Classification of symmetry breaking patterns in the theory of non-linear realizations

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Classification of symmetry breaking patterns in the theory of non-linear realizations

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List of Publications

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- D. Roest, D. Stefanyszyn and P. Werkman, "An Algebraic Classification of Exceptional EFTs," JHEP **08** (2019), 081 [arXiv:1903.08222 [hep-th]].
- D. Roest, D. Stefanyszyn and P. Werkman, "An Algebraic Classification of Exceptional EFTs Part II: Supersymmetry," JHEP **11** (2019), 077 [arXiv:1905.05872 [hep-th]].

Chapter 1

Introduction

Symmetries and the principle of relativity

The concept of symmetry has been crucial to the study of physics throughout the centuries. We may distinguish the symmetries of objects in the physical world from the symmetries that exist in the laws of nature. An object has a symmetry when a transformation exists that leaves it invariant. For example, we may rotate a sphere in any direction without changing its properties. A symmetry in the natural laws is a different kind of transformation. It changes the positions, velocities, etc. of all objects in the world simultaneously, thereby acting on a *history* of the world, represented in equations by dynamical variables and coordinates. Such a transformation is a symmetry if the newly obtained history abides by the same natural laws as the original.

In 1632, Galileo put forward the *principle of relativity*, which states that the laws of nature are the same for all inertial observers, or inertial *frames of reference*. Equivalently, the principle of relativity states that the laws of nature enjoy a symmetry that takes one inertial frame of reference into another. In an inertial frame of reference, an object with no force acting on it will move at a constant velocity. Thus, Galileo argued, any experiment we perform on board of a ship will yield the same results whether the ship is anchored at port or sailing at constant velocity. The principle of relativity was a central part of Galileo's argument for a heliocentric model of the solar system.

Newton and Leibnitz also understood the power and importance of symmetries. They both put forward a conserved *quantity of motion*. Leibnitz advocated conservation of kinetic energy, defined as $K = \frac{1}{2}\sum_i m_i v_i^2$ where i sums over the constituents of the system. Newton preferred conservation of momentum, the vector quantity $\sum_i m_i \vec{v}_i$. We now understand that both momentum and energy are conserved quantities. The fact that they are conserved is a consequence of translation symmetries in the laws of nature. Conservation of energy and momentum follows from the observation that the same laws of physics apply at all times and at all points in space, respectively. The general correspondence between conserved quantities and symmetries in the laws of nature is known as *Noether's first theorem.* [35]

In 1687, Newton published his three laws of motion, which together make up the formalism of classical mechanics. Newton's first law defines inertial frames of reference as those in which objects maintain their velocity unless a force acts upon them. Newton's second and third laws take the same form in any inertial frame of reference. Therefore, classical mechanics respects the principle of relativity. The transformations that take one inertial frame into another are known as *Galilean transformations.* Newton's laws of motion operate in a fixed background of space, which itself has no dynamical properties. Inertial observers may be rotated, moving, or translated with respect to each other, but all agree on the relative positions of objects. In addition, Newton postulated the existence of a universal notion of time, shared by all observers.

The discovery of the laws of electromagnetism, finalized by Maxwell in 1865, led to many new insights into the role of symmetries in physics. At first glance, Maxwell's equations seemed to violate the principle of relativity. For example, they admit vacuum wave solutions which propagate at a speed c defined by the electric and magnetic constants ϵ_0 and μ_0, $c = 1/\sqrt{\epsilon_0\mu_0}$. There are no free-space wave solutions that propagate at a different velocity. This observation is impossible to reconcile with the principle of relativity if inertial observers are defined by the familiar Galilean transformations. In actual fact, Maxwell's equations enjoy a group of symmetries - called the *Lorentz transformations* - that transforms electric and magnetic fields into each other. At the same time, they dilate the time between events and contract the spatial coordinate along the direction of a boost, in such a way that the velocity of electromagnetic waves is preserved. Accordingly, these transformations look nothing like Galilean boosts.

In 1905, Einstein gave the correct interpretation of the Lorentz symmetry in Maxwell's equations. By making the minimal assumptions of 1) the principle of relativity and 2) universality of the speed of light among inertial observers, he immediately derived that Lorentz transformations provide the correct coordinate redefinitions that map one inertial observer to another. According to Einstein's theory of *special relativity*, then, the Lorentz transformations furnish the true symmetry group of space and time. Because Lorentz transformations mix time and space coordinates, it is natural to consider them different parts of the same beast, a *space-time*, rather than completely separate concepts. This was an astonishing departure from the

Newtonian conception of fixed space and universal time, which physicists had adhered to since the 17th century. Einstein achieved fundamental new insights into the nature of space and time by assigning a *primary* importance to symmetry. The principle of relativity and the experimentally succesful theory of electromagnetism together carry greater weight than our intuitive preference for the fixed space/time and Galilean transformations of Newtonian mechanics. After Einstein's great achievements with the theories of special and general relativity, physicists began to think of theories as *defined* by their symmetries.

Electromagnetism, field theory, and symmetries

Maxwell's equations led to yet more discoveries about the role of symmetry in physics. Einstein's special relativity suggests that Maxwell's electric and magnetic fields - described by the 3-dimensional spatial vectors $\vec{E}$ and $\vec{B}$ are really different aspects of the same underlying force. In fact, they (or rather their scalar and vector potentials) are unified by the space-time 4-vector A_μ, called the *gauge potential*. The object A_μ transforms into itself under the relativistic symmetries of Maxwell's theory. In other words, the gauge potential furnishes a representation of the Lorentz group. In the *Lagrangian* field theory formalism, the relativistic symmetry of Maxwell's equations then becomes manifest when we switch from $\vec{E}$ and $\vec{B}$ to A_μ. There is a new subtlety in this formulation of electromagnetism, however. Maxwell's equations depend on A_μ only through the combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, called the *gauge field strength*. Therefore, Maxwell's equations are symmetric under the transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$, where $\alpha(x)$ is some function of the coordinates x. Such a symmetry, which depends on an arbitrary function of space-time, is known as a *gauge symmetry*. In modern practice, we think of Maxwell's theory of electromagnetism as defined by its $U(1)$ gauge symmetry.

A similar symmetry defines the theory of General Relativity (GR), published by Einstein in 1915. In GR, the Lorentz symmetry from the special theory, which connects inertial observers using the same coordinate system, is generalized to *diffeomorphism invariance*. This ensures that all observers, inertial or not, apply the same laws of physics, which take the same form in any coordinate system. This is known as the principle of *general coordinate invariance*.

The $U(1)$ gauge symmetry of Maxwell's equations and the diffeomorphism invariance of GR are fundamentally different from the other symmetries we have encountered. [1] Gauge transformations do not map different physical

[1] To be precise, the part of the gauge redundancies that truly depends on an arbitrary

states into each other. Rather, field configurations that are mapped into each other under a gauge transformations describe the same physical state. We can understand this fact by appealing to *Noether's second theorem* [35], which states that gauge symmetries are in one-to-one correspondence with differential relations among the equations of motion. Such a relation implies that the equations of motion are underdetermined: boundary and initial conditions do not determine a solution uniquely, but only up to a gauge transformation. Therefore, gauge-equivalent configurations must correspond to the same physical state.

For this reason, gauge symmetries are often referred to instead as *gauge redundancies*. They are an artifact of the (Lagrangian) field theory formalism used to describe the physical system. By describing the electromagnetic field as a Lagrangian field theory and with the gauge potential A_μ, we have accomplished to make the Lorentzian symmetries of Maxwell's equations manifest, but it has come at the cost of introducing a degree of redundancy. This is our first encounter with the complicated relationship between Lagrangian field theory and symmetries. For much of this thesis, we will be concerned with how one navigates the space of possible symmetry groups while avoiding the redundancies of ordinary classical and quantum field theory.

The Standard Model and Effective Field Theory

The concept of defining a theory by means of its symmetry group really took root later on in the 20th century. Yang and Mills generalized the $U(1)$ gauge theory of Maxwell to the non-Abelian groups $SU(N)$. [37] Thanks to a spectacular model-building effort [56–61], we now understand that all the interactions seen at colliders are well-described by a Yang-Mills gauge theory of $SU(3) \times SU(2) \times U(1)$. Each of these factor groups introduces a vector field A^a_μ for each of its generators T_a. Furthermore, there are fermionic particles called *quarks* and *leptons*, which are charged under (some of) the gauge groups. The pure Yang-Mills theory of $SU(3) \times SU(2) \times U(1)$ does a spectacular job of describing, for example, gauge boson self-interaction and the $SU(3)$ multiplet structure of the *hadrons* [60], but it has a peculiar property which seems to conflict with experiment. Yang-Mills theory does not allow the gauge potentials A^a_μ to obtain a mass. Additionally, the non-Abelian groups $SU(3) \times SU(2)$ forbid a mass for the quarks and leptons.

These facts can be reconciled with experiment by introducing a scalar field multiplet and adding cubic interactions with the quarks and leptons of the type $\phi\bar{\psi}\psi$, while maintaining gauge invariance. [58, 61] The fermions

function of space-time, rather than a (possibly infinite) series of constant coefficients. The former are *proper gauge redundancies*, the latter make up the *large gauge transformations*.

then obtain an effective mass when ϕ has a vacuum expectation value. A non-trivial vev is possible only if the scalar multiplet ϕ has a potential with a minimum away from the field space origin. The multiplet is charged under $SU(2) \times U(1)$, however. Therefore, the symmetries dictate that there exists a continuum of degenerate solutions with non-trivial vacuum expectation value, which transform into each other under the gauge group. Any one point along this continuum is not invariant under $SU(2) \times U(1)$ so the vacuum solution does not share all the symmetries of the Standard Model (recall that symmetries of objects (i.e. physical states) are defined as invariances). This is known as *spontaneous symmetry breaking.* [58, 61, 62]

Given a vev v, we can parametrize the multiplet ϕ in terms of fluctuations around v. We can distinguish fluctuations along the field space direction of the continuum of degenerate vacuum solutions and orthogonal fluctuations. We will refer to the former type of fluctuations as *Goldstone modes.* [65, 87] It is very important that the $SU(2) \times U(1)$ transformations are realized *non-linearly* on the Goldstone modes. This must be the case because none of the degenerate solutions are invariant. Naively, we can easily see that fluctuating in the Goldstone directions costs no energy. Therefore, those fluctuations should be associated with a massless degree of freedom. In fact, Goldstone, Salam and Weinberg [63] proved rigorously that broken *global* internal symmetry generators correspond to massless particles, called *Goldstone modes.* For spontaneously broken gauge symmetries, the interpretation is rather different. It is possible to redefine the vector fields A^a_μ such that the Goldstone mode g^a - associated to the broken generator T^a - becomes the longitudinal mode of a new massive vector field. Schematically, we define: $\tilde{A}^a_\mu = A^a_\mu + \partial_\mu g^a$. Thus, it is often said that the vector A^a_μ "eats" the would-be Goldstone mode for T^a. This is how the Higgs mechanism allows us to account also for the observed masses of force carrying bosons for the electroweak interaction.[2] The

