* in the set

$$\bigcap_{(i,j)\in I_p\times I_q} L_{(i,j)}.$$

Output: x^* is an element of $LRS(\alpha)$, where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$, such that $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$.

Remark 1.

(i) *For each $j \in I_q$, the vector $\hat{c}_j^* \in \mathbb{R}^p$ is found by finding the value of lexicographic optimization of the deterministic multi-objective mapping $worst_j(f(\cdot,\mathcal{U})) : \mathbb{R}^n \to \mathbb{R}^p$. For information on methods for finding the reference point as in Definition 2, one may see [28].*

(ii) *Observe that the computation of value $\alpha^{\inf}$ is finding the infimum of subset of real numbers. Thus, we can apply many elementary existing methods of finding this value.*

(iii) *Under the assumptions of $f_i(\cdot, s_j)$ being continuous for all $i \in I_p$ and $j \in I_q$, together with an assumption that the feasible set X is compact, by applying Proposition A1 (see the Appendix A.1), we have $\hat{c}_j^{(i)}(\cdot)$ is also continuous for all $i \in I_p$ and $j \in I_q$. Thus, since $\hat{c}_j^{(i)}(\cdot)$ is continous, we have that the level set $L_{(i,j)}$ as defined by (8) is also a closed set. Thus, in order to finding a point in Formulation (8) and complete Step 4, we can apply many existing algorithms, we refer the reader to [29–31].*

3. Case Study

In this study we consider data from [32] and solve the problem by using the lexicographic tolerlable robust solution concept. The original problem of Water Resources Master Plan for Serbia (WRMS) is to find a suitable plan for balancing water demands and available water resources. This problem is concerned with the six alternative solutions and eight objectives as follows:

Decision factors:

- The need for municipal water supply (d_1)
- The need for industrial water supply (d_2)
- Irrigation needs (d_3)
- Hydropower generation (d_4)
- Flood protection (d_5)
- Water quality control (d_6)

Objectives:

- Regional political interest (f_1)
- Local interest (communities) (f_2)
- Negative effects on the resettlement of people (f_3)
- System reliability (f_4)
- Positive environmental effects (f_5)
- Positive effects of alternative plans on water quality (f_6)
- Total cost (f_7)
- Energy consumption (f_8)

The modelling techniques of alternative solution and the measurement of the objective function, we refer the readers to see more details in Chapter 10 of [32]. Here, six alternative solutions were created by considering the above specific factors in the planning process for the WRMS problem. Thus, the decision space is $X := \{x^1, x^2, x^3, x^4, x^5, x^6\} \subseteq \mathbb{R}^6$ where $x^k := (d_1^k, d_2^k, d_3^k, d_4^k, d_5^k, d_6^k)$ for each $k \in I_6$.

Looking at the above eight objectives, one can see that the five objectives f_1, f_2, f_4, f_5, and f_6, relate to positive outcomes that the group of decision makers naturally wants to maximize. Meanwhile, the three objectives f_3, f_7, and f_8 relate to negative outcomes that they naturally want to minimize. Notice that the first six objectives are qualitative, while the remaining two are quantitative. The quality level of the first six objectives are divided by the relative scale into five levels from being bad to being excellent as 1 to 5.

In the solution selection process, the preferences of the decision makers were collected through a set of public meetings. Since the decision makers were not able or willing to express their preferences, the planning team have to generate a number of different sets of weights to cover a broad range of decision-making positions in accordance with the relative importance of the various objectives. Since the generation of weight sets in the WRMS is obtained from ranges of decision maker's preferences, it means that the weight sets are imprecise data. Hence, these imprecise data can be seen as an uncertainty in the WRMS. So, it is reasonable to consider a robustness concept for the WRMS that is quite sensitive to preference changes of the decision makers. Six different weight sets had been presented in the WRMS, and here these six weight sets will be considered as scenarios. That is, the uncertainty set is:

$$\mathcal{U} := \{s_1, s_2, \ldots, s_6\} \subseteq \mathbb{R}^8.$$

Therefore, the WRMS problem is formulated as an uncertain multi-objective optimization problem $\mathcal{MP}(\mathcal{U})$ where $\mathcal{MP}(\mathcal{U})$ is given as a family of $\{\mathcal{MP}(s_j) | s_j \in \mathcal{U}\}$ of deterministic multi-objective optimization problems:

$$\begin{aligned} (\mathcal{MP}(s_j)) \quad \min \quad & f(x^k, s_j) \\ \text{subject to } & x^k \in X \end{aligned} \tag{8}$$

where $f : X \times \mathcal{U} \to \mathbb{R}^8$, $X = \{x^1, x^2, x^3, x^4, x^5, x^6\}$, and $\mathcal{U} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The primary data of the outcome for each alternative solution x^k in the WRMS over all scenarios are shown in Table 1.

Table 1. The objective function f of each alternative solution x^k under all scenarios s_j.

