On two-valued minimal graphs and minimal surfaces arising from the Allen–Cahn equation

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Declaration

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Abstract

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This work is divided into two, largely independent parts. The first discusses so-called two-valued minimal graphs, and takes up the majority of the text. It concludes with a proof of a Bernstein-type theorem valid in dimension four: in this dimension entire two-valued minimal graphs are linear. In gross terms we follow a strategy similar to that used to prove the Bernstein theorem for single-valued graphs; for example we prove interior gradient and area estimates which echo those available in this classical theory. The main contrast with these historical results is the possible presence of a large set of singularities. This is exacerbated by the fact that two-valued minimal graphs do not minimise area, unlike their single-valued counterparts. As a consequence the space of surfaces which could arise as weak limits from them is potentially huge. This includes the so-called tangent and blowdown cones, which respectively approximate the infinitesimal behaviour near singular points and the asymptotic behaviour at large scales. Of special interest are a subclass we call classical cones, as they provide local models near particularly large sets of singularities. The classification of these, which we establish in dimensions up to seven, represents one of the main technical challenges of our work. In dimension four, we are able to push this further and give a proof of the aforementioned Bernstein-type theorem.

The second part deals with minimal surfaces arising from a semilinear elliptic PDE called the Allen–Cahn equation. There we prove a spectral lower bound for hypersurfaces that arise from sequences of critical points with bounded indices. In particular, the index of two-sided minimal hypersurfaces constructed using multi-parameter Allen–Cahn min-max methods is bounded above by the number of parameters used in the construction. Finally, we point out by an elementary inductive argument how the regularity of the hypersurface follows from the corresponding result in the stable case.

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Introduction

Two-valued minimal graphs. The study of minimal surfaces is one of the cornerstones of the calculus of variations, dating back at least to Euler and Lagrange. It lies at the intersection of several areas, chief among which are the analysis of PDE and geometry, including geometric measure theory. One of the results which spurred the growth in the field over the course of the twentieth century was proved by Bernstein in 1927. In [Ber27] he investigated entire solutions of the *minimal surface equation*

$$(1+|Du|^2)\Delta u - \sum_{i,j=1}^{n} D_i u D_j u D_{ij} u = 0$$

with $n = 2$, and proved that an entire solution $u \in C^2(\mathbf{R}^2)$ was necessarily affine linear. The surprising contrast with the loosely related harmonic functions is that this holds without any additional hypothesis, in particular no bound is assumed for the growth of the function.

This initial result sparked a profusion of work, among which we focus on attempts at generalisations to higher dimensions. Bernstein's approach revealed itself to be hard to adapt to cases where $n \geq 3$, so the field turned to other arguments. (In fact, his proof was later found to contain a gap, which was corrected by Hopf [Hop50a, Hop50b].) The modern treatment of the question heavily relies on geometric measure theory. Using tools from this field, one can show that the asymptotic behaviour at large scales of an entire minimal graph can be weakly approximated by its so-called *blowdown cones*. These blowdown cones—in principle there could be several—are obtained as measure-theoretic limits of homothetic rescalings of the graph. When working with single-valued functions, both the graph and its blowdown cones possess a certain area-minimising property which allows strong conclusions about the regularity of the latter [Fed70, Fle62, DG61]. In fact when $n \leq 7$ then the blowdown cones are smooth, and thus are planes [Alm66,Sim68]. This sufficiently constrains the behaviour of the graph to conclude that u is affine linear. It was subsequently shown by Bombieri–de Giorgi–Giusti [BDGG69] that there was no hope for a generalisation of this result to higher dimensions: for any $n \geq 8$ there exist non-linear entire solutions of the minimal surface equation defined on $\mathbf{R}^n$, with (necessarily) singular blowdown cones.

These methods fail when working with *two-valued functions*, the setting we are interested in. These are defined to be functions taking values in the set $\mathcal{A}_2(\mathbf{R})$ of unordered pairs of real numbers. Let $\Omega \subset \mathbf{R}^n$ be an open domain and $u : x \in \Omega \mapsto \{u_1(x), u_2(x)\} \in \mathcal{A}_2(\mathbf{R})$ be a two-valued function. By its *graph* we mean the set $G = \operatorname{graph} u \subset \Omega \times \mathbf{R}$ defined by $G = \{(x, X^{n+1}) \in \Omega \times \mathbf{R} \mid X^{n+1} = u_1(x) \text{ or } X^{n+1} = u_2(x)\}$. We say that this graph is *minimal* if it is a stationary point for the area functional, when taking deformations in a suitably large class of surfaces in $\Omega \times \mathbf{R}$. (Specifically we work with integer density countably rectifiable varifolds.) We are interested in the study of *entire* two-valued minimal graphs, that is where u is defined on the whole $\mathbf{R}^n$. Note that the existence of such entire graphs is without doubt, as the graphs of two-valued linear functions are minimal. We are therefore concerned not with the existence of these graphs, but instead with their classification with the ultimate aim to show that entire two-valued minimal graphs are necessarily linear. That being said, most of the results developed in anticipation of this hold in a local setting; for example we prove an interior gradient bound which mimics classical bounds valid for single-valued minimal graphs.

The definition above is readily generalised to the notion of Q-valued functions for any $Q \in \mathbf{Z}_{>0}$, taking values in the set $\mathcal{A}_Q(\mathbf{R})$ of unordered Q-tuples of real numbers. These multi-valued functions were introduced into geometric measure theory by Almgren, who used them in his monumental regularity theory, valid for area-minimising surfaces of any codimension [Alm00]. (This theory was more recently revisited and simplified by De Lellis–Spadaro [DLS11, DLS15, DLS14, DLS16a, DLS16b].) We exclusively use $Q = 2$, and additionally work only with scalar two-valued graphs, which have codimension one as subsets of $\mathbf{R}^{n+1}$. They are of interest mainly for the following two reasons. First off, they provide the simplest non-trivial setting in which so-called *branch points* appear. Furthermore Wickramasekera has shown that in a certain sense two-valued minimal graphs provide the canonical local picture for codimension one stationary stable integral varifolds near multiplicity two branch points, see [Wic08, Wic20].

In addition to these branch points, a two-valued minimal graph can also contain so-called *classical singularities*, in a neighbourhood of which it is immersed. These are easily seen to form an $n-1$-dimensional set. In contrast to this, obtaining a helpful bound on the size of the branch set is much harder. Using an approach based on a frequency functional, Simon–Wickramasekera [SW16] showed that the branch set has Hausdorff dimension at most $n-2$. This was later meaningfully strengthened by Krummel–Wickramasekera [KW20], who showed that in fact it is countably $n-2$–rectifiable. The presence of these singularities introduces significant challenges to the analysis of two-valued minimal graphs, which are entirely absent in the classical single-valued theory. These complications are compounded by the fact that two-valued minimal graphs do not minimise area. However they are stable, as we will prove.

It is interesting to ask whether despite these difficulties an analogue of Bernstein's theorem holds for two-valued minimal graphs, in dimensions up to $n+1=8$. (In higher dimensions this fails for what are essentially trivial reasons, using the single-valued example constructed in [BDGG69].) When $n+1=3$ this was proved by L. Rosales, who used a so-called logarithmic cutoff argument to show that the second fundamental form vanishes pointwise [Ros16]. However this argument crucially uses the quadratic area growth of the surfaces, which is false in all larger dimensions. This is the setting we set out to study, concluding with an extension of the two-valued Bernstein theorem to the case $n+1=4$.

In preparation for this we prove a number of results respectively valid in arbitrary dimensions, and in dimensions up to seven. In the first place we prove that two-valued minimal graphs are stable (that is, have non-negative second variation for the area functional) for a large class of perturbations, whose support may include the singular set of the graph. The resulting L^2-estimate for the second fundamental are then used to prove interior gradient estimates. These are obtained via an argument analogous to that given for the single-valued case in [GT98], although the proof is more involved due to the presence of the branch point singularities. After some additional work one obtains the following; see Lemma 5.1.1.

THEOREM. *Let $\alpha \in (0,1)$, $n \geq 1$ be arbitrary and $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2(\mathbf{R}))$ be an entire two-valued minimal graph. If u has bounded growth, that is*

$$\limsup_{R\to\infty} \big(\max_{D_R} \|u\|/R \big) < \infty,$$

then it is linear.

The proof of this is accomplished via a classification of possible blowdown cones of $G = \operatorname{graph} u$ at infinity. Indeed one can show that these are necessarily equal to the union of two, possibly equal n-dimensional planes. The conclusion then comes from the monotonicity formula for minimal surfaces.

In broad strokes the strategy is the same in the general case, when one does not assume the bounded growth of u, namely one proceeds by classifying the blowdown cones at infinity. However a much larger class of cones needs to be considered here. This is also a stark contrast with single-valued graphs, whose blowdown cones are greatly constrained by the requirement that they be area-minimising. In the present context the blowdown cones can be singular even in small dimensions, and generally could admit a large and complicated set of singularities, whereas in the single-valued theory the blowdown cones are necessarily smooth when $n \leq 7$. In that setting one moreover has that for $n \geq 8$ the Hausdorff dimension of the singular set is at most $n - 7$.

To study these singularities one has to consider all cones which appear as limits of arbitrary sequences of two-valued minimal graphs; we call these *limit cones*. These limit cones form a class even larger than those which are obtained by blowing down a single, fixed two-valued minimal graph. However general dimension reduction principles mean that we need only consider cones which satisfy additional structural constraints. A significant portion of our work is dedicated to the classification of so-called *classical limit cones*. By this we mean limit cones which are supported in a union of n-dimensional half-planes $\pi_1, \ldots, \pi_D$ meeting along a single $n-1$-dimensional axis. We usually denote such a cone $\mathbf{P} = \sum_i m_i |\pi_i|$, where the $m_i \in \mathbf{Z}_{>0}$ are the multiplicities of the half-planes. Our classification relies on preliminary improved area estimates. As these deteriorate with increasing dimension, our result is constrained by $n \leq 6$; see Corollary 4.4.6.

LEMMA. *Let $\alpha \in (0,1)$, $n \leq 6$ and $(u_j \mid j \in \mathbf{N})$ define a sequence of two-valued minimal graphs in $D_1 \times \mathbf{R}$, with $u_j \in C^{1,\alpha}(D_1; \mathcal{A}_2(\mathbf{R}))$. Suppose that their graphs G_j weakly converge to a classical cone $\mathbf{P} \llcorner D_1 \times \mathbf{R}$. Then $\mathbf{P}$ is a union of two n-dimensional planes.*

Via a diagonal extraction argument one deduces from this that any surface arising as a limit of two-valued minimal graphs is smoothly immersed away from a singular set of codimension two, which includes the branch set. This crucially relies on the work Wickramasekera [Wic20] and Krummel–Wickramasekera [KW20]. For those cones obtained by blowing down a single, entire two-valued minimal graph, we additionally prove a structure theorem, which demonstrates that they can be decomposed into a union of an n-dimensional plane and a cylindrical cone, which is invariant under translations in the vertical direction. This holds for arbitrary blowdown cones without any additional assumptions; in particular we do not assume they are classical. In dimension $n+1 = 4$, we combine this with integral curvature estimates and a combinatorial argument to prove the following Bernstein-type theorem for entire two-valued minimal graphs; see Theorem 5.2.1.

THEOREM. *Let $\alpha \in (0,1)$ and let $u \in C^{1,\alpha}(\mathbf{R}^3; \mathcal{A}_2(\mathbf{R}))$ be an entire two-valued minimal graph. Then u is linear.*

Minimal surfaces and the Allen–Cahn equation. In the second part we present our work on minimal surfaces arising from the so-called Allen–Cahn equation [Hie18]. This is a semilinear, second-order elliptic PDE whose solutions describe the behaviour of a two-phase solution. The equation is obtained as the Euler–Lagrange equation of the eponymous Allen–Cahn functional. It depends on a small parameter; as this tends to zero the transition region between the two phases converges (in a certain weak, measure-theoretic sense) to a minimal hypersurface. In our work we proceeded to compare the second-order variation of the Allen–Cahn functional on the one hand to that of the area functional of the limit interface on the other. We proved upper bounds for the Morse index of the limit minimal surface in terms of those of the approaching sequence of critical points of the Allen–Cahn equation. These Morse indices are obtained by counting the number of strictly negative eigenvalues of certain linear elliptic operators; for the limit this is the *Jacobi operator*. We can thus refine our estimates by translating them into bounds for the spectrum of this operator in terms of the spectra of the approaching sequence of solutions. These estimates have applications in so-called min-max constructions, which naturally produce sequences of critical points with bounded Morse index.

A classical theorem, due to the combined work of Almgren, Pitts and Schoen–Simon, asserts that for $n \geq 2$, every $(n+1)$-dimensional closed Riemannian manifold M contains a minimal hypersurface smoothly embedded away from a closed singular set of Hausdorff dimension at most $n-7$. The original proof of this theorem is based on a highly non-trivial geometric min-max construction due to Pitts [Pit81], which extended earlier work of Almgren [Alm65]. This construction is carried out directly for the area functional on the space of hypersurfaces equipped with an appropriate weak topology, and it yields in the first instance a critical point of area satisfying a certain almost-minimizing property. This property is central to the rest of the argument, and allows to deduce regularity of the min-max hypersurface from compactness of the space of uniformly area-bounded stable minimal hypersurfaces with singular sets of dimension at most $n-7$, a result proved for $2 \leq n \leq 5$ by Schoen–Simon–Yau [SSY75] and extended to arbitrary $n \geq 2$ by Schoen–Simon [SS81]. (The Almgren–Pitts min-max construction has recently been streamlined by De Lellis and Tasnady [DLT13] giving a shorter proof. However, their argument still follows Pitts' closely and is in particular based on carrying out the min-max procedure directly for the area functional on hypersurfaces.)

In recent years an alternative approach to this theorem has been developed, whose philosophy is to push the regularity theory to its limit in order to gain substantial simplicity on the existence part. Specifically, this approach differs from the original one in two key aspects: first, it is based on a strictly PDE-theoretic min-max construction that replaces the Almgren–Pitts geometric construction; second, for the regularity conclusions, it relies on a sharpening of the Schoen–Simon compactness theory for stable minimal hypersurfaces. The idea in this approach is to construct a minimal hypersurface as the limit-interface associated with a sequence of solutions $u = u_i$ to the Allen–Cahn equation

$$\Delta u - \epsilon_i^{-2} W'(u) = 0 \tag{0.1}$$

on the ambient space M, where $W\colon \mathbb{R} \to \mathbb{R}$ is a fixed double-well potential with precisely two minima at ± 1 with $W(\pm 1) = 0$. Roughly speaking, if the u_i solve (0.1) and satisfy appropriate bounds, then the level sets of u_i converge as $\epsilon_i \to 0^+$ to a stationary codimension 1 integral varifold V. This fact was rigorously established by Hutchinson–Tonegawa [HT00], using in part methods inspired by the earlier work of Ilmanen in the parabolic setting [Ilm93]. Note that u_i solves (0.1) if and only if it is a critical point of the Allen–Cahn functional

$$E_{\epsilon_i}(u) = \int_U \epsilon_i \frac{|\nabla u|^2}{2} + \frac{W(u_i)}{\epsilon_i}.$$

If the solutions u_i are additionally assumed stable with respect to E_{ϵ_i}, then Tonegawa and Wickramasekera [TW12] proved that the resulting varifold V is supported on a hypersurface smoothly embedded away from a closed singular set of Hausdorff dimension at most $n-7$, using an earlier result of Tonegawa [Ton05] which established the stability of the regular part $\operatorname{reg} V$ with respect to the area functional. Their proof of this regularity result uses the regularity and compactness theory for stable codimension 1 integral varifolds developed by Wickramasekera [Wic14a] sharpening the Schoen–Simon theory.

Stability of u_i means that the second variation of the Allen–Cahn functional E_{ϵ_i} with respect to $H^1(M)$ is a non-negative quadratic form. More generally the index $\operatorname{ind} u_i$ denotes the number of strictly negative eigenvalues of the elliptic operator $L_i = \Delta - \epsilon_i^{-2} W''(u_i)$, so that u_i is stable if and only if $\operatorname{ind} u_i = 0$. Using min-max methods for semi-linear equations, Guaraco [Gua18] recently gave a simple and elegant construction of a solution u_i to (0.1) with $\operatorname{ind} u_i \leq 1$ and $\|u_i\|_{L^\infty} \leq 1$, and such that as $\epsilon_i \to 0$, the energies $E_{\epsilon_i}(u_i)$ are bounded above and below away from 0. The lower energy bound guarantees that the resulting limit varifold V is non-trivial.

Since $\operatorname{ind} u_i \leq 1$, u_i must be stable in at least one of every pair of disjoint open subsets of M; similarly if $\operatorname{ind} u_i \leq k$ then u_i must be stable in at least one of every $(k+1)$-tuple of disjoint open sets. This elementary observation, originally due to Pitts in the context of minimal surfaces, together with a tangent cone analysis in low dimensions, allowed Guaraco to deduce the regularity of V from the results of [TW12]. More recently still, Gaspar and Guaraco [GG18] have used k-parameter min-max methods to produce sequences of critical points with Morse index at most k, for all positive integers k. Our results show that this index bound is inherited by the minimal surface arising as $\epsilon_i \to 0$, provided it has a trivial normal bundle. We also point out that the regularity follows in all dimensions from the corresponding result in the stable case via an inductive argument that avoids the tangent cone analysis used in [Gua18]. See Theorem 6.1.8, Corollary 6.1.10 and the discussion immediately following it.

COROLLARY. *Let M be a closed Riemannian manifold of dimension $n+1 \geq 3$. Let V be the integral varifold arising as the limit-interface of the sequence (u_i) of solutions to* (0.1) *constructed in* [Gua18] *(respectively in* [GG18] *using k-parameter min-max methods). Then $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$. If $\operatorname{reg} V$ is two-sided, then its Morse index with respect to the area functional satisfies $\operatorname{ind}_{\mathcal{H}^n} \operatorname{reg} V \leq 1$ (respectively $\operatorname{ind}_{\mathcal{H}^n} \operatorname{reg} V \leq k$).*

In min-max theory, one generally expects that the Morse index of the constructed critical point is no greater than the number of parameters used in the construction. The above corollary gives this result for the constructions of Guaraco and Gaspar–Guaraco, provided the arising hypersurface is two-sided. This was recently shown by Chodosh and Mantoulidis [CM20] to hold automatically when the ambient manifold M has dimension 3 and is equipped either with a bumpy metric or has positive Ricci curvature. Building on work of Wang and Wei [WW19], Chodosh–Mantoulidis prove curvature and strong sheet separation estimates, and use these to deduce that in this three-dimensional setting the convergence of the level sets occurs with multiplicity 1. They moreover show that in all dimensions, if the limiting surface has multiplicity 1, then its index is bounded *below* by the index of the u_ϵ.

This complements our upper bound for the index, which is a direct consequence of a lower bound for (λ_p), the spectrum of the elliptic operator $L_V = \Delta_V + |A|^2 + \operatorname{Ric}_M(\nu,\nu)$—the *scalar Jacobi operator*—in terms of (λ_p^i), the spectra of the operators (L_i). Establishing this spectral lower bound is our main result; see Theorem 6.1.8.

THEOREM. *Let M be a closed Riemannian manifold of dimension $n+1 \geq 3$. Let V be the integral varifold arising from a sequence (u_i) of solutions to* (0.1) *with* $\operatorname{ind} u_i \leq k$ *for some* $k \in \mathbf{N}$. *Then* $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$ *and*

(i) $\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$ *for all* $W \subset\subset M \setminus \operatorname{sing} V$ *and* $p \in \mathbf{N}$,

(ii) $\operatorname{ind}_{\mathcal{H}^n} C \leq k$ *for every two-sided connected component* $C \subset \operatorname{reg} V$.

REMARK. The spectral lower bound of (i) also holds if the assumptions on the u_i are weakened in a spirit similar to the work of Ambrozio, Carlotto and Sharp [ACS16], that is if instead of an index upper bound one assumes that for some $k \in \mathbf{N}$ there is $\mu \in \mathbf{R}$ such that $\lambda_k^i \geq \mu$ for all i. (Note that the index bound $\operatorname{ind} u_i \leq k$ is equivalent to $\lambda_{k+1}^i \geq 0$.)

REMARK. As similar result had previously been proved by Le [Le11] in ambient Euclidean space, under the additional assumption that the convergence to the limit surface occurs with multiplicity 1. Adapting the methods developed in [Le11, Le15] to ambient Riemannian manifolds, Gaspar generalised our results to the case where the limit varifold is one-sided, without any assumption on multiplicity [Gas20]. Their general approach is similar to ours but subtly different, in that they instead consider the second *inner* variation of the Allen–Cahn functional; see also the recent work of Le and Sternberg [LS19], where similar bounds are established for other examples of eigenvalue problems.

For the minimal hypersurfaces obtained by a direct min-max procedure for the area functional on the space of hypersurfaces (as in the Almgren–Pitts construction), index bounds have recently been established by Marques and Neves [MN16]. Both the Almgren–Pitts existence proof and the Marques–Neves proof of the index bounds are rather technically involved; in particular, the min-max construction in this setting has to be carried out in a bare-handed fashion in the absence of anything like a Hilbert space structure. In contrast, in the approach via the Allen–Cahn functional, Guaraco's existence proof is strikingly simple, and our proofs for the spectral bound and the regularity of V are elementary bar the fact that they rely on the highly non-trivial sharpening of the Schoen–Simon regularity theory for stable hypersurfaces as in [Wic14a].

Outline. Chapter 1 introduces basic notions from geometric measure theory which will be used throughout the remainder of the text. We start with the well-known notion of the Hausdorff measure, and use this to define Hausdorff dimension, countably rectifiable sets etc. We also briefly discuss Caccioppoli sets, as well as the capacity of sets. Our main references are [Sim84, EG15, Giu84]. In the second half of the chapter we move on to concepts like integral varifolds and currents. More specifically the focus lies on stationary varifolds, and accordingly we also list some of the consequences of the first variation formula, including the monotonicity formula. We conclude the chapter by defining so-called tangent and limit cones.

In Chapter 2 we move on to more advanced notions, mainly pertaining to the regularity theory of minimal surfaces. We include both results valid for stationary integral varifolds, chief among which are the Allard regularity theorem, and more recent work on the regularity theory of stable, stationary integral varifolds in codimension one. The result we use most frequently in the remainder is a branched sheeting theorem of Wickramasekera [Wic20] valid near branch point singularities. After this general theory, we give a short exposition of the theory of area-minimising currents, including an abbreviated history of the classical, single-valued Bernstein theorem, valid in dimensions $n + 1 \leq 8$. We conclude the chapter with a technical lemma inspired by the work of Jenkins–Serrin [JS66a, JS66b]. This is an original result, and a crucial ingredient in our classification of classical limit cones in Chapter 4.

Chapter 3 starts with some basic definitions and notation for two-valued functions in Section 3.1. The only place where we deviate slightly from the literature is by defining notation for integrals of two-valued functions, which are ubiquitous in our *a priori* estimates. In Section 3.2 we define minimality for two-valued graphs and describe some of their basic properties. Among the new results we establish is the stability of these graphs; the proof of this relies on the rectifiability of the branch set proved by Krummel–Wickramasekera [KW20]. In Section 3.3 we prove area estimates for two-valued minimal graphs. Our derivation of these inequalities echoes the argument for single-valued minimal graphs given in [GT98, Ch. 16]. These initial bounds show that homothetic rescalings of a fixed entire graph converge weakly to a so-called blowdown cone. For sequences of two-valued minimal graphs that converge in this weak sense, we obtain significantly improved estimates. These are not present in the above, and are crucial both for our classification of classical limit cones and for our proof of the two-valued Bernstein theorem in dimension four. The area bounds do not rely on the fine properties of the branch set proved by Krummel–Wickramasekera [KW20], instead the dimension bound derived by Simon–Wickramasekera [SW16] is more than sufficient. This is not so in Section 3.4, where we prove *a priori* interior gradient estimates for two-valued minimal graphs. These appear to be new in this context, although both the statements and their proofs mirror the classical, single-valued theory. We follow the presentation in [GT98, Ch. 16], and leverage the stability of the graphs to obtain integral estimates for an auxiliary function w, which measures the slope of the graph. However the singularities in the graph introduce significant complications which increase the length of the arguments. The function w is used again in the final part of the chapter. There we first prove a sort of maximum principle for w valid near branch points. This is used in the first place to establish the regularity of Lipschitz two-valued minimal graphs. This is interesting in its own right, but also underpins the analysis of entire graphs with bounded growth. In our proof we follow a strategy suggested to us by S. Becker-Kahn, whose results we also rely on in the inductive step, see [BK17].

In Chapter 4 we analyse cones which arise as weak limits of sequences of two-valued minimal graphs. From a technical point of view, the results presented in this chapter lie at the heart of our work. The simplest situation is where the limit is supported in a plane. This case is treated in Section 4.1, where we show that the plane has multiplicity at most two. This allows the application of the results of Krummel–Wickramasekera [KW20] via a diagonal convergence argument; this shows that any limit surface has a countably $n-2$–rectifiable branch set. We also study the situation where there is mass cancellation in the limit, and prove that in this case the sequence of approaching graphs must be unbranched. In Section 4.2 we make our first foray into the classical cones that may appear as limits of two-valued minimal graphs. As a first step we show that these must be equal to a sum of n-dimensional planes meeting along a common $n-1$-dimensional axis (rather than half-planes). This is proved using a method from [SS81], substituting the sheeting theorem employed there with that of [Wic20]. This drastically cuts down on the number of cases one needs to consider. Our complete classification of the classical limit cones distinguishes between cases according to whether the cone is vertical or not. The latter case is simpler, and is treated in Section 4.3. We use an entirely different argument to deal with vertical cones, of a more combinatorial nature; in particular this does not rely on a modification of the arguments of Schoen–Simon. Instead we proceed by marrying a local analysis near the limit cone with a counting argument. This is the place where we use the Jenkins–Serrin-type lemma for single-valued minimal graphs. Combined with the restrictions from our improved area estimates we can exclude all cases but that where the limit is a union of two planes each with multiplicity one. As these area estimates become ineffective when $n \geq 7$, our classification of cones is valid only for n smaller than that. In this range of dimensions, we may use a diagonal convergence argument to obtain fairly strong information about the structure of arbitrary limits of sequences of two-valued minimal graphs. Combining our classification with the results of [KW20, Wic20] we find that any such limit surface is smoothly immersed away from a countably $n-2$–rectifiable set of singularities.

Chapter 5 is the last chapter dealing with two-valued minimal graphs, concluding with a proof of the Bernstein theorem in dimension four. We start with Section 5.1, where we prove two results. The first holds in any dimension, and states that entire two-valued minimal graphs in $C^{1,\alpha}$ with bounded growth are automatically linear. The second holds in dimension at most seven, as it indirectly relies on our improved area estimates. It states that every blowdown cone can be decomposed into a sum of a plane and a cone which is translation-invariant in the vertical direction. In Section 5.2 we prove the two-valued Bernstein theorem in dimension $n+1=4$: an entire two-valued minimal graph defined on $\mathbf{R}^3$ must be linear. Our proof is built upon essentially all the results that precede it. Combining these with a well-known, so-called logarithmic cutoff argument we obtain an initial reduction which shows that all blowdown cones are sums of planes. This does not complete the proof, as these planes need not in principle meet along a common axis. To show that this is in fact the case, we use a combinatorial argument. Specifically we associate to the cone a finite directed graph, and prove that this can only contain short directed paths. This restriction excludes all but one case, where the cone is equal to a sum of two possibly equal planes. A direct application of the monotonicity formula concludes the proof.

Chapter 6 is the last chapter, where we present our work on the Allen–Cahn equation, and minimal surfaces arising from it. We show that one can derive Morse index bounds, and bounds for the eigenvalues of the Jacobi operator of a minimal surface, if one assumes analogous bounds for the sequence of solutions of the Allen–Cahn equation from which it arises. In Section 6.1 we give the statements of the main result and its corollaries. Their proof requires a number of preliminary results, which are contained in Section 6.2. The proof of the main result (Theorem 6.1.8) is in Section 6.3, and is split into two parts: in the first part we prove the spectral lower bound by an inductive argument on $\operatorname{ind} u_i$; this immediately implies the index upper bound. The proof of $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$ is given in the second part, and uses a similar inductive argument. There are two appendices: Appendix A contains two elementary lemmas from measure theory that are used repeatedly in Section 6.2. Appendix B gives a proof of Proposition 6.2.6, which is a straight-forward adaptation of an argument used by Tonegawa for the stable case.

Chapter 1

Geometric measure theory

1.1. Measure theory: basic notions

1.1.1. The Hausdorff measure. Let $k, m \in \mathbf{Z}_{\geq 0}$ be two non-negative integers, and $n \in \mathbf{Z}_{>0}$. Consider an arbitrary subset $A \subset \mathbf{R}^{n+k}$. We define

$$\mathcal{H}^m_\delta(A) = \inf \sum_{j=1}^{\infty} \omega_m (\operatorname{diam} C_j/2)^m,$$

where the infimum is taken over all countable covers $(C_j \mid j \in \mathbf{N})$ of A with $\operatorname{diam} C_j < \delta$ for all j and ω_m is the volume of the m-dimensional unit ball in $\mathbf{R}^m$. With decreasing δ, $\mathcal{H}^m_\delta(A)$ is non-decreasing. The *m-dimensional Hausdorff measure* is

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \mathcal{H}^m_\delta(A).$$

The monotonicity with respect to δ ensures the existence of this limit, and in fact means that $\mathcal{H}^m(A) = \sup_{\delta>0} \mathcal{H}^m_\delta(A)$. The Hausdorff measures are *Borel regular*, that is the Borel subsets of $\mathbf{R}^{n+k}$ are $\mathcal{H}^m$-measurable and for every subset $A \subset \mathbf{R}^{n+k}$ there is a Borel subset $B \supset A$ with $\mathcal{H}^m(A) = \mathcal{H}^m(B)$. However for $0 \leq s < n$ the Hausdorff measure $\mathcal{H}^s$ is not a Radon measure as it is not σ-finite.

One can extend this definition to non-integer exponents $s \geq 0$, essentially the only necessary change being to find a reasonable value for ω_s. For integer $m \in \mathbf{Z}_{\geq 0}$ we may notice that $\omega_m = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, where $\Gamma(s) = \int_0^\infty \mathrm{e}^{-x} x^{s-1}\,\mathrm{d}x$ is Euler's Γ-function. This formula extends to non-negative s without any changes. Let $s \leq t$. Note that when $\delta < 1$ then for any set $A \subset \mathbf{R}^{n+k}$, $\mathcal{H}^s_\delta(A) \geq \mathcal{H}^t_\delta(A)$. Passing to the limit we find that $\mathcal{H}^s(A) \geq \mathcal{H}^t(A)$.

Proposition 1.1.1 ([EG15]). *Let $0 \leq s < t < \infty$, and $A \subset \mathbf{R}^n$. Then:*

- (i) *$\mathcal{H}^0$ is the counting measure.*
- (ii) *$\mathcal{H}^n$ coincides with the Lebesgue measure on $\mathbf{R}^n$.*
- (iii) *$\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0$.*
- (iv) *$\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for every affine isometry $L : \mathbf{R}^n \to \mathbf{R}^n$.*
- (v) *If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.*
- (vi) *If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = \infty$.*

Hence we find that for any set $A \subset \mathbf{R}^{n+k}$ there is $d \geq 0$ so that $\mathcal{H}^s(A) = \infty$ for $s < d$ and $\mathcal{H}^s(A) = 0$ for $s > d$. We call $d = \inf\{0 \leq s < \infty \mid \mathcal{H}^s(A) = 0\}$ the *Hausdorff dimension* of A, and write $d = \dim_{\mathcal{H}} A$. At $s = d$ itself one might have any one of the following three possibilities: $\mathcal{H}^d(A)$ can be zero, positive or infinite. (It might come as a surprise that $\mathcal{H}^d(A) = 0$ is a possibility, although this is common in stochastic processes. For example, one can show that the trajectory of a Brownian motion in the plane almost surely is a Lebesgue null set with Hausdorff dimension two, see [MP10, Ch.4].)

The definition of a Hausdorff dimension readily generalises to an arbitrary metric space (X, d) say. The only case we are interested where X is not Euclidean space is the following. Let $\Sigma \subset \mathbf{R}^{n+k}$ be a C^1-embedded n-dimensional submanifold, with d its intrinsic distance function. Then one can take $X = \Sigma$ and define the n-dimensional Hausdorff measure on (Σ, d). This coincides with the usual notion of n-dimensional volume measure. In particular, one may write (as indeed we will in the remainder of the text) given any $f \in C^1_c(\mathbf{R}^{n+k})$, $\int_\Sigma f \,\mathrm{dvol} = \int_\Sigma f \,\mathrm{d}\mathcal{H}^n$ at least provided Σ has *locally bounded mass*, meaning $\mathcal{H}^n(\Sigma \cap K) < +\infty$ for all compact $K \subset \mathbf{R}^{n+k}$.

Similarly on $X = \mathbf{R}^n$ the n-dimensional Hausdorff measure agrees with the Lebesgue measure, and thus given $f \in C^1_c(\mathbf{R}^n)$ we write $\int f(x)\,\mathrm{d}x = \int f \,\mathrm{d}\mathcal{H}^n$.

Let (X, d) be a locally compact and separable metric space. We call *Radon measure* any measure μ on X which is Borel-regular and finite on compact subsets of X.

$$\mu(A) = \inf\{\mu(U) \mid A \subset U, U \text{ open}\}$$

for any subset $A \subset X$, and

$$\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ compact}\}$$

for any μ-measurable subset $A \subset X$. Remember we say that a subset $A \subset X$ is *μ-measurable* if for all subsets $S \subset X$, $\mu(S) = \mu(S \cap A) + \mu(S \setminus A)$.

Let $C_c(X)$ be the space of continuous functions $X \to \mathbf{R}$ with compact support. It is well-known that we can identify the Radon measures on X with the space of non-negative linear functionals on $C_c(X)$ by mapping $\mu \mapsto [f \mapsto \int f \,\mathrm{d}\mu]$. Using the Banach–Alaoglu theorem one obtains the following compactness result for Radon measures, see for example [Sim84].

THEOREM 1.1.2. *Let $(\mu_k \mid k \in \mathbf{N})$ be a sequence of Radon measures on X with $\sup_k \mu_k(U) < \infty$ for all open $U \subset X$ with compact closure. Then there is a subsequence $(\mu_{k'})$ which converges to a Radon measure μ on X in the sense that $\int f \,\mathrm{d}\mu_{k'} \to \int f \,\mathrm{d}\mu$ for all $f \in C_c(X)$.*

1.1.2. Lipschitz functions and rectifiable sets. On an arbitrary metric space (X, d) we say that a function $f : X \to \mathbf{R}$ is *Lipschitz* if there is $L \geq 0$ so that $|f(x) - f(y)| \leq Ld(x, y)$ for all $x, y \in X$. We say that a function $f = (f_1, \ldots, f_{n+k}) : X \to \mathbf{R}^{n+k}$ is Lipschitz if the $f_1, \ldots, f_{n+k}$ are all Lipschitz.

THEOREM 1.1.3 (Rademacher's theorem). *If $f : \mathbf{R}^n \to \mathbf{R}$ is Lipschitz, then f is differentiable $\mathcal{H}^n$-almost everywhere.*

This also holds for Lipschitz functions defined on a subset $A \subset \mathbf{R}^n$. Indeed, by [Sim84, Thm. 5.1] there is a Lipschitz function $\bar{f} : \mathbf{R}^n \to \mathbf{R}$ which restricts to f on A and has the same Lipschitz constant.

THEOREM 1.1.4 ([Sim84]). *If $f : \mathbf{R}^n \to \mathbf{R}$ is Lipschitz, then for all $\epsilon > 0$ there is a continuously differentiable $g : \mathbf{R}^n \to \mathbf{R}$ with $\mathcal{H}^n(\{x \in \mathbf{R}^n \mid f(x) \neq g(x), Df(x) \neq Dg(x)\}) \leq \epsilon$.*

Finally, we use the following weaker version of the well-known *Arzelà–Ascoli theorem*; here we write $\|f\|_1 = \sup_X |f| + |Df|$ for a Lipschitz function f on X.

THEOREM 1.1.5. *Let (X, d) be locally compact metric space. Let $(f_k \mid k \in \mathbf{N})$ be a sequence of Lipschitz functions with $\sup_k \|f_k\|_1 < \infty$. Then there is a subsequence $(f_{k'})$ and a Lipschitz function f so that $f_{k'} \to f$ locally uniformly.*

A subset $M \subset \mathbf{R}^{n+k}$ is called *countably n-rectifiable* if there exists a collection of Lipschitz functions $F_j : \mathbf{R}^n \to \mathbf{R}^{n+k}$ so that $\mathcal{H}^n(M \setminus \cup_j F_j(\mathbf{R}^n)) = 0$. We may apply the approximation theorem, Theorem 1.1.4 above to every F_j separately using a sequence of $\epsilon = \epsilon_i$ converging to zero to find the following equivalent characterisation of rectifiability. A set $M \subset \mathbf{R}^{n+k}$ is countably n-rectifiable if there is a countable collection of C^1 embedded submanifolds $(M_j \mid j \in \mathbf{N})$ so that $\mathcal{H}^n(M \setminus \cup_j M_j) = 0$.

We make two quick trivial observations: first, any set with $\mathcal{H}^n(A) = 0$ is automatically n-rectifiable. Hence we also find that any countably n-rectifiable set has Hausdorff dimension $\dim_{\mathcal{H}} A \leq n$, with equality if $\mathcal{H}^n(A) \neq 0$. As a quick aside, let us point out that there are sets with $\mathcal{H}^n(A) < \infty$ which are however are not countably n-rectifiable, see the examples given in [Fed69, 3.3.19] or [Mor16, Ch. 3].

Let $M \subset \mathbf{R}^{n+k}$ be a countably n-rectifiable set. Let $\theta : M \to [0, \infty)$ be a Borel-measurable function, locally integrable on M. Using a slight abuse of notation, we can then define the weight measure $\|M\|$ corresponding to the pair (M, θ) by integrating θ along M. This is a Radon measure defined by $\int f \,\mathrm{d}\|M\| = \int_M f\theta \,\mathrm{d}\mathcal{H}^n$ for all $f \in C^1_c(\mathbf{R}^{n+k})$. Combining several results from [Sim84, Ch.11], one finds that $\theta(X) = \lim_{\rho\to 0}\|M\|(B_\rho(X))/\omega_n\rho^n$ at $\mathcal{H}^n$-a.e. $X \in M$. The right-hand side can be defined for any Radon measure μ. Where this limit exists, it is called the *density* of μ at X, and is written $\Theta(\mu, X)$. (In general one cannot assume the existence of this limit outside some null set, and has to work with the upper and lower densities.) In analogy with this more general case, we often also also write $\theta(X) = \Theta(\|M\|, X)$ and calls this the *density* of M at the point X.

Pick now an arbitrary point $X \in \mathbf{R}^{n+k}$, and define the *homothety* with scale factor $\lambda > 0$ centered at this point, to be the map $\eta_{X,\lambda} : Y \in \mathbf{R}^{n+k} \mapsto \lambda^{-1}(Y - X)$. We say that an n-dimensional linear subspace $P \subset \mathbf{R}^{n+k}$ is is the *approximate tangent space* to M at X with respect to θ if given any $f \in C^1_c(\mathbf{R}^{n+k})$, $\int_{\eta_{X,\lambda}M} f(Y)\theta(X+\lambda Y)\,\mathrm{d}\mathcal{H}^n(Y) \to \theta(X)\int_P f \,\mathrm{d}\mathcal{H}^n$ as $\lambda \to 0$. Such an approximate tangent space exists at $\mathcal{H}^n$-a.e. point $X \in M$ [Sim84, Thm.11.6], and where it is defined we write $P = T_X M$.

Let $M \subset \mathbf{R}^{n+k}$ be a countably rectifiable set, and $f : \mathbf{R}^{n+k} \to \mathbf{R}$ be differentiable. Then, at all points $X \in M$ where $T_X M$ is defined we call the projection of $Df(X) \in \mathbf{R}^{n+k}$ onto $T_X M$ the *gradient* of f at X with respect to M, and denote it $\nabla_M f(X)$ or $\nabla f(X)$ if the context is clear. Similarly we write $\nabla^\perp_M f(X) = Df(X) - \nabla_M f(X)$, sometimes abbreviating this $\nabla^\perp f(X)$.

In terms of the pushforwards of the weight measure under the homothetic rescalings, this may be summarised by writing that $\eta_{X,\lambda\#}\|M\| \to \|\mathbf{P}\|$ as $\lambda \to 0$ in the topology of Radon measures, where $\|\mathbf{P}\|$ is the weight measure associated to the plane P with constant density equals $\theta(X)$.

1.1.3. BV functions and Caccioppoli sets. Let $U \subset \mathbf{R}^n$ be an open set, and let $f \in L^1(U)$. Let $\xi \in C^1_c(U; \mathbf{R}^n)$ be a compactly supported vector field, which we can integrate against f to obtain $\int_U f \operatorname{div}\xi \,\mathrm{d}\mathcal{H}^n = \int_U f \sum_i D_i\xi_i \,\mathrm{d}\mathcal{H}^n$. We say that f has *bounded variation*, and write $f \in BV(U)$, if

$$\sup\left\{ \int_U f \operatorname{div}\xi \mid \xi \in C^1_c(U;\mathbf{R}^n), |\xi| \le 1\right\} < +\infty.$$

Let $f \in BV(U)$. Then the gradient of f in the sense of distributions defines a (vector-valued) Radon measure on U, whose total variation we write $|Df|$. Using this notation we find that the supremum above is precisely equal to $\int_U |Df|$. The analogous construction for functions $f \in L^1_{\mathrm{loc}}(U)$ yields the space $BV_{\mathrm{loc}}(U)$ of functions with *locally bounded variation*.

We list a few important results concerning functions with bounded variation.

THEOREM 1.1.6 ([Giu84]). *Let $U \subset \mathbf{R}^n$ be an open set, and $(f_j \mid j \in \mathbf{N})$ be a sequence of functions in $BV(U)$, which converge to a function f in $L^1_{\mathrm{loc}}(U)$. Then $f \in BV(U)$ and $\int_U |Df| \le \liminf_{j\to\infty} \int_U |Df_i|$.*

When endowed with the norm $\|f\|_{BV} = |f|_{L^1} + \int_U |Df|$, the space $BV(U)$ is a Banach space. It is not hard to see that the Sobolev space $W^{1,1}(U) \subset BV(U)$. The Example 1.1.9 below shows that however $W^{1,1}(U)$ is not equal $BV(U)$.

Note that both spaces are endowed with the same norm, that is for $f \in W^{1,1}(U)$, $\|f\|_{W^{1,1}} = \|f\|_{BV}$. As $W^{1,1}(U)$ is a Banach space, it is a closed proper subset of $BV(U)$. Moreover, as $C^\infty(U) \cap W^{1,1}(U) \subset W^{1,1}(U)$ there is no hope to approximate a function $f \in BV(U) \setminus W^{1,1}(U)$ by a sequence of smooth functions convergent with respect to the norm on $BV(U)$. That being said, the following holds.

THEOREM 1.1.7 ([Giu84]). *Let $f \in BV(U)$. There exists a sequence of functions $(f_j \mid j \in \mathbf{N})$ in $C^\infty(U)$ so that $|f_j - f|_{L^1} \to 0$ and $\int_U |Df_j| \to \int_U |Df|$ as $j \to \infty$.*

From the Sobolev embeddings we obtain the following compactness result.

THEOREM 1.1.8 ([Giu84]). *Let $U \subset \mathbf{R}^n$ be a bounded open set with Lipschitz boundary. Let $(f_j \mid j \in \mathbf{N})$ be a sequence of functions with $\sup_j |f_j|_{BV} < \infty$. Then there is a subsequence $(f_{j'})$ and a function $f \in BV(U)$ so that $|f_{j'} - f|_{L^1} \to 0$.*

Let $U \subset \mathbf{R}^n$ be still an open set, and let $E \subset \mathbf{R}^n$ be a Borel-measurable set, not necessarily contained inside U. Let $\mathbb{1}_E$ be the indicator function of E. We define the *perimeter* of E in U as $\mathrm{Per}(E, \Omega) = \int_U |D\mathbb{1}_E| = \sup\{\int_E \operatorname{div}\xi \mid \xi \in C^1_c(U; \mathbf{R}^n), |\xi| \le 1\}$. In the particular case $U = \mathbf{R}^n$ we reserve the notation $\mathrm{Per}(E) = \mathrm{Per}(E, \mathbf{R}^n)$. Finally, we say that a set E has *locally finite perimeter* if for all bounded open $U \subset \mathbf{R}^n$, $\mathrm{Per}(E, U) < \infty$. Sets with locally finite perimeter are also called *Caccioppoli sets*.

EXAMPLE 1.1.9 ([Giu84]). Let $U \subset \mathbf{R}^n$ be open, and let $E \subset \mathbf{R}^n$ be a bounded open set, with Lipschitz boundary ∂E. Let $\xi \in C^1_c(U; \mathbf{R}^n)$ be a vector field with $|\xi| \le 1$. Then $\int_E \operatorname{div}\xi = \int_{\partial E} \langle \xi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$, where ν is the outward unit normal to ∂E, defined $\mathcal{H}^{n-1}$-a.e. along the boundary. Therefore we find $\int_E \operatorname{div}\xi \le \mathcal{H}^{n-1}(\partial E \cap U)$, and passing to the supremum over all such vector fields we get $\mathrm{Per}(E, U) \le \mathcal{H}^{n-1}(\partial E \cap U)$. In particular the set E is a Caccioppoli set. If in fact we knew the boundary was smooth (or even piecewise smooth) then we would have equality in the above, that is $\mathrm{Per}(E, U) = \mathcal{H}^{n-1}(\partial E \cap U)$.

PROPOSITION 1.1.10 ([Giu84]). *Let $E, E_1, E_2 \subset \mathbf{R}^n$ be a Caccioppoli sets, and $U \subset \mathbf{R}^n$ be open.*

(i) *If $V \supset U$ is open then* $\mathrm{Per}(E, U) \leq \mathrm{Per}(E, V)$.
(ii) $\mathrm{Per}(E_1 \cup E_2, U) \leq \mathrm{Per}(E_1, U) + \mathrm{Per}(E_2, U)$.
(iii) *If $\mathcal{H}^n(E) = 0$ then* $\mathrm{Per}(E) = 0$.
(iv) *If $\mathcal{H}^n(E_1 \triangle E_2) = 0$ then* $\mathrm{Per}\, E_1 = \mathrm{Per}\, E_2$.

Let $E \subset \mathbf{R}^n$ be a Caccioppoli set. Recall from the above that $D\mathbb{1}_E$ is a vector-valued Radon measure. One can show that $\mathrm{spt}\, D\mathbb{1}_E \subset \partial E$. Moreover the following formula holds. Let $U \subset \mathbf{R}^n$ be open, and $\xi \in C^1_c(U; \mathbf{R}^n)$ be a vector field. Then $\int_E \mathrm{div}\, \xi = -\int_{\partial E} \langle \xi, D\mathbb{1}_E \rangle$. If $|\xi| \leq 1$ then this means that $\mathrm{Per}(E \cap U) \leq \mathcal{H}^{n-1}(\partial E \cap U)$, with the caveat that the right-hand side could be infinite even while the perimeter is finite.

Theorems 1.1.6 and 1.1.8 can be combined to obtain an existence result for area-minimising Caccioppoli sets.

THEOREM 1.1.11 ([DG61]). *Let $U \subset \mathbf{R}^n$ be a bounded open set, and let $L \subset \mathbf{R}^n$ be a Caccioppoli set. Then there is a Caccioppoli set E which coincides with L outside U and so that* $\mathrm{Per}\, E \leq \mathrm{Per}\, F$ *for all sets with $F = L$ outside U.*

1.1.4. Capacity. We work with subsets of $\mathbf{R}^n$, and let $p \in [1, n)$.

REMARK 1.1.12. Most results will distinguish between the cases $p = 1$ and $p \in (1, n)$. The most common application of these results will be for $p = 2$, followed by the more complicated case where $p = 1$.

Following Section 4.7 of Evans–Gariepy [EG15] we define

$$K^p = \{f : \mathbf{R}^n \to \mathbf{R} \mid f \geq 0, f \in L^{p^*}, Df \in L^p(\mathbf{R}^n; \mathbf{R}^n)\},$$

where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.

DEFINITION 1.1.13. If $A \subset \mathbf{R}^n$, set

$$\mathrm{cap}_p(A) = \inf \Big\{ \int_{\mathbf{R}^n} |Df|^p \mid f \in K^p, A \subset \mathrm{int}\{f \geq 1\} \Big\},$$

and call this the *p-capacity* of A.

REMARK 1.1.14. If $f \in K^p$ then $\min\{f, 1\} \in K^p$ also, so that in Definition 1.1.13 we could additionally have imposed that $0 \leq f \leq 1$ $\mathcal{H}^n$-a.e. in $\mathbf{R}^n$.

If the set A is compact, then the infimum can be taken over smooth compactly supported functions,

$$\mathrm{cap}_p(A) = \inf \Big\{ \int_{\mathbf{R}^n} |Df|^p \mid f \in C^\infty_c(\mathbf{R}^n), f \geq \mathbb{1}_A \Big\}$$

We give a few elementary properties of capacity, taken from those listed in Theorem 2, Section 4.7 of [EG15].

THEOREM 1.1.15 ([EG15]). *Assume* $A, B \subset \mathbf{R}^n$. *Then*

(i) $\mathrm{cap}_p(A) = \inf\{\mathrm{cap}_p(U) \mid U \text{ open}, A \subset U\}$,
(ii) $\mathrm{cap}_p(\lambda A) = \lambda^{n-p}\,\mathrm{cap}_p(A)$ *for all* $\lambda > 0$,
(iii) $\mathrm{cap}_p(A) \leq C(n,p)\mathcal{H}^{n-p}(A)$ *for some constant* $C(n,p) > 0$,
(iv) $\mathrm{cap}_p(A \cup B) + \mathrm{cap}_p(A \cap B) \leq \mathrm{cap}_p(A) + \mathrm{cap}_p(B)$,
(v) *if* $(A_k \mid k \in \mathbf{N})$ *is increasing, then* $\lim_{k\to\infty} \mathrm{cap}_p(A_k) = \mathrm{cap}_p(\cup_k A_k)$,
(vi) *if* $(A_k \mid k \in \mathbf{N})$ *and every* A_k *is compact, then* $\lim_{k\to\infty} \mathrm{cap}_p(A_k) = \mathrm{cap}_p(\cap_k A_k)$.

Next we give two results which establish a link between capacity and Hausdorff dimension. We state the cases $1 < p < n$ and $p = 1$ separately to avoid any confusion. Both are again taken from Section 4.7 of [EG15], Theorems 3 and 4.

THEOREM 1.1.16 ([EG15]). *Assume* $1 < p < n$.

(i) *If* $\mathcal{H}^{n-p}(A) < +\infty$ *then* $\mathrm{cap}_p(A) = 0$.
(ii) *If* $\mathrm{cap}_p(A) = 0$ *then* $\mathcal{H}^s(A) = 0$ *for all* $s > n-p$, *that is* $\dim_{\mathcal{H}} A \leq n-p$.

We may next combine this result with the fact that countably unions of sets with capacity zero also have capacity zero, which can be seen for instance by Theorem 1.1.15 (iv) and (v). For a fixed $1 < p < n$, this yields the following sets of inclusions for subsets of $\mathbf{R}^n$:

$$\begin{aligned}&\{A \subset \mathbf{R}^n \mid \mathcal{H}^{n-p}(A) = 0\}\\ &\quad\subset \{A \subset \mathbf{R}^n \mid A \text{ countably } (n-p)\text{-rectifiable or } \mathcal{H}^{n-p}(A) < \infty\}\\ &\quad\subset \{A \subset \mathbf{R}^n \mid \mathrm{cap}_p(A) = 0\}\\ &\quad\subset \{A \subset \mathbf{R}^n \mid \dim_{\mathcal{H}} A \leq n-p\}.\end{aligned}$$

These inclusions are all strict. Let us quickly make a remark on the first inclusion. At first sight one might expect that a set with $\mathcal{H}^{n-p}(A) < \infty$ would automatically be countably $(n-p)$-rectifiable. This is not so, as one finds by considering the examples in [Fed69, 3.3.19] or [Mor16, Ch.3].

Although these inclusions remain valid in the case $p = 1$, in fact much more is true. The following result, which can be found as Theorem 2 in Section 5.6 of [EG15], justifies singling this case out.

THEOREM 1.1.17 ([EG15]). *Let* $A \subset \mathbf{R}^n$. *Then* $\mathrm{cap}_1(A) = 0$ *if and only if* $\mathcal{H}^{n-1}(A) = 0$.

Let us briefly return to the setting of Example 1.1.9 above, with the aim of comparing the 1-capacity and perimeter.

EXAMPLE 1.1.18. Let $E \subset \mathbf{R}^n$ be a bounded open set with piecewise smooth boundary. Recall from Example 1.1.9 that $\operatorname{Per} E = \mathcal{H}^{n-1}(\partial E)$. Here we want to supplement this with the observation that $\operatorname{cap}_1 E \leq \operatorname{Per} E$. We show this by explicitly giving a sequence of functions $(\eta_j \mid j \in \mathbf{N})$ in $C^1_c(\mathbf{R}^n)$ with $E \subset \operatorname{int}\{\eta_j \geq 1\}$ and $\int |D\eta_j| \to \operatorname{Per} E$. Note however that one should in general expect this inequality to be strict. (Indeed this is the key observation behind the improved estimates obtained in Corollary 3.3.6.)

Define a function $d : x \in \mathbf{R}^n \mapsto \operatorname{dist}(x, E)$, and consider its level sets $\{d = t\}$ for all $t \geq 0$. We can then pick $\delta_E > 0$ small enough in terms of E that the function $t \in (0, \delta_E) \mapsto \mathcal{H}^{n-1}(\{d = t\})$ is bounded and continuous, with limit $\mathcal{H}^{n-1}(\partial E)$ as $t \to 0$.

Next define a function $\eta : \mathbf{R} \to \mathbf{R}$

$$\eta(t) = \begin{cases} 1 & \text{if } t < 0 \\ 1 - t & \text{if } t \in [0, 1] \\ 0 & \text{if } t > 1 \end{cases}$$

and for large enough $j \in \mathbf{N}$ that $2^{-j} < \delta_E$ define $\eta_j : x \in \mathbf{R}^n \mapsto \eta(2^j d(x))$. These functions are all Lipschitz regular, and $\eta_j \in K^1$ for all j. Using the co-area formula we find

$$\begin{aligned} \int_{\mathbf{R}^n} |D\eta_j| &= \int_0^{2^{-j}} \int_{\{d=t\}} |D\eta_j| \,\mathrm{d}t \\ &\leq \int_0^{2^{-j}} 2^j \mathcal{H}^{n-1}(\{d = t\}) \,\mathrm{d}t. \end{aligned} \tag{1.1}$$

As the integrand is continuous near zero, given any $\epsilon > 0$ we can take $\delta \in (0, \delta_E)$ small enough that for all $t \in [0, \delta)$,

$$|\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(\{d = t\})| < \epsilon.$$

Next update j to ensure $2^{-j} < \delta$ and deduce from (1.1) that $\int_{\mathbf{R}^n} |D\eta_j| \leq \mathcal{H}^{n-1}(\partial E) + \epsilon$.

To conclude the section we consider a *compact* set $A \subset \mathbf{R}^n$ with $\operatorname{cap}_p A = 0$ for some $1 \leq p < n$ and list the properties we may impose for a sequence of functions $(\eta_j \mid j \in \mathbf{N})$ which 'cut out' A. For this purpose we write

$$(A)_r = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, A) < r\}$$

for the open tubular neighbourhood of A of radius r. (Less frequently we also use the notation $[A]_r = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, A) \leq r\}$ for the closed tubular neighbourhood of radius r.) We may impose that for all j,

(1) $\eta_j \in C_c^1(\mathbf{R}^n)$,
(2) $0 \leq \eta_j \leq 1$,
(3) $\eta_j \equiv 1$ on $(A)_{r_j}$ for some $r_j \to 0$,
(4) $\eta_j \to 0$ $\mathcal{H}^n$-a.e.,
(5) $\int_{\mathbf{R}^n} |D\eta_j|^p \to 0$.

REMARK 1.1.19. Properties (1) and (3) can only be mandated when A is compact, but if these are replaced respectively by $\eta_j \in K^p$ and $A \subset \operatorname{int}\{\eta_j = 1\}$, then the list of properties also holds for non-compact A.

The only property in the list that we did not see before is (4). This is a simple consequence of the Sobolev embeddings. Indeed they give a constant $C(n,p) > 0$ so that for all $f \in K^p$,

$$\|f\|_{L^{p^*}(\mathbf{R}^n)} \leq C(n,p)\|Df\|_{L^p(\mathbf{R}^n)}.$$

Applying this here with $f = \eta_j$ we obtain that $\|\eta_j\|_{L^{p^*}(\mathbf{R}^n)} \to 0$ as $j \to \infty$. Although this does not strictly imply that $\eta_j \to 0$ $\mathcal{H}^n$-a.e., it allows us to extract a subsequence for which this is true, confirming (4).

Note next that if U is an open set which compactly contains A, then we may additionally impose that $\operatorname{spt}\eta_j \subset U$. Indeed there exists $\varphi \in C_c^1(U)$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on A, so that given a sequence $(\eta_j \mid j \in \mathbf{N})$ we may instead simply take $(\eta_j\varphi \mid j \in \mathbf{N})$. To see this, simply note that $\int |D(\eta_j\varphi)|^p \leq 2^{p-1}\int \eta_j^p|D\varphi|^p + \varphi^p|D\eta_j|^p$. The justification that $\int |D(\eta_j\varphi)|^p \to 0$ is standard, using for example dominated convergence for the first term and Hölder's inequality for the second.

1.2. GEOMETRIC MEASURE THEORY: VARIFOLDS AND CURRENTS

1.2.1. Varifolds. Let $U \subset \mathbf{R}^{n+k}$ be an open set. We write $Gr_n(U)$ for the n-dimensional *Grassmann space* on U, that is $Gr_n(U) = U \times Gr(n, n+k)$ where we write $Gr(n, n+k)$ for the space of linear n-dimensional subspaces of $\mathbf{R}^{n+k}$. A *general varifold* V on U is a Radon measure on $Gr_n(U)$. We say that a sequence $(V_j \mid j \in \mathbf{N})$ *converges in the varifold topology* to another varifold V if for all $f \in C_c^1(Gr_n(U))$, $\int f \,\mathrm{d}V_j \to \int f \,\mathrm{d}V$. The *weight measure* of a varifold V is obtained by forgetting the Grassmann structure. Given any test function $f \in C_c(U)$ we obtain $\bar{f} \in C_c(Gr_n(U))$ by setting $\bar{f}(X, \Pi) = f(X)$ for all $(X, \Pi) \in Gr_n(U)$. Using this one defines a Radon measure $\|V\|$ on U by integrating $\int f \,\mathrm{d}\|V\| = \int \bar{f} \,\mathrm{d}V$. Denote the space of general varifolds on U by $\mathbf{V}(U)$. Note that the compactness of the space of Radon measures implies that $\mathbf{V}(U)$ is compact in the varifold topology. That is, if $(V_j \mid j \in \mathbf{N})$ if a sequence of general varifolds, with $\sup_j \|V_j\|(K) < \infty$ for all compact $K \subset\subset U$ then there is a subsequence $(V_{j'})$ and a general varifold $V \in \mathbf{V}(U)$ such that $V_{j'} \to V$ in the varifold topology.

Moreover, from the elementary property of Radon measures one deduces the following continuity results for the mass.

PROPOSITION 1.2.1. *Let $(V_j \mid j \in \mathbf{N})$ be a sequence of varifolds in U which converges weakly to $V \in \mathbf{V}(U)$. Then for all*

(i) *compact $K \subset U$, $\|V\|(K) \geq \limsup_{j\to\infty} \|V_j\|(K)$.*
(ii) *open $W \subset U$, $\|V\|(W) \leq \liminf_{j\to\infty} \|V_j\|(W)$,*
(iii) *Borel sets $A \subset U$ with $\|V\|(\partial A) = 0$, $\|V\|(A) = \lim_{j\to\infty} \|V_j\|(A)$.*

This general notion is somewhat too broad to be useful in practise. For this reason we define *countably n-rectifiable varifolds* as those varifolds in U which are supported in a countably n-rectifiable set M with $\int f \,\mathrm{d}V = \int_U f(X, T_X M)\theta(X)\,\mathrm{d}\mathcal{H}^n(X)$ for some non-negative density function $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^n \llcorner M)$. If we substitute a test function $f \in C_c(U)$ into this identity we find that the weight measures coincide, that is $\|V\| = \|M\|$.

If in fact $\theta(X) \in \mathbf{Z}_{>0}$ at $\mathcal{H}^n$-a.e. $X \in M$, then we say that V is an *integer-density* countably n-rectifiable varifold, or *integral varifold* (sometimes also *integer varifolds*) for short. This is the class of varifolds we use essentially exclusively in the remainder. We denote their space by $\mathbf{IV}_n(U)$. Moreover we say $V \in \mathbf{IV}_n(U)$ has *dimension n* and *codimension k*. We frequently write $V = |M|$ for the varifold with density one supported in the set M, and more generally write $V = \theta|M|$ for the varifold with constant density $\theta \in \mathbf{Z}_{>0}$. Of course this has corresponding weight measure $\theta\|M\|$.

The space of integral varifolds is not closed under varifold convergence. That is, although from a sequence $(V_j \mid j \in \mathbf{N})$ of $V_j \in \mathbf{IV}_n(U)$ with locally bounded mass one can extract a subsequence which converges weakly to a limit V, this limit will in general not have integer multiplicity. Example 1.2.6 gives a sequence of one-dimensional integer varifolds for which this fails.

Below we define what it means for a varifold to be *stationary*. Under this condition, that is if the $V_j \in \mathbf{IV}_n(U)$ are all additionally *stationary* in U, then the so-called *Allard–Almgren compactness theorem* holds [Alm65, All72].

THEOREM 1.2.2 ([Alm65, All72]). *Let $U \subset \mathbf{R}^{n+k}$ be an open set, and $(V_j \mid j \in \mathbf{N})$ be a sequence of stationary integral varifolds with $\sup_j \|V_j\|(K) < \infty$ for all compact $K \subset\subset U$. Then there is a subsequence $(V_{j'})$ and a stationary integral varifold $V \in \mathbf{IV}_n(U)$ s.t. $V_{j'} \to V$ in the varifold topology.*

Let $U, W \subset \mathbf{R}^{n+k}$ be open sets, and $\Phi : U \to W$ be an injective, proper Lipschitz map. Let $M \subset U$ be a rectifiable set, and $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^n \llcorner M)$ be a $\mathbf{Z}_{>0}$-valued density function. Write $V \in \mathbf{IV}_n(U)$ for the varifold associated to the couple (M, θ). Then we may define the *pushforward* of V by Φ as the integral varifold $\Phi_\# V \in \mathbf{IV}_n(W)$ corresponding to the pair $(\Phi(M), \theta \circ$

$\Phi^{-1})$. If $f \in C_c(Gr_n(W))$ then $\int f \,\mathrm{d}\Phi_\# V = \int_{\Phi(M)} f(X, T_X\Phi(M))(\theta \circ \Psi^{-1})(X)\,\mathrm{d}\mathcal{H}^n(X) = \int_M f(\Phi(Y), D\Phi(T_Y M))\theta(Y)\,\mathrm{Jac}\,\Phi(Y)\,\mathrm{d}\mathcal{H}^n(Y)$, where the integrand is weighted by the so-called *Jacobian determinant* $\mathrm{Jac}\,\Phi(Y) = (\det(D\Phi(Y)^t \circ D\Phi(Y))^{1/2}$ and $D\Phi(Y) : T_Y M \to \mathbf{R}^{n+k}$ is the $\mathcal{H}^n$-almost everywhere defined linear map induced by Φ.

EXAMPLE 1.2.3 (Cylindrical varifolds). Let $U \subset \mathbf{R}^{n+k}$ be an open set, and V be a general varifold in U, with dimension n. Let $l \in \mathbf{Z}_{>0}$ be a positive integer. Then we can form the product $V \times \mathbf{R}^l$, which defines a varifold in $U \times \mathbf{R}^l$, as follows. Let $f \in C_c(Gr_{n+l}(U \times \mathbf{R}^l))$. For any fixed point $Y \in \mathbf{R}^l$ write $f(\cdot, Y) \in C_c(U)$ for the restriction of f to $U \times \{Y\}$. Then $\int f \,\mathrm{d}(V \times \mathbf{R}^l) = \int_{\mathbf{R}^l} \{\int f(\cdot, Y)\,\mathrm{d}V\}\,\mathrm{d}\mathcal{H}^l(Y)$. (This essentially corresponds to the product measure of V with the Hausdorff measure $\mathcal{H}^l$.) For us this construction most often arises when V is a varifold in an open subset $U \subset \mathbf{R}^n$, which we identify with $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$. We then often denote the product $V \times \mathbf{R}$ by $V \times \mathbf{R}e_{n+1}$ to emphasise that V is invariant under translations in the vertical direction. Note moreover that if $V \in \mathbf{IV}_n(U)$ then automatically $V \times \mathbf{R}^l \in \mathbf{IV}_{n+l}(U)$. We call varifolds of this form *cylindrical*. More generally, let $E \subset \mathbf{R}^{n+k}$ be a vector subspace, and for each $e \in E$ consider the translation map $\tau_e : X \in \mathbf{R}^{n+k} \mapsto X - e \in \mathbf{R}^{n+k}$. Suppose that the open set $U \subset \mathbf{R}^{n+k}$ is invariant under τ_e for all $e \in E$. Take a varifold $V \in \mathbf{IV}_n(U)$ with the same property, $\tau_{e\#} V = V$ for all $e \in E$. Say E has dimension d. Then there exists a varifold $V' \in \mathbf{IV}_{n-d}(U)$ so that $V = V' \times E$, in the same sense as described above.

The case in codimension one where $E = \mathbf{R}e_{n+1} = \mathrm{span}\{e_{n+1}\}$ is of particular importance to us. Let $\Omega \subset \mathbf{R}^n \times \{0\}$ be open, and $U = \Omega \times \mathbf{R} \subset \mathbf{R}^{n+1}$. Let moreover $V \in \mathbf{IV}_n(\Omega \times \mathbf{R})$. If $\tau_{te_{n+1}\#} V = V$ for all $t \in \mathbf{R}$, then there is $V_0 \in \mathbf{IV}_{n-1}(\Omega)$ so that $V = V_0 \times \mathbf{R}e_{n+1}$. Varifolds with this property are called *cylindrical in the vertical direction*, or *vertical* for short.

Cylindrical varifolds are closed under weak convergence, and properties such as stationarity also behave well, that is $V = V' \times E$ is stationary if and only if V' is. Hence if $V \in \mathbf{IV}_n(U)$ is cylindrical, then so are tangent cones at the points $X \in U \cap \mathrm{spt}\|V\|$.

