Analysis of Structural Properties of Complex and Networked Systems

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DOI:
10.33612/diss.136303707

Publication date:
2020

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The research described in this dissertation has been carried out at the Faculty of Science and Engineering, University of Groningen, the Netherlands.

The research reported in this dissertation is part of the research program of the Dutch Institute of Systems and Control (DISC). The author has successfully complete the educational program of DISC.

This work was supported by Chinese Scholarship Council (CSC), the Chinese Ministry of Education.

Printed by: **Ipskamp Printing** || *https://www.ipskampprinting.nl/*

Cover picture is designed by: Cheng and Jiajia

ISBN: 978-94-6421-088-0

Analysis of Structural Properties of Complex and Networked Systems

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus, Prof. C. Wijmenga,
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

Monday 9 November, 2020 at 12.45 hour

by

Jiajia Jia

born on 18 February 1989
in Jiangsu, China

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To my family

Fiancee Kangwei,
Mother Yuxia, Father Xueyou and Sister Xiaofan

献给我的家人

未婚妻左康薇、
母亲刘玉侠、父亲贾学友、姐姐贾小凡

Acknowledgments

Just like yesterday, four years ago, I came to Groningen with hope for the future and perturbed in mind. How time flies, my Groningen time is drawing to an end. At this farewell moment, my most profound feeling is that it has been the wisest choice for me so far to be a PhD student in the Systems, Control, and Applied Analysis (SCAA) group at the University of Groningen. The bits and pieces of the past four years will become the most precious and warmest memories in my life. There are many people here that I will always miss and never forget. It is you who have been with me through the pain and confusion caused by the setbacks, and it is also you who have witnessed my joy and excitement after each success. Fortunately, I have finally completed the final challenge of my PhD, finishing my PhD thesis. I will start a new career in the near far. Therefore, looking back on the past and looking forward to the future, I would like to offer this thesis as my gift to you. At the same time, please allow me to express my most sincere gratitude to all of you.

Firstly, many thanks go to my two superiors: Prof. Harry L. Trentelman and Prof. M. Kanat Camlibel. I sincerely hope that our deep mutual affection between teacher and student can last forever.

Dear Harry, I am honored to be one of your PhD students. I still remembered our first meet in Beijing five years ago. At that time, your warm and cheerful personality attracted me deeply. Your recognition and encouragement also strengthened my courage to start a doctoral research career. At that moment, I felt fortunate to meet a mentor who was so knowledgeable and considerate. The past four years of getting along have repeatedly proved how lucky it is to be your student. In the past four years, you have used infinite patience and passion for gradually cultivating me from a toddler in scientific research to a progressively independent researcher. When I look back on the past, I surprisingly find that I have received hundreds of emails from you in the past four years besides the countless face-to-face conversation and discussions. There is no doubt that each step of my progress and completing my doctoral dissertation are inseparable from your detailed guidance. “Rich knowledge makes a teacher; high moral reputation makes a model.” Besides teaching me, your daily practices have also become a lifelong-learning model for me. More explicitly, you have shown me how to be rigorous, hardworking, open-minded, deep-thinking, and skeptical for academic work. While concerning daily life, you have set a perfect example of balancing work and rest and how to live life fully. In my future career, I will devote myself to continuing the codes of conduct and attitudes to work and life that I learned from you. I will also try to be a qualified researcher and an excellent teacher like you. Lastly, I sincerely wish you, my beloved supervisor, happy and healthy every day. I hope that we will have the opportunity to meet each other in the future.

Dear Kanat, thank you very much for your care and help for me over the past four years. I cherish all the memories of our past four years together. I still remember that the first time we met each other when I just arrived in Groningen. My first impression of you was that you were a gentleman but slightly critical. However, as I got in touch with you more and more, I realized that you are a warm and friendly person, and are always ready to help me in scientific research and daily life. You have consistently helped me find and guide me to overcome my deficiencies. For instance, it was you who have taught me how to write mathematics; it was you who warned me that I have to work hard on writing to become a scientist; it was also you who reminded me to read what I have written with a critical eye. There remain too many similar instructions. Therefore, please allow me to offer you my most sincere gratitude. At the same time, I wish you and your family the best of luck and happiness every day and sincerely hope to see you again in the future.

Also, I would like to thank Prof. Arjan van der Schaft, Prof. Mehran Mesbahi, and Prof. A. Stephen Morse for being part of the assessment committee.

Thanks also go to Henk, Wouter, and Nikos for our pleasant collaborations, which have led to several excellent papers and constituted most of my thesis.

Henk, in our continuous cooperation over the past four years, you have not only discussed and solved problems with me but also kept trying to help me improve my writing and presentation skills. One of my most precious gains in the past four years is that we have gradually formed a deep friendship through academic cooperations. Thanks very much for agreeing to act as my paranymph. Moreover, I want to thank you and your wonderful wife, Tessa, for translating the summary of my thesis. I hope you both have a happy life and prosperous careers in the future. Wouter, I always remember the time we spent discussing problems. You are so optimistic, lively, and enthusiastic that doing research with you is a pleasant activity that makes me forget the time. Therefore, each of our discussions ends with the word 'we can call it a day now.' Besides, thank you very much for helping me to translate the abstract of my thesis. I sincerely hope that you are always happy and maintain your infectious personality. Nikos, I miss the experience of working with you very much. You are

a hardworking and curious student. I remember that when you first came into our research questions, our discussions always gave me valuable teaching and learning experience. Besides, you kept giving me feedback about my shortcomings in teaching and presenting. I hope that you will be happy every day and achieve more tremendous success in your new job.

I would also like to thank all the members of my group, SCAA. You have provided a flexible, helpful, and comfortable research environment for me during the past four years. I enjoyed our lunchtime conversations and day-to-day interactions. Junjie, in my heart, apart from colleagues, we are also very destined friends. We already knew each other before we came to Groningen. Then, we came to the Netherlands on the same plane together. Besides, we have completed PhD research together under the guidance of two same supervisors. In the past four years, you have helped me solve many confusions in work and life. Thank you for your help and support all the time. I wish you a happy life and a successful career in Munich. Mark, you are a knowledgeable and enthusiastic mathematician. Thank you very much for your valuable comments and suggestions on my research questions in the past. It is hard to forget the many pleasant night chats we had when we worked overtime together. I wish you a successful completion of your doctoral thesis and look forward to seeing your music plan come true soon. Brayan, you are very talented and good at solving problems. It is my pleasure to collaborate with you in our recent research problem. I hope our research paper will be finished soon, and hope to cooperate with you again in the future. Jaap, thank you for giving me suggestions for the propositions of my thesis. Thanks also go to Arjan, Bart, Stephan, Alden, Yahao, Aska, Sumon, Koroosh, Anne-men, Paul, Di, Jiaming, Teke, Cheng, Pooya, Li Wang, Hidde-Jan, Noorma and Isil. I would also like to thank Eduardo, Marc Paul, Enis, and Francesca. It is pleasant sharing the same office together with you. Many thanks also go to our wonderful secretaries: Ineke, Elina, Anita, Sarah, Jan, Monique, and Renske.

I would like to thank my friends I met in Groningen. I would like first to thank the friends I met in Stationsplein 9. Firstly, I would like to thank Bei Guo, you are an enthusiastic and trustworthy brother. In the past four years, we have cared about and supported each other. We also have visited many cities and towns around the Netherlands together. Thanks very much for agreeing to act as my paranymph. I sincerely wish you a happy and successful life in Tianjin in the future. Fangyuan, you are a person who loves life so much, especially interested in travel. I fell delighted that you have invited me to travel together many times. These experiences made my PhD life more colorful and left me with many unforgettable memories. I wish you a smooth completion of your doctoral dissertation and expect you to acquire a satisfactory job as soon as possible. Qian, you gave me the impression of being optimistic and cheerful and full of humor. Thank you for the happiness and laughter you have brought to me in the past four years. I hope you finish your PhD research smoothly and keeps happy and optimistic in the future. Yu Yi, you are an outgoing, enthusiastic, and leadership person. Thank you for introducing me to various social activities in the past few years. These experiences have enriched my life and expanded my knowledge. I hope you will achieve more outstanding brilliance and achievements in the future. Wei Teng, you are a good friend who is willing to help others and are thoughtful and meticulous. I miss the time very much when I talked and laughed with you. I hope you can continue the success and brilliance of scientific research in Xi-an city. Thanks also goes to Lulu, Xiu Jia, Liping, Shuai Feng, Shun Fang, Jian Gao, Yafeng, Mengfan, Jianjun, Yukun. You swept my anxiety of being new in Groningen and made me feel like I was in a big family.

I would also like to thank some other friends that I met in Groningen. Firstly, thanks go to my ‘shixiong’ Xiaodong and Yuzhen. Xiaodong, you are a wise and insightful person. Thank you very much for your guidance and advice when I first started my PhD study. This has a profound impact on the progress of my PhD research. Yuzhen, thank you very much for your help in writing my doctoral dissertation, and I wish you success in your postdoctoral research. At the same time, I would like to express my most sincere thanks to Cheng Wang. Traveling and photography with you have provided me a lot of fun in my life in the past two years. Moreover, I thank you for your help in the cover design of my thesis. I would like to thank you and your wonderful wife, Yuequ, for sharing me with delicious dinners. I wish you both a successful doctoral study and a happy family. I hope your lovely daughter, Anna, a healthy growth. My thanks also go to Miao Guo and Bei Tian. The happy and sweet life of you two has left a deep impression on me. I am also very grateful to you for sharing the happy time with me. It is hard to forget our ‘crazy’ experience of partying at your home until the early hours of the morning. I sincerely wish you two keeping happy and optimistic forever. Haibin, I am delighted to meet you, such a bold and sincere friend. And I still remembered the happiness from chatting with you. Wish your doctoral research is smooth and fruitful. I hope to continue to drink and chat with you in the future. Thanks to my dear friends Hongyu, Weijia, Sha Luo, Tinghua, Xiaoyan, Siwei, Lanlin, Ningbo, Xuegang, and all the other friends that I met in this city, and wish you good luck in the future.

I would also like to thank the friends of the Groningen Chinese Choir. I cherished the experience of rehearsing and participating in the chorus performance of the Chinese New Year Gala in Utrecht this year. I wish all of you a happy and happy future, and I hope the choir will do better.

Moreover, I would like to acknowledge the financial support of the China Scholarship Council (CSC). Thanks also go to the people in DISC for the excellent courses, informative summer schools and good social chances in Benelux meeting. Bedankt

Groningen, you are a city so beautiful and peaceful that you have offered me a comfortable and suitable place to study and pursue scientific research. Besides, you also contain so many good memories of mine.

See you again, my supervisors! See you again, my colleagues! See you again, my friends! See you again Groningen!

我还要感谢我的身在家乡的亲友们，是你们的关怀和支持让我有了足够的勇气和信念完成了四年的博士研究学习。首先，感谢我的亲爱的叔叔、舅舅以及兄弟姐妹们，难以忘记当初离开家乡时你们的满含热泪不舍担忧的眼神，难以忘记你们殷切期待和谆谆嘱托。其次，感谢威姐、凡姐、姐夫和外甥们，感谢你们在我漂泊海外时对我的关心和支持，是你们在过去几年的时间里代我陪伴爸妈解决了我的后顾之忧，可以安心在外求学。同时，我要由衷地感谢我的爸爸妈妈，你们永远那么宠溺我，一直以来只有关怀支持默默守护而没有半点要求，是你们的鼓励使我得以轻装简行地追求自己的理想，可以醉心于学术的研究，完成了博士论文的撰写。最后，我想和我的挚爱康薇说，今生与君相识相爱何其幸哉。你的坚强，独立和善解人意，让我领会了爱情的真谛。虽然在过去的几年里我们聚少离多，相隔万里，我们的感情却进行了蜕变。正如歌词所说：两地相思苦，一世回望甜。我想将本文献给你作为向你求婚的礼物，并祝愿我们以后携手面对未来的生活的挑战，并且朝朝暮暮宿长风，白首不相移。

Jiajia Jia
Groningen,
November, 2020

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1 Introduction

‘*I think the next century will be the century of complexity.*’ *–Stephen Hawking, January 23, 2000. (San Jose Mercury News)*

As pointed out by Stephen Hawking, over the past decades, humankind has been experiencing systems that tend to be increasingly sophisticated and interconnected. There are several reasons for this. One of the reasons is that due to technological developments such as the emergence of the Internet and the growing relevance of smart power grids [1–3], more and more engineering systems consist of millions or even billions of subsystems. For example, the Internet integrates billions of computers and routers. Also, in natural and social science, deeper understanding of biological systems and society have contributed to this surge of large scale interconnected systems [2–6]. For instance, our biological existence relies on seamless interactions between thousands of genes and metabolites within our cells, and society requires cooperation between billions of individuals. As a result, we are now surrounded by systems that are inherently complex [7], which are referred to as *complex systems*, see, e.g., [1, 7–10] and references therein. Given the importance and universality of complex systems in modern society, and in science and economy, their understanding, mathematical description, prediction, and, eventually, control is one of the most significant intellectual and scientific challenges of the 21st century [7]. This challenge roots in the fact that in order to understand the behavior of a complex system, we must understand not only the action of the parts, but also how these parts act together to form the functioning of the whole.

1.1 Background

The I Ching (易经), one of China's oldest philosophical books, has provided the Chinese people with philosophical wisdom in order to deal with the complex and changing world for thousands of years. One of the highlights in this book is the three principles of I Ching stating the following:

I. *Changes (变易): stating that everything keeps changing through various rules.*

II. *Simplicity (简易): stating that no matter how complex the universe and changes that occur are, they turn out to be simple after we understand the principles behind them.*

III. *Invariant (不易): stating that even though things keep changing, there exist certain underlying patterns or functions that do not change.*

In this section, we will elaborate on a modern mainstream research idea and philosophy to deal with the challenges of complex systems. This philosophy and the classic wisdom in the I Ching turn out to be surprisingly consistent.

1.1.1 Complex networks–a skeleton of complex systems

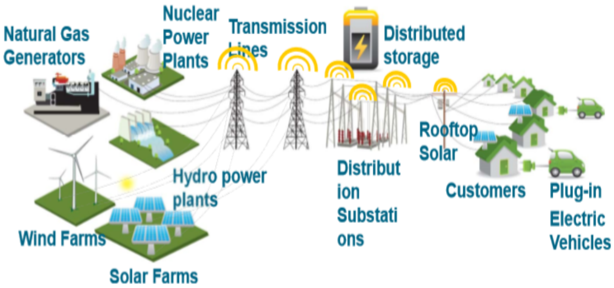
As we have mentioned before, the difficulty of understanding and controlling a complex system roots in the entanglement of the nontrivial and various dynamics of its parts, and the large-scale and complicated interconnection relations. Depending on the field, the parts of a complex system may represent different objects or subsystems, possessing different characteristics and dynamics. For example, the parts of friendship networks in social science are individuals, as depicted in Figure 1.1a, those of the Internet are computers and routers, shown in Figure 1.1b, while smart grids in Figure 1.1c consist of many kinds of components including smart meters, smart appliances, renewable energy resources, etc. In addition, most complex systems contain a large number of parts (or *agents* in some fields), and the topology interconnecting these parts might be irregular or even evolving in time.

In order to be able to analyze a complex system, naturally the question then arises on how to deal with this complicated entanglement of its parts. This is a nontrivial and challenging question, but keeping in mind that no matter how complex the universe and its changes are, they become simple after we understand the basic principles behind them.

Thanks to the emergence of network science [1, 3, 7, 11] in the first decade of the 21st century, researchers have found that notwithstanding the differences in form, size, nature, age, and scope of realistic complex systems, their underlying network structure

(a) Friendship network system.
https://www.infosystem.co.id/the-different-types-of-friendship-networks-and-how-they-can-influence-your-success/.

(b) The Internet.
https://askleo.com/whats-the-difference-between-the-web-and-the-internet/.



(c) Smart grid system.
https://forum.huawei.com/enterprise/en/smart%C2%A0grid-smart%C2%A0grid-introduction/thread/523687-100027.

Figure 1.1: Examples of complex systems from different fields.

is driven by common organizing principles [7]. A fundamental idea of network theory is that the network scheme is the principal research object, while the living part of the network, which is contained in the nodes, is kept as simple as possible [12]. Once we disregard the precise nature of the components and that of the interactions between them, networks are often more similar than different from each other. This observation enables us to study universal properties of different types of complex systems modeled as networks, called complex networks [3], which can be represented by directed graphs with a large number of nodes and complicated interconnections. For example, the brain system is one of the most complicated systems in the world. However, a brain system can be translated into a network through four steps, depicted in Figure 1.2, and can then be analyzed using graph-theoretic tools.

Following the above idea, many insights have occurred in the understanding of complex networks, such as, for example, the understanding that many complex systems display a surprising tolerance against errors [13]. Another example is the insight that most pairs of vertices in many realistic networks are connected by paths with quite a short length, often called the small-world effect [14]. An important observation is also the fact that the degree distributions of most networks are power-law distributions. Networks with this property are referred to as scale-free networks [15]. Furthermore, inspired by the aforementioned insights as well as motivated by the ubiquity of control problems in natural, social, and technological systems, more and more attention has been devoted to controlling complex systems and complex networks. For a detailed review of this research topic, we refer to [6] and the references therein.

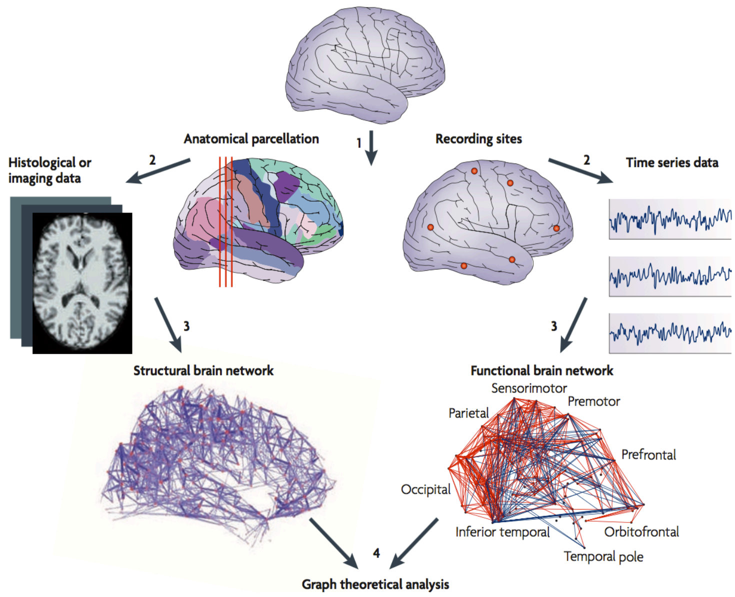


Figure 1.2: Example of modeling brain networks by graphs.
This figure is cited from the Box 1 of [10]. Structural and functional brain networks can be explored using graph theory through the above four steps.

1.1.2 Structural analysis for control of complex networks

This subsection aims to clarify the important role of network structure in control of complex systems. We will illustrate this role employing the concept of controllability, an essential notion in modern control theory that verifies the ability to steer a dynamical system from any initial state to any desired final state in finite time [16].

The current interest in control of complex networked systems was initiated by the pioneering work in [17]. In this paper, the authors first demonstrated that although most complex systems are driven by nonlinear dynamics, their controllability is structurally similar to that of linear systems. Indeed, while many complex systems are characterized by nonlinear interactions between the components, the first step in many control design problems is to establish controllability of a locally linearized system [18]. By this observation, research on controllability of complex networks can be based on a standard linear time-invariant (LTI) system model. Moreover, to allow zooming in on the role of the network graph, it is common to proceed with the simplest possible dynamics at the subsystems of the network and to take the agents in the network to be single integrators, with a one-dimensional state space. Consequently, the overall networked system can be represented by an LTI system of the form

$$\dot{x} = Ax + Bu, \tag{1.1}$$

where x is the state vector of the whole network which consists of the states of the n agents, and u is the control vector collecting direct external controls. The system matrix A represents interconnections among the agents, while the matrix B specifies the routing of the external controls to the state variables.

In the case that the values of the edge weights in the network are known precisely, the matrices A and B are given constant matrices, where A is often taken as the adjacency matrix of the graph [19], or the graph Laplacian matrix [20–25]. Then, by using the Kalman rank test or the Hautus test [26], one can verify whether the network is controllable or not. However, in many situations, the scale of the networks is prohibitively large, and hence the above controllability tests are impracticable. For example, in the Kalman rank test, one needs to check the rank of the so-called controllability matrix $C = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$, while there is no *efficient* algorithm to numerically determine the rank of such controllability matrix C of large-dimensions [17]. Similar problems arise for the other test, in which one needs to check the rank of the matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ for every eigenvalue of A, where I is an identity matrix of appropriate dimension. In addition to the computational complexity due to the large-scale nature, another obstacle is that, in most scenarios, the values of the edge weights in the network are not known exactly. To tackle the above difficulties, the authors in [17] have introduced the concept of *structural controllability* [27], which allows us to check whether a controlled network is structurally controllable or not by merely inspecting its network topology, avoiding expensive matrix operations and precise knowledge of the edge weights. In other words, structural controllability analysis allows us to decide a network's controllability even if we do not know the precise numerical values of the weights of the links among the agents. We only have to make sure that we acquire an accurate 'map' of the system's wiring diagram, i.e., knowledge of which components are linked and which are not.

Up to now, two types of structural controllability have been studied, namely *weak* structural controllability and *strong* structural controllability (SCC). A network is called weakly structurally controllable if there is at least one choice of values for the unknown entries in the system matrices such that the corresponding matrix pair (A, B) is controllable. Due to the generality of controllability, if a network is weakly structurally controllable, then for *almost all* choices of values for the unknown entries in the system matrices, the corresponding matrix pair (A, B) is controllable. On the other hand, the network is called strongly structurally controllable if for *all* choices of nonzero values for the unknown entries, the matrix pair (A, B) is controllable. Conditions for weak and strong structural controllability have been expressed entirely in terms of the underlying network graph, using concepts like cactus graphs, maximal matchings, and zero forcing sets, see [17, 27–32]. For details on the analysis of control principles of complex networks, see [6, 7] and the references therein.

In addition, following the seminal paper [27], many other structural control properties have been analyzed, which has led to the field of *structural control theory*, see, e.g., [33, 34] and the references therein. In [27] and also in subsequent work, the system matrix is not a known, given, matrix, but rather a matrix with a certain pattern, such as a zero/nonzero structure [27, 28], a sign pattern [35, 36] or mixed matrices [37], and so on. However, in the framework of complex systems, it is of particular interest to study the zero/nonzero structure, i.e., the elements of the system matrices are either fixed zeros or nonzero unknown entries. This is due to the following characteristics of the zero/nonzero structure:

1. it allows us to capture an essential part of the structural information in complex systems, i.e., the existence and absence of relations between the subsystems.

2. many control properties of systems can be expressed in terms of an associated directed graph and hence are often intuitive and easy to interpret physically.

3. conditions can be expressed in graph theoretic terms, and hence they can be checked by certain efficient polynomial algorithms.

In this thesis, a family of LTI systems sharing the same zero/nonzero structure is referred to as a *linear structured system* [33].

1.2 Problem statements

In this thesis, we will focus on the analysis of strong structural properties of linear structured systems. In the existing literature up to now, the rather restrictive assumption is usually made that for each of the entries of the system matrices, there are only two possibilities: it is either a fixed zero or an arbitrary nonzero value

[28, 29, 32, 38–41]. This means that, although we do not need the information on the exact values of the network links, the complete wiring topology is needed, i.e., we need to know exactly which connections there exist between the components of the complex system. However, often exact knowledge of the network graph is not available, in the sense that it is unknown whether certain edges in the graph exist or not. This issue of missing knowledge of the network graph appears, for example, in social networks [42], the world wide web [43], biological networks [44, 45] and ecological systems [46]. Another cause for uncertainty about the network graph might be malicious attacks and unintentional failures. This issue is encountered in transportation networks [47], sensor networks [48] and gas networks [49]. Therefore, the first research problem in this thesis is formulated as follows:

Problem 1.1. *Establish a new framework that captures missing knowledge of the wiring topology, and analyze strong structural properties of linear structured systems in this framework.*

On the other hand, in the framework of analysis of strong structural properties, another restrictive assumption up to now has been that the indeterminate entries in the system matrices take their values arbitrarily. However, often in realistic network systems the strength of the interconnection links might have constraints. These constraints can require that some of the nonzero entries have given values, see e.g. [50], or that there are given linear dependencies between some of the nonzero entries, see [51]. More examples can be found in [52–57] and the references therein. This observation leads to a need for a more detailed structure, namely that of a zero/nonzero structure with extra constraints, yielding to a subclass of the family of systems associated with a given zero/nonzero structure. Notice that, roughly speaking, strong structural properties can be regarded as *sufficient* but not *necessary* conditions for their corresponding classical control properties. This implies that the more information we use, the sharper the conditions we will obtain. Therefore, another question is the following.

Problem 1.2. *Establish a new framework allowing extra constraints on the unknown entries, and analyze strong structural properties of linear structured systems in this framework.*

1.3 Outline and contributions of the thesis

We will now explain how this thesis is structured and state its specific contributions, making a distinction between two parts: strong structural properties in a unifying framework of zero/nonzero/arbitrary patterns and zero/nonzero patterns with equality constraints.

In Chapters 2 and 3 we present our main contributions in the context of Problem 1.1. In Chapter 2, we first introduce a new framework for structured systems, namely structured systems with zero/nonzero/arbitrary structure, which capture the case that some of the entries are equal to zero, some of the entries are arbitrary but nonzero, and the remaining entries are arbitrary (zero or nonzero). We then formalize this in terms of pattern matrices whose entries are either fixed zero, arbitrary nonzero, or arbitrary. We establish necessary and sufficient algebraic conditions for strong structural controllability in terms of full rank tests on certain pattern matrices. Next, we provide a necessary and sufficient graph-theoretic condition for the full rank property of a given pattern matrix. This graph-theoretic condition makes use of a so-called color change rule that was introduced in [58]. Based on the above results, we establish a necessary and sufficient graph-theoretic condition for strong structural controllability. The material in this chapter is based on the journal paper [58].

Chapter 3 deals with the fault detection and isolation (FDI) problem for linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices. This chapter follows a geometric approach to verify solvability of the FDI problem for linear structured systems. We first develop a necessary and sufficient condition under which the FDI problem for a given particular linear time-invariant system is solvable. Next, we establish a necessary condition for solvability of the FDI problem for linear structured systems. In addition, we develop a sufficient algebraic condition for solvability of the FDI problem in terms of a rank test on an associated pattern matrix. To show that this condition is not a necessary condition, we provide a counterexample in which the FDI problem is solvable, while the aforementioned sufficient condition does not hold. Finally, we develop a graph-theoretic condition for solvability of the FDI problem. The material in this chapter is based on the journal paper [59].

In Chapters 4 and 5 we present our main contributions in the framework of Problem 1.2. In Chapter 4, we consider strong structural controllability of *leader-follower networks*. The system matrix defining the network dynamics is a pattern matrix in which a priori given entries are equal to zero, while the remaining entries take nonzero values. These nonzero entries correspond to edge weights in the network topology, which is represented by a *simple directed graph*, a graph without multiple edges. The novelty of the material in this chapter is that we consider the situation that prespecified nonzero entries in the system's pattern matrix are constrained to take identical (nonzero) values. These constraints can be caused by many reasons, such as symmetry properties or physical constraints on the network, and so on. Restricting the system matrices to those satisfying these constraints yields to a new notion of strong structural controllability. We then provide graph-theoretic conditions for this more general property of strong structural controllability. The material in this chapter is based on the conference and journal papers [55, 57].

Chapter 5 deals with strong structural controllability of linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices. Instead of assuming that the nonzero and arbitrary entries of the system matrices can take their values completely independently, in this chapter we allow equality constraints on these entries, in the sense that *a priori* given entries in the system matrices are restricted to take arbitrary but identical values. To formalize this general class of structured systems, we introduce the concepts of colored pattern matrices and colored structured systems. The main contribution of this chapter is that it generalizes both the classical results on strong structural controllability of structured systems as well as results on controllability of systems defined on colored graphs introduced in Chapter 4. Moreover, this chapter provides both algebraic and graph-theoretic conditions for strong structural controllability of this more general class of structured systems. The material in this chapter is based on the journal paper [60].

Finally, in Chapter 6 we formulate our conclusions and provide some suggestions for future work.

1.4 List of publications

Journal articles

1. J. Jia, H.J. van Waarde, H.L. Trentelman, and M.K. Camlibel, "A unifying framework for strong structural controllability," *To appear in IEEE Transactions on Automatic Control*, 2020, doi: 10.1109/TAC.2020.2981425. (Chapter 2)
2. J. Jia, H.L. Trentelman, W. Baar, and M.K. Camlibel, "Strong structural controllability of systems on colored graphs," *To appear in IEEE Transactions on Automatic Control*, 2020, doi:10.1109/TAC.2019.2948425. (Chapter 4)
3. J. Jia, H.L. Trentelman, and M.K. Camlibel, "Fault detection and isolation for linear structured systems," *IEEE Control Systems Letters*, vol. 4, no. 4, 874–879, 2020. (Chapter 3)
4. J. Jia, H.L. Trentelman, N. Charalampidis, and M.K. Camlibel, "Strong structural controllability of colored structured systems," 2020, under review. (Chapter 5)

Conference papers

1. J. Jia, H.L. Trentelman, W. Baar, and M.K. Camlibel, "A sufficient condition for colored strong structural controllability of networks," *IFAC-PapersOnLine*, vol. 51, no. 23, pp. 16–21, 2018. (Chapter 4)

1.5 Notation

Throughout this thesis, we will use standard notation. The most commonly used definitions and notation will be listed here, while specific notions and notation can be found in each of the chapters.

Sets

We denote by $\mathbb{C}$ and $\mathbb{R}$ the fields of complex and real numbers, respectively. The vector spaces of n-dimensional real and complex vectors are denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. Likewise, the spaces of $n \times m$ real and complex matrices are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$, respectively. For a given finite set S, its number of elements will be denoted by $|S|$. A finite collection $\{S_1, \ldots, S_k\}$ of subsets of S is called a partition of S if $S_i \cap S_j = \varnothing$ for all $i \neq j$ and $S_1 \cup \cdots \cup S_k = S$.

Matrices and vectors

For a given matrix $A \in \mathbb{R}^{m\times n}$, the entry in the ith row and jth column is denoted by A_{ij}. The ith column of A is denoted by A_i. For given subsets

$$S = \{s_1, \ldots, s_k\} \subseteq \{1, \ldots, m\} \quad \text{and} \quad T = \{t_1, \ldots, t_l\} \subseteq \{1, \ldots, n\}$$

we define the $k \times l$ submatrix of A associated with S and T by $A_{S,T}$, with

$$(A_{S,T})_{ij} := A_{s_i t_j}.$$

Similarly, for a given n-dimensional vector x, we denote by x_T the subvector of x consisting of the entries of x corresponding to T. For a given square matrix A, we denote its determinant by $\det(A)$. We denote by $A^\top$ the transpose of A . Furthermore, we define its image by

$$\operatorname{im} A := \{Ax \mid x \in \mathbb{R}^m\}$$

and its kernel by

$$\ker A := \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

If $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ then we define the image of $\mathcal{S}$ under A by

$$A\mathcal{S} := \{Ax \mid x \in \mathcal{S}\}.$$

Finally, the symbol I will denote the identity matrix of appropriate dimension.

Part I

Linear Structured Systems

2 A Unifying Framework for Strong Structural Controllability

This chapter deals with strong structural controllability of linear structured systems. In contrast to existing work, the structured systems studied in this chapter have a so-called zero/nonzero/arbitrary structure, which means that some of the entries are equal to zero, some of the entries are arbitrary but nonzero, and the remaining entries are arbitrary (zero or nonzero). We formalize this in terms of pattern matrices whose entries are either fixed zero, arbitrary nonzero, or arbitrary. We establish necessary and sufficient algebraic conditions for strong structural controllability in terms of full rank tests for certain pattern matrices. We also give a necessary and sufficient graph theoretic condition for the full rank property of a given pattern matrix. This graph theoretic condition makes use of a new color change rule that is introduced in this chapter. Based on these two results, we establish a necessary and sufficient graph theoretic condition for strong structural controllability. Moreover, we relate our results to those that exist in the literature and explain how our results generalize previous work.

2.1 Introduction

Controllability is a fundamental concept in systems and control. For linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

controllability can be verified using the Kalman rank test or the Hautus test [26]. Often, the exact values of the entries in the matrices A and B are not known, but the underlying interconnection structure between the input and state variables is known exactly.

In order to formalize this, Mayeda and Yamada have introduced a framework in which, instead of a fixed pair of real matrices, only the *zero/nonzero* structure of A and B is given [28]. This means that each entry of these matrices is known to be either a *fixed zero* or an *arbitrary nonzero* real number. Given such zero/nonzero structure, they then study controllability of the family of systems for which the state and input matrices have this zero/nonzero structure. In this setup, this family of systems is called *strongly structurally controllable* if all members of the family are controllable in the classical sense [28].

Most of the existing literature up to now has considered strong structural controllability under the above rather restrictive assumption that for each of the entries of the system matrices there are only *two possibilities*: it is either a fixed zero, or an arbitrary nonzero value [28, 29, 32, 38–41]. There are, however, many scenarios in which, in addition to these two possibilities, there is a third possibility, namely, that a given entry is not a fixed zero or nonzero, but can take *any real value*. In such a scenario, it is not possible to represent the system using a zero/nonzero structure, but a third possibility needs to be taken into account. To illustrate this, consider the following example.

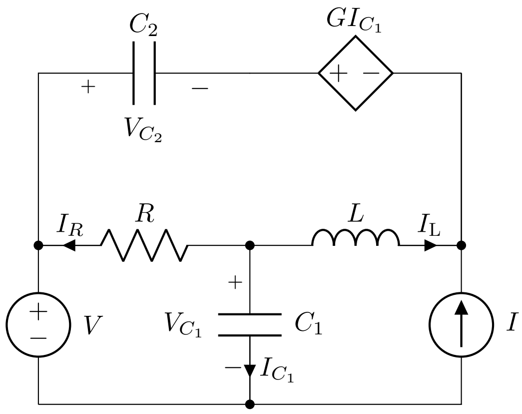


Figure 2.1: Example of an electrical circuit.

Example 2.1. The electrical circuit in Figure 2.1 consists of a resistor, two capacitors, an inductor, an independent voltage source, an independent current source and a current controlled voltage source. Assume that the parameters R, C_1, C_2 and L are positive but not known exactly. We denote the current through R, L, and C_1 by I_R, I_L, and I_{C_1}, respectively, and the voltage across C_1 and C_2 by V_{C_1} and V_{C_2}, respectively. The current controlled voltage source is represented by GI_{C_1} with gain G assumed to be positive. Define the state vector as $x = \begin{bmatrix} V_{C_1} & V_{C_2} & I_L \end{bmatrix}^\top$ and the input as $u = \begin{bmatrix} V & I \end{bmatrix}^\top$. By Kirchhoff's current and voltage laws, the circuit is represented by a linear time-invariant system (2.1) with

$$A = \begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ \frac{R-G}{RL} & \frac{1}{L} & -\frac{G}{L} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & -\frac{1}{C_2} \\ \frac{G-R}{RL} & 0 \end{bmatrix}. \tag{2.2}$$

Recall that the parameters $R, C_1, C_2, L > 0$ are not known exactly. This means that the matrices in (2.2) are not known exactly, but we do know that they have the following structure. Firstly, some entries are *fixed zeros*. Secondly, some of the entries are always *nonzero*, for instance, the entry with value $-\frac{1}{RC_1}$. The third type of entries, those with value $\frac{R-G}{RL}$ and $\frac{G-R}{RL}$, can be either zero (if $R = G$) or nonzero. Since the system matrices in this example do not have a zero/nonzero structure, the existing tests for strong structural controllability [28, 29, 32, 38–41] are not applicable.