[2]We can give a different interpretation of the Higgs mechanism. Owing to the fact that (proper) gauge transformations are redundancies rather than symmetries, we can often introduce them to a theory without changing any of its physical predictions. We can then arrive at the Higgs mechanism by applying the following (post-hoc) line of reasoning: we can describe the interactions of the Standard Model perfectly well at low energies with an effective theory of massive vectors with a global $SU(2) \times U(1)$ symmetry. We can restore gauge invariance by means of the *Stueckelberg trick* [64]: we define new vector fields $\tilde{A}^a_\mu$ according to $A^a_\mu = \tilde{A}^a_\mu + \partial_\mu g^a$, where $\tilde{A}_\mu$ is a massless gauge potential. This is the inverse of the field redefinition considered above: we now absorb the longitudinal mode of the massive vector by means of a scalar field. This theory works perfectly well at low energies, but in the UV the interactions of g^a become non-unitary. The easiest way to restore unitarity at high energies is to couple g^a to a massive scalar field. We have then effectively rediscovered the Higgs mechanism, which now has a more natural interpretation: the Higgs boson exists to unitarize the effective theory of massive vectors

remaining scalar excitation modes in ϕ are massive and remain as observable particles. In 2012, the massive Higgs boson was discovered by CERN, with a mass of 126 GeV. [67]

Although reasoning based on symmetry was instrumental in the discovery of the Standard Model, it was not the only criterion on the minds of physicists at the time. When calculating amplitudes in a generic perturbative quantum field theory, one encounters non-convergent integrals when calculating the contribution of loop diagrams. There exists several procedures to regularize these infinities, such as introducing a finite cut-off Λ in momentum space. Scattering amplitudes then depend on the cut-off, but the dependence on Λ may be absorbed either by a redefinition of coupling constants already assumed to exist or by introducing suitable *counterterms* into the Lagrangian. A *renormalizable* theory is one where the process of regularizing and introducing counterterms truncates, so that the theory may be completely defined by fixing a finite number of coupling constants to experiment. At the time, renormalizability was seen as a necessary condition to make sense of a quantum field theory. Therefore, physicists favored the Yang-Mills theory of $SU(3) \times SU(2) \times U(1)$ specifically for its renormalizability. [68, 69]

At the same time, however, Nambu, Weinberg and others [41–47] achieved great results in describing hadron-hadron interactions by means of a non-renormalizable theory of pions. This theory is based on the observation that the vacuum expectation values of the quadratic operators $\langle\bar{\psi}^a\psi^b\rangle$ do not vanish. This introduces a second form of spontaneous symmetry breaking in the Standard Model, which naively should come with a multiplet of Goldstone bosons. However, because the chiral symmetry is only an approximate symmetry in the Standard Model, the pseudo-Goldstone bosons can attain a small mass. Additionally, they are not fundamental particles but rather bound states of the strong interaction, called *pions*. Weinberg's theory is based on the assumption that the chiral symmetry is non-linearly realized on the pions. He then proceeded to write down interactions, up to suitable order in the fields and derivatives, consistent with the non-linear symmetry transformations. This leaves a number of coupling constants, to be fixed by experiments.

This procedure perfectly follows the modern paradigm of *Effective Field Theory* (EFT). [128] In EFT, a theory is assumed to be valid only in a given energy scale. The infinities of loop diagrams are assumed to be an artifact of assuming validity up to arbitrary energies. One defines a theory by selecting relevant degrees of freedom and a set of (linear and non-linear) symmetries.

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in the most simple way possible.

Then, one assumes that *all* interactions consistent with those symmetries exist, renormalizable or not. One can truncate the Lagrangian at a finite order, because all coupling constants are assumed to take order unity values in units defined by the energy scale at which the theory ceases to be valid. This is why an EFT does not need to be renormalizable: counterterms at higher order are assumed suppressed by the cut-off scale. Effective Field Theory represents the height of reasoning based on symmetry principles.

The ideas behind Weinberg's theory of pions were generalized by Callan, Coleman, Wess, and Zumino (CCWZ). [49, 50] They developed a general recipe for deriving the non-linear transformation laws associated with breaking a symmetry group G to a subgroup H. For much of this thesis, we will concern ourselves with the question of what sort of broken symmetry groups are compatible with an assumed set of degrees of freedom, according to the theory of CCWZ.

Navigating the space of symmetry groups

There is something special about the symmetry groups that define the Standard Model and Weinberg's theory: they are all internal symmetries, i.e. they commute with the symmetries of space-time. It is not immediately obvious why this should be the case. The full symmetry group of nature could be, a priori, a *hybrid symmetry* which combines non-trivially with the Poincaré group. In fact, in the years preceding the formulation of the full Standard Model, many such exotic symmetries were proposed to explain the spectrum of particles seen at colliders. [40] The (Lagrangian) field theory formalism is perfectly compatible with such symmetries at the classical level. It seemed there was no way to exhaust the space of possible symmetry groups, until the work of Coleman and Mandula. [38]

Coleman and Mandula discovered that, under rather general assumptions, theories with a hybrid symmetry lead to physically undesirable scattering amplitudes. A hybrid symmetry for instance leads to $2 \to 2$ amplitudes which vanish for values of momentum transfer that make up a continuous region in momentum space, rather than merely at a discrete set of points. [40] Such scattering behavior is not seen in experiments. The work of Coleman and Mandula put an end to much, but not all, of the search for physically realistic models with hybrid symmetry. It is possible to invalidate the assumptions behind the Coleman-Mandula theorem, but one has to consider symmetries which are quite different from what had been seen before.

One possibility is to consider that the symmetries of nature do not make up an ordinary Lie group, but a *supergroup*. A supergroup is infinitesimally characterized by a superalgebra, which consists of ordinary bosonic or *even* elements - which obey commutation relations similar to an ordinary Lie algebra - and fermionic or *odd* elements which follow anti-commutation relations. Haag, Lopuszanski, and Sohnius (HLS) extended the work of Coleman and Mandula to superalgebras. [39] They found that all possible superalgebras belong to the class of *supersymmetry*, which add to the Poincaré algebra a number of odd elements Q^i_α and $\bar{Q}^i_{\dot{\alpha}}$, which are Weyl spinors under the Lorentz group. The odd elements satisfy the following characteristic anti-commutation relation:

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}}\} = 2\delta^{ij}(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu \,. \tag{1.1}$$

Supersymmetric theories have many phenomenologically interesting properties. Each bosonic (fermionic) particle is assigned by supersymmetry to a partner fermionic (bosonic) particle. Together, the particle and all of its superpartners make up a representation of supersymmetry, called a *supermultiplet*. If supersymmetry is unbroken, all particles in a supermultiplet have the same mass. [122, 123] The particles observed in colliders do not organize into equal-mass supermultiplets. Thus, if supersymmetry is realized in nature, it must be (explicitly or spontaneously) broken.

There are other kinds of symmetries that are not ruled out by Coleman and Mandula's theorem. For example, the theorem does not apply to *dynamical symmetries*. These are symmetries that do not lead to an algebraic constraint on the S-matrix operator. They are outside the scope of Coleman and Mandula, because they explicitly assume the existence of a quantum charge operator that implements the algebraic condition. Spontaneously broken symmetries or non-linearly realizations live within the class of dynamical symmetries, because the existence of Goldstone modes prevents Noether currents from integrating to a quantum charge. [109, 137]

In Chapters 4 and 5, we will attempt to extend the work of Coleman-Mandula and HLS into the realm of the non-linearly realized symmetries. Like Coleman and Mandula, we will not make direct use of (Lagrangian) field theory, but address the question using algebraic methods and the CCWZ theory of non-linear realizations. We will explain how others have tackled the same issue using the structure of scattering amplitudes, also avoiding the complication of redundancies and field redefinitions that arise in Lagrangian field theory.

Symmetry and simplicity

The existence of a symmetry can lead to simplification in calculating physical quantities. In some cases, symmetries are so powerful that one can obtain exact results that are impossible to produce in generic quantum field theories. For example, in two- or three-dimensional conformal field theories, one can use the conformal symmetry to constrain, or sometimes exactly calculate, n-point correlation functions. This procedure is known as the *conformal bootstrap*. [70–73] Furthermore, it is possible to exactly calculate the Euclidean partition function (and expectation values of supersymmetric operators) in certain supersymmetric theories defined on compact manifolds. This is possible because supersymmetry sometimes allows one to reduce a path integral, which sums over the infinite-dimensional space of field configurations, to an ordinary integral. This is known as *supersymmetric localization*. [74] As a last example, there exist special theories that enjoy an infinite set of mutually commuting sequence and thus an infinite set of conserved charges. If these continue to exist at the quantum level, they can lead to a factorization of the full S-matrix in terms $2 \to 2$ scattering processes. Theories with an infinite set of commuting symmetries are known as *integrable systems*.

In other cases, a symmetry allows one to maintain control over calculations against, for example, quantum corrections. This is due to the simple fact that all corrections due to loop diagrams must also respect the symmetries of the theory. The protection of a symmetry against quantum corrections is usually maintained even when the symmetry is spontaneously or (weakly) explicitly broken. In the latter case, quantum corrections are naturally proportional to the small symmetry breaking parameter. These protections are particularly needed in theories that make use of scalar fields, such as the Higgs mechanism or the theory of inflation. Supersymmetry is often invoked to explain the small value of the Higgs boson mass. [89] In the absence of a symmetry to protect it, quantum corrections naturally generate a Higgs mass at the energy scale of new physics, rather than the observed $m_H = 126\,\text{GeV}$. [67] In the theory of inflation, one requires a scalar field to exist with a very flat potential, which is easily spoiled by quantum corrections. Many kinds of symmetry are often employed to protect the inflationary potential against such corrections, such as: simple shift symmetries, isometries of non-linear sigma models, or supersymmetry. [88] In the scenarios of DBI- and ambient inflation, the scalar field receives protection from a non-linearly realized space-time symmetry. In the former case, the symmetry is powerful enough to allow investigation of the theory at large time variation of the scalar field, a region of parameter space that is normally plagued by quantum corrections. We will come back to the case of the DBI scalar and its symmetries many times throughout this thesis.

In the examples just mentioned, symmetries are invoked as a *tool* to learn more about the nature of quantum field theory or about a broad physical idea like the theory of inflation. A symmetry can therefore have a use even if it not realized in nature.

Given the importance of symmetry in both phenomenology and formal physics theory, one would like to have an understanding of all the kinds of symmetry that can exist. A systematic exploration for the non-linearly realized symmetries was lacking until recently. This thesis is devoted to how one achieves such a classification using algebraic methods and the theory of non-linear realizations.

In Chapter 2, we will give precise definitions for the notions of symmetry in classical and quantum field theory, and introduce the concepts of algebraic and dynamical symmetries. In Chapter 3, we present a thorough review for the general theory of non-linear realizations for internal and space-time (super)-symmetries. Then, in Chapters 4 and 5, we will present a classification of exceptional EFTs using algebraic methods, comparing and contrasting the results from the approach based on scattering amplitudes.

Chapter 2

Symmetries in Effective Field Theory

In the Introduction, we gave a brief overview of symmetry and its application in physics. In this Chapter, we will give the technical definitions for the many classes of symmetries we have already encountered. We will define symmetries in both classical and canonical/path integral quantum field theory. Our definitions will differ slightly in each of these formalisms, but all definitions of symmetry fundamentally revolve around the same idea. Recall that both theories and the physical states within a theory may enjoy symmetry. A theory is symmetric whenever it is possible to define a mapping, satisfying certain requirements, that takes physical states into physical states. A physical state, on the other hand, enjoys a symmetry when the same mapping sends the state into itself. Thus, a physical state may preserve some, all, or none of the symmetries of the underlying theory. We will in this Chapter give the formal definitions for the mapping and for the notion of a "physical state", for each of the formalisms mentioned.