1.2.2. Currents. Let $U \subset \mathbf{R}^{n+k}$ be an open set. Write $\mathcal{D}^n(U)$ for the space of smooth compactly supported n-forms in U. Equivalently we could write $\mathcal{D}^n(U) = C_c^\infty(U, \Lambda^n \mathbf{R}^{n+k}) = \Omega_c^n(U)$. This space can be equipped with a family of seminorms, simultaneously indexed by the compact subsets $K \subset U$ and multi-indices $\alpha \in \mathbf{Z}_{\geq 0}^{n+k}$, setting $p_{K,\alpha}(\omega) = \sup_K |D^\alpha \omega|$ for all $\omega \in \mathcal{D}^n(U)$, where we write $D^\alpha \omega = D_1^{\alpha_1} \cdots D_{n+k}^{\alpha_{n+k}} \omega$.

A linear map $T : \mathcal{D}^n(U) \to \mathbf{R}$ is called *n-dimensional current* if it is continuous with respect to the family of seminorms $\{p_{K,\alpha} \mid K \subset U \text{ compact}, \alpha \in \mathbf{Z}^{n+k}_{\geq 0}\}$. Explicitly this means that for every compact $K \subset U$ there exist $C_K > 0$ and $N_K \in \mathbf{Z}_{\geq 0}$ so that $|T(\omega)| \leq C_K p_{K,\alpha}(\omega)$ for all $\omega \in \mathcal{D}^n(U)$ with $\operatorname{spt} \omega \subset K$ and all indices $\alpha \in \mathbf{Z}^{n+k}_{\geq 0}$ with $|\alpha| \leq N_K$.

The space of n-dimensional currents is sometimes denoted $\mathcal{D}_n(U)$. We will seldom use this notation as we essentially exclusively work with the more restrictive class of *integer multiplicity currents*, defined as follows.

Let M be a countably n-rectifiable set, and $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^n \llcorner M)$ be a density function taking values in $\mathbf{Z}_{>0}$. Suppose moreover that at $\mathcal{H}^n$-a.e. point $X \in M$ we are given $\xi(X) \in \Lambda_n \mathbf{R}^{n+k}$ so that $\xi(X) = \tau_1 \wedge \cdots \wedge \tau_n$, where $\tau_1, \ldots, \tau_n$ is orthonormal basis for $T_X M$. We can use this triplet (M, θ, ξ) to define a current T in U by setting $T(\omega) = \int_M \langle \xi, \omega \rangle \theta \, \mathrm{d}\mathcal{H}^n = \int_M \langle \xi(X), \omega(X) \rangle \theta(X) \, \mathrm{d}\mathcal{H}^n(X)$ for all $\omega \in \mathcal{D}^n(U)$. We call all currents obtained in this manner *integer multiplicity countably n-rectifiable currents*, or *integer multiplicity currents* for short. (In the literature these are sometimes also called *rectifiable* or *locally rectifiable currents*, see for example [FF60, Mor16, Fed69].) We also use the abbreviated notation $T = [\![M]\!]$ for a current supported in M with density one, and more generally $\theta[\![M]\!]$ for currents with constant density $\theta \in \mathbf{Z}_{>0}$.

This is endowed with the topology induced by the family of seminorms defined above. Equivalently we say that a sequence of currents $(T_j \mid j \in \mathbf{N})$ converges to a limit current $T \in \mathcal{D}_n(U)$ if for all $\omega \in \mathcal{D}^n(U)$, $T_j(\omega) \to T(\omega)$.

In the definition of the seminorms $p_{K,\alpha}$ above there was a slight ambiguity, as we did not give an explicit defining expression for the norm of the n-form ω and its partial derivatives. We rectify this now. At a point $X \in U$ we define the norm of $\omega(X) \in \Lambda^n \mathbf{R}^{n+k}$ by setting the orthonormal basis formed by vectors of the form $e^I = e^{i_1} \wedge \cdots \wedge e^{i_n}$ to be orthonormal. One easily extends this to arbitrary forms $\omega = \sum_I' \omega_I e^I$ by multilinearity, where we used notation indicates that the sum ranges only over increasing multi-indices. Finally one sets $|\omega| = \sup_{X \in U} |\omega(X)|$. Using this we define the *mass* of a current as follows. Let $W \subset\subset U$ be an open subset of U with compact closure, and let T be an n-dimensional current in U. We then set $M_W(T) = \sup\{T(\omega) \mid \omega \in \mathcal{D}^n(W), |\omega| \leq 1\}$, and call this the *mass in* W of T.

Sometimes in the literature a slightly different convention is used, starting by defining the so-called *co-mass* of n-forms, denoted $\|\omega\|$. This is equivalent to our norm, $\binom{n+k}{n}^{-1/2} |\omega| \leq \|\omega\| \leq |\omega|$ for all n-forms. One can then calculate the supremum of $T(\omega)$ taken over all forms with $\|\omega\| \leq 1$.

Of course this is related to the mass we define by a constant factor. See e.g. [Sim84, Ch. 26] for a more in-depth discussion of this.

Regardless of which definition of mass is being used, one cannot in general expect currents $T \in \mathcal{D}_n(U)$ to have even locally finite mass. Indeed this is the case if and only T can be represented by integration, that is if there is a Radon measure μ_T and a Borel-measurable function ξ taking values in $\Lambda_n \mathbf{R}^{n+k}$ with $|\xi| = 1$ holding $\mathcal{H}^n$-a.e. so that $T(\omega) = \int_U \langle \omega, \xi \rangle \, \mathrm{d}\mu_T$ for all n-forms $\omega \in \mathcal{D}^n(U)$. This is the case for example when T is an integer multiplicity current, when $\mu_T = \|T\| = \theta \llcorner \mathcal{H}^n$.

If T is an arbitrary n-dimensional current on U, we define its *boundary* to be the $n-1$-dimensional current on U with $\partial T(\omega) = T(\mathrm{d}\omega)$ for all $\omega \in \mathcal{D}^{m-1}(U)$. The discussion from the previous paragraph also applies here. Even if T is an integer multiplicity current, it is in general false that ∂T has locally finite mass. However, the following is true.

THEOREM 1.2.4 ([Sim84, Thm. 30.3]). *Let $T \in \mathcal{D}_n(U)$ be an integer multiplicity current. If $M_W(\partial T) < \infty$ for all open $W \subset\subset U$ with compact closure then ∂T is an integer multiplicity current.*

Such currents are called *integral currents*. That is, we say $T \in \mathcal{D}_n(U)$ is *integral*, and write $T \in \mathbf{I}_n(U)$ if both T and ∂T have integer multiplicity. This is the class of currents we mainly work with in the remainder. The space $\mathbf{I}_n(U)$ satisfies the following useful compactness theorem, proved by Federer and Fleming [FF60].

THEOREM 1.2.5 ([Sim84, Thm. 27.3]). *Let $(T_j \mid j \in \mathbf{N})$ be a sequence of integral currents in U with $\sup_j M_W(T_j) + M_W(\partial T_j) < \infty$ for all open $W \subset\subset U$ with compact closure. Then there is a subsequence $(T_{j'})$ and an integral current $T \in \mathbf{I}_n(U)$ such that $T_{j'} \to T$ weakly in the current topology.*

We can define a forgetful map $T \in \mathbf{I}_n(U) \mapsto |T| \in \mathbf{IV}_n(U)$ which integrates functions $f \in C_c(U)$ like $\int_M f\theta \, \mathrm{d}\mathcal{H}^n$. In fact this is defined for an integer multiplicity current and maps the triple $(M, \theta, \xi) \mapsto (M, \theta)$. One must be careful to keep in mind that this map is not continuous with respect to the respective weak topologies, even for sequences as in Theorem 1.2.5.

EXAMPLE 1.2.6. We briefly sketch an example to illustrate this; see for example [Whi14, Ex. 2.8] for a more thorough treatment. Consider a sequence of one-dimensional, integral currents $T_j \in \mathbf{I}_1(D_1^2)$ contained in the two-dimensional unit disc $D_1 = D_1^2 \subset \mathbf{R}^2$. Construct these to all have $\partial T_j = 0$ in D_1, as a broken line resembling a staircase, with step sizes going to zero as $j \to \infty$. Using Theorem 1.2.5 above, one finds that these converge weakly as currents to another current, which we denote $L \in I_1(D_1)$. One may set up this sequence in such a way that this limit current is a diagonal line, with multiplicity one. Note that this has total mass $\sqrt{2}$, whereas the jagged lines can be constructed to all have mass 2. In other words, the mass drops in the limit. If we consider these jagged lines as one-dimensional integral varifolds, $|T_j| \in \mathbf{IV}_1(D_1)$ then this sequence does not tend to the line $|L|$ with multiplicity one. In fact the sequence $|T_j|$ does not converge to an integral varifold at all. Instead one finds that $|T_j| \to \sqrt{2}|L|$, as the mass cannot be lost in varifold convergence (see Proposition 1.2.1). (As the varifolds are not stationary, there is no contradiction with the Allard–Almgren compactness theorem, as stated in Theorem 1.2.2.) Thus in this example note that while $T_j \to L$ weakly as currents, we do not have $|T_j| \to |L|$ weakly as varifolds. In fact White [Whi14] remarks that the example can be constructed in such a way that the mass diverges. This example conclusively demonstrates the lack of continuity of the forgetful map $T \mapsto |T|$ defined above.

We close this chapter by briefly comparing the topologies defined on the space $\mathbf{I}_n(U)$, starting with the so-called *flat metric topology*. Let $T_1, T_2 \in \mathbf{I}_n(U)$. For all bounded open subsets $W \subset\subset U$ we define the pseudometric $d_W(T_1, T_2) = \inf\{M_W(S) + M_W(R) \mid T_1 - T_2 = \partial R + S, R \in \mathbf{I}_{n+1}(U), S \in \mathbf{I}_n(U)\}$. Then $\mathbf{I}_n(U)$ can be equipped corresponding to the family d_W of pseudometrics indexed by all bounded open subsets $W \subset\subset U$. This topology is called the *flat metric topology*. This topology turns out to be equivalent to the weak topology of currents.

THEOREM 1.2.7 ([Sim84, Thm. 31.2]). *Let $T \in \mathbf{I}_n(U)$ be an integral current, and let $(T_j \mid j \in \mathbf{N})$ be a sequence of integral currents in U with $\sup_j M_W(T_j) + M_W(\partial T_j) < \infty$. Then $T \to T_j$ weakly in the current topology if and only if $d_W(T_j, T) \to 0$ for all bounded open $W \subset\subset U$.*

There is another notion of convergence, called convergence in the *mass topology*. This can in fact be defined in a more general setting, for integer multiplicity currents. Let $(T_j \mid j \in \mathbf{N})$ be a sequence of integer multiplicity currents, and $T \in \mathcal{D}_n(U)$. We say that the sequence converges to T in a strong sense if for all bounded open $W \subset\subset U$, $M_W(T_j - T) \to 0$. Moreover, one can show that the set of integer multiplicity currents in $\mathcal{D}_n(U)$ is complete with respect to this topology, meaning the limit T automatically has integer multiplicity [Sim84, Lem. 27.5]. We mainly state this for completeness of our summary, we will not make of use this notion of convergence in the remainder of the text.

EXAMPLE 1.2.8 (Cylindrical currents). We can repeat the same construction as we did for varifolds in Example 1.2.3 in the context of currents. Let

$E \subset \mathbf{R}^{n+k}$ be a vector subspace of dimension $d \in \mathbf{Z}_{>0}$. Let $U \subset \mathbf{R}^{n+k}$ be an open set invariant under the translations τ_e, where e ranges over E. Let moreover $T \in \mathcal{D}_n(U)$ be a current with the same property, that is $\tau_{e\#}T = T$ for all $e \in E$. (Recall that we write τ_e for the translation in $-e$, that is mapping $X \mapsto X - e$.) Then there is a current $T' \in \mathcal{D}_{n-d}(U)$ so that $T = T' \times E$. Currents of this form are called *cylindrical* in general, and *vertical* in the special case where $E = \mathbf{R}e_{n+1}$. The subclasses of currents we defined earlier behave well under this operation. For example T has integer multiplicity if and only if T' does, $\partial T = \partial T' \times E$ and $T \in \mathbf{I}_n(U)$ if and only if $T' \in \mathbf{I}_{n-d}(U)$. Let $T \in I_n(U)$ be a cylindrical current, with additionally $\partial T = 0$ and is stationary in U. Then the tangent cones in the current topology are also are cylindrical.

1.2.3. The first variation formula and stationary varifolds. Let $U \subset \mathbf{R}^{n+k}$ be an open set, and $\xi \in C^1_c(U; \mathbf{R}^{n+k})$ be an arbitrary vector field, with flow (Φ_t). Let $K \subset\subset U$ be a compact subset containing $\operatorname{spt}\xi$. Let $V \in \mathbf{IV}_n(U)$ be an integral varifold, corresponding as usual to a couple (M, θ). Its pushforwards $\Phi_{t\#}V$ have locally finite measure, and moreover we can compute the derivative of the function $t \mapsto \|\Phi_{t\#}V\|(K)$, and express this in terms of ξ as the *first variation formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \|\Phi_{t\#}V\|(K) = \int_M (\operatorname{div}_M \xi)\theta \,\mathrm{d}\mathcal{H}^n = \int_U \operatorname{div}_M \xi \,\mathrm{d}\|V\|, \tag{1.1}$$

where the divergence with respect to M is defined at $\mathcal{H}^n$-almost every point $X \in M$ by $(\operatorname{div}_M \xi)(X) = \sum_{\alpha=1}^{n+k}\langle e_\alpha, \nabla_M \xi_\alpha(X)\rangle$, where $(e_1, \ldots, e_{n+k})$ is the standard basis of $\mathbf{R}^{n+k}$ and $\xi_\alpha = \langle \xi, e_\alpha\rangle$. If $(\tau_1, \ldots, \tau_n)$ is an orthonormal basis for $T_X M$, then the divergence can also be expressed as $\operatorname{div}_M \xi = \sum_{i=1}^n \langle D_{\tau_i}\xi, \tau_i\rangle$. Note that one can easily justify using a Lipschitz vector field in (1.1) using a mollification argument for example. (Later we will need to plug in even less regular functions into the formula, justifying these substitutions as they become necessary.) We say that the varifold $V \in \mathbf{IV}_n(U)$ is *stationary* in U if $\int \operatorname{div}_M \xi \,\mathrm{d}\|V\| = 0$ for all vector fields $\xi \in C^1_c(U, \mathbf{R}^n)$.

The entire discussion above can also be made for general varifolds, essentially the only necessary modification being that the first variation would then be $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\|\Phi_{t\#}V\| = \int \operatorname{div}_S \xi(X)\,\mathrm{d}V(X,S)$, where the divergence with respect to the plane $S \in Gr(n, n+k)$ is defined using a formula analogous to the above.

Example 1.2.9. Let $U \subset \mathbf{R}^{n+k}$ be an open set and $M \subset U$ be a properly embedded smooth manifold, with $U \cap \overline{M} \setminus M = \emptyset$ and $\theta \equiv 1$. Then the varifold corresponding to the couple (M, θ) is stationary if and only if its mean curvature vanishes identically, $H \equiv 0$.

Using (1.1) it is not hard to see that stationarity of varifolds is preserved in the limit. In other words, let $V \in \mathbf{IV}_n(U)$ and $(V_j \mid j \in \mathbf{N})$ be a sequence of varifolds in $\mathbf{IV}_n(U)$ with $V_j \to V$ weakly. If the V_j are all stationary in U, then so V.

Let still $U \subset \mathbf{R}^{n+k}$ be open, and take a stationary integral varifold $V \in \mathbf{IV}_n(U)$. Fix some point $X \in U$, not necessarily contained in $\operatorname{spt}\|V\|$, and write $R = \operatorname{dist}(X, \partial U)$. Testing the first variation formula with a suitable test function one obtains the *monotonicity formula*:

$$\|V\|(B_\sigma(X))/(\omega_n \sigma^n) - \|V\|(B_\rho(X))/(\omega_n \rho^n) = \int_{B_\sigma(X)\setminus \overline{B}_\rho(X)} \frac{|\nabla^\perp r|^2}{r^n} \,\mathrm{d}\|V\|$$

for all $0 < \rho < \sigma < R$ where $r(Y) = |Y - X|$. Hence the function $\rho \in \mathbf{R}_{>0} \mapsto \|V\|(B_\rho(X))/(\omega_n \rho^n)$ is monotone increasing, with equality if and only if V is invariant under homothetic rescalings around X. When this is the case V is called a *cone* with vertex at X; we usually denote these cones by $\mathbf{C}$ or a variant thereof.

As $\rho \to 0$ the function has a finite limit, denoted

$$\Theta(\|V\|, X) = \lim_{\rho \to 0} \|V\|(B_\rho(X))/(\omega_n \rho^n)$$

and called the *density* of V at X. The monotonicity formula furthermore implies that the density function is upper semicontinuous, that is if $X \in U$ is a fixed point, then $\Theta(\|V\|, X) \geq \limsup_{Y \to X} \Theta(\|V\|, Y)$.

For a moment assume that $U = \mathbf{R}^{n+k}$, so that the monotonicity formula is defined for all $0 < \rho < \sigma$. Then we can take the limit as $\rho \to \infty$, and call this the *density at infinity* at V. Occasionally this is denoted somewhat colloquially as $\Theta(\|V\|, \infty)$. This does not depend on the point $X \in \mathbf{R}^{n+k}$ around which we may measure the area ratios, but it may of course be infinite. Even when it is finite, by monotonicity we have $\Theta(\|V\|, \infty) \geq \|V\|(B_\rho(X))/(\omega_n \rho^n)$ for all $X \in \mathbf{R}^{n+k}$ and $\rho > 0$.

Now return to the general case where $U \subset \mathbf{R}^{n+k}$ is an arbitrary open set. Let $(V_j \mid j \in \mathbf{N})$ be a sequence of stationary integral varifolds in U, which converge weakly to a limit $V \in \mathbf{IV}_n(U)$. Then their supports converge locally with respect to the Hausdorff distance. That is, for all compact $K \subset\subset U$, $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|V_j\| \cap K, \operatorname{spt}\|V\| \cap K) \to \infty$ as $j \to \infty$. This would fail if one were given an arbitrary sequence of varifolds, even if one knew that they were all countably rectifiable. Indeed if the V_j are rectifiable but θ_j are not integer-valued, then one would relinquish control over those portions where their density functions θ_j go to zero.

We state also the following result valid for sequences of currents $T_j \in \mathbf{I}_n(U)$ with $|T_j| \in \mathbf{IV}_n(U)$. One may compare this with the stronger conclusion of Proposition 2.2.1.

PROPOSITION 1.2.10. *Let $(T_j \mid j \in \mathbf{N})$ be a sequence of integral currents in U with $\partial T_j = 0$ and $\sup_j M_W(T_j) < +\infty$ for all bounded open $W \subset\subset U$. Suppose that $|T_j|$ is stationary in U for all j. Then there exist $T \in \mathbf{I}_n(U)$ and $V \in \mathbf{IV}_n(U)$ and a subsequence $(T_{j'})$ so that $T_{j'} \to T$ and $|T_{j'}| \to V$. Moreover $|T| \ll V$.*

Using Allard's regularity theorem for stationary integral varifolds near multiplicity one planes, we obtain the following special case.

PROPOSITION 1.2.11. *Let $(T_j \mid j \in \mathbf{N})$ be a sequence of integral currents as above. Suppose that there are integers $l_i, m_i \in \mathbf{Z}_{\geq 0}$ and distinct half-planes π_i so that $T_j \to \sum_i l_i [\![\pi_i]\!]$ and $|T_j| \to \sum_i m_i |\pi_i|$ as $j \to \infty$. Then $l_i \leq m_i$, and there is equality if $m_i = 1$.*

Of course equality in $l_i \leq m_i$ may also happen when $m_i \geq 2$, but the essence of this result is that there cannot be loss of mass when the convergence is with multiplicity one. Using terminology we introduce in the next chapter, one could make the same analysis for general limits. The analogue would then be the following. Under the same hypotheses as above, assume that $T_j \to T$ and $|T_j| \to V$ as $j \to \infty$. If a point $X \in U \cap \operatorname{reg} V$ has $\Theta(\|V\|, X) = 1$, then $\Theta(\|T\|, X) = 1$ also.

1.2.4. Tangent and limit cones. Let again $U \subset \mathbf{R}^{n+k}$ be an arbitrary open set, $V \in \mathbf{IV}_n(U)$ be stationary and $X \in U \cap \operatorname{spt}\|V\|$. Finally, let $(\lambda_j \mid j \in \mathbf{N})$ be a sequence of positive scalars with $\lambda_j \to 0$ as $j \to \infty$. Consider the sequence $(\eta_{X,\lambda_j\#} V \mid j \in \mathbf{N})$, defined on a sequence of open sets $\eta_{X,\lambda_j}(U)$. (Although the sequence is not necessarily increasing, note that every compact set $K \subset \mathbf{R}^{n+k}$ eventually lies inside these sets.) Applying the Allard–Almgren compactness theorem to this sequence, we find that there is a subsequence $(\lambda_{j'})$ so that $(\eta_{X,\lambda_{j'}\#} V)$ converges weakly in the varifold topology. Moreover by the monotonicity formula the limit must be a cone, say $\mathbf{C}$: $\eta_{X,\lambda_{j'}\#} V \to \mathbf{C}$ as $j' \to \infty$, weakly in the varifold topology of $\mathbf{IV}_n(W)$ for all bounded open $W \subset\subset U$.

More generally, we say that a cone $\mathbf{C} \in \mathbf{IV}_n(\mathbf{R}^{n+k})$ is *tangent to V at X* if there exists a sequence of positive scalars so that $\eta_{X,\lambda_j\#} V \to \mathbf{C}$ as $j \to \infty$. Apart from the usual terminology calling $\mathbf{C}$ a *tangent cone* to V at X we sometimes also call it a *blow-up cone*, and call the sequence $(\eta_{X,\lambda_j\#} V \mid j \in \mathbf{N})$ a *blow-up sequence* at the point $X \in \operatorname{spt}\|V\| \cap U$. The set of all *tangent cones* to V at X is denoted $\operatorname{VarTan}(V, X)$. In principle there could be several such cones, that is blowing V up at X along two different sequences $\lambda_j, \mu_j \to 0$ of positive scalars could produce different cones in the limit. To avoid any confusion, let us briefly return to the definition of rectifiable sets. Indeed if $V \in \mathbf{IV}_n(U)$ is any integer varifold with support $M = U \cap \operatorname{spt}\|V\|$ say, then by definition there is at $\mathcal{H}^n$-a.e. point $X \in M$ a plane $\Pi_X \in Gr(n, n+k)$ so that $\operatorname{VarTan}(V, X) = \{\theta(X)\Pi_X\}$, where $\theta(X) = \Theta(\|V\|, X)$. At such points the tangent cone is thus unique, and indeed is a plane with positive multiplicity $\Theta(\|V\|, X)$. Proving the *uniqueness of tangent cones* in general remains a major open problem in geometric measure theory, in which all but a few cases are unknown. For a non-exhaustive list of the positive results in this area, see among others [AA81, Whi83, Sim83, Sim94, BK17].

LEMMA 1.2.12. *Let $U \subset \mathbf{R}^{n+1}$ be an open set and $V_j \in \mathbf{IV}_n(U)$ be a sequence of integral varifolds with $V_j \to V$ as $j \to \infty$, weakly in the varifold topology. Let $X \in \operatorname{spt}\|V\| \cap U$, and $\mathbf{C}_X \in \operatorname{VarTan}(V, X)$. After extracting a subsequence we may find a sequence $\lambda_{j'} \to 0$ so that*

$$\eta_{X,\lambda_{j'}\#} V_{j'} \to \mathbf{C}_X \text{ as } j' \to \infty.$$

PROOF. Let $\mu_k \to 0$ be a sequence of positive scaling parameters along which $\eta_{X,\mu_k\#}V \to \mathbf{C}_X$ as $k \to \infty$. The cone $\mathbf{C}_X$ has bounded positive mass equal to $\Theta(\|V\|, X)$, so for all $\lambda > 0$ small enough and $j \geq J(\lambda)$, $1/2\Theta(\|V\|, X) \leq 1/(\omega_n\lambda^n)\|V_j\|(B_\lambda(X)) \leq 2\Theta(\|V\|, X)$. We may thus metrise both the convergence of the blow-up sequence to $\mathbf{C}$ and that of $V_j \to V$ by some unspecified distance function. For all $R, \epsilon > 0$ there is $K = K(R, \epsilon) \in \mathbf{N}$ with the property that $B_{\mu_K R}(X) \subset U$ and

$$\operatorname{dist}(\eta_{X,\mu_K R\#}(V \llcorner B_{\mu_K R}(X)), \eta_{0,R\#}(\mathbf{C}_X \llcorner B_R)) < \epsilon/2.$$

These being fixed, we find $J = J(R, \epsilon, K)$ so that

$$\operatorname{dist}(\eta_{X,\mu_K R\#}(V_j \llcorner B_{\mu_k R}(X)), \eta_{X,\mu_K R\#}(V \llcorner B_{\mu_k R}(X)) < \epsilon/2$$

for all $j \geq J$. To conclude let $\epsilon_i \to 0$, $R_i \to \infty$ be two independent positive sequences, let $K_i = K(\epsilon_i, R_i)$ as above and set $j' = J(R_i, \epsilon_i, K_i)$ and $\lambda_{j'} = \mu_{K_i}$. □

Let now $T \in \mathbf{I}_n(U)$ be an integral current with $\partial T = 0$, and suppose that the varifold $|T|$ is stationary. Using the monotonicity formula, we may again take weakly convergent blow-up sequences at any point $X \in U \cap \operatorname{spt}\|T\|$, whose limit in the current topology, say $T_X \in \mathbf{I}_n(\mathbf{R}^{n+k})$, is still invariant under homotheties. We still call T_X a *tangent cone* to T at X, and denote the set of all tangent cones by $\operatorname{VarTan}(T, X)$. Moreover we have the following analogue for the result above.

LEMMA 1.2.13. *Let* $T_j \in I_n(U)$ *be a sequence of integral currents with* $\partial T_j = 0 \in \mathbf{I}_{n-1}(U)$ *and so that* $|T_j|$ *is stationary in* U. *Suppose that* $T_j \to T$ *as* $j \to \infty$ *weakly in the current topology. Let* $X \in \operatorname{spt}\|T\| \cap U$, *and* $T_X \in \operatorname{VarTan}(T, X)$. *After extracting a subsequence we may find a sequence of scalars* $\lambda_{j'} \to 0$ *so that*

$$\eta_{X,\lambda_{j'}\#}T_{j'} \to T_X \text{ as } j' \to \infty.$$

These two lemmas motivate defining so-called *limit varifolds* and *currents*. Let $U \subset \mathbf{R}^{n+1}$ be open and $(V_j \mid j \in \mathbf{N})$ be a sequence of integer varifolds in $\mathbf{IV}_n(U)$ be a sequence of integral varifolds (resp. $(T_j \mid j \in \mathbf{N})$ be a sequence of integral currents in $\mathbf{I}_n(U)$). Then an integral varifold V (resp. an integral current T) is called a *limit varifold* (resp. a *limit current*) of the sequence if there is a subsequence $(V_{j'})$ (resp. $(T_{j'})$) so that for some sequence of points $(X_{j'}) \in U$ and of positive scalars $(\lambda_{j'})$ we have $\eta_{X_{j'},\lambda_{j'}\#}V_j \to V$ (resp. $\eta_{X_{j'},\lambda_{j'}\#}T_j \to T$). Often the varifold V (resp. the current T) will be invariant under homotheties, in which case we call them *limit cones* of the sequence. Note however that even if $\lambda_{j'} \to 0$ or $\lambda_{j'} \to \infty$ the monotonicity formula does not guarantee that the limit is scale-invariant, because to draw this conclusion one needs to consider it with a fixed base point. (This is true even if one considers a moving sequence of points $(X_j \mid j \in \mathbf{N})$ along a fixed varifold $V_j = V$, which one may moreover assume to converge to a point $X \in \operatorname{spt}\|V\|$. When this rescaling procedure converges to a cone, this is usually called a *pseudotangent cone* at the point X in the literature. We mention this mainly for sake of completeness, and will not use this in the remainder.)

Chapter 2

Regularity theory of minimal surfaces

2.1. Singular points and regularity of minimal surfaces

Let $U \subset \mathbf{R}^{n+k}$ be an open set, and $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. A point $X \in U \cap \operatorname{spt}\|V\|$ is called *regular* if there is a radius $\rho > 0$ so that $B_\rho(X) \cap \operatorname{spt}\|V\|$ is an embedded surface. A point which is not regular is called *singular*. We denote the *regular set* $\operatorname{reg} V$ and the *singular set* $\operatorname{sing} V$.

2.1.1. Regularity of stationary varifolds: Allard regularity. Allard's regularity theorem is the foundational result in the regularity theory of stationary varifolds in arbitrary codimension. (We will only use the codimension one version.)

Theorem 2.1.1 ([All72]). *Let $\alpha \in (0,1)$. There is $\epsilon = \epsilon(n,k,\alpha) > 0$ so that if $V \in \mathbf{IV}_n(B_2^{n+k})$ is stationary in $B_2 = B_2^{n+k}$ and $0 \in \operatorname{spt}\|V\|$ with*

$$\|V\|(B_2)/(\omega_n 2^n) < 2 - \alpha$$

$$\int_{B_2} \operatorname{dist}(X,\Pi)^2 \,\mathrm{d}\|V\|(X) < \epsilon$$

for some n-dimensional plane $\Pi \in Gr(n, n+k)$ then $\operatorname{spt}\|V\| \cap B_1 \subset \operatorname{reg} V$. In fact $\operatorname{spt}\|V\| \cap B_1 \subset \operatorname{graph} u \subset \operatorname{reg} V$ for some function $u \in C^{1,\gamma}(\Pi \cap B_1; \Pi^\perp)$ with $\gamma = \gamma(n,k,\alpha) \in (0,1)$. Moreover there is $C = C(n,k,\alpha) > 0$ so that

$$|u|_{1,\gamma;B_1\cap\Pi} \leq C\Big(\int_{B_2} \operatorname{dist}(X,\Pi)^2 \,\mathrm{d}\|V\|(X)\Big)^{1/2}.$$

Naturally under the same hypotheses elliptic regularity gives that the function u is smooth, and one obtains estimates of the form $|u|_{l,\gamma;B_1\cap\Pi} \leq C_l(\int_{B_2} \operatorname{dist}(X,\Pi)^2 \,\mathrm{d}\|V\|(X))^{1/2}$ for all $l \in \mathbf{N}$, where $C_l = C(n,k,\alpha,l)$.

The following is an equivalent version of the theorem above, with the formulation again taken from [Wic14b].

Theorem 2.1.2 ([All72]). *Let $U \subset \mathbf{R}^{n+k}$ be open and $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. Then there is $0 < \epsilon = \epsilon(n,k) < 1$ so that if $Y \in U \cap \operatorname{spt}\|V\|$ has $\Theta(\|V\|, Y) < 1 + \epsilon$ then $Y \in \operatorname{reg} V$.*

We will often apply Theorem 2.1.1 to sequences of integral varifolds, in the following way. Let $(V_j \mid j \in \mathbf{N})$ be a sequence of integral varifolds in $\mathbf{IV}_n(B_2)$, so that $|V_j| \to |\Pi| \llcorner B_2$ for some n-dimensional plane $\Pi \in$

$Gr(n, n+1)$. Then provided j is large enough, we get $\operatorname{spt}\|V_j\| \cap B_1 \subset \operatorname{reg} V_j$ and there exists a function $u_j \in C^\infty(\Pi \cap B_1; \Pi^\perp)$ so that $\operatorname{spt}\|V_j\| \cap B_1 \subset \operatorname{graph} u_j \subset \operatorname{reg} V_j$. Moreover, for all $l \in \mathbf{N}, \gamma \in (0,1)$ we get $|u_j|_{l,\gamma;B_1\cap\Pi} \to 0$ as $j \to \infty$.

This only requires a slight modification to hold for sequences convergent to a smooth minimal surface. Let $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. Let $W \subset\subset U \setminus \operatorname{sing} V$ be a bounded open set, lying a positive distance away from $\partial U \cup \operatorname{sing} V$. Suppose additionally that the normal bundle to $\operatorname{reg} V$, denoted $T^\perp \operatorname{reg} V$, is trivialisable on $W \cap \operatorname{reg} V$. Given small enough $\tau > 0$ the tubular neighbourhood W_τ of $W \cap \operatorname{reg} V$ with width τ is diffeomorphic to $\operatorname{reg} W \times B^k_\tau$, and in particular has $W_\tau \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V$.

COROLLARY 2.1.3. *Let $U \subset \mathbf{R}^{n+k}$ be open. Let $(V_j \mid j \in \mathbf{N})$ be a sequence of stationary integral varifolds in $\mathbf{IV}_n(U)$, and suppose $V_j \to V$. Let $W \subset\subset U \setminus \operatorname{sing} V$ and $\tau > 0$ be as above, and suppose that for all $Y \in W \cap \operatorname{spt}\|V\|$, $\Theta(\|V\|, Y) = 1$. Then for large enough $j \geq J(\tau)$, $W_\tau \cap \operatorname{spt}\|V_j\| \subset \operatorname{reg} V_j$ and there is a smooth function $u_j \in C^\infty(W \cap \operatorname{reg} V; T^\perp \operatorname{reg} V)$ so that $W_\tau \cap \operatorname{spt}\|V_j\| \subset \operatorname{graph} u_j \subset \operatorname{reg} V_j$. Moreover for all $l \in \mathbf{N}, \gamma \in (0,1)$, $|u_j|_{l,\gamma;W\cap\operatorname{reg} V} \to 0$ as $j \to \infty$.*

2.1.2. Stratification of the singular set. Let $U \subset \mathbf{R}^{n+k}$ be an open set, and $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. Consider a point $X \in U \cap \operatorname{sing} V$ and a tangent cone $\mathbf{C} \in \operatorname{VarTan}(V, X)$. This cone $\mathbf{C}$ may be invariant under translation by vectors $V \in \mathbf{R}^{n+1}$. The set $\mathcal{S}(\mathbf{C}) = \{V \in \mathbf{R}^{n+k} \mid \tau_{V\#}\mathbf{C} = \mathbf{C}\}$ forms a vector space called the *spine* of $\mathbf{C}$. For all $m = 0, \dots, n$ write $\mathcal{S}^m(V) = \{X \in U \cap \operatorname{sing} V \mid \dim \mathcal{S}(\mathbf{C}) \leq m$ for all $\mathbf{C} \in \operatorname{VarTan}(V, X)\}$. If $\dim \mathcal{S}(\mathbf{C}) = m$ then there is a stationary cone $\mathbf{C}' \in \mathbf{IV}_{n-m}(\mathbf{R}^{n-m+k})$ so that $\mathbf{C} = \mathbf{C}' \times \mathcal{S}(\mathbf{C})$. Let $\mathbf{C} \in \mathbf{IV}_n(\mathbf{R}^{n+k})$ be a stationary integral cone. Then for $X \in \operatorname{spt}\|\mathbf{C}\| \setminus \{0\}$ every tangent cone $\mathbf{C}_X \in \operatorname{VarTan}(\mathbf{C}, X)$ has $\mathcal{S}(\mathbf{C}_X) \neq \{0\}$ because $\mathbf{R} \cdot X \subset \mathcal{S}(\mathbf{C}_X)$. This means that there exists $\mathbf{C}'_X \in \mathbf{IV}_{n-1}(\mathbf{R}^{n-1+k})$ so that $\mathbf{C}_X = \mathbf{C}'_X \times \mathbf{R} \cdot X$. Then we have the so-called *Almgren–Federer stratification theorem.*

THEOREM 2.1.4 ([Alm00]). *Let $U \subset \mathbf{R}^{n+k}$ be open, and $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. Then $\dim_{\mathcal{H}} \mathcal{S}^m(V) \leq m$ for all $m = 0, \dots, n$.*

Under same hypotheses, Naber–Valtorta [NV15] improve this to the following.

THEOREM 2.1.5 ([NV15]). *Let $U \in \mathbf{R}^{n+k}$ be open, and $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold. Then $\mathcal{S}^m(V)$ is countably k-rectifiable for all $m = 0, \dots, n$.*

In the literature both $\mathcal{S}^k(V)$ and $\mathcal{S}^k \backslash \mathcal{S}^{k-1}(V)$ are sometimes called *strata* of the singular set of V. The singular set of a stationary $V \in \mathbf{IV}_n(U)$ can be divided like $\operatorname{sing} V = \mathcal{S}^{n-2}(V) \cup (\mathcal{S}^n \setminus \mathcal{S}^{n-2})(V)$. By Naber–Valtorta's result, the lower strata gathered into $\mathcal{S}^{n-2}(V)$ can be excised by by a suitable sequence of test functions using a capacity argument.

The top stratum $\mathcal{S}^n \setminus \mathcal{S}^{n-1}(V)$ is also called the *branch set* of V, and denoted $\mathcal{B}(V)$ or $\mathcal{B}_V$. A singular point X belongs to $\mathcal{B}(V)$, and is called a *branch point* if at least one cone $\mathbf{C} \in \operatorname{VarTan}(V, X)$ is of the form $\mathbf{C} = Q|\Pi|$ for some n-dimensional plane $\Pi \in Gr(n, n+k)$ with multiplicity $Q \in \mathbf{Z}_{>0}$. (By Allard regularity the multiplicity of such a branch point must be at least two.) Note that if V were to coincide with a regular minimal surface Σ taken with multiplicity two, that is $V = 2|\Sigma|$ then the support of V is embedded near all points in $U \cap \operatorname{spt}\|V\|$. In our convention these points do not belong to $\mathcal{B}_V$, although they are sometimes referred to as *false branch points*.

The next stratum $\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}(V)$ is formed by those points $X \in U \cap \operatorname{sing} V$ near which at least one tangent cone $\mathbf{C} \in \operatorname{VarTan}(V, X)$ is of the form $\mathbf{C} = \sum_{i=1}^{D} m_i |\pi_i|$ where the π_i are n-dimensional half-planes meeting along a common $n-1$–dimensional axis L, and $m_i \in \mathbf{Z}_{>0}$. We will call such cones *classical cones* for lack of a better word. Note however that a point $X \in \mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}(V)$ is not necessarily a *classical singularity.* Indeed so-called *classical singularities* are those where there are D embedded sheets with boundary $\Sigma_1, \dots, \Sigma_D$ with respective multiplicities $m_1, \dots, m_D$ which meet along a common $n-1$-dimensional axis Γ say. We denote the classical singular set by $\mathcal{C}(V)$ or $\mathcal{C}_V$. An example of particular relevance to us is where $D = 4$, $m_1, \dots, m_4 = 1$ and the half-planes $\pi_1, \dots, \pi_4$ form a union of two n-dimensional planes $\Pi_1, \Pi_2 \in Gr(n, n+1)$. In other words $\mathbf{C} = |\Pi_1| + |\Pi_2|$. Such tangent cones arise for example at *immersed singularities*, where there are two sheets Σ_1, Σ_2 which are separately embedded and meet transversely along an $n-1$–dimensional axis $\Gamma = \Sigma_1 \cap \Sigma_2 \ni X$. However, note that in general it is not known whether for stationary varifolds such cones can also arise at singularities which are not immersed. However in some cases this result is known, for example when one additionally imposes *stability* (as defined below) then this this is shown in [Wic20]. Moreover, in the class of two-valued Lipschitz graphs (and arbitrary codimension) the analogous result was proved by [BK17]. We will use both of these results in our arguments.

2.1.3. The second variation formula and stability. Here let $k = 1$, that is consider codimension one minimal surfaces. In the previous chapter we calculated the first variation of the area of a varifold $V \in \mathbf{IV}_n(U)$ when perturbed in the direction of a vector field $\xi \in C^1_c(U; \mathbf{R}^{n+1})$. In the same vein one can calculate the second derivative. When the field belongs to $C^2_c(U;\mathbf{R}^{n+1})$, there is a similar version of the first variation formula, expressing this in terms of only the first and second derivatives of the initial vector field, see [Sim84, 9.4]. We are most interested in a variant of this formula in which the second fundamental form of $\operatorname{reg} V$ appears. Suppose for this that the regular set of V is *orientable*, or equivalently that there is a smooth unit normal N defined on $\operatorname{reg} V$. For this, suppose that $\operatorname{spt}\xi \cap \operatorname{sing} V = \emptyset$. Then its restriction to $\operatorname{spt}\|V\| \cap U$ can be expressed as φN for some function $\varphi \in C^1_c(\operatorname{reg} V)$. Let $K \subset U$ be a compact set containing $\operatorname{spt}\varphi$. Deforming V in the direction of this vector field yields the *second variation formula*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \|\Phi_{t\#}V\|(K) = \int_{K\cap \operatorname{reg} V} |\nabla_V \varphi|^2 - |A_V|^2\varphi^2 \,\mathrm{d}\|V\|, \tag{2.1}$$

where ∇_V and A_V are the gradient operator and second fundamental form on $\operatorname{reg} V$ respectively.

A varifold $V \in \mathbf{IV}_n(U)$ is said to have *stable regular part* if this is non-negative for all perturbations $\varphi \in C^1_c(\operatorname{reg} V)$, that is $\int |A_V|^2\varphi^2 \,\mathrm{d}\|V\| \leq \int |\nabla_V\varphi|^2 \,\mathrm{d}\|V\|$. This is also called the *stability inequality*. Note that this automatically yields L^2-bounds for the curvature away from the singular set. Indeed, let $X \in U \cap \operatorname{reg} V$ be a point with $\operatorname{dist}(X, \operatorname{sing} V) > 2R$. If we let $\varphi \in C^1_c(B_{2R}(X))$ be a standard cutoff function, with $\varphi = 1$ on $B_R(X)$ and $|D\varphi| \leq 2/R$ then (2.1) yields $\int_{\operatorname{reg} V\cap B_R(X)} |A_V|^2 \,\mathrm{d}\|V\| \leq 4R^{-2}\|V\|(B_{2R}(X))$. Note however that if V has a large singular set, then these bounds can in general not be extended to hold across singular points, that is for balls with $B_R(X) \cap \operatorname{sing} V \neq \emptyset$.

We introduce a second notion of stability, which we call *ambient stability*. Suppose that we know *a priori* the varifold V to have locally bounded curvature. By this we mean that for all compact $K \subset U$, $\int_{K\cap \operatorname{reg} V} |A_V|^2 \,\mathrm{d}\|V\| < \infty$. Then we say that V is *ambient stable* if the inequality

$$\int_{U\cap \operatorname{reg} V} |A_V|^2\varphi^2 \,\mathrm{d}\|V\| \leq \int_{U\cap \operatorname{reg} V} |\nabla_V\varphi|^2 \,\mathrm{d}\|V\| \tag{S_V}$$

holds for all $\varphi \in C^1_c(U)$. We emphasise here that both the local bounds for the curvature and the stability inequality hold for any compact subsets of U, not just those that avoid the singularities of V. Though it appears similar, this notion of stability is significantly stronger, and thus more constraining, then those usually used in the literature. To give but one example, compare it with the stability inequality [Wic14a, (3.2)] in Wickramasekera's regularity theory, which is only mandated to hold for test functions $\varphi \in C^1_c(\operatorname{reg} V)$.

The advantage that ambient stability holds over this weaker notion of stability is that (S_V) yields local L^2-estimates for the second fundamental form, which hold across the singular set, depending essentially on the mass of the varifold.

This in turn allows the application of the theory developed by Hutchinson in [Hut86], from whence we derive that ambient stability is preserved under weak convergence of varifolds.

PROPOSITION 2.1.6 ([Hut86]). *Let $U \subset \mathbf{R}^{n+1}$ be open, and let $(V_j \mid j \in \mathbf{N})$ be a sequence of stationary varifolds in U satisfying (S_V). Suppose that $V_j \to V \in \mathbf{IV}_n(U)$ weakly in the varifold topology. Then V is stationary and ambient stable.*

Let us make a quick comment on the orientability of $\operatorname{reg} V$ we assumed above. Although we initially assumed this in the derivation of the stability inequality, note that (S_V) itself is well-defined regardless of whether the regular part is orientable or not.

2.1.4. The Jacobi operator. We stay in codimension $k = 1$ as in the previous section. Let $U \subset \mathbf{R}^{n+1}$ be a bounded set, and $V \in \mathbf{IV}_n(U)$ be a stationary integer varifold with regular part $\operatorname{reg} V$. In what follows we furthermore assume that $U \cap \operatorname{reg} V$ is connected and orientable. On $\operatorname{reg} V$ we can define the linear, elliptic operator $L_V = \Delta_V + |A_V|^2$, commonly called the *Jacobi operator*. Consider another open subset $W \subset\subset U \setminus \operatorname{sing} V$ which lies a positive distance away from $\operatorname{sing} V$. We may then study the operator L_V on the domain $W \cap \operatorname{reg} V$, and write $(\lambda_p(W) \mid p \in \mathbf{N})$ for the spectrum of L_V with zero Dirichlet eigenvalues on $\partial W \cap \operatorname{reg} V$. Recall from the previous section that $\operatorname{reg} V$ is called *stable* if for all $\varphi \in C^1_c(\operatorname{reg} V)$, $\int_{U \cap \operatorname{reg} V} |A_V|^2 \varphi^2 \, \mathrm{d}\|V\| \leq \int_{U \cap \operatorname{reg} V} |\nabla_V \varphi|^2 \, \mathrm{d}\|V\|$. As we assumed $U \cap \operatorname{reg} V$ is connected, the Constancy Theorem [Sim84, Thm 41.1] allows us to say that the density of V is constant everywhere on $\operatorname{reg} V$, and the above inequality becomes $\int_{U \cap \operatorname{reg} V} |A_V|^2 \varphi^2 \, \mathrm{d}\mathcal{H}^n \leq \int_{U \cap \operatorname{reg} V} |\nabla_V \varphi|^2 \, \mathrm{d}\mathcal{H}^n$. After integrating by parts it is not hard to see that this is equivalent to requiring that the spectrum of L_V is non-negative for all open $W \subset\subset U \setminus \operatorname{reg} V$, that is $\lambda_p(W) \geq 0$ for all $p \in \mathbf{N}$. The standard arguments for elliptic operators show that the eigenfunctions of L_V are smooth inside $W \cap \operatorname{reg} V$, and that the eigenfunction corresponding to the least eigenvalue is strictly positive in $W \cap \operatorname{reg} V$.

On the other hand note that the symmetries of V lead to *Jacobi fields*, that is to functions $f \in C^2(\operatorname{reg} V)$ which solve $\Delta_V f + |A_V|^2 f = 0$ in the classical, pointwise sense. This is easiest to see for the translations of V in the direction of the standard basis vectors $e_1, \ldots, e_{n+1}$ of $\mathbf{R}^{n+1}$. The correspondings flows are just translations at unit speed, which preserve the area of V as isometries. In particular the second variation is constant and equals to zero, so that we find $\Delta_V \langle N, e_i \rangle + |A_V|^2 \langle N, e_i \rangle = 0$ for all i.

Now consider the following special case. Let $D_1 \subset \mathbf{R}^n$ be the unit disc and $U = D_1 \times \mathbf{R} \subset \mathbf{R}^{n+1}$. Suppose also given on D_1 the smooth function $u \in C^2(D_1)$ whose graph $G = \{(x, u(x)) \mid x \in D_1\}$ defines a stationary varifold $|G| \in \mathbf{IV}_n(D_1 \times \mathbf{R})$. Note that in this case $|G|$ is completely regular, and we can orient $\operatorname{reg} G$ by the upward-pointing unit normal vector $N = \nu = (1 + |Du|^2)^{-1/2}(-Du, 1)$. In particular note that here $\langle \nu(X), e_{n+1}\rangle = (1+|Du|(x))^{-1/2} > 0$ at all points $X = (x, X^{n+1}) \in G$. Thus $f = \langle \nu, e_{n+1}\rangle$ defines a strictly positive solution of the equation $L_V f = 0$. A general principle for elliptic operators then shows that G is stable—see 3.2.5 for a second description of the argument. Assume, in order to obtain a contradiction, that this fails and for some open subset $W \subset\subset D_1 \times \mathbf{R}$ we have $\lambda_1 = \lambda_1(W) < 0$. There is a positive $\varphi_1 \in C^2(W \cap G) \cap C^0(\overline{W} \cap G)$ with $\varphi_1 = 0$ on $\partial W \cap G$ and $\Delta_G \varphi_1 + |A_G|^2 \varphi_1 = -\lambda_1 \varphi_1$. But then φ_1 must be a subsolution, because $\Delta_G \varphi_1 + |A_G|^2 \varphi_1 > 0$. Comparing this to f yields a contradiction with the maximum principle, and concludes our argument that G is necessarily stable.

2.1.5. Regularity of stable minimal surfaces in codimension one. Let $k = 1$, and let $U \subset \mathbf{R}^{n+1}$ be open. Then [SS81] established the following.

THEOREM 2.1.7 ([SS81]). *Let $U \subset \mathbf{R}^{n+1}$ be open and let $V \in \mathbf{IV}_n(U)$ be a stationary integral varifold $V \in \mathbf{IV}_n(U)$ with stable regular part. If for all compact $K \subset\subset U$,*

$$\mathcal{H}^{n-2}(\operatorname{sing} V \cap K) < \infty,$$

then the singular set $\operatorname{sing} V$

(i) *is empty if* $1 \leq n \leq 6$,
(ii) *is discrete if* $n = 7$,
(iii) *and has* $\dim_{\mathcal{H}} \operatorname{sing} V \leq n - 7$ *if* $n \geq 8$.

Technically in [SS81] the assumptions on the singular set were stated as $\mathcal{H}^{n-2}(\operatorname{sing} V) = 0$, although their arguments are also applicable if one only knows that $\operatorname{sing} V$ has locally finite $\mathcal{H}^{n-2}$-measure or alternatively that $\operatorname{sing} V$ is countably $n-2$–rectifiable. Combining this with the result of [NV15] cited above, this means that if V has stable regular part and $\mathcal{S}^n \setminus \mathcal{S}^{n-2}(V)$ is countably $n-2$–rectifiable then the conclusions of Theorem 2.1.7 hold.

Wickramasekera [Wic14a] later substantially generalised this result by weakening the *a priori* assumption on the singular set of V, only assuming that $U \cap \operatorname{sing} V$ does not contain any so-called classical singularities, which we defined above in (2.1.2). Of course this includes those stable varifolds which have $\mathcal{H}^{n-1}(\operatorname{sing} V) = 0$. That alone is a substantial generalisation over the assumption that $\operatorname{sing} V$ is countably $n-2$–rectifiable.

THEOREM 2.1.8 ([Wic14a]). *Let* $U \subset \mathbf{R}^{n+1}$ *be open and let* $V \in \mathbf{IV}_n(U)$ *be a stationary integral varifold with stable regular part. If* $\operatorname{sing} V$ *does not contain any classical singularities, then it*

(i) *is empty if* $1 \leq n \leq 6$,
(ii) *is discrete if* $n = 7$,
(iii) *and has* $\dim_{\mathcal{H}} \operatorname{sing} V \leq n - 7$ *if* $n \geq 8$.

Wickramasekera [Wic14c] next used this regularity theory to establish a sharp version of Ilmanen's singular maximum principle [Ilm96], which holds in the following setting. Let $V_1, V_2 \in \mathbf{IV}_n(U)$ be stationary integral varifold, so that $\operatorname{spt}\|V_2\|$ lies locally on one side of $\operatorname{reg} V_1$ in the following sense.

Hypothesis K. *For all* $Y \in \operatorname{reg} V_1 \cap \operatorname{spt}\|V_2\|$, *there is* $\rho > 0$ *so that*

(a) $\operatorname{sing} V_1 \cap B_\rho(Y) = \emptyset$,
(b) $B_\rho(Y) \setminus \operatorname{spt}\|V_1\|$ *is disconnected,*
(c) $\operatorname{spt}\|V_2\| \cap B_\rho(Y)$ *belongs to one of the two connected components in the complement of* $\operatorname{spt}\|V_1\|$.

THEOREM 2.1.9 ([Wic14c]). *Let* $U \subset \mathbf{R}^{n+1}$ *be open and* $V_1, V_2 \in \mathbf{IV}_n(U)$ *be stationary integral varifolds satisfying* Hypothesis K. *If* $\mathcal{H}^{n-1}(\operatorname{sing} V_1) = 0$ *then either* $\operatorname{spt}\|V_1\| \cap \operatorname{spt}\|V_2\| = \emptyset$ *or* $\operatorname{spt}\|V_1\| = \operatorname{spt}\|V_2\|$.

2.1.6. Wickramasekera's branched sheeting theorem. Here we are still working in codimension one, that is $k = 1$. Moreover recall $B_r = B_r^{n+1}$ is the open ball of radius $r > 0$. The nature of the results listed in this section requires us to use some notation for two-valued functions we introduce later in 3.1, notably given a two-valued function $u = \{u_1, u_2\}$ we abbreviate $u_1 \wedge u_2 = \max\{u_1, u_2\}$ and $u_1 \vee u_2 = \min\{u_1, u_2\}$.

THEOREM 2.1.10 ([Wic20]). *There is* $\epsilon = \epsilon(n) > 0$ *so that if* $V \in \mathbf{IV}_n(B_2)$ *is stationary with stable regular part,* $\|V\|(B_2)/(\omega_n 2^n) < 2 + \epsilon$, $\Theta(\|V\|, Y) \neq 3/2$ *for all* $Y \in B_2$ *and*

(i) *either* $\int_{B_2} \operatorname{dist}(X, \Pi)^2 \, \mathrm{d}\|V\|(X) < \epsilon$ *for some* $\Pi \in Gr(n, n+1)$,
(ii) *or* $\int_{B_2} \operatorname{dist}(X, \Pi_1)^2 \wedge \operatorname{dist}(X, \Pi_2)^2 \, \mathrm{d}\|V\|(X) + \int_{B_2} \operatorname{dist}(X, \operatorname{spt}\|V\|)^2 \, \mathrm{d}(\|\Pi_1\| + \|\Pi_2\|)(X) < \epsilon$ *for some* $\Pi_1 \neq \Pi_2 \in Gr(n, n+1)$.

Then

(i) *either* $B_1 \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V \cup \mathcal{C}_V \cup \mathcal{B}_V$ *and there is* $\gamma = \gamma(n, \epsilon) \in (0, 1)$ *and a two-valued function* $u \in C^{1,\gamma}(B_1 \cap \Pi; \mathcal{A}_2(\Pi^\perp))$ *so that* $B_1 \cap \operatorname{spt}\|V\| \subset \operatorname{graph} u \subset \operatorname{reg} V \cup \mathcal{C}_V \cup \mathcal{B}_V$. *Moreover there is* $C = C(n, \epsilon) > 0$ *so that* $|u|_{1,\gamma; B_1 \cap \Pi} \leq C\big(\int_{B_2} \operatorname{dist}(X, \Pi)^2 \, \mathrm{d}\|V\|(X)\big)^{1/2}$.
(ii) *or* $B_1 \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V \cup \mathcal{C}_V$ *and there is* $\gamma = \gamma(n, \epsilon) \in (0, 1)$ *and two single-valued functions* $u_i \in C^{1,\gamma}(B_1 \cap \Pi_i, \Pi_i^\perp)$ *so that* $B_1 \cap \operatorname{spt}\|V\| \subset$

$\cup_i \operatorname{graph} u_i \subset \operatorname{reg} V \cup \mathcal{C}_V$. *Moreover there is* $C = C(n,\epsilon) > 0$ *so that* $|u_i|_{1,\gamma;B_1\cap\Pi} \leq C\big(\int_{B_2} \operatorname{dist}(X,\Pi_1)^2 \wedge \operatorname{dist}(X,\Pi_2)^2 \,\mathrm{d}\|V\|(X) + \int_{B_2} \operatorname{dist}(X,\operatorname{spt}\|V\|)^{1/2}\,\mathrm{d}(\|\Pi_1\| + \|\Pi_2\|)(X).\big)^{1/2}$.

Elliptic regularity theory is not available for two-valued minimal graphs. However, Simon–Wickramasekera [SW16] have shown that necessarily $u \in C^{1,1/2}(B_1 \cap \Pi; \mathcal{A}_2(\Pi^\perp))$. Of course, in the second case described above, where V is close to $|\Pi_1| + |\Pi_2|$ and $V \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V \cup \mathcal{C}_V$ the two functions u_i are in fact smooth. Moreover the usual, single-valued elliptic regularity applies, and gives that for all $l \in \mathbf{Z}_{>0}$ there is $C_l = C(n,l)$ so that $|u_i|_{l,\gamma;B_1\cap\Pi_i} \leq C_l\big(\int_{B_2} \operatorname{dist}(X,\Pi_1)^2 \wedge \operatorname{dist}(X,\Pi_2)^2\,\mathrm{d}\|V\|(X) + \int_{B_2} \operatorname{dist}(X,\operatorname{spt}\|V\|)^{1/2}\,\mathrm{d}(\|\Pi_1\| + \|\Pi_2\|)(X).\big)^{1/2}$

THEOREM 2.1.11 ([Wic20]). *Let* $U \subset \mathbf{R}^{n+1}$ *be open and* $V \in \mathbf{IV}_n(U)$ *be stationary with stable regular part. Then there is* $\epsilon = \epsilon(n) \in (0,1)$ *so that if* $Y \in U \cap \operatorname{spt}\|V\|$ *has* $\Theta(\|V\|, Y) < 2 + \epsilon$ *then* $Y \in \operatorname{reg} V \cup \mathcal{B}_V \cup \mathcal{C}_V$.

Let us quickly comment on these three possibilities. Let $V \in \mathbf{IV}_n(U)$ be as above, and let $Y \in U \cap \operatorname{spt}\|V\|$ have $\Theta(\|V\|, Y) = 2$. Then either $Y \in \mathcal{B}_V$ or else there is $0 < \rho < \operatorname{dist}(Y, \partial U)$ so that

(1) either $B_\rho(X) \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V$ and there is a smooth embedded Σ so that $V \llcorner B_\rho(X) = 2|\Sigma|$,
(2) or $B_\rho(X) \cap \operatorname{spt}\|V\| \subset \operatorname{reg} V \cup \mathcal{C}_V$ is immersed and there are two smooth embedded surfaces Σ_1, Σ_2 which meet transversely along an axis of immersed, classical singularities, so that $V \llcorner B_\rho(Y) = |\Sigma_1| + |\Sigma_2|$ and $\operatorname{sing} V \cap B_\rho(Y) = \Sigma_1 \cap \Sigma_2$.

2.2. Single-valued minimal graphs and the Bernstein theorem

Let $\Omega \subset \mathbf{R}^n$ be an open, possibly unbounded domain. We say that a smooth function $u \in C^2(\Omega)$ defines a *minimal graph* over Ω if it is a classical solution to the minimal surface equation

$$(1 + |Du|^2)\Delta u - \sum_{i,j=1}^n D_i u D_j u D_{ij} u = 0.$$

To distinguish between these and the two-valued minimal graphs which we will introduce later, we will sometimes say that a minimal graph is *single-valued*. Instead of saying that $u \in C^2(\Omega)$ satisfies the minimal surface equation, we sometimes abbreviate this and say that u is a minimal graph. (Likewise in the two-valued setting we will sometimes say that a two-valued function $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ is a two-valued minimal graph.) By the elliptic regularity for quasilinear elliptic PDE, a single-valued function u as above is automatically smooth. This is not the case for two-valued minimal graphs, whose regularity cannot be improved beyond $C^{1,1/2}(\Omega;\mathcal{A}_2)$.

The asymptotic properties of entire, single-valued minimal graphs can be studied most efficiently using the theory of so-called area-minimising currents.

2.2.1. Area-minimising currents. Let $U \subset \mathbf{R}^{n+k}$ be an open set, and $T \in \mathcal{D}_n(U)$ be an integer multiplicity current. We say that T is *area-minimising* in U if for all bounded open $W \subset\subset U$, $M_W(T) \leq M_W(S)$ for all $S \in \mathcal{D}_n(U)$ with $\partial T = \partial S$ and $\operatorname{spt}(S-T)$ a compact subset of W. Recall that we write $|T| \in \mathbf{IV}_n(U)$ for the integer varifold obtained from T by 'forgetting' its orientation. Suppose that $\partial T = 0$ and T is area-minimising in U. Then $|T|$ is stationary and stable in U, the latter in the ambient sense where any compactly supported deformations are allowed. Area-minimising currents satisfy the following useful compactness property.

Proposition 2.2.1 ([Sim84, Thm 34.5]). *Let $(T_j \mid j \in \mathbf{N})$ be a sequence of area-minimising currents in U, with $\sup_j M_W(T_j) + M_W(\partial T_j) < \infty$ for all bounded open $W \subset\subset U$. Then there is a subsequence $(T_{j'})$ and a minimising current $T \in \mathcal{D}_n(U)$ so that $T_{j'} \to T$ and $|T_{j'}| \to |T|$ in the current and varifold topologies respectively.*

This is stronger than the general compactness theorem for sequences of integral currents $T_j \in \mathbf{I}_n(U)$. There we were also assured the convergence of a subsequence in both the current and varifold topologies, say $T_{j'} \to T$ and $|T_{j'}| \to V$, but in general one has only $|T| \ll V$, not $|T| = V$. (This is precisely what happens in Example 1.2.6.)

Moreover, for Proposition 2.2.1 it is not sufficient for the $|T_j|$ to be stationary, although we may recall the weaker result of Proposition 1.2.10, where one is guaranteed the existence of a convergent subsequence with $|T_{j'}| \to T$ and $|T_{j'}| \to V \in \mathbf{IV}_n(U)$, but in general one cannot improve upon $|V| \ll |T|$.

To see this one may simply consider a sequence of scaled-down catenoids defined above the horizontal plane $\Pi = \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$. In the varifold topology the catenoids converge to $2|\Pi|$, whereas in the current topology there is *mass cancellation.* In the limit the two sheets of the catenoid cancel each other out as they are oriented in opposite directions, and the limit is the zero current.

From the above obtains the existence of tangent cones to area-minimising currents, see for example [Sim84, Thm. 35.1]. We have that $\Theta(\|T\|, X)$ exists at all point $X \in U$, the density function is upper semicontinuous and for each $X \in U \cap \operatorname{spt}\|T\|$ and each sequence of positive scalars $\lambda_j \to 0$ there is a subsequence $(\lambda_{j'})$ so that $\eta_{X,\lambda_{j'}} T \to T_X$ to some integral current

$T_X \in \mathbf{I}_n(\mathbf{R}^{n+k})$ with $\partial T_X = 0$. (A similar point was made in the discussion preceding Lemma 1.2.13.)

In codimension one, that is when $k = 1$, one can obtain powerful regularity results, stronger even than those obtained for stable minimal hypersurfaces. This theory was developed during the 1960s, and was the result of the combined work of de Giorgi [DG61], Federer–Fleming [FF60], Federer [Fed70], Reifenberg [Rei60], Fleming [Fle62], Almgren [Alm66] and Simons [Sim68].

THEOREM 2.2.2 ([Fed69, Thm. 5.4.15]). *Let $2 \leq n \leq 6$, let $T \in \mathbf{I}_n(U)$ be area-minimising in $U \subset \mathbf{R}^{n+1}$ with $\partial T = 0$. Then $\operatorname{sing} T = \emptyset$ and $U \cap \operatorname{spt}\|T\|$ is a smooth embedded surface in U.*

The main technical result behind this theorem is a classification of area-minimising cones. In low dimensions, namely for $n \leq 6$ one can show that these can only be planes, see e.g. [Alm66] for $n = 3$ and [Sim68] for the cases $2 \leq n \leq 6$.

The connection between such cones and minimal graphs was first established by Fleming [Fle62], who showed that if there existed a non-linear, entire single-valued minimal graph $u \in C^2(\mathbf{R}^n)$, then there would exist a non-planar area-minimising cone in $\mathbf{R}^{n+1}$. Such cones can be obtained by taking blowdown sequences of the graph of u. (Indeed from a modern vantage point this follows from the compactness of area-minimising currents combined with the monotonicity formula.) Next de Giorgi [DG61] showed that such cones would necessarily be cylindrical of the form $\mathbf{C} = \mathbf{C}' \times \mathbf{R}e_{n+1} \in \mathbf{I}_n(\mathbf{R}^{n+1})$. As $\mathbf{C}' \in \mathbf{I}_{n-1}(\mathbf{R}^n)$ is also area-minimising, a non-linear entire minimal graph $u \in C^2(\mathbf{R}^n)$ in fact yields an area-minimising cone in $\mathbf{R}^n$. Thus Simons' classification of area-minimising cones yields the following result.

THEOREM 2.2.3 (Bernstein's theorem). *Let $n \leq 7$. Then every single-valued minimal graph $u \in C^2(\mathbf{R}^n)$ is linear.*

In larger dimensions area-minimising currents can develop singularities. The first example of this was the cone $\mathbf{C}_S \subset \mathbf{R}^8$ found by [BDGG69], with link $\operatorname{spt}\|\mathbf{C}_S\| \cap S^7 = \{(X,Y) \in \mathbf{R}^4 \times \mathbf{R}^4 \mid |X| = |Y|\} \cap S^7$. The authors also give a construction for an entire minimal graph defined on $\mathbf{R}^8$ which is asymptotic at infinity to $\mathbf{C}_S \times \mathbf{R}e_9$. The same methods work in every even dimension. Later, Simon [Sim89] constructed a plethora of new examples of entire minimal graphs.

Despite these examples of singular cones, a so-called dimension reduction argument shows that the singular set of an area-minimising current T say has codimension at least seven, that is $\dim_{\mathcal{H}} \operatorname{sing} T \leq n - 7$. For stable minimal hypersurfaces, such a result is only possible if one *a priori* excludes at least classical singularities, see Theorem 2.1.8 for a statement of the most general available result, proved by [Wic14a]. This is one of the complicating factors of working with two-valued minimal graphs, as these generally have both branch points and classical singularities.

2.2.2. Minimal graphs and calibrations. Let $\Omega \subset \mathbf{R}^n$ be an open set, and $u \in C^2(\Omega)$ define a minimal graph. By elliptic regularity such a function is automatically smooth. We define a smooth n-form on $\Omega \times \mathbf{R}$ by setting $\omega(x, X^{n+1}) = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n + \sum_{i=1}^n (-1)^i D_i u(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^{n+1}$ at all $X = (x, X^{n+1}) \in \Omega \times \mathbf{R}$. Equivalently this can be defined by imposing that for any vectors $V_1, \ldots, V_n \in \mathbf{R}^{n+1}$, $\omega(V_1, \ldots, V_n) = \det(V_1, \ldots, V_n, \nu)$ where recall $\nu = (1 + |Du|^2)^{-1/2}(-Du, 1)$ is the upward-pointing unit normal to G. (Note that the form ω is constant along lines of the form $\{x\} \times \mathbf{R}$ where $x \in \Omega$.)

Suppose that $\{V_1, \ldots, V_n\}$ form an orthonormal family of vectors. From the second characterisation we find that $|\omega(V_1, \ldots, V_n)| \leq 1$ with equality if and only $T_X G = \mathrm{span}\{V_1, \ldots, V_n\}$. Additionally from the coordinate definition one may check that ω is closed, that is $\mathrm{d}\omega = 0$ in $\Omega \times \mathbf{R}$. These two properties taken together make ω a so-called *calibration*. We will not discuss the rich theory of calibrated currents further, but just explain how to use ω to show that $G = \operatorname{graph} u$ is area-minimising. Define the current $[\![G]\!] \in \mathbf{I}_n(\Omega \times \mathbf{R})$ using the upward-pointing unit normal. This current has boundary $\partial[\![G]\!] = 0$. The following is a standard, well-known fact.