A similar problem as in Example 2.1 appears in the context of linear networked systems. Strong structural controllability of such systems has been well-studied [29, 30, 32, 50, 61]. In the setup of these references, the weights on the edges of the network graph are unknown, while the network graph itself is known. Under the assumption that the edge weights are arbitrary but nonzero, linear networked systems can thus be regarded as systems with a given zero/nonzero structure. This zero/nonzero structure is determined by the network graph, in the sense that nonzero entries in the system matrices correspond to edges in the network graph. However, often even exact knowledge of the network graph is not available, in the sense that it is unknown whether certain edges in the graph exist or not. This issue of missing knowledge appears, for example, in social networks [42], the world wide web [43], biological networks [44, 45] and ecological systems [46]. Another cause for uncertainty about the network graph might be malicious attacks and unintentional failures. This issue is encountered in transportation networks [47], sensor networks [48] and gas networks [49].

Example 2.2. Consider a network of three agents with single-integrator dynamics, represented by

$$\dot{x}_i(t) = v_i(t)$$

for $i = 1, 2, 3$. Here $x_i \in \mathbb{R}$ is the state of agent i and $v_i \in \mathbb{R}$ is its input. The communication between the agents is represented by the graph in Figure 2.2.

The links $(1, 1)$, $(2, 2)$, $(2, 3)$ and $(3, 1)$ are known to exist, while the link $(1, 2)$ is uncertain in the sense that it may or may not be present. This is represented by solid and dashed edges, respectively. Agents 1 and 2 are only affected by the states of their neighbors, while agent 3 is also influenced by an external input $u \in \mathbb{R}$. This means that

$$v_1 = w_{11}x_1 + w_{13}x_3, \quad v_2 = w_{21}x_1 + w_{22}x_2 \quad \text{and} \quad v_3 = w_{32}x_2 + u.$$

Here the weights w_{11}, w_{22}, w_{32} and w_{13} are *nonzero* since they correspond to existing edges, while the weight w_{21} that corresponds to the uncertain link is arbitrary (zero or nonzero). We can write the network system in compact form (2.2) by defining

$$A = \begin{bmatrix} w_{11} & 0 & w_{13} \\ w_{21} & w_{22} & 0 \\ 0 & w_{32} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.3}$$

Since w_{21} can be zero or nonzero, the system matrices in (2.3) do not have a zero/nonzero structure.

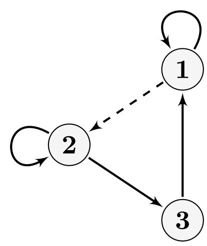


Figure 2.2: Example of a networked system.

To conclude, both in the context of modeling physical systems, as well as in representing networked systems, capturing the system simply by a zero/nonzero structure is not always possible, and a more general concept of system structure is required. The papers [30,50,52,62–64] study classes of zero/nonzero/arbitrary patterns in the context of strong structural controllability. However, necessary and sufficient conditions for strong structural controllability of general zero/nonzero/arbitrary patterns have not yet been established.

The goal of this chapter is to provide such general necessary and sufficient conditions. In particular, our main contributions are the following:

1. We extend the notion of zero/nonzero structure to a more general *zero/nonzero/arbitrary structure*, and formalize this structure in terms of suitable pattern matrices.

2. We establish necessary and sufficient conditions for strong structural controllability for families of systems with a given zero/nonzero/arbitrary structure.

These conditions are of an algebraic nature and can be verified by a rank test on two pattern matrices.

3. We provide a graph theoretic condition for a given pattern matrix to have full row rank. This condition can be verified using a new *color change rule*, that will be defined in this chapter.

4. We establish a graph theoretic test for strong structural controllability for the new families of structured systems.

5. Finally, we relate our results to those existing in the literature by showing how existing results can be recovered from those we present in this chapter. We find that seemingly incomparable results of [32] and [30] follow from our main results, which reveals an overarching theory. For these reasons, this chapter can be seen as a unifying approach to strong structural controllability of linear time-invariant systems without parameter dependencies.

We conclude this section by giving a brief account of research lines that are related to strong structural controllability but that will not be pursued in this chapter. The concept of weak structural controllability was introduced by Lin in [27] and has been studied extensively, see [17, 27, 33, 65–68]. Another, more recent, line of work focuses on structural controllability of systems for which there are *dependencies* among the arbitrary entries of the system matrices [51, 57]. An important special case of dependencies among parameters arises when the state matrix is constrained to be symmetric, which was considered in [50,53,54]. The problem of *minimal input selection* for controllability has also been well-studied, see, e.g., [69–72]. Strong structural controllability was also studied for *time-varying systems* in [73], and conditions for controllability were established both for discrete-time and continuous-time systems. Finally, weak and strong structural *targeted* controllability have been investigated in [74] and [62, 75], respectively.

The outline of the rest of the chapter is as follows. In Section 2.2, we present some preliminaries. Next, in Section 2.3, we formulate the main problem treated in this chapter. Then, in Section 2.4 we state our main results. Section 2.5 contains a comparison of our results with previous work. In Section 2.6 we state proofs of the main results. Finally, in Section 2.7 we formulate our conclusions.

2.2 Preliminaries

In this chapter, we will use so-called pattern matrices. By a pattern matrix we mean a matrix with entries in the set of symbols $\{0, *, ?\}$. These symbols will be given a meaning in the sequel.

The set of all $p \times q$ pattern matrices will be denoted by $\{0, *, ?\}^{p\times q}$. For a given $p \times q$ pattern matrix $\mathcal{M}$, we define the *pattern class* of $\mathcal{M}$ as

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p\times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \quad M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *\}.$$

This means that for a matrix $M \in \mathcal{P}(\mathcal{M})$, the entry M_{ij} is either (i) *zero* if $\mathcal{M}_{ij} = 0$, (ii) *nonzero* if $\mathcal{M}_{ij} = *$, or (iii) *arbitrary* (zero or nonzero) if $\mathcal{M}_{ij} = ?$. To illustrate the definition of pattern class, consider the following example.

Example 2.3. Consider the pattern matrix $\mathcal{M}$

$$\mathcal{M} = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & 0 & * & 0 & * \\ ? & * & * & ? & 0 \end{bmatrix}. \tag{2.4}$$

Then, $\mathcal{P}(\mathcal{M})$ consists of all matrices of the form

$$\begin{bmatrix} a_1 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & a_4 & 0 & a_5 \\ b_1 & a_6 & a_7 & b_2 & 0 \end{bmatrix} \tag{2.5}$$

where $a_1, \ldots, a_7$ are nonzero real numbers, and b_1 and b_2 are arbitrary (zero or nonzero) real numbers.

2.3 Problem formulation

Let $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$ be pattern matrices. Consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.6}$$

where the system matrix A is in $\mathcal{P}(\mathcal{A})$ and the input matrix B is in $\mathcal{P}(\mathcal{B})$, and where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input.

We will call the family of systems (2.6) a *linear structured system*. To simplify the notation, we denote this structured system by the ordered pair of pattern matrices $(\mathcal{A}, \mathcal{B})$.

Example 2.4. Consider the electrical circuit discussed in Example 2.1. Recall that this was modelled as the state space system (2.2) in which the entries of the system matrix and input matrix were either fixed zeros, strictly nonzero or undetermined. This can be represented as a structured system $(\mathcal{A}, \mathcal{B})$ with pattern matrices

$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}. \tag{2.7}$$

In this chapter we will study structural controllability of structured systems. In particular, we will focus on strong structural controllability, which is defined as follows.

Definition 2.1. *The system* $(\mathcal{A}, \mathcal{B})$ *is called* strongly structurally controllable *if the pair* (A, B) *is controllable for all* $A \in \mathcal{P}(\mathcal{A})$ *and* $B \in \mathcal{P}(\mathcal{B})$.

The problem that we will investigate in the this chapter is stated as follows.

Problem 2.2. *Given two pattern matrices* $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ *and* $\mathcal{B} \in \{0, *, ?\}^{n\times m}$, *provide necessary and sufficient conditions under which* $(\mathcal{A}, \mathcal{B})$ *is strongly structurally controllable.*

In the remainder of this chapter, we will simply call the structured system $(\mathcal{A}, \mathcal{B})$ *controllable* if it is strongly structurally controllable.

Remark 2.1. In addition to strong structural controllability, *weak structural controllability* has also been studied extensively. This concept was introduced by Lin in [27]. Instead of requiring *all* systems in a family associated with a given structured system to be controllable, weak structural controllability only asks for the existence of at least one controllable member of that family, see [27, 33, 65]. In these references, conditions were established for weak structural controllability of structured systems in which the pattern matrices only contain 0 or ? entries. The question then arises: is it possible to generalize the results from [27, 33, 65] to structured systems in the context of this chapter, with more general pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$. Indeed, it turns out that the results in [27, 33, 65] can immediately be applied to assess weak structural controllability of our more general structured systems. To show this, for given pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$ we define two new pattern matrices $\mathcal{A}' \in \{0, ?\}^{n\times n}$ and $\mathcal{B}' \in \{0, ?\}^{n\times m}$ as follows:

$$\mathcal{A}'_{ij} = 0 \iff \mathcal{A}_{ij} = 0 \quad \text{and} \quad \mathcal{B}'_{ij} = 0 \iff \mathcal{B}_{ij} = 0.$$

The new structured system $(\mathcal{A}', \mathcal{B}')$ is now a structured system of the form studied in [27, 33, 65]. Using the fact that weak structural controllability is a generic property [65], it can then be shown that weak structural controllability of $(\mathcal{A}', \mathcal{B}')$ is equivalent to that of $(\mathcal{A}, \mathcal{B})$. In other words, weak structural controllability of general $(\mathcal{A}, \mathcal{B})$ can be verified using the conditions established in previous work [27, 33, 65].

2.4 Main results

In this section, the main results of this chapter will be stated. Firstly, we will establish an algebraic condition for controllability of a given structured system. This condition states that controllability of a structured system is equivalent to full rank conditions on two pattern matrices associated with the system. Secondly, a graph theoretic condition for a given pattern matrix to have full row rank will be given in terms of a so-called *color change rule*. Finally, based on the above algebraic condition and graph theoretic condition, we will establish a graph theoretic necessary and sufficient condition for controllability of a structured system.

Our first main result is a rank test for controllability of a structured system. In the sequel, we say that a pattern matrix $\mathcal{M}$ *has full row rank* if every matrix $M \in \mathcal{P}(\mathcal{M})$ has full row rank.

Theorem 2.3. *The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the following two conditions hold:*

1. *The pattern matrix $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ has full row rank.*
2. *The pattern matrix $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ has full row rank where $\bar{\mathcal{A}}$ is the pattern matrix obtained from $\mathcal{A}$ by modifying the diagonal entries of $\mathcal{A}$ as follows:*

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \textit{if } \mathcal{A}_{ii} = 0, \\ ? & \textit{otherwise.} \end{cases} \tag{2.8}$$

We note here that the two rank conditions in Theorem 2.3 are independent, in the sense that one does not imply the other in general. To show that the first rank condition does not imply the second, consider the pattern matrices $\mathcal{A}$, the corresponding $\bar{\mathcal{A}}$, and $\mathcal{B}$ given by

$$\mathcal{A} = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{A}} = \begin{bmatrix} ? & * \\ 0 & * \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

It is evident that the pattern matrix $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ has full row rank. However, for the choice

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{P}(\bar{\mathcal{A}}) \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{P}(\mathcal{B}),$$

the matrix $\begin{bmatrix}\bar{A} & B\end{bmatrix}$ does not have full row rank.

To show that the second condition does not imply the first one, consider the pattern matrix $\mathcal{A}$, the corresponding $\bar{\mathcal{A}}$, and $\mathcal{B}$ given by

$$\mathcal{A} = \begin{bmatrix} ? & 0 \\ * & 0 \end{bmatrix}, \quad \bar{\mathcal{A}} = \begin{bmatrix} ? & 0 \\ * & * \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

Obviously, the pattern matrix $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ has full row rank. However, for the choice

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{P}(\mathcal{A}) \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{P}(\mathcal{B}),$$

we see that $\begin{bmatrix} A & B \end{bmatrix}$ does not have full row rank.

Next, we discuss a noteworthy special case in which the first rank condition in Theorem 2.3 is implied by the second one. Indeed, if none of the diagonal entries of $\mathcal{A}$ is zero, it follows from (2.8) that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{P}(\bar{\mathcal{A}})$. Hence, we obtain the following corollary to Theorem 2.3.

Corollary 2.4. *Suppose that none of the diagonal entries of $\mathcal{A}$ is zero. Let $\bar{\mathcal{A}}$ be as defined in* (2.8)*. The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ has full row rank.*

Note that both $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ appearing in Theorem 2.3 are $n \times (n+m)$ pattern matrices. Next, we will develop a graph theoretic test for checking whether a given pattern matrix has full rank. To do so, we first need to introduce some terminology.

Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leqslant q$. We associate a directed graph $\mathcal{G}(\mathcal{M}) = (V, E)$ with $\mathcal{M}$ as follows. Take as node set $V = \{1, 2, \ldots, q\}$ and define the edge set $E \subseteq V \times V$ such that $(j, i) \in E$ if and only if $\mathcal{M}_{ij} = *$ or $\mathcal{M}_{ij} = ?$. If $(i, j) \in E$, then we call j an *out-neighbor* of i. Also, in order to distinguish between $*$ and ? entries in $\mathcal{M}$, we define two subsets E_* and $E_?$ of the edge set E as follows: $(j, i) \in E_*$ if and only if $\mathcal{M}_{ij} = *$ and $(j, i) \in E_?$ if and only if $\mathcal{M}_{ij} = ?$. Then, obviously, $E = E_* \cup E_?$ and $E_* \cap E_? = \varnothing$. To visualize this, we use solid and dashed arrows to represent edges in E_* and $E_?$, respectively.

Example 2.5. As an example, consider the pattern matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & * & 0 & 0 \\ 0 & * & * & ? & * \\ * & 0 & ? & 0 & 0 \\ 0 & * & 0 & 0 & ? \end{bmatrix}.$$

The associated directed graph $\mathcal{G}(\mathcal{M})$ is then given in Figure 2.3.

Next, we introduce the notion of *colorability* for $\mathcal{G}(\mathcal{M})$:

1. Initially, color all nodes of $\mathcal{G}(\mathcal{M})$ white.
2. If a node i has exactly one white out-neighbor j and $(i, j) \in E_*$, we change the color of j to black.
3. Repeat step 2 until no more color changes are possible.

The graph $\mathcal{G}(\mathcal{M})$ is called *colorable* if the nodes $\{1, \ldots, p\}$ are colored black following the procedure above. Note that the remaining nodes $p+1, \ldots, q$ can never be colored black since they have no incoming edges.

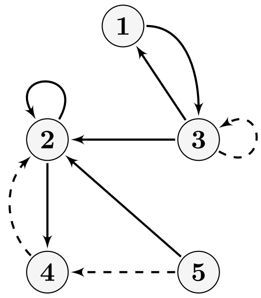


Figure 2.3: Example of a graph associated with a given pattern matrix.

We refer to step 2 in the above procedure as the *color change rule*. Similar color change rules have appeared in the literature before (see e.g. [30, 32, 76]). Unlike some of these rules, node i in step 2 does not need to be black in order to change the color of a neighboring node.

Example 2.6. Consider the pattern matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{bmatrix} * & 0 & 0 & 0 & * & 0 \\ 0 & ? & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ 0 & ? & * & * & 0 & 0 \end{bmatrix}.$$

The directed graph $\mathcal{G}(\mathcal{M})$ associated with $\mathcal{M}$ is depicted in Figure 2.4a. By repeated application of the color change rule as shown in Figure 2.4b to 2.4d, we obtain the derived set $\mathcal{D} = \{1, 2, 3, 4\}$. Hence, $\mathcal{G}(\mathcal{M})$ is colorable.

The following theorem now provides a necessary and sufficient graph theoretic condition for a given pattern matrix to have full row rank.

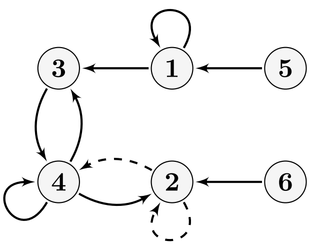
Theorem 2.5. *Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leqslant q$. Then, $\mathcal{M}$ has full row rank if and only if $\mathcal{G}(\mathcal{M})$ is colorable.*

It is clear from the definition of the color change rule that colorability of a given graph can be checked in polynomial time.

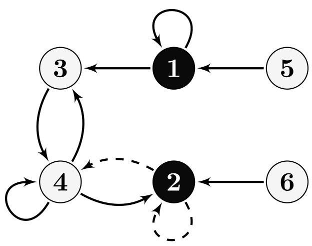
Finally, based on the rank test in Theorem 2.3 and the result in Theorem 2.5, the following necessary and sufficient graph theoretic condition for controllability of a given structured system is obtained.

Theorem 2.6. *Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Also, let $\bar{\mathcal{A}}$ be obtained from $\mathcal{A}$ by modifying the diagonal entries of $\mathcal{A}$ as follows:*

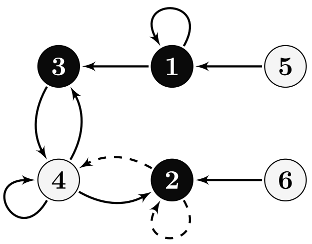
$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0, \\ ? & \text{otherwise.} \end{cases} \tag{2.9}$$



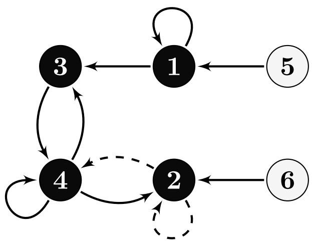
(a) The graph $\mathcal{G}(\mathcal{M})$.



(b) Node 5 colors 1 and 6 colors 2.



(c) Node 1 colors 3.



(d) Node 3 colors 4.

Figure 2.4: Example of a colorable graph.

Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are colorable.

As an example, we study controllability of the electrical circuit discussed in Example 2.1.

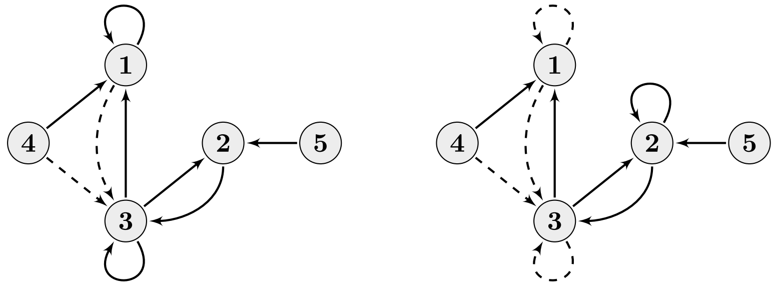
Example 2.7. According to Example 2.4, the electrical circuit depicted in Figure 2.1 can be modelled as a structured system of the form (2.6) where the pattern matrices $\mathcal{A}$ and $\mathcal{B}$ are given by:

$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}.$$

Then, we obtain

$$\bar{\mathcal{A}} = \begin{bmatrix} ? & 0 & * \\ 0 & * & * \\ ? & * & ? \end{bmatrix}.$$

The graphs $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are depicted in Figure 2.5a and Figure 2.5b, respectively. Both graphs are colorable. Indeed, node 5 colors 2, node 2 colors 3, and finally 3 colors 1 in both graphs. Therefore, the system $(\mathcal{A}, \mathcal{B})$ is controllable by Theorem 2.6.



(a) The graph $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$. (b) The graph $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$.

Figure 2.5: The graphs associated with the circuit in Example 2.1.

As a second example, we apply Theorem 2.6 to verify the controllability of the networked system in Example 2.2.

Example 2.8. The networked system in Example 2.2 can be represented as a structured system of the form (2.6), where the pattern matrices $\mathcal{A}$ and $\mathcal{B}$ are given by:

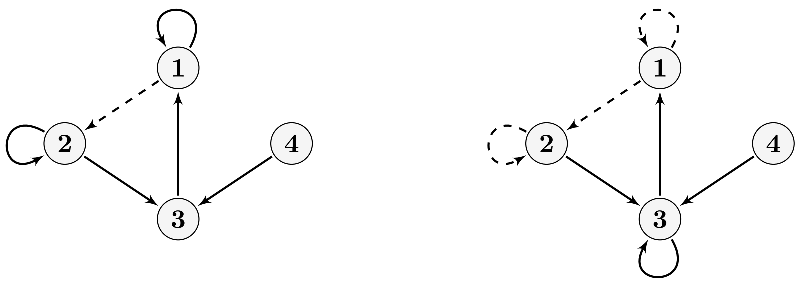
$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ ? & * & 0 \\ 0 & * & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}.$$

Clearly,

$$\bar{\mathcal{A}} = \begin{bmatrix} ? & 0 & * \\ ? & ? & 0 \\ 0 & * & * \end{bmatrix}.$$

The graphs $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are depicted in Figure 2.6a and Figure 2.6b, respectively. The graph in Figure 2.6 is colorable. Indeed, node 4 colors 3, node 2 colors 2, and finally 3 colors 1. However, the graph in Figure 2.6b is not colorable. Therefore, the system $(\mathcal{A}, \mathcal{B})$ is not controllable. However, if we would know that the edge $(1, 2)$ does exist in the graph, i.e. if $\mathcal{A}_{21} = *$, then it can be verified that $(\mathcal{A}, \mathcal{B})$ is controllable.

By applying Theorem 2.6 to the special case discussed in Corollary 2.4, we obtain the following.



(a) The graph $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$. (b) The graph $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$.

Figure 2.6: The graphs associated with the network in Example 2.2.

Corollary 2.7. *Suppose that none of the diagonal entries of $\mathcal{A}$ is zero. Let $\bar{\mathcal{A}}$ be defined as in* (2.9)*. Then, the system* $(\mathcal{A}, \mathcal{B})$ *is controllable if and only if* $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ *is colorable.*

To conclude this section, the results we have obtained for controllability lead to an interesting observation in the context of structural stabilizability. We say that a structured system $(\mathcal{A}, \mathcal{B})$ is *stabilizable* if the pair (A, B) is stabilizable for all $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$.

For a single linear system, controllability implies stabilizability, whereas the reverse implication does not hold in general. Interestingly, for structured systems controllability and stabilizability do turn out to be equivalent, as stated next.

Theorem 2.8. *The system* $(\mathcal{A}, \mathcal{B})$ *is stabilizable if and only if it is controllable.*

2.5 Discussion of existing results

In this section, we compare our results with those existing in the literature. We focus on the most relevant related work [28–30, 32, 38–41]. The structured systems studied in these references are all special cases of those we study in this chapter. In Table 2.1 we highlight the different types of pattern matrices $\mathcal{A}$ and $\mathcal{B}$ studied in these references. We also include the type of conditions that were developed, i.e., either graph theoretic, algebraic or or both. Note that the references [29, 30, 32] study controllability in a network context, where the pattern matrix $\mathcal{B}$ has a particular structure in the sense that each column has exactly one $*$-entry, and each row has at most one $*$-entry. Additionally, the paper [30] considers a particular class of systems where the diagonal entries of $\mathcal{A}$ are all ? and none of the off-diagonal entries is ?. In the following two subsections, we elaborate on the existing graph theoretic conditions

Ref.	$\mathcal{A}$	$\mathcal{B}$	Conditions	
			GTC	AC
[28]	$\{0,*\}^{n\times n}$	$\{0,*\}^{n\times 1}$	✓	–
[38]			–	✓
[39]		$\{0,*\}^{n\times m}$	✓	–
[40]			✓	✓
[41]			✓	✓
[29]		particular $\{0,*\}^{n\times m}$	–	✓
[32]			–	✓
[30]	particular $\{0,*,?\}^{n\times n}$		✓	✓

Table 2.1: A table summarizing prior work on strong structural controllability. graph theoretic condition s are abbreviated by 'GTC' and algebraic conditions by 'AC'.

and algebraic conditions, respectively. In both sections, we also compare these results to the present work.

2.5.1 Graph theoretic conditions

The graph theoretic conditions provided in [28, Theorem 1] for the single-input case ($m = 1$) and extended to the multi-input case in [39, Satz 3] are based on the graph $\mathcal{G} = (V, E)$ associated with a pattern matrix $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ where $\mathcal{A} \in \{0,*\}^{n\times n}$ and $\mathcal{B} \in \{0,*\}^{n\times m}$. Note that $V = \{1,\ldots,n+m\}$ in this case. The graph theoretic characterization in [39, Satz 3] (or in [28, Theorem 1] if $m = 1$) consists of three conditions. The first one requires checking the so-called accessibility of each node in $\{1,\ldots,n\}$ from the nodes in $\{n+1,\ldots,n+m\}$. The remaining two conditions require checking certain relations for *all* subsets of $\{1,\ldots,n\}$. As such, the computational complexity of checking these conditions is *at least* exponential in n. Note that, in contrast, the computational complexity of checking the colorability conditions of our Theorem 2.6 is polynomial in n.

The paper [28] provides another set of graph theoretic conditions, stated, more specifically, in [28, Theorem 2] (only for the case $m = 1$). As argued in [28, p. 135], this theorem performs better than [28, Theorem 1] for sparse graphs. Essentially, the conditions given in [28, Theorem 2] require checking the existence of a unique serial buds cactus as well as nonexistence of certain cycles within the graph $\mathcal{G}$. How these conditions can be checked in an algorithmic manner is not clear, whereas the colorability conditions given in Theorem 2.6 can be checked by a simple algorithm.

On top of the advantages of computational complexity, the conditions provided in Theorem 2.6 are more attractive because of their conceptual simplicity. Indeed, colorability is a simpler and more intuitive notion than those appearing in the results of [28] and [39].

Yet another graph theoretical characterization is provided in [41, Theorem 5]. In order to verify the conditions of [41, Theorem 5], one needs to check whether a unique spanning cycle family with certain properties exists in $\binom{n+m}{n}$ directed graphs obtained from the pattern matrices $\mathcal{A}$ and $\mathcal{B}$. Needless to say, checking the conditions of Theorem 2.6 is much easier than checking these conditions.

Also in the context of networked systems, graph theoretic conditions for strong structural controllability have been obtained (see, e.g., [29, 30, 32]). To elaborate further on the relationship between the work on networked systems and our work, we first need to explain the framework of the papers [29, 30, 32]. The starting point of these papers is a directed graph $\mathcal{H} = (W, F)$ where $W = \{1, \dots, n\}$ denotes the node set and F the edge set. The graphs considered in [29, 32] are so-called loop graphs, that are graphs which are allowed to contain self-loops, whereas [30] does not allow self-loops. Apart from the graph $\mathcal{H}$, these papers consider a subset of the node set W, the so-called leader set, say $W_L = \{w_1, \dots, w_m\}$. Based on the graph $\mathcal{H}$ and W_L, [29, 30, 32] introduce systems of the form (2.6) where the pattern matrix $\mathcal{B}$ is defined by

$$\mathcal{B}_{ij} = \begin{cases} * & \text{if } i = w_j \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$. In [29] and [32] the pattern matrix $\mathcal{A}$ is defined by

$$\mathcal{A}_{ij} = \begin{cases} * & \text{if } (j, i) \in F \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

whereas in [30] the pattern matrix $\mathcal{A}$ is defined by

$$\mathcal{A}_{ij} = \begin{cases} * & \text{if } (j, i) \in F \\ ? & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

for $i, j \in \{1, \dots, n\}$.

In [29], the authors first define two bipartite graphs obtained from the pattern matrices $\mathcal{A}$ and $\mathcal{B}$. Then, they show in [29, Theorem 5] that $(\mathcal{A}, \mathcal{B})$ is strongly structurally controllable if and only if there exist so-called constrained matchings with certain properties in these bipartite graphs. Later, in [32, Theorem 5.4] an equivalence between the existence of constrained matchings and so-called zero forcing sets for loop graphs was established. To explain this in more detail, we need to introduce the notion of zero forcing that was originally studied in the context of minimal rank problems (see e.g. [76]).

Let $\mathcal{H} = (W, F)$ be a directed loop graph and $S \subseteq W$. Color all nodes in S black and the others white.

If a node i (of any color) has exactly one white out-neighbor j, we change the color of j to black and write $i \to j$. If all the nodes in W can be colored black by repeated application of this color change rule, we say that S is a *loopy zero forcing set* for $\mathcal{H}$. Given a loopy zero forcing set, we can list the color changes in the order in which they were performed to color all nodes black. This list is called a chronological list of color changes.

In order to quote [32, Theorem 5.5], we need two more definitions. Define $W_{\text{loop}} \subseteq W$ to be the subset of all nodes with self-loops and let $\mathcal{H}^*$ be the graph obtained from $\mathcal{H}$ by placing a self-loop at every node.

Theorem 2.9. *[32, Theorem 5.5] Let $\mathcal{H}$ be a directed loop graph and W_L be a leader set. Consider the pattern matrices defined in* (2.10) *and* (2.11)*. Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the following conditions hold:*

1. *W_L is a loopy zero forcing set for $\mathcal{H}$.*
2. *W_L is a loopy zero forcing set for $\mathcal{H}^*$ for which there is a chronological list of color changes that does not contain a color change of the form $i \to i$ with $i \in W_{\text{loop}}$.*

A result similar to this theorem was obtained in [30] for controllability of pattern matrices defined by (2.10) and (2.12) that are obtained from a graph $\mathcal{H}$ *without* self-loops. However, in order to deal with this class of pattern matrices, [30] introduces a slightly different notion of zero forcing to be defined below.

Let $\mathcal{H} = (W, F)$ be a directed graph without self-loops and $S \subseteq W$. Color all nodes in S black and the others white. If a *black* node i has exactly one white out-neighbor j, we change the color of j to black. If all the nodes in W can be colored black by repeated application of this color change rule, we say that S is a *ordinary zero forcing set* for $\mathcal{H}$.

We now state the graph theoretic characterization of controllability established in [30].

Theorem 2.10. *[30, Theorem IV.4] Let $\mathcal{H}$ be a directed graph without self-loops and W_L be a leader set. Consider the pattern matrices given by* (2.10) *and* (2.12)*. Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if W_L is an ordinary zero forcing set for $\mathcal{H}$.*

Even though Theorems 2.9 and 2.10 present conditions that are similar in nature, it is not possible to compare these results immediately as they deal with two different and non-overlapping system classes. Indeed, the pattern matrices considered in [32] (given by (2.11)) *do not contain* any ? entries whereas those studied in [30] (given by (2.12)) contain *only* ? entries on their diagonals.

Next, we will show that the conditions of Theorem 2.6 are equivalent to those of Theorems 2.9 and 2.10 if specialized to the corresponding pattern matrices. This will shed light on the relationship between these results based on the different zero forcing notions.

We start with Theorem 2.9. According to our color change rule, the nodes belonging to W_L will be colored black in both $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ because $\mathcal{B}$ is a pattern matrix with structure defined by (2.10). Since $\mathcal{A}$ does not contain ? entries, $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ is colorable if and only if W_L is a loopy zero forcing set for $\mathcal{G}(\mathcal{A})$. By noting that $\mathcal{H} = \mathcal{G}(\mathcal{A})$, we see that the first condition in Theorem 2.6 is equivalent to that of Theorem 2.9. Now, let the pattern matrix $\mathcal{A}^*$ be such that $\mathcal{H}^* = \mathcal{G}(\mathcal{A}^*)$. Since $W_{\text{loop}} = \{i \mid \bar{\mathcal{A}}_{ii} = ?\}$, we see that $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ is colorable if and only if the second condition of Theorem 2.9 holds. Thus, the second condition of Theorem 2.6 is equivalent to that of Theorem 2.9.

Now, we turn attention to Theorem 2.10. It follows from (2.9) and (2.12) that $\bar{\mathcal{A}} = \mathcal{A}$, i.e., graphs $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ are the same. As in the discussion above, the nodes belonging to W_L will be colored black in $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ because $\mathcal{B}$ is a pattern matrix with structure defined by (2.10). According to our color change rule, a white node can never color any other white node in $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ since $(i,i) \in E_?$ for every node i of $\mathcal{G}(\bar{\mathcal{A}})$. This means that $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ is colorable if and only if W_L is an ordinary zero forcing set for $\mathcal{G}(\bar{\mathcal{A}})$. By noting that $\mathcal{H} = \mathcal{G}(\mathcal{A}) = \mathcal{G}(\bar{\mathcal{A}})$, we see that the conditions in Theorem 2.6 are equivalent to the single condition of Theorem 2.10.

2.5.2 Algebraic conditions

In this subsection, we will compare our rank tests for strong structural controllability with those provided in [29, 30, 40]. More precisely, we will show that the rank tests in Theorem 2.3 reduce to those in [29, 30, 40] for the corresponding special cases of pattern matrices.

An algebraic condition for controllability of $(\mathcal{A}, \mathcal{B})$ was provided in [40, Theorem 2] for $\mathcal{A} \in \{0, *\}^{n\times n}$ and $\mathcal{B} \in \{0, *\}^{n\times m}$. Later, these conditions were reformulated in [29, Theorem 3]. These conditions rely on a matrix property that will be defined next for pattern matrices that may also contain ? entries.

Definition 2.11. *Consider a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$ with $p \leqslant q$. The matrix $\mathcal{M}$ is said to be of Form III if there exist two permutation matrices P_1 and P_2*

such that

$$P_1 \mathcal{M} P_2 = \begin{bmatrix} \otimes & \cdots & \otimes & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \ddots & \ddots & \vdots \\ \otimes & \cdots & \otimes & \cdots & \otimes & * & 0 \\ \otimes & \cdots & \otimes & \cdots & \otimes & \otimes & * \end{bmatrix} \tag{2.13}$$

where the symbol $\otimes$ *indicates an entry that can be either* 0, $*$ *or* $?$.

The above-mentioned algebraic conditions are stated next.

Theorem 2.12. *[29, Theorem 3] Let* $\mathcal{A} \in \{0, *\}^{n \times n}$ *and* $\mathcal{B} \in \{0, *\}^{n \times m}$ *be two pattern matrices. Also, let* $\mathcal{A}^*$ *be the pattern matrix obtained from* $\mathcal{A}$ *by replacing all diagonal entries by* $*$. *The system* $(\mathcal{A}, \mathcal{B})$ *is controllable if and only if the following two conditions hold:*

1. *The matrix* $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ *is of Form III.*
2. *The matrix* $\begin{bmatrix} \mathcal{A}^* & \mathcal{B} \end{bmatrix}$ *is of Form III with the additional property that* $*$ *entries appearing in* (2.13) *do not originate from diagonal elements in* $\mathcal{A}$ *that are* $*$ *entries.*

It can be shown that our algebraic conditions in Theorem 2.3 are equivalent to those in Theorem 2.12 for the special case of pattern matrices that only contain 0 and $*$ entries. Recall that it follows from Theorem 2.3 that $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ have full row rank, where $\bar{\mathcal{A}}$ is given in (2.9). To relate our algebraic conditions with the ones in Theorem 2.12, we need the following lemma.