In classical field theory, symmetries have an immediate consequence on the form of the Euler-Lagrange equations. Noether discovered that each independent *global* and continuous symmetry G^i corresponds to a *conserved current* J_i^μ, which is divergence-free whenever the equations of motions are satisfied, $\partial_\mu J_i^\mu = 0$. A conserved current J_i^μ defines a time-independent conserved *charge* Q_i by integrating J_i^μ over a space-like hypersurface.

A continuous *local* symmetry, conversely, does not lead to a non-trivial conserved current. Instead, it implies a differential relation among the Euler-Lagrange equations. This means that the E-L equations are *underdetermined* whenever they admit a local symmetry. Its solutions are determined by the equations of motion and the boundary conditions only modulo the local symmetry transformation. Therefore, one should consider configurations linked by local symmetry transformations as physically equivalent.

Not all symmetries of a classical theory will survive in its "quantized" counterpart. A classical symmetry may be destroyed entirely by quantization, for instance when the path integral measure does not share a symmetry of the Lagrangian. Such a symmetry is called *anomalous*. However, even among the symmetries that survive in the quantum theory we can distinguish two important classes with very different physical implications. The first are those symmetries whose conserved charge Q_i promotes to a well-defined quantum operator. Such symmetries become symmetries of the *S-matrix*. We will refer to these as *algebraic symmetries*. The algebraic symmetries classify the single-particle states of the theory. An important example of an algebraic symmetry is the Poincaré group of relativistic field theories. Particle states are defined as representations of the Poincaré group.

The remaining symmetries (those whose current does not integrate into a well-defined quantum charge) play a very different role. They cannot be used to define the free single-particle states and do not become symmetries of the S-matrix. However, they still have *dynamical* consequences because they restrict the form that the interactions can take. We will call such symmetries *dynamical symmetries*. Understanding the distinction between algebraic and dynamical symmetries, and the very different consequences they have on the behavior of a theory, will be the focus of this Chapter.

2.1 Classical symmetries and redundancies

2.1.1 Symmetry transformations

We will first turn our attention to symmetry in classical Lagrangian field theory. In this context, a symmetry transformation is a bijective mapping from the space J_{EL} of solutions of the equations of motion onto J_{EL} itself. Consider a set of fields and coordinates (x, ϕ). The coordinates x parametrize the space-time manifold M and the fields $\phi(x)$ are functions that map M to the field space manifold U, $\phi : M \to U$. Then, the space-time derivatives of ϕ, $\phi^{(n)} = \partial^n \phi$ parametrize the spaces $U^{(n)}$. Together, the space-time manifold, the fields, and their derivatives up to n-th order define the *jet space* $J^n = M \times U \times U^{(1)} \times \ldots U^{(n)}$. [133–135][1]

Introduce an action functional $S[\phi]$ with equations of motion $E(x, \phi)$. A

[1] In this thesis, we will consider cases where some of the fields and coordinates are Grassmann-odd variables. The concepts generalize in a straightforward way, however, so we will ignore this complication for now.

symmetry (f, h) of the system $S[\phi]$ is a mapping $(f, g) : J^\infty \to J^\infty$

$$x \to x' = f(x, \phi, \partial\phi, \ldots), \quad \phi \to \phi' = h(x, \phi, \partial\phi, \ldots)\,, \tag{2.1}$$

such that $E(x, \phi) = 0$ if and only if $E(x', \phi') = 0$. In other words, $(x, \phi) \in J_{EL}$ if and only if $(x', \phi') \in J_{EL}$. The symmetry transformation must be invertible. More precisely, it should be a diffeomorphism on the infinite order jet space J^∞. Special symmetry transformations may be well-defined on jet spaces of finite order, for example when (f, h) do not involve derivatives of the fields. A generic transformation, however, changes the derivative order of the equations of motion.

An important class of symmetry transformations in classical field theory are the *variational symmetries*. A variational symmetry is a transformation $(x, \phi) \to (x', \phi')$ such that

$$\mathcal{L}(x, \partial^n\phi) = \mathcal{L}(x', \partial'^n\phi') + \nabla\mathcal{K}\,, \tag{2.2}$$

In other words, they are symmetries of the action functional itself, modulo boundary terms. Clearly, a variational symmetry maps the space of solutions of the equations of motion into itself, $E(x, \phi) = 0 \iff E(x', \phi') = 0$, as a total derivative makes no contribution to the equations of motion. Many of the important consequences of symmetry (such as Noether's and Goldstone's theorems) are applicable only to variational symmetries. However, not all classical symmetry transformations are variational symmetries. An important example of a symmetry that is not evident at the level of the action functional is the electric-magnetic duality one encounters in p-form gauge theories. For most of this thesis, we will be concerned with variational symmetries only. In most cases, we will work with trivial boundary conditions, so that (2.2) implies:

$$S[\phi] = S[\phi']\,. \tag{2.3}$$

We have defined a symmetry as a simultaneous transformation of the coordinates as well as the fields. However, every symmetry has a corresponding *active* form, where the coordinates do not change at all. Given the *passive* form of the transformation $x \to x' = f(x, \partial^n\phi)$, $\phi(x) \to \phi'(x') = h(x, \partial^n\phi)$, the corresponding active transformation is:

$$x \to x, \quad \phi \to \phi'(x) = h\big(f^{-1} \cdot x,\, (f^{-1} \cdot \partial)^n \phi(f^{-1} \cdot x)\big)\,. \tag{2.4}$$

where f^{-1} is the transformation that takes x' to x. The active form of a transformation law is a symmetry if and only if the passive form is a symmetry. Therefore, the active and passive transformations are equivalent. We will encounter both active and passive transformations in what follows.

2.1.2 Symmetry groups, algebras and invariant forms

The set of symmetry transformations of the system $S[\phi]$ forms a group. By definition, any transformation $g \cdot (x, \phi) = (x', \phi')$ has an inverse g^{-1} such that $g^{-1} \cdot g \cdot (x, \phi) = (x, \phi)$. Furthermore, a trivial transformation is clearly a symmetry. Thus, the symmetry transformations satisfy the group axioms.

We can now distinguish between continuous and discrete symmetry groups. Both of these play an important role in all areas of physics. Examples of discrete symmetries are charge conjugation, parity, and time reversal. The product of these three discrete transformations is a symmetry of any Lorentz-invariant quantum field theory. The most important class of continuous symmetries are those that form Lie groups. A Lie group G is a differentiable manifold on which a group operation $\cdot$ and an inversion mapping $^{-1}$ can be defined, [136]

$$\begin{aligned} &\cdot : G \times G \to G,\ (g_1, g_2) \to g_1 \cdot g_2 \,, \\ &^{-1} : G \to G,\ g \to g^{-1} \text{ such that } g \cdot g^{-1} = 1 \,, \end{aligned} \tag{2.5}$$

such that both the group operation and the inversion $^{-1}$ (i.e. $g^{-1} \cdot g = 1$) are differentiable in the usual sense.

Given two elements $g, a \in G$ of the Lie group G, we can define the *left-* and *right-translations* of g by a as follows:

$$\begin{aligned} R_a g &= ga \,, \\ L_a g &= ag \,. \end{aligned} \tag{2.6}$$

The diffeomorphisms $R_a, L_a : G \to G$ induce the pushforward mappings $L_{a\star} : T_gG \to T_{ag}G$ and $R_{a\star} : T_gG \to T_{ga}G$, from elements of the tangent space T_g at g to the tangent space of its right- or left-translation. There is a distinguished set of vector fields on G called the *left-* or *right-invariant* vector fields. As we will see, these vector fields make up the *Lie algebra* $\mathfrak{g}$ associated to the Lie group G. A left-invariant (LI) vector field X is a vector field that is invariant under left-translations. In other words, X must satisfy:

$$L_{a\star} X|_g = X|_{ag} \,. \tag{2.7}$$

The definition of right-invariant vector fields is of course analogous. Any vector V in the tangent space T_e of the identity element defines a unique left-invariant vector field V_L by way of the mappings $L_{g\star}$:

$$V_L|_g = L_{g\star} V \,. \tag{2.8}$$

Obviously, every left-invariant vector field V_L defines a unique element of T_e simply by evaluating it at the identity. Therefore, there is a one-to-one correspondence between elements of the tangent space T_e and the invariant vector fields on G. Furthermore, the set of left-invariant vector fields is closed under the *Lie bracket*. Let us see why this is the case. Given two vector fields X, Y the Lie bracket $[X, Y]$ is defined as follows:

$$[X, Y]f = X[Y[f]] - Y[X[f]] \,, \tag{2.9}$$

where f is some curve in G. Then, if X and Y are left-invariant, we find:

$$L_{a\star}[X, Y]|_g = [L_{a\star}X|_g, L_{a\star}Y|_g] = [X|_{ag}, Y|_{ag}] = [X, Y]|_{ag} \,, \tag{2.10}$$

so the Lie bracket of two left-invariant fields is itself a left-invariant field. Now we can define the *Lie algebra* $\mathfrak{g}$ of G as the set of left-invariant vector fields in G with the Lie bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We will sometimes refer to the elements of $\mathfrak{g}$ as the *generators* of G. The Lie algebra satisfies the *Jacobi identity*. Given three elements $X, Y, Z \in \mathfrak{g}$, we have:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \,. \tag{2.11}$$

We may label the elements of $\mathfrak{g}$ as G_i where $i = 0, 1, \ldots \dim\mathfrak{g}$. Then, for the Lie bracket we obtain: $[G_i, G_j] = f_{ij}{}^k G_k$. The $f_{ij}{}^k$ are the *structure constants* of the algebra and the group.

Conversely, we can now give a bottom-up definition of a Lie algebra: a Lie algebra $\mathfrak{g}$ is a vector space over a field with the anti-symmetric bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi identity.

The Lie algebra contains all the information about the local properties of the group. In fact, we may reconstruct the group G in a finite neighborhood of the identity by exponentiating the generators: $g(\epsilon) = e^{\epsilon^i X_i}$, where the parameters ϵ^i provide a local set of coordinates for G. When G is simply connected, the image of the exponential mapping is G, so that the entire group is characterized by the Lie algebra.

When the symmetries of the system $S[\phi]$ form the Lie group G, its *infinitesimal* transformations make up the Lie algebra $\mathfrak{g}$. Just like the Lie algebra contains all the local information about the Lie group, the infinitesimal transformations characterize the symmetry group locally. For most of our purposes, only the local properties are important. Let us see how a Lie algebra arises from the infinitesimal transformation laws. Parametrize the symmetry transformation close to the identity with the local coordinates ϵ^i: $g(\epsilon) \cdot (x, \phi) = F(\epsilon; x, \phi) = (x', \phi')$. Now take ϵ^i infinitesimal. We then find:

$$\phi'(x') = \phi(x) + \epsilon^i g_i(x, \phi, \ldots), \quad x' = x + \epsilon^i f_i(x, \phi, \ldots) \,. \tag{2.12}$$

We then define the infinitesimal transformation law associated to the generator G_i as:

$$\delta_i(x, \phi, \ldots) = (f_i(x, \phi, \ldots), g_i(x, \phi, \ldots)) \,. \tag{2.13}$$

It is then easy to see that the infinitesimal transformation laws realize the Lie algebra $\mathfrak{g}$:

$$[\delta_i, \delta_j](x, \phi, \ldots) = f_{ij}{}^k \delta_k(x, \phi, \ldots) \,. \tag{2.14}$$

We can now separate symmetry groups into two important classes: internal and space-time symmetries. Every theory that does not include a dynamical metric is defined on a certain background geometry. This background may have certain isometries. One usually requires that the isometries are represented as variational symmetry transformations of the action functional $S[\phi]$. The isometries then make up a subgroup of the total group of symmetries of $S[\phi]$. The most important example is the Poincaré symmetry group enjoyed by relativistic theories on the flat Minkowski background geometry. Then, a symmetry generator that in the Lie algebra commutes with each of the Poincaré generators corresponds to an *internal symmetry*. Each generator that fails to commute with any of the Poincaré generators is a *space-time symmetry*.