	$f(\cdot, s_1)$	$f(\cdot, s_2)$	$f(\cdot, s_3)$	$f(\cdot, s_4)$	$f(\cdot, s_5)$	$f(\cdot, s_6)$
$f(x^1, \cdot)$	$\begin{bmatrix}-0.70\\-1.40\\-1.50\\-3.60\\-5.50\\-3.00\\369.12\\19.30\end{bmatrix}$	$\begin{bmatrix}-0.80\\-1.00\\-0.90\\-2.40\\-6.50\\-2.40\\461.40\\28.95\end{bmatrix}$	$\begin{bmatrix}-1.00\\-2.00\\-3.00\\-2.00\\-5.00\\-3.00\\307.60\\19.30\end{bmatrix}$	$\begin{bmatrix}-1.40\\-2.60\\-3.00\\-2.40\\-1.50\\-2.10\\307.60\\19.30\end{bmatrix}$	$\begin{bmatrix}-0.70\\-1.40\\-1.50\\-2.20\\-9.00\\-3.00\\369.12\\19.30\end{bmatrix}$	$\begin{bmatrix}-0.70\\-1.40\\-1.50\\-3.60\\-5.50\\-3.00\\307.60\\23.16\end{bmatrix}$
$f(x^2, \cdot)$	$\begin{bmatrix}-1.40\\-1.40\\-2.00\\-3.60\\-5.50\\-3.00\\376.20\\17.60\end{bmatrix}$	$\begin{bmatrix}-1.60\\-1.00\\-1.20\\-2.40\\-6.50\\-2.40\\470.25\\26.40\end{bmatrix}$	$\begin{bmatrix}-2.00\\-2.00\\-4.00\\-2.00\\-5.00\\-3.00\\313.50\\17.60\end{bmatrix}$	$\begin{bmatrix}-2.80\\-2.60\\-4.00\\-2.40\\-1.50\\-2.10\\313.50\\17.60\end{bmatrix}$	$\begin{bmatrix}-1.40\\-1.40\\-2.00\\-2.20\\-9.00\\-3.00\\376.20\\17.60\end{bmatrix}$	$\begin{bmatrix}-1.40\\-1.40\\-2.00\\-3.60\\-5.50\\-3.00\\313.50\\21.12\end{bmatrix}$
$f(x^3, \cdot)$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-9.00\\-3.30\\-4.00\\475.08\\14.50\end{bmatrix}$	$\begin{bmatrix}-4.00\\-2.00\\-1.50\\-6.00\\-3.90\\-3.20\\593.85\\21.75\end{bmatrix}$	$\begin{bmatrix}-5.00\\-4.00\\-5.00\\-5.00\\-3.00\\-4.00\\395.90\\14.50\end{bmatrix}$	$\begin{bmatrix}-7.00\\-5.20\\-5.00\\-6.00\\-0.90\\-2.80\\395.90\\14.50\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-5.50\\-5.40\\-4.00\\475.08\\14.50\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-9.00\\-3.30\\-4.00\\395.90\\17.40\end{bmatrix}$
$f(x^4, \cdot)$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-7.20\\-4.40\\-2.00\\454.80\\13.70\end{bmatrix}$	$\begin{bmatrix}-4.00\\-1.50\\-1.50\\-4.80\\-5.20\\-1.60\\568.50\\20.55\end{bmatrix}$	$\begin{bmatrix}-5.00\\-3.00\\-5.00\\-4.00\\-4.00\\-2.00\\349.00\\13.70\end{bmatrix}$	$\begin{bmatrix}-7.00\\-3.90\\-5.00\\-4.80\\-1.20\\-1.40\\379.00\\13.70\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-4.40\\-7.20\\-2.00\\454.80\\13.70\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-7.20\\-4.40\\-2.00\\379.00\\16.44\end{bmatrix}$
$f(x^5, \cdot)$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-7.20\\-4.40\\-2.00\\446.16\\14.00\end{bmatrix}$	$\begin{bmatrix}-4.00\\-2.00\\-1.50\\-4.80\\-5.20\\-1.60\\557.70\\21.00\end{bmatrix}$	$\begin{bmatrix}-5.00\\-2.00\\-5.00\\-4.00\\-4.00\\-2.00\\371.80\\14.00\end{bmatrix}$	$\begin{bmatrix}-7.00\\-5.20\\-5.00\\-4.80\\-1.20\\-1.40\\371.80\\14.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-4.40\\-7.20\\-2.00\\446.16\\14.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.80\\-2.50\\-7.20\\-4.40\\-2.00\\371.80\\16.80\end{bmatrix}$
$f(x^6, \cdot)$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-9.00\\-3.30\\-5.00\\471.72\\14.70\end{bmatrix}$	$\begin{bmatrix}-4.00\\-1.50\\-1.50\\-6.00\\-3.90\\-4.00\\589.65\\22.05\end{bmatrix}$	$\begin{bmatrix}-5.00\\-1.50\\-5.00\\-5.00\\-3.00\\-5.00\\393.10\\14.70\end{bmatrix}$	$\begin{bmatrix}-7.00\\-3.90\\-5.00\\-6.00\\-0.90\\-3.50\\393.10\\14.70\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-5.50\\-5.40\\-5.00\\471.72\\14.70\end{bmatrix}$	$\begin{bmatrix}-3.50\\-2.10\\-2.50\\-9.00\\-3.30\\-5.00\\393.10\\17.64\end{bmatrix}$

4. Result Analysis

We will classify the above eight objectives into three groups. Group 1 concerns people ($G_1 := (f_1, f_2, f_3)$); Group 2 concerns the environment ($G_2 := (f_4, f_5, f_6)$); Group 3 concerns financial matters ($G_3 := (f_7, f_8)$). In each group, we will also consider the priority of the objectives in the group, for example in the group G_1, the regional interest (f_1) is considered as the most important one, the local interest (communities) (f_2) is considered as the second most important one, and the negative effects on the resettlement of people (f_3) is considered as the least important one of the group G_1. In this problem, we show the computation of the objective group (G_1, G_2, G_3). By applying Algorithm 1, we can obtain a lexicographic tolerable robust solution for the WRMS. Table 2 shows the information of function $\hat{c}^{(\cdot)}(x^k)$ of each alternative solution x^k which is obtained from sorting the vector of component function $\hat{c}^{(i)}(x^k)$ in nonincreasing way over all scenarios s_j, for each $i \in I_p$. Table 3 presents the *jth* worst performance vector of all alternative solutions and the reference point of this problem is $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_6^*) \in \mathbb{R}^{8\times 6}$. According to Theorem 1, we obtain the tolerance threshold $\alpha^{\inf} = 0$. Therefore, the resulting set of lexicographic robust solutions with respect to $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_6)$ is:

$$LRS(\alpha) = \{x^3\},$$

where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^8$, for all $j \in I_q$.

4.1. Solution Sets of Different Objective Priorities

In Table 4, the solution sets of the problem which are corresponding to each group of the ordered objective function are presented. One may observe that, different values for the priorities yield different solution sets.

Remark 2.

(i) *An important point to note is that by using the most robust compromise solution concept which was discussed in [32], the solution will be x^5. This means that the output from the lexicographic tolerable robust solution and the output from the most robust compromise solution concept can be quite different. Furthermore, note that the solution set derived from the most robust compromise concept will remain the same, regardless of the permutations of the components of the objective function.*

(ii) *Another solution concept is the robust efficiency, which was introduced by Ehrgott et al. [23]. Based on the data, which are considered in the WRMS problem, and following the concept of the robust efficiency concept we can see that the solution set is $\{x^1, x^2, x^3, x^4, x^5, x^6\}=\mathcal{S}$. In fact, in [23], each element of the solution set $\mathcal{S}$ can be found by applying the weighted sum scalarization method:*

$$(\mathcal{MP}(\mathcal{U}))_{w_l} \quad \min \max_{j\in\{1,2,\ldots,6\}} \sum_{i=1}^{8} w_l^{(i)} f_i(x^k, s_j) \qquad \text{subject to } x^k \in X, \tag{9}$$

where $w_l := (w_l^{(1)}, w_l^{(2)}, \ldots, w_l^{(8)}) \in \mathbb{R}^8_{\succ}$. One may use the following weight sets to consider the above single objective optimization problem $(\mathcal{MP}(\mathcal{U}))_{w_l}$:

$w_1 = (691.0782, 458.1161, 165.2403, 249.0968, 91.5001, 221.3457, 484.6561, 455.7014),$

$w_2 = (831.0456, 43.0179, 48.2109, 258.1919, 29.4128, 526.5054, 264.536, 716.3191),$

$w_3 = (224.5293, 605.5699, 649.4945, 864.5647, 341.5705, 106.62, 8.2109, 291.8441),$

$w_4 = (299.4271, 397.7614, 868.3355, 286.74, 781.3634, 129.4872, 9.4937, 891.1759),$

$w_5 = (952.469, 440.2029, 336.5277, 328.4372, 902.203, 627.7193, 22.8332, 125.6362),$

$w_6 = (733.3956, 693.8117, 796.0924, 198.2816, 8.0612, 979.1434, 37.3021, 228.8411),$

and find that the corresponding solutions of weights w_1, w_2, w_3, w_4, w_5, and w_6 are x^1, x^2, x^3, x^4, x^5, and x^6, respectively.

Table 2. The sorted vector, $\hat{c}^{(i)}(\cdot)$, of vector $c^{(i)}(\cdot)$.