Lemma 2.13. *Let* $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ *with* $p \leqslant q$. *Then,* $\mathcal{M}$ *has full row rank if and only if* $\mathcal{M}$ *is of Form III.*

From Lemma 2.13 it immediately follows that $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ has full row rank if and only if $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ is of Form III. Hence, the first condition of Theorem 2.3 is equivalent to that of Theorem 2.12. We will now also show that $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ has full row rank if and only if the second condition of Theorem 2.12 holds. From Lemma 2.13, we have that $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ has full row rank if and only if $\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}$ is of Form III. By definition of $\bar{\mathcal{A}}$ and $\mathcal{A}^*$, it follows that $\bar{\mathcal{A}}_{ij} = \mathcal{A}^*_{ij}$ for all $i \neq j$. If $\mathcal{A}_{ii} = 0$ then both $\bar{\mathcal{A}}_{ii} = *$ and $\mathcal{A}^*_{ii} = *$. On the other hand, if $\mathcal{A}_{ii} = *$ then $\bar{\mathcal{A}}_{ii} =?$ and $\mathcal{A}^*_{ii} = *$. To sum up, $\bar{\mathcal{A}}_{ij} \neq \mathcal{A}^*_{ij}$ if and only if $i = j$ and $\mathcal{A}_{ii} = *$. In other words, all entries of $\bar{\mathcal{A}}$ and $\mathcal{A}^*$ are *the same*, except for those that correspond to the diagonal elements of $\mathcal{A}$ that are $*$ entries. Hence, there exist two permutation matrices P_1 and P_2 such that all entries of the matrices $P_1 \begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix} P_2$ and $P_1 \begin{bmatrix} \mathcal{A}^* & \mathcal{B} \end{bmatrix} P_2$ are the same, except those that originate from diagonal elements of $\mathcal{A}$ that are $*$ entries. This implies that $\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}$ is of Form III if and only if $\begin{bmatrix}\mathcal{A}^* & \mathcal{B}\end{bmatrix}$ is of Form III with the additional property that the $*$ entries in (2.13) do not originate from diagonal elements in $\mathcal{A}$ that are $*$ entries. In other words, the second conditions of Theorem 2.3 and Theorem 2.12 are equivalent. Since also the first conditions in these theorems are equivalent, we conclude that the algebraic conditions in Theorem 2.3 are equivalent to those in Theorem 2.12 for the special case in which $\mathcal{A} \in \{0,*\}^{n\times n}$ and $\mathcal{B} \in \{0,*\}^{n\times m}$.

A different algebraic condition was introduced in [30] for systems defined on simple directed graphs. The pattern matrices of such systems can be represented by $\mathcal{A}$ and $\mathcal{B}$ given by (2.12) and (2.10), respectively. The algebraic condition referred to above is then stated as follows.

Theorem 2.14. *[30, Lemma IV.1] Consider the pattern matrices $\mathcal{A}$ and $\mathcal{B}$ given by* (2.12) *and* (2.10)*, respectively. Then, $(\mathcal{A},\mathcal{B})$ is controllable if and only if $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ has full row rank.*

In order to see that this theorem follows from Corollary 2.4, note that $\mathcal{A} = \bar{\mathcal{A}}$ since all diagonal entries of $\mathcal{A}$ are ?'s.

2.6 Proofs

2.6.1 Proof of Theorem 2.3

To prove the 'only if' part, assume that $(\mathcal{A},\mathcal{B})$ is controllable. By the Hautus test [26, Theorem 3.13] and the definition of strong structural controllability, it follows that $\begin{bmatrix}A-\lambda I & B\end{bmatrix}$ has full row rank for all $(A,B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$ and all $\lambda \in \mathbb{C}$. By substitution of $\lambda = 0$ we conclude that condition 1 is satisfied. To prove that condition 2 also holds, suppose that $x^\top \begin{bmatrix}\bar{A} & B\end{bmatrix} = 0$ for some pair $(\bar{A},B) \in \mathcal{P}(\bar{\mathcal{A}}) \times \mathcal{P}(\mathcal{B})$ and $x \in \mathbb{R}^n$. We want to prove that $x = 0$. Let $\alpha \in \mathbb{R}$ be a nonzero real number such that

$$\alpha \notin \{\bar{A}_{ii} \mid i \text{ is such that } \mathcal{A}_{ii} = *\}.$$

Then, define a nonsingular diagonal matrix $X \in \mathbb{R}^{n\times n}$ as

$$X_{ii} = \begin{cases} 1 & \text{if } \bar{\mathcal{A}}_{ii} = ? \\ \alpha/\bar{A}_{ii} & \text{if } \bar{\mathcal{A}}_{ii} = *. \end{cases}$$

It is clear that $\bar{A}X \in \mathcal{P}(\bar{\mathcal{A}})$ and $x^\top \begin{bmatrix}\bar{A}X & B\end{bmatrix} = 0$. Furthermore, by the choice of α and X we obtain $\hat{A} := \bar{A}X - \alpha I \in \mathcal{P}(\mathcal{A})$. By assumption, $\begin{bmatrix}\hat{A}+\alpha I & B\end{bmatrix}$ has full row rank (by substitution of $\lambda = -\alpha$). In other words, $\begin{bmatrix}\bar{A}X & B\end{bmatrix}$ has full row rank and therefore $x = 0$. We conclude that condition 2 is satisfied.

To prove the 'if' part, assume that conditions 1 and 2 are satisfied. Suppose that

$$z^H \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$$

for some $(A, B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$ and $(\lambda, z) \in \mathbb{C} \times \mathbb{C}^n$, and z^H denotes the conjugate transpose of z. We want to prove that $z = 0$. Note that if $\lambda = 0$, it readily follows that $z = 0$ by condition 1. Therefore, it remains to be shown that $z = 0$ if $\lambda \neq 0$. To this end, write $z = \xi + j\eta$, where $\xi, \eta \in \mathbb{R}^n$ and j denotes the imaginary unit. Next, let $\alpha \in \mathbb{R}$ be a nonzero real number such that

$$\alpha \notin \left\{ -\frac{\xi_i}{\eta_i} \mid \eta_i \neq 0 \right\} \cup \left\{ -\frac{(\xi^\top A)_i}{(\eta^\top A)_i} \mid (\eta^\top A)_i \neq 0 \right\}.$$

We define $x := \xi + \alpha\eta$. Now, we claim that

(a) $x_i = 0$ if and only if $z_i = 0$.

(b) $x_i = 0$ if and only if $(x^\top A)_i = 0$.

Note that (a) follows directly from the definition of x and the choice of α. To prove the 'only if' part of (b), suppose that $x_i = 0$. By (a), this implies that $z_i = 0$. Since $z^H A = \lambda z^H$, we see that $(z^H A)_i = 0$. Equivalently, $((\xi^\top - j\eta^\top)A)_i = 0$. Therefore, both $(\xi^\top A)_i = 0$ and $(\eta^\top B)_i = 0$. We conclude that $(x^\top A)_i = ((\xi^\top + \alpha\eta^\top)A)_i = 0$.

To prove the 'if' part of (b), suppose that $(x^\top A)_i = 0$. This means that $((\xi^\top + \alpha\eta^\top)A)_i = 0$. Equivalently, $(\xi^\top A)_i + \alpha(\eta^\top A)_i = 0$. By the choice of α, this implies that $(\xi^\top A)_i = (\eta^\top A)_i = 0$. We conclude that $(z^H A)_i = 0$. Recall that $z^H A = \lambda z^H$, where λ was assumed to be *nonzero*. This implies that $z_i = 0$. Again, using (a) we conclude that $x_i = 0$. This proves (b).

Next, we define the diagonal matrix $X' \in \mathbb{R}^{n \times n}$ as

$$X'_{ii} = \begin{cases} 1 & \text{if } x_i = 0 \\ \frac{(x^\top A)_i}{x_i} & \text{otherwise.} \end{cases}$$

We know that X' is nonsingular by (b). By definition of X' we have $x^\top A = x^\top X'$. Furthermore, as $z^H B = 0$ we obtain $\xi^\top B = \eta^\top B = 0$ and therefore $x^\top B = 0$. Hence $x^\top \begin{bmatrix} A - X' & B \end{bmatrix} = 0$. Since X' is nonsingular, it follows that $A - X' \in \mathcal{P}(\bar{\mathcal{A}})$. By condition 2, this means that $x = 0$. Finally, we conclude that $z = 0$ using (a). ∎

2.6.2 Proof of Theorem 2.5

To prove Theorem 2.5, we need the following auxiliary result.

Lemma 2.15. *Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leqslant q$. Consider the directed graph $\mathcal{G}(\mathcal{M})$. Suppose that each node is colored white or black. Let $D \in \mathbb{R}^{p \times p}$ be the diagonal matrix defined by*

$$D_{kk} = \begin{cases} 1 & \text{if node } k \text{ is black,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose further that $j \in \{1, \ldots, p\}$ is a node for which there exists a node $i \in \{1, \ldots, p\}$, possibly identical to j, such that j is the only white out-neighbor of i and $(i, j) \in E_*$. Then for all $M \in \mathcal{P}(\mathcal{M})$ we have that $\begin{bmatrix} M & D \end{bmatrix}$ has full row rank if and only if $\begin{bmatrix} M & D + e_j e_j^\top \end{bmatrix}$ has full row rank where e_j denotes the jth column of I.

Proof. The 'only if' part is trivial. To prove the 'if' part, suppose that $M \in \mathcal{P}(\mathcal{M})$ and $\begin{bmatrix} M & D + e_j e_j^\top \end{bmatrix}$ has full row rank. Let $z \in \mathbb{R}^p$ be such that $z^\top \begin{bmatrix} M & D \end{bmatrix} = 0$. Our aim is to show that $z_j = 0$. Indeed, if z_j is zero then $z^\top \begin{bmatrix} M & D + e_j e_j^\top \end{bmatrix} = z^\top \begin{bmatrix} M & D \end{bmatrix} = 0$ and hence z must be zero. This would prove that $\begin{bmatrix} M & D \end{bmatrix}$ has full row rank. We will distinguish two cases: $i = j$ and $i \neq j$. Suppose first that $i = j$. Let $\beta, \omega \subseteq \{1, \ldots, p\}$ be defined as the index sets

$$\beta = \{k \mid k \neq j \text{ and } k \text{ is black}\} \quad \text{and} \quad \omega = \{\ell \mid \ell \neq j \text{ and } \ell \text{ is white}\}.$$

In the sequel, to simplify the notations, for a given vector $z \in \mathbb{R}^p$ and a given index set $\alpha \subseteq \{1, \ldots, p\}$, we define

$$z_\alpha := \{x \in \mathbb{R}^{|\alpha|} \mid x_i = z_{\alpha(i)}, i \in \{1, \ldots, |\alpha|\}\}$$

where $|\alpha|$ is the cardinality of α. From $z^\top M = 0$, we get

$$z_j M_{jj} + z_\beta^\top M_{\beta j} + z_\omega^\top M_{\omega j} = 0. \tag{2.14}$$

Since j is the only white out-neighbor of itself, we must have that M_{jj} is nonzero and that $M_{\omega j}$ is a zero vector. Moreover, it follows from $z^\top D = 0$ that z_β must a zero vector. Therefore, (2.14) implies that z_j must be zero.

Next, suppose that $i \neq j$. Let $\beta, \omega \subseteq \{1, \ldots, p\}$ be defined as the index sets

$$\beta = \{k \mid k \neq i,\ k \neq j, \text{ and } k \text{ is black}\} \quad \text{and} \quad \omega = \{\ell \mid \ell \neq i,\ \ell \neq j, \text{ and } \ell \text{ is white}\}.$$

From $z^\top M = 0$, we now get

$$z_i M_{ii} + z_j M_{ji} + z_\beta^\top M_{\beta i} + z_\omega^\top M_{\omega i} = 0. \tag{2.15}$$

Since j is the only white out-neighbor of i, we must have that M_{ji} is nonzero and that $M_{\omega i}$ is a zero vector. Moreover, it follows from $z^\top D = 0$ that z_β must a zero vector. Therefore, (2.15) implies that

$$z_i M_{ii} + z_j M_{ji} = 0. \tag{2.16}$$

Now, we distinguish two cases: i is black and i is white. If i is black, then we have that z_i is zero because $z^\top D = 0$. Therefore, (2.16) implies that $z_j = 0$ as desired. Finally, if i is white, then we have that $M_{ii} = 0$ for otherwise i would have two white out-neighbors. Again, (2.16) implies that z_j is zero. This completes the proof. ■

Now, we can give the proof of Theorem 2.5.

Proof of Theorem 2.5. To prove the 'if' part, suppose that $\mathcal{G}(\mathcal{M})$ is colorable. Let $M \in \mathcal{P}(\mathcal{M})$. By repeated application of Lemma 2.15, it follows that M has full row rank if and only if $\begin{bmatrix} M & I \end{bmatrix}$ has full row rank, which is obviously true. Therefore, we conclude that M has full row rank.

To prove the 'only if' part, suppose that $\mathcal{M}$ has full row rank but $\mathcal{G}(\mathcal{M})$ is not colorable. Let C be the set of nodes that are colored black by repeated application of the color change rule until no more color changes are possible. Then, C is a *strict* subset of $\{1, 2, \ldots, p\}$. Thus, possibly after reordering the nodes, we can partition $\mathcal{M}$ as

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}$$

where the rows of the matrix $\mathcal{M}_1$ correspond to the nodes in C and the matrix $\mathcal{M}_2$ correspond to the remaining white nodes. Note that $C = \varnothing$ means that $\mathcal{M}_2 = \mathcal{M}$ and $\mathcal{M}_1$ is absent. Since no more color changes are possible, there is no column of $\mathcal{M}_2$ that has exactly one $*$ entry while all other entries are 0. Therefore, for any column of $\mathcal{M}_2$, we have one of the following three cases:

a. All entries are 0.

b. There exists exactly one ? entry while all other entries are 0.

c. At least two entries belong to the set $\{*, ?\}$.

Consequently, there exists a matrix $M_2 \in \mathcal{P}(\mathcal{M}_2)$ such that its column sums are zero, that is $\mathbb{1}^\top M_2 = 0$, where $\mathbb{1}$ denotes the vector of ones of appropriate size. Take any $M_1 \in \mathcal{P}(\mathcal{M}_1)$. Then

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \in \mathcal{P}(\begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}) = \mathcal{P}(\mathcal{M})$$

satisfies

$$\begin{bmatrix} 0^\top & \mathbb{1}^\top \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 0.$$

Hence, M does not have full row rank and we have reached a contradiction. ■

2.6.3 Proof of Theorem 2.6

By Theorem 2.3 and Theorem 2.5, we have that $\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}$ is controllable if and only if if and only if $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix})$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix})$ are colorable. ■

2.6.4 Proof of Theorem 2.8

The 'if' part is evident. Therefore, it is enough to prove the 'only if' part. Suppose that the system $(\mathcal{A}, \mathcal{B})$ is stabilizable. Let $(A, B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$. Then, (A, B) is stabilizable. Note that $A \in \mathcal{P}(\mathcal{A})$ if and only if $-A \in \mathcal{P}(\mathcal{A})$. Therefore, we have both (A, B) and $(-A, B)$ stabilizable. It follows from the Hautus test for stabilizability (see e.g. [26, Theorem 3.32]) that (A, B) is controllable. Consequently, the system $(\mathcal{A}, \mathcal{B})$ is controllable. ■

2.6.5 Proof of Lemma 2.13

Since the 'if' part is evident, it remains to prove the 'only if' part. Suppose that $\mathcal{M}$ has full row rank. From Theorem 2.5, it follows that $\mathcal{G}(\mathcal{M})$ is colorable. In particular, there exist $i \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, p\}$ such that $\mathcal{M}_{ji} = *$ and $\mathcal{M}_{ki} = 0$ for all $k \neq j$. Therefore, we can find permutation matrices P_1' and P_2' such that

$$P_1' \mathcal{M} P_2' = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathcal{M}' & & \vdots \\ & & & 0 \\ \hline \otimes & \cdots & \otimes & * \end{array}\right]$$

where the symbol $\otimes$ indicates an entry that can be either 0, $*$ or ?. Note that M has full row rank for all $M \in \mathcal{P}(\mathcal{M})$ if and only if M' has full row rank for all $M \in \mathcal{P}(\mathcal{M}')$. Therefore, repeated application of the argument above results in permutation matrices P_1 and P_2 such that

$$P_1 \mathcal{M} P_2 = \begin{bmatrix} \otimes & \cdots & \otimes & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \ddots & \ddots & \vdots \\ \otimes & \cdots & \otimes & \cdots & \otimes & * & 0 \\ \otimes & \cdots & \otimes & \cdots & \otimes & \otimes & * \end{bmatrix}.$$

■

2.7 Conclusions

In most of the existing literature on strong structural controllability of structured systems, a zero/nonzero structure of the system matrices is assumed to be given. However, in many physical systems or linear networked systems, apart from fixed zero entries and nonzero entries, we need to allow a third kind of entries, namely those that can take arbitrary (zero or nonzero) values. To deal with this, we have extended the notion of zero/nonzero structure to what we have called zero/nonzero/arbitrary structure. We have formalized this more general class of structured systems using pattern matrices containing fixed zero, arbitrary nonzero, and arbitrary entries. In this setup, we have established necessary and sufficient algebraic conditions for strong structural controllability of these systems in terms of full rank tests on two associated pattern matrices. Moreover, a necessary and sufficient graph theoretic condition for a given pattern matrix to have full row rank has been provided in terms of a new color change rule. We have then established a graph theoretic test for strong structural controllability of the new class of structured systems. Finally, we have shown how our results generalize previous work. We have also shown that some existing results [30, 32] that are seemingly incomparable to ours, can be put in our framework, thus unveiling an overarching theory.

In addition to strong structural controllability, weak structural controllability and strong structural stabilizability of structured systems with zero/nonzero/arbitrary structures have been briefly analyzed. We have shown that weak structural controllability of our structured systems can be checked using tests that already exist in the literature. We have also shown that a structured system with zero/nonzero/arbitrary structure is strongly structurally stabilizable if and only if it is strongly structurally controllable.

It would be interesting to adopt our new framework of structured systems to other problem areas in systems and control. In the next chapter, we will study the fault detection and isolation problem [77] for structured systems in this framework.

3 Fault Detection and Isolation for Linear Structured Systems

In this chapter, we follow a geometric approach to verify solvability of the fault detection and isolation (FDI) problem for linear structured systems. Firstly, we will develop a necessary and sufficient condition under which the FDI problem for a given particular linear time-invariant (LTI) system is solvable. Based on this condition, we will then establish a necessary condition for solvability of the FDI problem for linear structured systems. By assuming that this necessary condition holds, we develop a sufficient algebraic condition for solvability of the FDI problem in terms of a rank test on an associated pattern matrix. To illustrate that this condition is not necessary, we provide a counterexample in which the FDI problem is solvable while the condition is not satisfied. Finally, we develop a graph theoretic condition for the full rank property of a given pattern matrix, which leads to a graph theoretic condition for solvability of the FDI problem.

3.1 Introduction

This chapter is concerned with the FDI problem for LTI systems with faults. This problem has received considerable attention within the control community in the past decades and this has led to several approaches to FDI, see, e.g., [77–82] and the references therein. Among these references, those closer to the results presented in the current chapter are [79] and [77], in which FDI for LTI systems is performed using unknown input observers that enable so-called output separability of the fault subspaces. If such observers exist, then we say that for the given system the FDI problem is solvable.

Although conditions for solvability of the FDI problem for a given LTI system have been introduced in [79], their application relies on the exact knowledge of the dynamics of this system, meaning that precise information on the system matrices is required. However, in many scenarios, such knowledge is unavailable, and only the zero/nonzero/arbitrary structure can be acquired. This leads to the concept of linear structured system introduced in [58] which represents a family of LTI systems sharing the same structure. A large amount of literature has been devoted to analyzing system-theoretical properties for linear structured systems. For instance, strong structural controllability has been studied in [29, 30, 32, 58], strong targeted controllability in [62, 75], and identifiability in[83].

Roughly speaking, in the framework of linear structured systems, the research on the FDI problem can be subdivided into two directions. The first direction aims at providing conditions under which the FDI problem is solvable for at least one member of a given structured system, see, e.g., [82, 84, 85]. The other direction aims at establishing conditions to guarantee that the FDI problem is solvable for all members of a given structured system, see, e.g., [77]. In the present chapter, we will pursue the second research direction. For a given structured system, if the FDI problem for all systems in the structured system is solvable, then we say that the FDI problem for this structured system is *solvable*. To the best of our knowledge, in this direction the only existing work is [77], which has studied a special kind of linear structured system, named systems defined on graphs. The goal of the present chapter is to provide conditions under which the FDI problem is solvable for a general structured system. The main contributions of this chapter are the following:

1. We develop a necessary and sufficient condition under which the FDI problem is solvable for a given particular LTI system.
2. For linear structured systems we first establish a necessary condition for solvability of the FDI problem. Assuming that this necessary condition holds, we then establish a sufficient algebraic condition for solvability of the FDI problem. This condition is expressed in terms of a rank test on a pattern matrix associated with the structured system. Moreover, we provide a counterexample to show that this sufficient condition is not necessary.
3. Using the concept of colorability of a graph, we provide a graph theoretic condition for solvability of the FDI problem for a given structured system.

3.2 Preliminaries and problem statement

3.2.1 Geometric control theory

Geometric control theory plays a fundamental role in this chapter. Therefore, in this subsection, we will give a brief review of some basic concepts in this field. Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output, respectively, and A, B and C are matrices of appropriate dimensions. A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is called (C, A)-*invariant* if $A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$. This condition is equivalent to the existence of a matrix $G \in \mathbb{R}^{n \times p}$ such that $\mathcal{S}$ is $(A + GC)$-*invariant*, i.e., $(A + GC)\mathcal{S} \subseteq \mathcal{S}$. Such a G is called a *friend* of $\mathcal{S}$. A family $\{\mathcal{S}_i\}_{i=1}^k$ of (C, A)-invariant subspaces of $\mathbb{R}^n$ is called *compatible* if the subspaces $\mathcal{S}_i$ have a common friend. Given the system (3.1), a family of subspaces $\{\mathcal{S}_i\}_{i=1}^k$ is called *output separable* if for $i = 1, \ldots, k$

$$C\mathcal{S}_i \cap (\sum_{j \neq i} C\mathcal{S}_j) = \{0\}.$$

Any output separable family of (C, A)-invariant subspaces is compatible [79, Lemma 2]. Moreover, if it also satisfies the condition that $C\mathcal{S}_i \neq \{0\}$ for $i = 1, \ldots, k$, we say that the family $\{C\mathcal{S}_i\}_{i=1}^k$ is *independent*.

For a given subspace $\mathcal{D} \subseteq \mathbb{R}^n$, there exists a smallest (C, A)-invariant subspace containing $\mathcal{D}$, denoted by $\mathcal{S}^*$. Such a minimal subspace can be computed by the following subspace algorithm (see, e.g., the conditioned invariant subspace algorithm p.111 of [26]):

$$\begin{aligned} &\mathcal{S}^0 = \mathcal{D} \\ &\mathcal{S}^k = \mathcal{D} + A(\mathcal{S}^{k-1} \cap \ker C) \text{ for } k = 1, 2, \ldots. \end{aligned} \tag{3.2}$$

Denote the dimension of $\mathcal{D}$ by $\dim \mathcal{D}$. It follows from Theorem 5.8 of [26] that there exists $k \leqslant n - \dim \mathcal{D}$ such that $\mathcal{S}_k = \mathcal{S}_{k+1}$, and hence $\mathcal{S}^* = \mathcal{S}_k$.

3.2.2 The geometric approach to the FDI problem for LTI systems

In this subsection, we will review the geometric approach to the FDI problem for LTI systems. Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Lf \\ y &= Cx \end{aligned} \tag{3.3}$$

where $x \in \mathbb{R}^n$, $f \in \mathbb{R}^q$ and $y \in \mathbb{R}^p$ are the state, fault and output, respectively, and A, L and C are matrices of appropriate dimensions. We denote the system (3.3) by (A, L, C). We say that the ith fault occurs if $f_i \neq 0$ (i.e., not identically equal to 0), where f_i is the ith component of f. In existing literature, the general question has been studied how to detect whether faults occur or not, and if so then isolate which fault f_i occurs. This problem is refereed to as the fault detection and isolation (FDI) problem. Following the approach proposed in [79], the FDI problem for (3.3) amounts to finding $G \in \mathbb{R}^{n\times p}$ such that the family of subspaces $\{C\mathcal{V}_i\}_{i=1}^q$ is independent, where $\mathcal{V}_i$ is the smallest $(A+GC)$-invariant subspace containing $\operatorname{im} L_i$. Here, L_i denotes the ith column of L. If such G exists, then we say that the FDI problem is *solvable*. In what follows, we will briefly explain this approach. Suppose that we have found a G satisfying the above constraints. Consider the state observer

$$\dot{\hat{x}} = (A+GC)\hat{x} - Gy. \tag{3.4}$$

Define the *innovation* as

$$r \coloneqq C\hat{x} - y$$

and *error*

$$e \coloneqq \hat{x} - x.$$

By interconnecting (3.3) and (3.4), we obtain

$$\begin{aligned} \dot{e} &= (A+GC)e - Lf \\ r &= Ce. \end{aligned} \tag{3.5}$$

Note that in this chapter, we do not consider any stability requirement on the observer, which means that we do not require $e(t) \to 0$, and we assume that $e(0) = 0$. Under this assumption, for any fault f, the resulting error trajectory $e(t)$ lies in the reachable subspace of $(A+GC, L)$, which is clearly equal to $\mathcal{V}_1 + \cdots + \mathcal{V}_q$. For the corresponding innovation trajectory $r(t)$ we then have

$$r(t) \in C\mathcal{V}_1 + \cdots + C\mathcal{V}_q.$$

If the family $\{C\mathcal{V}_i\}_{i=1}^q$ is independent, then this is a direct sum, and $r(t)$ can be written uniquely as

$$r(t) = r_1(t) + \cdots + r_q(t) \tag{3.6}$$

with $r_i(t) \in C\mathcal{V}_i$ for all t. The unique representation (3.6) can be used to determine whether the ith fault occurs. Indeed in (3.6) $r_i \neq 0$ (i.e., not identically equal to 0) only if $f_i \neq 0$. To see this, note that $f_i(t) = 0$ for all t implies $e(t) \in \sum_{j\neq i} \mathcal{V}_j$, so $r(t) \in \sum_{j\neq i} C\mathcal{V}_j$, equivalently, $r_i(t) = 0$ for all t.

Let $\mathcal{S}_i^*$ be the smallest (C, A)-invariant subspace containing $\operatorname{im} L_i$. In [79] it has been shown that the FDI problem for the system (3.3) is solvable if and only if the family $\{C\mathcal{S}_i^*\}_{i=1}^q$ is independent, i.e., the family $\{\mathcal{S}_i^*\}_{i=1}^q$ is output separable and $C\mathcal{S}_i^* \neq \{0\}$ for $i = 1, \dots, q$.

3.2.3 Linear structured systems and problem formulation

Again, consider the LTI system (3.3). In many scenarios, the exact values of the entries in the system matrices are not known, but some entries are known to be always zero, some are nonzero, and the remaining entries are arbitrary real numbers. To describe such kind of matrices, the authors in [58] have introduced the definition of *pattern matrix* as follows.

A pattern matrix is a matrix with entries in the set of symbols $\{0, *, ?\}$. The set of all $r \times s$ pattern matrices is denoted by $\{0, *, ?\}^{r\times s}$. For a given $r \times s$ pattern matrix $\mathcal{M}$, we define the *pattern class* of $\mathcal{M}$ as

$$\begin{aligned}\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{r\times s} \mid & M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \\ & M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *\}.\end{aligned}$$

This means that for a matrix $M \in \mathcal{P}(\mathcal{M})$, the entry M_{ij} is either (i) *zero* if $\mathcal{M}_{ij} = 0$, (ii) *nonzero* if $\mathcal{M}_{ij} = *$, or (iii) *arbitrary* (zero or nonzero) if $\mathcal{M}_{ij} = ?$.

Let $\mathcal{A} \in \{0, *, ?\}^{n\times n}$, $\mathcal{L} \in \{0, *, ?\}^{n\times q}$ and $\mathcal{C} \in \{0, *, ?\}^{n\times p}$. The family of systems (A, L, C) with $A \in \mathcal{P}(\mathcal{A})$, $L \in \mathcal{P}(\mathcal{L})$ and $C \in \mathcal{P}(\mathcal{C})$ is called the *linear structured system associated with* $\mathcal{A}$*,* $\mathcal{L}$*, and* $\mathcal{C}$. Throughout this chapter, we use $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ to represent this structured system, and we write $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$ if $A \in \mathcal{P}(\mathcal{A})$, $L \in \mathcal{P}(\mathcal{L})$ and $C \in \mathcal{P}(\mathcal{C})$. Based on these notions and notations, we define the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ to be *solvable* if the FDI problem is solvable for every $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$. The research problem of this chapter is then formally stated as follows.

Problem 3.1. *Given $(\mathcal{A}, \mathcal{L}, \mathcal{C})$, find conditions under which the FDI problem is solvable for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$.*

3.3 Conditions for solvability of the FDI problem for (A, L, C)

In this section, we will establish a necessary and sufficient condition under which the FDI problem is solvable for a given LTI system (A, L, C) of the form (3.3). Recall that solvability of the FDI problem for (A, L, C) is equivalent to the independence of the family $\{C\mathcal{S}_i^*\}_{i=1}^q$, where $\mathcal{S}_i^*$ is the smallest (C, A)-invariant subspace containing $\operatorname{im} L_i$ $(i = 1, \ldots, q)$. Therefore, we will first provide a characterization of $\mathcal{S}_i^*$. Let d_i be a positive integer such that

$$CA^j L_i = 0 \text{ for } j = 0, 1, \ldots, d_i - 2 \quad \text{and} \quad CA^{d_i-1} L_i \neq 0.$$

Here and in the sequel, we define $A^0 := I$. It is obvious from the Cayley-Hamilton theorem that either $d_i \leqslant n$ or d_i does not exist. If this d_i exists, we then call it the *index* of (A, L_i, C).

We are now ready to state a characterization of $C\mathcal{S}_i^*$ in the following lemma.

Lemma 3.2. *Consider the system* (A, L, C) *of the form* (3.3). *Let* $i \in \{1, \ldots, q\}$. *Denote by* $\mathcal{S}_i^*$ *the smallest* (C, A)*-invariant subspace containing* $\operatorname{im} L_i$. *Then, we have that*

$$C\mathcal{S}_i^* = \begin{cases} \operatorname{im} CA^{d_i - 1}L_i & \text{if the index } d_i \text{ of } (A, L_i, C) \text{ exists,} \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. In this proof, we will employ the recurrence relation (3.2) to prove the statement. Let $\mathcal{S}_i^\ell$ be the sequence of subspaces given by

$$\begin{aligned} \mathcal{S}_i^0 &= \operatorname{im} L_i, \\ \mathcal{S}_i^\ell &= \operatorname{im} L_i + A(\mathcal{S}_i^{\ell-1} \cap \ker C) \text{ for } \ell = 1, 2, \ldots. \end{aligned} \tag{3.7}$$

We then distinguish two cases: (i) d_i exists, and (ii) d_i does not exist. In case (i), we have that

$$CA^k L_i = 0 \text{ for } k = 0, 1, \ldots, d_i - 2 \tag{3.8}$$

and

$$CA^{d_i-1}L_i \neq 0. \tag{3.9}$$

By combining (3.7) and (3.8), it can be verified directly that

$$\mathcal{S}_i^k = \operatorname{im} \begin{bmatrix} L_i & AL_i & \ldots & A^k L_i \end{bmatrix} \text{ for } k = 0, 1, \ldots, d_i - 1. \tag{3.10}$$

Now, we claim that:

(a) $\mathcal{S}_i^{d_i-1} = \mathcal{S}_i^{d_i}$,

(b) the dimension of $\mathcal{S}_i^{d_i-1}$ is strictly larger than that of $\mathcal{S}_i^{d_i-2}$.

If both claims (a) and (b) are true, then

$$\mathcal{S}_i^* = \mathcal{S}_i^{d_i-1} = \operatorname{im} \begin{bmatrix} L_i & AL_i & \ldots & A^{d_i-1}L_i \end{bmatrix},$$

and hence $C\mathcal{S}_i^* = \operatorname{im} CA^{d_i-1}L_i$. Note that (a) follows immediately from (3.9) and (3.10):

$$\begin{aligned} \mathcal{S}_i^{d_i-1} &\overset{(3.10)}{=} \operatorname{im} \begin{bmatrix} L_i & AL_i & \ldots & A^{d_i-1}L_i \end{bmatrix} \\ \mathcal{S}_i^{d_i} &= \operatorname{im} L_i + A(\mathcal{S}_i^{d_i-1} \cap \ker C) \overset{(3.9)}{=} \operatorname{im} \begin{bmatrix} L_i & AL_i & \ldots & A^{d_i-1}L_i \end{bmatrix}. \end{aligned}$$

To prove (b), we assume that (b) is not true, i.e.,

$$\mathcal{S}_i^{d_i-1} = \mathcal{S}_i^{d_i-2} = \operatorname{im}\begin{bmatrix} L_i & AL_i & \dots & A^{d_i-2}L_i \end{bmatrix}.$$

This implies

$$A^{d_i-1}L_i \in \operatorname{im}\begin{bmatrix} L_i & AL_i & \dots & A^{d_i-2}L_i \end{bmatrix} \subseteq \ker C,$$

which contradicts (3.9), and hence (b) is proved. For case (ii), we have

$$CA^kL_i = 0 \text{ for } k = 0, 1, \dots, n-1. \tag{3.11}$$

By combining (3.7) and (3.11), we obtain

$$\begin{aligned} \mathcal{S}_i^{n-1} &= \operatorname{im}\begin{bmatrix} L_i & AL_i & \cdots & A^{n-1}L_i \end{bmatrix} \subseteq \ker C \\ \mathcal{S}_i^{n} &= \operatorname{im}\begin{bmatrix} L_i & AL_i & \cdots & A^{n-1}L_i & A^nL_i \end{bmatrix} \end{aligned}.$$

It then follows from the Cayley-Hamilton theorem that

$$A^nL_i \in \mathcal{S}_i^{n-1}.$$

This means that $\mathcal{S}_i^{n-1} = \mathcal{S}_i^n$, and hence $\mathcal{S}_i^* = \mathcal{S}^{n-1} \subseteq \ker C$. Therefore, we have $C\mathcal{S}_i^* = \{0\}$. This completes the proof. ∎

By the above lemma, the family $\{C\mathcal{S}_i^*\}_{i=1}^q$ of subspaces is independent if and only if the index d_i exist for $i = 1, \dots, q$, and the vectors $\{CA^{d_i-1}L_i\}_{i=1}^q$ are linearly independent. Thus we arrive at the main result of this section which provides a necessary and sufficient condition under which the FDI problem for (A, L, C) is solvable.

Theorem 3.3. *Consider the system (A, L, C) of the form (3.3). The FDI problem for (A, L, C) is solvable if and only if the index d_i exists for $i = 1, \dots, q$, and the matrix R has full column rank, where R is defined by*

$$R := \begin{bmatrix} CA^{d_1-1}L_1 & \cdots & CA^{d_q-1}L_q \end{bmatrix}. \tag{3.12}$$

Proof. The proof follows immediately from Lemma 3.2 and is hence omitted. ∎

3.4 Algebraic conditions for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$

In this section, we will establish a necessary condition and a sufficient condition that enables the FDI problem for a given structured system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ to be solvable. Before presenting the results of this section, we first provide some background on operations on pattern matrices. More details can be found in [86]. Addition and multiplication within the set $\{0, *, ?\}$ are defined in Table 3.1 below.