Proposition 2.2.4. *Let $\Omega \subset \mathbf{R}^n$ be open, and $u \in C^2(\Omega)$ define a minimal graph. Then $[\![G]\!] \in \mathbf{I}_n(\Omega \times \mathbf{R})$ is area-minimising in $\Omega \times \mathbf{R}$.*

Proof. Let $W \subset\subset \Omega \times \mathbf{R}$ be a compactly contained bounded open set, $S \in \mathbf{I}_n(\Omega \times \mathbf{R})$ be an integral current with $\mathrm{spt}([\![G]\!] - S) \subset W$ and $\partial S = 0 = \partial[\![G]\!]$. On the one hand, as ω is closed we have $\int_W \omega\,\mathrm{d}S = \int_W \omega\,\mathrm{d}[\![G]\!] = \|G\|(W) = \mathcal{H}^n(G \cap W)$. On the other hand, $\int_W \omega\,\mathrm{d}S \leq \|S\|(W)$, which yields the desired inequality. □

In particular we find that the varifold $|G| = |\operatorname{graph} u| \in \mathbf{IV}_n(\Omega \times \mathbf{R})$ is stationary in $\Omega \times \mathbf{R}$. Let $A < B \in \mathbf{R}$, and note that $[\![G]\!]$ is also minimising in the bounded cylinder $\Omega \times (A, B) \subset \Omega \times \mathbf{R}$, essentially by definition. In what follows we need to slightly strengthen this property. Here we take $\Omega \subset \mathbf{R}^n$ a bounded open set with Lipschitz boundary, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We may consider $[\![G]\!]$ as a current in $\mathbf{R}^{n+1}$, which now has non-zero boundary. That being said, our hypotheses guarantee that $\partial[\![G]\!] \in I_{n-1}(\mathbf{R}^{n+1})$ is an integral current supported inside $\partial\Omega \times \mathbf{R}$. (To see this one need only check that the mass of the boundary is locally finite.) Let moreover $A < B \in \mathbf{R}$, and define the set $G_{A,B} = G \cap \Omega \times (A, B)$.

PROPOSITION 2.2.5. *Let* $\Omega \subset \mathbf{R}^n, u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *define a minimal graph,* $A < B \in \mathbf{R}$ *and* $[\![G_{A,B}]\!]$ *be as above. Let* $T \in \mathbf{I}_n(\mathbf{R}^{n+1})$ *be a current with* $\operatorname{spt}\|T\| \subset \overline{\Omega} \times \mathbf{R}$ *and* $\partial T = \partial[\![G_{A,B}]\!]$. *Then* $\mathcal{H}^n(G \cap \Omega \times (A, B)) \leq \|T\|(\mathbf{R}^{n+1})$.

REMARK 2.2.6. This result is perhaps slightly more subtle than meets the eye, as it is not true in general that $\mathcal{H}^n(G\cap\Omega\times(A, B)) \leq \|T\|(\Omega\times(A, B))$, nor even can it be bounded by $\|T\|(\Omega\times\mathbf{R})$, the mass of the current in the interior. To see this, consider the following simple example. Take $\Omega = D_1$ equal to the unit disc, and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be so that $\mathcal{H}^n(G) > \mathcal{H}^n(D_1)$. Let $M > \sup_{D_1} u$, and consider the open set $U = \{(x, X^{n+1}) \in D_1 \times \mathbf{R} \mid u(x) < X^{n+1} < M\}$. This is a Caccioppoli set, and the corresponding current $[\![U]\!]$ has boundary $\partial[\![U]\!] = -[\![G]\!] + S$, where $\operatorname{spt}(S - [\![D_1 \times \{M\}]\!]) \subset \partial D_1 \times \mathbf{R}$. Then $\partial(S - [\![G]\!]) = 0$, and yet $\|S\|(D_1 \times \mathbf{R}) = \mathcal{H}^n(D_1) < \mathcal{H}^n(G)$.

PROOF. We start by modifying T so as to ensure that it has compact support. As the slab $\mathbf{R}^n \times [A, B]$ is convex, the projection $P_{A,B} : \mathbf{R}^{n+1} \to \mathbf{R}^n \times [A, B]$ onto it does not increase mass. This is a proper Lipschitz map, so we can consider the pushforward $T_{A,B} = P_{A,B\#}T$. The current $T_{A,B}$ has total mass $\|T_{A,B}\|(\mathbf{R}^{n+1}) \leq \|T\|(\mathbf{R}^{n+1})$, and $\operatorname{spt} T_{A,B} \subset \overline{\Omega} \times [A, B]$.

Let ξ be a Lipschitz compactly supported vector field in $\mathbf{R}^{n+1}$ which vanishes on $\partial G_{A,B}$ and points into $\Omega \times \mathbf{R}$ at all other points of $\partial(\Omega \times (A, B))$. Then the flow Φ_t of ξ leaves the boundary invariant, and moves the remainder of $T_{A,B}$ inward. Therefore $\operatorname{spt}\|\Phi_{t\#}T_{A,B}\| \cap \partial(\Omega \times (A, B)) = \partial G_{A,B}$, and the total mass of $\Phi_{t\#}T_{A,B}$ lies in the interior of the region, that is $\|\Phi_{t\#}T_{A,B}\|(\mathbf{R}^{n+1}) = \|\Phi_{t\#}T_{A,B}\|(\Omega \times (A, B))$. Then repeating the calculations from the previous proof we find that $\|\Phi_{t\#}T_{A,B}\|(\Omega \times (A, B)) \geq \mathcal{H}^n(G\cap\Omega\times(A, B))$. We conclude that $\|T_{A,B}\|(\mathbf{R}^{n+1}) \geq \mathcal{H}^n(G\cap\Omega\times(A, B))$ by letting $t \to 0$. □

2.2.3. A Jenkins–Serrin type lemma. Let $\Omega \subset \mathbf{R}^n$ be a convex, bounded domain with piecewise smooth boundary. Let $(\Omega_j \mid j \in \mathbf{N})$ be a sequence of bounded open sets with $\operatorname{dist}_{\mathcal{H}}(\Omega_j, \Omega) \to 0$ as $j \to \infty$. These need neither be convex, nor as regular as Ω. Suppose given, on every Ω_j, a smooth function $u_j \in C^2(\Omega_j)$ whose graph G_j is minimal. Depending on whether we consider this endowed with an orientation or not, we can associate to G_j the varifold $|G_j|$ and the current $[\![G_j]\!]$ in $\mathbf{R}^{n+1}$. We have $|G_j| \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ but in general it is only stationary inside $\Omega_j \times \mathbf{R}$. In the same spirit $[\![G_j]\!] \in \mathbf{I}_n(\mathbf{R}^{n+1})$ but $\partial[\![G_j]\!]$ is only zero in $\Omega_j \times \mathbf{R}$. We saw previously that the current $[\![G_j]\!]$ is area-minimising inside the open set $\Omega_j \times \mathbf{R}$.

Let $T = T_0 \times \mathbf{R}e_{n+1} \in \mathbf{I}_n(\mathbf{R}^{n+1})$ be a vertical current with $\operatorname{spt} T_0 \subset \partial\Omega$. Suppose further T_0, and hence also T, has piecewise smooth support. We write $\operatorname{spt} T_0 = N_0 \cup \cup_{k\leq K}\Gamma_k^0$ where the $\Gamma_k^0 \subset \partial\Omega$ are smooth embedded and $N_0 = \cup_{k\leq K}\partial\Gamma_k^0$ has $\mathcal{H}^{n-1}(N_0) < \infty$. Likewise we write $\operatorname{spt} T = N \cup \cup_{k\leq K}\Gamma_k$ where $N = N_0 \times \mathbf{R}$ and $\Gamma_k = \Gamma_k^0 \times \mathbf{R}$. It is convenient, though slightly imprecise, to abbreviate these as $\operatorname{reg} T_0 = \cup_k \Gamma_k^0$ and $\operatorname{sing} T_0 = \cup_k \partial\Gamma_k^0$. We assume moreover that T, T_0 have constant multiplicity one on their regular part. We can write $T_0 = [\![\Gamma_0]\!]$ and $T = [\![\Gamma]\!] = T \times \mathbf{R}e_{n+1}$, where both are oriented to point inwards, that is into Ω.

The following result is inspired by the work of Jenkins–Serrin in dimension $n = 2$, see [JS66a, JS66b].

THEOREM 2.2.7. *Let* $\Omega, \Omega_j \subset \mathbf{R}^n \times \{0\}$*, and* $u_j \in C^2(\Omega_j)$ *be as above. Suppose that* $\operatorname{dist}_{\mathcal{H}}(\Omega_j, \Omega) \to 0$ *and* $[\![G_j]\!] \to T$ *as* $j \to \infty$*, and that for all* $0 < \tau < 1 < A$ *there is* $J(\tau, A) \in \mathbf{N}$ *so that for* $j \geq J(\tau, A)$*,*

$$(\operatorname{reg} T)_\tau \cap \{|X|^{n+1} < A\} \cap \operatorname{spt} \partial[\![G_j]\!] \subset (\operatorname{sing} T)_\tau.$$

Then

$$\|T_0\| \leq 1/2\mathcal{H}^{n-1}(\partial\Omega).$$

REMARK 2.2.8. The estimate is sharp, as can be seen by considering the following example. Let $n = 2$, and consider $\Omega = (-\pi/2, \pi/2)\times(-\pi/2, \pi/2) \subset \mathbf{R}^2$ and define a function

$$u(x_1, x_2) = \log\Big(\frac{\cos x_1}{\cos x_2}\Big).$$

(This is the fundamental piece used to construct *Scherk's doubly-periodic surface*, see [Sch35].) Note that this function u diverges to $+\infty$ near the sides $\{\pm\pi/2\} \times (-\pi/2, \pi/2)$ and to $-\infty$ near the other sides of the square, $(-\pi/2, \pi/2) \times \{\pm\pi/2\}$. Then the graphs associated to $u_j = u - j$, with $\Omega_j = \Omega$ kept fixed, converge to the union of the two planes $[\![\{\pm\pi/2\} \times (-\pi/2, \pi/2)]\!] \times \mathbf{R}e_3$ in the topology of currents. The length of the corresponding portion of the boundary is 2π, precisely half of the total length of the boundary $\partial\Omega$.

PROOF. We ultimately obtain the inequality by constructing a comparison surface. We begin the construction by making the following two assumptions:

(1) $\Omega \subset \Omega_j$ for all j,
(2) and for all $0 < \tau < 1 < A$ there is $J = J(\tau, A) \in \mathbf{N}$ so that $|u_j| > 2A$ on $\Omega \setminus (\operatorname{spt} T_0)_\tau$ for all $j \geq J$.

We will explain later why this may be done without restriction of generality.

Let $0 < \tau < \sigma < 1 < A$ be given, with the eventual aim of letting $\sigma, \tau \to 0$ and $A \to \infty$, and assume that $j \geq J(\tau, A)$ is large enough to guarantee the validity of (2). We may perturb both by a small amount to guarantee that the level sets $\{u_j = \pm A\}$ and the boundary of the tubular neighbourhood $(\operatorname{spt} T_0)_\tau$ are smooth inside Ω_j, justifying this by Sard's lemma. (Note however that both are allowed to meet the boundary of Ω_j along a set which is not in general embedded.) For the remainder of the proof whenever we adjust the values of τ, σ or A we assume that we do so in a way which preserves this property.

The first adjustment we make to τ, σ is to take them small enough that $(\operatorname{spt} T_0)_\tau \cap \partial\Omega \subset \operatorname{spt} T_0 \cup (\operatorname{sing} T_0)_\sigma$. This is possible because $\partial\Omega$ and $\operatorname{spt} T_0$ are piecewise smooth. Moreover we may impose that $(\Gamma_k)_\tau \cap (\Gamma_l)_\tau \subset (\operatorname{sing} T_0)_\sigma$ for all $k \neq l$ which means that we can decompose $\operatorname{spt} T_0 \cap E_{\tau,\sigma} = \cup_k (\Gamma_k)_\tau \cap E_{\tau,\sigma}$ into a union of pairwise disjoint sets. Here and in what follows we set $E_{\tau,\sigma} = (\operatorname{spt} T_0)_\tau \setminus [\operatorname{sing} T_0]_\sigma = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, \operatorname{spt} T_0) < \tau, \operatorname{dist}(x, \operatorname{sing} T_0) > \sigma\}$. Define also the open set $\Omega_{j,\tau,\sigma} = (\Omega \cup (\operatorname{spt} T_0)_\tau) \cap \Omega_j \setminus [\operatorname{sing} T_0]_\sigma$. We may take $J(\tau, \sigma, A) \in \mathbf{N}$ large enough that $|u_j| > 2A$ in $\Omega_{j,\tau,\sigma}$ provided $j \geq J(\tau, \sigma, A)$. (In fact even $u_j > 2A$ holds true in that region.) From a geometric point of view this is equivalent to saying that $\partial [\![G_j]\!] \cap \Omega_{j,\tau,\sigma} \cap (-2A, 2A) = \emptyset$. As a consequence we can restrict the graphs to smaller set, making sure that $|G_j| \llcorner \Omega_{j,\tau,\sigma} \times (-3/2A, 3/2A)$ is stationary. In particular $|G_j| \llcorner \Omega_{j,\tau,\sigma} \times (-3/2A, 3/2A)$ is stationary in the open set $(\Gamma_k^0)_\tau \setminus [\partial\Gamma_k^0]_\sigma \times (-3/2A, 3/2A)$ for all k. Using the Allard regularity theorem we know that there is a function $U_{j,k} = U_{j,k,\tau,\sigma,A} \in C^\infty(\Gamma_k, \Gamma_k^\perp)$ defined on some subset of Γ_k with values in the direction normal to it, so that $G_j \cap (\Gamma_k^0)_\tau \setminus [\partial\Gamma_k^0]_\sigma \times (-5/4A, 5/4A) \subset \operatorname{graph} U_{j,k}$. (In principle from Allard regularity one obtains only a function defined on an open domain of Γ_k with this property. As $\operatorname{graph} U_{j,k} \setminus \Omega_{j,\tau,\sigma} \times (-5/4A, 5/4A)$ has no further importance to us, we can extend this function smoothly to the whole surface Γ_k in an arbitrary way.) Given any $\epsilon > 0$ we can update $J(\tau, \sigma, A, \epsilon) \in \mathbf{N}$ to a larger value which ensures that $\sup_{\Gamma_k} |U_{j,k}| \leq \epsilon$ for all k and $j \geq J(\tau, \sigma, A, \epsilon)$. (By elliptic regularity this could be extended to hold in any Hölder norm, as indeed when one extends $U_{j,k}$ to Γ_k one can do so in a way that preserves the smallness of the function and its derivatives.)

Consider the open set $W_{j,\tau,\sigma,A} \subset \Omega_{j,\tau,\sigma} \times (-A, A)$ defined by

$$W_{j,\tau,\sigma,A} = \{(x, X^{n+1}) \in \Omega_{j,\tau,\sigma} \times \mathbf{R} \mid u_j(x) \vee -A < X^{n+1} < A\},$$

where we abbreviate $u_j \vee -A = \min\{u_j, -A\}$. This is a Caccioppoli set, and its topological boundary is contained inside, but in general not equal to, the union of the following sets:

(1) $(\{u_j < A\} \cap \Omega_{j,\tau,\sigma}) \times \{A\}$ and $(\{u_j < -A\} \cap \Omega_{j,\tau,\sigma}) \times \{-A\}$,
(2) the boundary away from $\operatorname{spt} T_0$, $\partial\Omega \setminus ([\operatorname{sing} T_0]_\sigma \cup [\operatorname{spt} T_0]_\tau) \times [-A, A]$,
(3) the boundaries of the tubular neighbourhoods, $\partial(\operatorname{sing} T_0)_\sigma \cap \Omega_{j,\tau,\sigma} \times [-A, A] \subset \partial(\operatorname{sing} T_0)_\sigma \times [-A, A]$.
(4) graphical portions near $\operatorname{reg} T_0$, $G_j \cap \Omega_{j,\tau,\sigma} \times [-A, A] = \cup_k G_j \cap (\Gamma_k)_\tau \setminus (\partial\Gamma_k)_\sigma \times [-A, A]$.

For a precise equality we decompose the current boundary into

$$\begin{aligned}
-\partial[\![W_{j,\tau,\sigma,A}]\!] &= [\![(\{u_j < A\} \cap \Omega_{j,\tau,\sigma}) \times \{A\}]\!] \\
&\quad + [\![(\{u_j < -A\}) \cap \Omega_{j,\tau,\sigma}) \times \{-A\}]\!] \\
&\quad + [\![\partial\Omega \times (-A, A) \setminus ([\operatorname{sing} T]_\sigma \cup \operatorname{spt} T)]\!] \\
&\quad + [\![\partial W_{j,\tau,\sigma,A} \cap \partial(\operatorname{sing} T)_\sigma]\!] \\
&\quad + [\![G_j \cap \Omega_{j,\tau,\sigma} \times (-A, A)]\!].
\end{aligned}$$

As we consider $-\partial[\![W_{j,\tau,\sigma,A}]\!]$ on the left-hand side, the currents on the right-hand side are oriented to point inwards into $W_{j,\tau,\sigma,A}$. This aligns with the orientation that we chose for the current T. We estimate the areas of all the summands separately. Let a small $\delta > 0$ be given. For the first two we respectively find

$$\mathcal{H}^n((\{u_j < \pm A\} \cap \Omega_{j,\tau,\sigma}) \times \{\pm A\}) \le \mathcal{H}^n(\Omega_{j,\tau,\sigma}) \le \mathcal{H}^n(\Omega_j).$$

Using the convergence $\Omega_j \to \Omega$ in the Hausdorff distance, we find that there is $J = J(\tau, \sigma, A, \delta) \in \mathbf{N}$ so that $\mathcal{H}^n(\Omega_j) \le \mathcal{H}^n(\Omega) + \delta$ for all $j \ge J$.

The third term, $[\![\partial\Omega \times (-A, A) \setminus ([\operatorname{sing} T]_\sigma \cup \operatorname{spt} T)]\!]$ does not actually depend on j, whence we find that taking σ, τ small enough in terms of δ we can ensure that

$$\begin{aligned}
&\mathcal{H}^n(\partial\Omega \times [-A, A] \setminus ([\operatorname{sing} T]_\sigma \cup [\operatorname{spt} T]_\tau)) \\
&\qquad\qquad \le 2A(\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\operatorname{spt} T_0)) + 2A\delta.
\end{aligned}$$

We bound $\mathcal{H}^n([\![\partial W_{j,\tau,\sigma,A} \cap \partial(\operatorname{sing} T)_\sigma]\!]) \le \mathcal{H}^n(\partial(\operatorname{sing} T)_\sigma \times \{|X|^{n+1} \le A\})$. To estimate this, take $\sigma > 0$ small enough that

$$\begin{aligned}
\mathcal{H}^n([\![\partial W_{j,\tau,\sigma,A} \cap \partial(\operatorname{sing} T)_\sigma]\!]) &\le 2A\mathcal{H}^{n-1}(\partial(\operatorname{sing} T_0)_\sigma) \\
&\le 4\pi A\sigma\mathcal{H}^{n-2}(\operatorname{sing} T_0) + 2A\delta.
\end{aligned}$$

Decompose the graph G_j inside $\Omega_{j,\tau,\sigma} \times [-5/4A, 5/4A]$ into a union of graphs like $[\![G_j \cap \Omega_{j,\tau,\sigma} \times [-5/4A, 5/4A]]\!] = \sum_{k=1}^K [\![\operatorname{graph} U_{j,k} \cap (\Gamma_k)_\tau \setminus [\partial\Gamma_k]_\sigma \cap$

$\{X^{n+1} < 5/4A\}]\!]$. Considering every set $(\Gamma_k)_\tau \setminus [\partial \Gamma_k]_\sigma \cap \{|X^{n+1}| < 5/4A\}$ separately we find that $|G_j| \llcorner (\Gamma_k)_\tau \setminus (\partial\Gamma_k)_\sigma \cap \{|X^{n+1}| < 5/4A\} \to |\Gamma_k| \llcorner (\Gamma_k)_\tau \setminus [\partial\Gamma_k]_\sigma \cap \{|X^{n+1}| < 5/4A\}$ as $j \to \infty$. Update $j \geq J(\tau, \sigma, A, \delta)$ so that for all k,

$$\mathcal{H}^n(G_j \cap (\Gamma_k)_\tau \setminus [\partial\Gamma_k]_\sigma \cap \{|X^{n+1}| \leq A\}) \geq 2A\mathcal{H}^{n-1}(\Gamma_k^0 \setminus (\partial\Gamma_k^0)_\sigma) - 2A\delta,$$

and summing these we obtain

$$\begin{aligned}
&\mathcal{H}^n(G_j \cap \Omega_{j,\tau,\sigma} \times [-A, A]) \\
&\qquad \geq 2A(\mathcal{H}^{n-1}(\operatorname{spt} T_0) - \mathcal{H}^{n-1}((\operatorname{sing} T_0)_\sigma)) - 2KA\delta \\
&\qquad \geq 2A(\mathcal{H}^{n-1}(\operatorname{spt} T_0) - 2\pi\sigma\mathcal{H}^{n-2}(\operatorname{sing} T_0) - \delta) - 2KA\delta,
\end{aligned}$$

provided we change σ to a suitably small value in terms of δ.

We use these area bounds to compare the currents $[\![G_j \cap \Omega_{j,\tau,\sigma} \times (-A, A)]\!]$ and

$$-T_{j,A} = -\partial[\![W_{j,\tau,\sigma,A}]\!] - [\![G_j \cap \Omega_{j,\tau,\sigma} \times (-A, A)]\!].$$

By construction these have $\partial T_{j,A} = \partial[\![G_j \cap \Omega_{j,\tau,\sigma} \times (-A, A)]\!]$. By Proposition 2.2.5, we then have

$$\mathcal{H}^n(G_j \cap \Omega_{j,\tau,\sigma} \cap (-A, A)) \leq \|T_{j,A}\|(\mathbf{R}^{n+1}) = \mathcal{H}^n(\operatorname{spt} T_{j,A}).$$

Substituting our term-by-term calculations into this inequality we find

$$\begin{aligned}
&2A(\mathcal{H}^{n-1}(\operatorname{spt} T_0) - 2\pi\sigma\mathcal{H}^{n-2}(\operatorname{sing} T_0) - \delta) - 2KA\delta \\
&\leq 2\mathcal{H}^n(\Omega) + 2\delta + 2A(\mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\operatorname{spt} T_0)) + 2A\delta + 4\pi A\sigma\mathcal{H}^{n-2}(\operatorname{sing} T_0) + 2A\delta,
\end{aligned}$$

whence after dividing by $2A$ we get

$$\begin{aligned}
&\mathcal{H}^{n-1}(\operatorname{spt} T_0) - 2\sigma\pi\mathcal{H}^{n-2}(\operatorname{sing} T_0) - \delta(1 + K) \\
&\leq \mathcal{H}^n(\Omega)/A + \mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\operatorname{spt} T_0) + 2\sigma\pi\mathcal{H}^{n-2}(\operatorname{sing} T_0) + \delta(2 + 1/A).
\end{aligned}$$

This simplifies to

$$\begin{aligned}
&2\mathcal{H}^{n-1}(\operatorname{spt} T_0) \\
&\qquad \leq \mathcal{H}^n(\Omega)/A + \mathcal{H}^{n-1}(\partial\Omega) + 4\sigma\pi\mathcal{H}^{n-2}(\operatorname{sing} T_0) + \delta(K + 3 + 1/A).
\end{aligned}$$

The desired inequality follows after letting $A \to \infty$, $\delta, \sigma, \tau \to 0$ and $j \geq J(\sigma, \tau, A, \delta) \to \infty$. To conclude it only remains to justify the two assumptions (1) and (2).

(1) After translating Ω we may assume that it contains the origin. The convexity of Ω and the piecewise regularity of its boundary ensure the following. For all $\tau > 0$ there is $\delta > 0$ so that $\eta_{0,(1+\delta)\#}\Omega \subset \Omega \setminus (\partial\Omega)_\tau$ and $(\Omega)_\tau \subset \eta_{0,(1+\delta)^{-1}\#}\Omega$. Moreover as $\tau \to 0$ we may impose that $\delta \to 0$ also.

Given $\tau > 0$ take $J(\tau) \in \mathbf{N}$ so that $\Omega \setminus (\partial\Omega)_\tau \subset \Omega_j \subset (\Omega)_\tau$. Taking $\delta > 0$ as above we get that $\eta_{0,1+\delta\#}\Omega \subset \Omega_j \subset \eta_{0,(1+\delta)^{-1}\#}\Omega$. After rescaling we find a sequence $(\eta_{0,1+\delta\#}[\![G_j]\!] \mid j \geq J(\tau))$ of single-valued minimal graphs respectively defined over $\eta_{0,1+\delta\#}\Omega_j$. Moreover $\eta_{0,1+\delta\#}[\![G_j]\!] \to T$ as $j \to \infty$ and $\eta_{0,1+\delta\#}T \to T$ as $\delta \to 0$. We may then diagonally extract a subsequence of graphs $[\![G_{j'}]\!]$ and find a sequence of positive scalars with $\delta_{j'} \to 0$ as $j' \to \infty$, so that $\Omega \subset \eta_{0,1+\delta_{j'}}\Omega_{j'}$ for all j' and $\eta_{0,1+\delta_{j'}\#}[\![G_{j'}]\!] \to T$ as $j' \to \infty$. Upon replacing our original sequence by this rescaled subsequence, we may assume throughout that $\Omega \subset \Omega_j$ without restriction of generality.

(2) Consider a compactly contained open subset $\Omega' \subset\subset \Omega$. We show that there is $J(\Omega', A) \in \mathbf{N}$ so that $|u_j| > 2A$ on Ω'. This follows from a simple contradiction argument. Suppose otherwise, and consider a sequence of points $X_j = (x_j, X_j^{n+1}) \in G_j\Omega' \times [-2A, 2A]$. We may then extract a subsequence converging to a point $X = (x, X^{n+1}) \in \overline{\Omega}' \times [-2A, 2A]$. Let $0 < \rho < \operatorname{dist}(\Omega', \partial\Omega)$, and note that $\|T\|(\overline{B}_\rho(X)) \geq \limsup_{j\to\infty}\|G_j\|(\overline{B}_\rho(X))$. Take $j \geq J(\rho)$ so that $|X_j - X| < \rho/2$, and note that because $\Omega \subset \Omega_j$ we have also that $B_{\rho/2}(X_j) \subset\subset \Omega_j \times \mathbf{R}$. By the monotonicity formula we find $\|G_j\|(B_{\rho/2}(X_j)) \geq \omega_n(\rho/2)^n$, whence also $\|T\|(\overline{B}_\rho(X)) \geq \omega_n(\rho/2)^n$. As $\operatorname{spt} T \subset \partial\Omega$, this is absurd. Now let $\delta > 0$ be given, and consider the open set $\Omega' = \Omega \setminus [\partial\Omega \setminus (\operatorname{spt} T_0)]_\delta$. Note that this is not compactly contained inside Ω anymore. Take any $0 < \tau < 1 < A$. We may assume that $\tau < \delta$, and then observe that $\Omega' \setminus (\operatorname{spt} T_0)_\tau \subset \Omega \setminus (\partial\Omega)_\tau$ lies a distance at least τ away from $\partial\Omega_j$, for all j. We can therefore argue in the same way as above. Namely, consider a sequence $X_j \in \Omega' \setminus (\operatorname{spt} T_0)_\tau \times [-2A, 2A]$, which again we assume convergent to a limit point X. This point lies in $\overline{\Omega}' \setminus (\operatorname{spt} T_0)_\tau \times [-2A, 2A]$, and arguing as above we find $\|T\|(\overline{B}_{\rho/2}(X)) \geq \omega_n(\rho/2)^n$ for all $0 < \rho < \tau$, from whence a contradiction immediately follows. Therefore we can apply the argument above to Ω' to get $\mathcal{H}^n(\operatorname{spt} T_0 \cap \Omega' \times \mathbf{R}) \leq 2\mathcal{H}^n(\partial\Omega')$, whence after letting $\tau \to 0$ we find that $\mathcal{H}^n(\operatorname{spt} T_0) \leq 1/2\mathcal{H}^n(\partial\Omega)$, as required. □

The following special case is of particular importance in what follows. Let $\pi = \pi_0 \times \mathbf{R}e_{n+1}, \pi' = \pi_0' \times \mathbf{R}e_{n+1}$ be two n-dimensional half-planes which do not form a plane. Suppose that they meet along an axis $L = L_0 \times \mathbf{R}e_{n+1}$ at which they form a positive angle $0 < \theta < \pi$ taken in the counterclockwise direction. Let N, N' be their respective unit normals, which we both take pointing in the counterclockwise direction. Further let $p, p' \in L^\perp$ be the vectors which direct π, π' respectively. Any point in $\mathbf{R}^n$ can be written $x = y + z = y + tp + t'p$ with $y \in L_0, z \in L_0^\perp$. Define $Q = \{x = y + z \in \mathbf{R}^n \mid |y| < 1, |z| < 1\}$ and the wedge-shaped region $V = \{x \in Q \mid \langle x, N\rangle > 0, \langle x, N'\rangle < 0\} = \{x = y + z = y + tp + t'p' \in \mathbf{R}^n \mid |y| < 1, t, t' > 0, t^2 + t'^2 < 1\}$. We then have the following result, for which the orientation of the half-planes is crucial.

LEMMA 2.2.9. *Let π, π' and $V \subset \mathbf{R}^n$ be as above, and let the current $T = [\![\pi \cup \pi' \cap \partial V \times \mathbf{R}]\!] \in \mathbf{I}_n(\mathbf{R}^{n+1})$ be oriented inward. Then there does not exist a sequence $(\Omega_j \mid j \in \mathbf{N})$ of bounded open domains $\Omega_j \subset \mathbf{R}^n$ and a sequence $(u_j \mid j \in \mathbf{N})$ of functions $u_j \in C^2(\Omega_j)$ with G_j minimal and*

$$\operatorname{dist}_{\mathcal{H}}(\Omega_j, V) \to 0 \text{ and } [\![G_j]\!] \to T$$

and so that for all $0 < \tau < 1 < A$ there is $J(\tau, A) \in \mathbf{N}$ so that for all $j \geq J(\tau, A)$,

$$(\pi \cup \pi')_\tau \cap \{|X^{n+1}| < A\} \cap \operatorname{spt} \partial [\![G_j]\!] \subset (\{|y| \leq 1, |z| = 0, 1\})_\tau.$$

PROOF. This is essentially a direct consequence of Theorem 2.2.7, although we need to construct a subdomain $\Delta_a \subset V$ to make the area comparison work, where $0 < a < 1/2$ is a small parameter whose value we leave undetermined for now. Let $\Delta_a = \{x = y + tp + t'p' \in V \mid |y| < 1/2, t, t' > 0, t + t' < a\}$ for $0 < a < 1/2$. When we fix $y_0 \in L_0$ with $|y_0| < 1/2$ then $\Delta_a \cap \{x = y_0 + z\}$ is an isosceles triangle with two sides of length a. The domain Δ_a is convex and has piecewise smooth boundary. Moreover, apart from $\pi \cap \partial\Delta_a$ and $\pi' \cap \partial\Delta_a$, the boundary of Δ_a contains only two subsets Γ_1, Γ_2 with positive area, namely $\Gamma_1 = \{|y| = 1/2\}$ and $\Gamma_2 = \{|y| < 1/2, t + t' = a\}$. On the one hand

$$\mathcal{H}^{n-1}(\Gamma_1) = a^2 \sin\theta \text{ and } \mathcal{H}^{n-1}(\Gamma_2) = a \sin(\theta/2)\omega_{n-2}2^{-n+3},$$

and on the other hand

$$\mathcal{H}^{n-1}(\pi \cap \partial\Delta_a) = a\omega_{n-2}2^{-n+2} = \mathcal{H}^{n-1}(\pi' \cap \partial\Delta_a).$$

Comparing the two we have

$$\begin{aligned} \mathcal{H}^{n-1}(\Gamma_1 \cup \Gamma_2) &= a(a \sin\theta + \sin(\theta/2)\omega_{n-2}2^{-n+3}) \\ &< a\omega_{n-2}2^{-n+3} = \mathcal{H}^{n-1}(\pi \cup \pi' \cap \partial\Delta_a) \end{aligned}$$

provided a is small enough. Explicitly it suffices to take it so small that $a \sin\theta < \omega_{n-2}2^{-n+3}(1 - \sin(\theta/2))$. If there were a sequence of minimal graphs $u_j \in C^2(\Omega_j)$ as in the statement, then by restricting them to $\Omega_j \cap \Delta_a$ and letting $j \to \infty$ we would obtain a contradiction to Theorem 2.2.7. This concludes the proof. □

CHAPTER 3

TWO-VALUED MINIMAL GRAPHS

3.1. TWO-VALUED FUNCTIONS

3.1.1. Unordered pairs. Let $\mathcal{A}_2(\mathbf{R})$ be the set of unordered pairs of real numbers, abbreviated $\mathcal{A}_2$ when no confusion is possible. An element of $\mathcal{A}_2(\mathbf{R})$ is written $\{x,y\} = \{y,x\}$. One can define $\mathcal{A}_2(\mathbf{R})$ as the set obtained by taking the quotient of $\mathbf{R}^2$ under the action by the group $\mathbf{Z}_2$ which transposes the two elements. The quotient map is then $\mathbf{R}^2 \to \mathcal{A}_2(\mathbf{R}) : (x,y) \mapsto \{x,y\}$. Alternatively one could define $\mathcal{A}_2(\mathbf{R})$ as the non-empty subsets of $\mathbf{R}$ counting at most two elements. More generally we can also take unordered pairs of elements of any set X, thus forming $\mathcal{A}_2(X)$. Here X is usually either a finite-dimensional vector space $\mathbf{E}$ say or a subset thereof. Apart from $\mathbf{E} = \mathbf{R}$ we also use $\mathbf{E} = \mathbf{R}^{n+1}$ and $\mathbf{E} = \mathbf{L}(\mathbf{R}^n;\mathbf{R})$, the space of linear functions $\mathbf{R}^n \to \mathbf{R}$.

For any set X one can define the *diagonal map* $\boldsymbol{\Delta} : x \in X \mapsto \{x,x\} \in \mathcal{A}_2(X)$. Moreover, we can extend a map $\Phi : X \to Y$ between two sets in the obvious way, by defining $\boldsymbol{\Phi} : \mathcal{A}_2(X) \to \mathcal{A}_2(Y) : \{x,y\} \mapsto \{\Phi(x), \Phi(y)\}$. For example we define $|\{u_1,u_2\}| = \{|u_1|,|u_2|\} \in \mathcal{A}_2(\mathbf{R}_{\geq 0})$. When $X = \mathbf{E}$ is a vector space, then one can define the *center map* $\boldsymbol{\eta} : \{x,y\} \in \mathcal{A}_2(\mathbf{E}) \mapsto x + y \in \mathbf{E}$. Given a pair $\{x_1,x_2\} \in \mathcal{A}_2(\mathbf{R})$ of real numbers, we define its *average* and *symmetric difference* by $x_a = \frac{1}{2}(x_1 + x_2)$ and $x_s = \{\pm(x_1 - x_2)\}$ respectively. Moreover write $x_+ = \max\{x_1,x_2\} = x_1 \vee x_2$ and $x_- = \min\{x_1,x_2\} = x_1 \wedge x_2$.

Let $X = \mathbf{E}$ be a normed vector space with norm $\|\cdot\|$. For a pair $\{x,y\} \in \mathcal{A}_2(\mathbf{E})$ we write $\|\{x,y\}\| = \|x\| + \|y\|$. We define a metric on $\mathcal{A}_2(\mathbf{E})$ by $\mathcal{G} : \mathcal{A}_2(\mathbf{E}) \times \mathcal{A}_2(\mathbf{E}) \to \mathbf{R}_{\geq 0}$ with

$$\mathcal{G}(u,v) = \min(\|u_1 - v_1\| + \|u_2 - v_2\|, \|u_1 - v_2\| + \|u_2 - v_1\|).$$

The analogous construction works for arbitrary metric spaces.

3.1.2. Two-valued functions. Let $A \subset \mathbf{R}^n$ be an arbitrary set. A *two-valued* function on A is a function $A \to \mathcal{A}_2(\mathbf{R})$. To u we can associate its average $u_a = \frac{1}{2}(u_1 + u_2)$ and symmetric difference $u_s = \{\pm\frac{1}{2}(u_1 - u_2)\}$. Similarly we write $u_+ = \max\{u_1,u_2\} = u_1 \wedge u_2$ and $u_- = \min\{u_1,u_2\} = u_1 \vee u_2$.

More generally we consider two-valued maps $A \to \mathcal{A}_2(X)$ where X may be arbitrary. Those taking values in $\mathcal{A}_2(\mathbf{R}^n)$ we shall call *two-valued vector fields*. Consider a map $\Phi : X \to \mathbf{R}$. Given $u : A \to \mathcal{A}_2(X)$ we can compose to $\Phi \circ u : A \to \mathcal{A}_2(\mathbf{R})$. If we are given two different functions $\Phi, \Psi : X \to \mathbf{R}$ then we may take the sum $(\Phi \circ u) + (\Psi \circ u) := (\Phi + \Psi) \circ u$, although in general for two functions $v_1, v_2 : A \to \mathcal{A}_2(\mathbf{R})$ their sum is not a well-defined two-valued function. The same of course holds for other binary operations.

As $\mathcal{A}_2$ is a metric space, endowing it with the corresponding topology and Borel σ-algebra allows us to define *measurable* and *continuous* two-valued functions. Explicitly, we say that $u : A \to \mathcal{A}_2$ is *continuous at* $x \in A$ if x is either isolated or for any sequence $(x_j \mid j \in \mathbf{N})$ in A with $x_j \to x$, $\mathcal{G}(u(x_j), u(x)) \to 0$ as $j \to \infty$. Let $\alpha \in (0,1)$. We say that a two-valued function on A is *α-Hölder continuous* if $\limsup_{x \neq y \in A} |x-y|^{-\alpha} \mathcal{G}(u(x), u(y)) < \infty$, and that u is *Lipschitz continuous* if this holds with $\alpha = 1$. Let $\Omega \subset \mathbf{R}^n$ be open. We say that u is *locally α-Hölder continuous* if the restriction of u to any compact subset $K \subset\subset \Omega$ is α-Hölder continuous. When this is the case we write $u \in C^{0,\alpha}(\Omega; \mathcal{A}_2)$, and also let $\mathrm{Lip}(\Omega; \mathcal{A}_2)$ be the similarly defined *locally Lipschitz functions*. Both notions are again defined for functions taking values in $\mathcal{A}_2(X)$ for any metric space (X, d), for example $X = \mathbf{L}(\mathbf{R}^n; \mathbf{R})$.

We say that a function $l : \mathbf{R}^n \to \mathcal{A}_2$ is *linear* if there exist two single-valued linear functions $l_i \in \mathbf{L}(\mathbf{R}^n; \mathbf{R})$ so that $l = \{l_1, l_2\}$. A two-valued function $u : \Omega \to \mathcal{A}_2$ is called *differentiable at x* if there is a two-valued linear function $l = \{l_1, l_2\}$ so that $t^{-1} \mathcal{G}(u(x + tv), \{u_i(x) + l_i(tv)\}) \to 0$ as $t \to 0$ for all $v \in \mathbf{R}^n$. If this exists, we write $Du(x) = l$ and call this the *derivative* of u at x. This defines a two-valued function $Du : \Omega \to \mathcal{A}_2(\mathbf{L}(\mathbf{R}^n; \mathbf{R}))$. Moreover we write $u \in C^1(\Omega; \mathcal{A}_2)$ if it is differentiable at all points $x \in \Omega$ and the function Du is continuous, and for $\alpha \in (0,1)$ we write $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ if $Du \in C^{0,\alpha}(\Omega; \mathcal{A}_2(\mathbf{L}(\mathbf{R}^n; \mathbf{R})))$.

Let $\alpha \in (0,1)$ and $u : D_1 \to \mathcal{A}_2(\mathbf{R})$ be a two-valued function. We define the following norms:

(1) $\|u\|_{0;D_1} = \sup_{D_1} \|u\|$,
(2) $\|u\|_{0,\alpha;D_1} = \|u\|_{0;D_1} + \sup_{x \neq y \in D_1} |x-y|^{-\alpha} \mathcal{G}(u(x), u(y))$,
(3) $\|u\|_{1;D_1} = \|u\|_{0;D_1} + \sup_{D_1} \|Du\|$,
(4) $\|u\|_{1,\alpha;D_1} = \|u\|_{1;D_1} + \sup_{x \neq y \in D_1} |x-y|^{-\alpha} \mathcal{G}(Du(x), Du(y))$.

For two-valued functions defined on an arbitrary bounded open domain $\Omega \subset \mathbf{R}^n$ one defines the analogous norms in precisely the same way. Although we shall not require this here, note that one defines scale-invariant analogues of these norms on discs $D_r \subset \mathbf{R}^n$ of any positive radius in the usual way, for example one would set $\|u\|_{1,\alpha;D_r} = \sup_{D_r} \|u\| + r \sup_{D_r} \|Du\| + r^{1+\alpha} \sup_{x \neq y} |x-y|^{-\alpha} \mathcal{G}(Du(x), Du(y))$.

One defines the functions spaces $C^0(\overline{D}_1;\mathcal{A}_2), C^{0,\alpha}(\overline{D}_1;\mathcal{A}_2), C^1(\overline{D}_1;\mathcal{A}_2)$ and $C^{1,\alpha}(\overline{D}_1;\mathcal{A}_2)$ in the usual way as those functions for which the norms are respectively finite, proceeding in the same way for arbitrary bounded open $\Omega\subset\mathbf{R}^n$.

Consider a two-valued function $u:\Omega\to\mathcal{A}_2$, and let $\Omega'\subset\Omega$ be an open subset. We say that two functions $u_1,u_2:\Omega\to\mathcal{A}_2$ define a *selection* for u on Ω' if $u=\{u_1,u_2\}$ on Ω'. Most often one seeks selections with favourable properties. We say that u_1,u_2 define a *continuous selection* if $u_1,u_2\in C^0(\Omega')$, a C^1 *selection* if $u_1,u_2\in C^1(\Omega')$, and a *smooth selection* if $u_1,u_2\in C^\infty(\Omega')$.

Let $\alpha\in(0,1)$ and consider $u\in C^{1,\alpha}(\Omega;\mathcal{A}_2)$. We define $\mathcal{K}_u=\{x\in\Omega\mid u_1(x)=u_2(x), Du_1(x)=Du_2(x)\}$. For every point $y\in\Omega\setminus\mathcal{K}_u$ there is $\sigma>0$ so that on $D_\sigma(y)$ there are two smooth functions $u_{1,y},u_{2,y}\in C^\infty(D_\sigma(y))$ so that $u=\{u_{1,y},u_{2,y}\}$. (By the above, we could equivalently state that u admits a smooth selection on $D_\sigma(y)$ when this is the case.) A point $x\in\Omega$ is called a *branch point* (sometimes also *true branch point*) if there is no radius $\sigma>0$ for which u admits a smooth selection on $D_\sigma(x)$. They form a set written $\mathcal{B}_u$, which is contained inside $\mathcal{K}_u$. Points in $\mathcal{K}_u\setminus\mathcal{B}_u$ are called *false branch points.* We also define the *touching set* $\mathcal{Z}_u=\{x\in\Omega\mid u_1(x)=u_2(x)\}$ and set of classical singularities $\mathcal{C}_u=\{x\in\Omega\mid u_1(x)=u_2(x), Du_1(x)\neq Du_2(x)\}$. By definition $\mathcal{C}_u=\mathcal{Z}_u\setminus\mathcal{K}_u$. Moreover, $\mathcal{C}_u$ is relatively open inside $\mathcal{Z}_u$, whereas $\mathcal{B}_u,\mathcal{K}_u\subset\mathcal{Z}_u$ are relatively closed in $\mathcal{Z}_u$, and hence closed subsets of Ω as well.

Let $u:\Omega\to\mathcal{A}_2$ be a two-valued function. Its *graph* $G\subset\Omega\times\mathbf{R}$ is

$$\operatorname{graph} u = G = \{(x,X^{n+1})\in\Omega\times\mathbf{R}\mid X^{n+1}\in\{u_1(x),u_2(x)\}\}.$$

This may be considered as a varifold inside $\Omega\times\mathbf{R}$, in which case we write $|G|=|\operatorname{graph} u|\in\mathbf{IV}_n(\Omega\times\mathbf{R})$ as is customary. (We emphasise that throughout we do not consider the graph as a subset of $\Omega\times\mathcal{A}_2(\mathbf{R})$, as one might expect by interpreting the term more literally.) In general of course the varifold $|G|$ will not be regular, and instead admit singular points which are related to the singularities of u as follows via the orthogonal projection P_0 onto $\mathbf{R}^n$. If $u\in C^1(\Omega;\mathcal{A}_2)$ then $P_0(\operatorname{sing} G\cup\{X\in\operatorname{reg} G\mid\Theta(\|G\|,X)=2\})=\mathcal{Z}_u$, $P_0(\mathcal{B}_G\cup\{X\in\operatorname{reg} G\mid\Theta(\|G\|,X)=2\})=\mathcal{K}_u$, and $P_0(\mathcal{C}_G)=\mathcal{C}_u$.

We close with a remark on the case where $n=1$ and $\Omega=I\subset\mathbf{R}$ is an open interval in the real line. Let $u\in C^1(I;\mathcal{A}_2)$. This automatically has $\mathcal{B}_u=\emptyset$, even while $\mathcal{K}_u$ may be non-empty. In other words, we can always find $u_1,u_2\in C^1(I)$ so that $u=\{u_1,u_2\}$ on I, although some arbitrary choices have to be made if $\mathcal{K}_u\neq\emptyset$. We will later use this elementary observation in the following context. Let $\Omega\subset\mathbf{R}^n$ be open, and $u\in C^1(\Omega;\mathcal{A}_2)$. Let

$y \in \Omega$ and $v \in \mathbf{R}^n$ be arbitrary. Write $l_y \subset \{y + tv \mid t \in \mathbf{R}\} \cap \Omega$ for the connected component containing y. This corresponds to an interval $I \subset \mathbf{R}$. Via this identification, the restriction of u to l_y defines a two-valued function in $C^1(I; \mathcal{A}_2)$. Hence we can find two functions $u_{1,y}, u_{2,y} \in C^1(I)$ so that $u(y + tv) = \{u_{1,y}(t), u_{2,y}(t)\}$ even though possibly $l_y \cap \mathcal{B}_u \neq \emptyset$. Moreover if $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ for some $\alpha \in (0, 1)$ then we can impose $u_{1,y}, u_{2,y} \in C^{1,\alpha}(I)$ as well.

3.1.3. Integrals of two-valued functions. Let $p \in [1, +\infty)$. We write $L^p(\Omega; \mathcal{A}_2)$ for the space of two-valued functions $u : \Omega \to \mathcal{A}_2$ with $\int_\Omega \|u\|^p < \infty$, and $L^\infty(\Omega; \mathcal{A}_2)$ for those functions for which $\|u\|$ is essentially bounded. We define the *integral* of a two-valued function $u \in L^1(\Omega; \mathcal{A}_2)$ by $\int_\Omega u = 2 \int_\Omega u_a = \int_\Omega u_1 + u_2$. Sometimes we also write $\int_\Omega u(x)\, \mathrm{d}x = \int_\Omega u$.

Let $u \in C^1(\overline{\Omega}; \mathcal{A}_2)$ and $\Phi : \mathbf{R} \to \mathbf{R}$ be so that $\Phi \circ u \in L^1(\Omega; \mathcal{A}_2)$. Then we have that $\int_\Omega (\Phi \circ u)(1 + |Du|^2) = \int_G \Phi \, \mathrm{d}\mathcal{H}^n$, where on the right-hand side the integral is over $G = \operatorname{graph} u \subset \Omega \times \mathbf{R}$.

Consider $u \in L^1(\Omega; \mathcal{A}_2)$, $A \subset \mathbf{R}$ be a Borel subset, and $\mathbb{1}_A$ be its indicator function. As u takes values in $\mathcal{A}_2$, the pre-image of A under u is not defined. However, we can define the integral $\int_{u \in A} u := \int u(\mathbb{1}_A \circ u) = \int u_1(\mathbb{1}_A \circ u_1) + u_2(\mathbb{1}_A \circ u_2)$. This can be generalised further if we consider $\Phi : \mathbf{R} \to \mathbf{R}$ so that $\Phi \circ u \in L^1(\Omega; \mathcal{A}_2)$, by setting $\int_{u \in A} \Phi \circ u = \int (\Phi \circ u)(\mathbb{1}_A \circ u)$, where recall the product of the two functions is a well-defined two-valued function in this specific context. We will often use a variant of this, where in fact $\Phi : \Omega \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is so that the two-valued function $x \in \Omega \mapsto \Phi(x, Du(x), u(x))$ is integrable, and we consider integrals of the form $\int_{u \in A} \Phi(x, Du(x), u(x))\, \mathrm{d}x$, or say $\int_{u \in A, Du \in B} \Phi(x, Du(x), u(x))\, \mathrm{d}x$ where $B \subset \mathbf{R}^n$ is another Borel subset.

3.1.4. Some results for two-valued functions. Using a general form of the Arzelà–Ascoli theorem, for example [Mun00, Thm. 47.1], we obtain a two-valued statement; note that we only use the following weaker version valid for Lipschitz functions.

LEMMA 3.1.1. *Let* $(f_j \mid j \in \mathbf{N})$ *be a sequence of two-valued Lipschitz functions on* D_1. *If* $\sup_j \|f_j\|_{1;D_1} < \infty$ *then there is a subsequence which converges locally uniformly to a Lipschitz function* $f \in \operatorname{Lip}(D_1; \mathcal{A}_2)$.

Moreover, if we are given a sequence $(f_j \mid j \in \mathbf{N})$ as in Lemma 3.1.1 then the single-valued sequences $(f_{j,\pm} \mid j \in \mathbf{N})$ inherit their Lipschitz bound. We may thus diagonally extract a common convergent subsequence using the classical, single-valued Arzela–Ascoli theorem with respective limits $f_\pm$.

LEMMA 3.1.2. *Let $f \in \mathrm{Lip}(D_1; \mathcal{A}_2)$ be a two-valued Lipschitz function. Then f is differentiable $\mathcal{H}^n$-a.e. in D_1.*

PROOF. The single-valued Rademacher theorem implies that $f_\pm$ are separately differentiable $\mathcal{H}^n$-a.e. in D_1. Let $E_\pm \subset D_1$ be the sets where they are respectively not differentiable. Then f is differentiable on $D_1 \setminus (E_+ \cup E_-)$. □

3.2. Two-valued minimal graphs

3.2.1. Definition and basic properties. Let $\Omega \subset \mathbf{R}^n$ be an open set, and let $\alpha \in (0, 1)$. We say that $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2(\mathbf{R}))$ defines a *two-valued minimal graph* if its graph

$$|G| = |\mathrm{graph}\, u| \in \mathbf{IV}_n(\Omega \times \mathbf{R})$$

is stationary as a varifold in the open cylinder $\Omega \times \mathbf{R}$. We will often abbreviate this by saying that $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ is a two-valued minimal graph. Ultimately we are interested in the classification of so-called *entire* two-valued minimal graphs, by which we mean that they are globally defined on $\mathbf{R}^n$. As the graphs of two-valued linear functions are minimal, the existence of such entire graphs is beyond question. Here instead we are concerned with their classification: we want to show that a two-valued minimal graph of $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ must be linear. However we often work with functions that are defined on domains $\Omega \subset \mathbf{R}^n$; in particular a lot of the results developed in preparation in anticipation of our end goal also hold in a local setting.

When an open subdomain $\Omega' \subset \Omega \setminus \mathcal{B}_u$ is simply connected then there is a selection $u_1, u_2 \in C^2(\Omega')$ so that $u = \{u_1, u_2\}$ in Ω'. Inside $\Omega' \times \mathbf{R}$ the graph can be decomposed like

$$|G| = |\mathrm{graph}\, u_1| + |\mathrm{graph}\, u_2|,$$

which we abbreviate as G_1 and G_2 from now on.

The stationarity of $|G|$ is inherited by $|G_1|$ and $|G_2|$. By elliptic regularity this in turn means that u_1, u_2 are both smooth, that is $u_1, u_2 \in C^\infty(\Omega')$ and they separately solve the minimal surface equation

$$\mathrm{div}\, T(Du_i) = 0 \text{ in } \Omega \text{ for } i = 1, 2, \tag{3.1}$$

where here and throughout we write, for all $p \in \mathbf{R}^n$

$$T_k(p) = \frac{p_k}{\sqrt{1 + |p|^2}} \text{ for } k = 1, \ldots, n.$$

The vector $T(Du) \in \mathbf{R}^n$ is the horizontal part of $-\nu$, the downward-pointing unit normal to the graph G. We habitually write $v = \sqrt{1 + |Du|^2}$, so that also $T(Du) = Du/v$.

If we interpret the equation (3.1) in a weak sense we can show that for all test functions $\phi \in C^1_c(\Omega \setminus \mathcal{B}_u)$

$$\int_\Omega \langle T(Du), D\phi\rangle = 0, \tag{3.2}$$

using a partition of unity argument to reduce to the case where ϕ is supported in a simply connected domain $\Omega' \subset \Omega \setminus \mathcal{B}_u$. Note that as u is a two-valued function, the integral above is understood to mean $\int_\Omega \langle T(Du), D\phi\rangle = \int_\Omega (\langle T(Du), D\phi\rangle)_a$ as explained in Subsection 3.1.3.

Using the properties of the branch set respectively developed by Simon–Wickramasekera [SW16] and Krummel–Wickramasekera [KW20], listed below in (3.2.3) we can extend this through the branch set. Let $\phi \in C^1_c(\Omega)$ be arbitrary, and $K \subset\subset \Omega$ be so that ϕ vanishes identically outside K. Take a sequence $(\eta_j \mid j \in \mathbf{N})$ of functions in $C^1_c(\Omega)$ with

(1) $0 \leq \eta_j \leq 1$ in Ω for all j,
(2) $\eta_j \equiv 1$ on $\mathcal{B}_u \cap K$ for all j,
(3) $\eta_j \to 0$ $\mathcal{H}^n$-a.e. in Ω,
(4) $\int_\Omega |D\eta_j| \to 0$ as $j \to \infty$.

Such a sequence exists because $\mathcal{H}^{n-1}(\mathcal{B}_u) = 0$ and $\mathcal{B}_u$ is a compact subset of Ω, see 3.2.3. Actually by [KW20] the branch set is countably $(n-2)$-rectifiable, and we use this below to find sequences of cut-off functions with finer properties than we just listed. Following the proof of Proposition 3.2.1 below one finds that

$$\int_\Omega \langle T(Du), D\phi\rangle = 0 \quad \text{for all } \phi \in C^1_c(\Omega)$$

whenever u is a two-valued minimal graph. We skip over the details of this, and move on to the proof of the more general identity, where the test function is allowed to depend on u.

PROPOSITION 3.2.1. *Let $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2) \cap C^0(\overline{\Omega}; \mathcal{A}_2)$ be a two-valued minimal graph and let $\Phi \in C^0(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega \times \mathbf{R} \times \mathbf{R}^n)$ have $\operatorname{spt} \Phi \subset \Omega' \times \mathbf{R} \times \mathbf{R}^n$ for some $\Omega' \subset\subset \Omega$. If*

$$\int_{\Omega \setminus \mathcal{B}_u} |D(\Phi(x, u, Du))| < +\infty \tag{3.3}$$

then

$$\int_\Omega \langle T(Du), D(\Phi(x, u, Du))\rangle = 0. \tag{3.4}$$

PROOF. Define a function $\phi \in C^1_c(\Omega)$ by setting $\phi(x) = \Phi(x, u, Du)$ for all $x \in \Omega$. This is smooth away from $\mathcal{B}_u$ and has support contained inside Ω'. Let $(\eta_j \mid j \in \mathbf{N})$ be a sequence with the same properties as above. As

$(1-\eta_j)\phi$ vanishes near $\mathcal{B}_u$ it is a valid test function in the integral identity (3.2), yielding $\int_\Omega \langle T(Du), D\phi\rangle(1-\eta_j) - \int_\Omega \langle T(Du), D\eta_j\rangle\phi = 0$.

That the second integral goes to zero is a direct application of Hölder's inequality. For the first integral, we can bound the integrand by $|\langle T(Du), D\phi\rangle(1-\eta_j)| \leq |D\phi|$ almost everywhere in Ω—on $\Omega \setminus \mathcal{B}_u$ to be precise. The bounding function is integrable by assumption, so that again we can use dominated convergence to let $j \to \infty$ and deduce that $\int_\Omega \langle T(Du), D(\Phi(x,u,Du))\rangle = 0$. □

3.2.2. Orientation and the current structure. Let $\Omega \subset \mathbf{R}^n$ be an open set, $\alpha \in (0,1)$ and let $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ define a two-valued minimal graph in $\Omega \times \mathbf{R}$. We saw above that $G = \operatorname{graph} u$ gives a well-defined and stationary integer varifold $|G| = |\operatorname{graph} u| \in \mathbf{IV}_n(\Omega \times \mathbf{R})$. We can also endow the graph with an orientation and a current structure. At all regular points $X = (x, X^{n+1}) \in \operatorname{reg} G \cap \Omega \times \mathbf{R}$ there is a well-defined upward-pointing unit normal written $\nu(X)$. In terms of a smooth selection $u = \{u_1, u_2\}$ for u on a small disc $D_\rho(x)$ we have $\nu(x, u_i(x)) = (1 + |Du_i(x)|^2)^{-1/2}(-Du_i(x), 1)$. Hence we find that the unit normal is also defined at all branch points $X \in \mathcal{B}_G \cap D_2 \times \mathbf{R}$, and ν defines a continuous vector field on $(\operatorname{reg} G \cup \mathcal{B}_G) \cap \Omega \times \mathbf{R}$, which is moreover smooth on $\operatorname{reg} G$. However ν cannot be continuously extended to the set of classical singularities $\mathcal{C}_G \cap \Omega \times \mathbf{R}$.

As this set has $\mathcal{H}^n(\mathcal{C}_G \cap \Omega \times \mathbf{R}) = 0$ we can however still define an integer multiplicity rectifiable current $[\![G]\!] \in \mathcal{D}_n(\Omega \times \mathbf{R})$ by integrating solely on $\operatorname{reg} G \cap \Omega \times \mathbf{R}$. To verify that this current has no boundary inside $\Omega \times \mathbf{R}$ we need to take the classical singular set into consideration, because it has codimension one inside the support of $[\![G]\!]$.

PROPOSITION 3.2.2. *Let $\Omega \subset \mathbf{R}^n$ be an open set and $\alpha \in (0,1)$. Let $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be a two-valued minimal graph, and $[\![G]\!]$ be the corresponding current. Then $\partial[\![G]\!] = 0$ and $[\![G]\!] \in \mathbf{I}_n(\Omega \times \mathbf{R})$.*

PROOF. The current boundary $\partial[\![G]\!]$ is necessarily supported inside $\mathcal{C}_G \cap \Omega \times \mathbf{R}$. Let $X = (x, X^{n+1}) \in \mathcal{C}_G$ and $\rho > 0$ be small enough that in the disc $D_\rho(x)$ there is a smooth selection $u_1, u_2 \in C^2(D_\rho(x))$ for u. Then $[\![G]\!] \llcorner D_\rho(x) \times \mathbf{R} = [\![G_1]\!] + [\![G_2]\!]$, where we write $G_i = \operatorname{graph} u_i$. Neither current has a non-zero boundary, that is $\partial[\![G_i]\!] = 0$. As $\partial[\![G]\!] \llcorner D_\rho(x) = \partial[\![G_1]\!] + \partial[\![G_2]\!] = 0$ and the point X was chosen arbitarily, this concludes the proof. □

REMARK 3.2.3. This result used the minimality of the graph only insofar as it allows both dimension bounds for the singular set and a precise description for the singular set near the classical singularities.

3.2.3. Properties of the branch set. Let $\alpha \in (0,1)$, and let $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be an arbitrary two-valued minimal graph. Using an approach based on a so-called frequency functional, Simon–Wickramasekera [SW16] proved the following.

THEOREM 3.2.4 ([SW16]). *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be a two-valued minimal graph. Then the branch set of u is either empty or $\dim_{\mathcal{H}} \mathcal{B}_u = n-2$ and $\mathcal{H}^{n-2}(\mathcal{B}_u) \neq 0$.*

In particular $\mathcal{H}^s(\mathcal{B}_u) = 0$ for all $s > n-2$. We use the fact that $\mathcal{H}^{n-1}(\mathcal{B}_u) = 0$ to derive our area estimates.

COROLLARY 3.2.5 ([SW16]). *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be a two-valued minimal graph. If G is not equal to the union of two distinct single-valued minimal graphs then $\dim_{\mathcal{H}} \mathcal{K}_u = n-2$ and $\mathcal{H}^{n-2}(\mathcal{K}_u) \neq 0$.*

Starting also from a frequency functional, this was taken further by Krummel–Wickramasekera [KW20], who proved the following.

THEOREM 3.2.6 ([KW20]). *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be a two-valued minimal graph defined on $\Omega \subset \mathbf{R}^n$. Then $\mathcal{B}_u$ is either empty or is countably $n-2$-rectifiable. Moreover if G is not equal to the union of two distinct single-valued minimal graphs then $\mathcal{K}_u$ is countably $n-2$-rectifiable.*

This represents a significant improvement over [SW16] as it means that the 2-capacity of $\mathcal{B}_u$ is zero. This in turn allows the branch set to be excised using cutoff functions with properties as described below in Lemma 3.2.7 and Corollary 3.2.8.

LEMMA 3.2.7. *Let $\alpha \in (0,1)$, and $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph. Then there is a sequence of functions $(\eta_j \mid j \in \mathbf{N})$ with for all j,*

(i) $\eta_j \in C^1_c(D_2)$,
(ii) $0 \leq \eta_j \leq 1$ *on* D_2,
(iii) $\eta_j \equiv 1$ *on* $(\mathcal{B}_u)_{r_j} \cap D_1$ *for some* $r_j \to 0$,
(iv) $\eta_j \to 0$ $\mathcal{H}^n$*-a.e. as* $j \to \infty$,
(v) $\int_{D_2} |D\eta_j|^2 \to 0$ *as* $j \to \infty$.

PROOF. The set $\mathcal{B}_u \cap \overline{D}_1$ is compactly contained inside the open disc D_2. The result then follows because $\operatorname{cap}_2 \mathcal{B}_u = 0$ by [KW20], see the discussion at the end of 1.1.4. □

In the same way as for the results cited above, one obtains a version of this valid for the set $\mathcal{K}_u$ provided the set G is not equal to a single-valued minimal graph. Moreover, In various places it is useful to modify the sequence from Lemma 3.2.7 and construct it on the graph itself.

COROLLARY 3.2.8. *Let $\alpha \in (0,1)$, and $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph. If $\mathcal{B}_G \neq \emptyset$ then there is a sequence of functions $(\eta_j \mid j \in \mathbf{N})$ with for all j,*

(i) $\eta_j \in C^1_c(D_2 \times \mathbf{R})$,
(ii) $0 \leq \eta_j \leq 1$ *on* $D_2 \times \mathbf{R}$,
(iii) $\eta_j \equiv 1$ *on* $(\mathcal{B}_G)_{r_j} \cap D_1 \times \mathbf{R}$ *for some* $r_j \to 0$,
(iv) $\eta_j \to 0$ $\mathcal{H}^n$-*a.e. on* $\operatorname{reg} G \cap D_2 \times \mathbf{R}$ *as* $j \to \infty$,
(v) $\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G \eta_j|^2 \, \mathrm{d}\mathcal{H}^n \to 0$ *as* $j \to \infty$.

PROOF. Let $(\eta^0_j \mid j \in \mathbf{N})$ be a sequence satisfying the properties listed in Lemma 3.2.7, except we additionally impose that $\operatorname{spt} \eta^0_j \subset D_{3/2}$ for all j. Inside this disc there is $A > 0$ so that $-A < \min_{D_{3/2}} u_- \leq \max_{D_{3/2}} u_+ < A$, where $u_- = \min\{u_1, u_2\}$ and $u_+ = \{u_1, u_2\}$. Let $\tau \in C^1_c(\mathbf{R})$ be a classical cutoff function with $\tau \equiv 1$ on $[-1,1]$, $\operatorname{spt} \tau \subset [-2,2]$ and $|\tau'| \leq 2$. For all $j \in \mathbf{N}$, extend η^0_j to $D_2 \times \mathbf{R}$ by setting $\eta_j(x, X^{n+1}) = \eta^0_j(x)\tau(X^{n+1}/A)$ at all $X = (x, X^{n+1}) \in D_2 \times \mathbf{R}$.

To obtain the last two properties, let $\delta > 0$ be given, and d_G be the unsigned distance function to $G \cap D_2 \times \mathbf{R}$. For any j we may replace η_j with $\bar{\eta}_j$, defined by $\bar{\eta}_{j,\delta}(X) = \eta_j(X)\tau(d_G(X)/\delta)$ at all $X \in D_2 \times \mathbf{R}$. This additionally has $\operatorname{spt} \bar{\eta}_{j,\delta} \subset (G)_{2\delta} \cap D_2 \times \mathbf{R}$. This function inherits most properties from η_j, but it is only Lipschitz regular because of d_G. This however can be easily remedied by a standard mollification argument, taking care to choose the mollification parameter small enough in terms of $r_j, \delta > 0$. Finally, we may pick any sequence $s_j \to 0$ and apply the construction above with $\delta = s_j/2$, letting $\bar{\eta}_j = \bar{\eta}_{j,s_j/2}$ to conclude. □

Let us conclude by pointing out the following consequence of [KW20], obtained by combining it with the results of [Wic20], see Theorem 2.1.10. Recall for this that $\mathcal{B}_V$ is the set of singular branch points of V, meaning the points $X \in \operatorname{sing} V$ at which there is at least one tangent cone of the form $2|\Pi_X|$ for some n-dimensional plane $\Pi_X \in Gr(n, n+1)$.

COROLLARY 3.2.9. *Let $V \in \mathbf{IV}_n(B_1)$ be a stationary varifold with stable regular part. Then $\mathcal{B}_V \cap \{\Theta(\|V\|, \cdot) \leq 2\}$ is countably $n-2$–rectifiable.*

3.2.4. Immersion away from the branch set. Let $\alpha \in (0,1)$ and $n \geq 1$ be arbitrary. Let $\Omega \subset \mathbf{R}^n$ be a connected open set, and let $u \in C^{1,\alpha}(\Omega; \mathcal{A}_2)$ be a two-valued function so that $|G| = |\operatorname{graph} u| \in \mathbf{IV}_n(\Omega \times \mathbf{R})$ is a stationary varifold.

In this section we construct a smooth n-dimensional manifold M and a minimal immersion $\iota : M \to \mathbf{R}^{n+1}$ with image

$$\iota(M) = \operatorname{reg} G \cup \mathcal{C}_G = G \setminus \mathcal{B}_G,$$

using a standard gluing construction.