The left-invariant vector fields of G are dual to its left-invariant one-forms[2]. Take the basis $(G_L)_i$ for the left-invariant vector fields defined by acting with (2.8) on the elements G_i of T_e which make up the Lie algebra. Now define the dual basis Θ^i such that $\langle (G_L)_i, \Theta^j \rangle = \delta_i{}^j$. The one-forms Θ^i span the set of left-invariant one-forms of G. The basis one-forms satisfy the Maurer-Cartan structure equation:

$$d\Theta^i = -\tfrac{1}{2} f_{jk}{}^i \Theta^j \wedge \Theta^k \,. \tag{2.15}$$

We can now define a special Lie algebra-valued one-form ω on G that will play an important role in the theory of non-linear realizations. The *Maurer-Cartan* form is a mapping $\omega : T_g \to T_e$ which acts on a vector field X at g as:

$$\omega(X)|_g = L_{g^{-1}\star} X \,. \tag{2.16}$$

It is easy to see that $\omega = G_i \otimes \Theta^i$ by expanding X into the basis $(G_L)_i$, using left-invariance and noting that $(G_L)_i|_e = G_i$. Furthermore, due to (2.15), the Maurer-Cartan form satisfies the *Maurer-Cartan equation*:

$$d\omega = -\tfrac{1}{2} \omega \wedge \omega \,. \tag{2.17}$$

2.1.3 Coset manifolds

Consider a Lie subgroup H of a Lie group G and define the equivalence relation $\sim$ such that $g \sim g'$ if and only if $g' = gh$ for some element $h \in H$.

[2]Similarly to an LI vector field, a left-invariant one-form is mapped to itself under the pullback of a left-translation.

The set of equivalence classes under $\sim$ forms the *coset space* G/H. If G and H are Lie groups, the coset space is always a manifold. It is a Lie group whenever H is a normal subgroup of G. Coset spaces of Lie (super)groups will play a very important role in the rest of this thesis, as they are the structure on which spontaneously broken symmetries are defined. For this reason, let us spend a few moments to see how some of the concepts of the previous section generalize to coset spaces.

To begin, let us define the notion of a *projectable p-form.* Consider a Lie group G with Lie subgroup H and the coset manifold $K = G/H$. We can define a projection mapping $\pi : G \to K$ that maps each element g of G to the corresponding equivalence class $\{gH\}$ in K,

$$\pi : gh \to \{gH\}\,, \tag{2.18}$$

where $h \in H$. Then, a p-form Ω in G is projectable if there is a corresponding p-form $\bar{\Omega}$ on K whose pullback by the projection map is Ω, i.e. $\Omega = \pi_\star(\bar{\Omega})$. A form Ω is projectable if and only if: [138, 139]:

- $\Omega(X_1, \ldots, X_p) = 0$ if any of the LI vector fields X_i is in $\mathfrak{h}$,
- $R_{h\star}\Omega = \Omega$, i.e. Ω is right-invariant under H.

We can now define a notion of cohomology that will prove important in finding *Wess-Zumino terms* in the theory of non-linear realizations, to be discussed in the next chapter. The relative Chevalley-Eilenberg (CE) cohomology of G and H is given by the p-forms in G which are: left-invariant, closed, projectable, and not the exterior derivative of an LI projectable $p-1$ form in G [3] The relative CE cohomology of G and H is related to the de Rham cohomology of K. In the next chapter, we will see that these cohomology groups classify the so-called Wess-Zumino terms.

2.1.4 Noether's first theorem

The most important consequence of a continuous variational symmetry is the existence of either a conserved *current* or a *gauge identity.* [35] The former arises from a *global* symmetry and the latter from a *local* or *gauge* symmetry.

[3] The LI forms on G are in one-to-one correspondence with the p-skew symmetric mappings $\mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \ldots \wedge \mathfrak{g}^p \to \mathbb{R}$. When such a mapping vanishes on $\mathfrak{h}$ and is ad$\mathfrak{h}$ invariant, the associated LI form is projectable. Moreover, the exterior derivative has a corresponding *coboundary* operator that takes $p-1$-skew symmetric mappings to p-skew symmetric mappings. Then, the relative *Lie algebra cohomology* is given by the space of p-skew symmetric mappings that satisfy the projectability conditions and are closed under the coboundary operator. Thus, CE cohomology is identical to Lie algebra cohomology.

A global symmetry is one that acts on the coordinates and the fields with the same group element at each space-time point. In other words, the parameter used to describe it is a constant ϵ. Conversely, the parameter of a local symmetry may be a function of space-time, $\epsilon(x)$.

In this section, we will deal with Noether's first theorem, which states that there is a one-to-one correspondence between the generators G_i of a continuous, global variational Lie group symmetry and conserved currents. The symmetry group is parametrized by the constants ϵ^i. We write the active infinitesimal transformation on the fields ϕ^a as $\delta\phi^a = \epsilon^i g_i^a(x, \phi, \ldots)$. Then, the variation of the Lagrangian is:

$$\delta\mathcal{L} = \epsilon^i \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu g_i^a + \frac{\partial\mathcal{L}}{\partial\phi^a} g_i^a \right] = \epsilon^i \partial_\mu K_i^\mu \,, \tag{2.19}$$

where the second equality is just the statement that $\delta\phi^a$ generates a symmetry. Using the Euler-Lagrange equations, one finds: [123]

$$\partial_\mu \left(- \frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi^a)} g_i^a + K_i^\mu \right) \equiv \partial_\mu J_i^\mu = 0 \,. \tag{2.20}$$

The bracketed quantity J_i^μ is the conserved current associated to the generator G_i. Note that we may redefine J_i^μ as $\tilde{J}_i^\mu = J_i^\mu + G^\mu$ if $G^\mu(x, \phi, \ldots)$ is either identically conserved (i.e. $\partial_\mu G^\mu = 0$ whether or not equations of motion are satisfied) or vanishes on-shell (i.e. $G^\mu(x, \tilde{\phi}, \partial\tilde{\phi}, \ldots) = 0$ whenever $\tilde{\phi}$ is a solution). Such a redefinition does not affect the fact that J^μ is conserved. We consider two currents equivalent if they differ by such a quantity G^μ.

Note that we have considered only an active transformation $\delta\phi^a(x) = \epsilon^i g_i^a(x, \phi, \ldots)$ here. However, there is no loss of generality as the current that follows from the associated passive transformation differs from J_i^μ by an identically conserved quantity. Therefore, as expected the active and passive forms of a symmetry give rise to equivalent conserved currents.

Given some foliation of D-dimensional space-time by space-like $(D-1)$-dimensional surfaces $\Sigma(\tau)$, we can integrate a conserved current to produce a *conserved charge*:

$$Q_i = \int_{\Sigma(\tau)} d\Sigma_\mu \, J_i^\mu \,. \tag{2.21}$$

Whether or not the integral converges depends on the behavior of the field configuration at infinity. When it is well-defined, Q_i is independent of the chosen surface thanks to the fact that J_i^μ is conserved. Choosing the fixed t surfaces $\Sigma(t)$, (2.21) reduces to:

$$Q_i = \int d^{D-1}\mathbf{x} \, J_i^0(\mathbf{x}, t) \,. \tag{2.22}$$

Then, Q_i is time independent, $\frac{dQ_i}{dt} = 0$. The conserved charges Q_i do not necessarily exist in the quantum theory, as we will see. Notably, the currents of spontaneously broken symmetries do not in general integrate to well-defined quantum charges.

With the generalization to the quantum theory in mind, let us examine charges in the classical Hamiltonian formalism. We specialize to the case where the Lagrangian depends on at most the first derivatives of the fields, $\mathcal{L} = \mathcal{L}(x, \phi, \partial\phi)$. The Poisson bracket of two quantities $A(\phi, \pi)$ and $B(\phi, \pi)$ is [123] defined as:

$$\{A, B\}_{PB} = \int d^{D-1}\mathbf{x} \left(\frac{\delta A}{\delta \phi^a} \frac{\delta B}{\delta \pi_a} - \frac{\delta A}{\delta \pi_a} \frac{\delta B}{\delta \pi^a} \right) . \tag{2.23}$$

where π_a is the canonical momentum conjugate to ϕ^a. Then, the Poisson bracket of the field ϕ^a and the Noether charge Q_i is the transformation law for the generator G_i:

$$\{\phi^a, Q_i\}_{PB} = g_i^a(x, \phi, \ldots) , \tag{2.24}$$

and the Poisson bracket of two charges realizes the Lie algebra of the symmetry group:

$$\{Q_i, Q_j\}_{PB} = f_{ij}{}^k Q_k . \tag{2.25}$$

2.1.5 Noether's second theorem

We now turn our attention to local symmetries and their consequences on the equations of motion. Noether's second theorem states that there is a one-to-one correspondence between a differential relation among the equations of motion and a symmetry depending on an arbitrary function of space-time. Consider a local symmetry which depends on the function $\epsilon(x)$ and its derivatives up to order n. Its active, infinitesimal form is:

$$\delta\phi(x) = \epsilon(x) g^{(0)}(x, \phi, \ldots) + \partial_\mu \epsilon(x) g^{(0)\mu}(x, \phi, \ldots) + \ldots + \partial^{(n)} g^{(n)}(x, \phi, \ldots) . \tag{2.26}$$

Once again, there is no loss of generality in considering an active transformation. By using the fact that $\delta\phi(x)$ generates a symmetry, we may write:

$$\delta\phi E(\phi) = \partial_\mu K^\mu , \tag{2.27}$$

where $E(\phi)$ represent the equations of motion of the system and for some $K^\mu(x, \phi, \ldots)$. We now assume that $\epsilon(x)$ vanishes at the boundary. Then, we can integrate over the internal volume and use partial integration to find: [133–135]

$$\int \epsilon(x)\left(\sum_{k=0}^{n}(-1)^k g^{(k)}\frac{d^k}{(dx)^k}E(\phi)\right)d^Dx = 0\,. \tag{2.28}$$

Since $\epsilon(x)$ is an arbitrary function of space-time, the bracketed quantity vanishes. Thus, we have derived an n-th order differential relation among the equations of motion from the existence of a local symmetry.

The existence of a differential relation implies that the equations of motion are underdetermined, i.e. given a set of boundary conditions, the equations of motion do not uniquely determine the evolution. For this reason, a gauge symmetry is often called a *redundancy* rather than a true symmetry. Field configurations which can be transformed into each other by a gauge transformation which preserves the boundary condition are considered physically equivalent. This interpretation is even more clear in the quantized theory, as one has to divide the path integral by the volume of gauge-equivalent field configurations in order to arrive at a well-defined theory.

We note that it was crucial here to assume that $\epsilon(x)$ vanish at the boundary. The subset of gauge transformations that do not leave the boundary conditions invariant, often called *large gauge transformations*, do not lead to differential relations and may be considered as an infinite set of global symmetries. For example, consider a $U(1)$ transformation of a gauge vector. Restricting to Taylor expandable functions $\epsilon(x)$, we find:

$$\delta A_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho + \dots\,. \tag{2.29}$$

Such large gauge transformations will play an important role in the rest of this thesis.