Table 3.1: Addition and multiplication within the set $\{0, *, ?\}$.

$+$	0	$*$	$?$
0	0	$*$	$?$
$*$	$*$	$?$	$?$
$?$	$?$	$?$	$?$

$\cdot$	0	$*$	$?$
0	0	0	0
$*$	0	$*$	$?$
$?$	0	$?$	$?$

Based on the operations in Table 3.1, multiplication of pattern matrices is then defined as follows.

Definition 3.4. *Let $\mathcal{M} \in \{0, *, ?\}^{r\times s}$ and $\mathcal{N} \in \{0, *, ?\}^{s\times t}$. The product of $\mathcal{M}$ and $\mathcal{N}$ is defined as $\mathcal{M}\mathcal{N} \in \{0, *, ?\}^{r\times t}$ given by*

$$(\mathcal{M}\mathcal{N})_{ij} := \sum_{k=1}^{q} (\mathcal{M}_{ik} \cdot \mathcal{N}_{kj}) \quad i = 1, \dots, r, \quad j = 1, \dots, t. \tag{3.13}$$

It is easily seen that $MN \in \mathcal{P}(\mathcal{M}\mathcal{N})$ for every pair of matrices $M \in \mathcal{P}(\mathcal{M})$ and $N \in \mathcal{P}(\mathcal{N})$. If $r = s$, we call $\mathcal{M}$ a square pattern matrix. For any given non-negative integer k, we define the kth power $\mathcal{M}^k$ recursively by

$$\mathcal{M}^0 = \mathcal{I}, \quad \mathcal{M}^i = \mathcal{M}^{i-1}\mathcal{M}, \quad i = 1, \dots, k$$

where $\mathcal{I}$ represents a square pattern matrix of appropriate dimensions with all diagonal entries equal to $*$ and all off-diagonal equal to 0. In the sequel, let $\mathcal{O}$ denote any pattern matrix of appropriate dimensions with all entries equal to 0.

Next, consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. Let $\mathcal{L}_i$ represent the ith column of $\mathcal{L}$ for $i = 1, \dots, q$. Let η_i be a positive integer such that

$$\mathcal{C}\mathcal{A}^j\mathcal{L}_i = \mathcal{O} \text{ for } j = 0, 1, \dots, \eta_i - 2 \quad \text{and} \quad \mathcal{C}\mathcal{A}^{\eta_i-1}\mathcal{L}_i \neq \mathcal{O}.$$

If η_i exists, then we call it the *index* of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. In the sequel, we will write $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ if $A \in \mathcal{P}(\mathcal{A})$, $L_i \in \mathcal{P}(\mathcal{L}_i)$ and $C \in \mathcal{P}(\mathcal{C})$. Before continuing to explore conditions for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$, we first provide the following lemma which states the relationship between the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and that of $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.

Lemma 3.5. *Consider the pattern matrix triple $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. Then the following holds:*

i. *Let $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. If both the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and the index d_i of (A, L_i, C) exist, then $d_i \geqslant \eta_i$.*

ii. *Suppose that the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ exists, and suppose further that at least one entry of $\mathcal{C}\mathcal{A}^{\eta_i-1}\mathcal{L}_i$ is equal to $*$. Let $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. Then, the index d_i of (A, L_i, C) exists and $d_i = \eta_i$.*

iii. *If the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ does not exist, then the index of (A, L_i, C) does not exist for any $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$.*

Proof. By Definition 3.4, it follows that the vector $CA^{\ell}L_i \in \mathcal{P}(\mathcal{C}\mathcal{A}^{\ell}\mathcal{L}_i)$ for $i = 0, 1, \dots$ and for all $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. In order to prove (i), suppose that both the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ and the index d_i of (A, L_i, C) exist. By the definition of η_i we have that $\mathcal{C}\mathcal{A}^{\ell}\mathcal{L}_i = \mathcal{O}$ for $\ell = 0, 1, \dots, \eta_i - 2$, and by the definition of d_i it follows that $CA^{d_i-1}L_i \neq 0$. Therefore, we obtain $d_i \geqslant \eta_i$. Next, to prove (ii), we assume that $\mathcal{C}\mathcal{A}^{\eta_i-1}\mathcal{L}_i$ contains at least one $*$ entry, which implies that all the vectors in the pattern class $\mathcal{P}(\mathcal{C}\mathcal{A}^{\eta_i-1}\mathcal{L}_i)$ are unequal to 0. Let $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. Clearly, the vector $CA^{\eta_i-1}L_i \in \mathcal{P}(\mathcal{C}\mathcal{A}^{\eta_i-1}\mathcal{L}_i)$, and hence $CA^{\eta_i-1}L_i \neq 0$. By definition, the index d_i of (A, L_i, C) must exist and $d_i \leqslant \eta_i$. Recalling (i), we conclude that $d_i = \eta_i$. The proof of (iii) is trivial. Indeed, suppose that the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ does not exist. It then follows that

$$\mathcal{C}\mathcal{A}^{\ell}\mathcal{L}_i = \mathcal{O} \text{ for } \ell = 0, 1, \dots$$

which implies that $CA^{\ell}L_i$ is equal to 0 for every $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. That is, the index of (A, L_i, C) does not exist for any $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. ∎

To illustrate the above lemma, we now provide an example.

Example 3.1. Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ with

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} ? & * & 0 \\ 0 & * & 0 \end{bmatrix}. \tag{3.14}$$

Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ denote the first, second and third column of $\mathcal{L}$. For $\mathcal{L}_1$ and $\mathcal{L}_2$ we compute

$$\mathcal{C}\mathcal{L}_1 = \begin{bmatrix} ? \\ 0 \end{bmatrix} \neq \mathcal{O} \quad \text{and} \quad \mathcal{C}\mathcal{L}_2 = \begin{bmatrix} * \\ * \end{bmatrix} \neq \mathcal{O}.$$

This implies that $\eta_1 = \eta_2 = 1$, where η_i is the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ for $i = 1, 2$. In addition, for $\mathcal{L}_3$ we compute

$$\mathcal{C}\mathcal{A}^{\ell}\mathcal{L}_3 = \mathcal{O} \quad \text{for} \quad \ell = 0, 1, \dots$$

which implies that the index of $(\mathcal{A}, \mathcal{L}_3, \mathcal{C})$ does not exists. Next, we will show that for some $(A, L_1, C) \in (\mathcal{A}, \mathcal{L}_2, \mathcal{C})$ the index d_1 of (A, L_1, C) is larger than η_1, for every $(A, L_2, C) \in (\mathcal{A}, \mathcal{L}_2, \mathcal{C})$ its index d_2 is equal to η_2, and for every $(A, L_3, C) \in (\mathcal{A}, \mathcal{L}_3, \mathcal{C})$ its index does not exists,. Indeed, for $A \in \mathcal{P}(\mathcal{A})$, $L \in \mathcal{P}(\mathcal{L})$ and $C \in \mathcal{P}(\mathcal{C})$ we have

$$A = \begin{bmatrix} 0 & 0 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & c_4 & c_5 \end{bmatrix}, \quad C = \begin{bmatrix} \lambda_1 & c_6 & 0 \\ 0 & c_7 & 0 \end{bmatrix} \tag{3.15}$$

where $c_1, \ldots, c_7$ are arbitrary nonzero real numbers, and λ_1 is an arbitrary real number. Next, we compute

$$\begin{bmatrix} CL_1 & CL_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_2 & c_3 c_6 \\ 0 & c_3 c_9 \end{bmatrix} \quad \text{and} \quad CAL_1 = \begin{bmatrix} c_1 c_2 c_6 \\ c_1 c_2 c_7 \end{bmatrix}. \tag{3.16}$$

Thus, for all choices of $c_1, \ldots, c_7$ and λ_1 we have

$$d_2 = 1 = \eta_2$$

while if $\lambda_1 = 0$ then $d_1 = 2 > \eta_1$ and otherwise $d_1 = 1 = \eta_1$. In addition, it is obvious that for all choices of $c_1, \ldots, c_7$ and λ_1 we have

$$CA^\ell L_3 = 0 \text{ for } \ell = 0, 1, \ldots$$

and hence the index of (A, L_3, C) does not exist.

Lemma 3.5 immediately yields a necessary condition for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$.

Theorem 3.6. *Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. Suppose that the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable. Then, the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ exists for all $i = 1, \ldots q$.*

Proof. Since the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable, the FDI problem is solvable for all $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$. Assume that for some $i \in \{1, \ldots, q\}$ the index η_i of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$ does not exist. By statement iii of Lemma 3.5, it follows that the index d_i of (A, L_i, C) does not exist for any $(A, L_i, C) \in (\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. It then follows from Theorem 3.3 that the FDI problem for (A, L, C) is not solvable for any $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$. Therefore, we reach a contradiction and complete the proof. ■

By the above theorem, in the sequel we will assume that for all $i = 1, \ldots q$ the indices η_i exist. Based on this assumption, we will continue to explore sufficient conditions for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. To do so, we first define the following pattern matrix associated with $(\mathcal{A}, \mathcal{L}, \mathcal{C})$:

$$\mathcal{R} := \begin{bmatrix} \mathcal{C}\mathcal{A}^{\eta_1 - 1}\mathcal{L}_1 & \cdots & \mathcal{C}\mathcal{A}^{\eta_q - 1}\mathcal{L}_q \end{bmatrix} \tag{3.17}$$

where η_i is the index of $(\mathcal{A}, \mathcal{L}_i, \mathcal{C})$. We say that $\mathcal{R}$ has *full column rank* if all the matrices in the pattern class $\mathcal{P}(\mathcal{R})$ have full column rank. We are now ready to establish a sufficient condition for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$.

Theorem 3.7. *Consider the system* $(\mathcal{A}, \mathcal{L}, \mathcal{C})$*. Let* $\mathcal{R}$ *be the pattern matrix given by* (3.17)*. The FDI problem for* $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ *is solvable if* $\mathcal{R}$ *has full column rank.*

Proof. Since $\mathcal{R}$ has full column rank, each column of $\mathcal{R}$ contains at least one $*$ entry. Let $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$. By (ii) of Lemma 3.5 it follows that $d_i = \eta_i$, where d_i is the index of (A, L_i, C) for $i = 1, 2, \ldots, q$. This implies that the matrix R given by (3.12) is in $\mathcal{P}(\mathcal{R})$, and hence R has full column rank. It then follows from Theorem 3.3 that the FDI problem is solvable. Since (A, L, C) is an arbitrary system in $(\mathcal{A}, \mathcal{L}, \mathcal{C})$, we conclude that the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable and complete the proof. ■

Note that the condition given in Theorem 3.7 is sufficient but not necessary. To show this, we provide the following counterexample.

Example 3.2. Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ with

$$\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the first and second column of $\mathcal{L}$. We compute

$$\mathcal{C}\mathcal{L}_1 = \begin{bmatrix} * \\ * \end{bmatrix} \quad \text{and} \quad \mathcal{C}\mathcal{L}_2 = \begin{bmatrix} ? \\ * \end{bmatrix}$$

and, by (3.17),

$$\mathcal{R} = \begin{bmatrix} * & ? \\ * & * \end{bmatrix}.$$

It then follows from

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{P}(\mathcal{R})$$

that $\mathcal{R}$ does not have full column rank. Next, we will show that, however, the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable. Due to Theorem 3.3, it suffices to show that for each $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$ the associated matrix R has full column rank. Clearly, every $(A, L, C) \in (\mathcal{A}, \mathcal{L}, \mathcal{C})$ has the form

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_4 & c_5 \\ c_6 & 0 \end{bmatrix}$$

where $c_1, \ldots, c_6$ are arbitrary nonzero real numbers. By (3.12), we obtain

$$R = CL = \begin{bmatrix} c_1c_4 & c_2c_4 + c_3c_5 \\ c_1c_6 & c_2c_6 \end{bmatrix}.$$

It turns out that R has full column rank. Indeed, the determinant of R is equal to $-c_1c_3c_5c_6$ which is always nonzero. Consequently, the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable. This provides a counterexample for the necessity of the condition in Theorem 3.7.

3.5 A graph theoretic condition for solvability of the FDI problem

So far, we have provided a sufficient condition for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ in terms of the full column rank property of its associated matrix $\mathcal{R}$. However, given such a matrix $\mathcal{R}$, it is not clear how to check its full column rank property. Hence, in this section, we will provide a graph theoretic condition under which a given pattern matrix $\mathcal{R}$ has full column rank. Clearly, by Theorem 3.7 this will immediately lead to a graph theoretic condition for solvability of the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$.

We will now first review the concept of graph associated with a given pattern matrix, and the color change rule that acts on this graph. For more details, see Section 2.4. For a given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{r\times s}$ with $r \leqslant s$, the graph $\mathcal{G}(\mathcal{M}) = (V, E)$ associated with $\mathcal{M}$ is defined as follows. Take as node set $V = \{1, \dots, r\}$ and define the edge set $E \subseteq V \times V$ such that $(j, i) \in E$ if and only if $\mathcal{M}_{ij} = *$ or $\mathcal{M}_{ij} =?$. Also, in order to distinguish between $*$ and ? entries in $\mathcal{M}$, we define two subsets E_* and $E_?$ of the edge set E as follows: $(j, i) \in E_*$ if and only if $\mathcal{M}_{ij} = *$ and $(j, i) \in E_?$ if and only if $\mathcal{M}_{ij} =?$. Then, obviously, $E = E_* \cup E_?$ and $E_* \cap E_? = \varnothing$. To visualize this, solid and dashed arrows are used to represent edges in E_* and $E_?$, respectively. We say that $\mathcal{M}$ has *full row rank* if the matrix M has full row rank for all $M \in \mathcal{P}(\mathcal{M})$. Next, we introduce a so-called *color change rule* which is defined as follows.

1. Initially, color all nodes in $\mathcal{G}(\mathcal{M})$ white.
2. If a node i has exactly one white out-neighbor j and $(i, j) \in E_*$, change the color of j to black.
3. Repeat step 2 until no more nodes can be colored black.

The graph $\mathcal{G}(\mathcal{M})$ is called *colorable* if the nodes $1, \dots, r$ are colored black following the procedure above. Note that the remaining nodes $r+1, \dots, s$ can never be colored black since they have no incoming edges.

Define the *transpose* of $\mathcal{R}$ as the pattern matrix

$$\mathcal{R}^\top \in \{0, *, ?\}^{s\times r} \quad \text{with} \quad (\mathcal{R}^\top)_{ij} = \mathcal{R}_{ji}$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, r$. Recalling the criterion for the full row rank property of $\mathcal{M}$ given by Theorem 2.5, we then obtain the following obvious fact:

Lemma 3.8. *Consider the system* $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. *Let* $\mathcal{R}$ *be the pattern matrix given by* (3.17) *and* $\mathcal{R}^\top$ *be its transpose. Then* $\mathcal{R}$ *has full column rank if and only if* $\mathcal{G}(\mathcal{R}^\top)$ *is colorable.*

This then immediately yields the main result of this section which provides a graph theoretic condition under which the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable.

Theorem 3.9. *Consider the system* $(\mathcal{A}, \mathcal{L}, \mathcal{C})$. *Suppose that the indices* η_i *exists for* $i = 1, \ldots, q$. *Let* $\mathcal{R}$ *be the pattern matrix given by* (3.17). *Then, the FDI problem for* $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ *is solvable if* $\mathcal{G}(\mathcal{R}^\top)$ *is colorable.*

Proof. The proof follows immediately from Theorem 3.7 and Lemma 3.8. ■

To conclude this section, we will provide an example to illustrate the application of Theorem 3.9.

Example 3.3. Consider the system $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ with

$$\mathcal{A} = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & ? & 0 & ? & 0 \\ 0 & * & * & ? & 0 \\ * & 0 & 0 & ? & * \\ 0 & 0 & * & 0 & * \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} * & 0 \\ ? & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

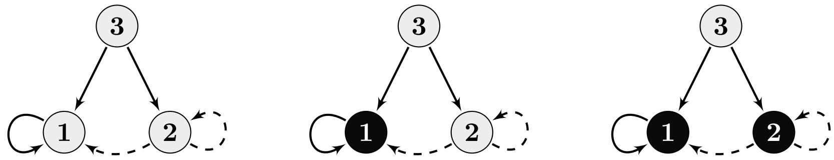
By multiplying the pattern matrices, we obtain that

$$\begin{bmatrix} \mathcal{C}\mathcal{L}_1 & \mathcal{C}\mathcal{A}\mathcal{L}_1 \end{bmatrix} = \begin{bmatrix} 0 & * \\ 0 & ? \\ 0 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathcal{C}\mathcal{L}_2 & \mathcal{C}\mathcal{A}\mathcal{L}_2 & \mathcal{C}\mathcal{A}^2\mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & ? \\ 0 & 0 & * \end{bmatrix}$$

where $\mathcal{L}_i$ is the ith column of $\mathcal{L}$. By (3.17), it follows that the associated matrix $\mathcal{R}$ and its transpose $\mathcal{R}^\top$ are given by

$$\mathcal{R} = \begin{bmatrix} \mathcal{C}\mathcal{A}\mathcal{L}_1 & \mathcal{C}\mathcal{A}^2\mathcal{L}_2 \end{bmatrix} = \begin{bmatrix} * & 0 \\ ? & ? \\ * & * \end{bmatrix} \quad \text{and} \quad \mathcal{R}^\top = \begin{bmatrix} * & ? & * \\ 0 & ? & * \end{bmatrix}.$$

As depicted in Figure 3.1 $\mathcal{G}(\mathcal{R}^\top)$ is colorable. Indeed, initially let all nodes in $\mathcal{G}(\mathcal{R}^\top)$ be colored white as shown in Figure 3.1(a). Node 1 then colors itself black as depicted in Figure 3.1(b), and finally node 3 colors 2 to black as in Figure 3.1(c). Therefore, by Theorem 3.9, the FDI problem for $(\mathcal{A}, \mathcal{L}, \mathcal{C})$ is solvable.



(a) The initial graph. (b) Node 1 colors 1. (c) Node 3 colors 2.

Figure 3.1: Example of a linear structured system for which FDI is solvable.

3.6 Conclusions

In this chapter, we have studied the FDI problem for linear structured systems. We have established a necessary and sufficient condition for solvability of the FDI problem for a given particular LTI system. Based on this, we have established a necessary condition under which the FDI problem for structured systems is solvable. Moreover, we have developed a sufficient condition for solvability of the FDI problem in terms of a rank test on a pattern matrix associated with the structured system. Next, we have provided a counterexample to show that this condition is not necessary. Finally, we have developed a graph theoretic condition for solvability of the FDI problem using the concept of colorability of a graph.

Part II

Colored Linear Structured Systems

4 Strong Structural Controllability of Systems Defined on Colored Graphs

This chapter deals with strong structural controllability of *leader/follower networks*. The system matrix defining the network dynamics is a pattern matrix in which a priori given entries are equal to zero, while the remaining entries take nonzero values. These nonzero entries correspond to edges in the network graph. We say that the network is strongly structurally controllable if the system is controllable in the classical sense for all choices of real values for the nonzero entries in the pattern matrix. The novelty of the material in this chapter is that we consider the situation that prespecified nonzero entries in the system's pattern matrix are constrained to take identical (nonzero) values. These constraints can be caused by symmetry properties or physical constraints on the network. Restricting the system matrices to those satisfying these constraints yields a new notion of strong structural controllability. This chapter aims to establish graph theoretic conditions for this more general property of strong structural controllability.

4.1 Introduction

The past two decades have shown an increasing research effort in networked dynamical systems. To a large extent this increase has been caused by technological developments such as the emergence of the Internet and the growing relevance of smart power grids. The spreading interest in social networks and biological systems have also contributed to this surge [2–4, 6].

A fundamental issue in networked systems is that of controllability. This issue deals with the question whether all parts of the global network can be adequately influenced or manipulated by applying control inputs only locally to the network. A vast amount of literature has been devoted to several variations on this issue, see

[17, 20–22, 62, 87] and the references therein. In most of the literature, a networked system is a collection of input-state-output systems, called agents, together with an interconnection structure between them. Some of these systems can also receive input from outside the network and are called leaders. The remaining systems are called followers. At a higher level of abstraction, a networked system can be described by a directed graph, called the network graph, where the vertices represent the input-state-output systems and the edges represent the interactions between them. Controllability of the networked system then deals with the question whether the states of all agents can be steered from any initial state to any final state in finite time by applying suitable input signals to the network through the leaders.

Based on the observation that the underlying graph plays an essential role in the controllability properties of the networked system [22], an increasing amount of literature has been devoted to uncovering this connection, see [88–90] and the references therein. In order to allow zooming in on the role of the network graph, it is common to proceed with the simplest possible dynamics at the vertices of the graph, and to take the agents to be single integrators, with a one-dimensional state space. These single integrators are interconnected through the network graph, and the interconnection strengths are given by the weights on the edges. Based on this, the overall networked system can be represented by a linear input-state system of the form

$$\dot{x} = Ax + Bu,$$

where the system matrix $A \in \mathbb{R}^{n \times n}$ represents the network structure with the given edge weights, and the matrix $B \in \mathbb{R}^{n \times m}$ encodes which m vertices are the leaders. The n-dimensional state vector x consists of the states of the n agents, and the m-dimensional vector u collects the input signals to the m leader vertices.

Roughly speaking, the research on network controllability based on the above model can be subdivided into three directions. The first direction deals with the situation that the values of the edge weights in the network are known exactly. In this case the matrix A is a given constant matrix, and specific dynamics are considered for the network. For example, the system matrix can be defined as the adjacency matrix of the graph [19], or the graph Laplacian matrix [20–25]. Furthermore, a framework for controllability was also introduced in [91], offering tools to treat controllability of complex networks with arbitrary structure and edge weights. Related results can be found in [92, 93]. We also refer to [94, 95].

A second research direction deals with the situation where the exact values of the edge weights are not known, but only information on whether these weights are zero or nonzero is available. In this case, the system matrix is not a known, given, matrix, but rather a matrix with a certain zero/nonzero pattern: some of the entries are known to be equal to zero, the other entries are unknown. This framework deals with the concept of *structural controllability.* Up to now, two types of structural controllability have been studied, namely *weak* structural controllability and *strong* structural controllability. A networked system of the form above is called weakly structurally controllable if there exists at least one choice of values for the nonzero unknown entries in the system matrices such that the corresponding matrix pair (A, B) is controllable. The networked system is called strongly structurally controllable if, roughly speaking, for *all* choices of nonzero values for the unknown entries the matrix pair (A, B) is controllable. Conditions for weak and strong structural controllability have been expressed entirely in terms of the underlying network graph, using concepts like cactus graphs, maximal matchings, and zero forcing sets, see [17, 27–32].

A third, more recent, research direction again deals with weak and strong structural controllability. However, the nonzero entries in the pattern matrices defining the networked system can no longer take arbitrary nonzero real values, independently of each other. Instead, this framework considers the situation that there are certain constraints on some of the nonzero entries. These constraints can require that some of the nonzero entries have given values, see e.g. [50], or that there are given linear dependencies between some of the nonzero entries, see [51]. In both cases, these constraints lead to a subclass of the family of systems dealt with in the second research direction mentioned above. A networked system with such constraints is called weakly (strongly) structurally controllable if almost all (all) members in the corresponding subclass are controllable. In [51] necessary and sufficient conditions for weak structural controllability were established in terms of *multi-colored subgraphs.* Later on, [50] studied weak and strong structural controllability of undirected networks. In addition, [54] studied weak structural controllability of networks with symmetric weights. In the present chapter, we will focus on a special constraint in which the values of certain a priori specified nonzero entries in the system matrix are constrained to be identical. In order to formalize this, the corresponding network structure is represented by so-called *colored graphs*, where edges with identical weights have identical colors.

Indeed, it is a typical situation that certain edge weights are equal, either by symmetry considerations or by the physics of the underlying problem. One application domain is provided by real world networks modeled as homogeneous multi-agent systems, such as those used in formation control. In such networks, agents can be considered as identical subnetworks of smaller order, which lead to identical edge weights in the overal network. Such situation can be considered as a so-called network-of-networks [96], which are obtained by taking the Cartesian product of smaller factor networks. For each factor network, the internal edge weights are independent. However, by applying the Cartesian product, some edge weights in the overall network will become identical.

Another application domain consists of physical networks such as power grids, traffic networks and water distribution networks. For example, in power networks certain physical components typically appear multiple times, leading to identical edge weights in the network models. The same holds for water distribution networks. As for traffic networks, two-directional traffic flow sharing the same channel leads to symmetry properties of the network models. An example is also provided by real world networks modeled as undirected networks [50, 51, 54], in which the network graph has to be symmetric.

In this chapter, strong structural controllability of networked systems defined on such colored graphs will be called *colored strong structural controllability*. This version of strong structural controllability has not been studied in the literature before. The aim of the present chapter is to establish graph theoretic tests for this property of networked systems.

The main contributions of this chapter are the following:

1. We introduce a new color change rule and define the corresponding notion of zero forcing set. To do this, we consider colored bipartite graphs and establish a necessary and sufficient graph theoretic condition for nonsingularity of the pattern class associated with this bipartite graph.

2. We provide a sufficient graph theoretic condition for our new notion of strong structural controllability in terms of zero forcing sets.

3. We introduce so-called elementary edge operations that can be applied to the original network graph and that preserve the property of strong structural controllability.

4. A sufficient graph theoretic condition for strong structural controllability is developed based on the notion of edge-operations-color-change derived set which is obtained by applying elementary edge operations and the color change rule iteratively.

The organization of this chapter is as follows. In Section 4.2, some preliminaries are presented. In Section 4.3, we give a formal definition of the main problem treated in this chapter in terms of systems defined on colored graphs. In Section 4.4, we establish our main result, which gives a sufficient graph theoretic condition for strong structural controllability of systems defined on colored graphs. Section 4.5 provides two additional sufficient graph theoretic conditions. For this, we introduce the concept of elementary edge operations and the associated notion of edge-operations-color-change derived set. This set is obtained from the initial coloring set by succesively applying elementary edge operations and the color change rule. Finally, Section 4.6 formulates the conclusions of this chapter.

4.2 Preliminaries

4.2.1 Elements of graph theory

Let $\mathcal{G} = (V, E)$ be a directed graph, with vertex set $V = \{1, \dots, n\}$, and the edge set E a subset of $V \times V$. In this chapter, we will only consider simple graphs, that is, the edge set E does not contain edges of the form (i, i). In this chapter, the phrase 'directed graph' will always refer to a simple directed graph. We call vertex j an out-neighbor of vertex i if $(i, j) \in E$. We denote the set of all out-neighbors of i by

$$N(i) := \{j \in V \mid (i, j) \in E\}.$$

Given a subset S of the vertex set V and a subset $X \subseteq S$, we denote by

$$N_{V \setminus S}(X) = \{j \in V \setminus S \mid \exists\ i \in X \quad \text{such that } (i, j) \in E\},$$

the set of all vertices outside S, but an out-neighbor of some vertex in X. A directed graph $\mathcal{G}_1 = (V_1, E_1)$ is called a subgraph of $\mathcal{G}$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Associated with a given directed graph $\mathcal{G} = (V, E)$ we consider the set of matrices

$$\mathcal{W}(\mathcal{G}) := \{W \in \mathbb{R}^{n \times n} \mid W_{ij} \neq 0 \text{ iff } (j, i) \in E\}.$$

For any such W and $(j, i) \in E$, the entry W_{ij} is called the weight of the edge (j, i) and W is called a *weighted adjacency matrix* of the graph. For a given directed graph $\mathcal{G} = (V, E)$, we denote the associated graph with weighted adjacency matrix W by $\mathcal{G}(W) = (V, E, W)$. This is then called the *weighted graph* associated with the graph $\mathcal{G} = (V, E)$ and weighted adjacency matrix W. Finally, we define the graph $\mathcal{G} = (V, E)$ to be an *undirected graph* if $(i, j) \in E$ whenever $(j, i) \in E$. In that case the order of i and j in (i, j) does not matter and we interpret the edge set E as the set of unordered pairs $\{i, j\}$ where $(i, j) \in E$.

An undirected graph $\mathcal{G} = (V, E)$ is called *bipartite* if there exist nonempty disjoint subsets X and Y of V such that $X \cup Y = V$ and $\{i, j\} \in E$ only if $i \in X$ and $j \in Y$. Such *bipartite graph* is denoted by $G = (X, Y, E_{XY})$ where we denote the edge set by E_{XY} to stress that it contains edges $\{i, j\}$ with $i \in X$ and $j \in Y$. In this chapter we will use the symbol $\mathcal{G}$ for arbitrary directed graphs and G for bipartite graphs.

A set of t edges $m \subseteq E_{XY}$ is called a *t-matching* in G, if no two distinct edges in m share a vertex. In the special case that $|X| = |Y| = t$, such a t-matching is called a *perfect matching*. For a bipartite graph $G = (X, Y, E_{XY})$, with vertex sets X and Y given by $X = \{x_1, \dots, x_s\}$ and $Y = \{y_1, \dots, y_t\}$, we define *the pattern class* of G by

$$\mathcal{P}(G) = \{M \in \mathbb{C}^{t \times s} \mid M_{ji} \neq 0 \text{ iff } \{x_i, y_j\} \in E_{XY}\}.$$

Note that, in the context of pattern classes for undirected bipartite graphs, we allow complex matrices.

4.2.2 Controllability of systems defined on graphs

For a directed graph $\mathcal{G} = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, the *qualitative class* of $\mathcal{G}$ is defined as the family of matrices

$$\mathcal{Q}(\mathcal{G}) = \{A \in \mathbb{R}^{n\times n} \mid \text{for } i \neq j : A_{ij} \neq 0 \text{ iff } (j, i) \in E\}.$$

Note that the diagonal entries of $A \in \mathcal{Q}(\mathcal{G})$ do not depend on the structure of $\mathcal{G}$ and can take arbitrary real values.

Next, we specify a subset $V_L = \{v_1, v_2, \ldots, v_m\}$ of V, called the *the leader set*, and consider the following family of leader/follower systems defined on the graph $\mathcal{G}$ with dynamics

$$\dot{x} = Ax + Bu, \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The systems (4.1) have the distinguishing feature that the matrix A belongs to $\mathcal{Q}(\mathcal{G})$ and $B = B(V; V_L)$ is defined as the $n \times m$ matrix given by

$$B_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

An important notion associated with systems defined on a graph $\mathcal{G}$ as in (4.1) is the notion of strong structural controllability.

Definition 4.1. *Let $\mathcal{Q}' \subseteq \mathcal{Q}(\mathcal{G})$. The system defined on the directed graph $\mathcal{G} = (V, E)$ with dynamics* (4.1) *and leader set $V_L \subseteq V$ is called* strongly structurally controllable with respect to $\mathcal{Q}'$ *if the pair (A, B) is controllable for all $A \in \mathcal{Q}'$. In that case we will simply say that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}'$.*

One special case of the above notion is that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}(\mathcal{G})$. In that case, we will simply say that $(\mathcal{G}; V_L)$ is controllable. Another special case is that $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}'$ where, for a given weighted adjacency matrix $W \in \mathcal{W}(\mathcal{G})$, $\mathcal{Q}'$ is the subclass of $\mathcal{Q}(\mathcal{G})$ defined by

$$\mathcal{Q}_W(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid \text{for } i \neq j : A_{ij} = W_{ij}\}.$$

This subclass is called the *weighted qualitative class* associated with W. Note that the off-diagonal elements of $A \in \mathcal{Q}_W(\mathcal{G})$ are fixed by those of the given adjacency matrix, while, again, the diagonal entries of $A \in \mathcal{Q}_W(\mathcal{G})$ can take arbitrary real values. Obviously

$$\mathcal{Q}(\mathcal{G}) = \bigcup_{W\in\mathcal{W}(\mathcal{G})} \mathcal{Q}_W(\mathcal{G}).$$

Since there is a unique weighted graph $\mathcal{G}(W) = (V, E, W)$ associated with the graph $\mathcal{G} = (V, E)$ and weighted adjacency matrix W, we will simply say that $(\mathcal{G}(W); V_L)$ *is controllable* if $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}_W(\mathcal{G})$.

4.2.3 Zero forcing set and controllability of $(\mathcal{G}; V_L)$

Let $\mathcal{G} = (V, E)$ be a directed graph with vertices colored either black or white. We now review the concept of color change rule [97]: if v is a black vertex in $\mathcal{G}$ with exactly one white out-neighbor u, then we change the color of u to black, and write $v \xrightarrow{c} u$. Such a color change is called a *force*. A subset C of V is called a *coloring set* if the vertices in C are initially colored black and those in $V \setminus C$ initially colored white. Given a coloring set $C \subseteq V$, the derived set $\mathcal{D}(C)$ is the set of black vertices obtained after repeated application of the color change rule, until no more changes are possible. It was shown in [97] that the derived set is indeed uniquely defined, in the sense that it does not depend on the order in which the color changes are applied to the original coloring set C. A coloring set $C \subseteq V$ is called a *zero forcing set for* $\mathcal{G}$ if $\mathcal{D}(C) = V$.

It was shown in [30] that controllability of $(\mathcal{G}; V_L)$ can be characterized in terms of zero forcing sets.

Proposition 4.2. *Let $\mathcal{G} = (V, E)$ be a directed graph and $V_L \subseteq V$ be the leader set. Then, $(\mathcal{G}; V_L)$ is controllable if and only if V_L is a zero forcing set.*

4.2.4 Balancing set and controllability of $(\mathcal{G}(W); V_L)$

Consider the weighted graph $\mathcal{G}(W) = (V, E, W)$ associated with the directed graph $\mathcal{G} = (V, E)$ and weighted adjacency matrix $W \in \mathcal{W}(\mathcal{G})$. For $i = 1, \ldots, n$, let x_i be a variable assigned to vertex i. For a given subset of vertices $C \subseteq V$ we put $x_j = 0$ for all $j \in C$. We call C *the set of zero vertices*. The values of the other vertices of $\mathcal{G}(W)$ are initially undetermined. To every vertex $j \in C$, we assign a so called *balance equation*:

$$\sum_{k \in N_{V \setminus C}(\{j\})} x_k W_{kj} = 0. \tag{4.3}$$

Note that for weighted undirected graphs, in which case $W = W^\top$, the balance equation (4.3) coincides with the one introduced in [50]. If for a given subset X of the set of zero vertices C, the system of $|X|$ balance equations corresponding to the vertices in X implies that $x_k = 0$ for all $k \in Y$ with $C \cap Y = \varnothing$, we say that *zeros extend from X to Y*. We denote this by $X \xrightarrow{z} Y$. The updated set of zero vertices is now defined as $C' = C \cup Y$.

This one step procedure of making the values of possibly additional vertices equal to zero is called *the zero extension rule*. We define the *derived set* $\mathcal{D}_z(C)$ to be the set of zero vertices obtained after repeated application of the zero extension rule until no more zero vertices appear. Although not explicitly stated in [50], it can be shown that the derived set is uniquely defined, in the sense that it does not depend on the particular zero extensions that are applied to the original set of zero vertices C. An initial zero vertex set $C \subseteq V$ is called a *balancing set* if the derived set $\mathcal{D}_z(C)$ is V.