	$\hat{c}^{(\cdot)}(x^1)$	$\hat{c}^{(\cdot)}(x^2)$	$\hat{c}^{(\cdot)}(x^3)$	$\hat{c}^{(\cdot)}(x^4)$	$\hat{c}^{(\cdot)}(x^5)$	$\hat{c}^{(\cdot)}(x^6)$
$\hat{c}^{(1)}(\cdot)$	$\begin{bmatrix} -0.70 \\ -0.70 \\ -0.70 \\ -0.80 \\ -1.00 \\ -1.40 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -1.40 \\ -1.40 \\ -1.60 \\ -2.00 \\ -2.80 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -3.50 \\ -3.50 \\ -4.00 \\ -5.00 \\ -7.00 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -3.50 \\ -3.50 \\ -4.00 \\ -5.00 \\ -7.00 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -3.50 \\ -3.50 \\ -4.00 \\ -5.00 \\ -7.00 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -3.50 \\ -3.50 \\ -4.00 \\ -5.00 \\ -7.00 \end{bmatrix}$
$\hat{c}^{(2)}(\cdot)$	$\begin{bmatrix} -1.00 \\ -1.40 \\ -1.40 \\ -1.40 \\ -2.00 \\ -2.60 \end{bmatrix}$	$\begin{bmatrix} -1.00 \\ -1.40 \\ -1.40 \\ -1.40 \\ -2.00 \\ -2.60 \end{bmatrix}$	$\begin{bmatrix} -2.00 \\ -2.80 \\ -2.80 \\ -2.80 \\ -4.00 \\ -5.20 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.10 \\ -2.10 \\ -2.10 \\ -3.00 \\ -3.90 \end{bmatrix}$	$\begin{bmatrix} -2.00 \\ -2.80 \\ -2.80 \\ -2.80 \\ -4.00 \\ -5.20 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.10 \\ -2.10 \\ -2.10 \\ -3.00 \\ -3.90 \end{bmatrix}$
$\hat{c}^{(3)}(\cdot)$	$\begin{bmatrix} -0.90 \\ -1.50 \\ -1.50 \\ -1.50 \\ -3.00 \\ -3.00 \end{bmatrix}$	$\begin{bmatrix} -1.20 \\ -2.00 \\ -2.00 \\ -2.00 \\ -4.00 \\ -4.00 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.50 \\ -2.50 \\ -2.50 \\ -5.00 \\ -5.00 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.50 \\ -2.50 \\ -2.50 \\ -5.00 \\ -5.00 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.50 \\ -2.50 \\ -2.50 \\ -5.00 \\ -5.00 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -2.50 \\ -2.50 \\ -2.50 \\ -5.00 \\ -5.00 \end{bmatrix}$
$\hat{c}^{(4)}(\cdot)$	$\begin{bmatrix} -2.00 \\ -2.20 \\ -2.40 \\ -2.40 \\ -3.60 \\ -3.60 \end{bmatrix}$	$\begin{bmatrix} -2.00 \\ -2.20 \\ -2.40 \\ -2.40 \\ -3.60 \\ -3.60 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -5.50 \\ -6.00 \\ -6.00 \\ -9.00 \\ -9.00 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -4.40 \\ -4.80 \\ -4.80 \\ -7.20 \\ -7.20 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -4.40 \\ -4.80 \\ -4.80 \\ -7.20 \\ -7.20 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -5.50 \\ -6.00 \\ -6.00 \\ -9.00 \\ -9.00 \end{bmatrix}$
$\hat{c}^{(5)}(\cdot)$	$\begin{bmatrix} -1.50 \\ -5.00 \\ -5.50 \\ -5.50 \\ -6.50 \\ -9.00 \end{bmatrix}$	$\begin{bmatrix} -1.50 \\ -5.00 \\ -5.50 \\ -5.50 \\ -6.50 \\ -9.00 \end{bmatrix}$	$\begin{bmatrix} -0.90 \\ -3.00 \\ -3.30 \\ -3.30 \\ -3.90 \\ -5.40 \end{bmatrix}$	$\begin{bmatrix} -1.20 \\ -4.00 \\ -4.40 \\ -4.40 \\ -5.20 \\ -7.20 \end{bmatrix}$	$\begin{bmatrix} -1.20 \\ -4.00 \\ -4.40 \\ -4.40 \\ -5.20 \\ -7.20 \end{bmatrix}$	$\begin{bmatrix} -0.90 \\ -3.00 \\ -3.30 \\ -3.30 \\ -3.90 \\ -5.40 \end{bmatrix}$
$\hat{c}^{(6)}(\cdot)$	$\begin{bmatrix} -2.10 \\ -2.40 \\ -3.00 \\ -3.00 \\ -3.00 \\ -3.00 \end{bmatrix}$	$\begin{bmatrix} -2.10 \\ -2.40 \\ -3.00 \\ -3.00 \\ -3.00 \\ -3.00 \end{bmatrix}$	$\begin{bmatrix} -2.80 \\ -3.20 \\ -4.00 \\ -4.00 \\ -4.00 \\ -4.00 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -1.60 \\ -2.00 \\ -2.00 \\ -2.00 \\ -2.00 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -1.60 \\ -2.00 \\ -2.00 \\ -2.00 \\ -2.00 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -4.00 \\ -5.00 \\ -5.00 \\ -5.00 \\ -5.00 \end{bmatrix}$
$\hat{c}^{(7)}(\cdot)$	$\begin{bmatrix} 461.40 \\ 369.12 \\ 369.12 \\ 307.60 \\ 307.60 \\ 307.60 \end{bmatrix}$	$\begin{bmatrix} 470.25 \\ 376.20 \\ 376.20 \\ 313.50 \\ 313.50 \\ 313.50 \end{bmatrix}$	$\begin{bmatrix} 593.85 \\ 475.08 \\ 475.08 \\ 395.90 \\ 395.90 \\ 395.90 \end{bmatrix}$	$\begin{bmatrix} 568.50 \\ 454.80 \\ 454.80 \\ 379.00 \\ 379.00 \\ 379.00 \end{bmatrix}$	$\begin{bmatrix} 557.70 \\ 446.16 \\ 446.16 \\ 371.80 \\ 371.80 \\ 371.80 \end{bmatrix}$	$\begin{bmatrix} 589.65 \\ 471.72 \\ 471.72 \\ 393.10 \\ 393.10 \\ 393.10 \end{bmatrix}$
$\hat{c}^{(8)}(\cdot)$	$\begin{bmatrix} 28.95 \\ 23.16 \\ 19.30 \\ 19.30 \\ 19.30 \\ 19.30 \end{bmatrix}$	$\begin{bmatrix} 26.40 \\ 21.12 \\ 17.60 \\ 17.60 \\ 17.60 \\ 17.60 \end{bmatrix}$	$\begin{bmatrix} 21.75 \\ 17.40 \\ 14.50 \\ 14.50 \\ 14.50 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} 20.55 \\ 16.44 \\ 13.70 \\ 13.70 \\ 13.70 \\ 13.70 \end{bmatrix}$	$\begin{bmatrix} 21.00 \\ 16.80 \\ 14.00 \\ 14.00 \\ 14.00 \\ 14.00 \end{bmatrix}$	$\begin{bmatrix} 22.05 \\ 17.64 \\ 14.70 \\ 14.70 \\ 14.70 \\ 14.70 \end{bmatrix}$

Table 3. The *jth* worst performance vector of each alternative solution x^k and the ideal point $\hat{c}_j^*$.