One can easily go through the calculation of the previous section for a local symmetry and find a quantity J^μ which is conserved. However, it turns out that for a gauge symmetry J^μ always either vanishes on-shell or is identically conserved. Therefore, the would-be Noether current is trivial.

2.2 Quantum symmetries

2.2.1 Symmetries of the path integral

Having addressed symmetries in classical Lagrangian field theory, we now move on to symmetries in quantum field theory. The natural generalization of the Lagrangian theory in quantum mechanics is the path integral formalism.

We will first make use of the path integral formalism to define symmetry transformations in QFT and to derive the quantum counterpart of Noether's theorem. Later, however, we will move to the canonical formalism to state the theorems of Coleman-Mandula [38] and Haag-Lopszanski-Sohnius. [39]

The fundamental object in the path integral formalism is the following generating functional, the *partition function*:

$$Z[J]|_{J=0} = \int \mathcal{D}\phi \, e^{-S[\phi]} \,, \tag{2.30}$$

where the argument of $Z[J]$ represents sources, which we have put to zero for the moment. The integration with the measure $\mathcal{D}\phi$ represents an integration over all field configurations $\phi(x)$. $S[\phi] = \int d^d x \mathcal{L}$ is the ordinary action, where $\mathcal{L}$ is again, in general, a mapping from the infinite order jet space J^∞ to $\mathbb{R}$. The expectation value of an operator $\mathcal{F}(\phi)$ is defined as:

$$\langle \mathcal{F}(\phi) \rangle = \int \mathcal{D}\phi \, e^{-S[\phi]} \mathcal{F}(\phi) \,, \tag{2.31}$$

Often, one calculates such expectation values by expanding $\mathcal{F}(\phi)$ in powers of $\phi(x)$ and applying functional derivatives with respect to $J(x)$ to $Z[J]$.

Let us see how various important notions from classical physics find their counterpart in the quantum field theory. By applying a functional derivative with respect to $\phi(x)$ inside the path integral, one finds the classical equations of motion $E(x, \phi)$:

$$\langle E(x,\phi) \rangle = \int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} e^{-S[\phi]} = \int \mathcal{D}\phi \, e^{-S[\phi]} E(x,\phi) \,. \tag{2.32}$$

Since the path integral over a total derivative is zero, assuming appropriate boundary conditions, we find that the expectation value of the classical equations of motion is equal to zero: $\langle E(x, \phi) \rangle = 0$. We can make an even more general statement. Insert into (2.32) a number of local operators $\mathcal{O}_1(x_1)$, $\mathcal{O}_2(x_2)$, ..., at points x_1, x_2, ... distinct from x:

$$\begin{aligned} \langle E(x,\phi) \, \mathcal{O}_1(x_1) \, \mathcal{O}_2(x_2) \ldots \rangle &= \int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} \Big(e^{-S[\phi]} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \ldots \Big) \,, \\ &= \int \mathcal{D}\phi \, e^{-S[\phi]} E(x,\phi) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \ldots = 0 \,. \end{aligned} \tag{2.33}$$

Thus, the equations of motion have vanishing expectation value also when inserting separated local operators. In other words, the equations of motion hold as *operator equations*. In the following, we will write operator equations like (2.33) as

$$\langle E(x,\phi)\ldots\rangle = 0\,, \tag{2.34}$$

with the ellipses representing insertions of local operators.

Let us turn our attention to symmetries. A variational symmetry of the path integral is an invertible transformation $(f,g) : J^\infty \to J^\infty$ where

$$x \to x' = f(x,\phi,\partial\phi,\ldots), \quad \phi \to \phi' = h(x,\phi,\partial\phi,\ldots)\,, \tag{2.35}$$

such that the product of measure and the quantity $e^{-S[\phi]}$ remains invariant: [130,131]

$$\mathcal{D}\phi'\, e^{-\int d^dx\mathcal{L}(x',\phi',\ldots)} = \mathcal{D}\phi\, e^{-\int d^dx\mathcal{L}(x,\phi,\ldots)}\,. \tag{2.36}$$

Clearly, any variational classical symmetry that leaves the measure invariant, $\mathcal{D}\phi' = \mathcal{D}\phi$, is also a symmetry of the path integral. Not all classical symmetries become symmetries of the path integral, however. Any classical variational symmetry which fails to become a path integral symmetry is known as an *anomalous symmetry*. In most cases, it is not a problem when a global symmetry becomes anomalous. An anomalous *gauge symmetry*, on the other hand, signals an inconsistency as one cannot make sense of the integration over field configurations which are classically gauge-invariant. The requirement that gauge symmetries should not be anomalous often leads to important restrictions on the Lagrangian. In the Standard Model, such *anomaly cancellation* conditions relate the quark electric charges to the electric charges of leptons in such a way to forbid bound states with fractional charge. In string theory, anomaly cancellation conditions fix the internal gauge symmetry groups of heterotic strings to either $SO(32)$ or $E_8 \times E_8$. [152]

Just like in the classical field theory, a continuous and global Lie group symmetry of the path integral leads to a conserved current. In addition, symmetries imply *Ward identities*, which relate products of currents and operators to the transformation laws of operators. To see this, consider the following active, infinitesimal and global symmetry transformation:

$$\phi(x) \to \phi(x) + \epsilon\Delta\phi(x)\,. \tag{2.37}$$

Then, localize the transformation with an arbitrary function $\rho(x)$:

$$\phi(x) \to \phi(x) + \epsilon\rho(x)\Delta\phi(x)\,, \tag{2.38}$$

Of course, (2.38) is not a symmetry transformation, but it reduces to (2.37) in the limit $\rho \to$ constant. Therefore, the variation of the quantity $\mathcal{D}\phi\, e^{-S[\phi]}$ is proportional to a derivative of ρ:

$$\mathcal{D}\phi'\, e^{-S[\phi']} = \mathcal{D}\phi\, e^{-S[\phi]}\left[1 + \epsilon\int_{\mathcal{M}} d^dx\,\sqrt{g}J^\mu(x)\partial_\mu\rho(x) + \mathcal{O}(\epsilon^2)\right]. \tag{2.39}$$

The quantity $J^\mu(x)$ is nothing but the Noether current for the symmetry transformation (2.37), as we will see. We now assume that $\rho(x)$ has support only in a submanifold $\mathcal{U}$ of $\mathcal{M}$. Then, consider the local operators $\mathcal{O}_1(x_1)$, $\mathcal{O}_2(x_2)$, ..., where x_1, x_2, ... lie outside of $\mathcal{U}$. Additionally, consider the local operators $\mathcal{A}_1(y_1)$, $\mathcal{A}_2(y_2)$, ..., where this time the coordinates y_1, y_2, ... lie *inside* of $\mathcal{U}$. By inserting the transformation (2.38) we then find, to first order in ϵ:

$$\begin{aligned} &\int \mathcal{D}\phi' \, e^{-S[\phi']} \Big(\mathcal{O}_1(x_1)\ldots\Big)\Big(\mathcal{A}_1(y_1)\ldots\Big) \\ &= \int \mathcal{D}\phi \, e^{-S[\phi]} \Big(\mathcal{O}_1(x_1)\ldots\Big)\Big(\mathcal{A}_1(y_1)\ldots - \delta\mathcal{A}_1(y_1)\mathcal{A}_2(y_2)\ldots - \\ &\ldots - \mathcal{A}_1(y_1)\delta\mathcal{A}_2(y_2)\ldots\Big)\,, \end{aligned} \tag{2.40}$$

where $\mathcal{A}_i(y_i) \to \mathcal{A}_i(y_i) + \rho(y_i)\delta\mathcal{A}_i(y_i)$ is the transformation law of $\mathcal{A}_i$ under (2.38). Equation (2.40) simply states that (2.38) is an invertible change of the jet space variables, which leaves the path integral invariant. The change of variables has no effect on the operators $\mathcal{O}_i$ either, as they lie outside $\mathcal{U}$. Now, insert equation (2.39) into (2.40) to find:

$$\begin{aligned} &\int_{\mathcal{M}} d^dx \sqrt{g}\, \epsilon \nabla_\mu J^\mu(x)\mathcal{A}_1(y_1)\mathcal{A}_2(y_2) + \ldots \\ &+ \delta\mathcal{A}_1(y_1)\mathcal{A}_2(y_2)\ldots + \mathcal{A}_1(y_1)\delta\mathcal{A}_2(y_2) + \ldots = 0\,, \end{aligned} \tag{2.41}$$

as an operator equation. We have integrated by parts once to move the derivative to J^μ. Equation (2.41) is known as the *Ward identity* for the symmetry (2.37). Let us remove the insertions $\mathcal{A}_i$ for now. We then find the operator equation:

$$\langle \nabla_\mu J^\mu(x)\ldots\rangle = 0\,. \tag{2.42}$$

This is the quantum version of Noether's first theorem. It is useful to rewrite (2.41) as:

$$\begin{aligned} \nabla_\mu J^\mu(x)\mathcal{A}_1(y_1)\mathcal{A}_2(y_2)\ldots = \frac{1}{\sqrt{g}}\Big(&\delta^d(x-y_1)\delta\mathcal{A}_1(y_1)\mathcal{A}_2(y_2)\ldots + \\ &+ \delta^d(x-y_2)\mathcal{A}_1(y_1)\delta\mathcal{A}_2(y_2)\ldots + \ldots\Big)\,. \end{aligned} \tag{2.43}$$

We have seen how the classical notions of equations of motion, symmetry transformations and Noether currents find their counterparts in quantum field theory. We wish to emphasize that we never assumed the transformation laws are simple linear functions of the fields.

2.2.2 Symmetries of the quantum effective action

We have seen that the classical equations of motion have meaning in the quantum theory as operator equations. However, the classical solutions in general do not correspond to quantum expectation values for the fields. There is a different functional, called the *quantum effective action*, whose stationary points do coincide with one-point functions for the fields. As we will see, the quantum effective action shares the symmetries of the path integral.