Let $(U_\alpha \mid \alpha \in A)$ be an open cover of $\Omega \setminus \mathcal{B}_u$, chosen so that every U_α is simply connected. For every open set U_α in this collection we can make a smooth selection $u_{\alpha,1}, u_{\alpha,2} \in C^1(U_\alpha)$ so that

$$u = \{u_{\alpha,1}, u_{\alpha,2}\} \text{ in } U_\alpha.$$

Write $G_{\alpha,1} = \operatorname{graph} u_{\alpha,1}$ and $G_{\alpha,2} = \operatorname{graph} u_{\alpha,2}$, so that accordingly

$$G \cap U_\alpha \times \mathbf{R} = G_{\alpha,1} \cup G_{\alpha,2}.$$

We then consider the disjoint union of these sets $(G_{\alpha,i} \mid \alpha \in A, i = 1,2)$, each of which is endowed with an obvious map $\iota_{\alpha,i} : G_{\alpha,i} \to \mathbf{R}^{n+1}$, which may also be composed with the projection $P_0 : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \{0\}$ to obtain bijections $P_0 \circ \iota_{\alpha,i} : G_{\alpha,i} \to U_{\alpha,i}$. We glue these together using the equivalence relation $\sim$ defined as follows.

> *Two points $X \in G_{\alpha,i}$ and $Y \in G_{\beta,j}$ are equivalent if $P_0 \circ \iota_{\alpha,i}(X) = P_0 \circ \iota_{\beta,j}(Y)$ and there is a neighbourhood of this point where $u_{\alpha,i}$ and $u_{\beta,j}$ coincide.*

Given this we simply set

$$M = \bigsqcup_{\substack{\alpha\in A\\ i=1,2}} G_{\alpha,i}/\sim.$$

Write p for the projection $\sqcup G_{\alpha,i} \to M$. By construction the map $\sqcup \iota_{\alpha,i} : \sqcup U_{\alpha,i} \to \mathbf{R}^{n+1}$ passes to the quotient by $\sim$, thus defining a map

$$\iota : M \to \mathbf{R}^{n+1}.$$

This map has the property that for every set $U_{\alpha,i}$ and all $x \in U_{\alpha,i}$, $\iota \circ p(x) = \iota_{\alpha,i}(x)$. Then it is not hard to see that M is a smooth n-dimensional manifold, with charts given by the collection $\{(p(U_{\alpha,i}), P_0 \circ \iota) \mid \alpha \in A, i = 1,2\}$.

Lemma 3.2.10.

(i) *The map ι is a smooth immersion, injective away from $\iota^{-1}(\mathcal{C}_G)$.*
(ii) *The immersion can be oriented by the upward unit normal ν.*
(iii) *The map ι is proper into $\mathbf{R}^{n+1} \setminus \mathcal{B}_G$, but not into $\mathbf{R}^{n+1}$ unless $\mathcal{B}_G = \emptyset$.*
(iv) *The manifold M is connected unless G is the union of two distinct single-valued graphs.*

Proof. The first two properties follow by construction.

(iii) Let $K \subset \mathbf{R}^{n+1} \setminus \mathcal{B}_G$ be a compact set, and consider a sequence of points $(X_j \mid j \in \mathbf{N})$ in $\iota^{-1}(K \cap G)$. Write $Y_j = \iota(X_j)$ for all j, and extract a convergent subsequence from this, with limit say $Y_{j'} \to Y \in \operatorname{reg} G \cup \mathcal{C}_G$. If

$Y \in \operatorname{reg} G$ then there is $\rho > 0$ so that the restriction of ι to $\iota^{-1}(B_\rho(Y) \cap G)$ is a homeomorphism onto $B_\rho(Y) \cap G$, from whence the property follows. If instead $Y \in \mathcal{C}_G$ then we can decompose $\iota^{-1}(B_\rho(Y) \cap G) = W_1 \cup W_2$ into two disjoint open sets, so that the restriction of ι to either of them is again a homeomorphism. As one of W_1, W_2 contains infinitely many terms in the sequence, the conclusion follows.

(iv) The map $P_0 \circ \iota : M \to \Omega$ is a double cover, so M has at most two connected components. Hence when indeed M is disconnected, then $M = M_1 \cup M_2$ and the restriction of $P_0 \circ \iota$ to either of them is a homeomorphism, and their images are two single-valued graphs. □

Of course we can pull back the metric on $\operatorname{reg} G$ to a metric defined on $\iota^{-1}(\operatorname{reg} G) \subset M$, which we can then extend to the entire manifold by continuity. Given this all associated quantities are defined also, e.g. the covariant derivative or the volume form. We use these in the following section.

3.2.5. Stability of two-valued minimal graphs. Let $|G|$ be the varifold associated to the graph of u. Its singular set admits a particularly simple form:

$$\operatorname{sing} G = \mathcal{B}_G \cup \mathcal{C}_G,$$

where $\mathcal{B}_G$ are the branch points of $|G|$, that is the points where $|G|$ admits a multiplicity two tangent plane, and $\mathcal{C}_G$ are the classical singularities. These take a much simpler form than in the setting of general varifolds. Indeed a point $X \in G$ belongs to $\mathcal{C}_G$ precisely if there is some radius $r > 0$ such that $G \llcorner B_r(X)$ is the union of two smooth embedded minimal graphs. In other words, $\mathcal{C}_G$ is the set of singular points of $|G|$ near which its support is smoothly immersed.

The minimal immersion constructed in the previous section, of M into $\mathbf{R}^{n+1}$ moreover admits a positive Jacobi field, namely $\langle \nu, e_{n+1} \rangle$. Thus it is stable, meaning that for all $\phi \in C^1_c(M)$,

$$\int_M |A_M|^2 \phi^2 \leq \int_M |\nabla_M \phi|^2. \tag{3.5}$$

Indeed, we may justify the stability via the following elementary argument, already used in 2.1.4. Pick any non-negative function $\varphi \neq 0 \in C^2_c(M)$. As φ is compactly supported, there is a positive $\delta > 0$ so that $\langle \nu, e_{n+1} \rangle \geq \delta > 0$ on $\operatorname{spt} \varphi$. Let $T = \max\{t \in \mathbf{R} \mid t\varphi \leq \langle \nu, e_{n+1} \rangle\}$. This is positive by the above, and by construction $\langle \nu, e_{n+1} \rangle - T\varphi$ is a non-negative function, attains the value zero at least once in M, but does not vanish identically. If φ had $(\Delta_M + |A_M|^2)\varphi \geq 0$, then $(\Delta_M + |A_M|^2)(\langle \nu, e_{n+1} \rangle - T\varphi) \leq 0$, which would contradict the strong maximum principle. This proves stability, because if there existed a function $\varphi \in C_c^1(M)$ with $\int_M |A_M|^2\varphi^2 > \int_M |\nabla_M \varphi|^2$ then we could mollify it to a smooth compactly supported satisfying the same inequality. This is impossible by our argument above, and confirms the stability inequality (3.5).

If φ is any function in $C_c^1(\mathbf{R}^{n+1} \setminus \mathcal{B}_G)$, then its pullback $\phi = \varphi \circ \iota$ by the immersion is compactly supported in M (because ι is proper away from the branch set). If we equate the integrals on M with ones on G we obtain that

$$\int_{\operatorname{reg} G} |A_G|^2 \varphi^2 \leq \int_{\operatorname{reg} G} |\nabla_G \varphi|^2$$

for all $\varphi \in C_c^1(\mathbf{R}^{n+1} \setminus \mathcal{B}_G)$. Extending this to arbitrary test functions requires a capacity argument.

Lemma 3.2.11. *Let $U \subset \mathbf{R}^{n+1}$ be an open set, and $G \subset U$ be a two-valued minimal graph. Then G is stable with respect to compactly supported ambient deformations in the sense that for all $\varphi \in C_c^1(U)$,*

$$(S_G) \qquad \int_{\operatorname{reg} G \cap U} |A_G|^2 \varphi^2 \leq \int_{\operatorname{reg} G \cap U} |\nabla_G \varphi|^2.$$

Proof. If $\varphi \in C_c^1(U)$ is so that

$$\operatorname{spt} \varphi \cap \mathcal{B}_G = \emptyset,$$

then $\varphi \circ \iota \in C_c^1(M)$ and it has

$$\int_M |A_M|^2 (\varphi \circ \iota)^2 \leq \int_M |\nabla_M (\varphi \circ \iota)|^2.$$

Translating this back to the graph we obtain the desired inequality (S_G).

If however the support of φ contains branch points, meaning $\operatorname{spt} \varphi \cap \mathcal{B}_G \neq \emptyset$ then we cut this part off using a sequence $(\eta_j \mid j \in \mathbf{N})$ of functions in $C_c^1(U)$ with properties as listed in Lemma 3.2.7. Indeed then $(1-\eta_j)\varphi \in C_c^1(U \setminus \mathcal{B}_G)$ and thus (S_G) holds with this test function,

$$\int_{\operatorname{reg} G \cap U} |A_G|^2 (1-\eta_j)^2 \varphi^2 \leq \int_{\operatorname{reg} G \cap U} |\nabla_G \{(1-\eta_j)\varphi\}|^2. \tag{3.6}$$

The right-hand side can be bounded uniformly in j because

$$\int_{\operatorname{reg} G \cap U} |\nabla_G \{(1-\eta_j)\varphi\}|^2 \leq 2 \int_{\operatorname{reg} G \cap U} |\nabla_G \eta_j|^2 \varphi^2 + (1-\eta_j)^2 |\nabla_G \varphi|^2$$

and as $j \to \infty$ we can separately estimate

$$\int_{\operatorname{reg} G \cap U} |\nabla_G \eta_j|^2 \varphi^2 \to 0 \tag{3.7}$$

and by dominated convergence

$$\int_{\operatorname{reg} G \cap U} (1-\eta_j)^2 |\nabla_G \varphi|^2 \to \int_{\operatorname{reg} G \cap U} |\nabla_G \varphi|^2. \tag{3.8}$$

In fact we can compute the bounding integral in (3.6) more precisely and show that the cross-term also has

$$2\int_{\operatorname{reg} G\cap U}(1-\eta_j)\varphi\langle\nabla_G(1-\eta_j),\nabla_G\varphi\rangle\to 0 \text{ as } j\to\infty. \tag{3.9}$$

On the left-hand side of (3.6) we may pass to the limit by Fatou's lemma, so that letting $j\to\infty$ we obtain the desired inequality

$$\int_{\operatorname{reg} G\cap U}|A_G|^2\varphi^2 \le \int_{\operatorname{reg} G\cap U}|\nabla_G\varphi|^2. \qquad \square$$

REMARK 3.2.12. In fact (S_G) holds for larger class of test functions, which allow us to move the two sheets near a classical singularity $X\in\mathcal{C}_G$ independently. Indeed if $r>0$ is small enough that we can decompose $G\cap B_r(X)=\Sigma_1\cup\Sigma_2$ into two smooth, single-valued graphs intersecting transversely along the classical axis $\operatorname{sing} G\cap B_r(X)=\mathcal{C}_G\cap B_r(X)=\Sigma_1\cap\Sigma_2$, then their respective pre-images $\iota^{-1}(\Sigma_i)\subset M$ are disjoint, and we could glue together any two $\varphi_i\in C^1_c(\iota^{-1}(\Sigma_i))$ to a function defined on the whole manifold M.

We refer to the inequality (S_G) in this lemma as the *stability inequality*. Using [Hut86] we have the following immediate corollary for sequences of two-valued minimal graphs; see (S_V) on page 25 for the definition of ambient stability.

COROLLARY 3.2.13. *Let $(u_j\mid j\in\mathbf{N})$ be a sequence of two-valued minimal graphs $u_j\in C^{1,\alpha}(D_2;\mathcal{A}_2)$, and suppose that their graphs converge weakly in the varifold topology to a limit varifold $V\in\mathbf{IV}_n(D_2\times\mathbf{R})$,*

$$|G_j|=|\operatorname{graph} u_j|\to V \text{ as } j\to\infty.$$

Then V is stationary and ambient stable.

3.3. Area estimates for two-valued minimal graphs

3.3.1. Area bounds for two-valued minimal graphs. Here we extend the classical area estimates, which are well-known for single-valued minimal graphs, to two-valued minimal graphs by adapting the arguments presented in [GT98, Ch. 16].

PROPOSITION 3.3.1. *Let $\alpha\in(0,1/2)$, and let $u\in C^{1,\alpha}(D_{2r};\mathcal{A}_2)$ be a two-valued minimal graph. Then*

$$\mathcal{H}^n(G\cap B_r)\le 2\omega_n(1+n)r^n,$$

where $\omega_n=\mathcal{H}^n(D_1)$.

PROOF. Let $\eta \in C^1_c(D_{2r})$ be a test function with $\eta \equiv 1$ on D_r and $|D\eta| \le 2/r$. Next define $\Phi(x,z) = \eta(x) z_r$, where

$$z_r = \begin{cases} r & \text{if } z > r \\ z & \text{if } -r \le z \le r \\ -r & \text{if } z < -r. \end{cases} \tag{3.1}$$

Then by the (two-valued) chain rule we get

$$D(\Phi(x,u)) = D\eta(x) u_r + \{D_z \Phi(x, u_i(x)) Du_i(x)\}.$$

This is clearly well-defined at any $x \in D_{2r} \setminus \mathcal{B}_u$; in fact the same is true at branch points $x \in \mathcal{B}_u$ because the two components of $Du(x) \in \mathcal{A}_2(\mathbf{L}(\mathbf{R}^n;\mathbf{R}))$ agree there. We evaluate this expression to be

$$D(\Phi(x,u)) = D\eta(x) u_r(x) + \eta(x) \mathbb{1}_{|u|<r} Du(x).$$

As η is compactly supported inside D_{2r} we get $\int_{D_{2r} \setminus \mathcal{B}_u} |D(\Phi(x,u))| < +\infty$. Then Proposition 3.2.1 justifies

$$\int_{D_{2r}} u_r \langle T(Du), D\eta \rangle + \eta \langle T(Du), Du \rangle \mathbb{1}_{|u|<r} = 0, \tag{3.2}$$

so that

$$\int_{D_r} \mathbb{1}_{|u|<r} \frac{|Du|^2}{v} \le 2r \int_{D_{2r}} |D\eta|. \tag{3.3}$$

The area of the graph is bounded by the integral $\mathcal{H}^n(G \cap B_r) \le \int v \mathbb{1}_{|u|<r}$, which we split as

$$\int_{D_r} \frac{1}{v} \mathbb{1}_{|u|<r} + \int_{D_r} \frac{|Du|^2}{v} \mathbb{1}_{|u|<r} \le 2(\mathcal{H}^n(D_r) + r \operatorname{cap}_1(D_r)),$$

whence we conclude by replacing $\operatorname{cap}_1(D_r) = \mathcal{H}^{n-1}(S^{n-1}_r) = n\omega_n r^{n-1}$. □

A similar argument yields area bounds in the cylinder above the disc D_r. A detailed proof in the single-valued case is given in [GT98, Ch. 16], to adapt it to two-valued graphs one makes the same modifications as above. These will be used in the proof of the gradient estimates (see the proof of Lemma 3.4.2).

LEMMA 3.3.2. *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(D_{2r};\mathcal{A}_2)$ be a two-valued minimal graph. Then*

$$\mathcal{H}^n(G \cap D_r \times \mathbf{R}) \le 2\omega_n r^n (1 + nr^{-1} \sup_{D_{2r}} \|u\|).$$

3.3.2. Improved estimates for convergent sequences. Let $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs $u_j \in C^{1,\alpha}(D_{2r};\mathcal{A}_2)$. Suppose additionally that the corresponding sequence of varifolds $|G_j| =$

$|\mathrm{graph}\, u_j|$ converges to a stationary integral varifold $V \in \mathbf{IV}_n(D_{2r} \times \mathbf{R})$. Suppose additionally that V is *vertical*, that is cylindrical of the form

$$V = V_0 \times \mathbf{R} e_{n+1}$$

for a stationary varifold $V_0 \in \mathbf{IV}_{n-1}(D_{2r})$. (This scenario typically—though not always—arises when the sequence $(u_j \mid j \in \mathbf{N})$ is obtained as the blow-down sequence of a single entire graph $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$.) When this arises we can sharpen the estimates of Proposition 3.3.1.

COROLLARY 3.3.3. *Let $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs $u_j \in C^{1,\alpha}(D_{2r}; \mathcal{A}_2)$. Suppose that they converge to a stationary varifold $V \in \mathbf{IV}_n(D_{2r} \times \mathbf{R})$,*

$$|G_j| = |\mathrm{graph}\, u_j| \to V = V_0 \times \mathbf{R} e_{n+1} \text{ as } j \to \infty.$$

Then

$$\limsup_{j \to \infty} \mathcal{H}^n(G_j \cap B_r) \leq 2n\omega_n r^n. \tag{3.4}$$

PROOF. The proof is the same as for Proposition 3.3.1, splitting the integral bound of $\mathcal{H}^n(G_j \cap B_r) \leq \int_{D_r} v_j \mathbb{1}_{|u_j|<r}$ into

$$\int_{D_r} v_j \mathbb{1}_{|u_j|<r} = \int_{D_r} \frac{1}{v_j} \mathbb{1}_{|u_j|<r} + \int_{D_r} \frac{|Du_j|^2}{v_j} \mathbb{1}_{|u_j|<r}.$$

The second integral can be bounded like $\int_{D_r} \frac{|Du_j|^2}{v_j} \mathbb{1}_{|u_j|<r} \leq 2r \int_{D_{2r}} |D\eta|$, where $\eta \in C^1_c(D_{2r})$ has $\eta = 1$ on D_r.

To confirm that the first integral goes to zero as $j \to \infty$ we use the varifold convergence $|G_j| \to V_0 \times \mathbf{R} e_{n+1}$. Let $K \subset D_{2r}$ be a compact subset containing D_r and $\tau > 0$ be a small constant. Then define a subset $E_{\tau,r} \subset Gr_n(D_{2r} \times \mathbf{R}) = D_{2r} \times \mathbf{R} \times Gr(n, n+1)$ by

$$E_{\tau,r} = \{(X, S) = (x, X^{n+1}, S) \in Gr_n(K \times \mathbf{R}) \mid |X^{n+1}| \leq r, |\langle N_S, e_{n+1} \rangle| \geq \tau\},$$

where we write N_S for the (unoriented) unit normal to S.

As $(V_0 \times \mathbf{R} e_{n+1})(E_{\tau,r}) = 0$ and the set $E_{\tau,r}$ is compact the weak convergence $|G_j| \to V$ means that given any $\epsilon > 0$ we can impose $j \geq J(\tau, r, \epsilon, K)$ to ensure $|G_j|(E_{\tau,r}) < \epsilon$. Define next a subset $E_{\tau,r,j} \subset D_{2r} \times \mathbf{R}$ by

$$E_{\tau,r,j} = \{X = (x, X^{n+1}) \in \mathrm{reg}\, G_j \cap K \times \mathbf{R} \mid |X^{n+1}| \leq r, \nu_j^{n+1}(X) \geq \tau\}$$

so that $\mathcal{H}^n(E_{\tau,r,j} \cap K \times \mathbf{R}) < \epsilon$ provided $j \geq J(\tau, r, \epsilon, K)$. Push forward the integral $\int_{D_r} \frac{1}{v} \mathbb{1}_{|u|<r}$ to an integral on the graph G_j, which we then divide by conditioning on the set $E_{\tau,r,j}$, meaning

$$\int_{D_r} \frac{1}{v_j} \mathbb{1}_{|u_j|<r} = \int_{E_{\tau,r,j} \cap D_r \times (-r,r)} \frac{1}{v_j^2} \, \mathrm{d}\mathcal{H}^n + \int_{(G_j \setminus E_{\tau,r,j}) \cap D_r \times (-r,r)} \frac{1}{v_j^2} \, \mathrm{d}\mathcal{H}^n.$$

These can be bounded separately; the first simply has

$$\int_{E_{\tau,r,j}\cap D_r\times(-r,r)} \frac{1}{v_j^2}\,\mathrm{d}\mathcal{H}^n \leq \mathcal{H}^n(E_{\tau,r,j}) < \epsilon$$

whereas for the second we can use the gradient lower bounds $v_j^2 > \tau^{-2}$ in $(G_j \setminus E_{\tau,r,j}) \cap D_r \times (-r,r)$ to deduce

$$\int_{(G_j\setminus E_{\tau,r,j})\cap D_r\times(-r,r)} \frac{1}{v_j^2}\,\mathrm{d}\mathcal{H}^n$$
$$\leq \mathcal{H}^n(G_j \cap \overline{D}_r \times [-r,r])\tau^2 \leq 2\|V\|(\overline{D}_r \times [-r,r])\tau^2$$

at least for $j \geq J(\tau,\epsilon,r,K)$, updated to ensure that the area of G_j inside $\overline{D_r} \times [-r,r]$ can be bounded in terms of $\|V\|$. Now letting $\tau,\epsilon \to 0$ and accordingly $J \to \infty$ we obtain that $\int_{D_r} \frac{1}{v_j}\mathbb{1}_{|u_j|<r} \to 0$ as $j \to \infty$.

Thus we obtain $\limsup_{j\to\infty} \int_{D_r} \frac{|Du_j|^2}{v_j}\mathbb{1}_{|u_j|<r} \leq 2\int_{D_{2r}}|D\eta|$. Varying over functions with $\eta \in C_c^1(D_{2r})$ with $\eta = 1$ on D_r we obtain $\limsup_{j\to\infty} \mathcal{H}^n(G_j \cap B_r) \leq 2r\,\mathrm{cap}_1(D_r) = 2r\mathcal{H}^{n-1}(S_r^{n-1}) = 2n\omega_n r^n$. □

This estimate is easily improved to strict inequality, valid under identical hypotheses. Indeed, revisiting the proofs of the original area bounds of Proposition 3.3.1 we notice that these in fact yield estimates for the area inside the cylinder $D_r \times (-r,r)$. The same is true for the sharpened estimates we just derived. As V is vertical, its weight measure $\|V\|$ charges the region $D_r \times (-r,r) \setminus B_r$ by a positive amount. This is precisely the amount by which the inequality can be improved. However, this improvement cannot be estimated precisely because the mass of V could be unevenly distributed, with a larger concentration near the origin.

This problem is avoided when it is known that the limit is a cone, that is $V = \mathbf{C} = \mathbf{C}^0 \times \mathbf{R}e_{n+1}$. Indeed then a simple calculation, the details of which are provided below in the proof of Corollary 3.3.4, shows that $\|\mathbf{C}\|(B_1)/\|\mathbf{C}\|(D_1 \times (-1,1)) = \int_0^{\pi/2} \cos^n\theta\,\mathrm{d}\theta$, which in what follows we denote $I_n \in (0,1)$. This can be expressed in terms of Euler's Γ-function, and comparing it with the expression for ω_n we find that $I_n = \omega_n/(2\omega_{n-1})$. The well-known values of these let us give exact expressions for the achievable improvement, given in Table 1.

Notice that the value of I_n decreases with n. Nonetheless the improved bounds, which we state in Corollary 3.3.4 increase with n. This is because the estimates previously obtained increase quicker than I_n. For large n one may use the well-known asymptotics for the Γ-function to find $I_n \sim \sqrt{\pi/(2n)}$, whence the estimates grows like $2nI_n \sim \sqrt{2\pi n}$ as $n \to \infty$. Luckily we are most interested in the values of $2nI_n$ for small values of n, up to seven. For this range the estimates obtained in Corollary 3.3.4 are effective, and crucial

TABLE 1. Values for the volume ω_n of the n-dimensional unit ball for $2 \le n \le 7$, along with the improvements I_n obtained in the area estimates and their approximations $[I_n]_2$ up to two decimal places.

n	2	3	4	5	6	7
ω_n	π	$4\pi/3$	$\pi^2/2$	$8\pi^2/15$	$\pi^3/6$	$16\pi^3/105$
I_n	$\pi/4$	$2/3$	$3\pi/16$	$8/15$	$5\pi/32$	$16/35$
$[I_n]_2$	0.79	0.67	0.59	0.53	0.49	0.46

TABLE 2. Values of the improved estimates $2nI_n$ for the area of two-valued minimal graphs in the unit ball B_1^{n+1} (up to multiplication by ω_n) for $2 \le n \le 7$ along with their approximations up to two decimal places.

n	2	3	4	5	6	7
$2nI_n$	π	4	$3\pi/2$	$16/3$	$15\pi/8$	$32/5$
$[2nI_n]_2$	3.14	4	4.71	5.33	5.89	6.4

for the remainder of the text. Their expressions and approximate values are given in Table 2.

COROLLARY 3.3.4. *Let* $(u_j \mid j \in \mathbf{N})$ *be a sequence of two-valued minimal graphs* $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$. *Suppose that*

$$|G_j| \to \mathbf{C} = \mathbf{C}_0 \times \mathbf{R}e_{n+1} \text{ as } j \to \infty$$

to a stationary cylindrical cone. Then

$$\limsup_{j\to\infty} \mathcal{H}^n(G_j \cap B_1) \le 2nI_n\omega_n \tag{3.5}$$

where $I_n = \int_0^{\pi/2} (\cos\theta)^n \,\mathrm{d}\theta$.

PROOF. Note first that the area bounds of Proposition 3.3.1, as well the sharpened bounds of Corollary 3.3.3 in fact yield bounds in the cylindrical region $D_r \times (-r, r)$. Thus from (3.4) we could in fact infer that $\limsup_{j\to\infty} \mathcal{H}^n(G_j \cap D_r \times (-r,r)) \le 2n\omega_n r^n$. This is because area bounds we prove here all start with the bound $\mathcal{H}^n(G_j \cap B_r) \le \int_{D_r} v_j \mathbb{1}_{|u_j|<r}$; in this case with the value $r = 1$. Simply noticing that in fact $\mathcal{H}^n(G_j \cap D_1 \times (-1,1)) \le \int_{D_1} v_j \mathbb{1}_{|u_j|<1}$ allows us to use the initial arguments of the proof of Corollary 3.3.3 to conclude the weak inequality

$$\|\mathbf{C}\|(D_1 \times (-1,1)) \le 2n\omega_n. \tag{3.6}$$

Next, as $\mathbf{C}$ is invariant under homothetic rescalings, we can precisely calculate the ratio to be

$$\|\mathbf{C}\|(B_1)/\|\mathbf{C}\|(D_1 \times (-1,1)) = I_n \tag{3.7}$$

provided this is not the zero measure. Indeed, we may use the co-area formula under the guise of Lemma 28.1 in [Sim84] to 'slice' the cone by the level sets of the function $X = (x, X^{n+1}) \mapsto |x|$, meaning the sets $\partial D_r \times \mathbf{R}$ where $0 < r \leq 1$ in our case. This yields a family of varifolds $(\mathbf{C}^{(r)} \mid 0 < r \leq 1)$ of codimension two so that for all open sets $U \subset D_1 \times \mathbf{R}$,

$$\|\mathbf{C}\|(U) = \int_0^1 \|\mathbf{C}^{(r)}\|(U \cap \partial D_r \times \mathbf{R})\, \mathrm{d}r.$$

As the limit cone is cylindrical of the form $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}e_{n+1}$, the same holds for the $\mathbf{C}^{(r)} = \mathbf{C}_0^{(r)} \times \mathbf{R}e_{n+1}$, and we may use the co-area formula again with respect to the height function $(x, X^{n+1}) \mapsto X^{n+1}$,

$$\|\mathbf{C}\|(U) = \int_0^1 \int_{-1}^1 \|\mathbf{C}_0^{(r)}\|(U \cap \partial D_r \times \{z\})\, \mathrm{d}z\, \mathrm{d}r,$$

where we use a slight abuse of notation to simplify the integrated expression.

We apply this formula with $U = B_1$ and $D_1 \times (-1,1)$ respectively, and use the homothetic invariance of $\mathbf{C}$ to rewrite

$$\|\mathbf{C}\|(B_1) = 2\|\mathbf{C}_0^{(1)}\|(\partial D_1) \int_0^1 r^{n-2}(1-r^2)^{1/2}\, \mathrm{d}r,$$

$$\|\mathbf{C}\|(D_1 \times (-1,1)) = 2\|\mathbf{C}_0^{(1)}\|(\partial D_1) \int_0^1 r^{n-2}\, \mathrm{d}r.$$

We may evaluate the former integral via a simple change of variable, yielding $\int_0^1 r^{n-2}(1-r^2)^{1/2}\, \mathrm{d}r = (n-1)^{-1} \int_0^{\pi/2} (\cos\theta)^n\, \mathrm{d}\theta$, which we also denote $(n-1)^{-1} I_n$. Comparing the two equations confirms the identity (3.7) that was claimed above.

Combining this with the initial bounds from (3.6) we obtain

$$\|\mathbf{C}\|(B_1) = I_n \|\mathbf{C}\|(D_1 \times (-1,1)) \leq 2n I_n \omega_n.$$

This is equivalent to (3.5) because the weight measure $\|\mathbf{C}\|$ of the limit cone does not charge ∂B_1. □

Although these improved estimates hold in the relatively general setting described above, we essentially exclusively apply them in situations where the structure of the limit $\mathbf{C} = \mathbf{C}^0 \times \mathbf{R}e_{n+1}$ is known *a priori*. Specifically we consider the case were the limit is $\mathbf{P} = \mathbf{P}^0 \times \mathbf{R}e_{n+1}$ is a sum of vertical planes $\Pi_j = \Pi_j^0 \times \mathbf{R}e_{n+1}$, where each plane has multiplicity m_j either one or two. We point out that we do not additionally assume that the planes meet along a common axis, although this case is included in our analysis. The proof of the improved estimates is somewhat more straightforward under these hypotheses. This motivates restating the estimates as a stand-alone result.

PROPOSITION 3.3.5. *Let $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a sequence of two-valued minimal graphs with $|G_j| \to \mathbf{P} = \sum_j m_j |\Pi_j^0| \times \mathbf{R}e_{n+1}$. Then $\sum_j m_j \leq \lfloor 2nI_n \rfloor$.*

PROOF. Every plane $\Pi_j = \Pi_j^0 \times \mathbf{R}e_{n+1}$ contributes precisely $2m_j\omega_{n-1}$ to the total weight $\|\mathbf{P}\|(D_1 \times (-1,1))$ in the cylinder. This in turn is bounded by the area estimates from Corollary 3.3.3, whence $\sum_j 2m_j\omega_{n-1} \leq 2n\omega_n$, or indeed $\sum_j m_j \leq n\omega_n/\omega_{n-1} = 2nI_n$. □

3.3.3. Further qualitative improvements. Let $\alpha \in (0,1)$ be fixed, and $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a sequence of two-valued minimal graphs. We return once again to the original setting of the previous section, where we obtained improved estimates under the *a priori* assumption that the sequence converges to a vertical varifold $V = V_0 \times \mathbf{R}e_{n+1} \in \mathbf{IV}_n(D_2 \times \mathbf{R})$. We obtain a tiny further improvement on the area estimates derived there.

COROLLARY 3.3.6. *Let $\alpha \in (0,1)$, and $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a sequence of two-valued minimal graphs with $|G_j| \to V = V_0 \times \mathbf{R}e_{n+1}$ as $j \to \infty$. Then there is $\delta_V > 0$ so that $\limsup_{j\to\infty} \mathcal{H}^n(G_j \cap D_1 \times (-1,1)) \leq (1-\delta_V)2n\omega_n$.*

PROOF. We first explain how to obtain these estimates, under the assumption that there is an open set $U \subset D_1$ with $\operatorname{spt}\|V_0\| \cap D_1 \subset U$ and $\operatorname{Per}(U) < \operatorname{Per}(D_1)$.

Given arbitrarily small $\delta > 0$, there is $\eta \in C_c^1(D_2)$ with $\eta \equiv 1$ on U and $\int_{D_2} |D\eta| \leq (1+\delta)\operatorname{Per}(U)$. Let $\tau > 0$ be small, and recall that we write $(U)_\tau = \{X \in \mathbf{R}^n \mid \operatorname{dist}(X, U) < \tau\}$ for the tubular neighbourhood of U with width τ. Now first take $\tau > 0$ small enough that $(U)_\tau \subset \{\eta = 1\}$, and next take the index $j \geq J(\tau)$ large enough that $G_j \cap D_1 \times (-1,1) \subset (U)_\tau \times (-1,1)$.

From here on we can proceed essentially in the same way as in the proof of Corollary 3.3.3. The first step is to substitute the test function η as above into the into the inequality (3.2). Having taking the index as large as above guarantees that the following inequality (3.3) remains valid, meaning that here we get $\int_{D_1} \mathbb{1}_{|u_j|<1} \frac{|Du_j|^2}{v_j} \leq 2\int_{D_2} |D\eta|$. Hence $\int_{D_1} \mathbb{1}_{|u_j|<1} \frac{|Du_j|^2}{v_j} \leq 2(1+\delta)\operatorname{Per}(U)$. Now let $\delta_V > 0$ be small enough that $(1+\delta_V)\operatorname{Per}(U) \leq (1-\delta_V)\operatorname{Per}(D_1)$, and take $\delta = \delta_V$ in the above.

The area of G_j inside the cylinder $D_1 \times (-1,1)$ can be split into the sum $\int_{D_1} \mathbb{1}_{|u_j|<1} \frac{1}{v_j} + \int_{D_1} \mathbb{1}_{|u_j|<1} \frac{|Du_j|^2}{v_j}$. Precisely as in the proof of Corollary 3.3.3 it suffices to note that the first of the integrals tends to zero as $j \to \infty$, whence we find that $\limsup_{j\to\infty} \mathcal{H}^n(G_j \cap D_1 \times (-1,1)) \leq (1-\delta_V)2n\omega_n$, precisely as required.

We now turn to the construction of an open set with properties as described above. This is particularly simple if one is not concerned about precise estimates for the ratio $\operatorname{Per} U/\operatorname{Per} D_1 = \operatorname{Per} U/\mathcal{H}^{n-1}(S^{n-1})$, as indeed here we are not. Pick some point $x \in \partial D_1$ lying a distance $2r > 0$ away from $\operatorname{spt}\|V_0\|$. Inside the disc $D_r(x)$ consider the Caccioppoli set $D_r(x) \cap D_1$. As $D_r(x)$ is mean convex, we may combine for example the results of [DG61] with [Whi00, Lem. 3.4] (see also [Giu84, Thm. 1.20]) to show that there is a Caccioppoli set E which agrees with D_1 outside $D_r(x)$ but has strictly smaller perimeter: $\operatorname{Per} E < \operatorname{Per} D_1$. Heuristically this corresponds to cutting out the portion $\partial D_1 \cap D_r(x)$ from ∂D_1 and replacing it by a flat cap with the same boundary. To conclude simply let U be the interior of E. The construction depends only on $\operatorname{spt}\|V\|$, so one can indeed find a constant $\delta_V > 0$ depending only on V so that $(1+\delta_V)\operatorname{Per} U \leq (1-\delta_V)\operatorname{Per} D_1$. □

REMARK 3.3.7. Should one so desire, one could make the construction above self-contained, at the price of producing a slightly less streamlined argument. A construction by hand could go as follows. Choose a point $x \in \partial D_1 \setminus \operatorname{spt}\|V_0\|$ as above, with $\operatorname{dist}(x, \operatorname{spt}\|V_0\|) = 2r$. Write Π for the plane tangent to ∂D_1 at x. Provided $r < 1$, one sees that $\partial D_1 \cap D_r(x)$ is a single-valued graph of a smooth function defined on an open, mean convex subset $\Omega \subset \Pi$, with values in $\Pi^\perp$. We denote this function $f \in C^\infty(\overline{\Omega}; \Pi^\perp)$, which thus has $\partial D_1 \cap D_r(x) = \operatorname{graph} f$. We may then solve the Dirichlet problem for the minimal surface equation on Ω, and find a function $u \in C^\infty(\overline{\Omega}; \Pi^\perp)$ with f on $\partial\Omega$, which by construction has $\mathcal{H}^{n-1}(\operatorname{graph} u) < \mathcal{H}^{n-1}(\operatorname{graph} f) = \mathcal{H}^{n-1}(\partial D_1 \cap D_r(x))$. As $\partial D_r(x)$ is mean convex, we can justify $\operatorname{graph} u \subset \partial D_r(x)$ by a maximum principle-type argument. Orient $\Pi^\perp$ to point outward, and let $V \subset D_r(x)$ be the set of points lying below $\operatorname{graph} u$. The set $U = V \cup D_1 \setminus \overline{D}_r(x)$ has the desired property, namely $\operatorname{Per} U < \operatorname{Per} D_1$.

The estimates of Corollary 3.3.6 are called qualitative because we allow the constant $\delta_V > 0$ to depend on V, rather trying to compute a value valid for all limits, for instance depending only on n. Nonetheless they allow for a rather useful improvement to the estimates of Proposition 3.3.5 for two-valued graphs in dimension $n+1=4$.

COROLLARY 3.3.8. *Let $n = 3$, and $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a sequence of two-valued minimal graphs with $|G_j| \to \mathbf{P} = \sum_j m_j |\Pi_j^0| \times \mathbf{R}e_4$. Then $\sum_j m_j \leq 3$.*

PROOF. Let $\delta > 0$ be the constant for $V = \mathbf{P}$ we obtain from Corollary 3.3.6. Using the same calculations as in the proof of Proposition 3.3.5 we find $\sum_j m_j \leq (1-\delta)3\omega_3/\omega_2 = 4(1-\delta)$. Taking integer values we find $\sum_j m_j \leq 3$. □

3.4. Gradient estimates for two-valued minimal graphs

Let $\alpha \in (0,1)$ and the dimension $n \geq 1$ be arbitrary. Let $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph. In this section we derive an interior gradient estimate analogous to the classical estimates for smooth, single-valued graphs. These can be found for example in Section 16.2 of [GT98], which we also follow for the structure of the argument. These gradient bounds stem from integral estimates for the function w, defined on $\operatorname{reg} G$ by the expression (3.1). To ensure the validity of these in the presence of branch points (which are absent in the single-valued case), we rely on the fine properties of the branch set proved by [KW20], specifically that it has zero 2-capacity. The main result in this section is the following.

LEMMA 3.4.1. *Let $\alpha \in (0,1)$ and let $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph. Then there exists $C = C(n) > 0$ so that*

$$\max_{D_1} \|Du\| \leq C \exp(C \sup_{D_2} \|u\|).$$

We prove the equivalent version below, for discs of arbitrary radius $r > 0$.

LEMMA 3.4.2. *Let $u \in C^{1,\alpha}(D_{3r}; \mathcal{A}_2)$ be a two-valued minimal graph. Then there is a constant $C = C(n) > 0$ so that*

$$\max\{|Du_1(0)|, |Du_2(0)|\} \leq C \exp(C \max_{D_{2r}} \|u\|/r).$$

3.4.1. Integral estimates and a mean value inequality for w.

Define a function w at all points $X \in \operatorname{reg} G \cap D_2 \times \mathbf{R}$ by

$$w(X) = \log v(X) = -\log\langle \nu(X), e_{n+1}\rangle, \tag{3.1}$$

where $\nu(X)$ is the upward-pointing unit normal to $\operatorname{reg} G$ at X and $v(X) = \langle \nu(X), e_{n+1}\rangle^{-1}$.

LEMMA 3.4.3. *For all compact $K \subset D_2 \times \mathbf{R}$,*

$$\sup_{K \cap \operatorname{reg} G} w + \int_{K \cap \operatorname{reg} G} |\nabla_G w|^2 < +\infty \tag{3.2}$$

and w satisfies $\Delta_G w = |\nabla_G w|^2 + |A_G|^2$ weakly in the sense that for all $\varphi \in C^1_c(D_2 \times \mathbf{R})$,

$$-\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla_G \varphi\rangle = \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} (|\nabla_G w|^2 + |A_G|^2)\varphi. \tag{3.3}$$

PROOF. Let $K \subset D_2 \times \mathbf{R}$ be an arbitrary compact subset. As the gradient of u is locally bounded, we get

$$\sup_{\operatorname{reg} G \cap K} w < +\infty.$$

To prove

$$\int_{\operatorname{reg} G \cap K} |\nabla_G w|^2 < +\infty$$

we use the minimal immersion $\iota : M \to \mathbf{R}^{n+1}$ with image $\iota(M) = G \setminus \mathcal{B}_G = \operatorname{reg} G \cup \mathcal{C}_G$ that we constructed in 3.2.4.

Pull back w to $w \circ \iota \in C^\infty(\iota^{-1}(\operatorname{reg} G))$, which we subsequently extend across $\iota^{-1}(\mathcal{C}_G)$ to yield a function in $C^\infty(M)$, still denoted by $w \circ \iota$. This function satisfies the PDE

$$\Delta_M(w \circ \iota) - |\nabla_M(w \circ \iota)|^2 - |A_M|^2 = 0 \tag{3.4}$$

pointwise (and thus also weakly) on M. From this we may deduce the bound

$$\int_M |\nabla_M(w \circ \iota)|^2 \phi^2 \leq 4 \int_M |\nabla_M \phi|^2, \tag{3.5}$$

valid for all $\phi \in C^1_c(M)$.

Indeed if we ignore the curvature term in (3.4)—as we may because it has a favourable sign—and integrate against an arbitrary $\phi \in C^1_c(M)$ we see that

$$\int_M |\nabla_M(w \circ \iota)|^2 \phi \leq -\int_M \langle \nabla_M(w \circ \iota), \nabla_M \phi \rangle.$$

Instead of ϕ we may also use its square as a test function, for which we obtain

$$\begin{aligned} &\int_M |\nabla_M(w \circ \iota)|^2 \phi^2 \\ &\leq -2 \int_M \phi \langle \nabla_M(w \circ \iota), \nabla_M \phi \rangle \leq 2\Big(\int_M \phi^2 |\nabla_M(w \circ \iota)|^2\Big)^{1/2} \Big(\int_M |\nabla_M \phi|^2\Big)^{1/2}. \end{aligned}$$

Unless $\int_M |\nabla_M(w \circ \iota)|^2 \phi^2 = 0$ we may divide both sides by its square root, yielding (3.5). (If the integral vanishes then the inequality is trivially satisfied.)

In particular, if we take $\varphi \in C^1_c(D_2 \times \mathbf{R} \setminus \mathcal{B}_G)$ and let $\phi = \varphi \circ \iota$ and translate (3.5) to the graph, we obtain

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G w|^2 \varphi^2 \leq 4 \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G \varphi|^2. \tag{3.6}$$

To extend this through the branch point singularities of G, we once again employ the sequence $(\eta_j \mid j \in \mathbf{N})$ with properties as described in Lemma 3.2.7.

Then proceed as in the proof of Lemma 3.2.11, namely take $\varphi \in C^1_c(D_2 \times \mathbf{R})$ and substitute the test function $\varphi(1-\eta_j)$ into (3.6), yielding

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G w|^2 \varphi^2 (1-\eta_j)^2$$
$$\leq 4 \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \varphi^2 |\nabla_G \eta_j|^2 + 2\varphi(1-\eta_j)\langle \nabla_G \varphi, \nabla_G (1-\eta_j)\rangle + |\nabla_G \varphi|^2 (1-\eta_j)^2.$$

The terms on the right-hand side are identical to those on the right-hand side of (3.6), so that we may justify passing to the limit $j \to \infty$ as in (3.7),(3.8) and (3.9). We thus obtain

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G w|^2 \varphi^2 \leq 4 \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G \varphi|^2, \tag{3.7}$$

justifying passing to the limit on the left-hand side by an application Fatou's lemma, again as in the proof of Lemma 3.2.11. This justifies (3.2).

To show that w is a weak solution of the PDE $\Delta_G w = |\nabla_G w|^2 + |A_G|^2$ we proceed in much the same way. The integral identity (3.3) is obtained for test functions $\varphi \in C^1_c(D_2 \times \mathbf{R} \setminus \mathcal{B}_G)$ by working with the immersed M instead, as above. Using the same sequence $(\eta_j \mid j \in \mathbf{N})$ of functions, we then obtain

$$-\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla\{(1-\eta_j)\varphi\}\rangle = \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} (|\nabla_G w|^2 + |A_G|^2)\varphi(1-\eta_j).$$

For the right-hand side of the identity, we may let $j \to \infty$ by dominated convergence, which we can justify using our previously established local L^2-bounds for A_G and $|\nabla_G w|$ from (S_G) and (3.2) respectively.

For the left-hand side of the identity, we expand

$$-\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla\{(1-\eta_j)\varphi\}$$
$$= \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla_G \eta_j\rangle \varphi - \langle \nabla_G w, \nabla_G \varphi\rangle (1-\eta_j),$$

and repeat calculations akin to those in the first part of the proof, noting that the bounds of (3.2) justify both the limit

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla_G \eta_j\rangle \varphi$$
$$\leq \sup_{D_2 \times \mathbf{R}} |\varphi| \Big(\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G w|^2 \Big)^{1/2} \Big(\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} |\nabla_G \eta_j|^2 \Big)^{1/2} \to 0 \text{ as } j \to \infty$$

and the application of dominated convergence to deduce that

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla_G \varphi\rangle (1-\eta_j) \to \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \langle \nabla_G w, \nabla_G \varphi\rangle \text{ as } j \to \infty.$$

Thus we have derived the identity (3.3), which concludes the proof of the lemma. □

REMARK 3.4.4. We could have derived the integral estimate for $|\nabla_G w|$ in (3.2) differently, using the local curvature bounds that follow from the stability inequality (S_G). Indeed at all regular points $\langle \nu, e_{n+1}\rangle > 0$, hence

$$|\nabla_G w|^2 \leq |A_G|^2(\langle \nu, e_{n+1}\rangle^{-2} - 1) \text{ on } \operatorname{reg} G, \tag{3.8}$$

where we used the fact that $|\nabla_G \langle \nu, e_{n+1}\rangle|^2 \leq |A_G|^2(1 - \langle \nu, e_{n+1}\rangle^2)$. Given any compact subset $K \subset D_2 \times \mathbf{R}$, the term $\langle \nu, e_{n+1}\rangle$ is bounded below, say $\langle \nu, e_{n+1}\rangle \geq \delta_K$ on $\operatorname{reg} G \cap K$. Then integrating (3.8) we obtain

$$\int_{\operatorname{reg} G\cap K} |\nabla_G w|^2 \leq (\delta_K^{-2} - 1)\int_{\operatorname{reg} G\cap K} |A_G|^2, \tag{3.9}$$

whence we get $\int_{\operatorname{reg} G\cap K} |\nabla_G w|^2 \leq C_K$ for some $C_K > 0$ using (S_G).

We chose to include the longer derivation in our proof, as it yields the more precise $\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} |\nabla_G w|^2\varphi \leq 4\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} |\nabla_G \varphi|^2$, valid for all $\varphi \in C_c^1(D_2 \times \mathbf{R})$. We will also use this in the derivation of the interior gradient estimates (see the proof of Lemma 3.4.2 below). Compare this with the less useful inequality derived by arguing as above, essentially combining (3.9) with the stability inequality (S_G) to yield $\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} |\nabla_G w|^2\varphi^2 \leq (\delta_K^{-2} - 1)\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} |\nabla_G \varphi|^2$, where $\varphi \in C_c^1(D_2 \times \mathbf{R})$ and $\operatorname{spt}\varphi \subset K$.

Let us quickly comment on a subtlety in the proof. The function $w = -\log\langle \nu, e_{n+1}\rangle$ is only defined on the regular part $\operatorname{reg} G \cap D_2 \times \mathbf{R}$, and cannot be extended continuously across $\mathcal{C}_G \cap D_2 \times \mathbf{R}$. However after pulling w back via the immersion $\iota : M \to G \setminus \mathcal{B}_G$ we obtain a function $w \circ \iota$ which we can extend smoothly through $\iota^{-1}(\mathcal{C}_G)$. This in turn allowed us to integrate by parts, yielding formulas which translate to G.

This way one obtains the following identity, valid for all $\varphi \in C_c^2(D_2 \times \mathbf{R} \setminus \mathcal{B}_G)$:

$$\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} (\Delta_G\varphi)w = -\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \langle \nabla_G\varphi, \nabla_G w\rangle = \int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \varphi\Delta_G w. \tag{3.10}$$

From this, we may verify using a capacity argument that for $\varphi \in C_c^2(D_2\times\mathbf{R})$,

$$-\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \langle \nabla_G\varphi, \nabla_G w\rangle = \int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \varphi\Delta_G w. \tag{3.11}$$

Let again $(\eta_j \mid j \in \mathbf{N})$ be a sequence of functions with properties as described in Lemma 3.2.7. By a mollification argument for example, we may additionally impose that $\eta_j \in C_c^\infty(D_2 \times \mathbf{R})$ for all j. If $\varphi \in C_c^2(D_2 \times \mathbf{R})$ then $(1 - \eta_j)\varphi \in C_c^2(D_2 \times \mathbf{R} \setminus \mathcal{B}_G)$ is a valid test function in (3.10).

Focus on the two integrals in (3.11). For the first, we may justify taking the limit

$$\int_{\operatorname{reg} G\cap U} \langle \nabla_G\{(1-\eta_j)\varphi\}, \nabla_G w\rangle \to \int_{\operatorname{reg} G\cap U} \langle \nabla_G\varphi, \nabla_G w\rangle \text{ as } j\to\infty$$

as usual, whereas for the second note that on $\operatorname{reg} G$,

$$|\varphi(1-\eta_j)\Delta_G w| \leq |\varphi|(|\nabla_G w|^2 + |A_G|^2) \text{ for all } j.$$

As $K = \operatorname{spt}\varphi$ is compact, by stability and (3.2) there is a $C_K > 0$ so that

$$\int_{\operatorname{reg} G\cap K} |\nabla_G w|^2 + |A_G|^2 \leq C_K.$$

Finally, dominated convergence allows taking the limit

$$\int_{\operatorname{reg} G\cap U} \varphi(1-\eta_j)\Delta_G w \to \int_{\operatorname{reg} G\cap U} \varphi\Delta_G w \text{ as } j\to\infty,$$

which confirms the identity (3.11) claimed above.

By dropping the curvature term from the (weakly satisfied) PDE $\Delta_G w = |\nabla_G w|^2 + |A_G|^2$ one sees that w is weakly subharmonic on G through the branch set. As a consequence it satisfies the following *mean value inequality.*

COROLLARY 3.4.5. *Let* $X = (x, X^{n+1}) \in D_2 \times \mathbf{R}$. *Then for all* $0 < \sigma < \rho < 2 - |x|$

$$\begin{aligned}(3.12)\quad \rho^{-n}\int_{\operatorname{reg} G\cap B_\rho(X)} w - \sigma^{-n}\int_{\operatorname{reg} G\cap B_\sigma(X)} w \\ \geq \int_{\operatorname{reg} G\cap B_\rho(X)\setminus \overline{B}_\sigma(X)} w|D^\perp r|^2 r^{-n} \geq 0.\end{aligned}$$

REMARK 3.4.6. Similar, though not identical, inequalities can be found for example in [Sim84, Ch.18]. However, the literature does not seem to contain a proof of the above under the hypotheses we consider, so we give a detailed derivation following the steps of Simon's proof of the monotonicity formula for area [Sim84, Ch.17].

PROOF. To simplify notation, we may assume without loss of generality that the point X lies at the origin and $\rho < 2$. We use a two-parameter family of Lipschitz cutoff functions $(\gamma_{\delta,s} \mid \delta \in (0,1), s \in (0,\rho))$ constructed by first setting

$$\gamma_\delta(t) = \begin{cases} 1 & \text{if } t \leq 1-\delta, \\ (1-t)/\delta & \text{if } 1-\delta < t < 1, \\ 0 & \text{if } t \geq 1 \end{cases}$$

and then rescaling

$$\gamma_{\delta,s}(t) = \gamma_\delta(t/s) \text{ for all } t \in \mathbf{R}.$$

Moreover we write $r = |X|$ and define the radial functions

$$\gamma_{\delta,s}(X) = \gamma_{\delta,s}(r) \text{ for all } X \in D_2 \times \mathbf{R},$$

which all have $\operatorname{spt}\gamma_{\delta,s} \subset\subset B_\rho \subset D_2 \times \mathbf{R}$. Fix $\delta \in (0,1)$ and $s \in (0,\rho)$, with the eventual aim of letting δ tend to zero.

Although the vector field $\gamma_{\delta,s}(r)Xw$ is not Lipschitz we can justify its use in the first variation formula using a quick capacity argument. Let $(\eta_j \mid j \in \mathbf{N})$ be a sequence of cutoff sequences with properties essentially as described in Corollary 3.2.8, namely $\eta_j \in C_c^1(D_2 \times \mathbf{R} \cap \operatorname{reg} G)$ with $0 \leq \eta_j \leq 1$ and $\eta_j \equiv 1$ on $(\mathcal{B}_G)_{r_j} \cap B_\rho$ for some $r_j \to 0$. Moreover as $j \to \infty$, $\eta_j \to 0$ $\mathcal{H}^n$-a.e. on $\operatorname{reg} G \cap B_\rho$ and $\int_{\operatorname{reg} G \cap B_\rho} |\nabla_G \eta_j|^2 \to 0$. As these cut out the branch set, the vector field $(1-\eta_j)\gamma_{s,\delta}Xw$ is a valid choice in the first variation formula, and yields $\int_{\operatorname{reg} G \cap B_\rho} \operatorname{div}_G\big((1-\eta_j)\gamma_{s,\delta}Xw\big) = 0$. We can expand this expression to get $|\int (1-\eta_j)\operatorname{div}_G(\gamma_{s,\delta}Xw)| \leq \int |\gamma_{s,\delta}Xw||\nabla_G\eta_j|$. The right-hand side tends to zero by the Cauchy–Schwarz inequality. On the left-hand side we justify the convergence $\int (1-\eta_j)\operatorname{div}_G(\gamma_{s,\delta}Xw) \to \int \operatorname{div}_G(\gamma_{s,\delta}Xw)$ by dominated convergence, after noticing that $\int |\operatorname{div}_G(\gamma_{s,\delta}Xw)| < \infty$. Hence we have

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \operatorname{div}_G(\gamma_{\delta,s}Xw) = 0. \tag{3.13}$$

Following the computations in [Sim84, p. 83] we see that $\operatorname{div}_G(\gamma_{\delta,s}X) = n\gamma_{\delta,s} + r\gamma_{\delta,s}'(1-|D^\perp r|^2)$ pointwise, and (3.13) leads to

$$\begin{aligned} n\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s}w + \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} r\gamma_{\delta,s}'w \\ = \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} r\gamma_{\delta,s}'w|D^\perp r|^2 - \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s}\langle \nabla_G w, X\rangle. \end{aligned} \tag{3.14}$$

To somewhat abbreviate this integral identity we define the two functions $I_\delta, J_\delta : (0,\rho) \to \mathbf{R}$ by setting, for all $s \in (0,\rho)$,

$$I_\delta(s) = \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s}w,$$

$$J_\delta(s) = \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s}w|D^\perp r|^2.$$

Notice that these are both differentiable in s with respective derivatives $I_\delta'(s) = \int \frac{\partial \gamma_{\delta,s}}{\partial s}w$ and $J_\delta'(s) = \int \frac{\partial\gamma_{\delta,s}}{\partial s}w|D^\perp r|^2$. Note $\gamma_{\delta,s}'(r) = 1/s\gamma_\delta'(r/s) = 1/(rs)\gamma_\delta'(r/s)$, so that $r\gamma_{\delta,s}'(r) = -s\frac{\partial}{\partial s}\gamma_{\delta,s}(r)$. Thus we can rewrite (3.14) as

$$nI_\delta(s) - sI_\delta'(s) = -sJ_\delta'(s) - \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s}\langle \nabla_G w, X\rangle \text{ for all } s \in (0,\rho).$$

Multiply this equation by s^{-n-1} and notice that this is

$$\frac{\mathrm{d}}{\mathrm{d}s}(s^{-n}I_\delta(s)) = s^{-n}J_\delta'(s) + s^{-n-1}\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \gamma_{\delta,s}\langle\nabla_G w, X\rangle.$$

Integrate this identity for $s \in (\sigma,\rho)$ to get

(3.15)

$$\rho^{-n}I_\delta(\rho) - \sigma^{-n}I_\delta(\sigma) = \int_\sigma^\rho s^{-n}J_\delta'(s) + \int_\sigma^\rho s^{-n-1}\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \gamma_{\delta,s}\langle\nabla_G w, X\rangle.$$

The eventual aim is to let $\delta \to 0$ in this identity; this will yield (3.12) and conclude the proof. Before we do this, we separately integrate both integrals on the right-hand side by parts. For the first, we obtain

(3.16)

$$\begin{aligned}\int_\sigma^\rho s^{-n}J_\delta'(s) &= \rho^{-n}J_\delta(\rho) - \sigma^{-n}J_\delta(\sigma) + n\int_\sigma^\rho s^{-n-1}J_\delta(s)\\ &= \int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} w|D^\perp r|^2\Big\{\rho^{-n}\gamma_{\delta,\rho} - \sigma^{-n}\gamma_{\delta,\sigma} + n\int_\sigma^\rho s^{-n-1}\gamma_{\delta,s}\Big\}\,\mathrm{d}\mathcal{H}^n.\end{aligned}$$

For the second integral, first fix $s \in (0,\rho)$ and notice that $\nabla_G(r^2 - s^2) = 2X^T$. The identity $\int_{D_2\times\mathbf{R}\cap\operatorname{reg} G}\operatorname{div}_G(\gamma_{\delta,s}(r^2-s^2)\nabla_G w) = 0$ is equivalent to

$$\begin{aligned}&2\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \gamma_{\delta,s}\langle\nabla_G w, X\rangle\\ &= -\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}}(\Delta_G w)\gamma_{\delta,s}(r^2-s^2) - \int_{\operatorname{reg} G\cap D_2\times\mathbf{R}}(r^2-s^2)\langle\nabla_G\gamma_{\delta,s},\nabla_G w\rangle.\end{aligned}$$

The first integral on the right-hand side is equal to $-\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}}(|A_G|^2 + |\nabla_G w|^2)\gamma_{\delta,s}(r^2-s^2)$, and is non-negative. From the identity we only retain the inequality

$$2\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}} \gamma_{\delta,s}\langle\nabla_G w, X\rangle \geq -\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}}(r^2-s^2)\langle\nabla_G\gamma_{\delta,s},\nabla_G w\rangle.$$

Notice that

$$\nabla_G\gamma_{\delta,s} = -(s\delta)^{-1}\frac{X^T}{r}\mathbb{1}_{B_s\setminus\overline{B}_{(1-\delta)s}},$$

so using the co-area formula we can estimate the integral on the right-hand side like

$$\begin{aligned}&\int_{\operatorname{reg} G\cap D_2\times\mathbf{R}}(r^2-s^2)\langle\nabla_G\gamma_{\delta,s},\nabla_G w\rangle\\ &\qquad= \int_{(1-\delta)s}^s (s\delta)^{-1}(s^2-\theta^2)\Big\{\int_{\partial B_\theta\cap\operatorname{reg} G}\langle\nabla_G w,\frac{X}{\theta}\rangle\Big\}\,\mathrm{d}\theta.\end{aligned}$$

As $\theta \in (s(1-\delta), s)$ we get $s^2 - \theta^2 \leq s^2\delta(2-\delta)$ and

$$\int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} (s^2 - r^2)|\langle \nabla_G \gamma_{\delta,s}, \nabla_G w\rangle| \\ \leq \int_{(1-\delta)s}^{s} \int_{\partial B_\theta \cap \operatorname{reg} G} (2-\delta)s|\nabla_G w| = (2-\delta)s \int_{\operatorname{reg} G \cap B_s \setminus \overline{B}_{(1-\delta)s}} |\nabla_G w|.$$

Then integrating this over $s \in (\sigma, \rho)$ we obtain a bound for the second integral in (3.15),

$$(3.17) \qquad \Big| \int_\sigma^\rho s^{-n-1} \int_{\operatorname{reg} G \cap D_2 \times \mathbf{R}} \gamma_{\delta,s} \Big| \leq 2 \int_\sigma^\rho s^{-n} \int_{\operatorname{reg} G \cap B_s \setminus \overline{B}_{(1-\delta)s}} |\nabla_G w|.$$

As we announced above, we may now let $\delta \to 0$ at the same time on the left-hand side of (3.15), in (3.16) and (3.17), justifying the convergence each time by dominated convergence. Thus

$$\rho^{-n} I(\rho) - \sigma^{-n} I(\sigma) \geq \lim_{\delta \to 0} \Big\{ \int_\sigma^\rho s^{-n} J_\delta'(s) \Big\},$$

where we write $I(s) = \int_{\operatorname{reg} G \cap B_s} w$ for all $s \in (0, \rho)$. We conclude by evaluating the limit on the left-hand side, which is

$$\int_{\operatorname{reg} G \cap B_\rho} w|D^\perp r|^2 \Big\{ \rho^{-n} - \sigma^{-n} \mathbb{1}_{r<\sigma} + (r \vee \sigma)^{-n} - \rho^{-n} \Big\} \\ = \int_{\operatorname{reg} G \cap B_\rho \setminus \overline{B}_\sigma} w|D^\perp r|^2 r^{-n}.$$

Therefore we obtain

$$\rho^{-n} I(\rho) - \sigma^{-n} I(\sigma) \geq \int_{\operatorname{reg} G \cap B_\rho \setminus \overline{B}_\sigma} w|D^\perp r|^2 r^{-n},$$

which establishes (3.12) and concludes the proof. □

3.4.2. Proof of the gradient bounds. We place ourselves in the situation described in Lemma 3.4.2. Let $u \in C^{1,\alpha}(D_{3r}; \mathcal{A}_2)$ be a two-valued minimal graph. Write $u_1(0), u_2(0)$ for the two values of u at $0 \in \mathbf{R}^n$, with corresponding $Du_1(0), Du_2(0) \in \mathbf{R}^n$. Note that we may consider w as a two-valued function defined on D_{3r}, including at the singular points of u.

Applying the mean value inequality at either point $X_i = (0, u_i(0))$ we obtain $\rho^{-n} \int_{G \cap B_\rho(X_i)} w - \sigma^{-n} \int_{G \cap B_\sigma(X_i)} w \geq 0$ for all $0 < \sigma < \rho < 3r$. Fixing $\rho = r$ and letting $\sigma \to 0$ we find

$$\lim_{\sigma \to 0} \Big\{ (\omega_n \sigma^n)^{-1} \int_{\operatorname{reg} G \cap B_\sigma(X_i)} w \Big\} \leq (\omega_n r^n)^{-1} \int_{\operatorname{reg} G \cap B_r(X_i)} w.$$

If X_i is regular, then the limit on the left-hand side is $w_i(0)$ whereas if X_i is singular then it is equal $w_1(0) + w_2(0)$. Translate the graph so that $0 \in G$

and w the larger of its two values there. In both cases

$$2\max\{w_1(0), w_2(0)\} \le (\omega_n r^n)^{-1} \int_{\operatorname{reg} G\cap B_r} w. \tag{3.18}$$

This allows us to reduce the proof of Lemma 3.4.2 to an estimation of the integral on the right-hand side of (3.18).

CLAIM 1. There is a constant $C = C(n)$ so that if $u \in C^{1,\alpha}(D_{3r}; \mathcal{A}_2)$ is a two-valued minimal graph with $0 \in G$ then

$$r^{-n} \int_{\operatorname{reg} G\cap B_r} w \le C(1 + r^{-1} \sup_{D_{2r}} \|u\|).$$

PROOF. We split the integral $\int_{\operatorname{reg} G\cap B_r} w = \int_{|x|^2+u^2<r^2} wv$ into the sum of $\int_{|x|^2+u^2<r^2} w$ and $\int_{|x|^2+u^2<r^2} \frac{|Du|^2}{v} w$. The former is easier to estimate, as $w \le v$. Therefore $\int_{|x|^2+u^2<r^2} w \le \mathcal{H}^n(G \cap B_r) \le C(n)r^n$, using the area bounds of Proposition 3.3.1.

We estimate the latter under the weaker restriction that $|x| < r, |u| < r$, for notational convenience. Let $\eta \in C^1_c(D_{2r})$ be a standard cutoff function with $\eta \equiv 1$ on D_r and $|D\eta| \le 2r^{-1}$. We write

$$\Phi(x, z, p) = \frac{1}{2}\log(1 + |p|^2)(z_r + r)\eta(x),$$

where z_r is defined as in (3.1), meaning that here

$$z_r + r = \begin{cases} 2r & \text{if } z > r, \\ z + r & \text{if } -r \le z \le r, \\ 0 & \text{if } z < -r. \end{cases}$$

One may then check that Φ satisfies the required hypotheses laid out in Proposition 3.2.1 to justify its use as a test function in the equation (3.4). The only hypothesis we check here is (3.3),

$$\int_{D_{3r}\setminus\mathcal{B}_u} |D(\Phi(x, u, Du))| < +\infty. \tag{3.19}$$

As $\Phi(x, u, Du) = (u_r + r)\eta w$ we calculate its derivative as

$$D\Phi(x, u, Du) = Du\mathbb{1}_{|x|<r,|u|<r}\eta w + D\eta(u_r + r)w + Dw(u_r + r)\eta. \tag{3.20}$$

Only the last term is not locally bounded. Instead we find the following integral bound. (The main observation underlying this bound, as proved in Claim 2, is the pointwise inequality (3.23). The unwieldiness of the proof is caused by the possibility of branch points in D_{3r}.)

CLAIM 2. For all compact $K \subset D_{3r}$ there is a constant $C_K > 0$ so that

$$\int_{K\setminus\mathcal{B}_u} |Dw| \le \int_{\operatorname{reg} G\cap K\times\mathbf{R}} |\nabla_G w| \le C_K. \tag{3.21}$$

PROOF. This essentially follows from the integral estimates for $|\nabla_G w|$ in (3.2). To justify this rigorously we proceed as follows. First we extend the two-valued function

$$w = \{w_1, w_2\} = \frac{1}{2}\{\log(1+|Du_1|^2), \log(1+|Du_2|^2)\}$$

to a two-valued function defined on the cylinder $D_{3r} \times \mathbf{R}$ by setting it constant in the vertical variable,

$$w(x, X^{n+1}) = w(x) = \{w_1(x), w_2(x)\} \tag{3.22}$$

for all $(x, X^{n+1}) \in D_{3r} \times \mathbf{R}$.

Near points $x \in D_{3r} \setminus \mathcal{B}_u$ we can make a local selection for u by say $u_1^x, u_2^x \in C^\infty(D_\sigma(x))$ so that $u = \{u_1^x, u_2^x\}$ in $D_\sigma(x)$. This selection is also valid for w, meaning that also $w = \{w_1^x, w_2^x\}$. Of course this relation also holds for the extension of w to $D_\sigma(x) \times \mathbf{R}$.

In the cylinder $D_\sigma(x) \times \mathbf{R}$ we split the graph into the transverse union

$$G \cap D_\sigma(x) \times \mathbf{R} = \operatorname{graph} u_1^x \cup \operatorname{graph} u_2^x.$$

Abbreviating $G_i^x = \operatorname{graph} u_i^x$ for $i = 1, 2$ we can separately calculate

$$|\nabla_{G_i^x} w_i^x(y, Y^{n+1})|^2 = |Dw_i^x(y, Y^{n+1})|^2 - |\langle \nu_{G_i^x}, Dw_i^x(y, Y^{n+1})\rangle|^2$$

at all $Y = (y, Y^{n+1}) \in D_\sigma(x) \times \mathbf{R}$. As the w_i^x are both independent of the vertical variable we obtain that for $i = 1, 2$,

$$|\langle \nu_{G_i^x}, e_{n+1}\rangle||Dw_i^x| \leq |\nabla_{G_i^x} w_i^x| \text{ on } D_\sigma(x) \times \mathbf{R}. \tag{3.23}$$

Next we split the integral $\int_{D_\sigma(x)} |Dw| = \int_{D_\sigma(x)} |Dw_1^x| + \int_{D_\sigma(x)} |Dw_2^x|$, and separately bound the two terms using (3.23), that is for $i = 1, 2$,

$$\int_{D_\sigma(x)} |Dw_i^x| = \int_{G_i^x \cap D_\sigma(x) \times \mathbf{R}} \frac{|Dw_i^x|}{v_i^x} \leq \int_{G_i^x \cap D_\sigma(x) \times \mathbf{R}} |\nabla_{G_i^x} w_i^x|.$$

where recall $v_i^x = (1 + |Du_i^x|^2)^{1/2} = \langle \nu_{G_i^x}, e_{n+1}\rangle^{-1}$. Thus we obtain

$$\int_{D_\sigma(x)} |Dw| \leq \int_{\operatorname{reg} G \cap D_\sigma(x)} |\nabla_G w|,$$

which is finite by (3.2) for example.