	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$	$worst_3(f(\cdot,\mathcal{U}))$	$worst_4(f(\cdot,\mathcal{U}))$	$worst_5(f(\cdot),\mathcal{U}))$	$worst_6(f(\cdot,\mathcal{U}))$
$worst_{(\cdot)}(f(x^1,\mathcal{U}))$	$\begin{bmatrix} -0.70 \\ -1.00 \\ -0.90 \\ -2.00 \\ -1.50 \\ -2.10 \\ 461.40 \\ 28.95 \end{bmatrix}$	$\begin{bmatrix} -0.70 \\ -1.40 \\ -1.50 \\ -2.20 \\ -5.00 \\ -2.40 \\ 369.12 \\ 23.16 \end{bmatrix}$	$\begin{bmatrix} -0.70 \\ -1.40 \\ -1.50 \\ -2.40 \\ -5.50 \\ -3.00 \\ 369.12 \\ 19.30 \end{bmatrix}$	$\begin{bmatrix} -0.80 \\ -1.40 \\ -1.50 \\ -2.40 \\ -5.50 \\ -3.00 \\ 307.60 \\ 19.30 \end{bmatrix}$	$\begin{bmatrix} -1.00 \\ -2.00 \\ -3.00 \\ -3.60 \\ -6.50 \\ -3.00 \\ 307.60 \\ 19.30 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -2.60 \\ -3.00 \\ -3.60 \\ -9.00 \\ -3.00 \\ 307.60 \\ 19.30 \end{bmatrix}$
$worst_{(\cdot)}(f(x^2,\mathcal{U}))$	$\begin{bmatrix} -1.40 \\ -1.00 \\ -1.20 \\ -2.20 \\ -1.50 \\ -2.10 \\ 470.25 \\ 26.40 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -1.40 \\ -2.00 \\ -2.20 \\ -5.00 \\ -2.40 \\ 376.20 \\ 17.60 \end{bmatrix}$	$\begin{bmatrix} -1.40 \\ -1.40 \\ -2.00 \\ -2.40 \\ -5.50 \\ -3.00 \\ 376.20 \\ 17.60 \end{bmatrix}$	$\begin{bmatrix} -1.60 \\ -1.40 \\ -2.00 \\ -2.40 \\ -5.50 \\ -3.00 \\ 313.50 \\ 17.60 \end{bmatrix}$	$\begin{bmatrix} -2.00 \\ -2.00 \\ -4.00 \\ -3.60 \\ -6.50 \\ -3.00 \\ 313.50 \\ 17.60 \end{bmatrix}$	$\begin{bmatrix} -2.80 \\ -2.60 \\ -4.00 \\ -3.60 \\ -9.00 \\ -3.00 \\ 313.50 \\ 17.60 \end{bmatrix}$
$worst_{(\cdot)}(f(x^3,\mathcal{U}))$	$\begin{bmatrix} -3.50 \\ -2.00 \\ -1.50 \\ -5.00 \\ -0.90 \\ -2.80 \\ 593.85 \\ 21.75 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -5.50 \\ -3.00 \\ -3.20 \\ 475.08 \\ 17.40 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -6.00 \\ -3.30 \\ -4.00 \\ 475.08 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -2.80 \\ -2.50 \\ -6.00 \\ -3.30 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -4.00 \\ -5.00 \\ -9.00 \\ -3.90 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -7.00 \\ -5.20 \\ -5.00 \\ -9.00 \\ -5.40 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$
$worst_{(\cdot)}(f(x^4,\mathcal{U}))$	$\begin{bmatrix} -3.50 \\ -1.50 \\ -1.50 \\ -4.00 \\ -1.20 \\ -1.40 \\ 568.50 \\ 20.55 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.10 \\ -2.50 \\ -4.40 \\ -4.00 \\ -1.60 \\ 454.80 \\ 16.44 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.10 \\ -2.50 \\ -4.80 \\ -4.40 \\ -2.00 \\ 454.80 \\ 13.70 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -2.10 \\ -2.50 \\ -4.80 \\ -4.40 \\ -2.00 \\ 379.00 \\ 13.70 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -3.00 \\ -5.00 \\ -7.20 \\ -5.20 \\ -2.00 \\ 379.00 \\ 13.70 \end{bmatrix}$	$\begin{bmatrix} -7.00 \\ -3.90 \\ -5.00 \\ -7.20 \\ -7.20 \\ -2.00 \\ 379.00 \\ 13.70 \end{bmatrix}$
$worst_{(\cdot)}(f(x^5,\mathcal{U}))$	$\begin{bmatrix} -3.50 \\ -2.00 \\ -1.50 \\ -4.00 \\ -1.20 \\ -1.40 \\ 557.70 \\ 21.00 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -4.40 \\ -4.00 \\ -1.60 \\ 444.16 \\ 16.80 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -4.80 \\ -4.40 \\ -2.00 \\ 446.16 \\ 14.00 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -2.80 \\ -2.50 \\ -4.80 \\ -4.40 \\ -2.00 \\ 371.80 \\ 14.00 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -4.00 \\ -5.00 \\ -7.20 \\ -5.20 \\ -2.00 \\ 371.80 \\ 14.00 \end{bmatrix}$	$\begin{bmatrix} -7.00 \\ -5.20 \\ -5.00 \\ -7.20 \\ -7.20 \\ -2.00 \\ 371.80 \\ 14.00 \end{bmatrix}$
$worst_{(\cdot)}(f(x^6,\mathcal{U})$	$\begin{bmatrix} -3.50 \\ -1.50 \\ -1.50 \\ -5.00 \\ -0.90 \\ -3.50 \\ 589.65 \\ 22.05 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.10 \\ -2.50 \\ -5.50 \\ -3.00 \\ -4.00 \\ 471.72 \\ 17.64 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.10 \\ -2.50 \\ -6.00 \\ -3.30 \\ -5.00 \\ 471.72 \\ 14.70 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -2.10 \\ -2.50 \\ -6.00 \\ -3.30 \\ -5.00 \\ 393.10 \\ 14.70 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -3.00 \\ -5.00 \\ -9.00 \\ -3.90 \\ -5.00 \\ 393.10 \\ 14.70 \end{bmatrix}$	$\begin{bmatrix} -7.00 \\ -3.90 \\ -5.00 \\ -9.00 \\ -5.40 \\ -5.00 \\ 393.10 \\ 14.70 \end{bmatrix}$
$\hat{c}_j^*$	$\begin{bmatrix} -3.50 \\ -2.00 \\ -1.50 \\ -5.00 \\ -0.90 \\ -2.80 \\ 593.85 \\ 21.75 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -5.50 \\ -3.00 \\ -3.20 \\ 475.08 \\ 17.40 \end{bmatrix}$	$\begin{bmatrix} -3.50 \\ -2.80 \\ -2.50 \\ -6.00 \\ -3.30 \\ -4.00 \\ 475.08 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -4.00 \\ -2.80 \\ -2.50 \\ -6.00 \\ -3.30 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -5.00 \\ -4.00 \\ -5.00 \\ -9.00 \\ -3.90 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$	$\begin{bmatrix} -7.00 \\ -5.20 \\ -5.00 \\ -9.00 \\ -5.40 \\ -4.00 \\ 395.90 \\ 14.50 \end{bmatrix}$

Table 4. The $LRS(\alpha)$ solution set for the WRMS problem in each ordered objective group where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_6)$ and $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^8$ for all $j \in I_q$.