A necessary and sufficient condition for strong structural controllability with respect to $\mathcal{Q}_W(\mathcal{G})$ for the special case that $W = W^T$ was given in [50]:

Proposition 4.3. *Let $\mathcal{G}$ be a simple undirected graph, $V_L \subseteq V$ be the leader set and $W \in \mathcal{W}(\mathcal{G})$ be a weighted adjacency matrix with $W = W^\top$. Then $(\mathcal{G}(W); V_L)$ is controllable if and only if V_L is a balancing set.*

4.3 Problem formulation

In this section we will introduce the main problem to be considered in this chapter. At the end of the section, we will also formulate two preliminary results that will be needed in the sequel. In order to proceed, we will now first formalize the constraint that the weights of a priori given edges in the network graph are equal. This is equivalent to saying that given off-diagonal entries in the matrices belonging to the qualitative class $\mathcal{Q}(\mathcal{G})$ are equal. To do this, we introduce a partition

$$\pi = \{E_1, \ldots, E_k\}$$

of the edge set E into disjoint subsets E_r whose union is the entire edge set E. The edges in a given cell E_r are constrained to have identical weights. We then define the *colored qualitative class* associated with π by

$$\mathcal{Q}_\pi(\mathcal{G}) = \{A \in \mathcal{Q}(\mathcal{G}) \mid A_{ij} = A_{kl} \text{ if } (j,i),(l,k) \in E_r \text{ for some } r\}.$$

In order to visualize the partition π of the edge set in the graph, two edges in the same cell E_r are said to have the same color. The colors will be denoted by the symbols $c_1, c_2, \ldots, c_k$ and the edges in cell E_r are said to have color c_r. This leads to the notion of *colored graph*. A *colored graph* is a directed graph together with a partition π of the edge set, which is denoted by $\mathcal{G}(\pi) = (V, E, \pi)$.

In the sequel, sometimes the symbols c_i will also be used to denote independent nonzero variables. A set of real values obtained by assigning to each of these variables c_i a particular real value is called a *realization* of the color set.

Example 4.1. Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ associated with the directed graph $\mathcal{G} = (V, E)$ and edge partition $\pi = \{E_1, E_2, E_3\}$, where $E_1 = \{(1,4),(1,6)\}$, $E_2 = \{(2,4),(2,5)\}$ and $E_3 = \{(3,5),(3,6)\}$ as depicted in Figure 4.1. Edges having the same color means that the weight of these edges are constrained to be equal. In this example, the edges in E_1 have color c_1 (blue), those in E_2 have color

c_2 (green), and those in E_3 have color c_3 (red). The corresponding colored qualitative class consists of all matrices of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & \lambda_4 & 0 & 0 \\ 0 & c_2 & c_3 & c_1 & \lambda_5 & c_3 \\ c_1 & 0 & c_3 & 0 & 0 & \lambda_6 \end{bmatrix}$$

where λ_i is an arbitrary real number for $i = 1, \ldots, 6$ and c_i is an arbitrary *nonzero* real number for $i = 1, 2, 3$.

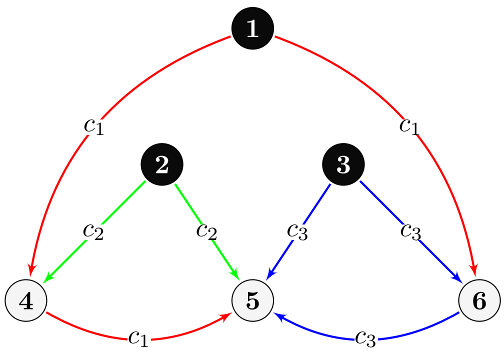


Figure 4.1: Example of a colored directed graph with its leader set.

Given a colored directed graph $\mathcal{G}(\pi) = (V, E, \pi)$ with edge partition $\pi = \{E_1, \ldots, E_k\}$, we define the corresponding family of weighted adjacency matrices

$$\mathcal{W}_\pi(\mathcal{G}) := \{W \in \mathcal{W}(\mathcal{G}) \mid W_{ij} = W_{kl} \text{ if } (j,i),(l,k) \in E_r \text{ for some } r\}.$$

Note that any weighted adjacency matrix $W \in \mathcal{W}_\pi(\mathcal{G})$ is associated with a unique realization of the color set. Obviously, the colored qualitative class $\mathcal{Q}_\pi(\mathcal{G})$ is equal to the union of all the subclasses $\mathcal{Q}_W(\mathcal{G})$ with $W \in \mathcal{W}_\pi(\mathcal{G})$, i.e,

$$\mathcal{Q}_\pi(\mathcal{G}) = \bigcup_{W \in \mathcal{W}_\pi(\mathcal{G})} \mathcal{Q}_W(\mathcal{G}). \tag{4.4}$$

If $(\mathcal{G}; V_L)$ is controllable with respect to $\mathcal{Q}' = \mathcal{Q}_\pi(\mathcal{G})$ (see Definition 4.1) we will simply say that $(\mathcal{G}(\pi); V_L)$ *is controllable*. In that case, we call the system *colored strongly structurally controllable*. For example, the system with graph depicted in

Figure 4.1 is colored strongly structurally controllable as will be shown later in this chapter.

The aim of this chapter is to establish graph theoretic tests for colored strong structural controllability of a given graph. In order to obtain these, we now first make the observation that conditions for strong structural controllability can be expressed in terms of balancing sets. Generalizing Proposition 4.3 to the case of weighted directed graphs, we have the following lemma:

Lemma 4.4. *Let $\mathcal{G} = (V, E)$ be a directed graph with leader set V_L and let $W \in \mathcal{W}(\mathcal{G})$. Then $(\mathcal{G}(W); V_L)$ is controllable if and only if V_L is a balancing set.*

Proof. By the Hautus test [98], $(\mathcal{G}(W); V_L)$ is controllable if and only if $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $A \in \mathcal{Q}_W(\mathcal{G})$ and all $\lambda \in \mathbb{C}$ with $B = B(V; V_L)$ given by (4.2). Let $V = \{1, \dots, n\}$.

We first prove the 'if' part. Suppose that V_L is a balancing set for $\mathcal{G}(W)$. Without loss of generality, we may assume that there is a chronological list of zero extensions

$$(C_1 \xrightarrow{z} Y_1, \dots, C_s \xrightarrow{z} Y_s),$$

where, for $r = 1, \dots, s$, C_r represents the current set of zero vertices before the rth zero extension and $Y_r \subseteq V \setminus C_r$, and $C_s \cup Y_s = V$. Assign variables $x_1, \dots, x_n$ to every vertex in V, with $x_i = 0$ if $i \in C_r$ and x_i undetermined otherwise. To every vertex $j \in C_r$, we then assign a balance equation given by (4.3). By definition of the zero extension rule, we have the following implications

$$x_{V \setminus C_i}^\top W_{V \setminus C_i, C_i} = 0 \ \Rightarrow \ x_{Y_i}^\top = 0 \quad \text{for} \ \ i = 1, \dots, s. \tag{4.5}$$

For any $A \in \mathcal{Q}_W(\mathcal{G})$ and $\lambda \in \mathbb{C}$, there exists a diagonal matrix $D \in \mathbb{C}^{n \times n}$ such that $A - \lambda I = W + D$. It then follows immediately that

$$(A - \lambda I)_{V \setminus C_i, C_i} = W_{V \setminus C_i, C_i} \quad \text{for} \ \ i = 1, \dots, s.$$

Recalling (4.5), we have that

$$x_{V \setminus C_i}^\top (A - \lambda I)_{V \setminus C_i, C_i} = 0 \ \Rightarrow \ x_{Y_i}^\top = 0 \quad \text{for} \ \ i = 1, \dots, s.$$

Since $x^\top B = 0 \ \Rightarrow \ x_{V_L}^\top = 0$ and $V_L \cup (\bigcup_{j=1}^s Y_j) = V$, we then have that

$$x^\top \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0 \ \Rightarrow \ x^\top = 0$$

which implies that $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank. Since the A and λ are arbitrary, $(\mathcal{G}(W); V_L)$ is controllable. Thus we have proved the 'if' part.

To prove the converse, suppose that $(\mathcal{G}(W); V_L)$ is controllable while V_L is not a balancing set. It follows immediately that $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $A \in \mathcal{Q}_W(\mathcal{G})$ and all $\lambda \in \mathbb{C}$, with $B = B(V; V_L)$ given by (4.2), and the derived set $D = \mathcal{D}_z(V_L)$ is not equal to V. Again assign variables x_i to the vertices $i \in V$ such that $x_i = 0$ if $i \in D$ and x_i is undetermined otherwise. Let $D' = V \setminus D$. By definition of the zero extension rule, we conclude that there exists a vector x such that $x_D = 0$, $x_{D'} \neq 0$ and $x^\top W = 0$, where x_D and $x_{D'}$ are the sub-vectors corresponding to the components in D and D', respectively. Recalling that $V_L \subseteq D$, it follows that $x^\top \begin{bmatrix} W & B \end{bmatrix} = 0$. This implies that the matrix $[W \; B]$ does not have full row rank. Thus we have reached a contradiction and the proof is completed. ■

The following lemma follows immediately from Lemma 4.4 by noting that (4.4) holds.

Lemma 4.5. *Let $\mathcal{G} = (V, E)$ be a directed graph with leader set V_L and let π be a partition of the edge set. Then $(\mathcal{G}(\pi); V_L)$ is controllable if and only if V_L is a balancing set for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$.*

Obviously, the necessary and sufficient conditions presented in Lemma 4.5 cannot be verified easily, as the set $\mathcal{W}_\pi(\mathcal{G})$ contains infinitely many elements. Therefore, we aim at establishing graph theoretic conditions under which $(\mathcal{G}(\pi); V_L)$ is controllable.

4.4 Zero forcing sets for colored graphs

In order to provide a graph theoretic condition for colored strong structural controllability, in this section we introduce a new color change rule and define the corresponding notion of zero forcing set. To do this, we first consider colored bipartite graphs and establish a necessary and sufficient graph theoretic condition for nonsingularity of the associated pattern class.

4.4.1 Colored bipartite graphs

Consider the bipartite graph $G = (X, Y, E_{XY})$, where the vertex sets X and Y are given by $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$. We will now introduce the notion of *colored* bipartite graph. Let $\pi_{XY} = \{E^1_{XY}, \ldots, E^\ell_{XY}\}$ be a partition of the edge set E_{XY} with associated colors $c_1, \ldots, c_\ell$. This partition is used to formalize that certain entries in the pattern class $\mathcal{P}(G)$ are constrained to take identical values. Again, the edges in a given cell E^r_{XY} are said to have the same color. The pattern class of the colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ is then defined as the following set of complex $t \times s$ matrices

$$\mathcal{P}_\pi(G) = \big\{ M \in \mathcal{P}(G) \mid M_{ji} = M_{hg} \text{ if } \{x_i, y_j\}, \{x_g, y_h\} \in E^r_{XY} \text{ for some } r \big\}.$$

Assume now that $|X| = |Y|$ and let $t = |X|$. Suppose that p is a perfect matching of $G(\pi)$. The *spectrum* of p is defined to be the set of colors (counting multiplicity) of the edges in p. More specifically, if the perfect matching p is given by $p = \big\{\{x_1, y_{\gamma(1)}\}, \ldots, \{x_t, y_{\gamma(t)}\}\big\}$, where γ denotes a permutation of $(1, \ldots, t)$, and $c_{i_1}, \ldots, c_{i_t}$ are the respective colors of the edges in p, then the spectrum of p is $\{c_{i_1}, \ldots, c_{i_t}\}$ where the same color can appear multiple times.

In addition, we define the *sign* of the perfect matching p as $\text{sign}(p) = (-1)^m$, where m is the number of swaps needed to obtain $(\gamma(1), \ldots, \gamma(t))$ from $(1, \ldots, t)$. Since every perfect matching is associated with a unique permutation, with a slight abuse of notation, we sometimes use the perfect matching p to represent its corresponding permutation.

Two perfect matchings are called *equivalent* if they have the same spectrum. Obviously this yields a partition of the set of all perfect matchings of $G(\pi)$ into *equivalence classes* of perfect matchings. We denote these equivalence classes of perfect matchings by $\mathbb{P}_1, \ldots, \mathbb{P}_l$, where perfect matchings in the same class $\mathbb{P}_i$ are equivalent. Clearly, $\mathbb{P}_i \cap \mathbb{P}_j = \varnothing$ for $i \neq j$. Correspondingly, we then define the *spectrum of the equivalence class* $\mathbb{P}_i$ to be the (common) spectrum of the perfect matchings in this class, and denote it by $\text{spec}(\mathbb{P}_i)$. Finally, we define the *signature of the equivalence class* $\mathbb{P}_i$ to be the sum of the signs of all perfect matchings in this class, which is given by

$$\text{sgn}(\mathbb{P}_i) = \sum_{p \in \mathbb{P}_i} \text{sign}(p).$$

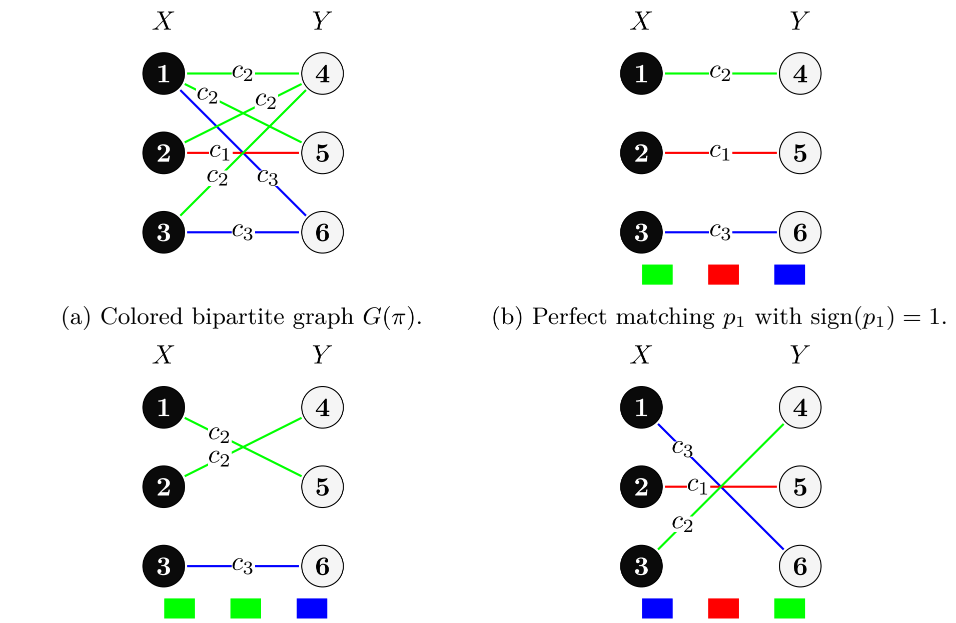
Example 4.2. Consider the colored bipartite graph $G(\pi)$ depicted in Figure 4.2a. It contains three perfect matchings, p_1, p_2 and p_3, respectively, depicted in Figures 4.2b-4.2d. Clearly, p_1 and p_3 are equivalent. The equivalence classes of perfect matchings are then $\mathbb{P}_1 = \{p_1, p_3\}$ and $\mathbb{P}_2 = \{p_2\}$. Clearly, $\text{sgn}(\mathbb{P}_1) = 0$ and $\text{sgn}(\mathbb{P}_2) = -1$.

We are now ready to state a necessary and sufficient condition for nonsingularity of all matrices in the colored pattern class $\mathcal{P}_\pi(G)$.

Theorem 4.6. *Let $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ be a colored bipartite graph and $|X| = |Y|$. Then, all matrices in $\mathcal{P}_\pi(G)$ are nonsingular if and only if there exists at least one perfect matching and exactly one equivalence class of perfect matchings has nonzero signature.*

Proof. Denote the cardinality of X and Y by t. Let $A \in \mathcal{P}_\pi(G)$. By the Leibniz Formula for the determinant, we have

$$\det(A) = \sum_{\gamma} \text{sign}(\gamma) \prod_{i=1}^{t} A_{i\gamma(i)},$$



(a) Colored bipartite graph $G(\pi)$. (b) Perfect matching p_1 with $\text{sign}(p_1) = 1$.

(c) Perfect matching p_2 with $\text{sign}(p_2) = -1$. (d) Perfect matching p_3 with $\text{sign}(p_3) = -1$.

Figure 4.2: Example of a colored bipartite graph and its perfect matchings.

where the sum ranges over all permutations γ of $(1, \dots, t)$ and where $\text{sign}(\gamma) = (-1)^m$ with m the number of swaps needed to obtain $(\gamma(1), \dots, \gamma(t))$ from $(1, \dots, t)$. Note that $\prod_{i=1}^t A_{i\gamma(i)} \neq 0$ if and only if there exists at least one perfect matching $p = \{\{x_1, y_{\gamma(1)}\}, \dots, \{x_{|X|}, y_{\gamma(t)}\}\}$ in $G(\pi)$. In that case, we have

$$\det(A) = \sum_p \text{sign}(p) \prod_{i=1}^t A_{ip(i)},$$

where p ranges over all perfect matchings and $\text{sign}(p)$ denotes the sign of the perfect matching (we now identify perfect matchings with their permutations). Suppose now there are l equivalence classes of perfect matchings $\mathbb{P}_1, \dots, \mathbb{P}_l$. Then we obtain

$$\det(A) = \sum_{j=1}^l \left(\text{sgn}(\mathbb{P}_j) \prod_{i=1}^t A_{ip(i)} \right), \tag{4.6}$$

where, for $j = 1, 2, \dots l$, in the product appearing in the jth term, p is an arbitrary matching in $\mathbb{P}_j$. We will now prove the 'if' part. Assume that there exists at least one perfect matching, and exactly one equivalence class of perfect matchings has nonzero signature. Without loss of generality, assume that the equivalence class $\mathbb{P}_1$ has nonzero signature. Obviously, for every $A \in \mathcal{P}_\pi(G)$, we then have

$$\det(A) = \operatorname{sgn}(\mathbb{P}_1) \prod_{i=1}^{t} A_{ip(i)} \neq 0,$$

where $p \in \mathbb{P}_1$ is arbitrary, in other words, every $A \in \mathcal{P}_\pi(G)$ is nonsingular.

Next, we prove the 'only if' part. For this, assume that all $A \in \mathcal{P}_\pi(G)$ are nonsingular, but one of the following holds:

(i) There does not exist any perfect matching.

(ii) No equivalence class of perfect matchings with nonzero signature exists.

(iii) There exist at least two equivalence classes of perfect matchings with nonzero signature.

We will show that all these cases lead to a contradiction.

In case (i), we must obviously have $\det(A) = 0$ for any $A \in \mathcal{P}_\pi(G)$ which gives a contradiction. For case (ii), it follows from (4.6) that $\det(A) = 0$ since all equivalence classes have zero signature. Therefore, we reach a contradiction again. Finally, consider case (iii). Without loss of generality, assume $\mathbb{P}_1$ and $\mathbb{P}_2$ have nonzero signature. The signatures of the remaining equivalence classes can be either zero or nonzero. In the sequel we associate the colors $c_1, \dots, c_\ell$ of the cells $E^1_{XY}, \dots E^\ell_{XY}$ with independent, nonzero, variables $c_1, \dots, c_\ell$ that can take values in $\mathbb{C}$. The spectrum of an equivalence class $\mathbb{P}_j$ then uniquely determines a monomial $c_1^{i_1} c_2^{i_2} \dots c_\ell^{i_\ell}$, where the powers $i_1, \dots i_k$ correspond to the multiplicities of the colors $c_1, \dots, c_\ell$ in the perfect matchings in $\mathbb{P}_j$. We also identify each entry of a matrix A in $\mathcal{P}_\pi(G)$ with the color of its corresponding edge. In particular, for such A we have

$$A_{ij} = \begin{cases} c_r & \text{if } (j,i) \in E_r \text{ for some } r \\ 0 & \text{otherwise.} \end{cases}$$

From the expression (4.6) for the determinant of A it can be seen that the perfect matchings in the equivalence class $\mathbb{P}_j$ yield a contribution $\operatorname{sgn}(\mathbb{P}_j) c_1^{i_1} c_2^{i_2} \dots c_\ell^{i_\ell}$, where the degrees correspond to the multiplicities of the colors of the perfect matchings in $\mathbb{P}_j$. By assumption we have that $\operatorname{spec}(\mathbb{P}_1)$ and $\operatorname{spec}(\mathbb{P}_2)$ are not equal. Without loss of generality, we assume that the multiplicity of c_1 as an element of $\operatorname{spec}(\mathbb{P}_1)$ is unequal to the multiplicity of c_1 as an element of $\operatorname{spec}(\mathbb{P}_2)$. Denote these multiplicities by j_1 and j_2, respectively, with $j_1 \neq j_2$. Then for all values of $c_2, \dots, c_\ell$, the determinant of A has the form

$$\det(A) = \operatorname{sgn}(\mathbb{P}_1) a_1 c_1^{j_1} + \operatorname{sgn}(\mathbb{P}_2) a_2 c_1^{j_2} + f(c_1), \tag{4.7}$$

where a_1 and a_2 depend on $c_2, \dots, c_k$ and $f(c_1)$ is a polynomial in c_1. The polynomial $f(c_1)$ corresponds to the remaining equivalence classes. It can happen that some of these equivalence classes also contain the color c_1 in their spectrum with multiplicity j_1 or j_2. By moving the corresponding monomials to the first two terms in (4.7) we obtain

$$\det(A) = b_1 c_1^{j_1} + b_2 c_1^{j_2} + f'(c_1), \tag{4.8}$$

with b_1 and b_2 depending on $c_2, \dots, c_k$. Note that the first term in (4.8) corresponds to the equivalence classes containing c_1 in their spectrum with multiplicity j_1, and likewise the second term with multiplicity j_2. The remaining polynomial $f'(c_1)$ does not contain monomials with $c_1^{j_1}$ and $c_1^{j_2}$. It is now easily verified that nonzero $c_2, \dots, c_\ell$ can be chosen such that $b_1 \neq 0$ and $b_2 \neq 0$. By the fundamental theorem of algebra we then have that the polynomial equation $b_1 c_1^{j_1} + b_2 c_1^{j_2} + f'(c_1) = 0$ has at least one nonzero root, since both b_1 and b_2 are nonzero. This implies that for some choice of nonzero complex values $c_1, \dots, c_\ell$ we have $\det(A) = 0$. In other words, not all $A \in \mathcal{P}_\pi(G)$ are nonsingular. This is a contradiction. ■

Example 4.3. Revisit the colored bipartite graph in Figure 4.2a. The corresponding pattern class consists of all matrices of the form

$$\begin{bmatrix} c_2 & c_2 & c_2 \\ c_2 & c_1 & 0 \\ c_3 & 0 & c_3 \end{bmatrix}$$

where c_1, c_2 and c_3 are arbitrary nonzero complex numbers. In Example 4.2 we saw that there is exactly one equivalence class of perfect matchings with nonzero signature. By Theorem 8 we thus conclude that all these matrices are nonsingular.

4.4.2 Color change rule and zero forcing sets

In this subsection, we will introduce a tailor-made zero forcing notion for colored graphs. Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored directed graph with $\pi = \{E_1, \dots, E_k\}$ the partition of E. For given disjoint subsets $X = \{x_1, \dots, x_s\}$ and $Y = \{y_1, \dots, y_t\}$ of V, we define an associated colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ as follows:

$$E_{XY} := \{\{x_i, y_j\} \mid (x_i, y_j) \in E,\ x_i \in X,\ y_j \in Y\}.$$

Obviously, the partition π induces a partition π_{XY} of E_{XY} by defining

$$E_{XY}^r := \{\{x_i, y_j\} \in E_{XY} \mid (x_i, y_j) \in E_r\} \quad r = 1, \dots, k.$$

Note that for some r, this set might be empty. Removing these, we get a partition

$$\pi_{XY} = \{E_{XY}^{i_1}, \ldots, E_{XY}^{i_\ell}\}$$

of E_{XY}, with associated colors $c_{i_1}, \ldots, c_{i_\ell}$, with $\ell \leqslant k$. Without loss of generality we renumber $c_{i_1}, \ldots, c_{i_\ell}$ as $c_1, \ldots, c_\ell$ and the edges in cell E_{XY}^r are said to have color c_r.

As before, a subset C of V is called a *coloring set* if the vertices in C are initially colored black and those in $V \setminus C$ initially colored white. We will now define the notion of *color-perfect white neighbor*.

Definition 4.7. *Let $X \subseteq C$ and $Y \subseteq V$ with $|Y| = |X|$. We call Y a* color-perfect white neighbor *of X if*

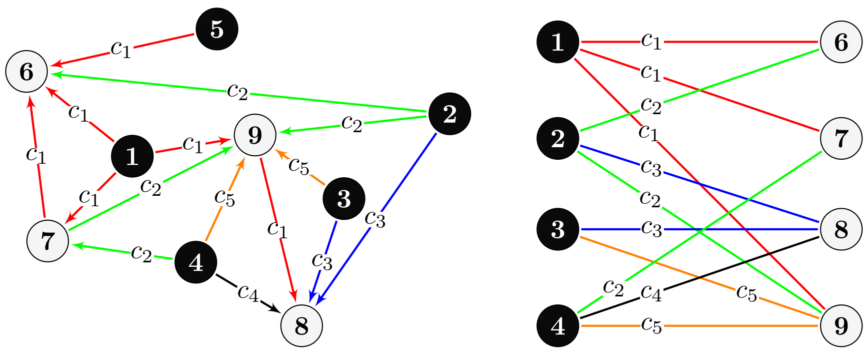
1. *$Y = N_{V\setminus C}(X)$, i.e. Y is equal to the set of white out-neighbors of X, and*
2. *in the associated colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$ there exists a perfect matching and exactly one equivalence class of perfect matchings has nonzero signature.*

Based on the notion of color-perfect white neighbor, we now introduce the following color change rule: if $X \subseteq C$ and Y is a color-perfect white neighbor of X, then we change the color of all vertices in Y to black, and write $X \xrightarrow{c} Y$. Such a color change is called a *force*. We define *a derived set $\mathcal{D}_c(C)$* as a set of black vertices obtained after repeated application of the color change rule, until no more changes are possible. In contrast with the original color change rule (see Section 4.2.3), under our new color change rule derived sets will no longer be uniquely defined, and may depend on the particular list of forces that is applied to the original coloring set C. This is illustrated by the following example.

Example 4.4. Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Figure 4.3a. Take as coloring set $C = \{1, 2, 3, 4, 5\}$. Consider the colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$ associated with $X = \{1, 2, 3, 4\}$ and $Y = \{6, 7, 8, 9\}$ as is depicted in Figure 4.3b. It can be shown that there exists exactly one equivalence class of perfect matchings in G with nonzero signature. Since $X \subset C$ and $Y = N_{V\setminus C}(X)$, we have that $X \xrightarrow{c} Y$. After applying this force we arrive at the derived set $\mathcal{D}_1(C) = V$.

On the other hand, obviously $X_1 \xrightarrow{c} Y_1$, with $X_1 = \{5\}$ and $Y_1 = \{6\}$. After applying this force, no other forces are possible. Indeed, it can be verified that there does not exist a subset of $\{1, 2, 3, 4, 5, 6\}$ that forces any subset of $\{7, 8, 9\}$. In this way we arrive at the derived set $\mathcal{D}_2(C) = \{1, \ldots, 6\}$.

We conclude that there exist two different derived sets in $\mathcal{G}(\pi)$ with coloring set C. Thus we have found an example for the non-uniqueness of derived sets for a given colored graph and coloring set.



(a) A colored directed graph $\mathcal{G}(\pi)$. (b) Colored bipartite graph G.

Figure 4.3: Example of non-uniqueness of derived sets.

A coloring set $C \subseteq V$ is called a *zero forcing set for* $\mathcal{G}(\pi)$ if there exits a derived set $\mathcal{D}_c(C)$ such that $\mathcal{D}_c(C) = V$.

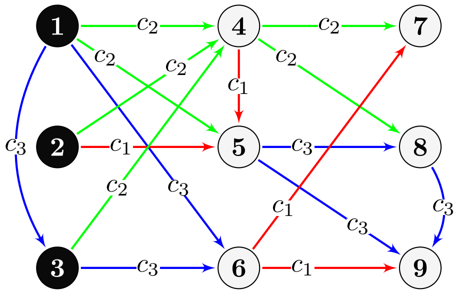
Before illustrating the new color change rule, we remark on its relation to the one defined earlier.

Remark 4.1. Given a directed graph $\mathcal{G} = (V, E)$, one can obtain a colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ by assigning to every edge a different color, i.e., $|\pi| = |E|$. Clearly, the colored qualitative class $\mathcal{Q}_\pi(\mathcal{G})$ coincides with the qualitative class $\mathcal{Q}(\mathcal{G})$. In addition, the original color change rule for $\mathcal{G}$ introduced in Section 4.2.3 can be seen to be a special case of the new one for $\mathcal{G}(\pi)$. This observation in mind, we will use the same terminology for these two color change rules and it will be clear from the context which one is employed.

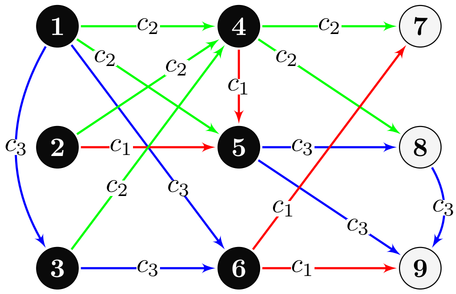
We now illustrate the new color change rule by means of an example.

Example 4.5. Figure 4.4 illustrates the repeated application of zero forcing in the context of colored graphs. In Figure 4.4a, initially, vertices $\{1, 2, 3\}$ are black and the remaining vertices are white. As shown in Example 4.2, $\{4, 5, 6\}$ is a color-perfect white neighbor of $\{1, 2, 3\}$. Therefore, we have $\{1, 2, 3\} \xrightarrow{c} \{4, 5, 6\}$ as shown in Figure 4.4b. Next, observe that the colored bipartite graph associated with $X = \{4, 5, 6\}$ and $Y = \{7, 8, 9\}$ has two perfect matchings, with identical spectrum and the same sign 1. Hence the single equivalence class has signature 2. As such, $\{7, 8, 9\}$ is a color-perfect white neighbor of $\{4, 5, 6\}$. Therefore, we have $\{4, 5, 6\} \xrightarrow{c} \{7, 8, 9\}$ as shown in Figure 4.4c. Consequently, we conclude that the vertex set $\{1, 2, 3\}$ is a zero forcing set for $\mathcal{G}(\pi)$.

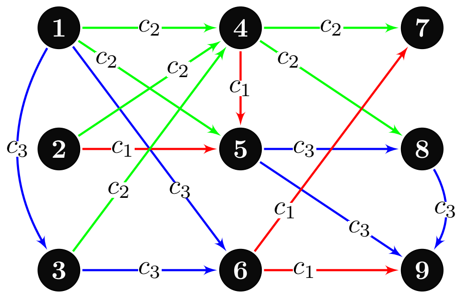
Next, we explore the relationship between zero forcing sets and controllability of



(a) Initial.



(b) $\{1, 2, 3\} \xrightarrow{c} \{4, 5, 6\}$.



(c) $\{4, 5, 6\} \xrightarrow{c} \{7, 8, 9\}$..

Figure 4.4: Example of a zero forcing set of a given colored graph.

$(\mathcal{G}(\pi); V_L)$. First we show that color changes do not affect the property of controllability. This is stated in the following theorem.

Theorem 4.8. *Let $\mathcal{G}(\pi)$ be a colored directed graph and let $C \subseteq V$ be a coloring set. Suppose that $X \xrightarrow{c} Y$ with $X \subseteq C$ and $Y \subseteq V \setminus C$. Then, $(\mathcal{G}(\pi); C)$ is controllable if and only if $(\mathcal{G}(\pi); C \cup Y)$ is controllable.*

Proof. Due to Lemma 4.5, it suffices to show that $\mathcal{D}_z(C) = V$ if and only if

$\mathcal{D}_z(C \cup Y) = V$ for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Here, C and $C \cup Y$ are taken as zero vertex sets.

Let $W \in \mathcal{W}_\pi(\mathcal{G})$ and $\mathcal{G}(W) = (V, E, W)$. By definition of the color change rule, $X \xrightarrow{c} Y$ means that $Y = N_{V \setminus C}(X)$ and there exists exactly one equivalence class of perfect matchings with nonzero signature in the colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$. By applying Theorem 4.6 we then find that all matrices in the pattern class of G are nonsingular. Now, let $x_1, \ldots, x_n$ be variables assigned to the vertices in V, with $x_j = 0$ for $j \in C$ and x_j undetermined for the remaining vertices. For the vertices $j \in C$, consider the balance equations (4.3). By the fact that $W_{kj} = 0$ for all $k \in V \setminus C$ with $k \notin N_{V \setminus C}(\{j\})$, the system of balance equations (4.3) for the vertices $j \in X$ can be written as

$$x_Y^\top W_{Y,X} = 0. \tag{4.9}$$

We now observe that the submatrix $W_{Y,X}$ of W belongs to the pattern class of G. Using the fact that all matrices in this pattern class are nonsingular, we obtain that $x_Y^\top = 0$. By the definition of the zero extension rule, we have that $X \xrightarrow{z} Y$ for $\mathcal{G}(W)$ with the set of zero vertices C. It then follows immediately that $C \cup Y \subseteq \mathcal{D}_z(C)$ and thus $\mathcal{D}_z(C \cup Y) = \mathcal{D}_z(C)$. As a consequence, C is a balancing set for $G(W)$ if and only if $C \cup Y$ is a balancing set for $G(W)$. Since this holds for arbitary choice of W in $W_\pi(G)$, the result follows immediately from Lemma 6.

■

By Theorem 4.8, colored strong structural controllability is invariant under application of the color change rule. We then obtain the following corollary.

Corollary 4.9. *Let $\mathcal{G}(\pi)$ be a colored directed graph, let $V_L \subseteq V$ be a leader set and let $\mathcal{D}_c(V_L)$ be a derived set. Then $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}(\pi); \mathcal{D}_c(V_L))$ is controllable.*

As an immediate consequence of Corollary 4.9 we arrive at the main result of this section which provides sufficient graph theoretic condition for controllability of $(\mathcal{G}(\pi); V_L)$.

Theorem 4.10. *Let $\mathcal{G}(\pi) = (V, E, \pi)$ be a colored directed graph with leader set $V_L \subseteq V$. If V_L is a zero forcing set, then $(\mathcal{G}(\pi); V_L)$ is controllable.*

Proof. The proof follows immediately from Corollary 4.9 and the fact that, trivially, $(\mathcal{G}(\pi); V)$ is controllable. ■

To conclude this section, we will provide a counter example to show that the condition in Theorem 4.10 is not a necessary condition.

Example 4.6. Consider the colored graph $\mathcal{G}(\pi)$ depicted in Figure 4.5 with leader set $V_L = \{1, 2\}$. It will turn out that $(\mathcal{G}(\pi); V_L)$ is controllable because V_L is a balancing set for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$ Yet, V_L is not a zero forcing set. Clearly, since none of the subsets $\{1, 2\}$, $\{1\}$ and $\{2\}$ have color-perfect white neighbors, there does not exist a derived set $\mathcal{D}_c(V_L)$ that equals V. Hence V_L is not a zero forcing set. We will show that, however, $(\mathcal{G}(\pi); V_L)$ is controllable. Due to Lemma 4.5, it is sufficient to show that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. To do this, let $W \in \mathcal{W}_\pi(\mathcal{G})$ correspond to a realization $\{c_1, c_2\}$ of the color set, with c_1 and c_2 nonzero real numbers. Assign variables $x_1, \ldots, x_5$ to the vertices in V. Let $x_1 = x_2 = 0$ and let x_3, x_4 and x_5 be undetermined. The system of balance equations (4.3) for the vertices 1 and 2 in V_L is then given by

$$\begin{aligned} c_1x_3 + c_1x_4 &= 0, \\ c_2x_3 + c_2x_4 + c_1x_5 &= 0. \end{aligned} \tag{4.10}$$

Since $c_1 \neq 0$ and $c_2 \neq 0$, the homogeneous system (4.10) is equivalent to the system

$$\begin{aligned} c_1x_3 + c_1x_4 &= 0, \\ c_1x_5 &= 0, \end{aligned} \tag{4.11}$$

which yields $x_5 = 0$. By the definition of the zero extension rule, we therefore have $\{1, 2\} \xrightarrow{z} \{5\}$. Repeated application of the zero extension rule yields that V_L is a balancing set. Since the matrix $W \in \mathcal{W}_\pi(\mathcal{G})$ was taken arbitrary, we conclude that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Thus we have found a counter example for the necessity of the condition in Theorem 4.10.