To define the quantum effective action, it will be necessary to introduce some new quantities. Let us restore the currents J in the partition function $Z[J]$:

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi]+i\int d^4x\, \phi^i(x)J_i(x)} \,, \tag{2.44}$$

where i runs over the number of fields in the theory and a summation over any group indices is implicit. The expectation value of the field ϕ^i, in the presence of the currents $J(x)$, is defined as:

$$\begin{aligned}\langle \phi^i(x)\rangle_J = \phi^i_J(x) &= \frac{1}{Z[J]}\int \mathcal{D}\phi \, e^{iS[\phi]+i\int d^4x\, \phi^i(x)J_i(x)}\phi^r(x) \\ &= -i\frac{1}{Z[J]}\frac{\delta}{\delta J_i(x)}Z[J] \,. \end{aligned}\tag{2.45}$$

We can rewrite the partition function $Z[J]$ as the exponential of a quantity $W[J]$:

$$Z[J] = \sum_{n=0}^{\infty}\frac{1}{n!}(iW[J])^n = \exp(iW[J]) \,. \tag{2.46}$$

Whereas the partition function $Z[J]$ is the sum of all vacuum-to-vacuum diagrams, $W[J]$ is the sum of all such *connected* diagrams. Clearly, $Z[J]$ is the sum of products of mutually disconnected diagrams. Each term in the sum is weighted by a symmetry factor $1/(n!)$ related to exchanging n connected subcomponents, leading to (2.46). In terms of $W[J]$, the expectation of ϕ^i becomes:

$$\phi^i_J(x) = \frac{\delta}{\delta J_i(x)}W[J] \,. \tag{2.47}$$

Let us now choose a particular field configuration $\bar{\phi}^i(x)$. Then, the we label the background current that leads to $\langle \phi^i(x)\rangle = \bar{\phi}^i(x)$ as $J_{i\bar{\phi}}(x)$. In other words:

$$\langle \phi^i(x)\rangle_{J_{\bar{\phi}}} = \frac{\delta}{\delta J_i(x)}W[J]|_{J=J_{\bar{\phi}}} = \bar{\phi}(x) \,. \tag{2.48}$$

The quantum effective action $\Gamma[\phi]$ is then defined as:

$$\Gamma[\phi] = -\int d^dx\, \phi^i(x)J_{\phi i}(x) + W[J_\phi] \,. \tag{2.49}$$

It is easy to see that stationary points of $\Gamma[\phi]$ are related to one-point functions. Acting with a functional derivative with respect to the field, we find:

$$\begin{aligned}\frac{\delta}{\delta\phi^i(x)}\Gamma[\phi] &= -J_{\phi i}(x) - \int d^d y\, \phi^j(y)\frac{\delta J_{\phi j}(y)}{\delta\phi^i(x)} + \int d^d y\, \frac{\delta J_{j\phi}(y)}{\delta\phi^i(x)}\frac{\delta W[J_\phi]}{\delta J_{j\phi}(y)} \\ &= -J_{\phi i}(x)\,, \end{aligned} \tag{2.50}$$

where in the second line we have used (2.48). Therefore, a stationary point $\phi^{(0)}(x)$ is the expectation value at zero background current. The equations (2.47) and (2.50) lead to important relations between second functional derivatives of $W[J]$ and $\Gamma[\phi]$:

$$\begin{aligned} P^{(ix,jy)} &= \frac{\delta^2 W[J]}{\delta J_i(x)\delta J_j(y)} = \frac{\delta\phi^i_J(x)}{\delta J_j(y)}\,, \\ \Pi_{(ix,jy)} &= \frac{\delta^2\Gamma[\phi]}{\delta\phi^i(x)\delta\phi^j(y)} = -\frac{\delta J_{\phi i}(x)}{\delta\phi^j(y)}\,. \end{aligned} \tag{2.51}$$

In other words, $P^{(ix,jy)}$ and $\Pi(ix,jy)$ are each other's inverse, in the sense:

$$\int d^d z\, P^{(ix,jz)}|_{J=J_\phi}\Pi_{(jz,ky)} = -\int d^d x\, \frac{\delta\phi^i(x)}{\delta J_{\phi j}(z)}\frac{\delta J_{\phi j}(z)}{\delta\phi^k(y)} = -\delta^{(d)}(x-y)\,\delta^i{}_k\,. \tag{2.52}$$

The quantity $P^{(ix,jy)}$, evaluated at $J=0$, is the complete interacting propagator. The relation (2.51) is important for proving Goldstone's theorem. [126]

The quantum effective action has other interesting properties. One can obtain, in principle, the full partition function $Z[J]$ from a tree level calculation using the quantum effective action in place of the ordinary action $S[\phi]$ (see for instance [125, 128, 129]):

$$W[J] = W_{\Gamma^{(0)}}[J]\,. \tag{2.53}$$

The quantum effective action shares the symmetries of the path integral. In particular, consider an active, infinitesimal transformation:

$$\phi(x) \to \phi(x) + \epsilon\Delta\phi(x)\,, \tag{2.54}$$

where $\Delta\phi(x)$ depends on the jet space variables. If this transformation is a symmetry of the path integral (in the sense defined in the previous section), then the quantum effective action is invariant under:

$$\phi(x) \to \phi(x) + \epsilon\langle\Delta\phi(x)\rangle_{J_\phi}\,. \tag{2.55}$$

If the symmetry of the path integral is linearly realized on ϕ, then (2.55) is the same as (2.54).

2.2.3 Symmetries in Hilbert space

Let us turn our attention to symmetries in the canonical Hilbert space formalism. In classical field theory, we defined a symmetry transformation as a mapping on the space of field configurations that takes physical states (i.e. solutions to the classical field equations) to physical states. In the canonical formalism, physical configurations are represented by *rays* $\mathcal{R}$ in Hilbert space. A ray is a set of elements in Hilbert space that are related to each other by multiplication with a phase factor. In other words, two elements Ψ_1 and Ψ_2 in Hilbert space belong to the same ray $\mathcal{R}$ if $\Psi_1 = e^{i\phi}\Psi_2$ for some $\phi \in \mathbb{R}$. Then, a natural definition of a symmetry in quantum mechanics is a transformation that maps rays $\mathcal{R}$ to different rays $\mathcal{R}'$. Furthermore, we require that the symmetry transformation preserves transition probabilities. In other words, given an element Ψ_1 of $\mathcal{R}_1$ and Ψ_2 of $\mathcal{R}_2$, we require:

$$|\langle\Psi_1|\Psi_2\rangle|^2 = |\langle\Psi_1'|\Psi_2'\rangle|^2 \,. \tag{2.56}$$

where Ψ_1' and Ψ_2' are of course elements of the transformed rays $\mathcal{R}_1'$ and $\mathcal{R}_2'$. In the classical theory, we required in addition that the symmetry transformation be a diffeomorphism on the jet space of coordinates and field variables. Similarly, in the quantum theory we must require that the transformation have an inverse that preserves transition probabilities. Clearly, with these requirements in place, the set of group transformations satisfies the group axioms. Finally, a symmetry transformation must act on the asymptotic in and out states of the interacting theory as it does on the free particle states. Because of this last requirement, our definition of symmetry transformations in Hilbert space is stronger than the one we gave in the path integral formalism. We are essentially restricting to *symmetries of the S-matrix*, as we will explain in more detail in section 2.2.5.

We want to represent the symmetry as a mapping between ray representatives, i.e. as an operator acting on states in Hilbert space. A fundamental theorem by Wigner states that any operator U which realizes a symmetry transformation is either:

- *linear* and *unitary*:

$$\begin{aligned} &U(a\Psi + b\Phi) = aU\Psi + bU\Phi \,, \\ &\langle U\Psi|U\Phi\rangle = \langle\Psi|\Phi\rangle \,, \end{aligned} \tag{2.57}$$

- or *anti-linear* and *anti-unitary*:

$$\begin{aligned} &U(a\Psi + b\Phi) = a^*U\Psi + b^*U\Phi \,, \\ &\langle U\Psi|U\Phi\rangle = (\langle\Psi|\Phi\rangle)^* \,. \end{aligned} \tag{2.58}$$

For a simple and detailed proof, see [125]. To accommodate the fact that an anti-linear symmetry operator is anti-unitary, the adjoint of an anti-linear operator U is defined a little differently than usual. We have:

$$\langle\Phi|U\Psi\rangle = (\langle U\Phi|\Psi\rangle)^* , \tag{2.59}$$

for any Φ and Ψ. Then, the condition of (anti-)unitarity becomes $U^{-1} = U^\dagger$. Most interesting symmetry transformations are represented by linear and unitary operators. Any set of symmetry transformations that is continuously connected to the trivial transformation has to be linear and unitary, because the identity matrix is. Conversely, the anti-linear and anti-unitary transformation always involve some sort of flip of the time coordinate, which is a discrete transformation.

There is an interesting subtlety regarding the symmetry group realized by the operators U. Since symmetry transformations are defined as mappings between rays rather than between states, there can be a phase shift in the group composition law. To be precise, given two group elements g_1 and g_2 of the symmetry group that corresponds to the transformation on rays, we have the operators $U(g_1)$ and $U(g_2)$. Then, we have:

$$U(g_1)U(g_2)\Psi = e^{i\phi_{12}}U(g_1g_2)\Psi \, . \tag{2.60}$$

for some $\phi_{12} \in \mathbb{R}$. It can be shown that this phase factor does not depend on the state Ψ. [125] [4] When $\phi_{12} \neq 0$, the operators $U(g)$ form a *projective representation* of the group realized by the ray transformations.

Let us introduce a set of coordinates θ that describe the symmetry group around the identity element. We may then describe the group law by a function $f(\theta, \theta')$ of the coordinates:

$$U(\theta)U(\theta') = U(f(\theta, \theta')) \, . \tag{2.61}$$

We then Taylor expand the operators $U(\theta)$, the function $f(\theta, \theta')$ and the phase factor in (2.60):

$$\begin{aligned} U(\theta) &= 1 + i\theta^i g_i + \tfrac{1}{2}\theta^i\theta^j g_{ij} + \dots \, , \\ f^i(\theta, \theta') &= \theta^i + \theta'^i + c^i{}_{jk}\theta'^j\theta^i + \dots \, , \\ \phi(\theta, \theta') &= h_{ij}\theta^i\theta'^j + \dots \, , \end{aligned} \tag{2.62}$$

Then, by using (2.60), one can easily see that:

$$[g_i, g_j] = if^k{}_{ij}g_k + if_{ij}1 \, , \tag{2.63}$$

[4] To be more precise, it does not depend on the choice of state within a *superselection sector*. States in the same superselection sector may be prepared in superposition, whereas states in different superselection sectors cannot.

where $f^k{}_i j = -c^k{}_{ij} + c^k{}_{ji}$ and $f_{ij} = -h_{ij} + h_{ji}$. Therefore, a non-zero phase factor is related to an identity element appearing on the right-hand side of a Lie bracket, i.e. a central charge. Whenever one can redefine the generators to remove central charges, one can similarly redefine $U(\theta)$ to remove the phase factor in (2.60), at least in a finite neighborhood around the identity. In the important case of the Poincaré algebra, one can indeed remove all central charges by redefining the basis of generators [125]. However, the Poincaré group is not simply connected. In a group that is not simply connected, there may be a topological obstruction to setting the phase factor to zero globally. From now on, we will ignore the possibility of projective representations and assume we have defined our operators such that $\phi(\theta, \theta') = 0$.

2.2.4 The Poincaré group and classification of single-particle states

We will now discuss the important Poincaré symmetry group of relativistic theories on flat $D = 4$ space-time. Our main aim here is to show how the existence of a Poincaré symmetry allows for a convenient definition of the particle species. The $D = 4$ Poincaré group $ISO(1, 3)$ is defined by the set of coordinate transformations which leave the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ invariant, i.e. it is the isometry group of Minkowski space. The coordinate transformations are characterized by the Lorentz transformations $\Lambda^\mu{}_\nu$ and the translations a^μ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,, \tag{2.64}$$

where $\Lambda^\mu{}_\nu$ satisfies:

$$\Lambda^\mu{}_\rho \, \eta_{\mu\nu} \, \Lambda^\nu{}_\lambda = \eta_{\rho\lambda} \,. \tag{2.65}$$

The Lie algebra of the Poincaré group is generated by the Lorentz generators $M_{\mu\nu}$ and translation generators P_μ. It is related by exponentiation to the subgroup formed by translations and the Lorentz transformations with $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geq 1$. The latter form what is known as the *proper orthochronous Lorentz group*. The commutation relations are:

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho}) \,, \\ &[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \quad [P_\mu, P_\nu] = 0 \,, \end{aligned} \tag{2.66}$$

Although we specialize to $D = 4$ in this section, these commutation relations define the Poincaré algebra $\mathfrak{iso}(1, D-1)$ for all dimension $D \geq 2$.