The Ordered Objective Groups	The $LRS(\alpha)$ Solution
(G_1, G_2, G_3)	x^3
(G_1, G_3, G_2)	x^5
(G_2, G_1, G_3)	x^6
(G_2, G_3, G_1)	x^6
(G_3, G_1, G_2)	x^1
(G_3, G_2, G_1)	x^1

4.2. Further Discussion

4.2.1. Ranking of Solution

In practice, the process of selecting a final solution for the considered problem usually involves multiple decision makers. Furthermore, there may occur the situation that some decision makers are not satisfied with the solution found by the lexicographic tolerable robust solution concept. Consequently, we may need to find more desirable solutions to offer those decision makers. Reasonably, in order to update the solution for fitting the preference or requirements of those decision makers, the monotonicity of the solution set is a vital property that the presented solution concept must satisfy. The following statement describes the monotonicity property of the solution set $LRS(\alpha)$.

Property 1 (Monotonicity). *The set $LRS(\alpha)$ is monotonic in the tolerance threshold set. That is, for $\alpha := (\alpha_1, \ldots, \alpha_q), \beta := (\beta_1, \ldots, \beta_q) \in \mathbb{R}^{p \times q}$ such that $\alpha_j \precsim \beta_j$, for all $j \in I_q$, we have*

$$LRS(\alpha) \subseteq LRS(\beta).$$

Proof. The proof directly follows from Definition 3. □

Remark 3. *Property 1 means that once the tolerance threshold set has been adjusted using small tolerance threshold values it will also function correctly with larger tolerance threshold values. In other words, a lexicographic robust solution correctly adjusted with low tolerance threshold values will remain a lexicographic robust solution even when the tolerance threshold values are high.*

Continuing from above discussion, in order to update the solution sets, the ranking concept needs to be considered. Here, we consider the common natural idea for the ranking of different sets as we shall begin with computing the smallest tolerance threshold such that the set $LRS(\alpha)$ is nonempty (see Theorem 1) and define it to be a tolerance threshold set of the first ranking of solution set. After that, the next ranking of the solution set can be computed by removing all elements that belong to the first ranking of solution set from the feasible set. This mentioned idea is encouraged by the following Theorem 3, which we present in the suitation that the feasible solution set X is finite.

Theorem 3. *Let X be a finite set and $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ be continuous functions for all $i \in I_p$. For each $m \in \{2, 3, \ldots, q\}$, let α^m defined by*

$$\alpha^m := \min_{x \in X \setminus LRS((\alpha^{m-1}, \ldots, \alpha^{m-1}), \ldots, (\alpha^{m-1}, \ldots, \alpha^{m-1}))} \max(\Delta_x), \tag{10}$$

where $\alpha^1 := \min_{x \in X} \max(\Delta_x)$. Then, for any $\beta \in [\alpha^m, \alpha^{m+1})$, we have

$$LRS((\beta, \ldots, \beta), \ldots, (\beta, \ldots, \beta)) = LRS((\alpha^m, \ldots, \alpha^m), \ldots, (\alpha^m, \ldots, \alpha^m)).$$

Proof. See in Appendix B.4 □

It should be noted that the lexicographic tolerable robust solution depending on the choice of tolerance threshold α. Theorem 3 provides a sufficient condition on how to choose the effective tolerance threshold for classifying the ranking on the solution set. The resulting solution set will remain the same set as the previous ranking of the solution set if a tolerance threshold does not reach to at least a value which was computed by (10). For more understanding on Theorem 3, we illustrate with the following remark.

Remark 4. *Considering again the data of the WRMS problem and suppose the situation that the solution choice x^3 does not satisfy the group of decision makers. Note that the alternative solution x^3 is considered as the first ranking of solution set. The other rankings of solution set are presented in the following Table 5.*

We now describe the computations for obtaining the results which are presented in Table 5. The value of the tolerance threshold for each ranking of solution set is computed according to Theorem 3. The tolerance threshold α_j^2 $(j \in \{1, 2, \ldots, 6\})$ for computing the second ranking is

$$\alpha_j^2 = (1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3) \in \mathbb{R}^8, \textit{ for all } j \in \{1, 2, \ldots, 6\}.$$

Subsequently, the solution set that is associated to the tolerance threshold α^2, where $\alpha^2 := (\alpha_1^2, \alpha_2^2, \ldots, \alpha_6^2) \in \mathbb{R}^{8\times 6}$ is $\{x^3, x^6\}$. Thus, as discussed above, we will say that the solution set of the second ranking is $\{x^6\}$.

Next, using again the Theorem 3, we can found that

$$\alpha_j^3 = (2, 2, 2, 2, 2, 2, 2, 2) \in \mathbb{R}^8, \textit{ for all } j \in \{1, 2, \ldots, 6\}.$$

Furthermore, the corresponding solution set of this tolerance threshold is $\{x^3, x^4, x^5, x^6\}$. So, we say that the third ranking of solution set is $\{x^4, x^5\}$. By continuing this idea, the rest of rankings of solution set can be computed and obtained as showing in Table 5.

Table 5. The set $LRS(\alpha^i)$ for the (G_1, G_2, G_3) objective group with respect to different tolerence threshold sets where $\alpha^i := (\alpha_1^i, \alpha_2^i, \ldots, \alpha_6^i)$.

Tolerence Threshold Set	$LRS(\alpha^i)$
$\{\alpha_j^1 = (0, 0, 0, 0, 0, 0, 0, 0) \mid \forall j = 1, \ldots, 6\}$	$\{x^3\}$
$\{\alpha_j^2 = (1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3) \mid \forall j = 1, \ldots, 6\}$	$\{x^3\} \cup \{x^6\}$
$\{\alpha_j^3 = (2, 2, 2, 2, 2, 2, 2, 2) \mid \forall j = 1, \ldots, 6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\}$
$\{\alpha_j^4 = (5.4, 5.4, 5.4, 5.4, 5.4, 5.4, 5.4, 5.4) \mid \forall j = 1, \ldots, 6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\} \cup \{x^2\}$
$\{\alpha_j^5 = (7.2, 7.2, 7.2, 7.2, 7.2, 7.2, 7.2, 7.2) \mid \forall j = 1, \ldots, 6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\} \cup \{x^2\} \cup \{x^1\}$

4.2.2. Refinement of the Tolerance Threshold

By choice of tolerance threshold α^3, the corresponding solution set of the third ranking is $\{x^4, x^5\}$, which was shown in Table 5., one may wonder whether $\{x^4, x^5\}$ are really in the same rank. Here, we consider the idea to sharpen the ranking of the solution.