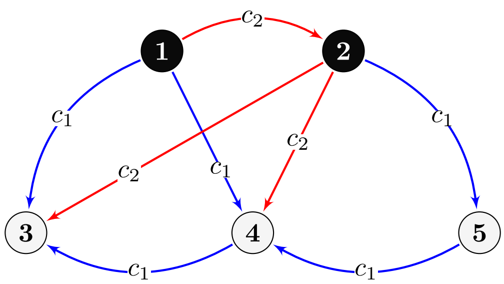


Figure 4.5: Example to show that the condition in Theorem 4.10 is not necessary.

Remark 4.2. In this remark we provide some intuition on why the colored graph of Example 4.6 leads to a controllable system, while $V_L = \{1, 2\}$ is not a zero forcing set for $\mathcal{G}(\pi)$. The main observation is that the balance equations (4.10) are equivalent to the equations (4.11), which correspond to a new colored graph $\mathcal{G}'(\pi)$ in which the edges $(2,3)$ and $(2,4)$ have been removed. In other words, we see that V_L is a balancing set for all weighted graphs associated with $\mathcal{G}(\pi)$ if and only if the same holds for $\mathcal{G}'(\pi)$. Since V_L is a zero forcing set for the new graph $\mathcal{G}'(\pi)$, we have that $(\mathcal{G}'(\pi), V_L)$ is controllable, so also $(\mathcal{G}(\pi), V_L)$ is controllable. In fact, we will generalize this idea in the next section.

4.5 Elementary edge operations and derived colored graphs

In the previous section, in Theorem 4.10, we have established a sufficient condition for colored strong structural controllability. In the present section we will establish another sufficient graph theoretic condition. This new condition is based on the so-called *elementary edge operations*. These are operations that can be performed on the original colored graph, and that preserve colored strong structural controllability. These edge operations on the graph are motivated by the observation that elementary operations on the systems of balance equations appearing in the zero extension rule do not modify the set of solutions to these linear equations. Indeed, in Example 4.6, we have verified that $\{1,2\} \xrightarrow{z} \{5\}$ for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. As explained in Remark 4.2, this is due to the fact that the system of balance equations (4.10) is equivalent to (4.11), implying that $x_5 = 0$ for all nonzero values c_1 and c_2. To generalize and visualize this idea on the level of the colored graph, we now introduce the following two types of elementary edge operations.

Let $C \subseteq V$ be a coloring set, i.e., the set of vertices initially colored black. The complement $V \setminus C$ is the set of white vertices. For two vertices $u, v \in C$ (where u and v can be the same vertex), we define

$$\mathcal{E}_u(v) := \{(v,j) \in E \mid j \in N_{V \setminus C}(u)\}$$

as the subset of all edges between v and white out-neighbors of u. We now introduce the following two elementary edge operations:

1. (*Turn color*) If all edges in $\mathcal{E}_u(u)$ have the same color, say c_i, then change the color of these edges to any other color in the color set.

2. (*Remove edges*) Assume $N_{V \setminus C}(u) \subseteq N_{V \setminus C}(v)$. If for any $k \in N_{V \setminus C}(u)$, the two edges (u,k) and (v,k) have the same color, then remove all edges in $\mathcal{E}_u(v)$.

The above elementary edge operations can be applied sequentially and, obviously, will not introduce new colors or add new edges. In the sequel, we will denote an edge operation by the symbol o. Applying the edge operation o to $\mathcal{G}(\pi)$, we obtain a new colored graph $\mathcal{G}'(\pi') = (V, E', \pi')$. We then call $\mathcal{G}'(\pi')$ a derived graph of $\mathcal{G}(\pi)$ associated with C and o. We denote such derived graph by $\mathcal{G}(\pi, C, o)$. An application of a sequence of elementary edge operations is illustrated in the following example.

Example 4.7. For the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Figure 4.6a, let $C = \{1, 2\}$ be the coloring set. For the vertex $1 \in C$, we have $\mathcal{E}_1(1) = \{(1, 3), (1, 4)\}$ in which both edges have the same color c_1. We apply the turn color operation to change the colors of $(1, 3)$ and $(1, 4)$ to c_2. Denote this operation by o_1. We then obtain the derived colored graph $\mathcal{G}(\pi, C, o_1)$ of $\mathcal{G}(\pi)$ with respect to C and o_1, which is denoted by $\mathcal{G}_1(\pi_1)$ and shown in Figure 4.6b. In addition, for the vertices 1 and 2 in $\mathcal{G}_1(\pi_1)$, we have $N_{V\setminus C}(1) \subseteq N_{V\setminus C}(2)$, where $N_{V\setminus C}(1) = \{3, 4\}$ and $N_{V\setminus C}(2) = \{3, 4, 5\}$. Besides, for any $k \in N_{V\setminus C}(1)$, the two edges $(1, k)$ and $(2, k)$ have the same color. Performing the edge removal operation denoted by o_2, we then remove all the edges in $\mathcal{E}_1(2) = \{(2, 3), (2, 4)\}$. Thus we obtain the derived colored graph $\mathcal{G}_1(\pi_1, C, o_2)$ of $\mathcal{G}_1(\pi_1)$ with respect to C and o_2, which is denoted by $\mathcal{G}_2(\pi_2)$ and depicted in Figure 4.6c.

Each elementary edge operation o corresponds to a single vertex $u \in C$ or a pair of vertices $u, v \in C$. In the sequel we will denote this subset of C corresponding to o by $C(o)$. Thus, $C(o)$ is either a singleton containing one vertex or a subset of V consisting of two vertices.

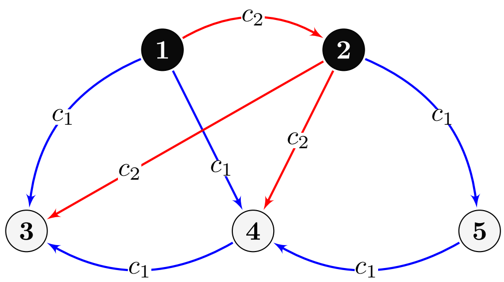
Next, we study the relationship between elementary edge operations and controllability of $(\mathcal{G}(\pi); V_L)$. We first show that elementary edge operations preserve zero extension. This issue is addressed in the following lemma.

Lemma 4.11. *Let $\mathcal{G}(\pi)$ be a colored directed graph and C be a coloring set. Let o represent an edge operation and let $\mathcal{G}'(\pi') = \mathcal{G}(\pi, C, o)$ be a derived graph with respect to C and o. Let $W \in \mathcal{W}_\pi(\mathcal{G})$ be a weighted adjacency matrix and let $W' \in \mathcal{W}_{\pi'}(\mathcal{G}')$ be the corresponding matrix associated with the same realization of the colors. Let $X \subseteq C \setminus C(o)$ and define $X' := C(o) \cup X$. Then, interpreting C as the set of zero vertices, for any $Y \subseteq V$ we have $X' \xrightarrow{z} Y$ in the weighted graph $\mathcal{G}(W)$ if and only if $X' \xrightarrow{z} Y$ in the weighted graph $\mathcal{G}'(W')$.*

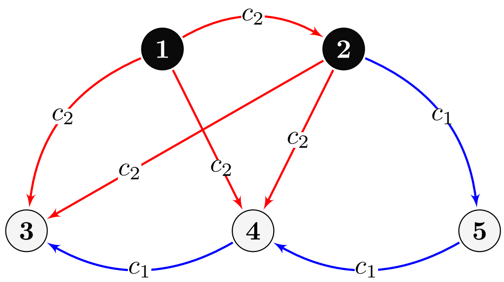
Proof. By suitably relabeling the vertices, we may assume that W has the form

$$W = \begin{bmatrix} W_{1,1} & \dots & W_{1,6} \\ \vdots & \ddots & \vdots \\ W_{6,1} & \dots & W_{6,6} \end{bmatrix},$$

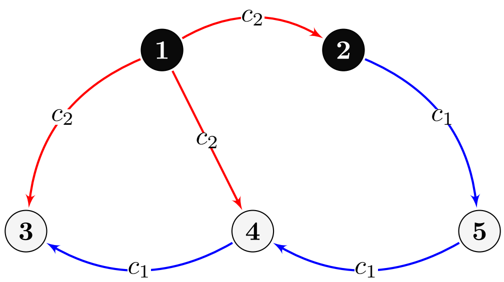
where the first row block corresponds to the vertices indexed by $C(o)$, the second row block corresponds to the vertices indexed by X, the third row block corresponds



(a) Initial colored graph $\mathcal{G}(\pi) = (V, E, \pi)$.



(b) Derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, C, o_1)$ where o_1 represents 'turning the colors of $(1, 3)$ and $(1, 4)$ to c_2'.



(c) Derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, C, o_2)$ where o_2 represents 'removing all the edges in $\mathcal{E}_1(2) = \{(2, 3), (2, 4)\}$'.

Figure 4.6: Example of performing elementary edge operations.

to the vertices indexed by $C \setminus X'$, the fourth row block corresponds to the vertices indexed by $N_{V \setminus C}(C(o))$, the fifth row block corresponds to the vertices indexed by $N_{V \setminus C}(X') \setminus N_{V \setminus C}(C(o))$ and the last row block corresponds to the remaining white vertices. The column blocks of W result from the same labeling. Correspondingly, the matrix W' must then be equal to

$$W' = \begin{bmatrix} W_{1,1} & W_{1,2} & \dots & W_{1,6} \\ W_{2,1} & W_{2,2} & \dots & W_{2,6} \\ W_{3,1} & W_{3,2} & \dots & W_{3,6} \\ W'_{4,1} & W_{4,2} & \dots & W_{4,6} \\ W_{5,1} & W_{5,2} & \dots & W_{5,6} \\ W_{6,1} & W_{6,2} & \dots & W_{6,6} \end{bmatrix}.$$

for some matrix $W'_{4,1}$. Since the fourth and fifth row blocks correspond to the vertices indexed by $N_{V\setminus C}(C(o))$ and $N_{V\setminus C}(X') \setminus N_{V\setminus C}(C(o))$, respectively, it follows easily that

$$W_{5,1} = 0, \quad W_{6,1} = 0 \quad \text{and} \quad W_{6,2} = 0$$

Consider the submatrices

$$W_{N_{V\setminus C}(X'),X'} = \begin{bmatrix} W_{4,1} & W_{4,2} \\ 0 & W_{5,2} \end{bmatrix} \quad \text{and} \quad W'_{N_{V\setminus C}(X'),X'} = \begin{bmatrix} W'_{4,1} & W_{4,2} \\ 0 & W_{5,2} \end{bmatrix}$$

of W and W', respectively. We then distinguish two cases:

(1) Suppose the edge operation o represents a color turn operation. In that case, $C(o)$ only contains one vertex, in other words, both $W_{4,1}$ and $W'_{4,1}$ consist of only one column. Hence, it follows that $W'_{4,1} = \alpha W_{4,1}$ for a suitable nonzero real number α.

(2) Suppose the edge operation o represents an edge removal operation. In that case $C(o)$ contains two vertices, say u and v, and both $W_{4,1}$ and $W'_{4,1}$ consist of two columns. We may assume that u and v correspond to the first and second column of these matrices, respectively, and the edges in $\mathcal{E}_u(v)$ are removed. This implies that

$$W'_{4,1} = W_{4,1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $W_{N_{V\setminus C}(X'),X'}$ and $W'_{N_{V\setminus C}(X'),X'}$ are column equivalent. Next, again assign variables $x_1, \dots, x_n$ to every vertex in V, where x_i is equal to 0 if $i \in C$ and otherwise undetermined. For the vertex $j \in C$ we consider the balance equation (4.3). By the fact that $W_{kj} = 0$ for all $k \in V \setminus C$ with $k \notin N_{V\setminus C}(\{j\})$ and $N_{V\setminus C}(\{j\}) \subseteq N_{V\setminus C}(X')$, the balance equation (4.3) is equivalent to

$$\sum_{k \in N_{V\setminus C}(X')} x_k W_{kj} = 0. \tag{4.12}$$

Again using the notation for the submatrix $W_{N_{V\setminus C}(X'),X'}$ and subvector $x_{N_{V\setminus C}(X')}$, we can rewrite the system of balance equations (4.12) for $j \in X'$ as

$$x^{\top}_{N_{V\setminus C}(X')} W_{N_{V\setminus C}(X'),X'} = 0. \tag{4.13}$$

Similarly, for the graph $\mathcal{G}'(W')$, we obtain the following system of balance equations for $j \in X'$:

$$x^{\top}_{N_{V\setminus C}(X')} W'_{N_{V\setminus C}(X'),X'} = 0. \tag{4.14}$$

Since $W'_{N_{V\setminus C}(X'),X'}$ and $W_{N_{V\setminus C}(X'),X'}$ are column equivalent, the solution sets of (4.13) and (4.14) coincide. By definition of the zero extension rule we therefore have that, for any vertex set Y, $X' \xrightarrow{z} Y$ in $\mathcal{G}(W)$ if and only if $X' \xrightarrow{z} Y$ in $\mathcal{G}'(W')$. This completes the proof. ■

It follows from the previous that colored strong structural controllability is preserved under elementary edge operations. Indeed, we have

Theorem 4.12. *Let $\mathcal{G}(\pi)$ be a colored directed graph, $V_L \subseteq V$ be a leader set, and o an elementary edge operation. Let $\mathcal{G}'(\pi') = \mathcal{G}(\pi, V_L, o)$ be a derived colored graph of $\mathcal{G}(\pi)$ with respect to V_L and o. Then we have that $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}'(\pi'); V_L)$ is controllable.*

Proof. The proof follows from Lemma 4.5 and Lemma 4.11. ■

As an immediate consequence of Theorem 4.12 and Theorem 4.10 we see that if the leader set V_L of the original colored graph $\mathcal{G}(\pi)$ is a zero forcing set for the derived graph $\mathcal{G}'(\pi') = \mathcal{G}(\pi, V_L, o)$, then $(\mathcal{G}(\pi); V_L)$ is controllable. Obviously, this result can be readily extended to derived graphs obtained by applying not only one, but a finite *sequence* of edge operations.

This immediately leads to the following sufficient graph theoretic condition for controllability of $(\mathcal{G}(\pi); V_L)$.

Corollary 4.13. *Let $\mathcal{G}(\pi)$ be a colored directed graph and let V_L be a leader set. Let $\mathcal{G}'(\pi')$ be a colored graph obtained by applying finitely many elementary edge operations. Then $(\mathcal{G}(\pi), V_L)$ is controllable if V_L is a zero forcing set for $\mathcal{G}'(\pi')$.*

Example 4.8. We now apply Corollary 4.13 to the colored graph depicted in Figure 4.5. We already saw that $V_L = \{1, 2\}$ is not a zero forcing set, but we showed that we do have strong structural controllability for this colored graph. This can now also be shown graph theoretically by means of Corollary 4.13: the leader set V_L is a zero forcing set for the derived graph in Figure 4.6c, so the original colored graph in Figure 4.6a yields a controllable system.

By combining Theorem 4.12 and Corollary 4.9 we are now in the position to establish yet another procedure for checking controllability of a given colored graph $(\mathcal{G}(\pi); V_L)$. First, distinguish the following two steps:

1. As the first step, apply the color change operation to compute a derived set $\mathcal{D}_c(V_L)$. If this derived set is equal to V we have controllability. If not, we can not yet decide whether we have controllability or not.

2. As a next step, then, apply an edge operation o to $\mathcal{G}(\pi)$ to obtain $\mathcal{G}_1(\pi_1)$, where $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, \mathcal{D}_c(V_L), o)$ is a derived graph of $\mathcal{G}(\pi)$ with coloring set $\mathcal{D}_c(V_L)$ and edge operation o.

By Theorem 4.12 and Corollary 4.9, it is straightforward to verify that $(\mathcal{G}(\pi); V_L)$ is controllable if and only if $(\mathcal{G}_1(\pi_1); \mathcal{D}_c(V_L))$ is controllable.

We can now repeat steps 1 and 2, applying them to $\mathcal{G}_1(\pi_1)$. Successive and alternating application of these two steps transforms the original leader set V_L using several color change operations associated with the several derived graphs appearing in the process. After finitely many iterations we thus arrive at a so called *edge-operations-color-change derived set* of V_L, that will be denoted by $\mathcal{D}_{ec}(C)$. This set will remain unchanged in case we again apply step 1 or step 2. Since controllability is preserved, we arrive at the following theorem which gives yet another sufficient condition for colored strong structural controllability.

Theorem 4.14. *Let $\mathcal{G}(\pi)$ be a colored directed graph and let $V_L \subseteq V$ be a leader set. Let $\mathcal{D}_{ec}(V_L)$ be an edge-operations-color-change derived set of V_L. We then have that $(\mathcal{G}(\pi); V_L)$ is controllable if $\mathcal{D}_{ec}(V_L) = V$.*

Remark 4.3. Obviously, a derived set $\mathcal{D}_c(V_L)$ of V_L in $\mathcal{G}(\pi)$ is always contained in an edge-operations-color-change derived set $\mathcal{D}_{ec}(V_L)$ of V_L. Hence the condition in Theorem 4.14 is weaker than the conditions in Theorem 4.10 and Corollary 4.13. However, note that $\mathcal{D}_{ec}(V_L)$ equal to V is still not necessary for controllability of $(\mathcal{G}(\pi); V_L)$.

We conclude this section by two examples. In the first one, we provide a counterexample in which $(\mathcal{G}(\pi); V_L)$ is strongly structurally controllable but $\mathcal{D}_c(V_L)$ is not equal to V. In the second example, we illustrate the application of Theorem 4.14 to check controllability of a given colored graph and leader set.

Example 4.9. Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Figure 4.7 with $V_L = \{1, 2, 3, 4, 5\}$ the leader set. It will turn out that $(\mathcal{G}(\pi); V_L)$ is controllable because V_L is a balancing set for all weighted graphs $\mathcal{G}(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. Yet, it can be checked that $\mathcal{D}_{ec}(V_L)$ is equal to V_L, since no subset of V_L

have color-perfect white neighbors, and no elementary edge operations are possible. Next, we will show that, however, $(\mathcal{G}(\pi); V_L)$ is controllable. Due to Lemma 4.5, it is sufficient to show that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$. To do this, let $W \in \mathcal{W}_\pi(\mathcal{G})$ correspond to a realization $\{c_1, c_2, c_3\}$ of the color set, with c_1, c_2 and c_3 nonzero real numbers. Assign variables $x_1, \dots, x_{10}$ to the vertices in V. Let $x_1 = \cdots = x_5 = 0$ and let $x_6, \dots, x_{10}$ be undetermined. The system of balance equations (4.3) for the vertices $1, 2$ in V_L is then given by

$$\begin{aligned} c_1 x_6 + c_2 x_7 &= 0, \\ c_3 x_6 + c_2 x_7 &= 0, \end{aligned} \tag{4.15}$$

and that for the vertices $1, 2$ in V_L is given by

$$\begin{aligned} c_1 x_8 + c_2 x_{10} &= 0, \\ c_1 x_8 + c_1 x_9 &= 0, \\ c_3 x_9 + c_2 x_{10} &= 0. \end{aligned} \tag{4.16}$$

We then distinguish two cases: $c_1 \neq c_3$ and $c_1 = c_3$. In the case that $c_1 \neq c_3$, the solution of system (4.15) is $x_6 = x_7 = 0$. By definition of the zero extension rule, we therefore have $\{1, 2\} \xrightarrow{z} \{6, 7\}$. Subsequently, one can verify that $\{7\} \xrightarrow{z} \{9\}$, $\{9\} \xrightarrow{z} \{10\}$ and $\{3\} \xrightarrow{z} \{8\}$. Therefore, V_L is a balancing set when $c_1 \neq c_3$. On the other hand, consider the case if $c_1 = c_3$. The homogeneous system (4.16) is equivalent to the system

$$\begin{bmatrix} x_8 & x_9 & x_{10} \end{bmatrix} \begin{bmatrix} c_1 & c_1 & 0 \\ 0 & c_1 & c_1 \\ c_2 & 0 & c_2 \end{bmatrix} = 0. \tag{4.17}$$

Since the determinant of the matrix $\begin{bmatrix} c_1 & c_1 & 0 \\ 0 & c_1 & c_1 \\ c_2 & 0 & c_2 \end{bmatrix}$ is equal to $2c_1^2 c_2$, $c_1 \neq 0$ and $c_2 \neq 0$, we have that $x_8 = x_9 = x_{10} = 0$. This implies that $\{3, 4, 5\} \xrightarrow{z} \{8, 9, 10\}$. Moreover, it is obvious that $\{8\} \xrightarrow{z} \{7\}$ and $\{1\} \xrightarrow{z} \{6\}$. Hence, the leader set V_L is a balancing set if $c_1 = c_3$. As a consequence, we conclude that V_L is a balancing set for all weighted graphs $\mathcal{G}(W)$ with $W \in \mathcal{W}_\pi(\mathcal{G})$, i.e.,$(\mathcal{G}(\pi); V_L)$ is controllable.

Example 4.10. Consider the colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ depicted in Figure 4.8a with $V_L = \{1, 2, 3\}$ the leader set. To start with, we compute a derived set $\mathcal{D}_c(V_L) = \{1, 2, 3, 4, 5, 6\}$ of V_L in $\mathcal{G}(\pi)$ as depicted in Figure 4.8b, and denote it by $\mathcal{D}_0$. For the vertices $5, 4 \in \mathcal{D}_0$, in $\mathcal{G}(\pi)$ we have $N_{V \setminus \mathcal{D}_0}(6) = \{8, 9\} \subseteq N_{V \setminus \mathcal{D}_0}(4)$. Since the edges $(6, 8)$ and $(6, 9)$ have the same color c_1, their color can be changed to

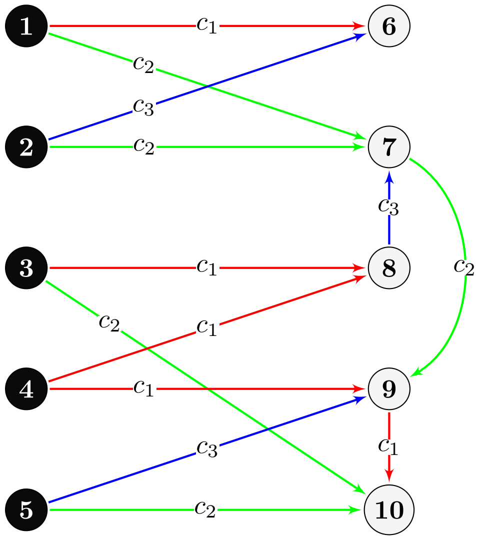


Figure 4.7: A counterexample in which $(\mathcal{G}(\pi); V_L)$ is strongly structurally controllable but $\mathcal{D}_{ec}(V_L)$ is not equal to V.

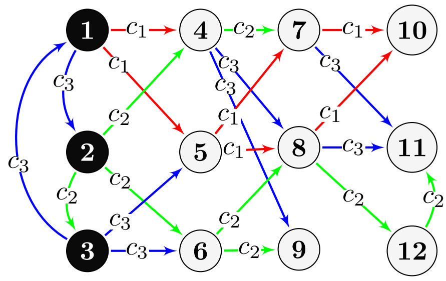
any arbitrary color. Here, we change the colors of $(6,8)$ and $(6,9)$ to c_3. Then, for any $k \in N_{V\setminus\mathcal{D}_0}(4)$, the two edges $(4,k)$ and $(6,k)$ have the the same color c_3. Thus we remove the edges in $\mathcal{E}_4(6) = \{(4,8),(4,9)\}$, and we denote the above two edge operations by o_0. In this way we obtain a derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}(\pi, \mathcal{D}_0, o_0)$ of $\mathcal{G}(\pi)$ with respect to $\mathcal{D}_0$ and o_0, that is depicted in Figure 4.8c. We proceed to compute a derived set $\mathcal{D}_c(\mathcal{D}_0) = \{1,2,3,4,5,6,7,8,9\}$ of $\mathcal{D}_0$ in $\mathcal{G}_1(\pi_1)$ as shown in Figure 4.8d and denote this derived set by $\mathcal{D}_1$. Since $\mathcal{D}_1 \neq V$ and $\mathcal{D}_1 \neq \mathcal{D}_0$, the procedure will continue. For the vertices $7,8 \in \mathcal{D}_1$ in the graph $\mathcal{G}_1(\pi_1)$, we have $N_{V\setminus\mathcal{D}_1}(7) \subseteq N_{V\setminus\mathcal{D}_1}(8)$, and for any $k \in N_{V\setminus\mathcal{D}_1}(7)$, the two edges $(7,k)$ and $(8,k)$ have the same color. Thus we remove the edges in $\mathcal{E}_7(8) = \{(8,10),(8,11)\}$ and denote this operation by o_1. We then obtain a derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, \mathcal{D}_1, o_1)$ of $\mathcal{G}_1(\pi_1)$ with respect to $\mathcal{D}_1$ and o_1, which is depicted in Figure 4.8e. Finally, we compute a derived set $\mathcal{D}_c(\mathcal{D}_1)$ of $\mathcal{D}_1$ in $\mathcal{G}_2(\pi_2)$ as shown in Figure 4.8f. This derived set is denoted by $\mathcal{D}_2$ and turns out to be equal to the original vertex set V. Thus we obtain that an edge-operations-color-change derived set $\mathcal{D}_{ec}(V_L)$ is equal to V, and conclude that $(\mathcal{G}(\pi); V_L)$ is controllable.

4.6 Conclusions

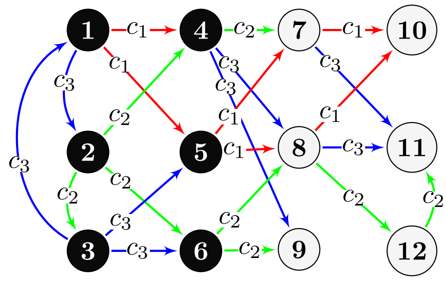
In this chapter, we have studied strong structural controllability of leader/follower networks. In contrast to existing work, in which the nonzero off-diagonal entries of matrices in the qualitative class are completely independent, in this chapter, we have studied the general case that there are equality constraints among these entries, in the sense that a priori given entries in the system matrix are restricted to take arbitrary but identical nonzero values. This has been formalized using the concept of colored graph and by introducing the new concept of colored strong structural controllability. In order to obtain conditions for colored strong structural controllability of leader/follower networks, we have introduced a new color change rule and a new concept of zero forcing set. These have been used to formulate a sufficient condition for controllability of the colored graph with a given leader set. We have shown that this condition is not necessary, by giving an example of a colored strong structurally controllable colored graph and leader set for which our sufficient condition is not satisfied.

Motivated by this example, we have established the concept of elementary edge operations on colored graphs. It has been shown that these edge operations preserve colored strong structural controllability. Based on these elementary edge operations and the color change rule, a second sufficient graph theoretic condition for colored strong structural controllability has been provided.

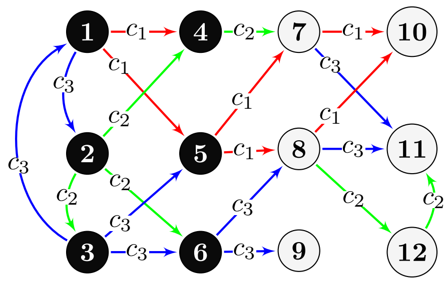
Finally, we have established a condition for colored strong structural controllability in terms of the new notion of edge-operations-color-change derived set. This derived set is obtained from the original leader set by applying edge operations and the color change rule sequentially in an alternating manner. This iterative procedure has been illustrated through an example.



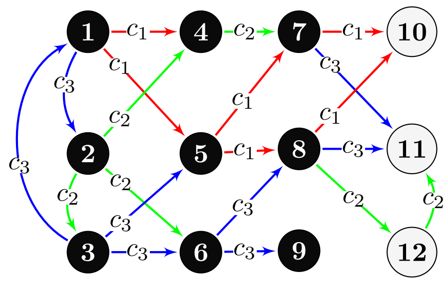
(a) Initial colored graph $\mathcal{G}(\pi) = (V, E, \pi)$ with coloring set $V_L = \{1, 2, 3\}$. Let $\mathcal{G}_0(\pi_0) = \mathcal{G}(\pi)$.



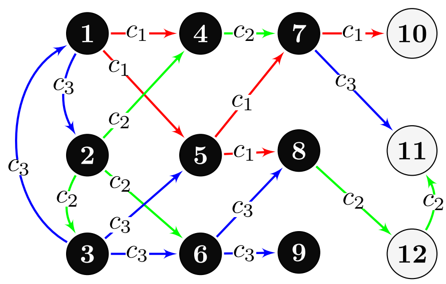
(b) Compute a derived set $\mathcal{D}_c(V_L) = \{1, \ldots, 6\}$ of V_L in $\mathcal{G}_0(\pi_0)$ and set $\mathcal{D}_0 = \mathcal{D}_c(V_L)$.



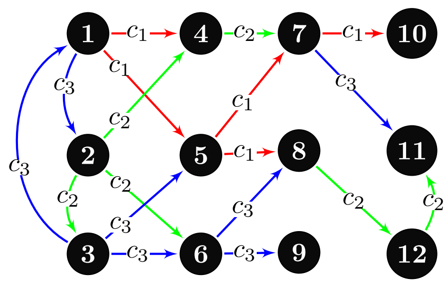
(c) Derived colored graph $\mathcal{G}_1(\pi_1) = \mathcal{G}_0(\pi_0, \mathcal{D}_0, o_0)$ with $\mathcal{D}_0 = \{1, \ldots, 6\}$ and o_0 representing 'turning colors of edges $(6, 8)$ and $(6, 9)$ to c_3 and removing edges $(4, 8)$ and $(4, 9)$'.



(d) Compute a derived set $\mathcal{D}_1 = \{1, \ldots, 9\}$ of $\mathcal{D}_0$ in the colored graph $\mathcal{G}_1(\pi_1)$.



(e) Derived colored graph $\mathcal{G}_2(\pi_2) = \mathcal{G}_1(\pi_1, \mathcal{D}_1, o_1)$ with $\mathcal{D}_1 = \{1, \ldots, 9\}$ and o_1 representing 'removing edges $(8, 10)$ and $(8, 11)$'.



(f) Compute a derived set $\mathcal{D}_2 = V$ of $\mathcal{D}_1$ in the colored graph $\mathcal{G}_2(\pi_2)$. Return that $(\mathcal{G}(\pi); V_L)$ is controllable.

Figure 4.8: Example of application of Theorem 4.14

5 Strong Structural Controllability of Colored Structured Systems

This chapter deals with strong structural controllability of linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices. Instead of assuming that the nonzero and arbitrary entries of the system matrices can take their values completely independently, we allow equality constraints on these entries, in the sense that *a priori* given entries in the system matrices are restricted to take arbitrary but identical values. To formalize this general class of structured systems, we introduce the concepts of colored pattern matrices and colored structured systems. These results generalize both the results in Chapter 2 and those in Chapter 4. In this chapter we will establish both algebraic and graph theoretic conditions for strong structural controllability of this more general class of structured systems.

5.1 Introduction

For LTI systems of the form

$$\dot{x} = Ax + Bu$$

controllability can be verified using the Kalman rank test or the Hautus test [26]. Often, the exact values of the entries in the matrices A and B are not known, but only the underlying interconnection structure between the input and state variables is known exactly. In order to formalize this, Mayeda and Yamada [28] have introduced a framework in which, instead of a fixed pair of real matrices, only the so-called *zero/nonzero* structure of A and B is given. This means that each entry of these matrices is known to be either a *fixed zero* or an *arbitrary nonzero real number*. In addition, to guarantee the controllability of all possible LTI systems with such a given zero/nonzero structure, in [28] they introduced the concept of *strong structural* *controllability.* Since then, many contributions have been made on the topic of strong structural controllability. (See [29, 30, 32, 40, 41, 99] and the references therein.)

Roughly speaking, two basic assumptions prevail in the aforementioned literature: (1) each entry of the system matrices is either a fixed zero or an arbitrary nonzero value, and (2) the nonzero entries take their values independently. Concerning the first of these assumptions, however, in many practical scenarios there might also be entries that can take arbitrary zero or nonzero values. Examples can be found in [45,58,63,64] and the references therein. In such scenarios, it is impossible to represent the system using a zero/nonzero structure. To deal with this, recently in [58] the notion of zero/nonzero structure has been extended to a more general *zero/nonzero/arbitrary structure*, and thus a unifying framework for strong structural controllability was established. Regarding the second of the above assumptions, we note that in physical systems it is often not the case that the nonzero entries in the system matrices can take their values independently. Indeed, some of the nonzero entries in the system matrices might have dependencies. (See [52–57] and the references therein.) In particular, in [55] and [57], the situation was considered that prescribed nonzero entries in the system matrices are constrained to take identical (nonzero) values. These constraints can be caused by symmetry properties ([52, 53]) or by physical constraints on the system [56].

In this chapter, we will explore the strong structural controllability of systems in which neither of the above two basic assumptions is satisfied. More explicitly, the present chapter will extend the approach taken in [55] and [57] using the newly introduced unifying framework from [58]. That is, we will study strong structural controllability of systems in which the zero/nonzero/arbitrary structure of the system matrices is given, and moreover, in which some of the entries in the system matrices are constrained to take identical values. Following the naming convention in [55], [57] and [58], we will call such kind of systems *colored structured systems*.

The main contributions of this chapter are the following:

1. We establish sufficient algebraic conditions for the strong structural controllability of a given colored structured system in terms of a full row rank test on two so-called *colored pattern matrices*.

2. We establish a test for the full row rank property of a given colored pattern matrix in terms of a new color change rule and colorability of the graph associated with the pattern matrix. In order to introduce this color change rule, we also establish a necessary and sufficient condition under which a given square colored pattern matrix is nonsingular.

3. Based on the above results, we establish sufficient graph theoretic conditions for strong structural controllability of colored structured systems.

The outline of the chapter is as follows. Section 5.2 presents some preliminaries. In Section 5.3, we formulate the problem treated in this chapter in terms of colored structured systems. In Section 5.4, we establish sufficient algebraic conditions for controllability of colored structured systems. We also provide a counterexample to show that these conditions are not necessary. Section 5.5 presents a necessary and sufficient graph theoretic condition under which a given square colored pattern matrix is nonsingular. In Section 5.6, we introduce a new color change rule and the concept of colorability of the graph associated with a given colored pattern matrix. Based on these concepts, we establish a graph theoretic condition under which a given colored pattern matrix has full row rank, and thus we obtain sufficient graph theoretic conditions for strong structural controllability of colored structured systems. Finally, in Section 5.7, we provide our conclusions.

5.2 Preliminaries

Here, we briefly review terminologies of graph theory and concepts of linear structured systems which will be used in this chapter. More details can be found in Section 2.2 and 4.2.