We may now return to the original problem of estimating the integral $\int_{K \setminus \mathcal{B}_u} |Dw|$. We may take a countable cover of the set $K \setminus \mathcal{B}_u$ by some collection of discs $(D_{\sigma_j}(x_j) \mid j \in \mathbf{N})$ centered at points $x_j \in K \setminus \mathcal{B}_u$ with

$$D_{\sigma_j}(x_j) \subset D_{3r} \setminus \mathcal{B}_u,$$

and let $(\rho_j \mid j \in \mathbf{N})$ be a partition of unity subordinate to the cover. Justifying the permutation of the sum and the integral by monotone convergence for example, we can decompose $\int_{K\cap D_{3r}\setminus\mathcal{B}_u}|Dw|$ like

$$\int_{K\cap D_{3r}\setminus\mathcal{B}_u}|Dw| = \sum_{j\in\mathbf{N}}\int_{K\cap D_{3r}\setminus\mathcal{B}_u}|Dw|\rho_j. \tag{3.24}$$

Arguing as in our derivation of the inequality (3.24) we can bound each of the integrals on the right-hand side by

$$\int_{K\setminus\mathcal{B}_u}|Dw|\rho_j \leq \int_{\operatorname{reg} G\cap D_{\sigma_j(x_j)}\times\mathbf{R}}|\nabla_G w|\rho_j,$$

where we extend ρ_j to a function defined on $D_{\sigma_j(x_j)}\times\mathbf{R}$ by setting it constant in the vertical variable, as we did for w in (3.22). Finally we may use monotone convergence once again to permute sums with integrals, yielding the desired

$$\int_{K\setminus\mathcal{B}_u}|Dw| \leq \int_{\operatorname{reg} G\cap K\times\mathbf{R}}|\nabla_G w| \leq C_K,$$

where the last inequality comes from (3.2). □

As we imposed that $\operatorname{spt}\eta\subset D_{2r}$, the inequality (3.21) from the claim applied with $K=\overline{D}_{2r}$ confirms the boundedness required by (3.19). We may thus substitute the expression (3.20) we calculated for $D\Phi(x,u,Du)$ into (3.4) to get

$$\int_{D_{3r}}\Big\langle\frac{Du}{v}, Du\mathbb{1}_{|x|<r,|u|<r}\eta w + D\eta(u_r+r)w + Dw(u_r+r)\eta\Big\rangle = 0,$$

so that

$$\int_{|x|<r,|u|<r}\frac{|Du|^2}{v}w \leq 2r\int_{|x|<2r,u>-r}|D\eta|w + |Dw|\eta.$$

To justify the expression on the left-hand side, simply note that for $i=1,2$, wherever $u_i>-r$ we have $(u_i)_r+r\leq 2r$, and if $u_i\leq -r$ then $(u_i)_r+r=0$. The proof then boils down to separately estimating the two integrals

$$\int_{|x|<2r,u>-r}|D\eta|w \quad\text{and}\quad \int_{|x|<2r,u>-r}|Dw|\eta.$$

The first integral is easier. Indeed using the fact that $w\leq v$ and the area bounds of Lemma 3.3.2 we get

$$\int_{|x|<2r,u>-r}|D\eta|w \leq 2r^{-1}\mathcal{H}^n(G\cap D_{2r}\times\mathbf{R}) \leq C(n)r^{n-1}(1+Mr^{-1}),$$

where we set $M=\sup_{D_{2r}}\|u\|$.

For the second integral $\int_{u>-r}|Dw|\eta$ we may start by arguing as in the proof of Claim 2 to justify that

$$\int_{u>-r}|Dw|\eta \leq \int_{\operatorname{reg} G\cap D_{2r}\times(-r,\infty)}|\nabla_G w|\eta. \tag{3.25}$$

To estimate this, recall the helpful integral inequality (3.7) we used in the proof of Lemma 3.4.3. To use it in the present context, we extend η to a test function compactly supported in the cylinder $D_{2r} \times \mathbf{R}$ by multiplying by a cutoff function $\tau \in C_c^1(\mathbf{R})$ in the vertical direction. We further impose that $\tau \equiv 1$ on $(-r, M)$ and $\operatorname{spt}\tau \subset (-2r, M+r)$ with $|\tau'| \leq 2r^{-1}$. Then from (3.7) we obtain

(3.26)

$$\int_{\operatorname{reg} G} |\nabla_G w|^2 \tau^2 \eta^2 \leq 8 \int_{\operatorname{reg} G} |\nabla_G \tau|^2 \eta^2 + \tau^2 |\nabla_G \eta|^2 \leq 64 r^{-2} \mathcal{H}^n(G \cap \operatorname{spt}\phi).$$

Estimate the right-hand side of (3.25) with Hölder's inequality,

$$\int_{\operatorname{reg} G \cap D_{2r} \times (-r,\infty)} |\nabla_G w| \eta$$
$$\leq (\mathcal{H}^n(G \cap D_{2r} \times (-2r, M+r))^{1/2} \Big(\int_{G \cap D_{2r} \times (-r,M)} |\nabla_G w|^2 \eta^2 \Big)^{1/2},$$

and combining this with (3.26),

$$\int_{\operatorname{reg} G \cap D_{2r} \times (-r,\infty)} |\nabla_G w| \eta \leq 8 r^{-1} \mathcal{H}^n(G \cap D_{2r} \times \mathbf{R}).$$

This in turn can be bounded using the area estimates from Lemma 3.3.2: $\mathcal{H}^n(G \cap D_{2r} \times \mathbf{R}) \leq C(n) r^n (1 + M r^{-1})$. This yields the desired bound for $\int_{u>-r} |Dw| \eta$ via (3.25), and concludes the proof of the claim. □

3.5. A regularity lemma for two-valued minimal graphs

3.5.1. A maximum principle near branch point singularities.

Lemma 3.5.1. *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph. Suppose that at the origin*

$$u(0) = \{0, 0\} \text{ and } Du(0) = \{0, 0\}.$$

Let $e \in \mathbf{R}^n \times \{0\}$ be a unit vector. If $\langle Du, e \rangle \leq 0$ on D_2 then $\langle Du, e \rangle = 0$.

Proof. To be precise by $\langle Du, e \rangle \leq 0$ we mean that $\langle Du_i(x), e \rangle \leq 0$ for $i = 1, 2$ and all $x \in D_2$. This is equivalent to $\langle \nu(X), e \rangle \geq 0$ for all $X \in \operatorname{reg} G \cap D_2 \times \mathbf{R}$, where ν is the upward-pointing unit normal to graph u. We argue by contradiction, assuming that $\langle \nu, e \rangle$ is non-negative but does not vanish identically. It is well-known that the function $\langle \nu, e \rangle$ is a Jacobi field for G, that is it satisfies the equation $\Delta \langle \nu, e \rangle + |A_G|^2 \langle \nu, e \rangle = 0$ both pointwise on $\operatorname{reg} G$ and weakly through singularities of G. (The justification for this can be made in essentially the same way as when working with $\langle \nu, e_{n+1} \rangle$ in the above.) Moreover, by the standard strong maximum principle, we know that $\langle \nu, e \rangle > 0$ on $\operatorname{reg} G$, meaning that we can define a smooth function

$w_e \in C^2(\operatorname{reg} G)$ by

$$w_e(X) = -\log\langle \nu(X), e\rangle \text{ at all } X \in \operatorname{reg} G.$$

The hypotheses of the claim ensure that $\langle \nu(0), e\rangle = 0$ at the origin, which also means that w_e diverges there. To avoid technical difficulties related to this, we perturb the vector e slightly. Let $\theta > 0$ be a small angle, through which we rotate e in the two-dimensional plane $\operatorname{span}\{e, e_{n+1}\}$, yielding the vector

$$e_\theta = (\cos\theta)e + (\sin\theta)e_{n+1}.$$

Unless we are in the pathological case where u diverges near the boundary of D_2, the function $\langle \nu, e_{n+1}\rangle = \frac{1}{\sqrt{1+|Du|^2}}$ is positive and bounded below, say

$$\langle \nu, e_{n+1}\rangle \geq \alpha > 0 \text{ on } \operatorname{reg} G \cap B_1.$$

Should u in fact diverge near the boundary, we can rescale it around the origin by a factor $\lambda > 1$ close to one, and restrict the resulting function to D_2 to reduce to the case where u is bounded near the origin.

As a consequence the rotated vector e_θ also has

$$\langle \nu, e_\theta\rangle \geq \alpha \sin\theta > 0 \text{ on } \operatorname{reg} G \cap B_1.$$

We can then define the function

$$w_\theta(X) = -\log\langle \nu(X), e_\theta\rangle \text{ at all } X \in \operatorname{reg} G$$

without running the risk of it diverging anywhere. This function has, locally for all compact $K \subset D_2 \times \mathbf{R}$ that

$$\int_{K\cap \operatorname{reg} G} w_\theta^2 + |\nabla w_\theta|^2 < +\infty,$$

and it satisfies the following equation weakly:

$$\Delta w_\theta - |\nabla w_\theta|^2 - |A_G|^2 = 0.$$

(These facts can be checked in much the same way as we did for the function $-\log\langle \nu, e_{n+1}\rangle$ in Section 3.4, see Lemma 3.4.3.)

In particular the function w_θ is weakly subharmonic on G, and thus by the mean value inequality, we get that for all $0 < r < s < 1$ and all points $X \in B_1$,

$$\frac{1}{\omega_n r^n}\int_{\operatorname{reg} G\cap B_r(X)} w_\theta \leq \frac{1}{\omega_n s^n}\int_{\operatorname{reg} G\cap B_s(X)} w_\theta.$$

Applying this at the origin and letting $r \to 0$ (as is justified by Fatou's lemma), we obtain that for all $0 < s < 1$,

$$2w_\theta(0) \le \frac{1}{\omega_n s^n} \int_{\operatorname{reg} G \cap B_s} w_\theta. \tag{3.1}$$

For all $\theta > 0$ we write

$$M_\theta = \sup_{\operatorname{reg} G \cap B_1} w_\theta \text{ and } m_\theta = \inf_{\operatorname{reg} G \cap B_1} \langle \nu, e_\theta \rangle,$$

which are related by $M_\theta = -\log m_\theta$. Then

$$w_\theta(0) \ge M_\theta + \log \alpha,$$

where recall $\alpha = \inf \langle \nu, e_{n+1} \rangle$. Indeed at the origin $\langle \nu(0), e_\theta \rangle = \sin \theta$, whereas $m_\theta \ge \alpha \sin \theta$. Therefore $\langle \nu(0), e_\theta \rangle \le m_\theta / \alpha$. Translating this to w_θ we obtain $w_\theta(0) \ge -\log(m_\theta/\alpha) = -\log m_\theta + \log \alpha$. Let $\delta > 0$ be given. As $M_\theta \to \infty$ as $\theta \to 0$, we may choose a small value $\theta_0 > 0$ in terms of α, δ so that for all $\theta \in (0, \theta_0)$ we also have

$$w_\theta(0) \ge (1 - \delta/2) M_\theta.$$

Substitute this into the mean value inequality (3.1) with radius $s = 1$, obtaining that

$$(2 - \delta) M_\theta \le \frac{1}{\omega_n} \int_{\operatorname{reg} G \cap B_1} w_\theta.$$

Let $\lambda > 0$ be a parameter whose value we will fix later. Then we may split the integral on the right-hand side by conditioning on the event that $\{w_\theta \ge \lambda\}$, obtaining two integrals which we can separately bound by

$$\frac{1}{\omega_n} \int_{\operatorname{reg} G \cap B_1 \cap \{w_\theta > \lambda\}} w_\theta \le M_\theta / \omega_n \mathcal{H}^n(\operatorname{reg} G \cap B_1 \cap \{w_\theta > \lambda\})$$
$$\frac{1}{\omega_n} \int_{\operatorname{reg} G \cap B_1 \cap \{w_\theta \le \lambda\}} w_\theta \le \lambda / \omega_n \mathcal{H}^n(\operatorname{reg} G \cap B_1 \cap \{w_\theta \le \lambda\}).$$

Recall that w is large only near $\{\langle \nu, e \rangle = 0\} \subset \operatorname{sing} G \cap B_1$, and hence is bounded away from the singular set. Since the functions w_θ converge to w pointwise on $\operatorname{reg} G \cap \overline{B}_1$, and uniformly in compact subsets $K \subset \operatorname{reg} G \cap \overline{B}_1$, we obtain uniform bounds for the sequence too: for all compact $K \subset \operatorname{reg} G \cap \overline{B}_1$ there exist $\theta_K > 0$ and $D_K > 0$ so that for all $\theta \in (0, \theta_K)$

$$w_\theta \le D_K \text{ on } K. \tag{3.2}$$

Working in the larger ball $B_{3/2}$, we see that $\mathcal{H}^n(\operatorname{sing} G \cap \overline{B}_{3/2}) = 0$. Therefore given any $\epsilon > 0$ we may find a finite open cover of $\operatorname{sing} G \cap \overline{B}_{3/2}$ by balls $B(X_1, r_1), \ldots, B(X_N, r_N)$ with $\sum_{k=1}^N r_k^n \le \epsilon$. Perhaps after slightly increasing the radii of the balls in the cover, we may arrange for $\operatorname{reg} G \cap B_1 \setminus$

$\cup_{k=1}^N B(X_k, r_k)$ to lie a positive distance away from the singular set. Using the bound above in (3.2), we see that there must be a constant $D > 0$ so that for all $\theta \in (0, \theta_0)$

$$w_\theta(X) \leq D \text{ at all } X \in \operatorname{reg} G \cap B_1 \setminus \cup_{k=1}^N B(X_k, r_k),$$

after adjusting θ_0 to a smaller value if necessary. We may then set $\lambda = D$, and see that for all $\theta \in (0, \theta_0)$,

$$\mathcal{H}^n(\operatorname{reg} G \cap B_1 \cap \{w_\theta > D\}) \leq \sum_{k=1}^N \omega_n r_k^n \leq \omega_n \epsilon. \tag{3.3}$$

The divergence of M_θ as $\theta \to 0$ additionally lets us impose that θ be small enough that $M_\theta \geq C(n)D$, where $C(n)$ is a constant so that $\mathcal{H}^n(\operatorname{reg} G \cap B_1) \leq \omega_n C(n)$, available via the area bounds of Proposition 3.3.1. With θ as small as this, we get

$$D/\omega_n \mathcal{H}^n(\operatorname{reg} G \cap B_1 \cap \{w_\theta \leq D\}) \leq M_\theta. \tag{3.4}$$

Substituting (3.3) and (3.4) into our decomposition for $1/\omega_n \int_{\operatorname{reg} G \cap B_1} w_\theta$ we obtain the inequality

$$(2 - \delta) M_\theta \leq \frac{1}{\omega_n} \int_{\operatorname{reg} G \cap B_1} w_\theta \leq \big(\epsilon + 1\big) M_\theta,$$

which is absurd provided δ, ϵ are small enough. □

The analogous statement is a lot easier to prove for single-valued, smooth minimal graphs. In fact, this is an immediate application of the classical, strong maximum principle, and we used this in the proof above to deduce that $\langle \nu(X), e \rangle > 0$ for all regular points $X \in \operatorname{reg} G \cap D_2 \times \mathbf{R}$.

3.5.2. Regularity by a geometric argument. Here we show that Lipschitz two-valued minimal graphs are automatically regular. This is somewhat well-known among experts in the field, although an explicit proof is absent from the literature. Here we follow a strategy suggested to us by S. Becker-Kahn, using the results developed in his thesis [BK17]. Our aim is to prove the following two results simultaneously, using an inductive argument on the dimension n.

THEOREM 3.5.2. *Let $u \in \operatorname{Lip}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph with Lipschitz constant L. Then there is $\alpha = \alpha(L, n) \in (0, 1)$ so that*

$$u \in C^{1,\alpha}(D_2; \mathcal{A}_2).$$

This turns out to be equivalent to the following, seemingly weaker lemma.

LEMMA 3.5.3. *Let $u \in \mathrm{Lip}(\mathbf{R}^n; \mathcal{A}_2)$ be a two-valued minimal graph with Lipschitz constant L. If additionally u is homogeneous,*

$$u(\lambda x) = \lambda u(x) \text{ for all } \lambda > 0, x \in \mathbf{R}^n, \tag{3.5}$$

then u is linear.

PROOF. Before moving on to the inductive argument, let us explain how Lemma 3.5.3 implies Theorem 3.5.2. Let $u \in \mathrm{Lip}(D_2; \mathcal{A}_2)$ be a two-valued minimal graph with Lipschitz constant L. This is smooth away from away from $\mathcal{B}_u$, so consider an arbitrary $x \in \mathcal{B}_u \cap D_2$. Write $X = (x, X^{n+1}) \in \mathcal{B}_G$ be the corresponding point in the graph, at height $X^{n+1} = u_1(x) = u_2(x)$. For any tangent cone $\mathbf{C}_X \in \mathrm{VarTan}(|G|, X)$—*a priori* these are not unique—, there is a two-valued Lipschitz function $U_X \in \mathrm{Lip}(\mathbf{R}^n; \mathcal{A}_2)$ with the same Lipschitz constant, so that $\mathbf{C}_X = |\mathrm{graph}\, U_X|$. This function U_X is homogeneous as in (3.5), and thus by Lemma 3.5.3 this must be linear. In other words there is $\Pi_X \in Gr(n, n+1)$ so that $\mathbf{C}_X = 2|\Pi_X|$. Then by [BK17] there is $0 < \gamma = \gamma(n, L) < 1$ so that for some $\rho > 0$, $u \in C^{1,\gamma}(D_\rho(x); \mathcal{A}_2)$. As the branch point x was chosen arbitrarily and $\alpha(n, L) := \gamma$ does not depend on it, we get $u \in C^{1,\alpha}(D_2; \mathcal{A}_2)$.

The base of the inductive argument is simple, as when $n = 1$ then two-valued minimal graphs are automatically linear. For the induction step, assume that Theorem 3.5.2 holds in dimension $n-1 \geq 1$. We prove Lemma 3.5.3 in dimension n, and for that purpose consider an arbitrary minimal graph $u \in \mathrm{Lip}(\mathbf{R}^n; \mathcal{A}_2)$, homogeneous as in (3.5). We claim that there is $\gamma = \gamma(n, L) > 0$ so that $u \in C^{1,\gamma}(\mathbf{R}^n \setminus \{0\}; \mathcal{A}_2)$. To see this, let $X = (x, X^{n+1}) \neq 0 \in \mathrm{sing}\, G \cap D_2 \times \mathbf{R}$. Every tangent cone $\mathbf{C}_X \in \mathrm{VarTan}(G, X)$ is the graph of a two-valued function $U_X \in \mathrm{Lip}(\mathbf{R}^n; \mathcal{A}_2)$ with the same Lipschitz constant. By a standard dimension reduction argument, $\mathbf{C}_X$ is invariant under translation by tX for all $t \in \mathbf{R}$. By the induction hypothesis U_X is linear, and $\mathbf{C}_X$ is

(1) either a sum of two multiplicity one planes, $\mathbf{C}_X = |\Pi_1^X| + |\Pi_2^X|$,
(2) or a single multiplicity two plane, $\mathbf{C}_X = 2|\Pi_1^X|$.

By [BK17] there exists $0 < \gamma = \gamma(n, L) < 1$ so that for some $0 < \rho < |x|$, $u \in C^{1,\gamma}(D_\rho(x); \mathcal{A}_2)$ regardless of whether $X = (x, X^{n+1})$ is a classical singularity or a branch point. As x is arbitrary and γ can be chosen independently of it, we get $u \in C^{1,\gamma}(\mathbf{R}^n \setminus \{0\}; \mathcal{A}_2)$.

To extend this across the origin, define a function w on the regular set,

$$w(X) = -\log\langle \nu(X), e_{n+1}\rangle \text{ for all } X \in \mathrm{reg}\, G. \tag{3.6}$$

This is non-negative and bounded, and we may let

$$M = \sup_{\operatorname{reg} G \cap \partial B_1} w > 0, \tag{3.7}$$

and consider a sequence of points $X_i \in \operatorname{reg} G \cap \partial B_1$ with $X_i \to Z \in G \cap \partial B_1$ and $w(X_i) \to M$ as $i \to \infty$.

CLAIM 3. If $Z = (z, Z^{n+1}) \notin \mathcal{B}_G \cap \partial B_1$ then u is locally linear near Z in the sense that there is $\rho > 0$ and a smooth selection $u_1, u_2 \in C^\infty(D_\rho(z))$ with u_1 linear and $Z \in \operatorname{graph} u_1$.

PROOF. As the graph $|G|$ is invariant under homotheties, so is w, whence if $X \in \operatorname{reg} G$ then $\lambda X \in \operatorname{reg} G$ and $w(\lambda X) = w(X)$ for all $\lambda > 0$. Therefore (3.7) also means $M = \sup_{\operatorname{reg} G \cap B_1} w$.

When $Z \in \operatorname{reg} G \cap \partial B_1$ then by the classical strong maximum principle w is locally constant near Z, and thus so is so $|Du|$. Pick a small radius $\rho > 0$ so that a smooth selection $\{u_1, u_2\}$ can be made for u on $D_\rho(z)$. We arrange for $Z \in \operatorname{graph} u_1$. As $|Du_1|$ is constant in $D_\rho(z)$, it is harmonic by inspection. By the Bochner formula, $|D^2 u_1|^2 = \Delta |Du_1|^2 \equiv 0$ on $D_\rho(z)$. Thus u_1 is affine linear, and the homothety-invariance of G means that it must in fact be linear.

The argument is similar when $Z \in \mathcal{C}_G \cap \partial B_1$. Make a smooth selection $\{u_1, u_2\}$ for u on $D_\rho(z)$ and write $G_i = \operatorname{graph} u_i$. Define the two functions w_1, w_2 on G_1, G_2 respectively, using the analogue of (3.6). Without loss of generality assume that G_1 contains infinitely many points of $\{X_i \mid i \in \mathbf{N}\}$. As w_1 is continuous at the point Z, we get $w_1(Z) = M$. From then on, one can argue in the same way as when Z is regular. □

Now suppose $Z \in \mathcal{B}_G \cap \partial B_1$. This argument is a bit more involved, but revolves around the same idea. Because $Du_1(z) = Du_2(z)$, the function w can be continuously extended to Z. (The same is true for all branch points.) Write $\nu(Z) \in \mathbf{R}^{n+1}$ for the unit normal, and $2|\Pi_Z| \in \operatorname{VarTan}(|G|, Z)$ for the tangent plane to G at Z. As $w(Z) = M > 0$, this plane is not horizontal, and $\nu(Z) \neq e_{n+1}$. Let $P = \operatorname{span}\{e_{n+1}, \nu(Z)\} \subset \mathbf{R}^{n+1}$. This intersects Π_Z in a one-dimensional line, from which we pick a vector $e \in \Pi_Z \cap \Pi$ with $\langle e, e_{n+1} \rangle > 0$. Write $e = a e_{n+1} + b\nu(Z)$ for $a, b \in \mathbf{R}$, which are constrained by $0 = \langle e, \nu(Z) \rangle = a \langle e_{n+1}, \nu(Z) \rangle + b$, or equivalently $b = -a \langle e_{n+1}, \nu(Z) \rangle$. Let $X \in \operatorname{reg} G$ be an arbitrary regular point near Z. Then

$$\begin{aligned} \langle \nu(X), e \rangle &= \langle \nu(X), a e_{n+1} - a \langle e_{n+1}, \nu(Z) \rangle \nu(Z) \rangle \\ &= a\big(\langle \nu(X), e_{n+1} \rangle - \langle e_{n+1}, \nu(Z) \rangle \langle \nu(X), \nu(Z) \rangle \big). \end{aligned} \tag{3.8}$$

Take X close enough to Z that $0 < \langle \nu(Z), \nu(X) \rangle \leq 1$, say this holds for $X \in \operatorname{reg} G \cap B_\rho(Z)$ for example. Moreover by construction $\langle e_{n+1}, \nu(Z) \rangle \leq$

$\langle e_{n+1}, \nu(X)\rangle$, and hence

$$\langle \nu(X), e\rangle \geq 0 \text{ for all } X \in \operatorname{reg} G \cap B_\rho(Z). \tag{3.9}$$

Upon decreasing $\rho > 0$ we find $U_Z \in C^{1,\gamma}(B_\rho(Z) \cap (Z + \Pi_Z); \Pi_Z^\perp)$ so that $G \cap B_\rho(Z) \subset \operatorname{graph} U_Z$. By construction $e \in \Pi_Z$, and we can apply Lemma 3.5.1 to U_Z to deduce from (3.9) that $\langle \nu(X), e\rangle = 0$ for all $X \in \operatorname{reg} G \cap B_\rho(Z)$. Returning to (3.8) we see that this is only possible if for these points we have both $\langle \nu(X), e_{n+1}\rangle = \langle \nu(Z), e_{n+1}\rangle$ and $\langle \nu(Z), \nu(X)\rangle = 1$. Either would suffice to conclude that $w(X) = w(Z)$ for all $X \in \operatorname{reg} G \cap B_\rho(Z)$. Once we have derived this, we may reason as in the proof of Claim 3 to draw the analogous conclusion.

We use this to show that G must be a union of planes, using an argument similar to that used to prove Lemma 5.1.4. Let $\mathcal{R}$ be the set of connected components of $\operatorname{reg} G$, of which there are at most countably many. Among them we write $\mathcal{R}_f \subset \mathcal{R}$ for those $\Sigma \in \mathcal{R}$ that are *flat*, meaning $|A_\Sigma| \equiv 0$. The rest is denoted $\mathcal{R}_c = \mathcal{R} \setminus \mathcal{R}_f$. We decompose $|G| = \mathbf{C}_f + \mathbf{C}_c \in \mathbf{IV}_n(\mathbf{R}^{n+1})$, respectively defined by $\mathbf{C}_f = \sum_{\Sigma \in \mathcal{R}_f} \Theta_\Sigma |\Sigma|$ and $\mathbf{C}_c = \sum_{\Gamma \in \mathcal{R}_c} \Theta_\Gamma |\Gamma|$. Here given $\Sigma \in \mathcal{R}$ we write $\Theta_\Sigma \in \mathbf{Z}_{>0}$ for its multiplicity, which is constant by [Sim84, Thm. 41.1]. Both $\mathbf{C}_f, \mathbf{C}_c$ are invariant under homotheties, and stationary. To justify the latter, it suffices to prove that $\mathbf{C}_f, \mathbf{C}_c$ are stationary near points in $\mathcal{C}_G$, as the other singularities do not contribute to the first variation. Pick some point $X \in \operatorname{spt}\|\mathbf{C}_f\| \cap \|\mathbf{C}_c\| \cap \mathcal{C}_G$, and let $\rho > 0$ be so that we can decompose $G \cap B_\rho(X) = \Sigma_1 \cup \Sigma_2$ into a union of two surfaces embedded in $B_\rho(X)$, which meet transversely along $\operatorname{sing} G \cap B_\rho(X)$. By a unique continuation argument we may arrange for $\Sigma_1 \subset \operatorname{spt}\|\mathbf{C}_f\|$ and $\Sigma_2 \subset \operatorname{spt}\|\mathbf{C}_c\|$. Both Σ_i are stationary in $B_\rho(X)$, whence $\mathbf{C}_f, \mathbf{C}_c$ are stationary inside $B_\rho(X)$ too. As X was arbitrary, they are stationary in $\mathbf{R}^{n+1}$. The argument above shows that $\mathcal{R}_f \neq \emptyset$ and $\mathbf{C}_f \neq 0$, and by Lemma 5.2.3 it is supported in a union of planes, say $\operatorname{spt}\|\mathbf{C}_f\| = \Pi_1 \cup \cdots \cup \Pi_D$ with $D \leq 2$. If $\mathbf{C}_c = 0$ then we are done, otherwise $D = 1$ and $\mathbf{C}_f = |\Pi_1|$. In this case too one ultimately finds that $|G| = |\Pi_1| + |\Pi_2|$, for instance using [Sim77]. □

Chapter 4

Limit cones

4.1. Multiplicity and branch points of limit cones

Let $\alpha \in (0,1)$ and $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs with $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$. Here we examine the situation in which these graphs converge to a plane weakly in the varifold topology, $|G_j| \to m|\Pi|$ as $j \to \infty$, where $\Pi \in Gr(n, n+1)$ and $m \in \mathbf{Z}_{>0}$.

4.1.1. An a priori multiplicity bound.

Lemma 4.1.1. *If* $|G_j| = |\operatorname{graph} u_j| \to m|\Pi| \llcorner D_2 \times \mathbf{R}$ *then* $m \leq 2$.

Proof. We prove first the easier case where the plane $\Pi \in Gr(n, n+1)$ is not vertical, that is where it is not cylindrical of the form $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$. The plane must have integer multiplicity $m \in \mathbf{Z}_{>0}$ say. There is nothing to prove if $m = 1$, so we may assume that $m \geq 2$. Let a small constant $0 < \tau < 1$ be given, and take $j \geq J(\tau)$ large enough that $G_j \cap (\Pi)_1 \subset (\Pi)_\tau$. In fact we have the same control over G_j in the whole cylinder, that is $G_j \cap D_2 \times \mathbf{R} \subset (\Pi)_\tau$. Indeed, if this were to fail then $G_j \cap D_2 \times \mathbf{R}$ would be disconnected, and we could write $G_j \cap D_2 \times \mathbf{R} = \Gamma_{j,1} \cup \Gamma_{j,2}$ where $\Gamma_{j,1} \cap (\Pi)_\tau \neq \emptyset$ and $\Gamma_{j,2} \cap (\Pi)_1 = \emptyset$. But then $\mathcal{H}^n(G_j \cap D_2 \times \mathbf{R} \cap (\Pi_1)_1) \leq \mathcal{H}^n(\Gamma_{j,1})$, and taking limits as $j \to \infty$ would yield $m = 1$. This is absurd as we initally assumed that m is at least two, and hence we have confirmed that $G_j \cap D_2 \times \mathbf{R} \subset (\Pi)_\tau$. Let $L = \max\{X^{n+1} \mid X = (x, X^{n+1}) \in \Pi \cap D_2 \times \mathbf{R}\}$, then $\|u_j(x)\| \leq 2(\tau + L)$ for all $x \in D_2$. Using the interior gradient estimates of Lemma 3.4.1 we find that there is a constant $C = C(n, L)$ so that eventually $\|u_j\|_{1;D_1} \leq C$. Up to extracting a subsequence we find that the u_j converge to a two-valued Lipschitz graph defined on D_1. As by assumption $|G_j| \to m|\Pi|$ we can conclude that $m = 2$.

We can now turn to the case where the limit plane Π is vertical, that is of the form $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$ for some $(n-1)$-dimensional plane $\Pi_0 \subset \mathbf{R}^n$. The argument is simplified by working with the open cubes of the form $I^k = (-1,1)^k \times \{0\}^{n-k}$, and in particular I^{n+1}, I^n and their respective closures. Moreover we assume without loss of generality that $\Pi = \{X_1 = 0\}$. Indeed we may reduce ourselves to this situation by rotating both the two-valued graphs and Π. Given $\tau > 0$ define

$$E_\tau = \{(X, S) \in Gr_n(\overline{I}^{n+1}) \mid |\langle N_S, e_{n+1}\rangle| \geq \tau\},$$

where N_S is the (unoriented) unit normal to S. As $|\Pi|(E_\tau) = 0$ and E_τ is compact, given any $\epsilon > 0$ we can take $j \geq J(\tau, \epsilon)$ large enough to guarantee that $|G_j|(E_\tau) < \epsilon$ by virtue of the weak convergence $|G_j| \to m|\Pi|$. In what follows the argument is notationally simpler if we define the sets

$$E_{\tau,j} = \{X \in \operatorname{reg} G_j \cap \overline{I}^{n+1} \mid |\langle \nu_j, e_{n+1}\rangle| \geq \tau\},$$

which provided $j \geq J(\tau, \epsilon)$ have $\mathcal{H}^n(E_{\tau,j}) < \epsilon$. Let additionally $\delta > 0$ be given, and update $j \geq J(\tau, \epsilon, \delta)$ to guarantee that also $G_j \cap \overline{I}^{n+1} \subset (\Pi)_\delta$. Split the graph G_j by conditioning on the set $E_{\tau,j}$,

$$\begin{aligned}\mathcal{H}^n(G_j \cap \overline{I}^{n+1}) &= \mathcal{H}^n(\operatorname{reg} G_j \cap \overline{I}^{n+1}) \\ &= \mathcal{H}^n(E_{\tau,j}) + \mathcal{H}^n(\operatorname{reg} G_j \cap \overline{I}^{n+1} \setminus E_{\tau,j}).\end{aligned}$$

Then $\mathcal{H}^n(E_{\tau,j}) < \epsilon$ and

$$\begin{aligned}&\mathcal{H}^n(\operatorname{reg} G_j \cap \overline{I}^{n+1} \setminus E_{\tau,j}) \\ &\qquad = \int_{I^n, |u_j| \leq 1, \langle \nu_j, e_{n+1}\rangle < \tau} \frac{1}{v_j} + \int_{I^n, |u_j| \leq 1, \langle \nu_j, e_{n+1}\rangle < \tau} \frac{|Du_j|^2}{v_j},\end{aligned}$$

where we abbreviate $v_j = (1 + |Du_j|^2)^{1/2}$. The first of the two is bounded like

$$\int_{I^n, |u_j| \leq 1, \langle \nu_j, e_{n+1}\rangle < \tau} \frac{1}{v_j} \leq \mathcal{H}^n(I^n)\tau = 2^n \tau.$$

For the second integral we do not need $\langle \nu_j, e_{n+1}\rangle < \tau$. Instead we bound $\int_{I^n, |u_j| \leq 1} \frac{|Du_j|^2}{v_j}$, using a modification of the arguments used to prove the *a priori* area estimates, starting with the integral identity (3.2) with $r = 1$. As $G_j \cap \overline{I}^{n+1} \subset (\Pi)_\delta$ the two-valued indicator function $\mathbb{1}_{|u_j| \leq 1}$ is supported inside the closure of $I^n \cap (\Pi_0)_\delta$. Instead of the usual test function, we may thus take $\eta \in C^1_c(D_2)$ with $0 \leq \eta \leq 1$ on D_2 and $\eta = 1$ on $I^n \cap (\Pi_0)_\delta$ and deduce

$$\int_{D_2, |u_j| \leq 1} \frac{|Du_j|^2}{v_j} \leq \int_{D_2, |u_j| \leq 1} \eta \frac{|Du_j|^2}{v_j} \leq 2 \int_{D_2} |D\eta|. \tag{4.1}$$

By varying η among all test functions identically equal to 1 on $(\Pi_0)_\delta \cap I^n$ the integral on the right-hand side can be made arbitrarily close to the perimeter of $(\Pi_0)_\delta \cap I^n$. We calculate this to be

$$\begin{aligned}\mathcal{H}^{n-1}(\partial\{(\Pi_0)_\delta \cap I^n\}) &= 2\mathcal{H}^{n-1}(I^{n-1}) + 2(n-1)(2\delta)\mathcal{H}^{n-2}(I^{n-2}) \\ &= 2^n + (n-1)2^n\delta,\end{aligned}$$

because $(\Pi_0)_\delta \cap I^n$ is an n-dimensional cube with $2n$ faces, all but two of which are isometric to $(-\delta, \delta) \times I^{n-2}$. These 'thin' faces each contribute $\mathcal{H}^{n-1}((-\delta,\delta) \times I^{n-2}) = 2\delta\mathcal{H}^{n-2}(I^{n-2})$ to the perimeter. The two remaining 'large' faces are $\{\pm\delta\} \times I^{n-1}$, each adding $\mathcal{H}^{n-1}(I^{n-1})$. Combining this with the inequality (4.1) we obtain

$$\int_{D_2, |u_j| \leq 1} \frac{|Du_j|^2}{v_j} \leq 2^{n+1} + (n-1)2^{n+1}\delta = 2\mathcal{H}^n(I^n) + C(n)\delta.$$

Summarising the estimates we obtain that for $j \geq J(\tau, \epsilon, \delta)$

$$\begin{aligned} \mathcal{H}^n(G_j \cap I^{n+1}) &= \mathcal{H}^n(E_{\tau,j}) + \mathcal{H}^n(\operatorname{reg} G_j \cap I^{n+1} \setminus E_{\tau,j}) \\ &\leq \epsilon + 2\mathcal{H}^n(I_n) + C(n)(\delta + \tau). \end{aligned}$$

As I^{n+1} is open we can let $\tau, \epsilon, \delta \to 0$ and $J(\tau,\epsilon,\delta) \to \infty$ to get

$$\begin{aligned} m\mathcal{H}^n(I^n) &= m|\Pi|(Gr_n(I^{n+1})) \\ &\leq \liminf_{j\to\infty} \mathcal{H}^n(G_j \cap I^{n+1}) = 2\mathcal{H}^n(I^n). \quad \square \end{aligned}$$

4.1.2. Multiplicity in limit varifolds. Combining the previous lemma with a diagonal extraction argument, we obtain the following result.

COROLLARY 4.1.2. *Let $\alpha \in (0,1)$, and let $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs in $C^{1,\alpha}(D_2; \mathcal{A}_2)$. Suppose that there are half-planes π_i and $m_i \in \mathbf{Z}_{>0}$ so that $|G_j| \to \sum_i m_i|\pi_i|$. Then $m_i \leq 2$.*

Similarly, though in a more general context, we can combine the estimate from Lemma 4.1.1 with the work of Krummel–Wickramasekera [KW20], quoted in Theorem 3.2.6. Recall here that for a stationary varifold V we write $\mathcal{B}_V$ for the top stratum of the singular set, that is $\mathcal{B}_V$ denotes the set of points $X \in \operatorname{sing} V$ where at least one tangent cone is of the form $2|\Pi_X|$ for an n-dimensional plane $\Pi_X \in Gr(n, n+1)$.

COROLLARY 4.1.3. *Let $G_j = \operatorname{graph} u_j$ be a sequence of two-valued minimal graphs, where $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ for all j for some $\alpha \in (0,1)$. Suppose that $|G_j| \to V \in \mathbf{IV}_n(D_2 \times \mathbf{R})$ weakly in the topology of varifolds. Then for all $Z \in \operatorname{reg} V$, $\Theta(\|V\|, Z) \leq 2$. If $Z \in \mathcal{B}_V$ then $\Theta(\|V\|, Z) = 2$, and the branch set is countably $n-2$-rectifiable.*

4.1.3. Local description near vertical planes. We return to the situation where $|G_j| \to 2|\Pi|$ to some vertical plane $\Pi = \Pi_0 \times \mathbf{R}e_{n+1} \in Gr(n, n+1)$. The limit in the current topology is supported in the same plane, $[\![G_j]\!] \to l[\![\Pi]\!]$ for some non-negative $l \in \mathbf{Z}$ with $l \leq 2$. By Allard's regularity theorem the multiplicity is either $l = 0$ or 2, see Proposition 1.2.11. The following result considers the case where the mass of the currents vanishes in the limit.

LEMMA 4.1.4. *Let $\alpha \in (0,1)$, $u_j \in C^{1,\alpha}(D_2;\mathcal{A}_2)$ be a sequence of two-valued minimal graphs, and let $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$ be a vertical plane. Suppose $|G_j| \to 2|\Pi|$ and $[\![G_j]\!] \to 0$ as $j \to \infty$. Then for large enough j,*

$$\mathcal{B}(G_j) \cap B_1 = \emptyset.$$

PROOF. Let N be either unit normal to Π, and define $f_j = \langle \nu_j, N\rangle$ on $\operatorname{reg} G_j$. Given any $\delta > 0$, f_j restricted to $\operatorname{reg} G_j \cap B_{3/2}$ takes values in $[-1,-1+\delta) \cup (1-\delta, 1]$ by [Wic20], at least for large enough $j \geq J(\delta)$. Write $\mathcal{R}_j$ for the connected components of $\operatorname{reg} G_j \cap B_{3/2}$, which we further divide into $\mathcal{R}_j^{\pm}$ according to the sign of f_j. Accordingly we may decompose $[\![G_j]\!] = T_j^+ + T_j^-$ into the sum of the two currents obtained by integrating over $\mathcal{R}_j^{\pm}$ respectively, in a way that $[\![G_j]\!] = T_j^+ + T_j^-$ and $|G_j| = |T_j^+| + |T_j^-|$. The two currents only meet along classical, immersed singularities of G_j, where they moreover intersect transversely. Therefore they are both separately stationary with $\partial T_j^{\pm} = 0$ in $B_{3/2}$. By assumption $T_j^+ + T_j^- \to 0$ and $|T_j^+| + |T_j^-| \to 2|\Pi| \llcorner B_{3/2}$ as $j \to \infty$ in the current and varifold topologies respectively. Moreover by Federer–Fleming compactness $T_j^{\pm} \to T^{\pm}$ separately as $j \to \infty$. The limit currents satisfy $T^+ + T^- = 0$, and thus they are equal to the plane Π with multiplicity, but with opposite orientations. By Allard regularity both $T_j^+ \llcorner B_1$ and $T_j^- \llcorner B_1$ can be written as smooth graphs defined on Π, and thus do not support any branch points. $\square$

Return to the general case, where $|G_j| \to m|\Pi|$ and $[\![G_j]\!] \to l[\![\Pi]\!]$ for some vertical plane $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$. If $l \neq 0$ then we let N be the unit normal to Π corresponding to the orientation induced on the plane by $[\![G_i]\!]$, and if $l = 0$ then we pick our orientation arbitrarily. Thus we can divide $D_1 \setminus \Pi_0 \subset \mathbf{R}^n$ into two connected components $D_1^{\pm} = \{x \in D_1 \mid \pm\langle x, N\rangle > 0\}$.

For each j define a function $F_j : D_1 \to \{0,1,2\}$ by

$$F_j(x) = \sum_{\substack{X \in P_0^{-1}(\{x\}) \\ X^{n+1} < -1}} \Theta(\|G_j\|, X). \tag{4.2}$$

This returns the number of points in G_j which lie below $x \in D_1$, counted with multiplicity. (We could equally well have worked with a function counting the points lying above $x \in D_1$, although formulas such as (4.3) would have the opposite sign.) These functions are eventually locally constant away from the plane Π_0, in the following sense.

CLAIM 4. Let $\tau > 0$ be arbitrary. Then there is $j \geq J(\tau)$ so that F_j is constant on the two components $D_1^{\pm} \setminus (\Pi_0)_\tau$.

PROOF. The proof is identical for both components, so we just work with D_1^+. By the convergence of the graphs G_j in the Hausdorff distance, we may take $j \geq J(\tau)$ large enough that $G_j \cap D_1 \times (-1,1) \subset (\Pi)_\tau$. Hence eventually $G_j \cap D_1^+ \times \mathbf{R} \setminus (\Pi)_\tau \subset \{|X^{n+1}| > 1\}$.

There are three possibilities:

(1) either $G_j \cap D_1^+ \times \mathbf{R} \setminus (\Pi)_\tau \cap \{X^{n+1} < -1\} = \emptyset$,
(2) or $G_j \cap D_1^+ \times \mathbf{R} \setminus (\Pi)_\tau \subset \{X^{n+1} < -1\}$,
(3) or $G_j \cap D_1^+ \times \mathbf{R} \setminus (\Pi)_\tau$ contains points with positive and negative values for X^{n+1}.

Going through these cases in the same order we find that at all points $x \in D_1^+ \setminus (\Pi)_\tau$ the function $F_j(x)$ takes the values $0, 2$ or 1. □

LEMMA 4.1.5. *Let $\alpha \in (0,1)$, $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ be a sequence of two-valued minimal graphs, and $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$ be a vertical plane. Suppose that $|G_j| \to m|\Pi|$ and $[\![G_j]\!] \to l[\![\Pi]\!]$ as $j \to \infty$. Then the F_j are eventually constant away from Π_0, taking the values $F^\pm$ on $D_1^\pm$ respectively, and*

$$F^+ - F^- = l. \tag{4.3}$$

PROOF. There are three possible cases:

(1) either $m = 2$ and $l = 2$,
(2) or $m = 2$ and $l = 0$,
(3) or $m = 1$ and $l = 1$.

The proof is basically the same in all three cases, so we only consider the first. Consider the line $L \subset \mathbf{R}^n \times \{0\}$ directed by N and passing through the origin. Identify L with $\mathbf{R}$ via a unit-speed parametrisation. Then there exist two functions $u_{j,L}^1, u_{j,L}^2 \in C^1(\mathbf{R})$ so that $u_j(tN) = \{u_{j,L}^1(tN), u_{j,L}^2(tN)\}$ for all $t \in \mathbf{R}$. Moreover, as $l = 2$ we get that $u_{j,L}^1(t) \wedge u_{j,L}^2(t) < -1$ on $(1/2, 1)$ and $u_{j,L}^2(t) \vee u_{j,L}^2(t) > 1$ for $t \in (-1, -1/2)$, provided j is large enough.

Now let $0 < \tau < 1$ be an arbitrary small constant. By Claim 4, the function F_j is constant on the two components of $D_1 \setminus (\Pi)_\tau$, at least provided $j \geq J(\tau)$ is chosen large enough. Combining this with our calculations above, we find that for points $x \in D_1 \setminus (\Pi)_\tau$, $F_j(x) = 2$ if $x \in D_1^+$ and $F_j(x) = 0$ if $x \in D_1^-$. These values do not change with large values of j, and we may set $F^+ = 2, F^- = 0$, which confirms that indeed $F^+ - F^- = 2 = l$. As explained above, the other cases can be argued similarly. □

4.2. Classical limit cones: initial analysis

Let $\alpha \in (0,1)$, and $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs, with $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ for all j. We assume that they converge to a *classical cone* in the varifold topology, say $|G_j| \to \mathbf{P}$. By this we mean that there are n-dimensional half-planes $\pi_1, \ldots, \pi_N$ meeting along a commmon $n-1$–dimensional axis $L \in Gr(n-1, n+1)$ and integers $m_1, \ldots, m_N \in \mathbf{Z}_{>0}$

so that $\mathbf{P} = \sum_i m_i|\pi_i|$. By the graphs are endowed with the orientation corresponding to their upward-pointing unit normal we obtain a sequence of currents which we may also assume convergent, say $[\![G_j]\!] \to T \in \mathbf{I}_n(D_2 \times \mathbf{R})$, extracting a subsequence if necessary. This limit too has a similar form to the above, namely $T = \sum_i l_i [\![\pi_i]\!]$ where $0 \leq l_i \leq m_i$. Here the half-planes are given the orientations induced by T, where we pick an arbitrary orientation for those π_i which have $l_i = 0$. Our main theorem in this section is the following.

THEOREM 4.2.1. *Let $\alpha \in (0,1)$, and let $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs with $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$. Suppose that $|G_j| \to \mathbf{P}$ and $[\![G_j]\!] \to T$ as $j \to \infty$, where $\mathbf{P}$ and T are classical cones. Then there exist planes $\Pi_1, \dots, \Pi_D \in Gr(n, n+1)$ and integers $0 \leq l_i \leq m_i \leq 2$ so that*

$$\mathbf{P} = \sum_{i=1}^{D} m_i|\Pi_i| \quad \textit{and} \quad T = \sum_{i=1}^{D} l_i [\![\Pi_i]\!]. \tag{4.1}$$

Our approach is similar to the arguments employed in the last section of [SS81], the main difference being the possible presence of classical and branch point singularities in the present setting. Thus one replaces their stable sheeting theorem with a new result of Wickramasekera [Wic20], which holds close to multiplicity two planes without the *a priori* exclusion of either of these singularities. We nonetheless repeat the details of their arguments here, trying to preserve the structure of the proof in [SS81] in order to highlight the similarities.

For the remainder of the proof the structure of the $|G_j|$ as two-valued graphs is of secondary importance. For this reason, we may simplify notation by rotating through some ambient isometry A of $\mathbf{R}^{n+1}$. We may do this in such a way as to map the axis L of $\mathbf{P}$ onto $\mathbf{R}^{n-1} \times \{0\}^2$, meaning that

$$A_{\#}|G_j| = \mathbf{R}^{n-1} \times \sum_{i=1}^{N} m_i|R_i|, \tag{4.2}$$

where each R_i is a ray of the form $R_i = \{\lambda p_i \mid \lambda > 0\}$ with $p_i \in S^1 \subset \mathbf{R}^2$. Write $V_j = A_{\#}|G_j|$, and $N_j = A_{\#}\nu_j$ for the image of the upward-pointing unit normal to the graphs, defined at every point of $\operatorname{reg} V_j$.

Let $\sigma > 0$ be arbitrary. For sufficiently large $j \geq J(\sigma)$,

$$A_{\#}G_j \cap \{x\} \times \{y \in \mathbf{R}^2 \mid \sigma/2 < |y| < \sigma\} = \cup_{k=1}^{\bar{M}} \gamma_{j,x}^k,$$

for each $x \in \mathbf{R}^{n-1}$ with $|x| \leq 1$, where $\bar{M} = \sum_{k=1}^{N} m_k$. The $\gamma_{j,x}^k$ are $C^{1,\alpha}$ embedded Jordan arcs with endpoints in $\{x\} \times \{y \in \mathbf{R}^2 \mid |y| = \sigma/2 \text{ or } \sigma\}$.

Moreover as $j \to \infty$ we have the uniform limits

$$\mathrm{dist}_{\mathcal{H}}(\cup_k \gamma^k_{j,x}, \{x\} \times \cup_{i=1}^N \{\lambda p_i \mid \sigma/2 < \lambda < \sigma\}) \to 0$$

and

$$\min_{i \in \{1,\ldots,N\}} \sup_{X \in \gamma^k_{j,x}} |\langle N_j(X), (0, p_i)\rangle| \to 0, \tag{4.3}$$

for all $x \in \mathbf{R}^{n-1}$ with $|x| \leq 1$, the latter being justified either by Allard regularity or [Wic20] depending on the multiplicity of the ray.

Let $\tau \in (0,1)$ be small enough that the tubular neighbourhoods $\big(\{\lambda p_i \mid \sigma/2 < \lambda < \sigma\}\big)_\tau$ are two-by-two disjoint. By then taking $j \geq J(\tau, \sigma)$ large enough that $\mathrm{dist}_{\mathcal{H}}(\gamma^k_{j,x}, \{x\} \times \cup_{i=1}^N \{\lambda p_i \mid \sigma/2 < \lambda < \sigma\}) < \tau$ we ensure that there lie precisely m_i Jordan arcs near every line segment $\{\lambda p_i \mid \sigma/2 < \lambda < \sigma\}$.

By [KW20] the branch set of the two-valued graphs is countably $n-2$-rectifiable for all j. Of course the same remains true for the V_j after rotating through by A. Let P_{n-1} be the orthogonal projection onto $\mathbf{R}^{n-1} \times \{0\}^2$. The projection of the branch set onto $\mathbf{R}^{n-1} \times \{0\}^2$ is a compact $\mathcal{H}^{n-1}$-null set, that is

$$\mathcal{H}^{n-1}(P_{n-1}(\mathcal{B}(V_j) \cap \overline{B}_1^{n-1}(0) \times \{y \in \mathbf{R}^2 \mid |y| \leq 1\})) = 0. \tag{4.4}$$

Combining this observation with Sard's theorem, we find that there is an open subset $\mathcal{U}_j \subset B_1^{n-1}(0) \times \{0\}^2$ of full $\mathcal{H}^{n-1}$-measure, so that for all $x \in \mathcal{U}_j$

$$\mathrm{spt}\|V_j\| \cap \{x\} \times \{y \in \mathbf{R}^2 \mid |y| < \sigma\} = \cup_{k=1}^P \Upsilon^k_{j,x} \cup \cup_{l=1}^Q \Delta^l_{j,x},$$

where the $\Upsilon^k_{j,x}$ are smooth properly embedded Jordan arcs with endpoints in $\{x\} \times \{y \in \mathbf{R}^2 \mid |y| = \sigma\}$ and the $\Delta^l_{j,x}$ are smooth properly embedded Jordan curves. We call the points in $\mathcal{U}_j$ *unbranched.*

LEMMA 4.2.2. *If* $\mathbf{P}$ *or* T *are not of the form* (4.1) *then there exist* $\varphi \in (0,1)$ *independent of* σ*, and* $J(\sigma) \in \mathbf{N}$ *so that if* $j \geq J(\sigma)$ *then for all* $x \in \mathcal{U}_j$*,*

$$\varphi \leq \max_{k \in \{1,\ldots,P\}} \sup_{X,Y \in \Upsilon^k_{j,x}} |N_j(X) - N_j(Y)|. \tag{4.5}$$

PROOF. A list of properties must hold for $\mathbf{P}$ and T to conform to (4.1). We go through them one by one, assuming every time that they fail and obtaining a lower bound akin to (4.5). First, we consider the case where there do not exist planes $\Pi_1, \ldots, \Pi_D$ so that

$$\mathrm{spt}\|\mathbf{P}\| = \cup_{i=1}^D \Pi_i. \tag{4.6}$$

Rotate through by an isometry A to consider $V_j = A_\#|G_j|$ instead as we did above in (4.2). Let $p_1, \ldots, p_M \in S^1 \subset \mathbf{R}^2$ be as above, and consider their indices modulo M for notational convenience. After doing this, we see that (4.6) fails precisely if there is no way of dividing these points into antipodal pairs. meaning that there exists p_i so that $-p_i \notin \{p_1, \dots, p_M\}$. Relabel the points in such a way that this is the case for p_0. Let $\Upsilon^{k_0}_{j,x}$ be a Jordan arc with at least one endpoint near the line segment $\{\lambda p_0 \mid \sigma/2 < \lambda < \sigma\}$. We may take $j \geq J(\tau, \sigma)$ large enough that this must exist, and moreover has

$$|\langle N_j(X), (0, p_0)\rangle| \leq \tau \tag{4.7}$$

for all $X \in \Upsilon^{k_0}_{j,x} \cap (\{\lambda p_0 \mid \sigma/2 < \lambda < \sigma\})_\tau$. The other endpoint of $\Upsilon^{k_0}_{j,x}$ must lie either near the same line segment, or near a line segment corresponding to some other $p_i \neq p_0$. By assumption also $p_i \neq -p_0$, and the normals to the rays R_0 and R_i must therefore be different. Thus if $\Upsilon^{k,0}_{j,x}$ had its endpoints lying near the rays corresponding to p_0, p_i then picking $\tau > 0$ small and $j \geq J(\tau, \sigma)$ large enough, we may impose that (4.7) also hold for points in $X \in \Upsilon^{k_0}_{j,x} \cap (\{\lambda p_i \mid \sigma/2 < \lambda < \sigma\})_\tau$. Therefore

$$\varphi_1 \leq \sup_{X,Y \in \Upsilon^{k_0}_{j,x}} |N_j(X) - N_j(Y)|, \tag{4.8}$$

where $\varphi_1 > 0$ is a positive constant depending only on the difference between the normals to R_0 and R_i. If instead both endpoints of $\Upsilon^{k_0}_{j,x}$ lie near R_0, then the value of this constant can be taken to be $\varphi_1 = 1$ for example.

The inequality (4.8) being established when (4.6) fails, we may for the remainder of the proof assume instead that in fact it holds. Next we show that in fact $\mathbf{P}$ is equal to a sum of planes. We proceed similarly in the next situation where we try to deduce (4.5). Note first that $N = 2D$, and we may decompose every plane in spt$\|\mathbf{P}\|$ into two closed half-planes lying on either side of L, so that for all $i = 1, \dots, D$, $\Pi_i = \pi_i \cup \pi_{i+D}$, possibly after relabelling the half-planes. Then the only way $\mathbf{P}$ could fail to be the sum of planes is that $m_i = m_{i+D}$ for at least one i. Choose a small $\tau > 0$ as above to ensure that the tubular neighbourhoods are disjoint. For large enough $j \geq J(\tau, \sigma)$, precisely m_i of the Jordan arcs lie $\{\gamma^1_{j,x}, \dots, \gamma^N_{j,x}\}$ lie near R_i. Similarly m_{i+D} of them lie near R_{i+D}. If the multiplicities disagree, $m_i \neq m_{i+D}$, then a pigeonhole argument demonstrates the existence of a Jordan arc $\Upsilon^{k_0}_{j,x} \in \{\Upsilon^1_{j,x}, \dots, \Upsilon^P_{j,x}\}$ satisfying one of the following. Either $\Upsilon^{k_0}_{j,x}$ has both endpoints lying near the ray R_i, or both endpoints lying near R_{i+D}, or else it has one endpoint lying near one of R_i, R_{i+D}, and the other lying near a third ray R'. Thus we obtain again the existence of a constant

$\varphi_2 > 0$ so that for all $x \in \mathcal{U}_j$,

$$\varphi_2 \leq \sup_{X,Y \in \Upsilon^{k_0}_{j,x}} |N_j(X) - N_j(Y)|$$

for some $k_0 \in \{1, \ldots, P\}$, at least if $j \geq J(\tau, \sigma)$ is large enough. In either of the first two cases we may take $\varphi_2 = 1$, and in the last case φ_2 only depends on the difference between the normals to R_i, R_{i+D} and the normal to the third ray R'.

In the remainder we may assume that $\mathbf{P} = \sum_i m_i|\Pi_i|$. We next prove the analogous results for the current T, that is show that we may find a positive constant φ_3 for which (4.5) holds unless $T = \sum_i l_i[\![\Pi_i]\!]$ for some integers $0 \leq l_i \leq m_i \leq 2$. There are two complicating factors in this case: uneven mass cancellation and misaligned orientations of the half-planes.

We exclude the former first. Recall that $0 \leq l_i \leq m_i$, and if $m_i = 2$ then l_i could be zero or two. Therefore, although $m_i = m_{i+D}$ one could have $l_i = 0$ but $l_{i+D} = 2$. This is what we seek to show is impossible. Label the half-planes as above, so that $\Pi_i = \pi_i \cup \pi_{i+D}$ for all $i = 1, \ldots, D$, and recall that for notational convenience we consider these indices modulo N. For all $i = 1, \ldots, D$ we write $n_i \in S^1 \subset \mathbf{R}^2$ for the common unit normal of the two rays R_i, R_{i+D}. (Whether we pick n_i or $-n_i$ has no bearing on what follows—remember that for now we cannot assume that the two half-planes π_i, π_{i+D} inherit the same orientation from the $[\![G_j]\!]$ as $j \to \infty$, nor indeed that they inherit any orientation at all.)

Without loss of generality, suppose that $l_0 = 0$ but $l_D = 2$. Let $\delta > 0$ be given. The uniform limit (4.3) ensures that if $j \geq J(\delta, \tau, \sigma)$ is taken large enough then $|\langle N_j(X), n_0\rangle| > 1 - \delta$ for all $X \in \operatorname{spt}\|V_j\| \cap (\{\lambda p \mid p = p_0, p_D, \sigma/2 < \lambda < \sigma\})_\tau$. Arguing as in the proof of Lemma 4.1.4 we find that $\langle N_j, n_0\rangle$ must take both positive and negative values near the ray R_0. For $\sigma, \tau > 0$ small and $j \geq J(\tau, \sigma)$ there lie precisely two of the Jordan arcs from $\{\gamma^1_{x,j}, \ldots, \gamma^{\bar{M}}_{x,j}\}$ inside $(\{\lambda p_0 \mid \sigma/2 < \lambda < \sigma\})_\tau$ and $(\{\lambda p_D \mid \sigma/2 < \lambda < \sigma\})_\tau$ respectively. Arguing as above, we may assume that there are two Jordan arcs $\Upsilon_\pm \in \{\Upsilon^1_{j,x}, \ldots, \Upsilon^P_{j,x}\}$ so that $\operatorname{spt}\|V_j\| \cap (\{\lambda p \mid p = p_0, p_D, \sigma/2 < \lambda < \sigma\})_\tau \subset \Upsilon_+ \cup \Upsilon_-$. Moreover we may assume Υ_+, Υ_- both have one endpoint near R_0 and another R_D, otherwise again we may find a positive constant $\varphi_3 > 0$ so that $\varphi_3 \leq \min_{\Upsilon \in \{\Upsilon_\pm\}} \sup_{X,Y \in \Upsilon} |N_j(X) - N_j(Y)|$. If there were no mass cancellation at π_D (that is, if $l_D = 2$) then $\langle N_j(X), n_0\rangle$ would take the same sign for all $X \in \operatorname{spt}\|V_j\| \cap (\{\lambda p_D \mid \sigma/2 < \lambda < \sigma)_\tau$. Without loss of generality, we may thus assume that $\langle N_j(X), n_0\rangle > 1 - \delta$ for these points. Label $\Upsilon_\pm$ in such a way that they respectively contain the points near R_0 with $\pm\langle N_j, n_0\rangle > 1 - \delta$. As $\langle N_j, n_0\rangle$ changes sign along Υ_-, the normal N_j

varies enough along Υ_- that $1 \le \sup_{X,Y \in \Upsilon_-} |N_j(X) - N_j(Y)|$. Thus we have established that (4.8) holds with some positive constant $\varphi_3 > 0$ unless $l_i = 0$ if and only if $l_{i+D} = 0$ for all $i = 1, \dots, D$.

For the last step we may assume that the limit current is of the form $T = \sum_{i=1}^{D} l_i([\![\pi_i]\!] + [\![\pi_{i+D}]\!])$, where we do not specify which orientation is chosen for π_i, π_{i+D}. Assume that there is at least one pair of half-planes, say π_0, π_D, whose orientations are misaligned. Let π_0 be oriented by n_0 and π_D by $-n_0$. There cannot be any mass cancellation at these half-planes, and for large $j \ge J(\delta, \tau, \sigma)$ we have $\langle N_j(X), n_0 \rangle > 1 - \delta$ for all $X \in \operatorname{spt}\|V_i\| \cap (\{\lambda p_0 \mid \sigma/2 < \lambda < \sigma\})_\tau$, and $\langle N_j, n_0 \rangle < -1 + \delta$ near R_D. We may then argue precisely as above, when we derived the lower bound (4.8) with the constant φ_3. Write φ_4 for the constant obtained for the lower bound here. This exhausts all the possible ways in which (4.1) can fail, and we conclude by letting $\varphi = \min\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} > 0$. □

Next we use this result to give a proof of Theorem 4.2.1.

Proof. We argue by contradiction, supposing that either $\mathbf{P}$ or T (or both) is not of the desired form. Using Lemma 4.2.2, we pick any unbranched point $x \in \mathcal{U}_j$ and find $\Upsilon \in \{\Upsilon^1_{j,x}, \dots, \Upsilon^P_{j,x}\}$ with $\varphi \le \sup_{X,Y \in \Upsilon} |N_j(X) - N_j(Y)|$. Identify Υ with a smooth parametrisation defined on $[0,1]$. We may then take $s_j < t_j \in (0,1)$ so that $\frac{\varphi}{2} \le |N_j(\Upsilon(t_j)) - N_j(\Upsilon(s_j))|$. Then

$$\frac{\varphi}{2} \le \int_{s_j}^{t_j} |(N_j \circ \Upsilon)'(t)| \, \mathrm{d}t = \int_{s_j}^{t_j} |\langle \nabla_{V_j} N_j(\Upsilon(t)), \Upsilon'(t) \rangle| \, \mathrm{d}t,$$

which in turn can be bounded in terms of the second fundamental form to yield

$$\frac{\varphi}{2} \le \int_{s_j}^{t_j} |A_{V_j}|(\Upsilon(t))|\Upsilon'(t)| \, \mathrm{d}t \le \int_{\Upsilon} |A_{V_j}| \, \mathrm{d}\mathcal{H}^1 \le \int_{\operatorname{reg} V_j \cap \{x\} \times B^2_\sigma} |A_{V_j}| \, \mathrm{d}\mathcal{H}^1. \tag{4.9}$$

As the point $x \in \mathcal{U}_j$ was chosen arbitrarily, we can use the coarea formula to integrate the lower bound (4.9) over $\mathcal{U}_j \subset B^{n-1}_1 \times \{0\}$, combining this with the Cauchy–Schwarz inequality to get

$$\begin{aligned} \varphi \omega_{n-1} &\le 2 \int_{\operatorname{reg} V_j \cap \mathcal{U}_j \times B^2_\sigma} |A_{V_j}| \, \mathrm{d}\mathcal{H}^n \\ &\le 2\mathcal{H}^n(\operatorname{reg} V_j \cap B^{n-1}_1 \times B^2_\sigma)^{1/2} \Big(\int_{\operatorname{reg} V_j \cap B^{n-1}_1 \times B^2_\sigma} |A_{V_j}|^2 \, \mathrm{d}\mathcal{H}^n \Big)^{1/2}. \end{aligned} \tag{4.10}$$

The last integral can be bounded via the stability inequality, using a test function $\phi \in C^1_c(B^{n-1}_{\sqrt{7}/2} \times B^2_{1/2})$ with $0 \le \phi \le 1, |D\phi| \le 8$ and $\phi \equiv 1$ on

$B^{n-1}_{(\sqrt{7}-1)/2} \times B^2_{1/4}$. Provided $\sigma \in (0, 1/4)$ we get

$$\begin{aligned}\int_{\operatorname{reg} V_j \cap B^{n-1}_1 \times B^2_\sigma} |A_{V_j}|^2 \, \mathrm{d}\mathcal{H}^n &\leq \int_{\operatorname{reg} V_j \cap B^{n-1}_{(\sqrt{7}-1)/2} \times B^2_{1/4}} |A_{V_j}|^2 \, \mathrm{d}\mathcal{H}^n, \\ &\leq \int_{\operatorname{reg} V_j \cap B^{n-1}_{\sqrt{7}/2} \times B^2_{1/2}} |\nabla_{V_j} \phi|^2 \, \mathrm{d}\mathcal{H}^n \\ &\leq 64 \mathcal{H}^n (\operatorname{reg} V_j \cap B^{n-1}_{\sqrt{7}/2} \times B^2_{1/2}) \\ &\leq 128 \|\mathbf{P}\| (B^{n-1}_{\sqrt{7}/2} \times B^2_{1/2}).\end{aligned}$$

This in turn is equal to $32\omega_{n-1}(\sqrt{7}/2)^{n-1} \sum_i m_i$, although from this we only retain that there is a constant $B > 0$ independent of j and σ so that $\int_{\operatorname{reg} V_j \cap B^{n-1}_1 \times B^2_\sigma} |A_{V_j}|^2 \leq B$ for all large enough j. A similar mass bound gives $\mathcal{H}^n(\operatorname{reg} V_j \cap B^{n-1}_1 \cap B^2_\sigma) \leq \|\mathbf{P}\|(B^{n-1}_1 \times B^2_\sigma) = 2\omega_{n-1}\sigma \sum_i m_i$. Substituting both back into (4.10) we get that

$$\int_{\operatorname{reg} V_j \cap B^{n-1}_1 \times B^2_\sigma} |A_{V_j}| \, \mathrm{d}\mathcal{H}^n \leq B\sigma^{1/2},$$

for some constant B independent of j and σ. This is absurd provided $\sigma > 0$ is small and j is large enough, which concludes the proof. □

4.3. Classical limit cones: non-vertical cones

Let $\alpha \in (0, 1)$ and $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs with $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$. Let $D \in \mathbf{Z}_{>0}$ and for $i = 1, \ldots, D$ let $\Pi_i \in Gr(n, n+1)$ be n-dimensional planes which meet along a common axis $L \in Gr(n-1, n+1)$. Recall a plane is called vertical if it is of the form $\Pi = \Pi_0 \times \mathbf{R}e_{n+1}$. Suppose that not all planes $\Pi_1, \ldots, \Pi_N$ are vertical. Equivalently, at most one of the planes is vertical. We may relabel the planes to arrange for Π_1 to be non-vertical. Assume that $|G_j| \to \sum_{i=1}^D m_i |\Pi_i| = \mathbf{P}$ and $[\![G_j]\!] \to \sum_{i=1}^D l_i [\![\Pi_i]\!] = T$ as $j \to \infty$. Note that we allow for the possibility that $D = 1$ and the cones are supported in a single plane, although technically these would not be called classical.