It is worth to remind that the lexicographic robust solutions sets depend on the considered tolerance threshold. Moreover, by Theorem 3, it has been asserted that there is no solution set that properly lies between the $LRS(\alpha^i)$ and $LRS(\alpha^{i+1})$ when these α^i are computed by the method presented in Proposition 1. Here, the computation of tolerance threshold to determine a sub-rank among elements in the i^{th} ranking of the solution set is presented. The first sub-rank of the i^{th} ranking can be determined by computing the following formulation of the tolerance threshold :

$$\alpha^{i_1} := \inf_{\text{with lex}} \left\{ (\alpha_1^{i_1}(x), \alpha_2^{i_1}(x), \ldots, \alpha_q^{i_1}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus LRS(\alpha^{i-1}) \right\}, \tag{11}$$

where

$$\alpha_j^{i_1}(x) = \left(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \dots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\right) \in \mathbb{R}^p,$$

and $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \dots, \hat{c}_j^{*(p)}\right) \in \mathbb{R}^p$, for all $j \in I_q$. The resulting solution set $LRS(\alpha^{i_1})$ corresponding to a tolerance threshold α^{i_1} is considered as a robust solution in the first sub-rank of the i^{th} ranking of the solution set.

The process of computing a tolerance threshold to determine the second sub-rank of the i^{th} ranking of the solution set will be continued if the remaining solution set $LRS(\alpha^i) \setminus \{LRS(\alpha^{i_1}) \cup LRS(\alpha^{i-1})\}$ is nonempty. Consequently, the second sub-rank of the i^{th} ranking of the solution set can be determined by computing the following formulation of the tolerance threshold:

$$\alpha^{i_2} := \inf_{\text{with lex}} \left\{ (\alpha_1^{i_2}(x), \alpha_2^{i_2}(x), \dots, \alpha_q^{i_2}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1})\} \right\}, \quad (12)$$

where

$$\alpha_j^{i_2}(x) = \left(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \dots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\right) \in \mathbb{R}^p,$$

where $j \in I_q$. The resulting solution set $LRS(\alpha^{i_2})$ corresponding to the tolerance threshold α^{i_2} is considered as a robust solution in the second sub-rank of the i^{th} ranking of the solution set.

We will continue this process of computing the third sub-rank if $LRS(\alpha^i) \setminus \{LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2}) \cup LRS(\alpha^{i-1})\}$ is nonempty. The third sub-rank of the i^{th} ranking of the solution set is determined by the following tolerance threshold:

$$\alpha^{i_3} := \inf_{\text{with lex}} \left\{ (\alpha_1^{i_3}(x), \alpha_2^{i_3}(x), \dots, \alpha_q^{i_3}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2})\} \right\}, \quad (13)$$

where

$$\alpha_j^{i_3}(x) = \left(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \dots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\right) \in \mathbb{R}^p,$$

where $j \in I_q$. We do continue the process of computing the next sub-rank until there is $m \in \mathbb{N}$ such that

$$LRS(\alpha^i) = LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2}) \cup \cdots \cup LRS(\alpha^{i_m}).$$

In general, the above formulation of computing the tolerance threshold to determine the k^{th} sub-rank of the i^{th} ranking can be expressed as follows:

$$\alpha^{i_k} := \inf_{\text{with lex}} \left\{ (\alpha_1^{i_k}(x), \alpha_2^{i_k}(x), \dots, \alpha_q^{i_k}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1}) \cup \cdots \cup LRS(\alpha^{i_{k-1}})\} \right\} \quad (14)$$

where

$$\alpha_j^{i_k}(x) = \left(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \dots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\right) \in \mathbb{R}^p,$$

where $k \in \mathbb{N}$ and $j \in I_q$.

Remark 5. *Observe that in Remark 4 by taking a tolerance threshold,* $\alpha_j^3 = (2,2,2,2,2,2,2,2)$*, for all* $j \in I_q$*, there are two members in the third ranking of solution set that are* x^4 *and* x^5*. By applying the formulation (14) to refine the tolerance threshold to classify the sub-rank between* x^4 *and* x^5*, we obtain the corresponding sub-rank of the* 3^{rd} *ranking of the solution set as follows:*

$$LRS(\alpha^{3_1}) = \{x^5\}$$

and

$$LRS(\alpha^{3_2}) = \{x^4\}.$$

These imply that the alternative solution x^5 is considered as a robust solution in the first sub-rank and the alternative solution x^4 is considered as a robust solution in the second sub-rank of the third ranking of the solution set, respectively. This means that x^5 is more desirable than x^4.

Observe that in Remark 5, the resulting solution sets with respect to the tolerance thresholds α^{3_1} and α^{3_2} which was computed by (14) are both singleton sets. This may raise an important question by the decision makers is that the refinement tolerance threshold as computed by (14) always provides a singleton solution set. The following example will provide an affirmative conclusion that this observation does not hold in general.

Example 1. *Let $X = \{x^1, x^2, x^3\}$, and the vector-valued function f under two possible scenarios s_1 and s_2 of each alternative solution x^k be presented as Table 6.*

The sort function of each component function f_i and the jth worst performance vector of each alternative solution x^k are provided in Table 7.

According to Theorem 3, the first ranking of the solution set and the second ranking of the solution set are $LRS(\alpha^1) = \{x^1\}$, and $LRS(\alpha^2) = \{x^2, x^3\}$, where $\alpha^1 := ((0,0),(0,0))$ and $\alpha^2 := ((4,4),(4,4))$. To refine the tolerance threshold α^2, we can now apply the formulation (14) and so the tolerance threshold for determining the first sub-rank of the 2^{nd} ranking of the solution set is:

$$\alpha^{2_1} := ((1,0),(1,4)).$$

Notice that

$$worst_1(f(x^2,\mathcal{U})) \in (\hat{c}_1^* + (1,0)) + \mathbb{R}^2_{\gtrapprox} \text{ and } worst_2(f(x^2,\mathcal{U})) \in (\hat{c}_2^* + (1,4)) + \mathbb{R}^2_{\gtrapprox},$$

and

$$worst_1(f(x^3,\mathcal{U})) \in (\hat{c}_1^* + (1,0)) + \mathbb{R}^2_{\gtrapprox} \text{ and } worst_2(f(x^3,\mathcal{U})) \in (\hat{c}_2^* + (1,4)) + \mathbb{R}^2_{\gtrapprox}.$$

This mean that the first sub-rank of the 2^{nd} ranking of the solution set is the set $LRS(\alpha^{2_1}) = \{x^2, x^3\}$. Therefore, we can conclude that by using the tolerance threshold which is computed by the formulation (14), cannot guarantee the corresponding singleton solution set of the sub-rank.

Table 6. The objective function $f = (f_1, f_2)$ for each alternative solution x^k under all scenarios s_j.

	Objective Function			
Alternatives	$f_1(\cdot,s_1)$	$f_1(\cdot,s_2)$	$f_2(\cdot,s_1)$	$f_2(\cdot,s_2)$
x^1	5	6	11	2
x^2	7	6	8	6
x^3	6	7	6	7

Table 7. The function $\hat{c}^{(i)}(\cdot)$ and $worst_j(f(\cdot,\mathcal{U}))$.