5.2.1 Elements of graph theory

Here, we briefly review notions of graph theory which will be used in this chapter. More details can be found in Section 4.2. We denote by $\mathcal{G} = (V, E)$ a directed graph with vertex set $V = \{1, \dots, n\}$ and edge set $E \subseteq V \times V$. We define the graph $\mathcal{G} = (V, E)$ to be *undirected* if $(i, j) \in E$ whenever $(j, i) \in E$. In that case, the order of i and j does not matter, and we interpret the edge set E as the set of unordered pairs $\{i, j\}$, where $(i, j) \in E$. Moreover, an undirected graph $\mathcal{G} = (V, E)$ is called a *bipartite graph* if there exist nonempty disjoint subsets X and Y of V such that $X \cup Y = V$ and $\{i, j\} \in E$ only if $i \in X$ and $j \in Y$. Such a bipartite graph is denoted by $G = (X, Y, E_{XY})$, where we denote the edge set by E_{XY} to stress that it contains edges $\{i, j\}$ with $i \in X$ and $j \in Y$. In this chapter, we will use the symbol $\mathcal{G}$ for directed graphs and G for bipartite graphs. A set of t edges $m \subseteq E_{XY}$ is called a *t-matching* in G, if no two distinct edges in m share the same vertex. In the special case that $|X| = |Y| = t$, such a t-matching is called a *perfect matching.*

5.2.2 Pattern matrices and structured systems

By a pattern matrix, we mean a matrix with entries in the set of symbols $\{0, *, ?\}$. The set of all $p \times q$ pattern matrices will be denoted by $\{0, *, ?\}^{p \times q}$. For a given $p \times q$

pattern matrix $\mathcal{M}$, we define the *pattern class* of $\mathcal{M}$ as

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p\times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0 \quad \text{and} \quad M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *\}.$$

This means that for a matrix $M \in \mathcal{P}(\mathcal{M})$, the entry M_{ij} is either (i) *zero* if $\mathcal{M}_{ij} = 0$, (ii) *nonzero* if $\mathcal{M}_{ij} = *$, or (iii) *arbitrary* (zero or nonzero) if $\mathcal{M}_{ij} = ?$.

Let $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$ be two pattern matrices. Consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the input. We will call the family of systems (5.1) with $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$ a *structured system*. We denote this structured system by the ordered pair of pattern matrices $(\mathcal{A}, \mathcal{B})$, and we denote by (A, B) a particular system of the form (5.1). Thus,

$$(\mathcal{A}, \mathcal{B}) = \{(A, B) \mid \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P}(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix})\}.$$

5.3 Problem formulation

In this section, we will introduce the problem to be considered in this chapter. Let $(\mathcal{A}, \mathcal{B})$ be the structured system associated with $\mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n\times m}$. The structured system $(\mathcal{A}, \mathcal{B})$ is called strongly structurally controllable if each (A, B) in this family is controllable. In Chapter 2 necessary and sufficient conditions for strong structural controllability were established. Note that in the set-up of Chapter 2, all $*$ and ?-entries in $\mathcal{A}$ and $\mathcal{B}$ take their values independently. In the present chapter we will extend the results of Chapter 2 and impose constraints on the $*$ and ?-entries. In particular, instead of considering the entire family $(\mathcal{A}, \mathcal{B})$, we will zoom in on a subclass of $(\mathcal{A}, \mathcal{B})$ containing those systems (A, B) that satisfy the condition that a prior given entries in $\begin{bmatrix} A & B \end{bmatrix}$ are equal. We will now formalize this equality constraints on the $*$ and ?-entries.

To do so, consider a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p\times q}$. Define the sets of locations of $*$ and ?-entries in $\mathcal{M}$ as

$$\mathcal{I}_{\mathcal{M}}(*) := \{(i, j) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \mid \mathcal{M}_{ij} = *\}$$

and

$$\mathcal{I}_{\mathcal{M}}(?) := \{(i, j) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \mid \mathcal{M}_{ij} =?\}.$$

Let $\pi^* := \{\mathcal{I}_1^*, \ldots, \mathcal{I}_k^*\}$ and $\pi^? := \{\mathcal{I}_1^?, \ldots, \mathcal{I}_l^?\}$ be partitions of $\mathcal{I}_{\mathcal{M}}(*)$ and $\mathcal{I}_{\mathcal{M}}(?)$, respectively. We then call $\pi := \pi^* \cup \pi^?$ a *coloring* of the pattern matrix $\mathcal{M}$ and the pair $(\mathcal{M}, \pi)$ a *colored pattern matrix*. Next, we define the so-called *colored pattern class* associated with $(\mathcal{M}, \pi)$ as

$$\mathcal{P}(\mathcal{M}, \pi) := \{M \in \mathcal{P}(\mathcal{M}) \mid M_{ij} = M_{kl} \text{ if } \exists\ r \in \{1, \dots, k\} \text{ or } s \in \{1, \dots, l\} \\ \text{such that } (i,j),(k,l) \in \mathcal{I}_r^* \text{ or } \mathcal{I}_s^?\}.$$

In order to visualize the coloring π, two $*$-entries in the same subset $\mathcal{I}_r^*$ are said to have the same color, which will be denoted by the symbol c_r. Likewise, two ?-entries in the subset $\mathcal{I}_s^?$ are said to have the same color, and this color will be denoted by the symbol g_s. In this chapter, we will always use symbols c_r $(r = 1, \dots, k)$ for colors associated with $*$-entries, and g_s $(s = 1, \dots, \ell)$ for colors associated with ?-entries. With a slight abuse of notation, sometimes we will also use the symbols c_i and g_i to denote nonzero or arbitrary real variables.

Example 5.1. Consider the colored pattern matrix $(\mathcal{M}, \pi)$ with

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & * & 0 & 0 & * & 0 \\ 0 & ? & 0 & * & ? & * & * \\ * & 0 & ? & 0 & 0 & 0 & 0 \\ ? & ? & * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi = \{\mathcal{I}_1^*, \mathcal{I}_2^*, \mathcal{I}_1^?, \mathcal{I}_2^?\} \tag{5.2}$$

where

$$\begin{aligned} \mathcal{I}_1^* &= \{(1,3),(1,6),(2,7),(3,1),(4,3),(4,4)\}, \\ \mathcal{I}_2^* &= \{(2,4),(2,6),(5,1),(5,2)\}, \\ \mathcal{I}_1^? &= \{(4,1),(4,2),(2,5)\}, \\ \mathcal{I}_2^? &= \{(2,2),(3,3)\}. \end{aligned}$$

In this example, the $*$-entries with locations in $\mathcal{I}_1^*$ have color c_1, and those with locations in $\mathcal{I}_2^*$ have color c_2. Besides, the ?-entries with locations in $\mathcal{I}_1^?$ have color g_1 and those with locations in $\mathcal{I}_2^?$ have color g_2.

Thus, $(\mathcal{M}, \pi)$ can be visualized by

$$\begin{bmatrix} 0 & 0 & c_1 & 0 & 0 & c_1 & 0 \\ 0 & g_2 & 0 & c_2 & g_1 & c_2 & c_1 \\ c_1 & 0 & g_2 & 0 & 0 & 0 & 0 \\ g_1 & g_1 & c_1 & c_1 & 0 & 0 & 0 \\ c_2 & c_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding colored pattern class consists of all real matrices of the form

$$\begin{bmatrix} 0 & 0 & c_1 & 0 & 0 & c_1 & 0 \\ 0 & g_2 & 0 & c_2 & g_1 & c_2 & c_1 \\ c_1 & 0 & g_2 & 0 & 0 & 0 & 0 \\ g_1 & g_1 & c_1 & c_1 & 0 & 0 & 0 \\ c_2 & c_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.3}$$

where the g_i are arbitrary real numbers, and the c_i are arbitrary nonzero real numbers.

A colored pattern matrix $(\mathcal{M}, \pi)$ is said to have *full row rank* if every matrix $M \in \mathcal{P}(\mathcal{M}, \pi)$ has full row rank.

Consider now the colored pattern matrix $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ associated with the pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ and the coloring

$$\pi = \{\mathcal{I}_1^*, \ldots, \mathcal{I}_k^*, \mathcal{I}_1^?, \ldots, \mathcal{I}_\ell^?\}.$$

We then define the *colored structured system* associated with $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ as

$$(\mathcal{A}, \mathcal{B}, \pi) := \{(A, B) \mid \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P}(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)\}.$$

We say that this colored structured system is *strongly structurally controllable* if (A, B) is controllable for all $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P}(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$. We will then simply say that $(\mathcal{A}, \mathcal{B}, \pi)$ is *controllable*. For example, the colored structured system $(\mathcal{A}, \mathcal{B}, \pi)$ with $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and π given by (5.2) is controllable, as will be shown later on in this chapter. The problem that we will investigate in this chapter can now be stated as follows.

Problem 5.1. *Given a colored structured system $(\mathcal{A}, \mathcal{B}, \pi)$, find conditions under which it is controllable.*

Remark 5.1. There is a strong relation between the work in this chapter and that in [55] and [57] on controllability of systems on colored graphs. Stated in terms of pattern matrices, the work in [55] and [57] deals with the very special case of colored structured systems $(\mathcal{A}, \mathcal{B}, \pi)$ in which

1. all diagonal entries of $\mathcal{A}$ are ?,
2. all off-diagonal entries of $\mathcal{A}$ are $*$ or 0,
3. in $\mathcal{B}$, each column contains exactly one $*$ and each row contains at most one $*$,
4. the coloring $\pi^? = \{(1, 1), \ldots, (n, n)\}$ of the ?-entries is given, i.e., the ?-entries have distinct colors.

In Chapter 4, conditions were obtained for controllability of this special class of systems. In the present chapter these results will be generalized to general colored structured systems.

5.4 Algebraic conditions for controllability

In this section, we will provide a sufficient algebraic condition for controllability. The condition states that a colored structured system is controllable if two particular colored pattern matrices associated with this system have full row rank.

Let $(\mathcal{A}, \mathcal{B}, \pi)$ be a colored structured system with

$$\mathcal{A} \in \{0, *, ?\}^{n\times n}, \quad \mathcal{B} \in \{0, *, ?\}^{n\times m} \quad \text{and} \quad \pi = \{\mathcal{I}_1^*, \ldots, \mathcal{I}_k^*, \mathcal{I}_1^?, \ldots, \mathcal{I}_\ell^?\}.$$

In order to state our first result, for the given $(\mathcal{A}, \mathcal{B}, \pi)$ we define an associated new colored pattern matrix $(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}, \bar{\pi})$ as follows.

Definition 5.2. *We define $\bar{\mathcal{A}}$ to be the pattern matrix obtained from $\mathcal{A}$ by modifying the diagonal entries of $\mathcal{A}$ as follows*

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0, \\ ? & \text{otherwise.} \end{cases}$$

The matrix $\mathcal{B}$ remains unchanged. Next, for $r = 1, \ldots, k$ and $s = 1, \ldots, l$ we remove the diagonal locations from $\mathcal{I}_r^*$ and $\mathcal{I}_s^?$ by defining

$$\bar{\mathcal{I}}_r^* := \{(i, j) \in \mathcal{I}_r^* \mid i \neq j\}$$

and

$$\bar{\mathcal{I}}_s^? := \{(i, j) \in \mathcal{I}_s^? \mid i \neq j\}.$$

Note that some of the $\bar{\mathcal{I}}_r^*$ or $\bar{\mathcal{I}}_s^?$ might be empty. Next, we partition the set of diagonal locations into n subsets. More explicitly, if $i_1, \ldots, i_w \in \{1, \ldots, n\}$ are the indices such that $\bar{\mathcal{A}}_{i_j i_j} = *$, then for $j = 1, \ldots, w$ we define

$$\bar{\mathcal{I}}_{k+j}^* := \{(i_j, i_j)\}.$$

Furthermore, if $t_1, \ldots, t_{n-w} \in \{1, \ldots, n\}$ are the indices such that $\bar{\mathcal{A}}_{t_j t_j} =?$ for $j = 1, \ldots, n-w$, then we define

$$\bar{\mathcal{I}}_{l+j}^? := \{(t_j, t_j)\} \quad \text{for } j = 1, \ldots, n-w.$$

Thus we obtain a partition

$$\bar{\pi} := \{\bar{\mathcal{I}}_1^*, \ldots, \bar{\mathcal{I}}_{k+w}^*, \bar{\mathcal{I}}_1^?, \ldots, \bar{\mathcal{I}}_{l+n-w}^?\}$$

of the sets $\mathcal{I}_{\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}}(*)$ and $\mathcal{I}_{\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}}(?)$ and define this to be the new coloring $\bar{\pi}$.

We are now ready to state our first result.

Theorem 5.3. *Let $(\mathcal{A}, \mathcal{B}, \pi)$ be a colored structured system, and let $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ be the colored pattern matrix obtained from $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ as in Definition 5.2. Then, $(\mathcal{A}, \mathcal{B}, \pi)$ is controllable if both $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ and $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ have full row rank.*

Proof. The proof of this theorem can be given by slightly adapting that of the sufficient condition in Theorem 2.3 and is hence omitted. ■

We will now illustrate Theorem 5.3 by an example.

Example 5.2. Consider $(\mathcal{A}, \mathcal{B}, \pi)$ with $\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}$ and π given by (5.2) in Example 5.1. Using Definition 5.2, we obtain the colored pattern matrix $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ with

$$\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} * & 0 & * & 0 & 0 & * & 0 \\ 0 & ? & 0 & * & ? & * & * \\ * & 0 & ? & 0 & 0 & 0 & 0 \\ ? & ? & * & ? & 0 & 0 & 0 \\ * & * & 0 & 0 & * & 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{\pi} = \{\bar{\mathcal{I}}_1^*, \bar{\mathcal{I}}_2^*, \bar{\mathcal{I}}_3^*, \bar{\mathcal{I}}_4^*, \bar{\mathcal{I}}_1^?, \mathcal{I}_2^?, \mathcal{I}_3^?, \mathcal{I}_4^?, \mathcal{I}_5^?\}, \tag{5.4}$$

where

$$\begin{aligned} &\bar{\mathcal{I}}_1^* = \{(1,3), (1,6), (2,7), (3,1), (4,3)\}, \\ &\bar{\mathcal{I}}_2^* = \{(2,4), (2,6), (5,1), (5,2)\}, \ \bar{\mathcal{I}}_3^* = \{(1,1)\}, \ \bar{\mathcal{I}}_4^* = \{(5,5)\}, \\ &\bar{\mathcal{I}}_1^? = \{(4,1), (4,2), (2,5)\}, \ \bar{\mathcal{I}}_2^? = \varnothing, \ \bar{\mathcal{I}}_3^? = \{(2,2)\}, \\ &\bar{\mathcal{I}}_4^? = \{(3,3)\}, \ \bar{\mathcal{I}}_5^? = \{(4,4)\}. \end{aligned}$$

It turns out that both $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ and $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ have full row rank. Indeed, let M be a matrix in $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$. Then, M is of the form (5.3). Let M' be the submatrix of M obtained by removing the fourth and fifth column from M. It is easily seen that $\det(M') = -c_1^4 c_2$, which is nonzero for c_1 and c_2. Hence, all matrices M given by (5.2) have full row rank, so all $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ has full row rank. Similarly, one can verify that $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ has full row rank, so, by Theorem 5.3, we conclude that $(\mathcal{A}, \mathcal{B}, \pi)$ is controllable.

Remark 5.2. In Chapter 2, necessary and sufficient conditions for controllability of structured systems $(\mathcal{A}, \mathcal{B})$ without a coloring were established, also in terms of two rank tests. We note that Theorem 5.3 generalizes this result to colored systems. The conditions obtained in the present chapter are however only sufficient. To illustrate that the conditions in Theorem 5.3 are not necessary for controllability, we will provide a counterexample of a colored structured system that is controllable while one of the conditions does not hold.

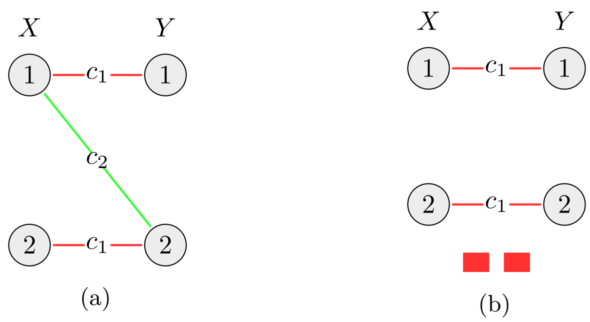


Figure 5.1: Example of a perfect matching associated with a colored bipartite graph. (a) Colored bipartite graph $G(X, Y, E_{XY}, \pi_{XY})$ associated with $X = \{6, 7\}$ and $Y = \{1, 2\}$. (b) Perfect matching p_1 with spectrum $\{c_1, c_1\}$ and $\mathrm{sign}(p_1) = 1$.

Example 5.3. Consider $(\mathcal{A}, \mathcal{B}, \pi)$ with

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & 0 & * \end{bmatrix} \quad \text{and} \quad \pi = \{\mathcal{I}_1^*, \mathcal{I}_2^*\}$$

where $\mathcal{I}_1^* = \{(1,1), (1,2), (2,1)\}$ and $\mathcal{I}_2^* = \{(1,3), (2,3)\}$. The corresponding colored pattern class consists of all matrices of the form

$$\begin{bmatrix} c_1 & c_1 & c_2 \\ c_1 & 0 & c_2 \end{bmatrix}$$

where c_1, c_2 are nonzero real numbers. The matrix $\begin{bmatrix} B & AB \end{bmatrix}$ is equal to

$$\begin{bmatrix} c_2 & 2c_1c_2 \\ c_2 & c_1c_2 \end{bmatrix}$$

which has full row rank for every choice of c_1 and c_2. By the Kalman rank test, we conclude that (A, B) is controllable for all $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{P}(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$, i.e., $(\mathcal{A}, \mathcal{B}, \pi)$ is controllable.

Next, we will show that the second condition in Theorem 5.3 is not satisfied. Let $(\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix}, \bar{\pi})$ be the colored pattern matrix obtained from $(\begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}, \pi)$ as in Definition 5.2 with

$$\begin{bmatrix} \bar{\mathcal{A}} & \mathcal{B} \end{bmatrix} = \begin{bmatrix} ? & * & * \\ * & * & * \end{bmatrix}, \quad \bar{\pi} = \{\bar{\mathcal{I}}_1^*, \bar{\mathcal{I}}_2^*, \bar{\mathcal{I}}_3^*, \bar{\mathcal{I}}_1^?\},$$

where $\bar{\mathcal{I}}_1^* = \{(1,2), (2,1)\}$, $\bar{\mathcal{I}}_2^* = \{(1,3), (2,3)\}$, $\bar{\mathcal{I}}_3^* = \{(2,2)\}$ and $\bar{\mathcal{I}}_1^? = \{(1,1)\}$. Now, consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly, $M \in \mathcal{P}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}, \bar{\pi})$ while it does not have full row rank. Hence, we conclude that the second condition in Theorem 5.3 is not satisfied.

Checking whether a colored pattern matrix has full row rank is in general not an easy task. Therefore, in the sequel we will develop a test for this in terms of a so-called color change rule on the graph associated with the colored pattern matrix. In order to do this, in the next section, we will consider *square* colored pattern matrices and establish graph theoretic conditions under which all matrices in the associated pattern class are nonsingular.

5.5 Conditions for nonsingularity of square colored pattern matrices

Let $\mathcal{N} \in \{0, *, ?\}^{t \times t}$ be a square pattern matrix. We define the *pattern class* of $\mathcal{N}$ as

$$\mathcal{P}(\mathcal{N}) := \{N \in \mathbb{C}^{t \times t} \mid N_{ij} = 0 \text{ if } \mathcal{N}_{ij} = 0, \quad N_{ij} \neq 0 \text{ if } \mathcal{N}_{ij} = *\}.$$

Note that here and in the sequel, in the context of pattern classes for square pattern matrices, we will allow complex matrices. Let $\pi = \{\mathcal{I}_1^*, \dots, \mathcal{I}_k^*, \mathcal{I}_1^?, \dots, \mathcal{I}_l^?\}$ be a coloring of $\mathcal{N}$. Again, two $*$-entries in the same subset I_r^* are said to have the same color, visualized by a symbol c_r, and two ?-entries in the same subset $I_s^?$ are said to have the same color, visualized by a symbol g_s. As before, $(\mathcal{N}, \pi)$ denotes the colored pattern matrix associated with $\mathcal{N}$ and π. The corresponding *pattern class* of $(\mathcal{N}, \pi)$ is given by

$$\begin{aligned}\mathcal{P}(\mathcal{N}, \pi) = \{N \in \mathcal{P}(\mathcal{N}) \mid N_{ij} = N_{mn} & \text{ if } \exists\, r \in \{1, \dots, k\} \text{ or } s \in \{1, \dots, l\} \\ & \text{ such that } (i,j), (m,n) \in \mathcal{I}_r^* \text{ or } \mathcal{I}_s^?\}.\end{aligned}$$

We say that $(\mathcal{N}, \pi)$ is *nonsingular* if all matrices in $\mathcal{P}(\mathcal{N}, \pi)$ are nonsingular. In this section, we will establish necessary and sufficient conditions for nonsingularity in terms of bipartite graphs.

We define the bipartite graph $G = (X, Y, E_{XY})$ associated with the $t \times t$ pattern matrix $\mathcal{N}$ as follows. Take as vertex sets $X = \{x_1, \dots, x_t\}$ and $Y = \{y_1, \dots, y_t\}$. An edge $\{x_i, y_j\}$ belongs to the edge set E_{XY} if $\mathcal{N}_{ji} = *$ or ?. To distinguish the edges corresponding to entries equal to ? and $*$, we introduce the two subsets

$$E_{XY}^* := \{\{x_i, y_j\} \in E_{XY} \mid \mathcal{N}_{ji} = *\}$$

and

$$E_{XY}^? := \{\{x_i, y_j\} \in E_{XY} \mid \mathcal{N}_{ji} = ?\}.$$

To visualize these different kinds of edges, we use solid lines to represent the edges in E^*_{XY} and dashed lines to represent the edges in $E^?_{XY}$. In addition, the coloring π induces a partition of the edge set E_{XY}:

$$\pi_{XY} := \{E^{*1}_{XY}, \ldots, E^{*k}_{XY}, E^{?1}_{XY}, \ldots, E^{?l}_{XY}\}$$

in which for $r = 1, \ldots, k$

$$E^{*r}_{XY} := \{\{x_i, y_j\} \in E^*_{XY} \mid (j, i) \in \mathcal{I}^*_r\},$$

and for $s = 1, \ldots, l$

$$E^{?s}_{XY} := \{\{x_i, y_j\} \in E^?_{XY} \mid (j, i) \in \mathcal{I}^?_s\}.$$

The partition π_{XY} is a coloring of the edge set E_{XY}. The edges in the same subset E^{*r}_{XY} inherit the color c_r corresponding to $\mathcal{I}^*_r$. Similarly, the edges in the subset $E^{?s}_{XY}$ inherit the color g_s corresponding to $\mathcal{I}^?_s$. Thus we define the *colored bipartite graph* associated with $(\mathcal{N}, \pi)$ as $G(\mathcal{N}, \pi) = (X, Y, E_{XY}, \pi_{XY})$.

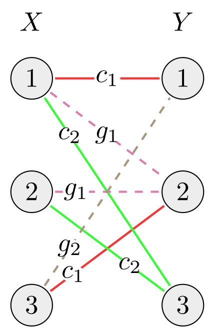


Figure 5.2: Example of a colored bipartite graph.

Example 5.4. Consider the square colored pattern matrix $(\mathcal{N}, \pi)$ with

$$\mathcal{N} = \begin{bmatrix} * & 0 & ? \\ ? & ? & * \\ * & * & 0 \end{bmatrix}, \quad \pi = \{\mathcal{I}^*_1, \mathcal{I}^*_2, \mathcal{I}^?_1, \mathcal{I}^?_2\}$$

where $\mathcal{I}^*_1 = \{(1,1), (2,3)\}$, $\mathcal{I}^*_2 = \{(3,1), (3,2)\}$, $\mathcal{I}^?_1 = \{(2,1), (2,2)\}$ and $\mathcal{I}^?_2 = \{(1,3)\}$. The associated colored bipartite graph $G(\mathcal{N}, \pi)$ is depicted in Figure 5.2.

In order to proceed, we will now review some concepts associated with perfect matchings in bipartite graphs. (See also Chapter 4.) Let p be a perfect matching in

$G(\mathcal{N}, \pi)$. The *spectrum* of p is defined as the set of colors of the edges in p. More explicitly, if the perfect matching p is given by

$$p = \{\{x_1, y_{\gamma(1)}\}, \ldots, \{x_t, y_{\gamma(t)}\}\}, \tag{5.5}$$

where γ denotes a permutation of $(1, \ldots, t)$, and

$$c_{i_1}, \ldots, c_{i_j}, g_{i_{j+1}}, \ldots, g_{i_t} \text{ with } j \leqslant t$$

are the respective colors of the edges in p, then the spectrum of p is defined as

$$\{c_{i_1}, \ldots, c_{i_j}, g_{i_{j+1}}, \ldots, g_{i_t}\},$$

where the same color can appear multiple times. We say that two perfect matchings are *equivalent* if they have the same spectrum. Obviously, this leads to a partition of the set of all perfect matchings of $G(\mathcal{N}, \pi)$ into *equivalence classes.* We denote these equivalence classes of perfect matchings by $\mathbb{P}_1, \ldots, \mathbb{P}_r$ in which perfect matchings in the same class $\mathbb{P}_i$ are equivalent. Define the *spectrum of the equivalence class* $\mathbb{P}_i$ to be the (common) spectrum of the perfect matchings in this class and denote it by $\operatorname{spec}(\mathbb{P}_i)$. Clearly, for $i \neq j$, we have $\mathbb{P}_i \cap \mathbb{P}_j = \varnothing$ and $\operatorname{spec}(\mathbb{P}_i) \neq \operatorname{spec}(\mathbb{P}_j)$. The *sign* of the perfect matching p given by (5.5) is defined as

$$\operatorname{sign}(p) := (-1)^m$$

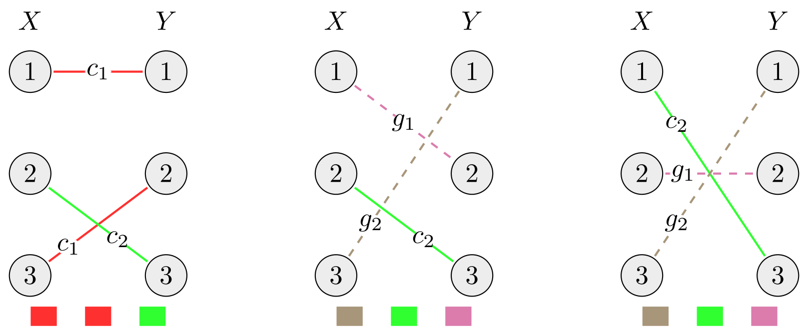
where m is the number of swaps required to permute $(1, \ldots, t)$ to $(\gamma(1), \ldots, \gamma(t))$. Finally, we define the signature of $\mathbb{P}_i$ as

$$\operatorname{sgn}(\mathbb{P}_i) := \sum_{p \in \mathbb{P}_i} \operatorname{sign}(p).$$

In other words, the signature of the equivalence class $\mathbb{P}_i$ is equal to the sum of the signs of the perfect matchings contained in $\mathbb{P}_i$. In order to illustrate the above definitions, we now give an example.

Example 5.5. Revisit the colored bipartite graph $G(\mathcal{N}, \pi) = (X, Y, E_{XY}, \pi_{XY})$ depicted in Figure 5.2. It has three perfect matchings p_1, p_2 and p_3 in $G(\mathcal{N}, \pi)$, which are depicted in Figures 5.3a, 5.3b and 5.3c, respectively. Clearly, p_2 and p_3 are equivalent. Thus, the equivalence classes are $\mathbb{P}_1 = \{p_1\}$ and $\mathbb{P}_2 = \{p_2, p_3\}$ with signature $\operatorname{sgn}(\mathbb{P}_2) = \operatorname{sign}(p_2) + \operatorname{sign}(p_3) = 0$ and $\operatorname{sgn}(\mathbb{P}_1) = -1$.

We are now ready to state a necessary and sufficient condition for a square colored pattern matrix to be nonsingular.



(a) (b) (c)

Figure 5.3: Example of perfect matchings associated with a colored bipartite graph. (a) Perfect matching p_1 with spectrum $\{c_1, c_1, c_2\}$ and $\text{sign}(p_1) = -1$. (b) Perfect matching p_2 with spectrum $\{g_2, c_2, g_1\}$ and $\text{sign}(p_2) = 1$. (c) Perfect matching p_3 with spectrum $\{g_2, c_2, g_1\}$ and $\text{sign}(p_3) = -1$.

Theorem 5.4. *Let $(\mathcal{N}, \pi)$ be a square colored pattern matrix and $G(\mathcal{N}, \pi) = (X, Y, E_{XY}, \pi_{XY})$ its associated bipartite graph. Then, $(\mathcal{N}, \pi)$ is nonsingular if and only if in $G(\mathcal{N}, \pi)$ the following three conditions hold:*

(1) There exists at least one perfect matching.

(2) Exactly one equivalence class of perfect matchings has a nonzero signature.

(3) The spectrum of this equivalence class contains only colors corresponding to edges in E^*_{XY}, i.e., solid edges.

Proof. Denote the dimension of $(\mathcal{N}, \pi)$ by t. Let $N \in \mathcal{P}(\mathcal{N}, \pi)$. From the well-known Leibniz formula for the determinant, we have

$$\det(N) = \sum_{\gamma} \left(\text{sign}(\gamma) \prod_{i=1}^{t} N_{\gamma(i)i} \right)$$

where the sum ranges over all permutations γ of $(1, \ldots, t)$, and $\text{sign}(\gamma) = (-1)^m$ with m the number of swaps necessary to permute $(1, \ldots, t)$ to $(\gamma(1), \ldots, \gamma(t))$. Clearly,

$$\prod_{i=1}^{t} N_{\gamma(i)i} \neq 0$$

only if there exists at least one perfect matching

$$p = \{\{1, \gamma(1)\}, \ldots, \{t, \gamma(t)\}\}$$

in $G(\mathcal{N}, \pi)$. Therefore, we rewrite the Leibniz formula as

$$\det(N) = \sum_{p} \left(\text{sign}(p) \prod_{i=1}^{t} N_{p(i)i} \right),$$

where p ranges over all perfect matchings in $G(\mathcal{N}, \pi)$ and $\text{sign}(p)$ is the sign of the perfect matching p (We now identify perfect matchings with their permutations). Suppose that there exist r equivalence classes of perfect matchings $\mathbb{P}_1, \ldots, \mathbb{P}_r$. Then, we have that

$$\det(N) = \sum_{\rho=1}^{r} \left(\text{sgn}(\mathbb{P}_\rho) \prod_{i=1}^{t} N_{p(i)i} \right) \tag{5.6}$$

where, for $\rho = 1, \ldots, r$, in the product appearing in the ρ-th term, the p is an arbitrary matching in $\mathbb{P}_\rho$.

We now move to prove the 'if' part. Suppose that there exists at least one perfect matching, exactly one equivalence class of perfect matchings with a nonzero signature, and the spectrum of this equivalence class contains only colors corresponding to solid edges. Without loss of generality, we denote that equivalence class by $\mathbb{P}_1$. Clearly, for every $N \in \mathcal{P}(\mathcal{N}, \pi)$, we then have

$$\det(N) = \text{sgn}(\mathbb{P}_1) \prod_{i=1}^{t} N_{p(i)i} \neq 0 \tag{5.7}$$

where $p \in \mathbb{P}_1$ is arbitrary. Since the spectrum of $p \in \mathbb{P}_1$ only contains colors associated with solid edges (whose symbols correspond to nonzero values), this implies that $(\mathcal{N}, \pi)$ is nonsingular. Thus, we have proved the 'if' part.

Next, we prove the 'only if' part. To do so, suppose that $(\mathcal{N}, \pi)$ is nonsingular and, for $G(\mathcal{N}, \pi)$, at least one of the following statements holds:

(i) There does not exist any perfect matching in $G(\mathcal{N}, \pi)$.

(ii) There does not exist an equivalence class of perfect matchings with a nonzero signature.

(iii) There exist at least two equivalence classes of perfect matchings with nonzero signature.

(iv) There exists exactly one equivalence class of perfect matchings with a nonzero signature, but its spectrum contains at least one color corresponding to a dashed edge.

Clearly, both in case (i) and (ii), it is obvious that $\det N = 0$ for all $N \in \mathcal{P}(\mathcal{N}, \pi)$. This leads to a contradiction. Consider case (iii). Without loss of generality, suppose

$\mathbb{P}_1$ and $\mathbb{P}_2$ have nonzero signature. The signatures of the remaining equivalence classes can be either zero or nonzero. Suppose $c_1, \ldots, c_k, g_1, \ldots, g_\ell$ are the colors associated with the partition π_{XY}. Introduce matrices $N \in \mathbb{C}^{t\times t}$ as follows:

$$N_{ij} := \begin{cases} a_r & \text{if } (j,i) \text{ has color } c_r \text{ for some } r, \\ a_{k+r} & \text{if } (j,i) \text{ has color } g_r \text{ for some } r, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_1, \ldots, a_{k+l}$ are independent, nonzero, variables that can take values in $\mathbb{C}$. Clearly, for all choices of the complex values $a_1, \ldots, a_{k+l}$, we have $N \in \mathcal{P}(\mathcal{N}, \pi)$. From formula (5.6) for the determinant of N, it is clear that the perfect matchings in the equivalence class $\mathbb{P}_\rho$ give a contribution

$$\operatorname{sgn}(\mathbb{P}_\rho) a_1^{j_1} a_2^{j_2} \cdots a_{k+\ell}^{j_{k+\ell}},$$

where the degrees correspond to the multiplicities of the colors of the perfect matchings in $\mathbb{P}_\rho$. By construction, we have $\operatorname{spec}(\mathbb{P}_1) \neq \operatorname{spec}(\mathbb{P}_2)$. Without loss of generality, assume that the multiplicity ϵ_1 of c_1 in $\mathbb{P}_1$ is different from that in $\mathbb{P}_2$, which is denoted by ϵ_2. Then, $\det(N)$ can be expressed as

$$\det(N) = \operatorname{sgn}(\mathbb{P}_1)\phi_1 a_1^{\epsilon_1} + \operatorname{sgn}(\mathbb{P}_2)\phi_2 a_1^{\epsilon_2} + f(a_1), \tag{5.8}$$

where ϕ_1 and ϕ_2 are determined by $a_2, \ldots, a_{k+\ell}$, and the polynomial $f(a_1)$ corresponds to the remaining equivalence classes. It may happen that some monomials in $f(a_1)$ contain a_1 with multiplicity ϵ_1 or ϵ_2. By taking common factors in (5.8), we then have

$$\operatorname{sgn}(\mathbb{P}_1)\psi_1 a_1^{\epsilon_1} + \operatorname{sgn}(\mathbb{P}_2)\psi_2 a_1^{\epsilon_2} + f'(a_1) = 0, \tag{5.9}$$

where ψ_1 and ψ_2 depend on $a_2, \ldots, a_{k+t}$. In addition, the polynomial $f'(a_1)$ does not contain the monomials with $a_1^{\epsilon_1}$ and $a_1^{\epsilon_2}$. Clearly, the variables $a_2, \ldots, a_{k+\ell}$ can be chosen such that $\psi_1 \neq 0$ and $\psi_2 \neq 0$. By the fundamental theorem of algebra, we conclude that the polynomial equation (5.9) has at least one nonzero complex solution. This implies that for some choice of nonzero complex values $a_1, \ldots, a_{k+\ell}$, we have that $\det N = 0$, and hence we reach a contradiction.