4.3.1. Slicing at an acute angle. Broadly speaking we again use an approach based on [SS81]. However from a technical standpoint the arguments from the previous section are maladapted to the current situation, as we want to exploit the two-valued graphicality of the G_j. Instead of taking slices orthogonal to the axis L of the cone, we proceed as follows. Let $v \in \mathbf{R}^{n+1}$ be a unit vector with $\langle v, e_{n+1} \rangle = 0$, so that v, e_{n+1}, L span $\mathbf{R}^{n+1}$. Let $V = \operatorname{span}\{v, e_{n+1}\}$, and write $Z = tv + ze_{n+1} \in V$. Then $\mathbf{R}^{n+1} = L + V$, and every point $X \in \mathbf{R}^{n+1}$ can uniquely by written $X = Y + Z = y + Y^{n+1}e_{n+1} + tv + ze_{n+1}$. We emphasise that in general L and V do not meet at a right angle, and this decomposition is not orthogonal. (The exception being the case where $L = \mathbf{R}^n \times \{0\}$.) The slices we take are adapted to this decomposition, using sets of the form $\{Y + Z \mid Z \in V, |Z| < \sigma\}$, where $0 < \sigma < 1$ and $Y = (y, Y^{n+1}) \in L$ is a fixed point with $|y| < 1$. We abbreviate this $\{Y + Z \mid |Z| < \sigma\}$, and in the same vein write for example $\{|y| < 1, |Z| < \sigma\} = \{Y + Z \mid Y \in L, Z \in V, |y| < 1, |Z| < \sigma\}$.

The corresponding projection map is written $Q : X = Y + Z \in \mathbf{R}^{n+1} \mapsto Y$. Analogously to (4.4) we have $\mathcal{H}^{n-1}\big(Q(\mathcal{B}(G_j) \cap \{|y| \leq 1, |Z| \leq 1\})\big) = 0$. Let $\sigma > 0$ be given. Together with Sard's theorem we find that for all j there is an open subset $\mathcal{V}_{j,\sigma} \subset L \cap \{|y| \leq 1\} \setminus Q(\mathcal{B}(G_j) \cap \{|y| \leq 1, |Z| \leq 1\})$ of full measure, so that for all $Y \in \mathcal{V}_{j,\sigma}$, $G_j \cap \{Z + Y \mid |Z| < \sigma\} = \cup_{k=1}^{P} \Upsilon_{j,Y}^k$ where the $\Upsilon_{j,Y}^k$ are smooth properly embedded Jordan arcs with endpoints in the set $\{Y + Z \mid |Z| = \sigma\}$. (This cannot contain any Jordan curves because of the graphicality of G_j.) In the same vein, given $\kappa > 0$ we write $\mathcal{V}_{j,\sigma}(\kappa) \subset \mathcal{V}_{j,\sigma}$ for the measurable subset of $\mathcal{V}_{j,\sigma}$ formed by those points $Y \in \mathcal{V}_{j,\sigma}$ with $\int_{\operatorname{reg} G_j \cap \{Z+Y \mid |Z|<\sigma\}} |A_{G_j}| \, \mathrm{d}\mathcal{H}^1 < \kappa$.

Lemma 4.3.1. *For all $\kappa > 0$, $\mathcal{H}^{n-1}(\mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa)) \to 0$ as $\sigma \to 0$ and $j \geq J(\sigma) \to \infty$.*

Proof. By definition of $\mathcal{V}_{j,\sigma}(\kappa)$, we can integrate over points $Z \in \mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa)$ to obtain the inequality

$$\mathcal{H}^{n-1}(\mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa))\kappa \leq \int_{\mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa)} \Big\{ \int_{\operatorname{reg} G_j \cap \{Z+Y \mid |Z|<\sigma\}} |A_{G_j}| \, \mathrm{d}\mathcal{H}^1 \Big\} \, \mathrm{d}\mathcal{H}^{n-1}(Y). \tag{4.1}$$

On the right-hand side we may increase the domain of integration to now be over the set $\{|Y| < 1, |Z| < \sigma\}$. By the Cauchy–Schwarz inequality, this larger integral is bounded like

$$\int_{\operatorname{reg} G_j \cap \{|Y|<1, |Z|<\sigma\}} |A_{G_j}| \, \mathrm{d}\mathcal{H}^n \leq \mathcal{H}^n(\operatorname{reg} G_j \cap \{|Y| < 1, |Z| < \sigma\}) \Big(\int_{\operatorname{reg} G_j \cap \{|Y|<1, |Z|<\sigma\}} |A_{G_j}|^2 \, \mathrm{d}\mathcal{H}^n \Big)^{1/2} \tag{4.2}$$

From here on, although the notation is different, the proof is essentially identical to the argument used to conclude Theorem 4.2.1. Using the stability inequality along with area estimates in terms of the limit cone, we find a constant $B > 0$ independent of j, σ so that provided $\sigma \in (0, 1/4)$ and $j \geq J(\sigma)$,

$$\int_{\operatorname{reg} G_j \cap \{|Y|<1, |Z|<\sigma\}} |A_{G_j}|^2 \, \mathrm{d}\mathcal{H}^n \leq B.$$

Combining this with the fact that there is another constant $C > 0$, also independent of j, σ so that

$$\mathcal{H}^n(\operatorname{reg} G_j \cap \{|Y| < 1, |Z| < \sigma\}) \leq C\sigma$$

provided again $j \geq J(\sigma)$, we find combining the two with (4.2) that there is a constant $D > 0$ so that for all $j \geq J(\tau)$,

$$\int_{\operatorname{reg} G_j \cap \{|Y|<1, |Z|<\sigma\}} |A_{G_j}| \, \mathrm{d}\mathcal{H}^n \leq B\sigma^{1/2}.$$

Comparing this with the initial inequality (4.1) we find that

$$\mathcal{H}^{n-1}(\mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa))\kappa \leq B\sigma^{1/2},$$

at least for $j \geq J(\sigma)$ large enough. To force $\mathcal{H}^{n-1}(\mathcal{V}_{j,\sigma} \setminus \mathcal{V}_{j,\sigma}(\kappa)) \to 0$, it suffices to let $\sigma \to 0$ and $j \geq J(\sigma) \to \infty$. □

Using this lemma, we see in particular that given any $\kappa > 0$, we can choose j large enough to ensure that $\mathcal{V}_{j,\sigma}(\kappa) \neq \emptyset$. The resulting integral curvature estimates in the slice $\{Z + Y \mid |Z| < \sigma\}$ at one of these points $Y \in \mathcal{V}_{j,\sigma}(\kappa)$ give us good control over G_j in the same region. This in turn, we will later see, is the first step of our contradiction arguments.

LEMMA 4.3.2. *Let $G \to \mathbf{P} = \sum_i m_i |\Pi_i|$. There is a constant $\varphi > 0$ depending only in $\mathbf{P}$ so that given any $\tau > 0$ and large enough $j \geq J(\tau, \sigma)$ the following holds. If $Y \in \mathcal{V}_{j,\sigma}(\varphi)$ then for each $\Upsilon \in \{\Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y}\}$ there is a plane $\Pi \in \{\Pi_1, \ldots, \Pi_D\}$ so that*

$$\operatorname{dist}_{\mathcal{H}}(\Upsilon, \Pi \cap \{Z + Y \mid |Z| < \sigma\}) \leq \tau. \tag{4.3}$$

PROOF. Given some $\tau > 0$, no harm is done by adjusting its value to ensure that $\tau < \sigma/2$. For $j \geq J(\tau, \sigma)$ sufficiently large we find that for all $Y \in \mathcal{V}_{j,\sigma}$,

$$G_j \cap \{Y + Z \mid \tau < |Z| < \sigma\} = \cup_{k=1}^{\bar{M}} \gamma^k_{j,Y},$$

where $\bar{M} = \sum_{k=1}^{D} m_i$ and the $\gamma^k_{j,Y}$ are smooth embedded Jordan arcs with endpoints in $\{Y + Z \mid |Z| = \tau \text{ or } \sigma\}$. Let $\delta > 0$ be a small constant, depending only on the cone, so that the regions $(\Pi_i)_{\delta\tau} \cap \{Y + Z \mid \tau < |Z| < \sigma\}$ are two-by-two disjoint, and each has two connected components, which respectively contain m_i curves.

Perhaps after updating $j \geq J(\tau, \sigma)$ to a yet larger value, we may write

$$\operatorname{dist}_{\mathcal{H}}(\cup_k \gamma^k_{j,Y}, \{Y + Z \mid \tau < |Z| < \sigma\}) < \delta\tau.$$

Moreover, using either Allard regularity near those planes in the support with multiplicity one, or Wickramasekera's stable sheeting theorem [Wic20] near those with multiplicity two, we find that we may assume that say

$$\min_{i\in\{1,\dots,D\}} \sup_{X\in\gamma^k_{j,Y}} |\nu_j(X) - N_i| < \tau,$$

where N_i is a unit normal to Π_i.

If $\varphi > 0$ is picked small enough in terms of $\mathbf{P}$, then arguing as in the proof of Lemma 4.2.2 we find that for all $\Upsilon \in \{\Upsilon^1_{j,Y}, \dots, \Upsilon^P_{j,Y}\}$ there is a plane $\Pi \in \{\Pi_1, \dots, \Pi_N\}$ so that $\mathrm{dist}(\Upsilon \cap \{Z + Y \mid \tau < |Z| < \sigma\}, \Pi \cap \{Z + Y \mid \tau < |Z| < \sigma\}) < \delta\tau$. (For example $\tau, \varphi < 1/3 \min_{i\neq j\in\{1,\dots,D\}} |N_i \pm N_j|$ ensures that this is true.) We relinquish precise control over Υ close to the axis, but there automatically $\Upsilon \cap \{Z + Y \mid |Z| < \tau\} \subset (\Pi)_\tau$. Finally, conclude by noting that the reverse inclusion required for (4.3) holds essentially by construction, as the curves $\Upsilon \in \{\Upsilon^1_{j,Y}, \dots, \Upsilon^P_{j,Y}\}$ are all connected. □

4.3.2. Initial reduction. If at least one of the planes in $\mathrm{spt}\|\mathbf{P}\|$ is non-vertical, then the axis L along which the planes meet cannot be vertical either. However, any vertical $\Pi \in \{\Pi_1, \dots, \Pi_D\}$ must contain L, and thus be of the form $\Pi = \mathrm{span}\{L, e_{n+1}\}$. This uniquely determines the plane, and thus at most one plane in the support of $\mathbf{P}$ is vertical.

LEMMA 4.3.3. *Suppose $G_j \to \sum_i m_i|\Pi_i|$, and that Π_1 is not vertical.*

(i) *If Π_2 is vertical then $\mathbf{P} = |\Pi_1| + m_2|\Pi_2|$.*
(ii) *If Π_2 is not vertical then $\mathbf{P} = |\Pi_1| + |\Pi_2|$.*

PROOF. We start by proving that

$$\mathbf{P} = m_1|\Pi_1| + m_2|\Pi_2| \tag{4.4}$$

regardless of whether or not Π_2 is vertical, arguing by contradiction. Suppose that there are at least three distinct planes $\Pi_1, \Pi_2, \Pi_3 \subset \mathrm{spt}\|\mathbf{P}\|$. Let the constant $\varphi > 0$ be as in Lemma 4.3.2, let $\tau > 0$ be small and take $j \geq J(\tau, \sigma)$ large enough that $\mathcal{H}^{n-1}(\mathcal{V}_{j,\sigma}(\varphi)) > 0$ as per Lemma 4.3.1 and so that additionally the conclusions of Lemma 4.3.2 hold. We may thus take any point $Y \in \mathcal{V}_{j,\sigma}(\varphi)$ and decompose $G_j \cap \{Z + Y \mid Z \in V, |Z| < \sigma\}$ as above. Next let $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \{\Upsilon^1_{j,Y}, \dots, \Upsilon^P_{j,Y}\}$ be three of the curves, lying respectively near Π_1, Π_2, Π_3, that is

$$\Upsilon_i \subset (\Pi_i)_\tau \cap \{Z + Y \mid Z \in V, |Z| < \sigma\} \text{ for } i = 1, 2, 3.$$

Possibly after extending Υ_1 slightly beyond its endpoints, we have that $\{tv \mid |t| < \sigma\} \subset P_0(\Upsilon_1)$. Now if $\tau > 0$ is small enough in terms of $\sigma > 0$ then Υ_2 and Υ_3 intersect in at least one point, say $Y + Z_0 = Y + t_0v + z_0e_{n+1}$ with $|Z_0| < \sigma$. This is absurd because the density of $\|G_j\|$ at such a point is at least two, and hence writing $Y = (y, Y^{n+1})$ we get

$$\sum_{X \in P_0^{-1}(\{y+t_0 v\})} \Theta(\|G_j\|, X) \geq \sum_{X \in P_0^{-1}(\{y+t_0 v\}) \cap \Upsilon_1} \Theta(\|G_j\|, X) + \Theta(\|G_j\|, Z_0 + Y) \geq 3.$$

This is impossible for a two-valued graph.

This proves (4.4); when we additionally know that Π_2 is not vertical then proving ((ii)) is even easier. Indeed, without resorting to the decomposition of the graph G_j inside the slice $\{Z + Y \mid Z \in V, |Z| < \sigma\}$ we may reason as follows. Pick any point $x \in D_1 \setminus P_0(L)$. Then there exist two distinct points X_1, X_2 in Π_1, Π_2 respectively so that $P_0(X_1) = x = P_0(X_2)$. As both planes have multiplicity two at most, we can apply the regularity theory of Wickramasekera if either of the two planes has multiplicity two, and Allard regularity otherwise, to guarantee that when j is large enough, then

$$\sum_{X \in P_0^{-1}(\{x\})} \Theta(\|G_j\|, X) \geq m_1 + m_2 \geq 2 \tag{4.5}$$

where m_1, m_2 are the respective multiplicities of Π_1, Π_2. Anything but equality in (4.5) would be absurd for a two-valued graph, whence $m_1 = 1 = m_2$ as required.

It remains to prove (i), namely that the multiplicity of Π_1 is one even when the second plane is vertical. This turns out to be relatively easy also. Taking values for $\sigma, \tau > 0$ as in the earlier stages of the proof above, and accordingly large $j \geq J(\tau, \sigma)$, we may find two Jordan arcs $\Upsilon_1, \Upsilon_2 \in \{\Upsilon^1_{j,Y}, \dots, \Upsilon^P_{j,Y}\}$ from among curves into which G_j decomposes in the slice $\{Z + Y \mid Z \in V, |Z| < \sigma\}$ at a point $Y \in \mathcal{V}_{j,\sigma}(\varphi)$, so that

$$\Upsilon_1, \Upsilon_2 \subset (\Pi_1)_\tau \cap \{Z + Y \mid Z \in V, |Z| < \sigma\}.$$

Arguing as above, we may extend the two arcs slightly beyond their respective endpoints and get that $\{y + tv \mid |t| < \sigma\} \subset P_0(\Upsilon_1) \cap P_0(\Upsilon_2)$, where we recall $Y = (y, Y^{n+1})$. If there were another curve Υ_3 say in that slice, then at any point $Z_0 + Y = tv_0 + z_0 e_{n+1} + Y \in \Upsilon_3 \setminus (\Upsilon_1 \cup \Upsilon_2)$ we would obtain a contradiction, as

$$\sum_{X \in P_0^{-1}(\{y+tv_0\})} \Theta(\|G_j\|, X) \geq \sum_{X \in P_0^{-1}(\{y+tv_0\}) \cap \Upsilon_1 \cup \Upsilon_2} \Theta(\|G_j\|, X) + \Theta(\|G_j\|, Z_0 + Y) \geq 3. \quad \square$$

4.3.3. Horizontal multiplicity one. We introduce some useful additional notation. As the arguments only simplify when $L \subset \mathbf{R}^n \times \{0\}$, we assume throughout that this is not the case. For an arbitrary point $X \in \mathbf{R}^{n+1}$ we write $X = Y + sf + ze_{n+1}$ where $Y \in L$ and $s, z \in \mathbf{R}$. Sometimes it is also convenient to write $Y = (y, Y^{n+1})$ where $y = P_0(Y)$.

We define open domains $Q, Q_\tau \subset \mathbf{R}^n \times \{0\}$ by $Q = \{y + sf \mid |y| < 1, s^2 < 1\}$ and $Q_\tau = Q \cap (\Pi_2)_\tau = \{y + sf \mid |y| < 1, s^2 < \tau^2\}$. A point $X = Y + sf + ze_{n+1} \in Q \times \mathbf{R}$ is said to lie *north* of Π_1 if $z > 1$ and *south* of Π_1 if $z < -1$. In the same vein we say that a set $E \subset Q \times \mathbf{R}$ lies *north* (resp. *south*) of Π_1 if all points in E lie north (resp. south) of Π_1.

Lemma 4.3.4. *Let $\Pi_1, \Pi_2 \in Gr(n, n+1)$ be so that Π_1 is not vertical, but Π_2 is. Suppose $|G_j| \to |\Pi_1| + 2|\Pi_2|$. For all $\tau > 0$ there is $J(\tau) \in \mathbf{N}$ so that for all $j \geq J(\tau)$, $\operatorname{sing} G_j \cap Q \times \mathbf{R} \subset Q_\tau \times \mathbf{R}$ and we can decompose $G_j \cap (Q \setminus \overline{Q}_\tau) \times \mathbf{R} = \Sigma^1_{j,-} \cup \Sigma^1_{j,+} \cup \Sigma^2_{j,-} \cup \Sigma^2_{j,+}$, into four embedded connected surfaces with*

$$\Sigma^1_{j,\pm} \subset (\Pi_1)_\tau \text{ and } \Sigma^2_{j,-} \cup \Sigma^2_{j,+} \subset \{z > 1\} \text{ or } \{z < -1\}. \tag{4.6}$$

Moreover if $\Sigma^2_{j,-} \cup \Sigma^2_{j,+} \subset \{z > 1\}$ then

$$\operatorname{sing} G_j \cap (Q \times \mathbf{R}) \subset \{s^2 < \tau^2, z \geq -1\}, \tag{4.7}$$

and likewise if $\Sigma^2_{j,\pm} \subset \{z < -1\}$.

Before we give a proof of the lemma, we use its conclusions to obtain the following corollary, showing that in fact $|\Pi_1| + 2|\Pi_2|$ cannot arise as a limit of two-valued minimal graphs.

Corollary 4.3.5. *Let $\Pi_1, \Pi_2 \in Gr(n, n+1)$ be so that Π_1 is not vertical, but Π_2 is. Then $|G_j| \to |\Pi_1| + 2|\Pi_2|$ is impossible.*

Proof. Fix a small value for $\tau > 0$, depending only on Π_1, Π_2 and $J(\tau) \in \mathbf{N}$ so that without loss of generality, $G_j \cap Q \times \mathbf{R} \cap \{z < -1\}$ is a non-empty subset of $Q_\tau \times \mathbf{R} \cap \{z < -1\}$ and $\operatorname{sing} G_j \cap Q_\tau \times \mathbf{R} \cap \{z < -1\} = \emptyset$. Then $G_j \llcorner Q \times \mathbf{R} \cap \{z < -1\}$ is equal to the graph of a single-valued, smooth function $u_{j,S}$ defined on some subset $\Omega_{j,S} \subset Q_\tau$. From this we only retain that the current $[\![G_j]\!] \llcorner Q \times \mathbf{R} \cap \{z < -1\}$ is area-minimising. As $j \to \infty$ we get that $|G_j| \llcorner Q \times \mathbf{R} \cap \{z < -1\} \to 2|\Pi_2| \llcorner Q \times \mathbf{R} \cap \{z < -1\}$ in the varifold topology. At the same time, by inspection $[\![G_j]\!] \llcorner Q \times \mathbf{R} \cap \{z < -1\} \to 0$ as $j \to \infty$ in the current topology. This mass cancellation is absurd in light of the compactness of area-minimising currents, quoted in Proposition 2.2.1. □

We now give the proof of Lemma 4.3.4.

PROOF. Note first that every singular point $X = Y + sf + ze_{n+1}$ in $\operatorname{sing} G_j \cap Q \times \mathbf{R}$ automatically belongs to $Q_\tau \times \mathbf{R}$, that is has $s^2 < \tau^2$. Indeed the Allard regularity theorem can be applied near Π_1 because it has multiplicity one in the limit, which guarantees that away from L the G_j converge to the plane like smooth single-valued graphs. Counting the pre-images of points $x \in Q \setminus \overline{Q}_\tau$ we find that $\operatorname{sing} u_j \cap Q \setminus \overline{Q}_\tau = \emptyset$, or equivalently $\operatorname{sing} G_j \cap Q \times \mathbf{R} \subset \overline{Q}_\tau \times \mathbf{R}$.

Although $Q \setminus \overline{Q}_\tau$ is not simply connected, its two connected components, which lie on either side of Π_2, both are. We may thus make a smooth selection $u^1_{j,\tau}, u^2_{j,\tau} \in C^\infty(Q \setminus \overline{Q}_\tau)$ for u_j, arranging for the graph of $u^1_{j,\tau}$ to lie near Π_1. Both graphs are disconnected, and we write $\operatorname{graph} u^1_{j,\tau} = \Sigma^1_{j,-} \cup \Sigma^1_{j,+}$ and $\operatorname{graph} u^2_{j,\tau} = \Sigma^2_{j,-} \cup \Sigma^2_{j,+}$. As the graphs G_j locally converge to $\Pi_1 \cup \Pi_2$ in the Hausdorff distance, we may take an even larger $j \geq J(\tau)$ to get

$$G_j \cap \overline{Q} \times \mathbf{R} \cap \{z^2 \leq 1\} \subset (\Pi_1 \cup \Pi_2)_\tau.$$

Thus $\Sigma^1_{j,\pm} \subset \{z^2 < (1 - \langle e, e_{n+1}\rangle^2)^{-1}\tau^2\}$ and

$$\Sigma^2_{j,\pm} \subset \{z^2 > 1\} = \{z > 1\} \cup \{z < -1\}.$$

We show that in fact either $\Sigma^2_{j,\pm} \subset \{z > 1\}$ or $\Sigma^2_{j,\pm} \subset \{z < -1\}$. Recall from our initial analysis that using Sard's theorem one finds an open subset of 'unbranched' points $\mathcal{U}_j \subset L \cap \{|y| < 1\}$ with $\mathcal{H}^{n-1}(L \cap \{|y| < 1\} \setminus \mathcal{U}_j) = 0$ so that for all $Y \in \mathcal{U}_j$,

$$G_j \cap \{Y + sf + ze_{n+1} \mid s^2 < 1, z^2 < 1\} \cap \mathcal{B}_{G_j} = \emptyset,$$

and in fact can be decomposed into a union of three, smooth properly embedded Jordan arcs $\Upsilon^1_{j,Y}, \Upsilon^2_{j,Y}, \Upsilon^3_{j,Y}$ with endpoints in $\{Y + sf + ze_{n+1} \mid s^2 = 1 \text{ or } z^2 = 1\}$.

Given $\kappa > 0$ we define the subset $\mathcal{U}_j(\kappa) \subset \mathcal{U}_j$ by

$$\mathcal{U}_j(\kappa) = \Big\{Y \in \mathcal{U}_j \Big| \int_{\operatorname{reg} G_j \cap \{Y + sf + ze_{n+1} | s^2, z^2 < \tau^2\}} |A_{G_j}| \, \mathrm{d}\mathcal{H}^n < \kappa.\Big\}$$

Arguing as in Lemma 4.3.1 we can show that here we can take $\tau > 0$ small enough (independently of j) and $j \geq J(\tau)$ to guarantee that $\mathcal{H}^{n-1}(\mathcal{U}_j \setminus \mathcal{U}_j(\kappa))$ is as small as we like, regardless of the value for κ we had originally chosen. For our purposes we may take for example $\kappa = 1/2$, $\tau > 0$ small and $j \geq J(\tau)$ large enough that $\mathcal{H}^{n-1}(\mathcal{U}_j \setminus \mathcal{U}_j(\kappa)) < \omega_{n-1}$, as then automatically $\mathcal{U}_j(\kappa) \neq \emptyset$. If we then take a point $Y \in \mathcal{U}_j(\kappa)$, then we may relabel the curves so that $\Upsilon^1_{j,Y} \subset (\Pi_1)_\tau$, and $\Upsilon^2_{j,Y} \cup \Upsilon^3_{j,Y} \subset (\Pi_2)_\tau$.

Consider $Y = (y, Y^{n+1}) \in \mathcal{U}_j$ and let $l_y = \{y + sf \mid s^2 < 1\} \subset \mathbf{R}^n \times \{0\}$. On this line segment we can make a smooth selection $u^1_{j,y}, u^2_{j,y} \in C^\infty(-1, 1)$, where we identify l_y with $(-1, 1) \subset \mathbf{R}$. Then $G_j \cap \{Y + sf + ze_{n+1} \mid s^2 <$

$1, z^2 < 1\} \subset \operatorname{graph} u^1_{j,y} \cup \operatorname{graph} u^2_{j,y}$. We may furthermore make our selection in such a way that $\Upsilon^1_{j,Y} \subset \operatorname{graph} u^1_{j,y} \subset (\Pi_1)_\tau$ and $\Upsilon^2_{j,Y} \cup \Upsilon^3_{j,Y} \subset \operatorname{graph} u^2_{j,y}$. In what follows we also write $l_y(a,b) = \{y + sf \mid a < s < b\}$, where $-1 < a < b < 1$. The graph of $u^2_{j,y}$ restricted to the short segment $l_y(-2\tau, 2\tau)$ is a single smooth curve, which by inspection has both its endpoints lying on the same side of Π_1, that is either both lie north or both lie south. But $\operatorname{graph} u^2_{j,y} \cap l_y(-2\tau, -\tau) \times \mathbf{R} \subset \Sigma^2_{j,-}$ and likewise $\operatorname{graph} u^2_{j,y} \cap l_y(\tau, 2\tau) \times \mathbf{R} \subset \Sigma^2_{j,+}$ whence we find that $\Sigma^2_{j,-}$ and $\Sigma^2_{j,+}$ too must lie on the same side of Π_1—that is either both lie north or both lie south of Π_1. This concludes the proof of (4.6).

To prove (4.7) we start by making a few general observations. First, by (4.6) we may assume without loss of generality that $\Sigma_{j,\pm} \subset \{z > 1\}$. Let $Y = (y, Y^{n+1}) \in L$ be an arbitrary point with $|y| < 1$, not necessarily in $\mathcal{U}_j$. By the two-valued sheeting theorem of [Wic20] applied in the region $Q \times \mathbf{R} \cap \{\tau^2 < s^2 \vee z^2 < 1\}$, we find that in this slice the graph can be decomposed into six differentiable curves,

$$G_j \cap \{Y + sf + ze_{n+1} \mid \tau^2 < s^2 \vee z^2 < 1\} = \cup_{k=1}^2 \gamma^1_{j,Y,k} \cup \cup_{l=1}^4 \gamma^2_{j,Y,l}, \tag{4.8}$$

where $\gamma^1_{j,Y,k} \subset (\Pi_1)_\tau$ and $\gamma^2_{j,Y,l} \subset (\Pi_2)_\tau$. (Were the slice $\{Y + sf + ze_{n+1} \mid \tau^2 < s^2 \vee z^2 < 1\}$ to contain a branch point of $\mathcal{B}(G_j)$, then some arbitrary choices would have to be made in this decomposition, with no impact on the argument.) These curves taken together have two endpoints $\{z = -1, s^2 < \tau^2\}$, counted with multiplicity. Hence $\# G_j \cap \{Y + sf - e_{n+1} \mid s^2 < \tau^2\} \leq 2$ and

$$\sum_{s^2 < \tau^2} \Theta(\|G_j\|, Y + sf - e_{n+1}) = 2. \tag{4.9}$$

There exist two functions $u^1_{j,y}, u^2_{j,y} \in C^1(-1,1)$ so that

$$G_j \cap \{Y + sf + ze_{n+1} \mid s^2 < 1, z \in \mathbf{R}\} = \operatorname{graph} u^1_{j,y} \cup \operatorname{graph} u^2_{j,y}.$$

Moreover their graphs $\operatorname{graph} u^i_{j,y}$ are two differentiable curves which do not meet the region south of Π_1, except in the thin strip near Π_2 where $s^2 < \tau^2$. In other words, $(\operatorname{graph} u^1_{j,y} \cup \operatorname{graph} u^2_{j,y}) \cap \{s^2 \geq \tau^2, z < -1\} = \emptyset$.

Notice that $\operatorname{sing} G_j \cap \{Y + sf + ze_{n+1} \mid s^2 < 1, z \in \mathbf{R}\} = \operatorname{graph} u^1_{j,y} \cap \operatorname{graph} u^2_{j,y}$. Hence, if $\{Y + sf + ze_{n+1} \mid s^2 < 1, z < -1\}$ contained a singular point of G_j, then $\operatorname{graph} u^1_{j,y}, \operatorname{graph} u^2_{j,y}$ would both contain portions lying in that region. This is impossible, because if both curves intersected that region they would both need to pass through the set $\{Y + s^2 - e_{n+1} \mid s^2 < \tau^2\}$ twice each, because neither meets the set $\{s^2 \geq \tau^2, z < -1\}$. This in turn would be a contradiction to the decomposition of the graph as we obtained in (4.8) above, and (4.9) in particular. As the point Y was chosen arbitrarily, we have $\operatorname{sing} G_j \cap \{s^2 < 1, z < -1\} = \emptyset$, which concludes the proof. □

4.4. Classical limit cones: vertical cones

Let $\alpha \in (0,1)$ and $(u_j \mid j \in \mathbf{N})$ be a sequence of two-valued minimal graphs $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$. Let $D \in \mathbf{Z}_{>0}$ and for $i = 1, \dots, D$, let $\Pi_i = \Pi_i^0 \times \mathbf{R}e_{n+1} \in Gr(n, n+1)$ be vertical planes which meet along a common $(n-1)$-dimensional axis $L = L_0 \times \mathbf{R}e_{n+1}$, and suppose that as $j \to \infty$, $|G_j| \to \sum_i m_i |\Pi_i|$ and $[\![G_j]\!] \to \sum_i l_i [\![\Pi_i]\!]$. Here $0 \leq l_i \leq m_i \leq 2$ are integers and we pick arbitrary orientations for those planes with $l_i = 0$. When this notation is convenient we write $\mathbf{P} = \sum_i m_i |\Pi_i|$ and $T = \sum_i l_i [\![\Pi_i]\!]$. To make statements less awkward, we allow the possibility that $\mathbf{P}$ and T are supported in a single plane, although technically these would not be called classical cones.

Label the planes $\Pi_1, \dots, \Pi_D$ so that they lie in counterclockwise order around L. From now on we consider their indices modulo D, and write $\Pi_i = \pi_i \cup \pi_{i+D}$, where π_i, π_{i+D} are two half-planes which meet along L. The indices of the half-planes are considered modulo $2D$. For every π_i let N_i be its unit normal pointing in the counterclockwise direction; note that $N_i = -N_{i+D}$. Write n_i for the unit normal induced on Π_i as limits of the $[\![G_j]\!]$, and let $s_i = \langle n_i, N_i \rangle$, equal to ± 1 depending on whether or not n_i agrees with the counterclockwise orientation. With this notation, $s_i = -s_{i+D}$. We say that two half-planes π_i, π_j are oriented in the same direction if they are both oriented in the clockwise or counterclockwise direction, or equivalently if $s_i = s_j$.

Let $Q = \{x \in \mathbf{R}^n \mid \operatorname{dist}(x, L) < 1, \operatorname{dist}(x, L^\perp) < 1\}$. Extend the functions F^j from (4.2) to Q using the same formula, counting the number of points in G_j lying below x with multiplicity. These functions are eventually constant away from $\cup_i \Pi_i$, that is given $\tau > 0$ we can take $j \geq J(\tau)$ large enough that the F^j are constant on every connected component of $Q \setminus (\cup_i \Pi_i)_\tau$. Write $Q \setminus \cup_i \Pi_i$ as a disjoint union of wedge-shaped connected components $V_1, \dots, V_{2D}$. Each V_i lies between π_i, π_{i+1},

$$V_i = \{x \in Q \mid \langle x, N_i \rangle > 0, \langle x, N_{i+1} \rangle < 0\}.$$

By the above there is $F_i = F(V_i) \in \{0, 1, 2\}$ so that $F^j(x) = F_i$ at all $x \in V_i \setminus (\pi_i \cup \pi_{i+1})_\tau$ provided $j \geq J(\tau)$. Although the notation is slightly ambiguous, no confusion should arise between the value $F_i = F(V_i)$ and the functions F^j.

Remark 4.4.1. Notice that $[\![\operatorname{graph} -u_j]\!] \to -T$. If we counted the number of sheets of $\operatorname{graph} -u_j$ eventually lying below V_i, we would obtain $2 - F_i$.

4.4.1. Results in arbitrary dimensions. Applying Lemma 4.1.5 in the present context, we can relate consecutive values of F_i.

LEMMA 4.4.2. *For all i, $F_i - F_{i-1} = s_i l_i$.*

Similarly using Lemma 2.2.9, we obtain the following result.

LEMMA 4.4.3. *If $F_i = 1$ and $\pi_i \neq \pi_{i+1}$ then both half-planes have multiplicity one and are oriented in the same direction.*

PROOF. Start with the observation that $1 = F_i = F_{i+1} - s_{i+1}l_{i+1} = F_{i-1} + s_i l_i$, so $l_i, l_{i+1} \in \{0, 1\}$. The possibilities are as follows:

(1) $m_i = 1 = m_{i+1}$ and $l_i = 1 = l_{i+1}$,
(2) $m_i = 2, m_{i+1} = 1$ and $l_i = 0$, $l_{i+1} = 1$,
(3) $m_i = 1, m_{i+1} = 2$ and $l_i = 1$, $l_{i+1} = 0$,
(4) $m_i = 2 = m_{i+1}$ and $l_i = 0 = l_{i+1}$.

The proof is similar in all four cases. Every case is argued by contradiction, eventually reaching a conclusion which is forbidden by Lemma 2.2.9. We give a detailed proof for $m_i = 1 = m_{i+1}$ and $l_i = 1 = l_{i+1}$ and explain the necessary modifications for the remaining cases.

Assume without loss of generality that π_i and π_{i+1} both point into V_i, that is $n_i = N_i$ and $n_{i+1} = -N_{i+1}$. (Otherwise we may consider graph $-u_j$ instead, and use Remark 4.4.1.) Let two small $0 < \tau < \sigma < 1$ and a large $A > 1$ be given, and define the open subset $U_{j,i} \subset Q \setminus [L]_\sigma$ by

$$\begin{aligned} U_{j,i} &= U_{j,i}(\tau, \sigma, A) \\ &= (V_i)_\tau \setminus [L]_\sigma \cap \{u_j^- < 2A\} \\ &= \{x \in Q \setminus [L]_\sigma \mid \langle N_i, x\rangle > -\tau, \langle N_{i+1}, x\rangle < \tau, u_j^-(x) < 2A\}, \end{aligned}$$

where recall $u_j^- = u_j^1 \wedge u_j^2$.

As $F_i = 1$ we know that for large enough $j \geq J(\tau, \sigma, A)$,

$$V_i \setminus ([\pi_i]_\tau \cup [\pi_{i+1}]_\tau \cup [L]_\sigma) \subset U_{j,i}$$

and $\operatorname{sing} G_j \cap U_{j,i} \cap (-\infty, -2A) = \emptyset$. Hence any singular points of u_j would have to lie in $U_{j,i} \cap [\pi_i \cup \pi_{i+1}]_\tau$. As $m_i = 1 = m_{i+1}$ we may use Allard regularity inside $Q \times (-9/4A, 9/4A) \cap (\pi_i \cup \pi_{i+1})_{2\tau}$ and find that in fact

$$\operatorname{sing} u_j \cap U_{j,i} = \emptyset,$$

at least provided $j \geq J(\tau, \sigma, A)$ is large enough.

Let us rename $u_{j,i,S} = u_{j,i}^-$ and $u_{j,i,N} = u_{j,i}^+$. These two functions give a smooth selection for u_j on $U_{j,i}$. By construction

$$G_j \cap (V_i)_\tau \setminus [L]_\sigma \times (-\infty, A) \subset \operatorname{graph} u_{j,i,S}. \tag{4.1}$$

From this we obtain a contradiction with Lemma 2.2.9. Indeed as we let $\sigma, \tau \to 0$, $A \to \infty$ and $j \geq J(\tau, \sigma, A) \to \infty$, we have $\mathrm{dist}_{\mathcal{H}}(U_{j,i}, V_i) \to 0$ and

$$\llbracket \mathrm{graph}\, u_{j,i,S} \rrbracket \to \llbracket \pi_i \rrbracket + \llbracket \pi_{i+1} \rrbracket \llcorner Q \times \mathbf{R}. \tag{4.2}$$

This concludes the proof in the first case. In the next case $m_i = 2$, $m_{i+1} = 1$ and $l_i = 0$, $l_{i+1} = 1$. As $l_i = 0$ the half-plane π_i has no well-defined orientation induced by T. The half-plane π_{i+1} may be assumed oriented in the clockwise direction without loss of generality, that is $n_{i+1} = -N_{i+1}$. (This can be justified in the same way as above, working with $-u_j$ if necessary.) By Lemmas 4.1.5 and 4.1.4 we may take $j \geq J(\tau, \sigma)$ large enough that

$$\begin{aligned} &\mathrm{sing}\, u_j \cap (V_i \cup V_{i-1}) \setminus ([\pi_{i-1} \cup \pi_i \cup \pi_{i+1}]_\tau \cup [L]_\sigma) = \emptyset, \\ &\mathcal{B}_{u_j} \cap (V_{i-1} \cup V_i \cup \pi_i) \setminus ([\pi_{i-1} \cup \pi_i]_\tau \cup [L]_\sigma) = \emptyset. \end{aligned}$$

Additionally the set $(V_{i-1} \cup V_i \cup \pi_i) \setminus ([\pi_{i-1} \cup \pi_i]_\tau \cup [L]_\sigma)$ is simply connected, so we can make a smooth selection $\{u_{j,i,S}, u_{j,i,N}\}$ for u_j on it, arranging the indices in a way that $\mathrm{graph}\, u_{j,i,S}$ lies south of V_i. (Here $u_{j,i,S}$ is not equal $u^-_{j,i}$ anymore.) Define the region $U_{j,i} = (V_i)_\tau \setminus [L]_\sigma \cap \{u_{j,i,S} < 2A\} \subset (V_i)_\tau \setminus [L]_\sigma$. As above (4.1) holds, and near π_i we have that given any $\delta > 0$,

$$\langle \nu_j(X), N_i \rangle > 1 - \delta \tag{4.3}$$

for all $X \in \mathrm{graph}\, u_{j,i,S} \cap \mathrm{reg}\, G_j \cap (\pi_i)_\tau \cap \{|X^{n+1}| < A\}$, at least after updating $j \geq J(\tau, \sigma, A, \delta)$. As we let $\tau, \sigma, \delta \to 0$ and $A \to \infty$ both $\mathrm{dist}_{\mathcal{H}}(U_{j,i}, V_i) \to 0$ and (4.2) hold as $j \geq J(\tau, \sigma, A, \delta) \to \infty$. The last inequality (4.3) guarantees that the limit current has the right orientation to apply Lemma 2.2.9, which immediately yields a contradiction. The third case, when $m_i = 1, m_{i+1} = 2$ and $l_i = 1, l_{i+1} = 0$ can be argued in precisely the same way, with reversed roles of π_i and π_{i+1}.

The last remaining case is $m_i = 2 = m_{i+1}$ and $l_i = 0 = l_{i+1}$. Arguing as above we find

$$\begin{aligned} &\mathrm{sing}\, u_j \cap (V_{i-1} \cup V_i \cup V_{i+1}) \setminus ([\pi_{i-1} \cup \pi_i \cup \pi_{i+1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma) = \emptyset, \\ &\mathcal{B}_{u_j} \cap (V_{i-1} \cup V_i \cup V_{i+1} \cup \pi_i \cup \pi_{i+1}) \setminus ([\pi_{i-1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma) = \emptyset. \end{aligned}$$

We may make a smooth selection $\{u_{j,i,S}, u_{j,i,N}\}$ for u_j on the latter set $(V_{i-1} \cup V_i \cup V_{i+1} \cup \pi_i \cup \pi_{i+1}) \setminus ([\pi_{i-1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma)$. We arrange for $u_{j,i,S}$ to lie south of V_i, and moreover

$$\langle \nu_j(X), N_i \rangle > 1 - \delta \text{ and } \langle \nu_j(X), -N_{i+1} \rangle > 1 - \delta$$

at all points $X \in \mathrm{graph}\, u_{j,i,S} \cap \mathrm{reg}\, G_j \cap (\pi_i)_\tau \cap \{|X^{n+1}| < A\}$ and $\mathrm{graph}\, u_{j,i,S} \cap \mathrm{reg}\, G_j \cap (\pi_{i+1})_\tau \cap \{|X^{n+1}| < A\}$ respectively. From that point on we can argue in the same way as above, ultimately leading to a contradiction with with Lemma 2.2.9. This exhausts the list of possible cases, and concludes the proof. □

Using this, we can immediately conclude mass cancellation in all but one case. Indeed $|T| \neq \mathbf{P}$ if and only if $l_i = 0$ for at least one plane. But then $F_{i-1} = 1 = F_i$ by Lemma 4.1.5, which contradicts Lemma 4.4.3 unless $\pi_i = \pi_{i+1}$ and $D = 1$. For the remainder of this section, we may assume that $l_i = m_i$ for all i. As a consequence also $F_i - F_{i-1} = s_i m_i$ for all i.

Lemma 4.4.4. *Let* $|G_j| \to \sum_{i=1}^{D} m_i |\Pi_i|$. *Then*

(i) *any consecutive* $\pi_{i+1}, \dots, \pi_{i+J}$ *have* $|\sum_{j=1}^{J} s_{i+j} m_{i+j}| \leq 2$,

(ii) *if* π_i *has multiplicity two then* $\sum_{j=1}^{D-1} s_{i+j} m_{i+j} = 0$.

Proof. (i) Iterating Lemma 4.4.2 we find that $F_{i+J} - F_i = \sum_{j=1}^{J} s_{i+j} m_{i+j}$. As $F_{i+J}, F_i \in \{0, 1, 2\}$ we get $|\sum_j s_{i+j} m_{i+j}| \leq 2$ as desired.

(ii) Here we consider the $D-1$ consecutive planes $\pi_{i+1}, \dots, \pi_{i+D-1}$, which stop just shy of π_i and π_{i+D}. Because $\pi_i \cup \pi_{i+D} = \Pi$ is a plane, they must have $s_i = -s_{i+D}$; without loss of generality $s_i = 1$. By Lemma 4.4.2, $F_i = F_{i-1} + 2$ and $F_{i+D} = F_{i+D-1} - 2$, so $F_i = 2 = F_{i+D-1}$. Iterating the same lemma over the half-planes $\pi_{i+1}, \dots, \pi_{i+D-1}$ we find $F_{i+D-1} = F_i + \sum_{j=1}^{D-1} s_{i+j} m_{i+j}$, whence $\sum_{j=1}^{D-1} s_{i+j} m_{i+j} = 0$, as desired. □

Corollary 4.4.5. *Let* $|G_j| \to \sum_{i=1}^{D} m_i |\Pi_i|$. *Then*

(i) *if* $m_1 = \cdots = m_D = 1$ *then* $D \equiv 2 \pmod 4$,

(ii) *if* $m_1 = \cdots = m_D = 2$ *then* D *is odd,*

(iii) *if* $m_1 = 2$, $m_2 = \cdots = m_D = 1$ *then* $D \equiv 1 \pmod 4$.

Proof. The result is obtained by listing the orientations of the half-planes $\pi_1, \dots, \pi_{2D}$ weighted by their respective multiplicities, and excluding certain subsequences from this. We start by considering three consecutive half-planes $\pi_{i-1}, \pi_i, \pi_{i+1}$ with multiplicities $m_{i-1}, m_i, m_{i+1} = 1$, and show that then $(s_{i-1}, s_i, s_{i+1}) \in \{\pm(1, -1, -1), \pm(1, 1, -1)\}$. First note that the two sequences $(s_{i-1}, s_i, s_{i+1}) = \pm(1, 1, 1)$ are excluded by Lemma 4.4.2. The remaining cases are argued by contradiction, assuming that $(s_{i-1}, s_i, s_{i+1}) = (1, -1, 1)$. Then on the one hand by Lemma 4.4.3, applied between π_{i-1}, π_i and π_i, π_{i+1} respectively we find that $F_{i-1}, F_i \neq 1$. This is absurd, as on the other hand $F_i = F_{i-1} + 1$. One reasons similarly when $(s_{i-1}, s_i, s_{i+1}) = (-1, 1, 1)$.

(i) When the multiplicities are all equal $m_1, \dots, m_D = 1$ then the only possibility is that $(m_1 s_1, \dots, m_{2D} s_{2D}) = (s_1, \dots, s_{2D}) = (1, 1, -1, -1, \dots)$ or a cyclic permutation thereof. Hence D must be even. As $s_1, s_2 = 1$ we get $s_{D+1}, s_{D+2} = -1$, whence $s_{D-1}, s_D = 1$ and $D \not\equiv 0 \pmod 4$.

TABLE 1. The possibilities for the multiplicities afforded by the improved area bounds of Corollary 3.3.4, up to cyclic permutations.

D	$(m_1, \dots, m_D)$
2	$(2,2), (2,1)$
3	$(2,1,1), (1,1,1), (2,2,1)$
4	$(2,1,1,1), (1,1,1,1)$
5	$(1,1,1,1,1)$

(ii) From Lemma 4.4.4 (i) either $(s_1, \dots, s_{2D}) = (1, 1, -1, -1, \dots)$ or a cyclic permutation thereof, so D must be odd.

(iii) Without loss of generality $s_1 = 1$, and by Lemma 4.4.2 the sequence $(m_i s_i)$ starts $(2s_1, s_2, s_3) = (2, -1, -1)$. The orientations of half-planes with multiplicity one alternate in pairs, so this continues $(s_4, s_5, s_6, s_7, \dots) = (1, 1, -1, -1, \dots)$. As $2s_{D+1} = -2$ we get $(s_{D-1}, s_D, 2s_{D+1}) = (1, 1, -2)$. Combining the two observations, $D - 1 \equiv 0 \pmod 4$. □

4.4.2. Classification in dimensions up to seven. Here too, as in the previous section, we consider a sequence of two-valued minimal graphs $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ which converge in the varifold topology, $|G_j| = |\operatorname{graph} u_j| \to \sum_i m_i |\Pi_i|$ as $j \to \infty$. These planes are assumed to meet along a single $n-1$-dimensional vertical axis $L = L_0 \times \mathbf{R}e_{n+1} \in Gr(n-1, n+1)$.

COROLLARY 4.4.6. *Let* $|G_j| \to \sum_{i=1}^D m_i |\Pi_i|$. *If* $2 \leq n \leq 6$ *then this is either* $2|\Pi_1|$ *or* $|\Pi_1| + |\Pi_2|$.

PROOF. As the graphs have dimension up to six, the area estimates of Proposition 3.3.5 give $\sum_j m_j \leq \lfloor n\omega_n / \omega_{n-1} \rfloor \leq 5$. Hence $D \leq 5$, and the possibilities for $(m_1, \dots, m_D)$ are listed in Table 1 up to cyclic permutation. Of these, the only not forbidden by Corollary 4.4.5 is $(m_1, m_2, m_3) = (2, 2, 1)$. However note $2s_2 + s_3 \neq 0$, which contradicts Lemma 4.4.2. □

4.4.3. Multiplicity and mass cancellation.

COROLLARY 4.4.7. *Suppose* $2 \leq n \leq 6$. *Let*

$$|G_j| \to V \neq 0 \in \mathbf{IV}_n(D_1 \times \mathbf{R}) \text{ as } j \to \infty.$$

Then either $\Theta(\|V\|, X) = 2$ *for* $\mathcal{H}^n$*-a.e.* $X \in \operatorname{reg} V$ *and there is a smooth, stable minimal surface* Σ *so that*

$$V = 2|\Sigma|,$$

or $\Theta(\|V\|, X) = 1$ *for* $\mathcal{H}^n$*-a.e.* $X \in \operatorname{reg} V$ *and*

(i) $\operatorname{spt}\|V\|$ *is immersed near points of* $\mathcal{S}^{n-1}(V) \setminus \mathcal{S}^{n-2}(V)$,

(ii) *the set* $\mathcal{S}^{n-2}(V) \cup \mathcal{B}(V)$ *is countably* $(n-2)$*-rectifiable.*

Proof. We only need to show that the multiplicity of regular points of V is either one or two; the conclusion follows by combining our limit cone classification with the results of [SS81, KW20, Wic20]. We may assume without loss of generality that $G_j \cap D_1 \times \mathbf{R}$ is connected for all j. As the graphs converge locally in $D_1 \times \mathbf{R}$ with respect to Hausdorff distance, $\operatorname{spt}\|V\| \cap D_1 \times \mathbf{R}$ is also connected. Let $\mathcal{R}$ be the set of connected components of $\operatorname{reg} V$, which we group into the two sets $\mathcal{R}_1$ and $\mathcal{R}_2$ according to their respective multiplicities. (The multiplicities are constant on every component by [Sim84, Thm. 41.1].)

We use a contradiction argument to show that one of the two is empty. Let $V_1 = \sum_{\Sigma \in \mathcal{R}_1} |\Sigma|$ and $V_2 = \sum_{\Gamma \in \mathcal{R}_2} 2|\Gamma|$. These are both stationary in $D_1 \times \mathbf{R}$ away from $\operatorname{spt}\|V_1\| \cap \operatorname{spt}\|V_2\| \cap D_1 \times \mathbf{R}$. By our classification of limit cones, $\mathcal{C}(V) \cap \operatorname{spt}\|V_2\| = \emptyset$. As $\mathcal{B}(V) \cup \mathcal{S}^{n-2}(V)$ is countably $(n-2)$-rectifiable, this means $\mathcal{H}^{n-1}(\operatorname{spt}\|V_1\| \cap \operatorname{spt}\|V_2\| D_1 \times \mathbf{R}) = 0$, whence V_1, V_2 are in fact stationary in $D_1 \times \mathbf{R}$ without restrictions. One argues in the same way to justify their stability in $D_1 \times \mathbf{R}$. As the support of V_2 contains neither genuine branch points nor classical singularities, [SS81] implies that there is a smooth embedded minimal surface $\Sigma_2 \subset D_1 \times \mathbf{R}$ so that $V_2 = 2|\Sigma_2|$. Take $X \in \operatorname{spt}\|V_1\| \cap \Sigma_2 \cap D_1 \times \mathbf{R}$, and $\rho > 0$ small enough enough that $B_\rho(X) \setminus \Sigma_2$ has two connected components, say $U_\pm$. Let $V_1^\pm$ be the two varifolds made up of the portions of V_1 lying in $U_\pm$ respectively. Arguing as above we find that $V_1^\pm$ are both stationary in $B_\rho(X)$. Without loss of generality $X \in \operatorname{spt}\|V_1^+\|$. As V_1^+ lies above Σ_2, we have reached a contradiction with the maximum principle of Wickramasekera [Wic14c] quoted in Theorem 2.1.9. □

Corollary 4.4.8 (No mass cancellation). *Let* $2 \le n \le 6$. *Suppose that as* $j \to \infty$,

$$|G_j| \to V \in \mathbf{IV}_n(D_1 \times \mathbf{R})$$
$$[\![G_j]\!] \to T \in \mathbf{I}_n(D_1 \times \mathbf{R})$$

If $T \neq 0$ *then* $|T| = V$.

Proof. The limit current T necessarily has $|T| \ll V$, with equality if and only if it there is no mass cancellation. This cannot occur at points of multiplicity one, so by Corollary 4.4.7 we may assume that $\Theta(\|V\|, X) = 2$ for $\mathcal{H}^n$-a.e. $X \in \operatorname{reg} V$ and there is a smooth embedded minimal surface Σ with $V = 2|\Sigma|$. Thus also $\operatorname{spt}\|T\| \subset \Sigma$, and by the Constancy Theorem [Sim84, Thm. 41.1] there is $m \in \mathbf{Z}_{>0}$ so that $|T| = m|\Sigma|$. As $m \le 2$, and $m \neq 1$ as otherwise we could use Allard's regularity theory, we have that either $m = 2$ and $|T| = V$ or $m = 0$ and $T = 0$. □

Chapter 5

Blowdown cone analysis and the Bernstein theorem

5.1. Blowdown cones and asymptotic analysis

In this section we derive two different results, which respectively hold for graphs of arbitrarily large dimension and when $n \leq 6$. For now let $n \in \mathbf{Z}_{>0}$ be arbitrary, and consider a two-valued minimal graph $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$. Without loss of generality we may assume that $u(0) = 0$ and that G is singular there. Let $\mathbf{C} \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ be a stationary varifold obtained as a blowdown cone of $|G|$ at infinity, along some sequence of positive scalars $\lambda_j \to \infty$. By this we mean that for every $j \in \mathbf{N}$ we set

$$u_j(x) = \lambda_j^{-1} u(\lambda_j(x)) \text{ for all } x \in \mathbf{R}^n,$$

write $G_j = \operatorname{graph} u_j$ and take their weak limit in the varifold sense,

$$|G_j| \to \mathbf{C} \text{ as } j \to \infty.$$

Passing to the limit, the cone inherits both the stationarity, and stability with respect to compactly supported ambient deformations from the two-valued graphs.

5.1.1. Entire graphs with bounded growth.

Lemma 5.1.1. *Let $\alpha \in (0,1)$ and $n \geq 2$ be arbitrary. Let $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ be an entire two-valued minimal graph. If*

$$\limsup_{r \to \infty} \left(\|u\|_{0;D_r}/r \right) < +\infty$$

then u is linear. Otherwise the support of every blowdown cone at infinity contains the half-line $L_+ = \{te_{n+1} \mid t \geq 0\}$ or its reflection $-L_+$.

Proof. Suppose first that u has bounded growth, say $\sup_r r^{-1}\|u\|_{0;D_r} \leq C$ for some $C > 0$. Let $(\lambda_j \mid j \in \mathbf{N})$ be a sequence of positive scalars with $\lambda_j \to \infty$, along which we blow-down u. For all $j \in \mathbf{N}$, define $u_j \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ by setting $u_j(x) = \lambda_j^{-1} u(\lambda_j x)$ for all $x \in \mathbf{R}^n$. Using the interior gradient estimates, we find that for all $r > 0$ there is a constant $C(r)$ so that $\sup_j \|u_j\|_{1;D_r} \leq C(r)$. Next, by the two-valued Lipschitz theorem we can extract a subsequence which guarantees that there is a two-valued Lipschitz function $U \in \mathrm{Lip}(\mathbf{R}^n; \mathcal{A}_2)$ so that simultaneously $u_{j'} \to U$ locally uniformly and $|G_{j'}| \to |\mathrm{graph}\, U|$. By the monotonicity formula graph U is a cone, and hence U is homogeneous. Next by Lemma 3.5.3, U must be linear, and its graph is a union of two possibly equal planes. In particular $\mathcal{H}^n(\mathrm{graph}\, U \cap B_1) = 2\omega_n$. Thus $\Theta(\|G\|, \infty) = 2$. But by assumption $\Theta(\|G\|, 0) = 2$, so by the monotonicity formula the graph G is the union of two possibly equal planes and u is linear. This concludes the proof of the first half of the lemma.

Now assume instead that this diverges. Up to considering $-u$ we may assume that $\sup_{r>0} r^{-1} \max_{D_r}\{u_1, u_2\} = +\infty$ which will ultimately allow us to conclude that, if $\mathbf{C}$ is a blowdown cone of u at infinity, then necessarily $\{te_{n+1} \mid t \geq 0\} \subset \mathrm{spt}\|\mathbf{C}\|$.

We argue as follows. As above, let $(\lambda_j \mid j \in \mathbf{N})$ be an arbitrary sequence of positive scalars with $\lambda_j \to +\infty$ along which we blow down $|\mathrm{graph}\, u|$ to obtain a sequence with limiting behaviour $|\mathrm{graph}\, u_j| \to \mathbf{C}$ as $j \to \infty$. It is well-known that the supports of these graphs converge to $\mathrm{spt}\|\mathbf{C}\|$ in the Hausdorff distance, $\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|G_j\| \cap K, \mathrm{spt}\|\mathbf{C}\| \cap K) \to 0$ as $j \to \infty$ for all compact $K \subset \mathbf{R}^{n+1}$. Here we take $K = \overline{D}_\delta \times [-1, 1]$ depending on a small parameter $\delta \to 0$ which we eventually let go to zero.

For now however let us fix a value for δ. Inside the disc D_δ the functions u_j have

$$\max_{\overline{D}_\delta}\{u_1^j, u_2^j\} \to +\infty \text{ as } j \to +\infty,$$

so that for large enough $j \geq J(\delta)$ we get

$$\max_{\overline{D}_\delta}\{u_1^j, u_2^j\} \geq 1.$$

Hence there exists a sequence of points $X_j = (x_j, X_j^{n+1}) \in \mathrm{spt}\|G_j\| \cap D_\delta \times \mathbf{R}$ with $X_j^{n+1} > 1$ for all j. As $\mathrm{spt}\|G_j\| \cap D_\delta \times \mathbf{R}$ is connected there is a continuous path $\gamma_j : [0, 1] \to \mathbf{R}^{n+1}$ with image $\gamma_j([0, 1]) \subset \mathrm{spt}\|G_j\| \cap D_\delta \times \mathbf{R}$ and endpoints $\gamma_j(0) = 0$ and $\gamma_j(1) = X_j$. This path must cross the solid disc $D_\delta \times \{1\}$ at height one, so that by picking a point in this intersection we can construct a sequence of points $(Y_{j,\delta} \mid j \geq J(\delta))$ each of which belongs to

$$Y_{j,\delta} \in \gamma_j([0, 1]) \cap D_\delta \times \{1\}.$$

Now of course we may proceed similarly regardless of the size of δ, so that taking a positive sequence $(\delta_m \mid m \in \mathbf{N})$ with $\delta_m \to 0$ as $m \to \infty$ we obtain via a diagonal extraction argument a subsequence of indices $(j_m \mid m \in \mathbf{N})$ and $(Y_m \mid m \in \mathbf{N})$ with

$$Y_m = Y_{j_m, \delta_m} \in G_{j_m} \cap D_{\delta_m} \times \{1\}.$$

In particular we obtain convergence $Y_m \to (0,1) \in \mathbf{R}^{n+1}$, which thus necessarily belongs to $\operatorname{spt}\|\mathbf{C}\|$. As $\mathbf{C}$ is a cone, its support in fact contains the entire line directed by $(0,1)$, that is $\{te_{n+1} \mid t \geq 0\} \subset \operatorname{spt}\|\mathbf{C}\|$. □

5.1.2. General results in low dimensions. Combining the results from 3.5.1 with the work of [Wic20] we obtain the following.

COROLLARY 5.1.2. *Let $V \in \mathbf{IV}_n(D_2 \times \mathbf{R})$ be the limit of a sequence of two-valued graphs $G_j = \operatorname{graph} u_j$, where there is $\alpha \in (0,1)$ so that $u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)$ for all j. Suppose that at the point $Z \in \operatorname{spt}\|V\|$, there is a tangent cone of the form*

$$|\Pi_1^0| \times \mathbf{R}e_{n+1}, 2|\Pi_1^0| \times \mathbf{R}e_{n+1}, \text{ or } (|\Pi_1^0| + |\Pi_2^0|) \times \mathbf{R}e_{n+1} \in \operatorname{VarTan}(V, Z),$$

where Π_1^0, Π_2^0 are two distinct $n-1$-dimensional planes in $\mathbf{R}^n$. Then there is $\rho > 0$ so that

$$\langle \nu(X), e_{n+1} \rangle = 0 \text{ for all } X \in \operatorname{reg} V \cap B_\rho(Z).$$

This result holds in arbitrary dimensions, but in the remainder we consider $2 \leq n \leq 6$. Nonetheless this is a useful prerequisite for the following theorem, which is the main result of this section.

THEOREM 5.1.3. *Let $\alpha \in (0,1)$ and $2 \leq n \leq 6$. Let $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ be an entire two-valued minimal graph and $\mathbf{C}$ be a blowdown cone of $|G|$ at infinity. Then*

- (i) *either $\mathbf{C}$ is cylindrical, that is of the form $\mathbf{C} = \mathbf{C}^0 \times \mathbf{R}e_{n+1}$,*
- (ii) *or $\mathbf{C} = |\Pi| + \mathbf{C}^0 \times \mathbf{R}e_{n+1}$ where $\Pi \in Gr(n, n+1)$,*
- (iii) *or $\mathbf{C}$ is the sum of two possibly equal planes $\Pi_1, \Pi_2 \in Gr(n, n+1)$, $\mathbf{C} = |\Pi_1| + |\Pi_2|$.*

The remainder is dedicated to proving this theorem, starting by decomposing the blowdown cone $\mathbf{C}$ into a vertical and a horizontal part. We construct this decomposition as follows. We consider the set $\mathcal{R}$ of connected components of $\operatorname{reg}\mathbf{C}$. (This set has at most countably many elements by a classical separability argument.) By [Sim84, Thm. 41.1] every $\Sigma \in \mathcal{R}$ has has constant multiplicity $\Theta_\Sigma \in \mathbf{Z}_{>0}$. We say that Σ is *vertical* if $\langle \nu, e_{n+1} \rangle \equiv 0$ on Σ, and *horizontal* if instead $\langle \nu, e_{n+1} \rangle > 0$. Thus $\mathcal{R} = \mathcal{R}_v \cup \mathcal{R}_h$, either of which is allowed to empty.

LEMMA 5.1.4. *Let $\alpha \in (0,1)$ and $2 \leq n \leq 6$. Let $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ be an entire two-valued minimal graph and $\mathbf{C} \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ be a blowdown cone of $|G|$ at infinity. Then $\mathbf{C}_v = \mathbf{C}_v^0 \times \mathbf{R}e_{n+1} = \sum_{\Sigma \in \mathcal{R}_v} \Theta_\Sigma |\Sigma|$ and $\mathbf{C}_h = \sum_{\Gamma \in \mathcal{R}_h} \Theta_\Gamma |\Gamma|$ are stationary integral varifolds, and*

$$\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h \in \mathbf{IV}_n(\mathbf{R}^{n+1}). \tag{5.1}$$

PROOF. First, the convergence of the two sums can be justified because their weight measures are bounded by $\|\mathbf{C}\|$. Let us write the argument out explicitly for $\mathbf{C}_v$. Assume that $\mathcal{R}_v$ is countably infinite, enumerated by $\mathcal{R}_v = \{\Sigma_i \mid i \in \mathbf{N}\}$ say. For every compact subset $K \subset \mathbf{R}^{n+1}$, $\sum_{\Sigma\in\mathcal{R}_v} \Theta_\Sigma \|\Sigma\|(K) \leq \|\mathbf{C}\|(K) \leq C_K$, so that the partial sums $\sum_{i=k}^{\infty} \Theta_{\Sigma_i}|\Sigma_i| \to 0$ when $k \to \infty$ as varifolds. Thus the sum $\sum_{i=1}^{\infty} \Theta_{\Sigma_i}|\Sigma_i|$ is convergent, with limit $\mathbf{C}_v \in \mathbf{IV}_n(\mathbf{R}^{n+1})$. As every $|\Sigma_i| \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ is invariant under homotheties, the same holds for their limit, which we are thus justified in denoting by $\mathbf{C}_v$. Similarly one may check that indeed $\mathbf{C}_v$ is vertical, meaning it is of the form $\mathbf{C}_v = \mathbf{C}_v^0 \times \mathbf{R}e_{n+1}$ for some $\mathbf{C}_v^0 \in \mathbf{IV}_{n-1}(\mathbf{R}^n)$.

Proceeding similarly one can justify the construction of $\mathbf{C}_h$, and confirm that $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h$ as in (5.1). Moreover, the stationarity of $\mathbf{C}$ means that $\mathbf{C}_v$ is stationary in the open set $\mathbf{R}^{n+1} \setminus \operatorname{spt}\|\mathbf{C}_h\|$ and vice-versa for $\mathbf{C}_h$. The only way either of the two cones could fail to be stationary in $\mathbf{R}^{n+1}$ is if

$$\mathcal{H}^{n-1}(\operatorname{spt}\|\mathbf{C}_v\| \cap \operatorname{spt}\|\mathbf{C}_h\| \cap B_1) > 0.$$

By construction $\operatorname{spt}\|\mathbf{C}_v\| \cap \operatorname{spt}\|\mathbf{C}_h\| \subset \operatorname{sing}\mathbf{C}$, which is stratified like

$$\mathcal{S}^0 \subset \cdots \subset \mathcal{S}^{n-2} \subset \mathcal{S}^{n-1} \subset \mathcal{S}^n,$$

where we abbreviate $\mathcal{S}^i = \mathcal{S}^i(\mathbf{C})$. By Corollary 4.4.7, we further have $\mathcal{H}^{n-1}(\mathcal{B}(\mathbf{C}) \cup \mathcal{S}^{n-2}) = 0$, whence

$$\mathcal{H}^{n-1}(\operatorname{spt}\|\mathbf{C}_v\| \cap \operatorname{spt}\|\mathbf{C}_h\| \setminus (\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})) = 0.$$

Now assume $\mathcal{H}^{n-1}(\operatorname{spt}\|\mathbf{C}_v\| \cap \operatorname{spt}\|\mathbf{C}_h\| \cap B_1) > 0$, and take a point

$$X_0 \in (S^{n-1} \setminus \mathcal{S}^{n-2}) \cap \operatorname{spt}\|\mathbf{C}_v\| \cap \operatorname{spt}\|\mathbf{C}_h\| \cap B_1.$$

The classification of classical tangent cones established in the previous section (e.g. see Corollary 4.4.7 again), and valid for the range of dimensions $2 \leq n \leq 6$ prescribed in the hypotheses, implies that $\operatorname{spt}\|\mathbf{C}\|$ must be immersed near X_0. Therefore both $\mathbf{C}_v$ and $\mathbf{C}_h$ must be embedded near X_0, say $B(X_0, \rho_0) \cap \operatorname{spt}\|\mathbf{C}\| \subset \operatorname{reg}\mathbf{C}_h \cup \operatorname{reg}\mathbf{C}_v$ for some $\rho_0 > 0$, which are transversely intersecting. Both $\operatorname{reg}\mathbf{C}_h$ and $\operatorname{reg}\mathbf{C}_v$ have separately pointwise vanishing mean curvature, and in particular they are both stationary near X_0. As X_0 was chosen arbitrarily, this proves that both $\mathbf{C}_v$ and $\mathbf{C}_h$ are stationary as varifolds in $\mathbf{IV}_n(\mathbf{R}^{n+1})$. $\square$

The three cases (i), (ii) and (iii) listed in Theorem 5.1.3 correspond to the following situations. In the first case $\mathbf{C}_h = 0$, and the conclusion is immediate, while in the last $\mathbf{C}_v = 0$ and we conclude using the uniform gradient bounds for u, as demonstrated in Lemma 5.1.1. Probably the most complicated case of the three remains, in which both $\mathbf{C}_v \neq 0$ and $\mathbf{C}_h \neq 0$.