Alternatives	$\hat{c}^{(1)}(\cdot)$	$\hat{c}^{(2)}(\cdot)$	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$
x^1	$(6,5)$	$(11,2)$	$(6,11)$	$(5,2)$
x^2	$(7,6)$	$(8,6)$	$(7,8)$	$(6,6)$
x^3	$(7,6)$	$(7,6)$	$(7,7)$	$(6,6)$
$\hat{c}_j^*$			$\hat{c}_1^* = (6,11)$	$\hat{c}_2^* = (5,2)$

The observation from Example 1 is that even we refine the tolerance threshold by using the formulation (14), the corresponding solution set with respect to such tolerance threshold can sometime be not singleton set. Indeed, the robust solutions which belong to the k^{th} sub-rank of the i^{th} ranking of the solution set are indifferent because the quality of these robust solutions which are computed by the lexicographic tolerable robust solution concept are the same, mean that the worst performance vectors of these robust solutions are located in an acceptable area corresponding to the tolerance threshold α^{i_k}.

Remark 6. *Notice that by choice of α^i and the formulation (14), of computing k^{th} sub-rank of the i^{th} ranking, we can see that $\alpha^{i_k} \precapprox \alpha^i$.*

5. Conclusions

This research has extended the concept of lexicographic α-robustness proposed by Kalai et al. [27] from its original use for uncertain single objective optimization problems to new uses for uncertain multi-objective optimization problems. This new concept of lexicographic robust solution works in situations of uncertainty in which the uncertainty is modelled on a discrete set of scenarios. This new approach is introduced to overcome drawbacks of the minmax robustness approach in the sense of limiting the degree of conservatism of the minmax robustness approach by introducing a tolerance threshold $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q)$. Accordingly, the resulting solution set is obtained from the proposed approach can be guaranteed the immunization of the solution when the decision-making facing of uncertainty and also each performance vector is close to the reference point within the acceptable tolerance threshold.

A numerical example, in water resources management planning based on [32] was shown to illustrate the implementation of a lexicographic tolerable robust approach to practical problems. This new approach of robust solution can be represent the actual needs of the decision makers by applying priorities to objectives, according to their priorities. Obviously, the solution sets obtained by the proposed approach depend on the order of the priority which we put on each objective. As we have seen in Section 2, the results derived from the implementation of different approaches to the same data are often provide different of solution sets.

In real world problems, the process of selecting a final solution is usually involves several decision makers. To obtain a satisfactory solution for all decision makers, the ranking of solution should be considered. Conceptually, the implementation of how to apply the mathematical formulation of computing and defining the ranking of solution set was shown in Section 4.2.1. This result of lexicographic tolerable robust solution approach could be helping the decision makers to obtain a cooperative solution in the discussions around negotiation table. In future research, it would be interesting to study the property and analysis of the performance of a lexicographic tolerable robust solution. Furthermore, the comparison between the proposed approach and other existing robustness approaches.

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Appendix A

Table A1. Notations.

Notation	Meaning
I_p	The index set $\{1, 2, \ldots, p\}$, for each $p \in \mathbb{N}$
$\mathbb{R}^p$	The vector space with p dimension
$\mathbb{N}$	Set of natural numbers
X	Feasible set in $\mathbb{R}^n$
$\mathcal{U}$	Set of uncertianty set
f	Objective function
$x \in \mathbb{R}^p$	A vector x with p coordinates, that is $x = (x_1, x_1, \ldots, x_p)$
$x \precapprox y$	$x_i \leqslant y_i$ for all $i \in I_p$
$x \preccurlyeq y$	$x_i \leqslant y_i$ for all $i \in I_p$ and $x \neq y$
$x \prec y$	$x_i < y_i$ for all $i \in I_p$
$x \leq_{lex} y$	$x_m < y_m$ where $m = \min\{k \mid x_k \neq y_k\}$
$\mathbb{R}^p_{\succapprox}$	$\{x \in \mathbb{R}^p \mid x \succapprox 0\}$
$\mathbb{R}^p_{\succcurlyeq}$	$\{x \in \mathbb{R}^p \mid x \succcurlyeq 0\}$
$\mathbb{R}^p_{\succ}$	$\{x \in \mathbb{R}^p \mid x \succ 0\}$
$\mathbb{R}^p_{\geq_{lex}}$	$\{x \in \mathbb{R}^p \mid x \geq_{lex} 0\}$
$A \subseteq \mathbb{R}^p$	A subset A of vector space $\mathbb{R}^p$
$\sup\limits_{\text{with lex}} A$	The supremum of a set A with respect to a lexicographic order; $x^{\sup} := \sup\limits_{\text{with lex}} A$ if $x^{\sup} \geq_{lex} x$, for all $x \in A$.

Appendix A.1 The Mathematical Results

Proposition A1. *Let $\mathcal{U} = \{s_1, s_2, \ldots, s_q\}$ and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a single objective function such that $f(\cdot, s_j) : \mathbb{R}^n \to \mathbb{R}$ is continuous on X, for each $j \in I_q$. Then, for each $j \in I_q$, the function $\hat{c}_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous on X.*

Proof. We will prove by induction. The result is true for the case $q = 2$, since $\hat{c}_1(\cdot) = \max\{f(\cdot, s_1), f(\cdot, s_2)\}$, and $\hat{c}_2(\cdot) = \min\{f(\cdot, s_1), f(\cdot, s_2)\}$.

Next, we assume that $f(\cdot, s_1), f(\cdot, s_2), \ldots, f(\cdot, s_k)$ are also continuous functions on $\mathbb{R}^n$ such that their corresponding sorting functions, $\hat{c}_1(\cdot), \hat{c}_2(\cdot), \ldots, \hat{c}_k(\cdot)$, are also continuous. Now, let $f(\cdot, s_{k+1})$ be a continuous function on $\mathbb{R}^n$.

Let us define the function $g_i : \mathbb{R}^n \to \mathbb{R}$ by

$$g_1(\cdot) = \max\{\hat{c}_1(\cdot), f(\cdot, s_{k+1})\} \text{ and } g_i(\cdot) = \max\{\hat{c}_i(\cdot), \min\{\hat{c}_{i-1}(\cdot), f(\cdot, s_{k+1})\}\}, \text{ for each } i \in \{2, 3, \ldots, k\}$$

and

$$g_{k+1}(\cdot) = \min\{\hat{c}_k(\cdot), f(\cdot, s_{k+1})\}.$$

Observe that we have

$$g_1(x) \geq g_2(x) \geq \cdots \geq g_k(x) \geq g_{k+1}(x),$$

for all $x \in \mathbb{R}^n$. This means $\{g_1(\cdot), g_2(\cdot), \ldots, g_k(\cdot), g_{k+1}(\cdot)\}$ is the set of sort functions for $f(\cdot, s_1), f(\cdot, s_2), \ldots, f(\cdot, s_k), f(\cdot, s_{k+1})$. Moreover, from the induction hypothesis together with the continuity of $f(\cdot, s_{k+1})$, we have g_i is a continuous function for all $i \in \{1, 2, \ldots, k, k+1\}$. This completes the proof. □

Appendix B

Appendix B.1

Proof of Proposition 1. (i) Let $\varepsilon > 0$ be given. By the definition of $\alpha^{\inf}$, there exists $x_\varepsilon \in X$ such that

$$\alpha^{\inf} \leq \max(\Delta_{x_\varepsilon}) < \alpha^{\inf} + \varepsilon.$$

For each $j \in I_q$, we write $\hat{c}_j^* = \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$. It follows that,

$$\max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(x_\varepsilon) - \hat{c}_j^{*(i)} \right\} < \alpha^{\inf} + \varepsilon,$$

for each $j \in I_q$. Subsequently, for each $j \in I_q$, we have

$$\hat{c}_j^{(i)}(x_\varepsilon) - \hat{c}_j^{*(i)} < \alpha^{\inf} + \varepsilon, \text{ for all } i \in I_p.$$

This implies that,

$$worst_j(f(x_\varepsilon, \mathcal{U})) \precsim \hat{c}_j^* + (\alpha_j + \varepsilon), \text{ for all } j \in I_q.$$

This shows that, $x_\varepsilon \in LRS(\alpha + \varepsilon)$, and the item (i) is proved.