Finally, consider the case (iv). Suppose that exact one equivalence class of perfect matchings has a nonzero signature, and its spectrum contains at least one color corresponding to some dashed edge. Let $p = \{\{1, \gamma(1)\}, \ldots, \{t, \gamma(t)\}\}$ be a perfect matching in $\mathbb{P}_1$, where γ denotes a permutation on $(1, \ldots, t)$. Without loss of generality, assume that the edge $\{1, \gamma(1)\}$ is a dashed edge with color g_r for some r. The remaining edges in p can be either solid or dashed. This implies that

$$\det(N) = \operatorname{sgn}(\mathbb{P}_1) \prod_{i=1}^{t} N_{\gamma(i)i} = \operatorname{sgn}(\mathbb{P}_1) \cdot g_r \cdot \prod_{i=2}^{t} N_{\gamma(i)i},$$

where g_r represents an arbitrary complex value. It is obvious that $\det(N) = 0$ if g_r is chosen as zero. Again, we reach a contradiction. This completes the proof. ■

Theorem 5.4 is a generalization of Theorem 4.6 which provides a necessary and sufficient condition for the special case of square colored pattern matrices only containing 0 and $*$-entries.

Example 5.6. Reconsider the square colored pattern matrix $(\mathcal{N}, \pi)$ given in Example 5.4 and its associated graph $G(\mathcal{N}, \pi)$ depicted in Figure 5.2. In Example 5.5, it has already been shown that $G(\mathcal{N}, \pi)$ admits exactly one equivalence class, $\mathbb{P}_1 = \{p_1\}$, with nonzero signature. Moreover, $\text{spec}(\mathbb{P}_1) = \{c_1, c_1, c_2\}$, which only contains colors associated with solid edges. Therefore, by Theorem 5.4, we conclude that $(\mathcal{N}, \pi)$ is nonsingular.

5.6 Color change rule and graph theoretic conditions

In this section, we will establish a graph theoretic test for checking whether a colored pattern matrix has full row rank. This test will be in terms of a so-called color change rule on the associated graph. Color change rules for checking the rank of a pattern matrix have been studied before, see e.g., [97],[30],[32],[58], [57]. Here, we will start off with introducing a new color change rule tailored for our purpose.

Let $(\mathcal{M}, \pi)$ be the colored pattern matrix associated with $\mathcal{M} \in \{0, *, ?\}^{p\times q}$ $(p \leqslant q)$ and $\pi = \{\mathcal{I}_1^*, \ldots, \mathcal{I}_k^*, \mathcal{I}_1^?, \ldots, \mathcal{I}_\ell^?\}$. Define a directed graph $\mathcal{G}(\mathcal{M}, \pi) = (V, E)$ associated with $(\mathcal{M}, \pi)$ as follows. Take the vertex set V equal to $\{1, \ldots, q\}$. Define the edge set $E \subseteq V \times V$ as

$$E := \{(i, j) \mid \mathcal{M}_{ji} = * \text{ or } \mathcal{M}_{ji} =?\}.$$

The coloring π gives the following partition of the edge set E:

$$\pi_E := \{E_1^*, \ldots, E_k^*, E_1^?, \ldots, E_\ell^?\}$$

in which for $r = 1, \ldots, k$

$$E_r^* := \{(i, j) \in E \mid (j, i) \in \mathcal{I}_r^*\},$$

and for $s = 1, \ldots, \ell$

$$E_s^? := \{(i, j) \in E \mid (j, i) \in \mathcal{I}_s^?\}.$$

We call the partition π_E a coloring of the edge set E. To visualize the coloring π_E, for $r = 1, \ldots, k$ we represent the edges in E_r^* by solid arrows with color c_r (inherited from $\mathcal{I}_r^*$). For $s = 1, \ldots, \ell$ we represent the edges in $E_s^?$ by dashed arrows with color g_s

(inherited from $\mathcal{I}_s^?$). Thus, we obtain a colored graph $\mathcal{G}(\mathcal{M},\pi) = (V, E, \pi_E)$ associated with $(\mathcal{M},\pi)$. Colored graphs were studied before in Chapter 4. In order to illustrate the above, we provide an example.

Example 5.7. Consider the colored pattern matrix $(\mathcal{M},\pi)$ of Example 5.1. The associated graph $\mathcal{G}(\mathcal{M},\pi)$ is depicted in Figure 5.4.

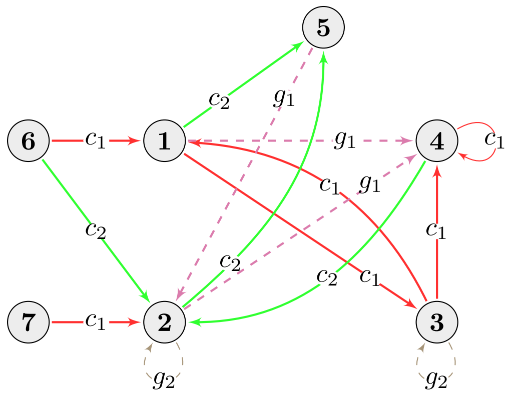


Figure 5.4: Example of a graph associated with a given colored pattern matrix.

We will now introduce a color change rule for $\mathcal{G}(\mathcal{M},\pi)$. In this graph, initially all vertices are colored "white." The color change rule will prescribe under what conditions vertices will change their color to "black." In earlier work, color change rules usually deal with conditions under which a *single* vertex colors a *single* white neighboring vertex black. (See [30], [32], [58] and the references therein.) In the present chapter we will deal with *sets* of vertices that color *sets* of vertices black. (See also [57].) We will now describe this rule. Let X and Y be two nonempty subsets of the vertex set V, containing the same number of vertices, i.e., $|X| = |Y|$. Define an associated colored bipartite graph $G(\pi) = (X, Y, E_{XY}, \pi_{XY})$ as follows:

$$E_{XY} := \{\{x_i, y_j\} \mid x_i \in X, y_j \in Y, (x_i, y_j) \in E\}.$$

Obviously, the partition π_E of E induces a partition

$$\pi_{XY} = \{E_{XY}^{*1}, \ldots, E_{XY}^{*k}, E_{XY}^{?1}, \ldots, E_{XY}^{?\ell}\}$$

of E_{XY} by defining for $r = 1, \ldots, k$

$$E_{XY}^{*r} := \{\{x_i, y_j\} \in E_{XY} \mid (x_i, y_j) \in E_r^*\},$$

and for $s = 1, \ldots, \ell$

$$E_{XY}^{?s} := \{\{x_i, y_j\} \in E_{XY} \mid (x_i, y_j) \in E_s^?\}.$$

Note that some of these sets might be empty. Removing all the empty sets, we then obtain a partition

$$\pi_{XY} = \{E_{XY}^{*i_1}, \ldots, E_{XY}^{*i_w}, E_{XY}^{?j_1}, \ldots, E_{XY}^{?j_v}\}$$

of E_{XY} with $w \leqslant k$ and $v \leqslant \ell$. The edges in $E_{XY}^{*i_r}$ have color c_{i_r}, and the edges in $E_{XY}^{?j_r}$ have color g_{j_r}. Without loss of generality, we renumber $c_{i_1}, \ldots, c_{i_w}$ as $c_1, \ldots, c_w$ and $g_{j_1}, \ldots, g_{j_v}$ as $g_1, \ldots, g_v$.

Next, return to the colored graph $\mathcal{G}(\mathcal{M}, \pi) = (V, E)$. Suppose that all vertices in V are colored either black or white. Take two nonempty subsets X and Y of the vertex set V. We say that Y is a *color-perfect white neighbor* of X if:

- Y and X contain the same number of vertices, i.e., $|Y| = |X|$;
- Y is equal to the set of white out-neighbors of X, i.e.,

$$Y = \{y_j \in V \mid y_j \text{ is white and } (x_i, y_j) \in E \text{ for some } x_i \in X\};$$

- in the associated bipartite graph $G(X, Y, E_{XY}, \pi_{XY})$, there exists a perfect matching, exactly one equivalence class of perfect matchings has a nonzero signature, and the spectrum of this equivalence class only contains colors corresponding to edges in E_{XY}^*, i.e., solid edges in $G(X, Y, E_{XY}, \pi_{XY})$.

Based on the notion of color-perfect white neighbor, we now introduce a *color change procedure* as follows.

1 Initially, all vertices in V are colored white.

2 If there exist two vertex sets $Y \subseteq \{1, \ldots, p\}$ and $X \subseteq \{1, \ldots, q\}$ such that Y is a color-perfect white neighbor of X, then change the color of all vertices in Y to black.

3 Repeat step 2 until no further color changes are possible.

We define a *derived set* $\mathcal{D}$ as a set of black vertices in V obtained by following the procedure above. Note that derived sets are not unique but may depend on the subsequent choices of Y and X in the second step above. An illustrative example can be found in Example 4.4. The graph $\mathcal{G}(\mathcal{M}, \pi)$ is called *colorable* if there exists a derived set $\mathcal{D}$ such that $\mathcal{D} = \{1, \ldots, p\}$. Of course, the remaining vertices $\{p+1, \ldots, q\}$ can never be colored black, since they have no incoming edges.

Example 5.8. Consider $(\mathcal{M}, \pi)$ given by (5.2) and its associated graph $\mathcal{G}(\mathcal{M}, \pi)$ depicted in Figure 5.4. Initially, color all vertices white. First, let $X = \{6, 7\}$ and $Y = \{1, 2\}$. It turns out that Y is a color-perfect white neighbor of X. Indeed, in the associated colored bipartite graph $G = (X, Y, E_{XY}, \pi_{XY})$ depicted in Figure 5.1, there exists exactly one equivalence class $\mathbb{P}_1 = \{p_1\}$ with nonzero signature and $\operatorname{spec}(\mathbb{P}_1) = \{c_1, c_1\}$. Consequently, we change the color of vertices 1 and 2 to black. Next, let $X' = \{1, 2, 3\}$ and $Y' = \{3, 4, 5\}$. Then Y' is a color-perfect white neighbor of X'. Indeed, the associated colored bipartite graph $G = (X', Y', E_{X'Y'}, \pi_{X'Y'})$ is depicted in Figure 5.2. In Example 5.5, we have shown that in the bipartite graph $G = (X', Y', E_{X'Y'}, \pi_{X'Y'})$ there exists exactly one equivalence class of perfect matchings with nonzero signature and the spectrum of this equivalence class contains only colors corresponding to solid edges. Therefore, vertices $3, 4, 5$ are colored black, and we conclude that $\mathcal{G}(\mathcal{M}, \pi)$ is colorable.

We now arrive at the main result of this section.

Theorem 5.5. *Let $(\mathcal{M}, \pi)$ be the colored pattern matrix associated with $\mathcal{M} \in \{0, *, ?\}^{p\times q}$ $(p \leqslant q)$ and*

$$\pi = \{\mathcal{I}_1^*, \ldots, \mathcal{I}_k^*, \mathcal{I}_1^?, \ldots, \mathcal{I}_\ell^?\}.$$

Then, $(\mathcal{M}, \pi)$ has full row rank if its associated graph $\mathcal{G}(\mathcal{M}, \pi)$ is colorable.

In order to prove this theorem, we need the following instrumental result.

Lemma 5.6. *Let $(\mathcal{M}, \pi)$ be a colored pattern matrix and $\mathcal{G}(\mathcal{M}, \pi)$ its associated colored graph. Suppose that the vertices $1, \ldots, p$ are black or white, and those in $p+1, \ldots, q$ are white. Define the diagonal matrix $D \in \mathbb{R}^{p\times p}$ by*

$$D_{ii} := \begin{cases} 1 & \text{if the vertex } i \text{ is black,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose further that $Y = \{y_1, \ldots, y_r\} \subseteq \{1, \ldots, p\}$ is a color-perfect white neighbor of $X = \{x_1, \ldots, x_r\} \subseteq \{1, \ldots, q\}$. Define the diagonal matrix $\Delta \in \mathbb{R}^{p\times p}$ by

$$\Delta := \sum_{i=1}^{r} e_{y_i} e_{y_i}^\top,$$

where e_{y_i} denotes the y_i-th column of the $p \times p$ identity matrix I. Then for every $M \in \mathcal{P}(\mathcal{M}, \pi)$ we have that $\begin{bmatrix} M & D \end{bmatrix}$ has full row rank if and only if $\begin{bmatrix} M & D + \Delta \end{bmatrix}$ has full row rank.

Proof. The 'only if' part is trivial and is hence omitted. To prove the 'if' part, suppose that, for all $M \in \mathcal{P}(\mathcal{M}, \pi)$ the matrix $\begin{bmatrix} M & D + \Delta \end{bmatrix}$ has full row rank. Let $z \in \mathbb{R}^p$ be such that $z^\top \begin{bmatrix} M & D \end{bmatrix} = 0$. In the sequel, for a given vector $z \in \mathbb{R}^p$ and a given index set $\alpha = \{\alpha_1, \ldots, \alpha_r\} \subseteq \{1, \ldots, p\}$ we define the vector

$$z_\alpha := (z_{\alpha_1}, \ldots, z_{\alpha_r})^\top.$$

Analogously, for a given matrix $M \in \mathbb{R}^{p \times q}$ and two given index sets

$$\alpha = \{\alpha_1, \ldots, \alpha_r\} \subseteq \{1, \ldots, p\} \quad \text{and} \quad \beta = \{\beta_1, \ldots, \beta_s\} \subseteq \{1, \ldots, q\}$$

we define the matrix $M_{\alpha\beta}$ by $(M_{\alpha\beta})_{ij} := M_{\alpha_i \beta_j}$. In what follows, we aim to show that $z_Y = 0$. Indeed, if $z_Y = 0$ then

$$z^\top \begin{bmatrix} M & D + \Delta \end{bmatrix} = z^\top \begin{bmatrix} M & D \end{bmatrix} = 0,$$

which would prove that $z = 0$. So, $\begin{bmatrix} M & D \end{bmatrix}$ has full row rank. To show that, indeed, $z_Y = 0$, let α be the set of black vertices. Clearly, it holds that

$$\alpha \subseteq \{1, \ldots, p\} \quad \text{and} \quad \alpha \cap Y = \varnothing.$$

Since $z^\top \begin{bmatrix} M & D \end{bmatrix} = 0$, we then obtain

$$z_Y^\top M_{YX} + z_\alpha^\top M_{\alpha X} + z_\beta^\top M_{\beta X} = 0 \quad \text{and} \quad z_\alpha = 0,$$

where $\beta = \{1, \ldots, p\} \setminus (Y \cup \alpha)$. Since Y is a color-perfect white neighbor of X, by Theorem 5.4 we have that M_{YX} is nonsingular and $M_{\beta X} = 0$. This implies that z_Y must be equal to 0. This completes the proof. ∎

We are now ready to prove Theorem 5.5.

Proof of Theorem 5.5. Suppose that $\mathcal{G}(\mathcal{M}, \pi)$ is colorable. Let $M \in \mathcal{P}(\mathcal{M}, \pi)$. By repeatedly applying Lemma 5.6, it follows that M has full row rank if and only if $\begin{bmatrix} M & I \end{bmatrix}$ has full row rank, which is obviously true. Therefore, we conclude that M has full row rank, which completes the proof. ∎

To show that the condition in Theorem 5.5 is not a necessary condition, we provide the following counterexample.

Example 5.9. Consider the colored pattern matrix $(\mathcal{M}, \pi)$ with

$$\mathcal{M} = \begin{bmatrix} * & * & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & * \end{bmatrix}, \quad \pi = \{\mathcal{I}_1^*, \mathcal{I}_2^*, \mathcal{I}_3^*\}$$

where $\mathcal{I}_1^* = \{(1,1),(3,1)\}$, $\mathcal{I}_2^* = \{(1,2),(2,2),(2,4),(3,3)\}$ and $\mathcal{I}_3^* = \{(1,3),(3,4)\}$. It will turn out that the associated graph $\mathcal{G}(\mathcal{M},\pi)$ depicted in Figure 5.5 is not colorable, while $(\mathcal{M},\pi)$ has full row rank. Indeed, one can verify that none of the subsets of $\{1,2,3,4\}$ has a color-perfect white neighbor. Hence, the graph $\mathcal{G}(\mathcal{M},\pi)$ is not colorable. However, all matrices of the form

$$\begin{bmatrix} c_1 & c_2 & c_3 & 0 \\ 0 & c_2 & 0 & c_2 \\ c_1 & 0 & c_2 & c_3 \end{bmatrix} \quad \text{with } c_i \neq 0 \text{ for } i = 1,2,3$$

have full row rank. Indeed, by taking

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

we obtain

$$PMQ = \begin{bmatrix} c_1 & c_2 & c_3 & 0 \\ 0 & c_2 & 0 & c_2 \\ 0 & 0 & 2c_2 & c_2 + c_3 \end{bmatrix},$$

which clearly has full row rank for all choices of $c_i \neq 0$. This provides a counterexample as claimed.

Finally, based on the rank tests in Theorem 5.3 and the result in Theorem 5.5, we obtain the following sufficient graph theoretic condition for controllability of colored structured systems.

Theorem 5.7. *Consider the colored structured system $(\mathcal{A},\mathcal{B},\pi)$. Let $(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$ be the colored pattern matrix associated with $(\mathcal{A},\mathcal{B},\pi)$ given by Definition 5.2. Then, $(\mathcal{A},\mathcal{B},\pi)$ is controllable if both graphs $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix},\pi)$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$ are colorable.*

To conclude this section, we illustrate the above theorem by an example.

Example 5.10. Consider the colored structured system $(\mathcal{A},\mathcal{B},\pi)$ given in Example 5.2. Denote by $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix},\pi)$ and $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$ the colored graphs associated with $(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix},\pi)$ and $(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$. In Example 5.8, we have already shown that $\mathcal{G}(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix},\pi)$ depicted in Figure 5.4 is colorable. It remains to show that the graph $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$, depicted in Figure 5.6, is also colorable. Clearly, the set $\{1,2\}$ is a color-perfect white neighbor of $\{6,7\}$. Hence, we color vertices 1 and 2 black. Subsequently, $\{3,4,5\}$ is a color-perfect white neighbor of $\{1,2,3\}$. This means that the vertices $3,4$ and 5 are also colored black. Therefore, we find that $\mathcal{G}(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix},\bar{\pi})$ is colorable. By Theorem 5.7, we conclude that $(\mathcal{A},\mathcal{B},\pi)$ is controllable.

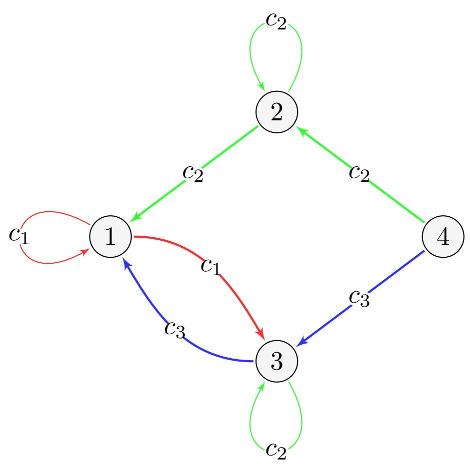


Figure 5.5: Example to show that the conditions in Theorem 5.5 are not a necessary condition.

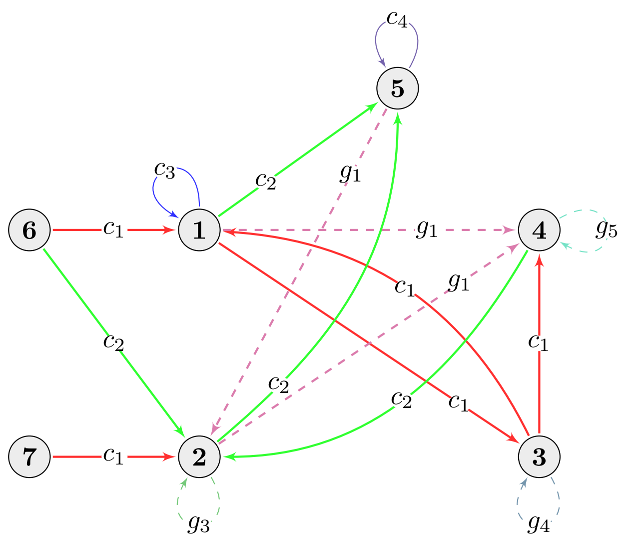


Figure 5.6: Example of a graph associated with a given $(\begin{bmatrix}\bar{\mathcal{A}} & \mathcal{B}\end{bmatrix}, \bar{\pi})$.

Remark 5.3. Theorem 5.7 can of course be applied to the special case described in Remark 5.1. Indeed, if the colored system $(\mathcal{A}, \mathcal{B}, \pi)$ satisfies the conditions 1 to 4 in Remark 5.1, then it is easily verified that $\bar{\mathcal{A}} = \mathcal{A}$ and the new coloring $\bar{\pi}$ coincides with the original coloring π. Thus we find that $(\mathcal{A}, \mathcal{B}, \pi)$ is controllable if the single colored pattern matrix $(\begin{bmatrix}\mathcal{A} & \mathcal{B}\end{bmatrix}, \pi)$ is colorable.

5.7 Conclusions

In this chapter, we have studied strong structural controllability of linear structured systems in which the structure of the system matrices is assumed to be given by zero/nonzero/arbitrary pattern matrices. In contrast to the work in Chapter 2 in which the nonzero and arbitrary entries of the system matrices are completely independent, in the present chapter we have studied the more general case that certain equality constraints among these entries are given, in the sense that a *priori* given entries in the system matrices are restricted to take arbitrary but identical values. We have formalized this general class of structured systems by introducing the concepts of colored pattern matrices and colored structured systems. In this setup, we have established sufficient algebraic conditions for strong structural controllability of a given colored structured system. These conditions are in terms of a rank test on two colored pattern matrices associated with this system. We have shown that these conditions are not necessary by providing an example in which a colored structured system is controllable while the conditions are not satisfied. Next, we established a necessary and sufficient graph theoretic condition for the nonsingularity of a given square colored pattern matrix. Based on the above condition, we obtained a graph theoretic condition for a given colored pattern matrix to have full row rank, which involves a new color change rule and the concept of colorability of the graph associated with this pattern matrix. Finally, we have established sufficient graph theoretic conditions under which a given colored structured system is strongly structurally controllable.

6 Conclusions and Outlook

In this thesis, we have considered structural analysis of control of complex and networked systems. More explicitly, this thesis focuses on the analysis of so-called linear structured systems, which, as argued in Chapter 1, are very important in understanding properties of complex and networked systems related to control design such as, for example, controllability and observability. In this chapter, we summarize the main contributions and results that have been obtained in this thesis. Finally, we provide an outlook on some potential research directions.

6.1 Conclusions

The past few decades have witnessed the emergence of complex and networked systems in many fields, from natural to social science and from economy to engineering. Although it would be challenging to understand and control complex systems fully, the analysis and control of such systems can be partially realized only after applying some reasonable simplifications. In particular, for the analysis of certain control properties, such as controllability, a complex system can be simplified to a linear structured system capturing an essential part of the structural information in that system, such as the existence or absence of relations between components of the system. This thesis has studied the effect of the interconnection structure of complex systems on their control properties following a structural analysis approach. More explicitly, we have analyzed strong structural properties of complex systems. The main contributions have been split into two parts corresponding to Problem 1.1 and Problem 1.2 as formulated in Chapter 1.

In Part I, we have introduced a new framework for linear structured systems in which the relations between the components of the systems are allowed to be unknown. This kind of systems has been formalized in terms of pattern matrices whose entries are either fixed zero, arbitrary nonzero, or arbitrary. Then, on the one hand, in Chapter 2, we have dealt with strong structural controllability of this kind of linear structured systems. More explicitly, Chapter 2 has first provided necessary and sufficient algebraic conditions under which a given structured system is controllable. Secondly, in Chapter 2, we have established a necessary and sufficient graph theoretic condition for the full rank property of a given pattern matrix, which yielded to a necessary and sufficient graph theoretic condition for strong structural controllability. On the other hand, in Chapter 3, we have studied the solvability of the FDI problem for linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices. We have provided both algebraic and graph theoretic conditions for solvability of the FDI problem.

In Part II, we have introduced a new framework for structured systems in which a priori given entries in the system matrices are restricted to take arbitrary but identical values. As initial work, Chapter 4 considered strong structural controllability of systems defined on colored graphs, which can be regarded as a special kind of linear structured systems. In Chapter 4, several sufficient graph theoretic conditions were established under which the systems are strongly structurally controllable. Generalizing the results in Chapter 2 and Chapter 4, Chapter 5 has studied the controllability of so-called colored linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices and where *a priori* given entries in the system matrices are restricted to take arbitrary but identical values. Finally, in Chapter 5, we have provided both algebraic and graph theoretic conditions for strong structural controllability of this more general class of structured systems.

6.2 Outlook

In this outlook, we will suggest some future research problems concerning the analysis of strong structural properties of complex systems. These suggestions for future research can be divided into three directions.

A first possible direction is to make a further analysis within the framework proposed in this thesis. For instance, in Chapter 3, we have provided a necessary condition as well as a sufficient criterion for solvability of the FDI problem, so to find both necessary *and* sufficient conditions is still a possible future research problem. Similarly, in Chapter 4 and Chapter 5, for controllability of colored linear structured systems, we have provided certain sufficient but not necessary conditions, and therefore finding conditions that are both necessary and sufficient is still an open problem. Also, in this thesis, we have established theoretical conditions under which a given structured system is controllable or the FDI problem is solvable. Algorithms to verify these conditions still need to be studied further. Lastly, analyzing other system properties, such as targeted controllability [62, 75], and identifiability [83] within the framework of this thesis is also a possible future research problem.

As a second direction, this thesis has focused on zero/nonzero structures. The advantages of this type of pattern are apparent. It allows us, for example, to capture an essential part of the structural information of complex systems, and conditions can be expressed in graph theoretic terms. However, it has some unavoidable drawbacks. For example, only having zero/nonzero information on the entries of the system matrices does not provide any information on, for example, the signs of these entries, or the location of eigenvalues. Thus certain significant system properties, such as stability [33], cannot be analyzed. Therefore, introducing some new framework for linear structured systems that allows analyzing properties like stability is yet another open problem.

Finally, in this thesis and other related literature [17, 29, 30, 32], after modeling a complex system as a complex network, in order to zoom in on the role of the network topology the dynamics of the components of the complex systems are simplified as much as possible. Indeed, the components are in general modeled as single integrators. However, in many scenarios, the dynamics of these subsystems can be much more complicated and will affect the system properties dramatically, see, e.g., [12, 68, 100] and the references therein. For example, if the network nodes contain self-loops, strong structural controllability properties of the network will change significantly. Besides, the impact of higher-dimensional subsystem dynamics on the controllability of complex systems will be more difficult to analyze. Therefore, for future research, setting up a new structured system framework that allows more complicated subsystem dynamics is an important open problem.

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Summary

This thesis studies structural properties of complex and networked systems. By a high level of abstraction, this kind of systems can be regarded as standard linear time-invariant (LTI) systems of very high dimensions. However, in many cases, it is impossible to obtain the exact values of all entries in the system matrices of these LTI systems. To deal with this problem, instead of particular LTI systems, this thesis focuses on so-called structured systems which consist of a family of LTI systems in which the system matrices contain entries that depend on the existence or the absence of information about the interconnections between the system components. We will now give a summary of the contributions and results discussed in this thesis. These can be divided into two parts.

Firstly, note that in many scenarios, complete information on the existence or absence of interconnections between components is unavailable. By that, we mean that for some components we do not know whether there exist interconnections between them or not. To deal with this problem, we introduce a new framework of structured systems in which the system matrices can contain three kind of entries: zero, arbitrary nonzero, and arbitrary. In this framework, the 'zero' entry in the system matrices represents that the interconnection between the corresponding system components is absent. On the other hand, the 'arbitrary nonzero' entry represents that the interconnection between the corresponding system components must exist. Finally, the 'arbitrary' entry means that it is uncertain whether the interconnection between the corresponding system components exists. In this thesis, for this new kind of structured systems, we analyze two control properties, namely controllability and solvability of the fault detection and isolation (FDI) problem. With respect to controllability, we first establish algebraic necessary and sufficient conditions under which a given structured system is strongly structurally controllable. As another contribution, we provide a necessary and sufficient graph theoretic condition for the full rank property of a given pattern matrix. This condition is expressed in terms of a so-called color change rule. Based on these results, we obtain a necessary and sufficient graph theoretic condition for strong structural controllability. With respect

to solvability of the FDI problem, we first provide a necessary condition for solvability of the FDI problem for linear structured systems. Assuming that this necessary condition holds, we then develop a sufficient algebraic condition for solvability of the FDI problem in terms of a rank test on an associated pattern matrix. Recalling the graph theoretic condition for the full rank property of a given pattern matrix, we then obtain a graph theoretic condition for solvability of the FDI problem.

Secondly, in many realistic complex and networked systems, the strength of the interconnections between some components might have dependencies rather than being independent of each other. In particular, some interconnection links might be constrained to take identical weights either by symmetry considerations or by the physics of the underlying problem. Motivated by this observation, we introduce another kind of structured system, in which a priori given entries in the system matrices are restricted to take arbitrary but identical values. To begin with, we first consider strong structural controllability of a special case of the systems mentioned above, which are called *systems defined on colored graphs*. In this kind of systems, the system matrices are zero/nonzero/arbitrary pattern matrices in which all entries on the diagonal are arbitrary, while the remaining entries can only be zero or arbitrary nonzero. In addition, we consider the situation that prespecified arbitrary nonzero entries in the system's pattern matrix are constrained to take identical (nonzero) values. Then, we establish graph theoretic conditions for strong structural controllability of this kind of systems. Finally, we deal with strong structural controllability of linear structured systems in which the system matrices are given by zero/nonzero/arbitrary pattern matrices while, in addition, a priori given entries of these matrices are restricted to take arbitrary but identical values. To formalize this general class of structured systems, we introduce the concepts of colored pattern matrices and colored structured systems. Based on these concepts, we provide both algebraic and graph theoretic conditions for strong structural controllability of this more general class of structured systems.

Samenvatting

In dit proefschrift worden structurele eigenschappen van complexe systemen en netwerksystemen onderzocht. In een hoog abstractieniveau kunnen dit soort systemen worden beschouwd als standaard LTI-systemen met erg hoge dimensies. In veel gevallen is het echter onmogelijk om de exacte waarden van alle elementen van de systeemmatrices van deze LTI-systemen te achterhalen. Om dit probleem te kunnen oplossen, focussen wij in dit proefschrift niet op LTI-systemen an sich, maar op zogenaamde gestructureerde systemen die bestaan uit een familie van LTI-systemen, waarin de systeemmatrices elementen bevatten die afhankelijk zijn van de aan- of afwezigheid van informatie over de verbindingen tussen de componenten van het systeem. We zullen nu een samenvatting geven van de bijdragen en resultaten in dit proefschrift. Deze kunnen worden onderscheiden in twee delen.

In de eerste plaats is het goed om op te merken dat in veel scenario's complete informatie over de aan- of afwezigheid van verbindingen tussen componenten niet beschikbaar is. Daarmee bedoelen we dat we voor sommige componenten niet weten of er verbindingen tussen bestaan of niet. Om dit probleem te kunnen aanpakken, introduceren we een nieuw framework van gestructureerde systemen waarin de systeemmatrices drie soorten elementen kunnen bevatten: nul, willekeurig niet-nul, en willekeurig. Binnen dit framework betekent het 'nul'-element dat er geen verbindingen tussen de corresponderende systeemcomponenten zijn. Het 'willekeurig niet-nul'-element betekent dat er een verbinding tussen de corresponderende systeemcomponenten moet bestaan. Ten slotte betekent het 'willekeurige' element dat het onzeker is of er verbindingen bestaan tussen de corresponderende systeemcomponenten. In dit proefschrift analyseren we voor deze nieuwe soort gestructureerde systemen twee systeemeigenschappen, namelijk regelbaarheid en oplosbaarheid van het FDI-probleem. Met betrekking tot regelbaarheid stellen we eerst algebraïsche noodzakelijkheid en voldoende voorwaarden vast waaronder een gegeven gestructureerd systeem sterk structureel regelbaar is. Een andere bijdrage is dat we een noodzakelijke en voldoende graaftheoretische voorwaarde bieden voor de volle rang-eigenschap van een gegeven pattern matrix. Deze voorwaarde is uitgedrukt in termen van een zogenaamde kleurveranderregel. Op basis van deze resultaten kunnen we een noodzakelijke en voldoende graaftheoretische voorwaarde voor sterke structurele regelbaarheid vinden. Met betrekking tot oplosbaarheid van het FDI-probleem bieden we in de eerste plaats een noodzakelijke voorwaarde voor de oplosbaarheid van het FDI-probleem voor lineair gestructureerde systemen. Aangenomen dat deze noodzakelijke voorwaarde standhoudt, ontwikkelen we vervolgens een voldoende algebraïsche voorwaarde voor oplosbaarheid van het FDI-probleem in termen van een rangtest van een geassocieerde pattern matrix. Teruggrijpend op de graaftheoretische voorwaarde voor de volle rang-eigenschap van een gegeven pattern matrix, kunnen we vervolgens een graaftheoretische conditie voor oplosbaarheid van het FDI-probleem vinden.

In de tweede plaats kunnen er in veel realistische complexe systemen en netwerksystemen afhankelijkheden bestaan tussen de sterktes van interconnecties tussen componenten. In het bijzonder zouden bepaalde verbindingen identieke gewichten kunnen hebben, ofwel vanwege symmetrieën, ofwel door de fysica van het onderliggende probleem. Door deze observatie gemotiveerd, introduceren we een ander soort gestructureerd systeem, waarin a priori gegeven elementen in de systeemmatrices beperkt zijn tot het aannemen van willekeurige, maar identieke waarden. Om te beginnen onderzoeken we sterk gestructureerde regelbaarheid van een specifiek geval van de hierboven genoemde systemen, namelijk systemen die gedefinieerd zijn door gekleurde grafen. In dit soort systemen zijn de systeemmatrices nul/niet-nul/willekeurige pattern matrices waarin alle elementen op de diagonaal willekeurig zijn, terwijl de overige elementen alleen nul of willekeurig niet-nul kunnen zijn. Daarnaast onderzoeken we de situatie waarin van tevoren gespecificeerde willekeurige niet-nul elementen in de pattern matrix van het systeem beperkt zijn tot het aannemen van identieke (niet-nul) waarden. Vervolgens stellen we graaftheoretische voorwaarden vast voor sterke gestructureerde regelbaarheid van dit soort systemen. Ten slotte behandelen we sterke gestructureerde regelbaarheid van lineair gestructureerde systemen waarin de systeemmatrices gegeven zijn door nul/niet-nul/willekeurige pattern matrices terwijl daarnaast a priori gegeven elementen van deze matrices beperkt zijn tot het aannemen van willekeurige, maar identieke waarden. Om deze algemene klasse gestructureerde systemen te formaliseren introduceren we de concepten van gekleurde pattern matrices en gekleurde gestructureerde systemen. Op basis van deze concepten bieden we zowel algebraïsche als graaftheoretische voorwaarden voor sterke gestructureerde regelbaarheid van deze algemenere klasse gestructureerde systemen.