The Poincaré group is realized on the Hilbert space by unitary operators

$U(\Lambda, a)$ which satisfy the composition law [5]:

$$U(\Lambda, a)U(\Lambda', a') = U(\Lambda\Lambda', \Lambda a' + a)\,. \tag{2.67}$$

The generators $M_{\mu\nu}$ and P_μ appear in the Taylor expansion of these operators around the origin:

$$U(1+\omega, \epsilon) = 1 + \frac{1}{2}i\omega^{\mu\nu}M_{\mu\nu} - i\epsilon^\mu P_\mu + \ldots\,. \tag{2.68}$$

Because $U(\Lambda, a)$ is unitary, P_μ and $M_{\mu\nu}$ are both Hermitian as quantum operators. Note that $U(\Lambda, a)$ and $M_{\mu\nu}$ cannot be respectively unitary or Hermitian in any finite-dimensional *matrix* representation, as the Lorentz group is non-compact. We can now use the generators $(P_\mu, M_{\mu\nu})$ to define multiplets of single-particle states as (infinite dimensional) representations of the Poincaré group. Firstly, consider the eigenstates $\Psi_{p,\sigma}$ of the four-momentum operators P_μ:

$$P_\mu\Psi_{p,s} = p_\mu\Psi_{p,s}\,. \tag{2.69}$$

The subscript s indicates a set of discrete variables that label the state. The finite translation $U(1, a)$ is then represented as $U(1, a)\Psi_{p,\sigma} = e^{-ia^\mu P_\mu}\Psi_{p,s} = e^{-ia^\mu p_\mu}\Psi_{p,s}$. Then, the Lorentz transformations $U(\Lambda, 0) = U(\Lambda)$ send the state with momentum eigenvalue p to a linear combination of states with momentum eigenvalue Λp:

$$U(\Lambda)\Psi_{p,s} = \sum_{s'} C_{s,s'}(\Lambda)\Psi_{\Lambda p,s'}\,. \tag{2.70}$$

Now, the state space of a particular particle species coincides with an irreducible Lorentz-invariant subspace of the space of states $\Psi_{p,s}$. In other words, the particle species are identified with the irreducible representations of the Lorentz group contained in $C_{s,s'}$. More generally, one would identify particle species according to irreducible representation of the full symmetry group of the theory. For example, the state space of gluons in the Standard Model coincides with a multiplet of massless vector representations of the Lorentz group that together form the adjoint representation of $SU(3)$.

We can learn a lot about the matrices $C_{s,s'}$ by using the *little group* method, more generally known as the *method of induced representations.* [36] We first separate the space of momentum vectors p^μ into equivalence classes

[5]We assume that we are dealing with bosonic representations. For fermions a minus sign can appear, signaling a projective representation.

formed by the orbits of proper orthochronous (homogeneous) Lorentz transformations. In other words, two momenta p^μ and k^μ belong to the same class if and only if they are related by a homogeneous Lorentz transformation L: $p^\mu = L^\mu{}_\nu k^\nu$. Proper orthochronous Lorentz transformations do not change the invariant mass $p^\mu p_\mu$ or the sign of p^0. Therefore, the equivalence classes are characterized by those properties. Then, we pick a representative *standard momentum* from each equivalence class. The subgroup of Lorentz transformations $W^\mu{}_\nu$ that leaves the standard momentum k^μ invariant, $W^\mu{}_\nu k^\nu = k^\mu$, is known as the *little group* of the class. For most physical purposes, there are three classes of interest:

- $p^2 = 0$, $p^0 = 0$. Representative $k^\mu = (0, 0, 0, 0)$. The little group is the full $SO(1, 3)$. The vacuum state in unbroken Lorentz-invariant theories belongs to this class.
- $p^2 = -M^2$, $p^0 < 0$. Representative $k^\mu = (-M, 0, 0, 0)$. Little group is $SO(3)$. This is the class of massive particles.
- $p^2 = 0$, $p^0 > 0$. Representative $k^\mu = (k, k, 0, 0)$. Little group is $ISO(2)$. Massless particles belong to this class.

For each momentum p^μ in the class of a particular k^μ, we define a standard Lorentz transformation $L(p)$ that takes k^μ to p^μ: $L(p)^\mu{}_\nu k^\nu = p^\mu$. Now define the orthonormalized eigenstates $\Psi_{k,s}$ of representative momenta:

$$\langle \Psi_{k,s} | \Psi_{k',s'} \rangle = \delta^{(3)}(\vec{k} - \vec{k}') \delta_{s,s'} \,. \tag{2.71}$$

We now assume that $p^0 > 0$. Define the states of arbitrary momentum p^μ as:

$$\Psi_{p,s} = \sqrt{\frac{p^0}{k^0}} U(L(p)) \Psi_{k,s} \,. \tag{2.72}$$

This definition then fixes all the transformation properties of the states $\Psi_{p,s}$:

$$U(\Lambda)\Psi_{p,s} = \sum_{s'} D_{s,s'}(W(\Lambda, p)) \Psi_{\Lambda p,s'} \,, \tag{2.73}$$

where $D_{s,s'}$ is an irreducible representation of the little group in the class of p^μ. For every p, $W(\Lambda, p)$ is a mapping from the Lorentz group to the little group in the equivalence class of p. It is defined as follows:

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) \,, \tag{2.74}$$

where $L(p)$ is the mapping defined above that takes a standard momentum k^μ to the argument p^μ. The mapping $W(\Lambda, p)$ is known as the *Wigner rotation* [36].

For massive particles, the Wigner rotation acts like an ordinary rotation when Λ is restricted to the $SO(3)$ subgroup. This means that all the familiar properties of $SO(3)$ representations in quantum mechanics carry over to relativistic quantum field theory. The representations are labeled by the spin quantum number $j \in \mathbb{N}$. They act on a $(2j+1)$-dimensional vector space. To include fermionic representations, one either has to allow for projective representations or enhance the Lorentz group to the universal cover $SL(2,\mathbb{C})$. Then, the little group for massive particles is $SU(2)$, which admits half-integer j representations.

The $ISO(2)$ little group has rather different properties. As $ISO(2)$ is not semi-simple, it allows in principle for *continuous spin representations* (CSR), which have a continuous spin degree of freedom. As these are not observed, one has to impose by hand that the eigenvalues of two of the generators of $ISO(2)$ are zero for all physical states. That leaves one generator, whose eigenvalue is known as the *helicity* of the massless particle state. This restriction on the massless degrees of freedom is implemented on the Lagrangian field theoy side by gauge redundancy. It is interesting to note that CSRs do not exist in low-energy models coming from string theory. [144] Ordinary field theory, on the other hand, does not explain why nature does not make use of CSRs.

2.2.5 Symmetries of the S-matrix

In the beginning of this section, we defined a quantum symmetry as an invertible mapping from the space of rays to itself that preserves transition probabilities. This definition ensures that symmetry transformations are represented on the Hilbert space as (anti)-linear and (anti)-unitary operators. We were then able to define single free particle states according to their transformation law under Lorentz transformations. We also required that a symmetry transformation act on the in and out states of the S-matrix in the same way that it does on free particle states. Let us take a moment to clarify this last requirement.

The *S-matrix* contains all the information about scattering experiments in a QFT. In a scattering experiment, a number of particles that are initially separated by large distances come together to interact in a small region, producing a possibly different set of particles which then separate again to large distances. As an approximation, the incoming particles (represented by the *in states* Ψ^-_α) come in from the infinite past $t = -\infty$ and the outgoing particles (the *out states* Ψ^-_α) travel into the infinite future $t = \infty$. We assume that the particles do not interact at all when they are separated by large distances, so that the in and out states approximate free particle states.

The S-matrix is then defined according to the matrix elements between in and out states:

$$\langle \Psi_\alpha^- | \Psi_\beta^+ \rangle = S_{\alpha\beta} \,, \tag{2.75}$$

where α, β labels stand for discrete variables (e.g. particle number, Lorentz or color indices) as well as momenta.

Let us give a proper definition of the in and out states. Consider a Hamiltonian $H = H_0 + H_{int}$ decomposed into a free Hamiltonian H_0 and an interaction Hamiltonian H_{int}. We have the following eigenstates of H and H_0:

$$H\Psi_\alpha = E_\alpha \Psi_\alpha \,, \quad H_0 \Phi_\alpha = E_\alpha \Phi_\alpha \,. \tag{2.76}$$

Note that H_0 is defined in such a way that H and H_0 have the same spectrum E_α. Now consider a probability $g(\alpha)$ distribution over the (discrete and continuous) particle labels α. Assume that $g(\alpha)$ is smooth over the continuous variables in a finite range. Otherwise, $g(\alpha)$ is arbitary. Then, the in and out states $\Psi_\alpha^\pm$ are those eigenstates of H which satisfy:

$$\lim_{t\to\pm\infty} \exp(-iHt) \int d\alpha \, g(\alpha) \Psi_\alpha^\pm = \lim_{t\to\pm\infty} \int d\alpha \, e^{-iE_\alpha t} g(\alpha) \Phi_\alpha \,. \tag{2.77}$$

In other words, the in and out states are energy eigenstates of the full Hamiltonian which approximate single free particle states in the distant past or future. Note that the states Ψ_α themselves are time-independent in the Heisenberg picture. The equation (2.77) implies that one will find a particle of the type represented by Φ_α if one observes the state Ψ_α in the infinite past or future. The distribution function $g(\alpha)$ is a necessary part of the definition because the operator $\exp(-iHt)$ acting on an eigenstates produces only an unphysical phase factor. The S-matrix is unitary: $S = S^\dagger$. It is useful to define a unitary operator S which represents the S-matrix on the free particle states, in the sense that:

$$\langle \Phi_\alpha \,|\, S\Phi_\beta \rangle = S_{\alpha\beta} \,. \tag{2.78}$$

Now let us come back to our definition of symmetries: a symmetry transformation acts on in and out states in the same way as it does on free particle states. For simplicity's sake, we consider here an internal symmetry group G, which only acts on internal discrete labels. For more complicated examples, see again [125]. The symmetry is realized as a unitary, linear operator $U(g)$ where g is a group element. On free particle states, we have:

$$\begin{aligned} g \cdot \Phi_{p_1\sigma_1 l_1;\, p_2\sigma_2 l_2;\, \dots} &= U(g) \Phi_{p_1\sigma_1 l_1;\, p_2\sigma_2 l_2;\, \dots} \\ &= \sum_{L_1, L_2, \dots} D^{(g)}_{l_1 L_1} D^{(g)}_{l_2 L_2} \dots \Phi_{p_1\sigma_1 L_1;\, p_2\sigma_2 L_2;\, \dots} \,, \end{aligned} \tag{2.79}$$

where $D^{(g)}$ is some unitary representation of the symmetry group. Now, our definition of a quantum symmetry implies that:

$$S_{p_1\sigma_1 l_1,\ldots;p_1\sigma_1 l'_1,\ldots} = \sum_{L_1,L_2,\ldots} \sum_{L'_1,L'_2,\ldots} (D^{(g)}_{l_1 L_1})^*(D^{(g)}_{l_2 L_2})^* \ldots D^{(g)}_{l'_1 L'_1} D^{(g)}_{l'_2 L'_2} \ldots S_{p_1\sigma_1 L_1,\ldots;p_1\sigma_1 L'_1,\ldots} \tag{2.80}$$

This is the case if and only if $U(g)^{-1}SU(g) = S$. In other words, the generators of the symmetry group G_i must commute with S:

$$[G_i, S] = 0\,. \tag{2.81}$$

Thus, the existence of a quantum symmetry leads to an *algebraic condition* on the S-matrix. As we stated in the beginning of this section, our definition for symmetries in Hilbert space is stricter than for symmetries of the path integral. Indeed, not every symmetry of the path integral leads to a condition of the type (2.81). However, we saw in the previous section that symmetries of the path integral lead to the important operator equation (2.42).