(ii) Let $x \in X$ be arbitrary but fixed. By definition of $\alpha^{\inf}$, we know that

$$\alpha^{\inf} \leq \hat{c}_j^{(i)}(x) - \hat{c}_j^{*(i)}, \text{ for all } j \in I_q \text{ and } i \in I_p.$$

Thus, for each $\varepsilon > 0$, we must have

$$\alpha^{\inf} - \varepsilon < \hat{c}_j^{(i)}(x) - \hat{c}_j^{*(i)},$$

for all $j \in I_q$ and $i \in I_p$. This implies that $x \notin LRS(\alpha - \varepsilon)$. Since x is an arbitary element of X, we can conclude that the item (ii) is proved. □

Appendix B.2

Proof of Theorem 1. Let $n \in \mathbb{N}$ be fixed. By choosing a threshold valued $\alpha^{\inf}$ as (7), we can find $x^n \in X$ such that

$$\alpha^{\inf} \leq \max(\Delta_{x^n}) < \alpha^{\inf} + \frac{1}{n}.$$

For each $j \in I_q$, we write $\hat{c}_j^* = \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$. It follows that,

$$\max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(x^n) - \hat{c}_j^{*(i)} \right\} < \alpha^{\inf} + \frac{1}{n},$$

for each $j \in I_q$. Accordingly, for each $j \in I_q$, we have

$$\hat{c}_j^{(i)}(x^n) - \hat{c}_j^{*(i)} < \alpha^{\inf} + \frac{1}{n}, \text{ for all } i \in I_p. \tag{A1}$$

This means that,

$$worst_j(f(x^n, \mathcal{U})) \precsim \hat{c}_j^* + (\alpha_j + \frac{1}{n}), \text{ for all } j \in I_q.$$

It follows that,

$$x^n \in LRS(\alpha^{\inf} + \frac{1}{n}), \text{ for all } n \in \mathbb{N}.$$

Moreover, since X is compact and $\{x^n\} \subseteq X$, we let $\tilde{x} \in X$ and a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ be such that $x^{n_k} \to \tilde{x}$, as $k \to \infty$.

Since, for each $i \in I_p$, we have $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ are continuous functions, we know that $\hat{c}_j^{(i)}(\cdot)$ are also continuous functions, for each $j \in I_q$ (see; Appendix A.1 The Mathematical Results). These imply,

$$\hat{c}_j^{(i)}(x^{n_k}) \to \hat{c}_j^{(i)}(\tilde{x}) \text{ as } k \to \infty, \tag{A2}$$

for all $i \in I_p$ and $j \in I_q$. Using this one together with the continuity of maximum function, in view of (A1), we have

$$\max_{j \in I_q} \max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(x^{n_k}) - \hat{c}_j^{*(i)} \right\} \to \alpha^{\inf}, \text{ as } k \to \infty.$$

Thus by (A2), we obtain that

$$\max_{j \in I_q} \max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(\tilde{x}) - \hat{c}_j^{*(i)} \right\} = \alpha^{\inf}.$$

This guarantees that $\tilde{x} \in LRS(\alpha)$. This completes the proof. □

Appendix B.3

Proof of Theorem 2. Let $x \in \bigcap_{(i,j) \in I_p \times I_q} L_{(i,j)}$. This means that,

$$x \in \{z \in X | \hat{c}_j^{(i)}(z) \leqslant \hat{c}_j^{*(i)} + \alpha_j^{(i)}\}, \text{ for all } i \in I_p \text{ and } j \in I_q.$$

This implies that,

$$x \in \{z \in X | worst_j(f(z, \mathcal{U})) \precsim \hat{c}_j^* + \alpha_j\}, \text{ for all } j \in I_q.$$

Thus, it follows directly that $x \in LRS(\alpha)$ and the theorem is proved. □

Appendix B.4

Proof of Theorem 3. By the monotonicity of a solution set, the "$\supseteq$" inclusion is obvious. So, we need to show that

$$LRS((\beta, \ldots, \beta), \ldots, (\beta, \ldots, \beta)) \subseteq LRS((\alpha^m, \ldots, \alpha^m), \ldots, (\alpha^m, \ldots, \alpha^m)).$$

Suppose on the contrary, there is $\bar{x} \in LRS((\beta, \ldots, \beta), \ldots, (\beta, \ldots, \beta))$, but

$$\bar{x} \notin LRS((\alpha^m, \ldots, \alpha^m), \ldots, (\alpha^m, \ldots, \alpha^m)).$$

It means that $\bar{x} \in X \backslash LRS((\alpha^m, \ldots, \alpha^m), \ldots, (\alpha^m, \ldots, \alpha^m))$ and by the definition of α^{m+1},

$$\alpha^{m+1} \leqslant \max(\Delta_{\bar{x}}). \tag{A3}$$

This implies that, there are $j_0 \in I_q$ and $i_0 \in I_p$ such that

$$\max(\Delta_{\bar{x}}) = \hat{c}_{j_0}^{(i_0)}(\bar{x}) - \hat{c}_{j_0}^{*(i_0)}.$$

Since $\bar{x} \in LRS((\beta, \ldots, \beta), \ldots, (\beta, \ldots, \beta))$,

$$worst_{j_0}(f(\bar{x}, \mathcal{U})) \in \hat{c}_{j_0}^* + (\beta, \beta, \ldots, \beta) - \mathbb{R}^p_{\gtrapprox}.$$

By the definition of $\mathbb{R}^p_{\precsim\approx}$, it follows that,

$$worst_{j_0}(f(\bar{x},\mathcal{U})) \precsim\approx \hat{c}^*_{j_0} + (\beta, \beta, \ldots, \beta).$$

This implies that, for any $i \in I_p$,

$$\hat{c}^{(i)}_{j_0}(\bar{x}) \leqslant \hat{c}^{*(i)}_{j_0} + \beta.$$

So, for fixed $i_0 \in I_p$,

$$\hat{c}^{(i_0)}_{j_0}(\bar{x}) \quad \leqslant \quad \hat{c}^{*(i_0)}_{j_0} + \beta. \tag{A4}$$

From Equations (A3) and (A4), it follows that

$$\alpha^{m+1} \leqslant \hat{c}^{(i_0)}_{j_0}(\bar{x}) - \hat{c}^{*(i_0)}_{j_0} \leqslant \beta.$$

Which leads to a contradiction with the definition of β. Therefore, we obtain the inclusion and so

$$LRS((\beta, \ldots, \beta), \ldots, (\beta, \ldots, \beta)) = LRS((\alpha^m, \ldots, \alpha^m), \ldots, (\alpha^m, \ldots, \alpha^m)).$$

□

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Sample Availability: Samples of the compounds are available from the authors.