LEMMA 5.1.5. *Let* $\alpha \in (0,1)$ *and* $2 \leq n \leq 6$. *Let* $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ *be a two-valued minimal graph and* $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ *be a blowdown cone of* $|G|$ *at infinity. If* $\mathbf{C}_v \neq 0$ *and* $\mathbf{C}_h \neq 0$ *then* $\mathbf{C}_h = |\Pi|$ *for some plane* $\Pi \in Gr(n, n+1)$.

In proving this lemma we will use a function $Q : D_1 \to [0, +\infty]$ defined by

$$Q(y) = \sum_{Y \in P_0^{-1}(\{y\})} \Theta(\|\mathbf{C}_h\|, Y) \text{ for all } y \in D_1, \tag{5.2}$$

where we convene that $Q(y) = +\infty$ at points $y \in D_1$ for which the set $\{Y \in P_0^{-1}(\{y\}) \mid \Theta(\|\mathbf{C}_h\|, Y) > 0\}$ is infinite, should they exist.

LEMMA 5.1.6. *Let* $\alpha \in (0,1)$ *and* $2 \leq n \leq 6$. *Let* $u \in C^{1,\alpha}(\mathbf{R}^n; \mathcal{A}_2)$ *be a two-valued minimal graph and* $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ *be a blowdown cone of* $|G|$ *at infinity. Then the set* $K = P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v) \cap D_1 \subset D_1$ *is closed and has* $\mathcal{H}^{n-1}(K) = 0$. *Moreover* Q *is finite and constant on* $D_1 \setminus K$, *with value either* 1 *or* 2.

PROOF. Recall that by Corollary 4.4.7 the support of $\mathbf{C} = \mathbf{C}_h + \mathbf{C}_h$ is immersed away from $\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C})$. This is true in particular at singularities in $(\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C})$, which are all classical, immersed singularities. For the remainder we thus write $\mathcal{C}(\mathbf{C}) = (\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C})$. These form an open subset of $\operatorname{sing} \mathbf{C}$, meaning that the projection of the set of non-immersed singularities onto $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ yields a closed subset of D_1 with

$$\mathcal{H}^{n-1}(P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C})) \cap D_1) = 0. \tag{5.3}$$

This last identity (5.3) is preserved if we additionally project the singularities of $\mathbf{C}_v$. Indeed the only singularities of $\mathbf{C}_v$ not contained in the set $\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C})$ are those in $(\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C}_v)$. Now, because the cone has the form $\mathbf{C}_v = \mathbf{C}_v^0 \times \mathbf{R}e_{n+1}$, the same holds for its singular set, meaning that $(\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C}_v) = [(\mathcal{S}^{n-2} \setminus \mathcal{S}^{n-3})(\mathbf{C}_v^0)] \times \mathbf{R}e_{n+1}$. Therefore

$$\mathcal{H}^{n-1}(P_0((\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C}_v))) = \mathcal{H}^{n-1}((\mathcal{S}^{n-2} \setminus \mathcal{S}^{n-3})(\mathbf{C}_v^0)) = 0.$$

The same is true for the closedness of $P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v) \cap D_1$ inside D_1, because the set $P_0(\operatorname{sing} \mathbf{C}_v) = \operatorname{sing} \mathbf{C}_v^0$ is closed. Then an argument similar to the one used in [SW16] to prove their Lemma A.1 can be applied here to conclude that the complement of $K = P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v) \cap D_1$ in the unit disc D_1 is path-connected.

Let us assume for now the validity of the conclusions of Claim 5 below, namely that Q is locally constant on the set $D_1 \setminus K$, with value either 1 or 2. As this set is path-connected, the function Q must in fact be constant, precisely what was to be shown. □

CLAIM 5. The function Q is finite and locally constant on $D_1 \setminus K$, with value either 1 or 2.

PROOF. Consider a point y in this set $D_1 \setminus K$. Let $Y_1, \ldots, Y_M$ be any finite collection of points contained in $P_0^{-1}(\{y\}) \cap \operatorname{spt}\|\mathbf{C}_h\|$. Every point $Y_i \in \{Y_1, \ldots, Y_M\}$ is either a regular point of $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h$ or else is singular, in which case by construction $Y_i \in \mathcal{C}(\mathbf{C})$ and its unique tangent cone is of the form

$$|\Pi^1_{Y_i}| + |\Pi^2_{Y_i}| \in \operatorname{VarTan}(\mathbf{C}, Y_i),$$

where $\Pi^1_{Y_i}, \Pi^2_{Y_i} \in Gr_n$. There are then two possibilities, namely either $Y_i \in \mathcal{C}(\mathbf{C}) \cap \operatorname{reg}\mathbf{C}_h$, or $Y_i \in \mathcal{C}(\mathbf{C}) \cap \mathcal{C}(\mathbf{C}_h)$. In these two cases the density with respect to $\mathbf{C}_h$,

$$\Theta_i = \Theta(\|\mathbf{C}_h\|, Y_i)$$

is equal to one or two respectively. By Corollary 5.1.2 this corresponds precisely to whether one or both of the planes $\Pi^1_{Y_i}, \Pi^2_{Y_i}$ one or two of the planes $\Pi^1_{Y_i}, \Pi^2_{Y_i}$ are horizontal respectively. When only one is horizontal, it will simplify our notation somewhat to denote this plane $\Pi_{Y_i} \in \{\Pi^1_{Y_i}, \Pi^2_{Y_i}\}$.

Moreover, when $Y_i \in \operatorname{reg}\mathbf{C}$ then automatically $Y_i \in \operatorname{reg}\mathbf{C}_h$ as well, so

$$\Theta_{Y_i} = \Theta(\|\mathbf{C}_h\|, Y_i) \in \{1, 2\}$$

and there is a unique plane $\Pi_{Y_i} \in Gr(n, n+1)$ so that

$$\operatorname{VarTan}(\mathbf{C}_h, Y_i) = \Theta_{Y_i}|\Pi_{Y_i}|.$$

(When $\Theta_{Y_i} = 2$ the point Y_i is a false branch point, a possibility we allow in our analysis.) When this is the case, then the tangent plane Π_{Y_i} is horizontal, meaning its normal is not orthogonal to e_{n+1}, regardless of whether $\Theta_i = 1$ or 2. We write N_{Y_i} for the upward-pointing unit normal to this plane.

Next, consider one of the preimages $Y_i \in \{Y_1, \ldots, Y_M\}$ say, and take some small radius $\rho > 0$ so that the balls $B(Y_1, \rho), \ldots, B(Y_N, \rho)$ are two-by-two disjoint and $\cup_i \overline{B}(Y_i, \rho) \cap \operatorname{spt}\|\mathbf{C}\| \subset \operatorname{reg}\mathbf{C} \cup \mathcal{C}(\mathbf{C})$. For every $Y_i \in \{Y_1, \ldots, Y_M\}$ either $Y_i \in \operatorname{reg}\mathbf{C}_h$ and we write Σ_{Y_i} for the smooth minimal surface so that

$$\Theta_{Y_i}|\Sigma_{Y_i}| = |\mathbf{C}_h| \llcorner B(Y_i, \rho),$$

or else $Y_i \in \mathcal{C}(\mathbf{C}_h)$ and we write $\Sigma^1_{Y_i}, \Sigma^2_{Y_i}$ for the two smooth minimal surfaces so that

$$|\Sigma^1_{Y_i}| + |\Sigma^2_{Y_i}| = |\mathbf{C}_h| \llcorner B(Y_i, \rho).$$

Before continuing our argument, let us relabel these surfaces to avoid having to distinguish between different cases depending on whether or not

$Y_i \in \operatorname{reg}\mathbf{C}_h$. We simply write $\Sigma_1, \dots, \Sigma_D$ for the entire collection

$$\cup_{i=1}^{M}\{\Sigma_{Y_i}^j \mid Y_i \in \mathcal{C}(\mathbf{C}_h), j = 1, 2\} \cup \{\Sigma_{Y_i} \mid Y_i \in \operatorname{reg}\mathbf{C}_h\}$$

and $\Theta_1, \dots, \Theta_D$ for their respective multiplicities in $\mathbf{C}_h$. With this notation we have that, for example, $\cup_i \Sigma_{Y_i} = \cup_k \Sigma_k$ and $|\mathbf{C}_h| \llcorner \cup_i B(Y_i, \rho) = \sum_k \Theta_k |\Sigma_k|$. Further write $\Pi_1, \dots, \Pi_D$ for the corresponding tangent planes, and $N_1, \dots, N_D$ for their respective unit normals, which we all choose upward-pointing.

Next let $m = \min_k \langle N_k, e_{n+1}\rangle > 0$, and adjust $\rho > 0$ so that

$$\langle \nu(X), e_{n+1}\rangle \geq m/2 \tag{5.4}$$

for all $X \in \cup_k \Sigma_k = \operatorname{spt}\|\mathbf{C}_h\| \cap \cup_i B(Y_i, \rho)$, where we write ν for the upward-pointing unit normal to $\operatorname{reg}\mathbf{C}_h$.

Write $G_j = \operatorname{graph} u_j$ for the two-valued minimal graphs obtained by blowing down G, where we have already extracted a subsequence to ensure that $|G_j| \to \mathbf{C}$. Then using our relabelled notation, we have $|G_j| \llcorner \cup_i B(Y_i, \rho) \to \sum_k \Theta_k |\Pi_k|$ as $j \to \infty$. For every surface $\Sigma_k \in \{\Sigma_1, \dots, \Sigma_D\}$ write Y_{i_k} for the unique corresponding point in $\Sigma_k \cap \{Y_1, \dots, Y_M\}$. Then, potentially after adjusting the radius $\rho > 0$ we may take j large enough so that, depending on the density of the point Y_{i_k} we may distinguish between the following cases.

(1) If $\Theta(\|\mathbf{C}\|, Y_{i_k}) = 1$ then by Allard regularity, we can take j large enough that there is a smooth function $U_{jk} \in C^\infty(B_{2\rho}(Y_{i_k}) \cap \Pi_k; \Pi_k^\perp)$ so that $|G_j| \llcorner B_\rho(Y_{i_k}) = |\operatorname{graph} U_{jk}| \llcorner B_\rho(Y_{i_k}) \to |\Sigma_k|$ as $j \to \infty$.
(2) If $\Theta(\|\mathbf{C}\|, Y_{i_k}) = 2$ and $Y_{i_k} \in \mathcal{C}(\mathbf{C})$ then write $\Sigma_l \in \{\Sigma_1, \dots, \Sigma_D\}$ for the other surface with $Y_{i_k} \in \Sigma_l$. Then there exist two smooth functions $U_{jk}, U_{jl} \in C^\infty(B_{2\rho}(Y_{i_k}) \cap \Pi_k; \Pi_k^\perp)$ so that $|G_j| \llcorner B_\rho(Y_{i_k}) = \big(|\operatorname{graph} U_{jk}| + |\operatorname{graph} U_{jl}|\big) \llcorner B_\rho(Y_{i_k})$. Moreover $|\operatorname{graph} U_{jk}| \llcorner B_\rho(Y_{i_k}) \to |\Sigma_k|$ as $j \to \infty$.
(3) If $\Theta(\|\mathbf{C}\|, Y_{i_k}) = 2$ and $Y_{i_k} \in \operatorname{reg}\mathbf{C}$ then there is a two-valued function $U_{jk} \in C^{1,\beta}(B_{2\rho}(Y_{i_k}) \cap \Pi_k; \mathcal{A}_2(\Pi_k^\perp))$ for some $\beta \in (0,1)$ not depending on j, so that $|G_j| \llcorner B_\rho(Y_{i_k}) = |\operatorname{graph} U_{jk}| \llcorner B_\rho(Y_{i_k}) \to 2|\Sigma_k|$ as $j \to \infty$.

In all three of these cases, we may additionally impose that

$$\langle \nu_j(X), e_{n+1}\rangle \geq m/3 \text{ for all } X \in \operatorname{reg} G_j \cap \cup_i B_\rho(Y_i), \tag{5.5}$$

potentially after taking j even larger if necessary, as this is a strictly weaker version of the lower bound (5.4) that holds in the limit. Once this is done, we know there is $\theta = \theta(m) > 0$ so that for all $x \in D_{\theta\rho}(y)$ and all $k \in \{1, \dots, D\}$, $\operatorname{graph} U_{jk}$ contains Θ_k points lying above x (counted with multiplicity), and thus

$$\sum_{X\in P_0^{-1}(\{x\})} \Theta(\|G_j\|, X) \geq \sum_{k=1}^{M} \Theta_k. \tag{5.6}$$

On the other hand the graphs G_j are all two-valued, and therefore we get $\sum_k \Theta_k \leq 2$ from (5.6). As $\Theta_k \geq 1$ for all k this means that $M \leq 2$.

This way we have established that for all points $y \in D_1 \setminus P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v)$ there lie at most two points of $\operatorname{spt}\|G_j\|$ above y, meaning precisely that $Q(y) \leq 2$ for all such points. Our analysis shows that then the only two possible values are $Q(y) \in \{1, 2\}$.

To show that Q is locally constant in this set $D_1 \setminus P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v)$, pick an arbitrary point y lying in it. Although the arguments are essentially identical, we distinguish between cases depending on whether $P_0(\operatorname{sing} \mathbf{C})$ or not.

If $y \in D_1 \setminus P_0(\operatorname{sing} \mathbf{C})$ then there lie $q = Q(y) \in \{1, 2\}$ points, say $Y_1, \ldots, Y_q$ in $\operatorname{spt}\|\mathbf{C}\| \cap P_0^{-1}(\{y\})$. Then there is $\rho > 0$ so that $D_\rho(y) \subset D_1 \setminus P_0(\operatorname{sing} \mathbf{C})$, and consequently we can find q smooth minimal surfaces $\Sigma_1, \ldots, \Sigma_q$ so that $\mathbf{C}_h \llcorner D_\rho(y) \times \mathbf{R} = \sum_i |\Sigma_i|$. Then clearly for all other $z \in D_\rho(y)$ we also have $Q(z) = q = Q(y)$. (Note that in this analysis we allow for the possibility that there lies a unique, regular point $Y \in P_0^{-1}(\{y\}) \cap \operatorname{spt}\|\mathbf{C}\|$ which however has $\Theta(\|\mathbf{C}\|, Y) = 2$. Such a point is called a *false branch point*, and we would count such a point twice, meaning $q = 2$ and $Y_1 = Y = Y_2$.)

If instead $y \in D_1 \setminus P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v)$ is the projection of a singular point, then automatically $P_0^{-1} \cap \operatorname{spt}\|\mathbf{C}\| = \{Y\}$ and $Y \in (\mathbf{C})$. Moreover its density with respect to $\mathbf{C}_h$ is exactly $\Theta(\|\mathbf{C}_h\|, Y) = q = Q(y)$. Similar to the above, we may pick $\rho > 0$ so that $D_\rho(y) \subset D_1 \cap P_0(\operatorname{reg} \mathbf{C} \cup \mathcal{C}(\mathbf{C}))$. There are then q smooth embedded minimal surfaces $\Sigma_1, \ldots, \Sigma_q$ so that $\mathbf{C}_h \llcorner D_\rho(y) \times \mathbf{R} = \sum_i |\Sigma_i|$, and thus for all points $z \in D_\rho(y)$ we get $Q(z) = q = Q(y)$. □

Proof (of Lemma 5.1.5). As we are working with cones, we work over the unit disc without restricting our conclusions. Using the same notation as in Lemma 5.1.5, we let $K = P_0(\mathcal{S}^{n-2}(\mathbf{C}) \cup \mathcal{B}(\mathbf{C}) \cup \operatorname{sing} \mathbf{C}_v) \cap D_1$, which is a closed subset of D_1 with $\mathcal{H}^{n-1}(K) = 0$. In the same lemma we showed that the function Q, defined on the unit disc D_1 by the expression (5.2) is constant either equal to one or two.

We start by showing that necessarily $Q = 1$ on $D_1 \setminus K$, using a contradiction argument assuming that instead $Q = 2$ on the same set. Speaking in very rough terms, the conclusion will follow by examining the intersection of $\operatorname{spt}\|\mathbf{C}_v\|$ and $\operatorname{spt}\|\mathbf{C}_h\|$. By a blow-up argument we will derive a contradiction with our classification of tangent cones from the previous section. To obtain this we distinguish between the following two cases:

$$\text{either } \operatorname{reg}\mathbf{C}_v^0 \setminus P_0(\operatorname{sing}\mathbf{C}_h) \neq \emptyset \text{ or } \operatorname{reg}\mathbf{C}_v^0 \subset P_0(\operatorname{sing}\mathbf{C}_h). \tag{5.7}$$

We treat the former case first, and pick a point

$$y \in \operatorname{reg}\mathbf{C}_v^0 \cap D_1 \setminus P_0(\operatorname{sing}\mathbf{C}_h).$$

Because $Q(y) = 2$, this has two pre-images

$$Y_i \in \operatorname{reg}\mathbf{C}_h \cap P_0^{-1}(\{y\}) \quad i = 1, 2$$

which by assumption also belong to $\operatorname{reg}\mathbf{C}_v$. Using Corollary 5.1.2 we see that they must both lie on an axis along which of $\operatorname{reg}\mathbf{C}_h$ and $\operatorname{reg}\mathbf{C}_v$ intersect transversely, and we may write

$$|\Pi_i^v| + |\Pi_i^h| \in \operatorname{VarTan}(\mathbf{C}, Y_i)$$

where as our notation suggests

$$|\Pi_i^v| = |\Pi_i^{0,v}| \times \mathbf{R}e_{n+1} \in \operatorname{VarTan}(\mathbf{C}_v, Y_i)$$

and

$$|\Pi_i^h| \in \operatorname{VarTan}(\mathbf{C}_h, Y_i).$$

Recall that we write $|G_j| = |\operatorname{graph} u_j|$ for the blowdown sequence which converges to $\mathbf{C}$. As the graphs are stable, we may use the results of [Wic20] to conclude that there is a small radius $\rho > 0$ so that $B(Y_1, 2\rho) \cap B(Y_2, 2\rho) = \emptyset$ and there exist $U_{ji} \in C^\infty(\Pi_{Y_i}^h \cap B(Y_i, 2\rho))$ so that

$$|G_j| \llcorner B(Y_i, \rho) = |\operatorname{graph} U_{ji}| \llcorner B(Y_i, \rho) \quad i = 1, 2.$$

Write N_i^h for the upward unit normal to Π_i^h and let $m = \min_{i=1,2}\langle N_i^h, e_{n+1}\rangle > 0$. Then for large enough j we get

$$\langle \nu_j(X), e_{n+1}\rangle \geq m/3 \quad \text{ for all } X \in \operatorname{reg} G_j \cap \{B_\rho(Y_1) \cup B_\rho(Y_2)\},$$

arguing as above when we derived (5.5). As neither of the planes is vertical, we can find $\theta = \theta(m) > 0$ so that $\operatorname{graph} U_{ji} \cap D(y_i, \theta\rho) \times \mathbf{R} \subset B(Y_i, \rho)$ at least provided j is large enough, where we write $Y_i = (y_i, Y_i^{n+1})$ for both $i = 1, 2$. We moreover know that with respect to Hausdorff distance $\operatorname{sing} G_j \cap B_\rho(Y_i) \to (\Pi_i^h \cap \Pi_i^v) \cap B(Y_i, \rho)$ as $j \to \infty$. Focus on the first graph, pick a singularity $Z_1 \in \operatorname{sing} G_j \cap D_{\theta\rho}(Y_1) \times \mathbf{R}$ which also belongs to $Z_1 \in \operatorname{graph} U_{j1} \cap B_\rho(Y_1)$. Then there is also a point $Z_2 \in \operatorname{graph} U_{j2} \cap D_{\theta\rho}(y)$ with $P_0(Z_1) = P_0(Z_2)$. Adding their densities we obtain the absurd inequality

$$\sum_{Z \in P_0^{-1}(\{z\})} \Theta(\|G_j\|, Z) \geq \Theta(\|G_j\|, Z_1) + \Theta(\|G_j\|, Z_2) \geq 3.$$

We may now turn to the second case described in (5.7), where $\operatorname{reg}\mathbf{C}_v^0 \subset P_0(\operatorname{sing}\mathbf{C}_h)$. As $\mathcal{H}^{n-1}(\operatorname{reg}\mathbf{C}_v^0 \cap D_1) > 0$ but $\mathcal{H}^{n-1}(P_0(\operatorname{sing}\mathbf{C}_h \setminus \mathcal{C}(\mathbf{C}_h))) = 0$ we may pick a point

$$y \in \operatorname{reg}\mathbf{C}_v^0 \setminus P_0(\operatorname{sing}\mathbf{C}_h \setminus \mathcal{C}(\mathbf{C}_h)),$$

whence

$$P_0^{-1}(\{y\}) \cap \operatorname{spt}\|\mathbf{C}_h\| \subset \mathcal{C}(\mathbf{C}_h).$$

Indeed by assumption there lies at least one singular point $Y_1 \in P_0^{-1}(\{y\}) \cap \mathcal{C}(\mathbf{C}_h)$. If we assumed the existence of a point $Y_2 \in P_0^{-1}(\{y\}) \cap \operatorname{reg}\mathbf{C}_h$, we could derive a contradiction along much the same lines as above, in the first case of (5.7). Therefore $P_0^{-1}(\{y\}) \cap \operatorname{spt}\|\mathbf{C}_h\| \subset \operatorname{sing}\mathbf{C}_h$. In fact one can even use the same argument to prove the disjointness

$$P_0(\mathcal{C}(\mathbf{C}_h)) \cap P_0(\operatorname{sing}\mathbf{C}_h \setminus \mathcal{C}(\mathbf{C}_h)) = \emptyset, \tag{5.8}$$

and deduce that in particular $P_0^{-1}(\{y\}) \cap \operatorname{spt}\|\mathbf{C}_h\| \subset \mathcal{C}(\mathbf{C}_h)$.

Indeed, if $z \in D_1 \cap P_0(\mathcal{C}(\mathbf{C}_h))$ then automatically $P_0^{-1}(\{z\}) \cap \operatorname{spt}\|\mathbf{C}_h\| \subset \operatorname{sing}\mathbf{C}_h$, and by assumption there must be at least one classical, immersed singularity Z in this preimage. Again arguing as above we obtain that $P_0^{-1}(\{z\}) \cap \mathcal{C}(\mathbf{C}_h) = \{Z\}$. Write $|\Pi_{Z,1}| + |\Pi_{Z,2}| \in \operatorname{VarTan}(\mathbf{C}_h, Z)$ for the unique tangent cone to $\mathbf{C}_h$ at Z, and let

$$m_Z = \min\{\langle N_{Z,1}, e_{n+1}\rangle, \langle N_{Z,2}, e_{n+1}\rangle\} > 0.$$

Then pick $\rho > 0$ for large enough j, there are two smooth functions $U_{ji} \in C^\infty(\Pi_{Z,i} \cap B(Z, 2\rho); \Pi_{Z_i}^\perp)$ so that $|G_j| \llcorner B_\rho(Z) = |\operatorname{graph} U_{j1}| + |\operatorname{graph} U_{j2}| \llcorner B_\rho(Z)$. Next take $\theta = \theta(m) > 0$ so that above all $x \in D_{\theta\rho}(z)$ there lie a point in each of the two graphs, say $X_{ji} \in \operatorname{graph} U_{ji} \cap P_0^{-1}(\{x\})$. (Should $x \in P_0(\mathcal{C}(\mathbf{C}_h))$ then we allow them to coincide, $X_{j1} = X_{j2}$.)

Counting the densities of these points, we see that

$$D_{\theta\rho}(z) \times \mathbf{R} \cap \operatorname{spt}\|G_j\| \subset \operatorname{graph} U_{j1} \cup \operatorname{graph} U_{j2}. \tag{5.9}$$

To confirm that (5.8) holds, write $Z_1 = Z$ and assume that there exists a point $Z_2 \in P_0^{-1}(z) \cap \operatorname{sing}\mathbf{C}_h \setminus \mathcal{C}(\mathbf{C}_h)$. Then necessarily $\Theta(\|\mathbf{C}_h\|, Z_2) > 1$, and thus no matter how small $\sigma > 0$ is chosen we must have, at least for j large enough, that $G_j \cap B_\sigma(Z_2) \neq \emptyset$. If we choose $\sigma < \theta\rho$ so small that $B_\sigma(Z_2) \cap B_\rho(Z_1) = \emptyset$, then this contradicts (5.9). Thus there could be no point in $P_0^{-1}(z) \cap \operatorname{sing}\mathbf{C}_h \setminus \mathcal{C}(\mathbf{C}_h)$, and instead we must have (5.8).

Thus we may take a radius $\rho > 0$ small enough that

$$\operatorname{reg}\mathbf{C}_v^0 \cap D_\rho(y) \subset P_0(\mathcal{C}(\mathbf{C}_h)). \tag{5.10}$$

Next let $Y \in P_0^{-1}(\{y\}) \cap \mathcal{C}(\mathbf{C}_h)$ be the unique singular point lying above y, and denote its tangent cone

$$|\Pi_{Y,1}| + |\Pi_{Y,2}| \in \mathrm{VarTan}(\mathbf{C}_h, Y),$$

with planes $\Pi_{Y,1}, \Pi_{Y,2}$ horizontal. By assumption the point Y also belongs to $\operatorname{reg} \mathbf{C}_v$, with respect to which it has the tangent cone

$$|\Pi_Y^v| = |\Pi_y^{v,0}| \times \mathbf{R} e_{n+1} \in \mathrm{VarTan}(\mathbf{C}_v, Y).$$

Blowing up the cone $\mathbf{C}$ at Y, we see that the splitting $\mathbf{C} = \mathbf{C}_v + \mathbf{C}_h$ is reflected at the tangent level in the sense that the tangent cone is

$$|\Pi_{Y,1}| + |\Pi_{Y,2}| + |\Pi_Y^v| \in \mathrm{VarTan}(\mathbf{C}, Y).$$

We may deduce a contradiction with our classification of classical limit cones from this provided we can show that this indeed is a classical tangent cone. For this we need to prove that the three planes $\Pi_{Y,1}, \Pi_{Y,2}, \Pi_Y^v$ pairwise intersect along the same $(n-1)$-dimensional axis L_Y say. This is the case precisely if setting $L_Y = \Pi_{Y,1} \cap \Pi_{Y,2}$ we have

$$\Pi_y^{v,0} = P_0(L_Y). \tag{5.11}$$

As both spaces have the same dimension, this is equivalent to the inclusion $\Pi_y^{v,0} \subset P_0(L_Y)$, which can be obtained by blowing up the inclusion $\operatorname{reg} \mathbf{C}_v^0 \cap D(y,r) \subset P_0(\operatorname{imm} \mathbf{C})$. To confirm this rigorously, take a sequence $(\mu_k \mid k \in \mathbf{N})$ of positive rescaling factors $\mu_k \to 0$ so that simultaneously

$$(\eta_{\mu_k,y})_\#(\mathbf{C}_v^0 \llcorner D_{\theta\mu_k}(y)) \to |\Pi_y^{v,0}| \llcorner D_\theta$$
$$(\eta_{\mu_k,Y})_\#(\mathbf{C}_h \llcorner B_{\mu_k}(Y)) \to (|\Pi_{Y,1}^h| + |\Pi_{Y,2}^h|) \llcorner B_1,$$

where the constant θ is chosen in terms of the planes $\Pi_{Y,1}, \Pi_{Y,2}$ in the following manner, analogous to have been proceeding above. Adjust the value picked for $\rho > 0$ in (5.10) to have it small enough that we can find two smooth functions $U_{ji} \in C^\infty(\Pi_{Y,i} \cap B_{2\rho}(Y); \Pi_{Y_i}^\perp)$ so that $\mathbf{C}_h \llcorner B_\rho(Y) = (|\mathrm{graph}\, U_{j1} + \mathrm{graph}\, U_{j2}|) \llcorner B_\rho(Y)$. Decrease the value of $\rho > 0$ again to get that $\langle \nu(X), e_{n+1} \rangle \geq m_Y/2$, for all $X \in \operatorname{reg} \mathbf{C}_h \cap B_\rho(Y)$ where $m_Y = \min\{\langle N_{Y,1}, e_{n+1}\rangle, \langle N_{Y,2}, e_{n+1}\rangle\}$. By smoothness of the two graphs, there is a constant $\theta = \theta(m_Y) > 0$ so that for all points $x \in D_r(y)$ each graph contains precisely one preimage $Y_i \in P_0^{-1}(y) \cap \mathrm{graph}\, U_{ji} \cap B(Y_i, \rho)$. Arguing as above using the convergence of the two-valued graphs $|G_j| \llcorner D_{\theta\rho}(y) \to (|\mathrm{graph}\, U_{j1}| + |\mathrm{graph}\, U_{j2}|) \llcorner D_{\theta\rho}(y)$ as $j \to \infty$, we deduce that

$$\mathrm{spt}\|\mathbf{C}_h\| \cap D_{\theta r}(y) \times \mathbf{R} \subset (\mathrm{graph}\, U_{j1} \cup \mathrm{graph}\, U_{j2}) \cap B_r(Y)$$

for all $r \in (0, \rho)$. In particular this holds for $r = \mu_k$, provided this is small enough. Moreover, an analogous inclusion holds for their respective singular sets, namely

$$\operatorname{sing}\mathbf{C}_h \cap D_{\theta\mu_k}(y) \times \mathbf{R} \subset (\operatorname{graph} U_{j1} \cap \operatorname{graph} U_{j2}) \cap B_{\mu_k}(Y), \tag{5.12}$$

where perhaps it is useful to recall that these are all immersed, classical singularities.

Let a small $\delta > 0$ be given. Abbreviate

$$\widehat{\mathbf{C}}_{h,k} = (\eta_{\mu_k,Y})_\# \mathbf{C}_h \text{ for all } k \in \mathbf{N},$$

whence by assumption $\widehat{\mathbf{C}}_{h,k} \to |\Pi_{Y,1}| + |\Pi_{Y,2}|$ as $k \to \infty$. Because this limit is regular with multiplicity away from the axis L_Y where the two planes intersect, by Allard regularity we can choose k large enough to ensure that

$$\operatorname{sing}\widehat{\mathbf{C}}_{h,k} \cap B_1 = \mathcal{C}(\widehat{\mathbf{C}}_{h,k}) \cap B_1 \subset (L_Y)_\delta \cap B_1. \tag{5.13}$$

Abbreviate as well, for all $k \in \mathbf{N}$,

$$\widehat{\mathbf{C}}^0_{v,k} = (\eta_{\mu_k,y})_\# \mathbf{C}^0_v \in \mathbf{IV}_{n-1}(\mathbf{R}^n).$$

Again by assumption $\widehat{\mathbf{C}}^0_{v,k} \to |\Pi^{v,0}_y|$ as $k \to \infty$ because $y \in \operatorname{reg}\mathbf{C}^0_v$. In particular, their supporting sets converge locally with respect to Hausdorff distance, and we may take large enough k that

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\widehat{\mathbf{C}}^0_{v,k}\| \cap D_\theta, \Pi^{v,0}_y \cap D_\theta) \leq \delta.$$

Equivalently at the original scale

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}^0_v\| \cap D_{\theta\mu_k}(y), (y + \Pi^{v,0}_y) \cap D_{\theta\mu_k}(y)) \leq \delta\mu_k,$$

where $y + \Pi^{v,0}_y$ is the affine plane parallel to $\Pi^{v,0}_y$ attached to the point y. If we combine this with (5.10) and (5.13) we get that

$$(y + \Pi^{v,0}_y) \cap D_\theta \subset (\operatorname{reg}\mathbf{C}^0_v \cap D_{\theta\mu_k}(y))_{\delta\mu_k} \subset (P_0(\mathcal{C}(\mathbf{C}_h)) \cap D_{\theta\mu_k}(y) \times \mathbf{R})_{\delta\mu_k}. \tag{5.14}$$

(Note here that, as $y \in \operatorname{reg}\mathbf{C}^0_v$ the support of $\mathbf{C}^0_v$ coincides with its regular set near the point, $\operatorname{reg}\mathbf{C}^0_v \cap D_{\theta\mu_k}(y) = \operatorname{spt}\|\mathbf{C}^0_v\| \cap D_{\theta\mu_k}(y)$ at least provided $\mu_k\theta < \rho$. We may thus equivalently use either expression in our identities.)

By (5.12) the right-most set of (5.14) satisfies $P_0(\mathbf{C}_h) \cap D_{\theta\mu_k}(y) = P_0(\mathbf{C}_h \cap D_{\theta\mu_k}(y) \times \mathbf{R}) \subset P_0(\mathcal{C}(\mathbf{C}_h) \cap B_{\mu_k}(Y))$, whence this becomes

$$(y + \Pi^{v,0}_y) \cap D_1 \subset (P_0(\mathcal{C}(\mathbf{C}_h) \cap B_{\mu_k}(Y)))_{\delta\mu_k}. \tag{5.15}$$

The convergence of the singular set we obtain from (5.13) says that, for k as large as imposed there,

$$\mathcal{C}(\mathbf{C}_h) \cap B_{\mu_k}(Y) \subset (Y + L_Y)_{\delta\mu_k} \cap B_{\mu_k}(Y), \tag{5.16}$$

where recall we explained above that the singularities of $\mathbf{C}_h$ in $\operatorname{sing}\mathbf{C}_h \cap B_{\mu_k}(Y)$ are automatically classical singularities. Note moreover that this is the main step where (5.12) was useful, because it meant we can use the local convergence in the Hausdorff distance valid in compact domains to draw the conclusion above, namely (5.16).

As the projection P_0 is a continuous linear map of norm one we get combining (5.15) and (5.16) that

$$(y + \Pi_y^{v,0}) \cap D_1(y) \subset (P_0(Y + L_Y))_{2\delta} \cap D_1(y),$$

whence after translating to the origin

$$\Pi_y^{v,0} \cap D_1 \subset (P_0(L_Y))_{2\delta} \cap D_1.$$

As $\delta > 0$ was an arbitrarily small constant, we may let it tend to zero to obtain the desired inclusion

$$\Pi_y^{v,0} \cap D_{1/2} \subset P_0(L_Y) \cap D_1,$$

which confirms the validity of (5.11).

As we have already explained in the paragraph preceding this identity, this means that we have shown that the tangent cone to $\mathbf{C}$ at the point $Y \in \operatorname{sing} \mathbf{C}$ is a classical cone. This is a contradiction, regardless of whether $\Pi_Y^{v,0}$ is distinct from the other two cones or not, because in any case one would have

$$\| |\Pi_{Y,1}^h| + |\Pi_{Y,2}^h| + |\Pi_Y^v| \|(B_1)/\omega_n = 3,$$

which our classification of classical limit cones of two-valued graphs demonstrates is impossible, see Corollary 4.4.6.

This concludes the argument in the second of the two cases described in (5.7). Therefore, if both $\mathbf{C}_v, \mathbf{C}_h \neq 0$ then

$$Q = 1 \text{ on } D_1 \setminus K, \tag{5.17}$$

with $K \subset D_1$ as in Lemma 5.1.6. From this we can draw the conclusion that the support of $\mathbf{C}_h$ is smooth embedded inside the open set $(D_1 \setminus K) \times \mathbf{R}$. In fact we can go further.

Claim 6. There is a smooth function $U_h \in C^\infty(D_1 \setminus K)$, satisfying the minimal surface equation, so that $|\mathbf{C}_h| \llcorner (D_1 \setminus K) \times \mathbf{R} = |\operatorname{graph} U_h|$.

Proof. Let $y \in D_1 \setminus K$ be any point. Because of (5.17) there lies precisely one point $Y \in P_0^{-1}(\{y\}) \operatorname{spt}\|\mathbf{C}_h\|$. Moreover, as $\Theta(\|\mathbf{C}_h\|, Y) = 1$ this point must be regular, and we may write $|\Pi_Y| \in \operatorname{VarTan}(\mathbf{C}_h, Y)$ for its unique tangent plane at the point. Either by inspection or using Corollary 5.1.2 we see that this cannot be vertical, meaning we may write N_Y for the upward-pointing normal. Thus, using the Allard regularity theorem there is a radius $\rho = \rho_Y > 0$ and a smooth function $U_Y \in C^\infty(\Pi_Y \cap B_{2\rho}; \Pi_Y^\perp)$

so that at least in a small ball $|\mathbf{C}_h| \llcorner B_\rho(Y) = |\operatorname{graph} U_Y| \llcorner B_\rho(Y)$. Write moreover $m = m_Y = \langle N_Y, e_{n+1}\rangle > 0$, and take $\theta = \theta_Y > 0$ small enough that above every $x \in D_{\theta\rho}(y)$ there lies precisely one point of $\operatorname{graph} U_Y$. Write $Y = (y, Y^{n+1})$ and define a function $U_h : D_1 \setminus K \to \mathbf{R}$ by setting $U_h(y) = Y^{n+1}$. Then at least locally we are guaranteed to have $|\operatorname{graph} U_h| \llcorner D_{\theta\rho}(y) \times \mathbf{R} = |\operatorname{graph} U_Y| \llcorner D_{\theta\rho}(y) \times \mathbf{R}$. This shows that U_h is smooth. Patching these local observations together we see that $|\operatorname{graph} U_h| = \mathbf{C}_h \llcorner (D_1 \setminus K) \times \mathbf{R}$. In particular the varifold $|\operatorname{graph} U_h|$ is stationary inside this region, and U_h is a classical solution of the minimal surface equation on $D_1 \setminus K$. □

Recall moreover from Lemma 5.1.6 that $\mathcal{H}^{n-1}(K) = 0$. Moreover K is a closed subset of D_1, and thus is locally compact. This places us precisely in the situation described by L. Simon in [Sim77]. The results proved there demonstrate that u can be extended smoothly across K, and thus there is a smooth, classical solution of the minimal surface equation defined on D_1 which restricts to U_h on $D_1 \setminus K$. Denote his extension of U_h to D_1 by U_h also. Since U_h is homothety-invariant on $D_1 \setminus K$, this must in fact hold in the entire unit disc. It is then a standard fact that U_h must in fact be linear, and its image is a plane, say $|\operatorname{graph} U_h| = |\Pi_h|$ where $\Pi_h \in Gr(n, n+1)$. Then also $\mathbf{C}_h = |\Pi_h|$, which is precisely what was to prove in Lemma 5.1.5. □

REMARK 5.1.7. The work of Simon in [Sim77] is a generalisation of an earlier result of de Giorgi and Stampacchia, who in [DGS65] manage to draw the same conclusion, but only under the stronger condition that K is compact. Allowing locally compact K, as Simon does, means that the subset $K \subset D_1$ can extend all the way up to the boundary of the domain. This is crucial in the application above. Both of these results could have been replaced by the regularity theory of Wickramasekera [Wic14a], or indeed that of Schoen–Simon [SS81]. We briefly sketch this alternative argument. If we had allowed ourselves to use the (significantly stronger) former results, then we simply need to make the following observation. If $\mathcal{C}(\mathbf{C}_h)$ were non-empty, then necessarily $\mathcal{C}(\mathbf{C}_h) \cap (D_1 \setminus K) \times \mathbf{R}$ because $\mathcal{H}^{n-1}(P_0(\mathbf{C}_h) \cap D_1) > 0$ but $\mathcal{H}^{n-1}(K) = 0$. This however would contradict would contradict the fact that $Q = 1$ on $D_1 \setminus K$, derived in (5.17). Therefore necessarily $\mathcal{C}(\mathbf{C}_h) = \emptyset$, and the results of [Wic14a] yield immediately that $\mathbf{C}_h$ is smooth, and thus must be a plane. If we had tried instead to only use the weaker version of these results from [SS81], we only need to notice the following. The singular set of $\mathbf{C}_h$ stratifies as usual, like $\mathcal{S}^{i-1}(\mathbf{C}_h) \subset \mathcal{S}^i(\mathbf{C})$ for all $i = 1, \ldots, n$. The conclusions of Corollary 4.4.7 do not quite apply here, because $\mathbf{C}_h$ itself is not a limit of two-valued graphs, only $\mathbf{C} = \mathbf{C}_h + \mathbf{C}_v$ is. This complication notwithstanding, we already know that $(\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathbf{C}_h) = \emptyset$. Moreover by inspection $\mathcal{B}(\mathbf{C}_h) \subset \mathcal{B}(\mathbf{C}) \cup \mathcal{S}^{n-2}(\mathbf{C})$ and is thus countably $n-2$-rectifiable, and for all the lower strata, that is $i \in \{0, \ldots, n-2\}$, we have $\mathcal{S}^i(\mathbf{C}_h) \subset \mathcal{S}^i(\mathbf{C})$. The statements of [SS81] still do not quite apply, because they call for $\mathcal{H}^{n-2}(\operatorname{sing} \mathbf{C}_h) = 0$, but going through their argument one sees that it suffices to have $\operatorname{sing} \mathbf{C}_h$ countably $n-2$–rectifiable, as is the case here.

5.2. The Bernstein theorem in four dimensions

The aim of this section is to prove the following result.

THEOREM 5.2.1. *Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(\mathbf{R}^3; \mathcal{A}_2)$ define an entire two-valued minimal graph. Then u is linear.*

5.2.1. Stability and the logarithmic cutoff trick. Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(\mathbf{R}^3; \mathcal{A}_2)$ be an entire two-valued minimal graph. If we had $\operatorname{sing} G = \emptyset$ then G would decompose into a disjoint union of two smooth, single-valued entire minimal graphs. Moreover by the classical, single-valued Bernstein theorem they would be two parallel planes. From now on $\operatorname{sing} G \neq \emptyset$. In fact we may assume $0 \in \operatorname{sing} G$ after translating the graph, in which case $\Theta(\|G\|, 0) = 2$. Let $\mathbf{C} \in \mathbf{IV}_3(\mathbf{R}^4)$ be a blowdown cone of $|G|$ at infinity. By the monotonicity formula, the desired conclusion follows by showing $\Theta(\|\mathbf{C}\|, 0) = \lim_{R \to \infty} \|G\|(B_R)/(\omega_3 R^3) = 2$. Indeed then G is a union of two, possible equal planes and u is linear.

By Theorem 5.1.3 the cone $\mathbf{C}$ must take one of the following three forms:

$$\mathbf{C}^0 \times \mathbf{R}e_4, |\Pi_1| + \mathbf{C}^0 \times \mathbf{R}e_4 \text{ or } |\Pi_1| + |\Pi_2|, \tag{5.1}$$

where Π_1, Π_2 are two non-vertical, possibly equal planes, and $\mathbf{C}^0 \in \mathbf{IV}_2(\mathbf{R}^3)$ is a stationary integral cone. The remainder of this section is dedicated to excluding the first two cases, that is necessarily $\mathbf{C} = |\Pi_1| + |\Pi_2|$. The proof starts with the observation that $\mathbf{C}^0$ inherits the ambient stability from $\mathbf{C}$, which allows the application of the so-called *logarithmic cutoff trick*.

LEMMA 5.2.2. *Let $V \in \mathbf{IV}_2(\mathbf{R}^3)$ be a stationary integral varifold. Suppose that V is ambient stable in the sense of (S_V), and that it has quadratic area growth,*

$$\|V\|(B_R) \leq \Lambda R^2 \textit{ for all } R > 0$$

for some constant $\Lambda > 0$. Then $|A_V|(X) = 0$ at all $X \in \operatorname{reg} V$.

PROOF. This follows from a classical application of the so-called logarithmic cut-off trick. The details of the argument used below are taken essentially verbatim from [CM11], and are reproduced solely for the convenience of the reader. Let $N \in \mathbf{N}$ be a large integer, and define the radial compactly supported function

$$\eta(r) = \begin{cases} 1 & \text{if } r \leq \mathrm{e}^N, \\ 2 - \log r/N & \text{if } \mathrm{e}^N < r \leq \mathrm{e}^{2N}, \\ 0 & \text{if } r > \mathrm{e}^{2N}. \end{cases}$$

We may use this function in the stability inequality (S_V), which yields $\int |A_V|^2 \eta^2 \,\mathrm{d}\|V\| \leq \int |\nabla_V \eta|^2 \,\mathrm{d}\|V\|$. Then note that for all $l = N+1, \ldots, 2N$, $\sup_{B_{\mathrm{e}^l} \setminus B_{\mathrm{e}^{l-1}}} |\nabla_V \eta|^2 \leq N^{-2} \mathrm{e}^{2-2l}$. Therefore

$$\int_{B_{\mathrm{e}^{2N}} \setminus B_{\mathrm{e}^N}} |\nabla_V \eta|^2 \,\mathrm{d}\|V\| \leq \sum_{l=N+1}^{2N} N^{-2} \mathrm{e}^{2-2l} \|V\|(B_{\mathrm{e}^l}) \leq \Lambda \mathrm{e}^2 N^{-1}.$$

Substituting this estimate into the stability inequality we get

$$\int_{B_{\mathrm{e}^N}} |A_V|^2 \,\mathrm{d}\|V\| \leq \Lambda \mathrm{e}^2 N^{-1}$$

Finally, letting $N \to \infty$ one obtains that $\int_{\mathbf{R}^3} |A_V|^2 \,\mathrm{d}\|V\| = 0$, and therefore that $A_V \equiv 0$ on $\operatorname{reg} V$. □

Applying this to $\mathbf{C}^0 \in \mathbf{IV}_2(\mathbf{R}^3)$ we find that it is totally geodesic, that is $A_{\mathbf{C}^0} \equiv 0$ on $\operatorname{reg} \mathbf{C}^0$.

LEMMA 5.2.3. *Suppose $n \geq 2$. Let $\mathbf{C} \in \mathbf{IV}_n(\mathbf{R}^{n+1})$ be a stationary integral cone, with support immersed outside of the origin and $|A_{\mathbf{C}}| \equiv 0$ on $\operatorname{reg} \mathbf{C}$. Then $\mathbf{C}$ is supported in a finite union of n-dimensional planes.*

PROOF. Let $\Pi \subset \mathbf{R}^{n+1}$ be any n-dimensional linear plane with $\operatorname{reg} \mathbf{C} \cap \Pi \neq \emptyset$. The set $\Pi \setminus \{0\}$ is connected because $n \geq 2$. As $\operatorname{spt}\|\mathbf{C}\| \cap \Pi \setminus \{0\}$ is a relatively closed subset of $\Pi \setminus \{0\}$, we only need to show that it is open to obtain $\Pi \subset \operatorname{spt}\|\mathbf{C}\|$. A point $X \in \operatorname{spt}\|\mathbf{C}\| \cap \Pi \setminus \{0\}$ is either regular or an immersed classical singularity. In both cases the fact that $|A_{\mathbf{C}}| \equiv 0$ on $\operatorname{reg} \mathbf{C}$ means that near X the support of $\mathbf{C}$ is either a plane or a union of planes. By assumption $X \in \Pi$, so one of these planes must be Π itself. This shows that $\operatorname{spt}\|\mathbf{C}\| \cap \Pi \setminus \{0\}$ is open inside $\Pi \setminus \{0\}$, and thus $\Pi \subset \operatorname{spt}\|\mathbf{C}\|$. Repeating this, we find a finite collection of planes $\Pi_1, \ldots, \Pi_D$ so that $\mathcal{H}^n(\operatorname{spt}\|\mathbf{C}\| \setminus \cup_i \Pi_i) = 0$, and hence $\operatorname{spt}\|\mathbf{C}\| \subset \cup_i \Pi_i$ using the monotonicity formula. □

To apply Lemma 5.2.3 to $\mathbf{C}^0$ we first need to show that it is immersed outside the origin. This is essentially a consequence of Corollary 4.4.7, which lays out the following two possibilities.

(1) Either $\Theta(\|\mathbf{C}\|, X) = 2$ for $\mathcal{H}^3$-a.e. $X \in \operatorname{reg} \mathbf{C}$, and then $\mathbf{C}$ is smooth embedded. This makes it impossible that $\mathbf{C} = \mathbf{C}^0 \times \mathbf{R}e_4 + \Pi_1$, and in the case where $\mathbf{C} = \mathbf{C}^0 \times \mathbf{R}e_4$ we find that $\mathbf{C}^0 = 2|\Pi^0|$ for some

two-dimensional plane $\Pi^0 \subset \mathbf{R}^3$. Thus $\mathbf{C} = 2|\Pi^0 \times \mathbf{R}e_4|$, and we can conclude by the monotonicity formula.

(2) The second possibility is that the density of $\mathbf{C}$ is $\mathcal{H}^3$-a.e. equal one, and thus automatically also $\Theta(\|\mathbf{C}^0\|, X) = 1$ for $\mathcal{H}^2$-a.e. $X \in \operatorname{reg}\mathbf{C}^0$. Recall that the singular set of $\mathbf{C}^0$ is stratified like $\mathcal{S}^0(\mathbf{C}^0) \subset \mathcal{S}^1(\mathbf{C}^0) \subset \mathcal{S}^2(\mathbf{C}^0)$. Invoking Corollary 4.4.7 again we find that that $\operatorname{spt}\|\mathbf{C}^0\|$ is immersed near points of $\mathcal{S}^1(\mathbf{C}^0) \setminus \mathcal{S}^0(\mathbf{C}^0)$, and the remaining singularities necessarily have $\mathcal{S}^0(\mathbf{C}^0) \cup \mathcal{B}(\mathbf{C}^0) \subset \{0\}$.

In both cases $\mathbf{C}^0$ is immersed outside the origin, and by Lemma 5.2.3 we find that $\mathbf{C}^0$ is supported in a union of planes. In the remainder we need only consider the second possibility, where the density of $\mathbf{C}^0$ at all regular points is one, and the cone is equal to a sum of planes, all of which are vertical and have multiplicity one. (Note Lemma 5.2.3 only gives that $\mathbf{C}^0$ is supported in a union of planes.) In the remainder we write $\mathbf{C}^0 = \mathbf{P}^0 = \sum_{j=2}^{D} |\Pi_j^0|$ to reflect this. At this stage of the proof we have reduced the possible forms of the blowdown cone given in (5.1) to

$$\mathbf{P}^0 \times \mathbf{R}e_4 \text{ or } |\Pi_1| + \mathbf{P}^0 \times \mathbf{R}e_4.$$

The two require different approaches, and we treat the latter first.

5.2.2. Non-vertical blowdown cones. Write $\mathbf{L}^0 \subset \mathbf{R}^3$ for the union of one-dimensional lines along which the planes in the support of $\mathbf{P}^0$ meet. The singularities of the blowdown cone are $\operatorname{sing}\mathbf{C} = (\Pi_1 \cap \operatorname{spt}\|\mathbf{P}^0 \times \mathbf{R}e_4\|) \cup \mathbf{L}^0 \times \mathbf{R}e_4$, and those lying in $\Pi_1 \cap \operatorname{spt}\|\mathbf{C}\| \setminus (\mathbf{L}^0 \times \mathbf{R}e_4)$ are all immersed. Let $\tau > 0$ be given. Using Allard's regularity theorem near the points of $\Pi_1 \cap \operatorname{reg}\mathbf{C}$ and Wickramasekera's stable sheeting theorem near those in $\Pi_1 \cap \operatorname{sing}\mathbf{C} \setminus (\mathbf{L}^0 \times \mathbf{R}e_4)$ we find the existence of a smooth function $u_{j,1} \in C^\infty(D_1 \setminus [\mathbf{L}^0]_\tau)$ with $G_{j,1} = \operatorname{graph} u_{j,1} \subset G_j \cap (\Pi_1)_\tau$, at least provided $j \geq J(\tau)$ is large enough. We obtain a smooth selection $u_{j,1}, u_{j,2} \in C^\infty(D_1 \setminus [\mathbf{L}^0]_\tau)$ by picking the remaining value of u_j for $u_{j,2}$ above every point. (It is not enough to observe that eventually $\mathcal{B}_{u_j} \cap D_1 \subset [\mathbf{L}^0]_\tau$, as the set $D_1 \setminus [\mathbf{L}^0]_\tau$ is not simply connected regardless of how small $\tau > 0$ is.) As $\tau \to 0$ and $j \geq J(\tau) \to \infty$ we find that by construction $|G_j^1| \to |\Pi_1|$ and $|G_j^2| \to \mathbf{P}^0 \times \mathbf{R}e_4$. As the G_j^2 are all single-valued graphs their limit $\mathbf{P}^0$ is supported in a single plane, say $\mathbf{P}^0 = |\Pi_2^0|$. (There are various ways of confirming this in more detail, all boiling down to the fact that $\mathbf{P}^0$ cannot be the limit of a sequence of area-minimising currents if it is supported in more than one plane. To give but one example, revisiting the arguments used to prove the improved area estimates we obtain that there is $\delta > 0$ so that $\|\mathbf{P}^0\|(D_1) \leq (2-\delta)\omega_2$.) Therefore $|G_j| \to |\Pi_1| + |\Pi_2|$, as desired.

5.2.3. Vertical blowdown cones: the adjacency graph. Here the blowdown sequences converges to a vertical cone, $|G_j| \to \mathbf{P} = \mathbf{P}^0 \times \mathbf{R}e_4 = \sum_{j=1}^{D} |\Pi_j^0| \times \mathbf{R}e_4$. In the current topology $[\![G_j]\!] \to \sum_{j=1}^{D} [\![\Pi_j^0]\!] \times \mathbf{R}e_4$ where the planes are respectively oriented by unit normals $n_1, \dots, n_D$. The improved area estimates give $D \leq 3$, see Corollary 3.3.4.

The only problematic value is $D = 3$. We exclude this by a combinatorial argument, constructing what we call the *adjacency graph* by a kind of dual cellular decomposition. The planes Π_1, Π_2, Π_3 divide $\mathbf{R}^3$ into a finite number of connected components $\Omega_1, \dots, \Omega_N \subset \mathbf{R}^3$. These are all polyhedral, with respective boundaries $\partial\Omega_1, \dots, \partial\Omega_N \subset \Pi_1 \cup \Pi_2 \cup \Pi_3$. We decompose these into faces, edges and vertices, and say that two regions $\Omega \neq \Omega'$ are adjacent if they meet along a face. To every component Ω we associate a vertex v, forming a set V. Connect two distinct vertices $v, v' \in V$ by an edge e if the corresponding regions Ω, Ω' are adjacent. If Ω, Ω' are adjacent then they meet along a single plane Π_i. We orient e so that it agrees with the orientation of this plane, meaning if n_i points away from Ω and into Ω' then e is directed from v to v' and vice-versa. Thus we obtain a set of directed edges denoted E. We call the finite, directed graph $H = (V, E)$ the *adjacency graph* of $\mathbf{P}$. Label the vertices of the graph by a function $F : V \to \{0, 1, 2\}$ which returns the number $F(v)$ of sheets of G_j eventually lying over the corresponding region $\Omega \cap D_1$. This is well-defined by Lemma 4.1.5 for example. Let $v, v' \in V$ be two adjacent vertices, and suppose that e points from v to v'. By Lemma 4.4.2, $F(v') = F(v) + 1$. As an immediate consequence we find that H cannot contain directed paths of lengths more than two. Indeed if H contained three edges e_1, e_2, e_3 so that e_i points from v_i to v_{i+1} then $F(v_4) = F(v_1) + 3$, which is absurd.

There are essentially only two ways in which the planes Π_1, Π_2, Π_3 can be arranged. Let $\Pi_3^1 = \{x \in \mathbf{R}^3 \mid \langle x, n_3 \rangle \equiv 1\} \subset \mathbf{R}^3$ be the affine plane parallel to Π_3^0 at height one. The two planes Π_1^0 and Π_2^0 intersect this transversely in a pair of affine lines l_1, l_2. If these lines were parallel, then the planes Π_1, Π_2, Π_3 would meet along a common axis, making $\mathbf{P} = \mathbf{P}^0 \times \mathbf{R}e_4$ a classical cone. As we have already dealt with these, we may assume this is not the case.

Hence we may assume the two lines l_1, l_2 intersecting, and compute the adjacency graph. The set $\Pi_3^1 \setminus (l_1 \cup l_2)$ has four connected components. Each of these leads to a pair of adjacent vertices in V, which correspond to regions meeting along a face in Π_3. Thus H contains eight vertices, arranged as four pairs of vertices lying on either side of Π_3. Additionally the four vertices corresponding to the regions contained inside $\{x \in \mathbf{R}^3 \mid \langle x, n_3 \rangle > 0\}$ are arranged in a square in H, with parallel edges oriented in the same direction. The same holds for the regions lying in the half-space $\{x \in \mathbf{R}^3 \mid \langle x, n_3 \rangle < 0\}$.

In short, H is a cube with eight vertices and twelve edges, with parallel edges pointing in the same direction. As this graph contains a directed path of length three, we have reached a contradiction.

CHAPTER 6

MORSE INDEX, MINIMAL SURFACES, AND THE ALLEN–CAHN EQUATION

6.1. STABILITY AND STATEMENT OF THE MAIN THEOREM

The setting is as follows: (M^{n+1}, g) is a closed (that is, compact without boundary) Riemannian manifold of dimension $n+1 \geq 3$, and $U \subset M$ is an arbitrary open subset, possibly equal to M itself.

6.1.1. Stability and the scalar Jacobi operator. Throughout this section V will be a stationary integral n-varifold in $U \subset M$. We call V *two-sided* if its regular part $\operatorname{reg} V$ is two-sided, that is if the normal bundle $NV := N(\operatorname{reg} V)$ admits a continuous non-vanishing section. When this fails, V is called *one-sided.* (Recall that when the ambient manifold M is orientable, then $\operatorname{reg} V$ is two-sided if and only if it is orientable.)

Suppose that V is two-sided, and fix a unit normal vector field $N \in C^1(NV)$, so that every function $\phi \in C^1_c(\operatorname{reg} V)$ corresponds to a section $\phi N \in C^1_c(NV)$ and vice-versa. After extending the vector field ϕN to $C^1_c(U, TM)$—the chosen extension will not matter for our purposes—we can deform $\operatorname{reg} V$ with respect to its flow (Φ_t). As V is stationary, the first variation vanishes: $\delta V(\phi N) = 0$. A routine calculation, the details of which can be found for instance in [Sim84, Ch. 2] shows that the second variation satisfies

$$\delta^2 V(\phi N) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \|(\Phi_t)_* V\|(U)$$
$$= \int_U |\nabla_V \phi|^2 - (|A|^2 + \operatorname{Ric}_M(N,N))\phi^2 \,\mathrm{d}\|V\|,$$

where ∇_V is the Levi-Civita connection on $\operatorname{reg} V$, A is the second fundamental form of $\operatorname{reg} V \subset M$, and Ric_M is the Ricci curvature tensor on M.

The expression on the right-hand side can be defined for one-sided V by replacing N by an arbitrary measurable unit section $\nu : \operatorname{reg} V \to NV$, but it loses its interpretation in terms of the second variation of the area.

DEFINITION 6.1.1 (Scalar second variation). The *scalar second variation* of a stationary integral varifold V is the quadratic form B_V defined for

$\phi \in C^1_c(\operatorname{reg} V)$ by

$$B_V(\phi,\phi) = \int_{\operatorname{reg} V} |\nabla_V \phi|^2 - (|A|^2 + \operatorname{Ric}_M(\nu,\nu))\phi^2 \, \mathrm{d}\|V\|.$$

REMARK 6.1.2. When V is one-sided, the second variation of its area has to be measured with respect to variations in $C^1_c(NV)$—we refer to [Sim84, Ch. 2] or [CM11, Sec. 1.8] for further information on this. We called B_V *scalar* in order to highlight its difference with the second variation of area in this case, but emphasise that for the remainder *second variation* refers exclusively to the quadratic form B_V from Definition 6.1.1. (For the same reasons we also call the Jacobi operator L_V *scalar* in Definition 6.1.3 below, but omit this adjective in the remainder of the text.)

One can consider $\operatorname{reg} V$ as a stationary integral varifold in its own right by identifying it with the corresponding varifold with constant multiplicity 1. Its scalar second variation

$$B_{\operatorname{reg} V}(\phi,\phi) = \int_{\operatorname{reg} V} |\nabla_V \phi|^2 - (|A|^2 + \operatorname{Ric}_M(\nu,\nu))\phi^2 \, \mathrm{d}\mathcal{H}^n$$

differs from B_V only in that the integral is with respect to the n-dimensional Hausdorff measure instead of $\|V\|$. This means exactly that while B_V is *weighted* by the multiplicity of V, the quadratic form $B_{\operatorname{reg} V}$ measures the variation of 'unweighted' area; we will briefly use this in Section 6.2.2.

After integrating by parts on $\operatorname{reg} V$, the form B_V corresponds to the second-order elliptic operator $L_V = \Delta_V + |A|^2 + \operatorname{Ric}_M(\nu,\nu)$, where Δ_V is the Laplacian on $\operatorname{reg} V$.

DEFINITION 6.1.3 (Scalar Jacobi operator). The *scalar Jacobi operator* of V, denoted L_V, is the second-order elliptic operator

$$L_V \phi = \Delta_V \phi + (|A|^2 + \operatorname{Ric}_M(\nu,\nu))\phi \quad \text{for all } \phi \in C^2(\operatorname{reg} V),$$

where $\nu : \operatorname{reg} V \to NV$ is an arbitrary measurable unit normal vector field.

The curvature of $\operatorname{reg} V$ can blow up as one approaches $\operatorname{sing} V$, in which case the coefficients of the operator L_V would not be bounded. To avoid this, we restrict ourselves to a compactly contained open subset $W \subset\subset U \setminus \operatorname{sing} V$; moreover we require $W \cap \operatorname{reg} V \neq \emptyset$ to avoid vacuous statements.

We use the sign convention for the spectrum defined in [GT98, Ch. 8], where $\lambda \in \mathbf{R}$ is an eigenvalue of L_V in W if there is $\varphi \in H^1_0(W \cap \operatorname{reg} V)$ such that $L_V \varphi + \lambda \varphi = 0$. By standard elliptic PDE theory the spectrum

$$\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$$

of L_V in W is discrete and bounded below. We will sometimes also write $\lambda_p(W)$ instead of λ_p in order to highlight the dependence of the spectrum on the subset W. The eigenvectors of L_V span the space $H^1_0(W \cap \operatorname{reg} V) = W^{1,2}_0(W \cap \operatorname{reg} V)$, which we abbreviate throughout by H^1_0.

The *index of* B_V in W is the maximal dimension of a subspace of H^1_0 on which B_V is negative definite; equivalently

$$\operatorname{ind}_W B_V = \operatorname{card}\{p \in \mathbf{N} \mid \lambda_p(W) < 0\}.$$

Moreover $\operatorname{ind} B_V := \sup_W(\operatorname{ind}_W B_V)$, where the supremum is taken over all $W \subset\subset U \setminus \operatorname{sing} V$ with $W \cap \operatorname{reg} V \neq \emptyset$.

REMARK 6.1.4. We will see in Section 6.2.2 that the index of B_V coincides with the Morse index of $\operatorname{reg} V$ with respect to the area functional, at least when $\operatorname{reg} V$ is two-sided.

6.1.2. Statement of main theorem. Let (ϵ_i) be a sequence of positive parameters with $\epsilon_i \to 0$ and consider an associated sequence of functions (u_i) in $C^3(U)$ satisfying the following hypotheses:

(A) Every $u_i \in C^3(U)$ is a critical point of the Allen–Cahn functional

$$E_{\epsilon_i}(u) = \int_U \epsilon_i \frac{|\nabla u|^2}{2} + \frac{W(u)}{\epsilon_i} \,\mathrm{d}\mathcal{H}^{n+1},$$

that is u_i satisfies the equation $-\epsilon_i^2 \Delta u_i + W'(u_i) = 0$ in U.

(B) There exist constants $C, E_0 < \infty$ such that

$$\sup_i \|u_i\|_{L^\infty(U)} \leq C \text{ and } \sup_i E_{\epsilon_i}(u_i) \leq E_0.$$

(C) There exists an integer $k \geq 0$ such that the Morse index of each u_i is at most k, i.e. any subspace of $C^1_c(U)$ on which the second variation

$$\delta^2 E_{\epsilon_i}(u_i)(\phi,\phi) = \int_U \epsilon_i |\nabla \phi|^2 + \frac{W''(u_i)}{\epsilon_i}\phi^2 \,\mathrm{d}\mathcal{H}^{n+1}$$

is negative definite has dimension at most k. We write this $\operatorname{ind} u_i \leq k$, and if $k = 0$, say that u_i is *stable in* U.

REMARK 6.1.5. More generally $\operatorname{ind}_{U'} u_i$ denotes the index of $\delta^2 E_{\epsilon_i}(u_i)$ with respect to variations in $C^1_c(U')$ (or equivalently in $H^1_0(U')$) for all open subsets $U' \subset U$. When $\operatorname{ind}_{U'} u_i = 0$, we say that u_i is *stable in* U'.

We follow Tonegawa [Ton05], using an idea originally developed by Ilmanen [Ilm93] in a parabolic setting to 'average the level sets' of $u_i \in C^3(U)$ and define a varifold V^i by

$$V^i(\phi) = \frac{1}{\sigma}\int_{U \cap \{\nabla u_i \neq 0\}} \epsilon_i \frac{|\nabla u_i(x)|^2}{2} \phi(x, T_x\{u_i = u_i(x)\}) \,\mathrm{d}\mathcal{H}^{n+1}(x) \tag{6.1}$$

for all $\phi \in C_c(Gr_n(U))$. Here $T_x\{u_i = u_i(x)\}$ is the tangent space to the level set $\{u_i = u_i(x)\}$ at $x \in U$, and $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\,\mathrm{d}s$ is a constant.

REMARK 6.1.6. In [HT00,Gua18] the varifold V^i is defined by the expression $V^i(\phi) = \frac{1}{\sigma}\int_{U\cap\{\nabla u_i \neq 0\}} |\nabla w_i(x)| \phi(x, T_x\{u_i = u_i(x)\})\,\mathrm{d}\mathcal{H}^{n+1}(x)$, with w_i as in Theorem 6.1.7. The 'equipartition of energy' (6.3) from Theorem 6.1.7 shows that the two definitions give rise to the same limit varifold V as $i \to \infty$.

The weight measures $\|V^i\|$ of these varifolds satisfy

$$\|V^i\|(A) = \frac{1}{\sigma}\int_{A\cap\{\nabla u_i \neq 0\}} \epsilon_i \frac{|\nabla u_i|^2}{2}\,\mathrm{d}\mathcal{H}^{n+1} \leq \frac{E_0}{2\sigma} \tag{6.2}$$

for all Borel subsets $A \subset U$, where the inequality follows from the energy bound in Hypothesis (B). The resulting bound $V^i(Gr_n(U)) \leq \frac{E_0}{2\sigma}$ allows us to extract a subsequence that converges to a varifold V, with properties laid out in the following theorem by Hutchinson–Tonegawa [HT00].

THEOREM 6.1.7 ([HT00]). *Let (u_i) be a sequence in $C^3(U)$ satisfying Hypotheses (A) and (B). Passing to a subsequence $V^i \to V$ as varifolds, and*

(a) *V is a stationary integral varifold,*
(b) $\|V\|(U) = \liminf_{i\to\infty} \frac{1}{2\sigma} E_{\epsilon_i}(u_i)$,
(c) *for all $\phi \in C_c(U)$:*

$$\lim_{i\to\infty}\int_U \epsilon_i \frac{|\nabla u_i|^2}{2}\phi = \lim_{i\to\infty}\int_U \frac{W(u_i)}{\epsilon_i}\phi = \lim_{i\to\infty}\int_U |\nabla w_i|\phi, \tag{6.3}$$

where $w_i := \Psi \circ u_i$ and $\Psi(t) := \int_0^t \sqrt{W(s)/2}\,\mathrm{d}s$.

Up to a factor of ϵ_i the second variation $\delta^2 E_{\epsilon_i}$ corresponds to the second-order elliptic operator $L_i := \Delta - \epsilon_i^{-2} W''(u_i)$. As in the discussion for the Jacobi operator, L_i has discrete spectrum $\lambda_1^i \leq \lambda_2^i \leq \cdots \to +\infty$, which we denote by $\lambda_p^i(W)$ when we want to emphasise its dependence on the subset W. The following theorem is our main result.

THEOREM 6.1.8. *Let M^{n+1} be a closed Riemannian manifold, and $U \subset M$ an open subset. Let (u_i) be a sequence in $C^3(U)$ satisfying Hypotheses (A), (B) and (C), and $V^i \to V$. Then $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$ and*

(i) *$\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$ for all open $W \subset\subset U \setminus \operatorname{sing} V$ with $W \cap \operatorname{reg} V \neq \emptyset$ and all $p \in \mathbf{N}$,*
(ii) *$\operatorname{ind} B_V \leq k$.*

REMARK 6.1.9. The spectral lower bound remains true if the assumptions are weakened and one assumes that for some $k \in \mathbf{N}$ there is $\mu \in \mathbf{R}$ such that

$$\lambda_k^i(U) \geq \mu \quad \text{for all } i \in \mathbf{N}$$

instead of an index bound—this observation is inspired the work of Ambrozio–Carlotto–Sharp [ACS16], where a similar generalisation is made in the context of minimal surfaces. One obtains the spectral bound via an inductive argument on k similar to the argument in Section 6.3, noting for the base case of the induction that bounds as in Corollary 6.2.5 hold if $\lambda_1^i \geq \mu$.

The following corollary is an immediate consequence of Theorem 6.1.8.

COROLLARY 6.1.10. *If* $\operatorname{reg} V$ *is two-sided, then its Morse index with respect to the area functional satisfies* $\operatorname{ind}_{\mathcal{H}^n} \operatorname{reg} V \leq k$.

If V is the stationary varifold arising from Guaraco's 1-parameter min-max construction [Gua18] (resp. from the k-parameter min-max construction of Gaspar–Guaraco [GG18]) and its regular part is two-sided, then by Corollary 6.1.10 its Morse index is at most 1 (resp. at most k).

6.2. Preliminary results

The preliminary results are divided into three parts. In the first, following [Ton05] we introduce 'second fundamental forms' A^i for the varifolds V^i and relate them to the second variation of the Allen–Cahn functional. The last two sections are dedicated to the spectra of the operators $L_V = \Delta_V + |A|^2 + \operatorname{Ric}_M(\nu, \nu)$ and $L_i = \Delta - \epsilon_i^{-2} W''(u_i)$.

6.2.1. Stability and L^2–bounds of curvature. To simplify the discussion fix for the moment a critical point $u \in C^3(U) \cap L^\infty(U)$ of the Allen–Cahn functional E_ϵ, with associated varifold V^ϵ defined by (6.1).

Let $x \in U$ be a regular point of u, that is $\nabla u(x) \neq 0$. In a small enough neighbourhood of x, the level set $\{u = u(x)\}$ is embedded in M. Call $\Sigma \subset M$ this embedded portion of the level set, and let A^Σ be its second fundamental form. We use this to define a 'second fundamental form' for V^ϵ.

DEFINITION 6.2.1. The function A^ϵ is defined at all $x \in U$ where $\nabla u(x) \neq 0$ by $A^\epsilon(x) = A^\Sigma(x)$.

REMARK 6.2.2. Second fundamental forms can be generalised to the context of varifolds via the integral identity (B.5)—see Appendix B, or [Hut86] for the original account of this theory. Strictly speaking it is an abuse of language to call A^ϵ the 'second fundamental form' of V^ϵ, as it satisfies this identity only up to a small error term (B.3).

By definition $\nabla_X Y = \nabla_X^\Sigma Y + A^\Sigma(X, Y)$ for all $X, Y \in C^1(T\Sigma)$. Making implicit use of the musical isomorphisms here and throughout the text, write $\nu^\epsilon(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$, so that

$$A^\Sigma(X, Y) = \langle \nabla_X Y, \nu^\epsilon \rangle \nu^\epsilon = -\langle Y, \nabla_X \nu^\epsilon \rangle \nu^\epsilon.$$

LEMMA 6.2.3. *Let $x \in U$ be a regular point of u. Then*

$$|A^\epsilon|(x)^2 \leq \frac{1}{|\nabla u|^2(x)}(|\nabla^2 u|^2(x) - |\nabla|\nabla u||^2(x)), \tag{6.1}$$

where $\nabla^2 u(x)$ is the Hessian of u at x.

PROOF. The second fundamental form A^Σ is expressed in terms of the covariant derivative $\nabla\nu^\epsilon$ by

$$A^\Sigma = -\nabla\nu^\epsilon|_{T\Sigma\otimes T\Sigma} \otimes \nu^\epsilon.$$

We can express $\nabla\nu^\epsilon$ as

$$\nabla\nu^\epsilon = \frac{\nabla^2 u}{|\nabla u|} - \nu^\epsilon \otimes \frac{\nabla|\nabla u|}{|\nabla u|},$$

whence after restriction to $T\Sigma \otimes T\Sigma$ we get

$$A^\Sigma = -\frac{1}{|\nabla u|}\nabla^2 u|_{T\Sigma\otimes T\Sigma} \otimes \nu^\epsilon.$$

On the other hand $\nabla|\nabla u| = \langle\nabla^2 u, \nu^\epsilon\rangle$ where $\nabla u \neq 0$, so after decomposing the Hessian $\nabla^2 u$ in terms of its action on $T\Sigma$ and $N\Sigma$, we obtain

$$|\nabla^2 u|^2 - |\nabla|\nabla u||^2 = |\nabla u|^2|A^\Sigma|^2 + |\nabla^2 u|_{T\Sigma\otimes N\Sigma}|^2 \geq |\nabla u|^2|A^\epsilon|^2. \qquad \square$$

When considering the second variation, it somewhat simplifies notation to rescale the energy as $\mathcal{E}_\epsilon = \epsilon^{-1}E_\epsilon$. Its second variation is $\delta^2\mathcal{E}_\epsilon(u)(\phi,\phi) = \int_U |\nabla\phi|^2 + \frac{W''(u)}{\epsilon^2}\phi^2$, defined for all $\phi \in C^1_c(U)$, which by a density argument can be extended to $H^1_0(U)$. The following identity will be useful throughout; a proof can be found in either of the indicated sources.

LEMMA 6.2.4 ([FSV13, Ton05]). *Let $u \in C^3(U) \cap L^\infty(U)$ be a critical point of E_ϵ. For all $\phi \in C^1_c(U)$,*

$$\delta^2\mathcal{E}_\epsilon(u)(|\nabla u|\phi, |\nabla u|\phi) = \int_U |\nabla u|^2|\nabla\phi|^2 - \left(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 + \mathrm{Ric}_M(\nabla u, \nabla u)\right)\phi^2 \,\mathrm{d}\mathcal{H}^{n+1}. \tag{6.2}$$

Combining (6.2) with the $\|V^\epsilon\|$-a.e. bound (6.1) yields for all $\phi \in C^1_c(U)$

$$\frac{\epsilon}{2\sigma}\delta^2\mathcal{E}_\epsilon(u)(|\nabla u|\phi, |\nabla u|\phi) \leq \int |\nabla\phi|^2 - (|A^\epsilon|^2 + \mathrm{Ric}_M(\nu^\epsilon, \nu^\epsilon))\phi^2 \,\mathrm{d}\|V^\epsilon\|. \tag{6.3}$$

When u is stable, that is when $\delta^2\mathcal{E}_\epsilon(u)$ is non-negative, then this identity yields $L^2(V^\epsilon)$–bounds for A^ϵ.

COROLLARY 6.2.5. *There is a constant $C = C(M) > 0$ such that if $u \in C^3(U) \cap L^\infty(U)$ is a critical point of E_ϵ and is stable in an open ball*

$B(x,r) \subset U$ of radius $r \leq 1$ then

$$\int_{B(x,\frac{r}{2})} |A^\epsilon|^2 \, \mathrm{d}\|V^\epsilon\| \leq \frac{C}{r^2} \|V^\epsilon\|(B(x,r)). \tag{6.4}$$

PROOF. The Ricci curvature term in (6.3) can be bounded by some constant $C(M) \geq 1$ as the manifold M is closed, so $\int_{B(x,r)} |A^\epsilon|^2 \phi^2 \, \mathrm{d}\|V^\epsilon\| \leq C(M) \int \phi^2 + |\nabla \phi|^2 \, \mathrm{d}\|V^\epsilon\|$ for all $\phi \in C^1_c(B(x,r))$. Plug in a cut-off function $\eta \in C^1_c(B(x,r))$ with $\eta = 1$ in $B(x, \frac{r}{2})$ and $|\nabla \eta| \leq 3r^{-1}$ to obtain the desired inequality. □

We now turn to a sequence (u_i) of critical points satisfying Hypotheses (A)–(C). If the u_i are stable in a ball as in Corollary 6.2.5, then the uniform weight bounds (6.2) imply uniform $L^2(V^i)$–bounds of the second fundamental forms, which we denote A^i from now on. Under these conditions the A^i converge weakly to the second fundamental form A (in the classical, smooth sense) of $\operatorname{reg} V$.

PROPOSITION 6.2.6. *Let $W \subset\subset U \setminus \operatorname{sing} V$ be open with $W \cap \operatorname{reg} V \neq \emptyset$. If $\sup_i \int_W |A^i|^2 \, \mathrm{d}\|V^i\| < +\infty$, then passing to a subsequence $A^i \, \mathrm{d}V^i \to A \, \mathrm{d}V$ weakly as Radon measures on $Gr_n(W)$, and*

$$\int_W |A|^2 \, \mathrm{d}\|V\| \leq \liminf_{i \to \infty} \int_W |A^i|^2 \, \mathrm{d}\|V^i\|,$$

where A is the second fundamental form of $\operatorname{reg} V \subset M$.

The weak subsequential convergence follows immediately from compactness of Radon measures; the main difficulty is to show that the weak limit is $A \, \mathrm{d}V$. The proof is a straight-forward adaptation of the argument used for the stable case in [Ton05]; we present a complete argument in Appendix B for the reader's convenience.

6.2.2. Spectrum of L_V and weighted min-max. Throughout we restrict ourselves to a compactly contained open subset $W \subset\subset U \setminus \operatorname{sing} V$ to avoid blow-up of the coefficients of L_V near the singular set, and assume $W \cap \operatorname{reg} V \neq \emptyset$ to avoid vacuous statements. As $W \cap \operatorname{reg} V$ is compactly contained in $\operatorname{reg} V$, it can intersect only finitely many connected components $C_1, \ldots, C_N$ of $\operatorname{reg} V$. By the constancy theorem [Sim84, Thm. 41.1] the multiplicity function Θ of a stationary integral varifold V is constant on every connected component of $\operatorname{reg} V$; we write $\Theta_1, \ldots, \Theta_N$ for the respective multiplicities of $C_1, \ldots, C_N$.

By classical theory for elliptic PDE [GT98, Ch. 8], the spectrum of L_V has the following min-max characterisation:

$$\lambda_p = \inf_{\dim S = p} \max_{\phi \in S \setminus \{0\}} \frac{B_{\operatorname{reg} V}(\phi, \phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } p \in \mathbf{N}, \tag{6.5}$$

where the infimum is taken over linear subspaces S of H_0^1 (recall this is our abbreviated notation for $H_0^1(W \cap \operatorname{reg} V)$). From this we easily obtain a min-max characterisation that is 'weighted' by the multiplicities $\Theta_1, \ldots, \Theta_N$ in the sense that

$$\lambda_p = \inf_{\dim S = p} \max_{\phi \in S \setminus \{0\}} \frac{B_V(\phi, \phi)}{\|\phi\|_{L^2(V)}^2} \quad \text{for all } p \in \mathbf{N}. \tag{6.6}$$

To see this, observe the following: as functions $\phi \in H_0^1$ vanish near the boundary of every connected component $C \subset \operatorname{reg} V$, the function ϕ_C on $W \cap \operatorname{reg} V$ defined by

$$\phi_C = \begin{cases} \phi & \text{on } C \\ 0 & \text{on } W \cap \operatorname{reg} V \setminus C \end{cases}$$

also belongs to H_0^1. Moreover

$$B_V(\phi_C, \phi_C) = \Theta_C B_{\operatorname{reg} V}(\phi_C, \phi_C) \text{ and } \|\phi_C\|_{L^2(V)}^2 = \Theta_C \|\phi_C\|_{L^2}^2,$$

where Θ_C denotes the multiplicity of C. We then define a linear isomorphism of H_0^1 via normalisation by the respective multiplicities of the components. This sends $\phi \mapsto \bar{\phi} := \sum_{j=1}^N \Theta_j^{-1/2} \phi_{C_j}$; then

$$\frac{B_V(\bar{\phi}, \bar{\phi})}{\|\bar{\phi}\|_{L^2}^2} = \frac{B_{\operatorname{reg} V}(\phi, \phi)}{\|\phi\|_{L^2(V)}^2}.$$

Therefore the 'unweighted' and 'weighted' min-max characterisations (6.5) and (6.6) are in fact equivalent. In the remainder we mainly use (6.6), and abbreviate this as $\lambda_p = \inf_{\dim S = p} \max_{S \setminus \{0\}} J_V$, where J_V denotes the 'weighted' Rayleigh quotient

$$J_V(\phi) = \frac{B_V(\phi, \phi)}{\|\phi\|_{L^2(V)}^2} \quad \text{for all } \phi \in H_0^1 \setminus \{0\}.$$

The min-max characterisation implies the following lemma, which highlights the dependence of the spectrum $\lambda_p(W)$ on the subset W.

Lemma 6.2.7.

(a) *If* $W_1 \subset W_2 \subset\subset U \setminus \operatorname{sing} V$, *then* $\lambda_p(W_1) \geq \lambda_p(W_2)$: *the spectrum is monotone decreasing.*

(b) *If* $W_1, W_2 \subset\subset U \setminus \operatorname{sing} V$ *have* $W_1 \cap W_2 = \emptyset$, *then* $\operatorname{ind}_{W_1} B_V + \operatorname{ind}_{W_2} B_V = \operatorname{ind}_{W_1 \cup W_2} B_V$.

(c) *If* $W \subset\subset U \setminus \operatorname{sing} V$ *and* $y \in W \cap \operatorname{reg} V$, *then* $\lambda_p(W) = \lim_{R\to 0} \lambda_p(W \setminus \overline{B}(y,R))$.

REMARK 6.2.8. The same properties hold for the spectrum and index of L_i, and the proof is easily modified to cover this case.

PROOF. (a) This is immediate from the min-max characterisations, or simply by definition of the spectrum. Similarly for (b).

(c) By monotonicity of the spectrum we have

$$\lambda_p(W \setminus \overline{B}(y,R)) \geq \lambda_p(W \setminus \overline{B}(y,R')) \geq \lambda_p(W)$$

for all $R > R' > 0$. The limit as $R \to 0$ therefore exists and is bounded below by $\lambda_p(W)$; it remains only to show that $\lim_{R\to 0} \lambda_p(W \setminus \overline{B}(y,R)) \leq \lambda_p(W)$.

By monotonicity of the spectrum it is equivalent to show that for a fixed radius $R > 0$, $\lim_{m\to\infty} \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)) \leq \lambda_p(W)$. Let $(\rho_m)_{m\in\mathbf{N}}$ be a sequence in $C^1_c(B(y,R) \cap \operatorname{reg} V)$ with the following properties (such a sequence exists provided $n \geq 2$, see Remark 6.2.9 below):

(1) $\rho_m|_{B(y,2^{-m}R)\cap \operatorname{reg} V} \equiv 0$ and $\rho_m \to 1$ $\mathcal{H}^n$-a.e. in $W \setminus \{y\} \cap \operatorname{reg} V$,
(2) $\|\nabla_V \rho_m\|_{L^2(W\cap \operatorname{reg} V)} \to 0$.

Let a small $\delta > 0$ be given and choose a family $(\phi_1, \dots, \phi_p)$ in $C^1_c(W \cap \operatorname{reg} V)$ whose $\operatorname{span}(\phi_1,\dots,\phi_p) =: S$ has $\max_{S\setminus 0} J_V \leq \lambda_p(W) + \delta$. Write $\rho_m S$ for $\operatorname{span}(\rho_m\phi_1, \dots, \rho_m\phi_p) \subset C^1_c(W \setminus \overline{B}(y, 2^{-m}R) \cap \operatorname{reg} V)$—for m large enough the functions $\rho_m\phi_i$ are indeed linearly independent. By the weighted min-max formula (6.6),

$$\max_{\rho_m S\setminus 0} J_V \geq \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)).$$

Let $t_m \in \mathbf{S}^{p-1} \subset \mathbf{R}^p$ denote the coefficients of the linear combination $t_m \cdot \rho_m\phi := \rho_m \sum_{j=1}^p t_{mj}\phi_j \in \rho_m S$ that realises $\max_{\rho_m S\setminus 0} J_V$. Passing to a convergent subsequence $t_m \to t \in \mathbf{S}^{p-1} \subset \mathbf{R}^p$ we get $J_V(t_m \cdot \rho_m\phi) \to J_V(t\cdot\phi)$, and hence

$$\lim_{m\to\infty} J_V(t_m \cdot \rho_m\phi) \leq \max_{S\setminus 0} J_V.$$

On the one hand $\max_{S\setminus 0} J_V \leq \lambda_p(W) + \delta$ by our choice of S, on the other hand $\lim_{m\to\infty} \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)) \leq \lim_{m\to\infty} J_V(t_m \cdot \rho_m\phi)$ by our choice of t_m. The conclusion follows after combining these two observations and letting $\delta \to 0$. □

REMARK 6.2.9. A sequence of functions (ρ_m) with properties (1) and (2) exists provided $n \geq 2$, as we assume throughout. When $n \geq 3$ one can use the standard cutoff functions; for $n = 2$ a more precise construction is necessary, described for instance in [EG15, Sec. 4.7].

6.2.3. Spectrum of L_i and conditional proof of Theorem 6.1.8. The main result in this section is Lemma 6.2.12; essentially it says that

$$\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$$

holds under the condition that $\sup_i \int_W |A^i|^2 \, \mathrm{d}\|V^i\| < +\infty$. What precedes it in this section are technical results required for its proof.

Again, by classical elliptic PDE theory the eigenvalues $\lambda_p^i(W)$ of $L_i = \Delta - \epsilon_i^{-2} W''(u_i)$ on $H_0^1(W)$ have the following min-max characterisation in terms of the rescaled Allen–Cahn functional $\mathcal{E}_{\epsilon_i} = \epsilon_i^{-1} E_{\epsilon_i}$:

$$\lambda_p^i(W) = \inf_{\dim S = p} \max_{\phi\in S\setminus\{0\}} \frac{\delta^2 \mathcal{E}_{\epsilon_i}(u_i)(\phi,\phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } p \in \mathbf{N},$$

where the infimum is over p-dimensional linear subspaces $S \subset H_0^1(W)$. Define the *Rayleigh quotient* J_i by

$$J_i(\phi) = \frac{\delta^2 \mathcal{E}_{\epsilon_i}(u_i)(\phi,\phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } \phi \in H_0^1(W) \setminus \{0\},$$

so that we can write the min-max characterisation more succinctly as $\lambda_p^i = \inf_{\dim S=p} \max_{S\setminus\{0\}} J_i$.

To compare the spectrum of L_V in H_0^1 with those of the operators L_i in $H_0^1(W)$, extend functions in $C_c^1(W \cap \operatorname{reg} V)$ to $C_c^1(W)$ in the standard way, which we now describe to fix notations. Pick a small enough $0 < \tau < \operatorname{inj}(M)$ so that $B_\tau V := \exp N_\tau V$ is a tubular neighbourhood of $W \cap \operatorname{reg} V$, where $N_\tau V := \{s_p \in NV \mid p \in W \cap \operatorname{reg} V, |s_p| < \tau\}$. We abuse notation slightly to denote points in $B_\tau V$ by s_p, and identify the fibre $N_p V$ with $(\exp_p)_* N_p V \subset T_{s_p}(B_\tau V)$. The distance function $d_V : x \in B_\tau V \mapsto \operatorname{dist}(x, \operatorname{reg} V)$ is Lipschitz and smooth on $B_\tau V \setminus \operatorname{reg} V$. By the Gauss lemma $\operatorname{grad} d_V(s_p) = -s_p/|s_p|$ for all $s_p \in B_\tau V \setminus \operatorname{reg} V$. A function $\phi \in C^1(B_\tau V)$ is constant along geodesics normal to $\operatorname{reg} V$ if $\phi(s_p) = \phi(0_p)$ for all $s_p \in B_\tau V$, or equivalently if $\langle \nabla\phi, \nabla d_V\rangle \equiv 0$ in $B_\tau V \setminus \operatorname{reg} V$.

Lemma 6.2.10. *Any $\phi \in C_c^1(W \cap \operatorname{reg} V)$ can be extended to $C_c^1(W)$ with $\langle \nabla\phi, \nabla d_V\rangle \equiv 0$ in $B_{\frac{\tau}{2}} V \setminus \operatorname{reg} V$ for some $\tau = \tau(\phi) > 0$.*

Proof. Extend $\phi \in C_c^1(W \cap \operatorname{reg} V)$ to $B_\tau V$ by setting $\tilde\phi(s_p) = \phi(p)$, so that $\langle \nabla d_V, \nabla\tilde\phi\rangle \equiv 0$ in $B_\tau V \setminus \operatorname{reg} V$. Let $\eta \in C^1[0,\infty)$ be a cutoff function with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $[0, 1/2)$ and $\operatorname{spt} \eta \subset [0, 1)$. Then

$$(\eta \circ d_V/\tau)\tilde\phi \in C_c^1(B_\tau V) \text{ and } (\eta \circ d_V/\tau)\tilde\phi = \tilde\phi \text{ on } B_{\tau/2} V.$$

Moreover even though $B_\tau V \not\subset W$ in general, as $\operatorname{spt}\phi$ is compactly contained in $W \cap \operatorname{reg} V$ we still have $(\eta \circ d_V/\tau)\tilde\phi \in C_c^1(W)$ provided $0 < \tau < \operatorname{dist}(\operatorname{spt}\phi, \partial W)$. $\square$

The following lemma gives an asymptotic lower bound for the Rayleigh quotient J_V in terms of the J_i.

LEMMA 6.2.11. *Let $B_\tau V$ be a tubular neighbourhood of $W \cap \operatorname{reg} V$ with width $\tau > 0$, and let $(\phi_i \mid i \in \mathbf{N})$ be a sequence of functions in $C^1_c(W)$ with*

(a) *$\langle \nabla\phi_i, \nabla d\rangle \equiv 0$ in $W \cap B_{\tau/2}V$ for all i,*

(b) *$\phi_i \to \phi$ in $C^1_c(W)$ as $i \to \infty$, where $\phi \neq 0$ in $W \cap \operatorname{reg} V$.*

If $\sup_i \int_W |A^i|^2 \,\mathrm{d}\|V^i\| < +\infty$, then $J_V(\phi) \geq \limsup_{i\to\infty} J_i(|\nabla u_i|\phi_i)$.

PROOF. Before we start the proof proper, note that for all $\phi \in H^1_0(U)$, dividing both sides of (6.3) by $\frac{\epsilon_i}{2\sigma}\int \phi^2 |\nabla u_i|^2 \,\mathrm{d}\mathcal{H}^{n+1} = \int \phi^2 \,\mathrm{d}\|V^i\|$ yields

$$\|\phi\|^{-2}_{L^2(V^i)} \int |\nabla\phi|^2 - (|A^i|^2 + \operatorname{Ric}_M(\nu_i,\nu_i))\phi^2 \,\mathrm{d}\|V^i\| \geq J_i(|\nabla u_i|\phi), \tag{6.7}$$

provided of course that $\|\phi\|^2_{L^2(V^i)} \neq 0$.

We treat the terms on the left-hand side separately in the calculations (1)–(4) below. Once these are completed, we combine (1) with our assumption that $\|\phi\|^2_{L^2(V)} \neq 0$ to obtain that $\|\phi_i\|_{L^2(V^i)} \neq 0$ for large enough i. The conclusion follows by combining (6.7) with the remaining calculations:

(1) $\int \phi^2 \,\mathrm{d}\|V\| = \lim_{i\to\infty} \int \phi_i^2 \,\mathrm{d}\|V^i\|$

(2) $\int |\nabla_V \phi|^2 \,\mathrm{d}\|V\| = \lim_{i\to\infty} \int |\nabla\phi_i|^2 \,\mathrm{d}\|V^i\|$,

(3) $\int |A|^2\phi^2 \,\mathrm{d}\|V\| \leq \liminf_{i\to\infty} \int |A^i|^2 \phi_i^2 \,\mathrm{d}\|V^i\|$,

(4) $\int \operatorname{Ric}_M(\nu,\nu)\phi^2 \,\mathrm{d}\|V\| = \lim_{i\to\infty} \int \operatorname{Ric}_M(\nu_i,\nu_i)\phi_i^2 \,\mathrm{d}\|V^i\|$.

(1) By assumption $\phi_i^2 \to \phi^2$ in $C_c(W)$, whence by Corollary A.3 we get $\int \phi^2 \,\mathrm{d}\|V\| = \lim_{i\to\infty} \int \phi_i^2 \,\mathrm{d}\|V^i\|$. The same argument proves (2), after noticing that $\langle \nabla\phi_i, \nabla d_V\rangle \equiv 0$ implies $|\nabla\phi|^2 = |\nabla_V\phi|^2$ on $W \cap \operatorname{reg} V$.

(3) The sequence $(A^i\phi_i \,\mathrm{d}\|V^i\|)$ converges weakly to $A\phi \,\mathrm{d}\|V\|$, as we can show by testing against an arbitrary $\varphi \in C_c(U)$:

$$\int A^i\phi_i\varphi \,\mathrm{d}\|V^i\| - \int A\phi\varphi \,\mathrm{d}\|V\| = \int A^i(\phi_i - \phi)\varphi \,\mathrm{d}\|V^i\| + \int A^i\phi\varphi \,\mathrm{d}\|V^i\| - \int A\phi\varphi \,\mathrm{d}\|V\|.$$

The first integral is bounded by

$$\left| \int A^i(\phi_i - \phi)\varphi \,\mathrm{d}\|V^i\| \right| \leq \|\phi_i - \phi\|_{L^\infty} \|\varphi\|_{L^2(V^i)} \|A^i\|_{L^2(V^i)} \to 0$$

because $\phi_i \to \phi$ in $C_c(W)$ as $i \to \infty$. The remaining terms tend to 0 by the weak convergence of $A^i \,\mathrm{d}\|V^i\| \to A \,\mathrm{d}\|V\|$ tested against $\phi\varphi \in C_c(W)$. Then inequality (A.1) gives $\int |A|^2\phi^2 \,\mathrm{d}\|V\| \leq \liminf_{i\to\infty} \int |A^i|^2\phi_i^2 \,\mathrm{d}\|V^i\|$.

(4) For each $S \in Gr_n(T_pM)$ pick a unit vector ν_S in T_pM orthogonal to S, and define a smooth function R_M on $Gr_n(U)$ by $R_M \colon S \mapsto \operatorname{Ric}_M(\nu_S,\nu_S)$.

Then $\phi_i^2 R_M \to \phi^2 R_M$ in $C_c(Gr_n(U))$ as $i \to \infty$, and by Corollary A.3,

$$\int \phi_i^2 \operatorname{Ric}_M(\nu_i, \nu_i) \,\mathrm{d}\|V^i\| \\ = \int \phi_i^2 R_M \,\mathrm{d}V^i \to \int \phi^2 R_M \,\mathrm{d}V = \int \phi^2 \operatorname{Ric}_M(\nu, \nu) \,\mathrm{d}\|V\|. \quad \square$$

We conclude the section with a proof of Theorem 6.1.8(i) in the case where there is a uniform $L^2(V^i)$–bound on the second fundamental forms (A^i).

LEMMA 6.2.12. *Let $W \subset\subset U \setminus \operatorname{sing} V$ be open with $W \cap \operatorname{reg} V \neq \emptyset$. If $\sup_i \int_W |A^i|^2 \,\mathrm{d}\|V^i\| < +\infty$, then $\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$ for all p.*

PROOF. We may assume that every connected component of W intersects $\operatorname{spt}\|V\|$ (or $\operatorname{reg} V$, equivalently as $W \cap \operatorname{sing} V = \emptyset$) without restricting generality: if C is a connected component of W with $C \cap \operatorname{reg} V = \emptyset$, then $\lambda_p(W \setminus C) = \lambda_p(W)$, and by monotonicity $\lambda_p^i(W \setminus C) \geq \lambda_p^i(W)$ for all i.

Given $\delta > 0$ there is a p-dimensional linear subspace $S = \operatorname{span}(\phi_1, \ldots, \phi_p)$ of $C_c^1(W \cap \operatorname{reg} V)$ with

$$\lambda_p(W) + \delta \geq \max_{S\setminus\{0\}} J_V. \tag{6.8}$$

Extend the functions ϕ_i to $C_c^1(W)$ as in Lemma 6.2.10; for large enough i the family $(|\nabla u_i|\phi_1, \ldots, |\nabla u_i|\phi_p)$ is still linearly independent. Indeed, otherwise we could extract a subsequence such that $(|\nabla u_{i'}|\phi_1, \ldots, |\nabla u_{i'}|\phi_p)$ has a linear dependence, with coefficients $a_{i'} \neq 0 \in \mathbf{R}^p$ say. Then notice that

$$|\nabla u_{i'}| a_{i'} \cdot \phi = 0 \Leftrightarrow \|a_{i'} \cdot \phi\|_{L^2(V^{i'})} = 0,$$

where we abbreviated $a_{i'} \cdot \phi := \sum_{j=1}^p a_{i'j} |\nabla u_{i'}| \phi_j$. We may normalise the coefficients $a_{i'}$ so as to guarantee $|a_{i'}| = 1$ and then, possibly after extracting a second subsequence, assume that $a_{i'} \to a \in \mathbf{S}^{p-1}$ as $i' \to \infty$. The resulting strong convergence $a_{i'} \cdot \phi \to a \cdot \phi$ in $C_c(W)$ combined with $\|V^i\| \to \|V\|$ yield $\|a_{i'} \cdot \phi\|_{L^2(V^{i'})} \to \|a \cdot \phi\|_{L^2(V)}$; this contradicts $\|a \cdot \phi\|_{L^2(V)} > 0$.

From now on take i large enough so that $(|\nabla u_i|\phi_1, \cdots, |\nabla u_i|\phi_p)$ is linearly independent. For such large i, we may let $t_i \in \mathbf{S}^{p-1} \subset \mathbf{R}^p$ be the (normalised) coefficients of a linear combination $t_i \cdot |\nabla u_i|\phi = \sum_{j=1}^p t_{ij} |\nabla u_i| \phi_j$ that maximises the Rayleigh quotient J_i:

$$J_i(t_i \cdot |\nabla u_i|\phi) = \max_{|\nabla u_i| S \setminus\{0\}} J_i \geq \lambda_p^i(W).$$

Extract a convergent subsequence $t_{i'} \to t \in \mathbf{S}^{p-1}$, so that $t_{i'} \cdot \phi \to t \cdot \phi$ in $C_c^1(W)$ as $i' \to \infty$. Lemma 6.2.11 gives $J_V(t \cdot \phi) \geq \limsup_{i\to\infty} \max_{|\nabla u_i| S\setminus\{0\}} J_i$, which in turn is greater than $\limsup_{i\to\infty} \lambda_p^i(W)$. Using (6.8),

$$\lambda_p(W) + \delta \geq J_V(t \cdot \phi) \geq \limsup_{i\to\infty} \lambda_p^i(W),$$

and we conclude by letting $\delta \to 0$. □

Lemma 6.2.12 has the following immediate corollary.

COROLLARY 6.2.13. *Under the hypotheses of Lemma 6.2.12,*

$$\operatorname{ind}_W B_V \leq \liminf_{i\to\infty}(\operatorname{ind}_W u_i).$$

6.3. PROOF OF THE MAIN THEOREM (THEOREM A)

We briefly recall the context of the proof: M^{n+1} is a closed Riemannian manifold and $U \subset M$ is an arbitrary open subset. The sequence of functions (u_i) in $C^3(U)$ satisfies Hypotheses (A), (B) and (C)—the last hypothesis says that $\operatorname{ind} u_i \leq k$ for all i. To every u_i we associate the n–varifold V^i from (6.1). By Theorem 6.1.7, we may pass to a subsequence of (V^i) converging weakly to a stationary integral varifold V.

6.3.1. Spectrum and index of V: proof of (i) and (ii). The main idea, inspired by an argument of Bellettini–Wickramasekera [BW18], is to fix a compactly contained open subset $W \subset\subset U \setminus \operatorname{sing} V$ and study the stability of u_i in open balls covering $W \cap \operatorname{reg} V \neq \emptyset$. We then shrink the radii of the covering balls to 0, and prove the spectral lower bound of Theorem 6.1.8(i) by induction on k. The upper bound on $\operatorname{ind} B_V$ of Theorem 6.1.8(ii) is then an immediate consequence.

In the base of induction the u_i are stable in U. Let $\eta \in C^1_c(U)$ be a cutoff function constant equal to 1 on W. The stability inequality (6.3) gives that

$$\int_W |A^i|^2 \,\mathrm{d}\|V^i\| \leq C(M)\operatorname{dist}(W, \partial U)^{-2}\|V^i\|(U) \quad \text{for all } i.$$

Combining this with (6.2) we get $\sup_i \int_W |A^i|^2 \,\mathrm{d}\|V^i\| < +\infty$, and thus $\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$ by Lemma 6.2.12.

For the induction step, let $k \geq 1$ and assume that Theorem 6.1.8(i) holds with $k-1$ in place of k. Consider an arbitrary $W \subset\subset U \setminus \operatorname{sing} V$ that intersects $\operatorname{reg} V$. Fix a radius $0 < r < \operatorname{dist}(W, \operatorname{sing} V)$, and pick points $x_1, \ldots, x_N \in W \cap \operatorname{reg} V$ such that $W \cap \operatorname{reg} V \subset \cup_{j=1}^N B(x_j, \frac{r}{2})$. We define the following *Stability Condition* for the cover $\{B(x_j, \frac{r}{2})\}_{1\leq j\leq N}$:

(SC) *For large i, each u_i is stable in every ball $B(x_1, r), \ldots, B(x_N, r)$.*

CLAIM 7. If for the cover $\{B(x_j, \frac{r}{2})\}_{1\leq j\leq N}$:

(a) (SC) holds, then $\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$;

(b) (SC) fails, then $\lambda_p(W \setminus \overline{B_r}) \geq \limsup_{i\to\infty} \lambda_p^i(W \setminus \overline{B_r})$ for some ball $B_r \in \{B(x_j, r)\}$.

PROOF. (a) Let $W_r = W \cap \cup_{j=1}^N B(x_j, \frac{r}{2})$, so that $W_r \cap \operatorname{reg} V = W \cap \operatorname{reg} V$ and hence $\lambda_p(W_r) = \lambda_p(W)$. Moreover $W_r \subset W$, so $\lambda_p^i(W_r) \geq \lambda_p^i(W)$ for all i by monotonicity of the spectrum. Therefore it is enough to show that $\lambda_p(W_r) \geq \limsup_{i\to\infty} \lambda_p^i(W_r)$.

Because (SC) holds, summing (6.4) over all balls we get

$$\int_{W_r} |A^i|^2 \, \mathrm{d}\|V^i\| \leq \frac{NC}{r^2}\|V^i\|(W_r) \leq \frac{NCE_0}{2r^2\sigma} \quad \text{for all } i,$$

so $\lambda_p(W_r) \geq \limsup_{i\to\infty} \lambda_p^i(W_r)$ by Lemma 6.2.12.

(b) If (SC) fails, then some subsequence $(u_{i'})$ must be unstable in one of the balls $B_r \in \{B(x_j, r)\}$, in other words $\operatorname{ind}_{B_r} u_{i'} \geq 1$ for all i'. On the other hand

$$\operatorname{ind}_{B_r} u_{i'} + \operatorname{ind}_{W\setminus\overline{B_r}} u_{i'} \leq \operatorname{ind}_W u_{i'}$$

because B_r and $W \setminus \overline{B_r}$ are disjoint open sets. As $\operatorname{ind}_W u_{i'} \leq k$, we get $\operatorname{ind}_{W\setminus\overline{B_r}} u_{i'} \leq k-1$ for all i', and we conclude after applying the induction hypothesis to $(u_{i'})$ in $W \setminus \overline{B_r}$. □

REMARK 6.3.1. This argument shows that when (SC) fails there is a ball $B_r \in \{B(x_j, r)\}$ with $\lambda_p(W \setminus \overline{B_r}) \geq \limsup_{i\to\infty} \lambda_{p+1}^i(W)$ for $p \geq k$, and thus also $\operatorname{ind}_{W\setminus\overline{B_r}} B_V \leq k-1$, but the induction step only requires the weaker conclusion from Claim 7.

Now consider a decreasing sequence $r_m \to 0$ with $0 < r_m < \operatorname{dist}(W, \operatorname{sing} V)$ and reason as above with $r = r_m$. For each m, pick points $x_1^m, \ldots, x_{N_m}^m \in W \cap \operatorname{reg} V$ such that

$$W \cap \operatorname{reg} V \subset \cup_{j=1}^{N_m} B\big(x_j^m, \frac{r_m}{2}\big).$$

If (SC) holds for a cover $\{B(x_j^m, \frac{r_m}{2})\}$, then $\lambda_p(W) \geq \limsup_{i\to\infty} \lambda_p^i(W)$ by Claim 7, and the induction step is completed.

Otherwise (SC) fails for all constructed covers, and by Claim 7 there is a sequence (y_m) in $W \cap \operatorname{reg} V$ with

$$\lambda_p(W \setminus \overline{B}(y_m, r_m)) \geq \limsup_{i\to\infty} \lambda_p^i(W \setminus \overline{B}(y_m, r_m)),$$

and thus by monotonicity of the spectrum

$$\lambda_p(W \setminus \overline{B}(y_m, r_m)) \geq \limsup_{i\to\infty} \lambda_p^i(W). \tag{6.1}$$

Passing to a subsequence if necessary, we may assume that (y_m) converges to a point $y \in \overline{W} \cap \operatorname{reg} V$. If we fix a radius $R > 0$, then $B(y_m, r_m) \subset B(y, R)$

for large enough m, so by monotonicity and (6.1),

$$\lambda_p(W \setminus \overline{B}(y, R)) \geq \limsup_{m\to\infty} \lambda_p(W \setminus \overline{B}(y_m, r_m)) \geq \limsup_{i\to\infty} \lambda_p^i(W).$$

The conclusion follows after combining this with $\lambda_p(W) = \lim_{R\to 0} \lambda_p(W \setminus \overline{B}(y, R))$ from Lemma 6.2.7.

Together with the base of induction, we have proved Theorem 6.1.8(i) for all sequences (u_i) with $\sup_i \operatorname{ind} u_i \leq k$ for some $k \in \mathbf{N}$. The index bound $\operatorname{ind}_W B_V \leq k$ follows immediately: the spectral lower bound implies that L_V must have fewer negative eigenvalues than the L_i as $i \to \infty$. Therefore

$$\operatorname{ind}_W B_V = \operatorname{card}\{p \in \mathbf{N} \mid \lambda_p(W) < 0\} \leq k.$$

As the subset W was arbitrary we get $\operatorname{ind} B_V \leq k$; this proves Theorem 6.1.8(ii).

6.3.2. Regularity of V: proof of $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$. The approach is the same as in the proof of Theorem 6.1.8(i)–(ii) with one difference: we again proceed by induction on k, but we now cover the entire support $\operatorname{spt}\|V\|$ (including the singular set), instead of constructing covers a positive distance away from $\operatorname{sing} V$.

The base of induction, where the u_i are stable in U, was proved in [TW12].

For the induction step, suppose that $\dim_{\mathcal{H}}(U' \cap \operatorname{sing} V) \leq n-7$ holds with $k-1$ in place of k, and for arbitrary open subsets $U' \subset U$. Fix $r > 0$, and choose points $x_1, \dots, x_N \in U \cap \operatorname{spt}\|V\|$ such that $U \cap \operatorname{spt}\|V\| \subset \cup_{j=1}^N B(x_j, r)$. The *Stability Condition* for the cover $\{B(x_j, r)\}_{1\leq j\leq N}$ is defined in the same way as above, except the radii need not be doubled:

(SC) *For large i, each u_i is stable in every ball $B(x_1, r), \dots, B(x_N, r)$.*

CLAIM 8. If for the cover $\{B(x_j, r)\}_{1\leq j\leq N}$:

(a) (SC) holds, then $\dim_{\mathcal{H}} \operatorname{sing} V \leq n-7$;
(b) (SC) fails, then $\dim_{\mathcal{H}} \operatorname{sing} V \setminus \overline{B_r} \leq n-7$ for some ball $B_r \in \{B(x_j, r)\}$.

PROOF. (a) The results from [TW12] give $\dim_{\mathcal{H}} B(x_j, r) \cap \operatorname{sing} V \leq n-7$ for every $j = 1, \dots, N$. As the balls $\{B(x_j, r)\}$ cover $U \cap \operatorname{spt}\|V\|$, the same holds for $\operatorname{sing} V$.

(b) When (SC) fails, there must be a subsequence $(u_{i'})$ that is unstable in one of the balls B_r of the cover, so that in its complement

$$\operatorname{ind}_{U\setminus\overline{B_r}} u_{i'} \leq k-1 \quad \text{for all } i'.$$

The conclusion follows from the induction hypothesis applied to $(u_{i'})$ in $U \setminus \overline{B_r}$. □

Now consider a decreasing sequence $r_m \to 0$. For every m, choose points $x_1^m, \dots, x_{N_m}^m \in U \cap \operatorname{spt}\|V\|$ such that $U \cap \operatorname{spt}\|V\| \subset \cup_{j=1}^{N_m} B(x_j^m, r_m)$. Then either (SC) holds for the cover $\{B(x_j^m, r_m)\}$ constructed for some m, in which case we can conclude from Claim 8, or else there is sequence of points (y_m) in $U \cap \operatorname{spt}\|V\|$ for which

$$\dim_{\mathcal{H}} \operatorname{sing} V \setminus \overline{B}(y_m, r_m) \leq n - 7.$$

Possibly after extracting a subsequence, the sequence (y_m) converges to a point $y \in \overline{U} \cap \operatorname{spt}\|V\|$. As $U \setminus \{y\} \subset \cup_{m \geq 0} U \setminus \overline{B}(y_m, r_m)$, we get $\dim_{\mathcal{H}} (\operatorname{sing} V \setminus \{y\}) \leq n - 7$.

If $n \geq 7$, then $\dim_{\mathcal{H}} \operatorname{sing} V \leq n - 7$ holds whether or not $y \in \operatorname{sing} V$, as points are zero-dimensional. If however $2 \leq n \leq 6$ then we need $\operatorname{sing} V = \emptyset$, which amounts to the following claim.

Claim 9. If $2 \leq n \leq 6$ then $y \notin \operatorname{sing} V$.

Proof. Choose $B(y, R) \subset U$, and consider balls $\{B(y, R_m)\}_{m \in \mathbf{N}}$ with shrinking radii $R_m := 2^{-m} R$. If for some m there is a subsequence $(u_{i'})$ with

$$\operatorname{ind}_{B(y, R_m)} u_{i'} \leq k - 1 \quad \text{for all } i',$$

then we can conclude from the induction hypothesis. Otherwise for all m

$$\operatorname{ind}_{B(y, R_m)} u_i = k \quad \text{for } i \text{ large enough},$$

and the u_i are eventually stable in the annulus $B(y, R) \setminus \overline{B}(y, R_m)$. By Theorem 6.1.8(ii)

$$\operatorname{ind}_{B(y,R) \setminus \overline{B}(y, R_m)} B_V = 0 \text{ for all } R_m \to 0,$$

and thus $\operatorname{ind}_{B(y,R) \setminus \{y\}} B_V = 0$.

By contradiction, suppose that $y \in \operatorname{sing} V$. Then $\operatorname{ind}_{B(y,r)} B_V = 0$ holds in the whole ball $B(y, r)$ away from $\operatorname{sing} V$, and the regularity results of [Wic14a] give $\dim_{\mathcal{H}} B(y, R) \cap \operatorname{sing} V \leq n - 7$, so that $y \notin \operatorname{sing} V$. □

Claim 9 concludes the induction step; together with the base of induction, we have proved that $\dim_{\mathcal{H}} \operatorname{sing} V \leq n - 7$. This finishes the proof of Theorem 6.1.8.

APPENDIX A

MEASURE-FUNCTION CONVERGENCE

In this appendix we give two elementary measure-theoretical lemmas that are used in the proofs of Lemma 6.2.11 and in Appendix B. Essentially they give a weak compactness result for sequences of the form $(f_i \, \mathrm{d}\mu_i \mid i \in \mathbf{N})$, with μ_i Radon measures and $f_i \in L^2(\mu_i)$. The weak convergence in question is sometimes called *measure-function convergence* in the literature. It appears in the work of Hutchinson [Hut86] on so-called *curvature varifolds*; there one also finds a proof of Lemma A.1 under more general hypotheses on the sequence (f_i).

LEMMA A.1 ([Hut86, Ton05]). *Let X be a locally compact Hausdorff space, let $(\mu_i \mid i \in \mathbf{N})$ be a sequence of Radon measures on X, and $(f_i \mid i \in \mathbf{N})$ be a sequence of real-valued Borel-measurable functions. Suppose that*

$$\sup_i \mu_i(X) < +\infty,$$
$$\sup_i \int_X f_i^2 \, \mathrm{d}\mu_i < +\infty.$$

Then there is a Radon measure μ and $f \in L^2(\mu)$ such that for some subsequence $\mu_{i'} \to \mu$ and $f_{i'} \, \mathrm{d}\mu_{i'} \to f \, \mathrm{d}\mu$ weakly as Radon measures, i.e.

$$\int_X f_{i'} \phi \, \mathrm{d}\mu_{i'} \to \int_X f \phi \, \mathrm{d}\mu \quad \textit{for all } \phi \in C_c(X).$$

Moreover, the weak limit $f \, \mathrm{d}\mu$ satisfies

$$\int_X f^2 \, \mathrm{d}\mu \leq \liminf_{i \to \infty} \int_X f_i^2 \, \mathrm{d}\mu_i. \tag{A.1}$$

REMARK A.2. In our applications X is either an open subset of $U \subset M$ or its Grassmannian $Gr_n(U)$, and μ_i is either $\|V^i\|$ or V^i.

PROOF. The signed Radon measures $\nu_i := f_i \, \mathrm{d}\mu_i$ have bounded total variation, so that the sequences (μ_i) and (ν_i) have convergent subsequences, with limits the Radon measures μ and ν respectively. We extract these subsequences without relabelling their indices.

Consider an arbitrary $\phi \in C_c(X)$. By the weak convergence $\nu_i \to \nu$,

$$\int \phi \, \mathrm{d}\nu = \lim_{i \to \infty} \int \phi f_i \, \mathrm{d}\mu_i \leq \|\phi\|_{L^2(\mu)} \liminf_{i \to \infty} \|f_i\|_{L^2(\mu_i)},$$

where we used the weak convergence $\mu_i \to \mu$ to get $\lim_{i\to\infty}\|\phi\|_{L^2(\mu_i)} = \|\phi\|_{L^2(\mu)}$. As $C_c(X)$ is dense in $L^2(\mu)$, the measure ν defines a bounded linear functional on $L^2(\mu)$, and by duality there is $f \in L^2(\mu)$ with $\|f\|_{L^2(\mu)} \le \liminf_{i\to\infty}\|f_i\|_{L^2(\mu_i)}$ such that $\nu = f\,\mathrm{d}\mu$. □

If the densities f_i are in $C_c(X)$ and converge strongly, then their limit coincides with the density of the weak limit of $(f_i\,\mathrm{d}\mu_i)$.

COROLLARY A.3. *Additionally to the hypotheses of Lemma A.1, assume that $f_i \in C_c(X)$, and that $\|f_i - f\|_{L^\infty} \to 0$ for some $f \in C_c(X)$. Then, additionally to the conclusions of Lemma A.1,*

$$\int_X f^2\,\mathrm{d}\mu = \lim_{i\to\infty}\int_X f_i^2\,\mathrm{d}\mu_i. \tag{A.2}$$

PROOF. We first show that $f_i\,\mathrm{d}\mu_i \to f\,\mathrm{d}\mu$. Let $\varphi \in C_c(X)$ be arbitrary, then

$$\int f_i\varphi\,\mathrm{d}\mu_i - \int f\varphi\,\mathrm{d}\mu = \int (f_i - f)\varphi\,\mathrm{d}\mu_i + \int f\varphi\,\mathrm{d}\mu_i - \int f\varphi\,\mathrm{d}\mu.$$

The first term $|\int (f_i - f)\varphi\,\mathrm{d}\mu_i| \le \|f_i - f\|_{L^\infty}\|\varphi\|_{L^1(\mu_i)} \to 0$ as $i \to \infty$. The remaining terms $\int f\varphi\,\mathrm{d}\mu_i - \int f\varphi\,\mathrm{d}\mu \to 0$ as $i \to \infty$ by the weak convergence $\mu_i \to \mu$. We reason similarly to show (A.2):

$$\left|\int_X f^2\,\mathrm{d}\mu - \int_X f_i^2\,\mathrm{d}\mu_i\right| \le \left|\int_X f^2\,\mathrm{d}\mu - \int_X f^2\,\mathrm{d}\mu_i\right| + \mu_i(X)\|f_i^2 - f^2\|_{L^\infty}.$$

The first term goes to 0 by the weak convergence $\mu_i \to \mu$, and so does the second as $\sup_i \mu_i(X) < +\infty$ and $\|f_i^2 - f^2\|_{L^\infty} \to 0$. □

Appendix B

Generalised second fundamental forms

Our main aim in this appendix is to give a proof of Proposition 6.2.6. We follow the approach of [Ton05], where the case of stable u_i is treated using notions from [Hut86]. Our account is self-contained but for the fact that we refer to these two works for some technical, but routine calculations.

Throughout this section, we assume that $U \subset M$ is isometrically embedded in some $\mathbf{R}^q$, and $W \subset\subset U \setminus \operatorname{sing} V$ is an open subset with $W \cap \operatorname{reg} V \neq \emptyset$. The fibre of the Grassmannian $Gr_n(U)$ at $x \in U$ is identified with the subspaces

$$\{S \subset \mathbf{R}^q \mid S \subset T_x M, \dim S = n\} \subset U \times Gr(n,q),$$

where $Gr(n,q) = \{S \subset \mathbf{R}^q \mid \dim S = n\}$ is the set of n-dimensional linear subspaces of $\mathbf{R}^q$. We furthermore identify an element $S \in U \times G(n,q)$ with the corresponding orthogonal projection $\mathbf{R}^q \to S$, so that $Gr_n(U) \subset U \times \mathbf{R}^{q^2}$. Throughout, $P(x) \in \mathbf{R}^{q^2}$ represents the orthogonal projection $\mathbf{R}^q \to T_x M$ and $(e_1, \dots, e_q)$ is the standard basis of $\mathbf{R}^q$; ∂_i and ∂^*_{ij} denote differentiation with respect to e_i and $e_i \otimes e_j \in \mathbf{R}^{q^2}$ respectively.

Consider first a smooth embedded hypersurface $\Sigma \subset U$, which we implicitely identify with the varifold $V_\Sigma := V_{\Sigma,1}$ with constant multiplicity. We consider test functions $\phi \in C^1(U \times \mathbf{R}^{q^2})$ with compact spatial support, that is with compact support in the first variable. We associate to it a function $\varphi \in C^1_c(\Sigma)$ defined by $\varphi(x) = \phi(x, S^\Sigma(x))$, where $S^\Sigma(x) \in \mathbf{R}^{q^2}$ is the orthogonal projection $\mathbf{R}^q \to T_x\Sigma$. Define a vector field $X \in C^1_c(\Sigma, TM)$ by $X = \varphi P(e_j)$, where e_j is one of the standard basis vectors. Its component tangential to Σ is $\varphi S^\Sigma(e_j)$, and by the standard divergence theorem we get $\int_\Sigma \operatorname{div}_\Sigma(\varphi S^\Sigma e_j) = 0$. A routine calculation shows that in coordinates

$$\operatorname{div}_\Sigma(\varphi S^\Sigma e_j) = S^\Sigma_{rj}\partial_r \phi + \phi S^\Sigma_{ri}\partial_i S^\Sigma_{jr} + S^\Sigma_{ji}\partial_i S^\Sigma_{kr}\partial^*_{kr}\phi, \tag{B.1}$$

with summation over repeated indices [Hut86]. Abbreviate $B^\Sigma_{jkr} = S^\Sigma_{ji}\partial_i S^\Sigma_{kr}$ and substitute this into the divergence formula:

$$\begin{aligned} 0 &= \int_\Sigma S^\Sigma_{rj}\partial_r\phi + B^\Sigma_{rjr}\phi + B^\Sigma_{jkr}\partial^*_{kr}\phi \,\mathrm{d}\mathcal{H}^n \\ &= \int_{Gr_n(U)} S_{rj}\partial_r\phi + B^\Sigma_{rjr}\phi + B^\Sigma_{jkr}\partial^*_{kr}\phi \,\mathrm{d}V_\Sigma(x,S). \end{aligned}$$

This identity is the basis of the following definition by Hutchinson [Hut86].

DEFINITION B.1 (Generalised curvature, [Hut86]). An n-dimensional integral varifold V in U is said to have *generalised curvature* if there exists a function $B = (B_{ijk})$ with values in $\mathbf{R}^{q^3}$ defined V-a.e. on $Gr_n(U)$ with

(a) $B \in L^1_{\text{loc}}(V)$,
(b) $\int_{Gr_n(U)} S_{rj}\partial_r\phi + B_{rjr}\phi + B_{jkr}\partial^*_{kr}\phi \,\mathrm{d}V(x,S) = 0$ for all $\phi \in C^1(U \times \mathbf{R}^{q^2})$ with compact spatial support.

The following lemma shows that the function B is well-defined V-a.e. on $Gr_n(U)$; it is taken from [Hut86].

LEMMA B.2 ([Hut86]). *Any two B and $\widetilde{B}$ satisfying (a) and (b) coincide V-a.e. on $Gr_n(U)$.*

PROOF. Let $\phi(x,S) = \alpha(x)\beta(S)$, where $\alpha \in C^1_c(U)$ and $\beta \in C^1(\mathbf{R}^{q^2})$. Letting $\beta \equiv 1$ we see that $\int B_{rjr}\alpha \,\mathrm{d}V = \int \widetilde{B}_{rjr}\alpha\,\mathrm{d}V$, and thus $B_{rjr} = \widetilde{B}_{rjr}$ V-a.e. on $Gr_n(U)$. If we now let $\beta(S) = 1$ if $S = S_{kr}$ and 0 otherwise then $\int B_{jkr}\alpha\,\mathrm{d}V = \int \widetilde{B}_{jkr}\alpha\,\mathrm{d}V$, whence the conclusion follows. □

In particular, applied to the smooth hypersurfaces, the following is an immediate consequence.

COROLLARY B.3. *If Σ is a smoothly embedded hypersurface, then $B_{ijk}(x,S) = B^\Sigma_{ijk}(x,S)$ for V_Σ-a.e. $(x,S) \in Gr_n(U)$, where $B^\Sigma_{ijk}(x,S) = S_{il}\partial_l S^\Sigma_{jk}$.*

The following elementary calculation relates B^Σ to the second fundamental form A^Σ.

LEMMA B.4. *Let A^Σ be the second fundamental form of a smoothly embedded hypersurface $\Sigma \subset U$. Then*

$$\langle A^\Sigma(S^\Sigma e_i, S^\Sigma e_j), Pe_k\rangle = P_{kr}S^\Sigma_{js}S^\Sigma_{il}\partial_l S^\Sigma_{rs} = P_{kr}S^\Sigma_{js}B^\Sigma_{irs}(x,S^\Sigma).$$

PROOF. Write A instead of A^Σ in this proof to simplify notation. The covariant derivative on M is the component of $D = \nabla^{\mathbf{R}^q}$ tangent to M, so $A = (D^{T_M})^{\perp_\Sigma} = (D^{\perp_\Sigma})^{T_M}$. As $e_k^{T_M} = P_{kr}e_r$, we get

$$\begin{aligned} A^k_{ij} &:= \langle A(S^\Sigma e_i, S^\Sigma e_j), e_k^{T_M}\rangle \\ &= \langle (D_{S^\Sigma e_i}S^\Sigma e_j)^{\perp_\Sigma}, Pe_k\rangle = P_{kr}\langle D_{S^\Sigma e_i}S^\Sigma e_j, e_r^{\perp_\Sigma}\rangle. \end{aligned}$$

Similarly $e_r^{\perp_\Sigma} = (\delta_{rs} - S^\Sigma_{rs})e_s$, so

$$A^k_{ij} = P_{kr}(\delta_{rs} - S^\Sigma_{rs})\langle D_{S^\Sigma e_i}S^\Sigma e_j, e_s\rangle = P_{kr}(\delta_{rs} - S^\Sigma_{rs})D_{S^\Sigma e_i}S^\Sigma_{js}.$$

As $S^\Sigma_{rs}S^\Sigma_{js} = S^\Sigma_{rj}$, we finally get $A^k_{ij} = P_{kr}S^\Sigma_{js}D_{S^\Sigma e_i}S^\Sigma_{rs} = P_{kr}S^\Sigma_{js}S^\Sigma_{il}\partial_l S^\Sigma_{rs}$, as required. □

This expression can then be used to generalise second fundamental forms from the smooth to the varifold setting.

DEFINITION B.5 (Generalised second fundamental forms, [Hut86]). Let V be an integral n-varifold with generalised curvature. Then its *generalised second fundamental form* is the function $A = (A_{ij}^k)$ with values in $\mathbf{R}^{q^3}$ and defined at V-a.e. $(x, S) \in Gr_n(U)$ by

$$A_{ij}^k(x, S) = P_{kr} S_{js} B_{irs}.$$

For a smoothly embedded $\Sigma \subset U$, we see after combining Corollary B.3 with Lemma B.4 that the generalised second fundamental form A of V_Σ is equal to the classical second fundamental form A^Σ in the sense that

$$A_{ij}^k(x, S) = \langle A^\Sigma(Se_i, Se_j), Pe_k \rangle \quad \text{for } V_\Sigma\text{-a.e. } (x, S) \in Gr_n(U).$$

We now want to relate these notions to the varifolds V^i defined in the main body. To simplify notation, we fix a critical point $u = u_i$ with associated varifold $V^\epsilon = V^i$. We define a 'second fundamental form' for V^ϵ using the coordinate expressions from Lemma B.4.

DEFINITION B.6. Define the functions $A^\epsilon = (A_{ij}^{\epsilon,k})$ and $B^\epsilon = (B_{ijk}^\epsilon)$ with values in $\mathbf{R}^{q^3}$ at all $(x, S) \in Gr_n(U)$ where $\nabla u(x) \neq 0$ by

$$\begin{aligned} B_{ijk}^\epsilon(x, S) &= S_{il} \partial_l S_{jk}^\epsilon, \\ A_{ij}^{k,\epsilon}(x, S) &= P_{kr} S_{js} S_{il} \partial_l S_{rs}^\epsilon = P_{kr} S_{js} B_{irs}^\epsilon, \end{aligned}$$

where $S^\epsilon = S^\epsilon(x) \in \mathbf{R}^{q^2}$ represents the projection $\mathbf{R}^q \to T_x\{u = u(x)\}$, and $P = P(x) \in \mathbf{R}^{q^2}$ the projection $\mathbf{R}^q \to T_x M$.

Technically speaking the function A^ϵ is not the second fundamental form of V^ϵ, as B^ϵ satisfies the integral identity of Definition B.1 only up to a small error term. This can be seen as follows: take an arbitrary vector field $X \in C_c^1(U, TM)$, multiply the Allen–Cahn equation by $\langle X, \nabla u \rangle$ and integrate by parts twice to obtain

$$\int_U |\nabla u|^2 \operatorname{div} X - \langle \nabla_{\nabla u} X, \nabla u \rangle = \int_U \left(\frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon^2} \right) \operatorname{div} X,$$

which using integration with respect to V^ϵ is equivalent to

$$\text{(B.2)} \qquad \int_{Gr_n(U)} \operatorname{div}_S X \, \mathrm{d}V^\epsilon(x, S) = \frac{1}{2\sigma} \int_U \left(\epsilon \frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon} \right) \operatorname{div} X,$$

where $\operatorname{div}_S X = \sum_{i=1}^q \langle D_{Se_i} X, Se_i \rangle$. As before let $X = \phi(x, S^\epsilon) S^\epsilon(e_j)$, where $\phi \in C^1(U \times \mathbf{R}^{q^2})$ has compact spatial support. Substitute this into (B.2) and perform routine coordinate computations as before in (B.1) to get

(B.3) $$\int_{Gr_n(U)} S_{rj}\partial_r\phi + B^\epsilon_{rjr}\phi + B^\epsilon_{jkr}\partial^*_{kr}\phi \, \mathrm{d}V^\epsilon(x,S) = \frac{1}{2\sigma}\int_U \left(\epsilon\frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon}\right)\operatorname{div} X.$$

The integral on the right-hand side goes to 0 as $\epsilon \to 0$—this is (6.3) in Theorem 6.1.7. This justifies the abuse of language that is calling A^ϵ the 'second fundamental form' of V^ϵ.

We now compare A^ϵ with the second fundamental form A^Σ of the level sets of u near regular points: take a point $x \in U$ with $\nabla u(x) \neq 0$. Then the level set $\{u = u(x)\}$ is embedded in a neighbourhood B around x; write $\Sigma = \{u = u(x)\} \cap B$. The calculations from Lemma B.4 show that $A^\epsilon(x, S^\epsilon) = A^\Sigma(x)$, so the second fundamental forms from Definitions 6.2.1 and B.6 agree V^ϵ-a.e. Combining this observation with (6.1), we get

$$|A^\epsilon|^2(x,S^\epsilon) = |A^\Sigma|^2(x) \leq \frac{1}{|\nabla u|^2}(|\nabla^2 u|^2 - |\nabla|\nabla u||^2).$$

Therefore, when $\delta^2 E_\epsilon(u)(|\nabla u|\phi, |\nabla u|\phi) \geq 0$ for some $\phi \in C^1_c(U)$, then $\int_{Gr_n(U)}|A^\epsilon|^2\phi^2\,\mathrm{d}V^\epsilon \leq \int_U|\nabla\phi|^2 - \mathrm{Ric}(\nu^\epsilon,\nu^\epsilon)\phi^2\,\mathrm{d}\|V^\epsilon\|$ as in (6.3), and Corollary 6.2.5 also remains valid.

The results in this appendix are valid for every term in the sequence (u_i) satisfying Hypotheses (A)–(C). Let (V^i) be the corresponding varifolds as in (6.1), and let (A^{ϵ_i}) be their second fundamental forms as in Definition B.6. We restate Proposition 6.2.6 in the following equivalent form, with A^{ϵ_i} in place of A^i.

Proposition B.7. *If* $\sup_i \int_W |A^{\epsilon_i}|^2\,\mathrm{d}\|V^i\| < +\infty$, *then some subsequence* $A^{\epsilon_{i'}}\,\mathrm{d}V^{\epsilon_{i'}} \to A\,\mathrm{d}V$ *weakly as Radon measures on* $Gr_n(W)$, *and*

$$\int_W |A|^2\,\mathrm{d}\|V\| \leq \liminf_{i\to\infty}\int_W |A^{\epsilon_i}|^2\,\mathrm{d}\|V^i\|,$$

where A is the classical second fundamental form of $\operatorname{reg} V \subset M$.

Proof. Routine calculations as above in the proof of Lemma B.4 show that B^{ϵ_i} is related to A^{ϵ_i} as follows for all i:

(B.4) $$B^{\epsilon_i}_{jkl}(x,S^{\epsilon_i}) = A^{l,\epsilon_i}_{jk} + A^{k,\epsilon_i}_{jl} + S^{\epsilon_i}_{ks}S^{\epsilon_i}_{jr}\partial_r P_{sl} + S^{\epsilon_i}_{ls}S^{\epsilon_i}_{jr}\partial_r P_{ks}.$$

If we square (B.4) and sum over $j,k,l = 1,\ldots,q$, we get

$$|B^{\epsilon_i}|^2 \leq 8(|A^{\epsilon_i}|^2 + |DP|^2) \quad V^i\text{-a.e. in } Gr_n(U)$$

The term $|DP|^2 := \sum_{j,k,l}^q(\partial_j P_{kl})^2$ can be bounded by some constant $C(M)$, so that $\sup_i \int_W |B^{\epsilon_i}|^2\,\mathrm{d}\|V^i\| < +\infty$ as well.

By Lemma A.1 we can pass to convergent subsequences $A^{\epsilon_i}\,\mathrm{d}V^i \to A\,\mathrm{d}V$ and $B^{\epsilon_i}\,\mathrm{d}V^i \to B\,\mathrm{d}V$ with limits related by $A^l_{jk} = P_{lr}S_{ks}B_{jrs}$ V-a.e. in $Gr_n(W)$. The limit $A\,\mathrm{d}V$ also satisfies

$$\int_{Gr_n(W)} |A|^2\,\mathrm{d}V \le \liminf_{i\to\infty} \int_{Gr_n(W)} |A^{\epsilon_i}|^2\,\mathrm{d}V^i,$$

and the analogous inequality holds for $B\,\mathrm{d}V$. Moreover the error term on the right-hand side of (B.3) tends to 0 as $i \to \infty$, so the weak limit $B\,\mathrm{d}V$ satisfies

$$\int_{Gr_n(W)} S_{rj}\partial_r\phi + B_{rjr}\phi + B_{jkr}\partial^*_{kr}\phi\,\mathrm{d}V(x,S) = 0 \tag{B.5}$$

for all $\phi \in C^1(W \times \mathbf{R}^{q^2})$ with compact spatial support. By Corollary B.3 we have $B = B^{\mathrm{reg}\,V}$ and thus also $A = A^{\mathrm{reg}\,V}$ V-a.e. in $Gr_n(W)$. This concludes the proof. □

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