Let us return to our discussion in 2.1.4. There, we indicated that some symmetries which have a Noether current J_i^μ may not integrate into a well-defined charge Q_i. This may occur for special classical field configurations, but it is a particularly important point in QFT. There may be symmetries of the Lagrangian with well-defined quantum currents which nevertheless are not represented by a unitary operator in the Hilbert space. There is then no operator G_i which imposes (2.81). Such a symmetry does not lead to an algebraic condition on the S-matrix, but still has physical implications on the dynamics of the theory. We have already come across an example of such symmetries, namely proper (i.e. not large) gauge redundancies. For the simple example of the $U(1)$ Maxwell theory, the gauge redundancy implies that the gauge potential A_μ appears in the Lagrangian only in the field strength $F_{\mu\nu}$. The associated current, however, is trivial and leads to a vanishing charge. However, the gauge parameter is constant (i.e. performing a "large gauge transformation"), we of course obtain conservation of electric charge, which imposes an algebraic condition on the S-matrix.

We are therefore led to the following classification of symmetries, following Weinberg's contribution to [137]. We will call symmetry transformations that lead to algebraic conditions on the S-matrix (i.e. quantum symmetries as we have defined them) *algebraic symmetries*. Symmetries of the Lagrangian that do not lead to such algebraic conditions are called *dynamical symmetries*. Note that we are not talking about symmetries which are destroyed by the quantization process (*anomalous symmetries)*, but symmetries of the full path integral that do not lead to algebraic conditions on the S-matrix.

In the next section, we will turn our attention to a very important class of dynamical symmetries: the spontaneously broken global symmetries. Such symmetries do not lead to algebraic conditions due to the appearance of massless *Goldstone modes* which can render the integral (2.21) divergent.

2.3 Spontaneous symmetry breaking and non-linear realizations

Not only theories, but also physical states may enjoy symmetry. A state $\bar{\phi}$ inherits a symmetry of the action whenever that symmetry sends $\bar{\phi}$ into itself. It is clear that a generic symmetry transformation does not leave all solutions to the field equations invariant, so most physical states break at least some of the symmetries of the action. [6] A particularly important case is when the *vacuum*, the state of lowest energy, breaks a symmetry of the action.

Let us address symmetry breaking by the vacuum state first in classical field theory. Consider the action $S[\phi]$ with a variational global symmetry group G and a vacuum solution $\phi_0(x)$. In the active form, the symmetry group G acts on the fields $\phi(x)$ as

$$g \cdot \phi(x) = \phi'(x) = f(x, \phi, \ldots)\,, \tag{2.82}$$

while the coordinates do not transform. A symmetry transformation for which $g \cdot \phi_0(x) \neq \phi_0(x)$ is broken by the vacuum field configuration. Conversely, the set of symmetry transformations h that do leave the vacuum invariant ($h \cdot \phi_0(x) = \phi_0(x)$), forms the *unbroken* subgroup H of G. Any two elements g_1 and g_2 have an equivalent action on the vacuum whenever $g_1 = g_2 h$ for some $h \in H$. The set of symmetry transformations broken by the vacuum state therefore makes up the right-coset manifold G/H. The coset manifold is itself a Lie group if and only if H is an invariant subgroup of G.

Now, let us rearrange our field basis such that $\phi_0(x) = 0$. In this field basis, any transformation broken by the vacuum is realized as a non-linear transformation on the fields. We may therefore use the theory of *non-linear realizations* to describe symmetries broken by the vacuum state. In fact, we

[6] A symmetry transformation that leaves all solutions invariant is called a *zilch symmetry*. Any action with two fields or more enjoys a zilch symmetry. [123] The Noether current associated to a zilch symmetry always vanishes on-shell, so it has no immediate physical implications. However, there are infinitesimal symmetry transformations whose algebra only closes up to zilch symmetries. Alternatively, one can say that such algebras are closed only on-shell.

can choose to dispense with the initial vacuum field configuration $\phi_0(x)$ and symmetry transformation (2.82) altogether. In this bottom-up approach, we keep only the fluctuations around the vacuum in the $\phi_0(x) = 0$ field basis and fix the symmetry transformations using non-linear realizations. We will have much more to say about this in Chapter 3.

One often starts with a field basis such that the origin $\phi(x) = 0$ is invariant under the full symmetry group: $g \cdot 0 = 0$. This is only possible if there are field configurations $\bar{\phi}(x)$ such that $g \cdot \bar{\phi}(x) = \bar{\phi}(x)$ for all $g \in G$. If there are no such field configurations, a subset of the symmetry transformations is non-linearly realized in any field basis. A symmetry transformation which admits a linear realization in some field basis, but which does not map ϕ_0 into itself, is known as a *spontaneously broken symmetry.* [44–47, 58, 61, 62]

Let us illustrate spontaneous symmetry breaking in a simple example. Consider a relativistic theory with a Poincaré invariant vacuum state with a spontaneously broken internal Lie group symmetry. Then, in the vacuum all time and space derivatives vanish so that the problem of finding the vacuum state reduces to finding extrema $\bar{\phi}$ of the potential $V(\phi)$. Because the symmetry is internal, it maps the vacuum into another Lorentz-invariant state of the same energy. In addition, because the symmetry transformation is continuous, the potential $V(\phi)$ must have flat directions at the point $\bar{\phi}$. Of course, any of the field configurations along the flat direction is an equally valid choice for vacuum state. We discover that spontaneous symmetry breaking is related to *vacuum degeneracy.* In addition, we observe that fluctuating the field configuration in one of the flat directions costs no potential energy. In other words, we find that the spontaneously broken symmetry corresponds to a massless mode. This is known as *Goldstone's theorem*: [63,87] every generator of a spontaneously broken, global Lie group symmetry corresponds to a massless particle with the same quantum numbers.

Clearly, fluctuating in the flat direction is equivalent to locally performing an infinitesimal transformation with the broken symmetry generator. Therefore, given a vacuum field configuration $|0\rangle$ and a broken symmetry generator G_i, we can identify the classical Goldstone modes with the following field configurations: [90]

$$\phi^i(x) G_i |0\rangle \, , \tag{2.83}$$

where $\phi^i(x)$ is some slowly varying function of the space-time coordinates.

To develop these ideas further, let us address spontaneous symmetry breaking in the quantum theory.

Let us now present a proof of Goldstone's theorem, due to Weinberg [126], that does not make use of the integrated quantum charge. Assume that we have the following path integral symmetry, linearly realized on the scalar fields $\phi_i(x)$:

$$\phi^i(x) \to \phi^i(x) + i\epsilon\, G^{ij}\phi^j(x)\,, \tag{2.84}$$

where G^{ij} is a matrix, providing a representation of a single generator of an internal symmetry group. Since the transformation is linear in the fields, the quantum effective action is invariant under the same transformation. We assume that the vacuum of our theory is Poincaré-invariant. In other words, the expectation values of $\phi^i(x)$ do not depend on the coordinates. Then, evaluated on the vacuum state, the effective action is just a potential:

$$\Gamma[\Phi] = -\mathcal{V}V[\phi]\,, \tag{2.85}$$

where we split off a factor $\mathcal{V}$ equal to the volume of space-time. Invariance of the effective action now leads to:

$$\frac{\partial V[\phi]}{\partial \phi^i} G^{ij}\phi^j = 0\,. \tag{2.86}$$

Differentiating this equation with respect to ϕ and evaluating at a stationary point $\bar{\phi}$ of the effective potential yields:

$$\left(\frac{\partial^2 V[\bar{\phi}]}{\partial\bar{\phi}^i\partial\bar{\phi}^j}\right) G^{ik}\bar{\phi}^k = 0\,. \tag{2.87}$$

The second derivative of the effective potential is also the second derivative of the quantum effective action, evaluated on constant field configurations. Therefore, we can identify $-\mathcal{V}\frac{\partial^2 V[\phi]}{\partial\phi^i\partial\phi^j} = \Pi_{(i,j)}(0)$. The quantity $\Pi_{(i,j)}(0)$ is the momentum-space version of $\Pi_{(ix,jy)}$ from (2.51), evaluated at zero momentum because we are dealing with constant field configurations. We find that $\Pi_{(i,j)}(0)$ has a non-zero eigenvector $G^{ij}\bar{\phi}^j$ with vanishing eigenvalue. The eigenvector here is non-zero, of course, precisely because we assume that the symmetry transformation does not leave the vacuum field configuration invariant. Since $P^{(i,j)}(q^2)$, the complete propagator, is the inverse of $\Pi_{(i,j)}$, it must have a pole at $q^2 = 0$. By the standard arguments regarding the non-perturbative structure of n-point functions (see e.g. [129]) we can identify this pole with a massless particle: the Goldstone mode of the broken symmetry transformation generated by G^{ij}.

Throughout this section, we have been careful to distinguish spontaneously broken symmetries from the larger set of symmetries broken by the vacuum state. Although one often assumes that a non-linear symmetry is an IR consequence of a linear symmetry in the UV, we can entertain the idea that a non-linear symmetry exists *period*. Therefore, there may be symmetries broken by the vacuum which are not, strictly speaking, spontaneously broken. In the rest of this work, we will be less careful with the term *Goldstone mode*. In our nomenclature, any field which non-linearly realizes a symmetry as a *preferred field* (as opposed to unpreferred *matter fields*, see Chapter 3), is a Goldstone field. As we will see, a generalized Goldstone field is not always protected from obtaining a mass. Indeed, not all of the symmetries we examine admit a linear realization, at least not by an ordinary local field theory living in the same number of dimensions.

2.4 Classification of algebraic symmetries

We have now defined many different classes of symmetries: global and local, internal and space-time, algebraic and dynamical, and so on. It is now time to ask what symmetry groups of each class can actually be realized in a physical theory. Let us begin to answer this question in the context of the algebraic symmetries, i.e. symmetries of the S-matrix. In 1976, Coleman and Mandula [38] proved the following important theorem: in a relativistic interacting theory in $D = 4$, all possible algebraic symmetry Lie groups are a direct product of the Poincaré group and an internal symmetry group.

The work of Coleman and Mandula put an end to much speculation about enhancements of the Poincaré group. It is not possible to introduce *hybrid symmetries*, which mix particle flavor and Lorentz indices. Nor can one embed the Poincaré group in a larger space-time symmetry group. It is not always impossible to define single-particle states which are representations of hybrid symmetries or enhanced space-time symmetry groups. However, either of these possibilities will introduce physically unrealistic behavior in scattering experiments. To make this clear, let us state the Coleman-Mandula theorem. [40]

The Coleman-Mandula theorem: Assume that:

- Algebraic symmetries are represented in Hilbert space as unitary operators that commute with the S-matrix operator S.
- The Poincaré group is a subgroup of the algebraic symmetry group.
- All particles have a positive mass. The number of particle types with mass below any given energy scale is finite.
- The T-matrix, $T_{\alpha\beta} = S_{\alpha\beta} - \mathbb{1}_{\alpha\beta}$ for $2 \to 2$ scattering is a product of an energy-momentum conservation Dirac delta and an analytic function of the kinematic Mandelstam variables s and t.

- The T-matrix vanishes only for a discrete set of values of the kinematic variables.

- The matrix elements $\langle \Psi_\alpha | G_i \Psi_\beta \rangle$ of the generators G_i are distributions in momentum space.

Then, the most general algebraic symmetry group is a direct product of the Poincaré group and an internal symmetry group. For a detailed proof, see [40].

The last assumption listed stands out from the others as technical rather than physical. This is because the Coleman-Mandula theorem was proved without assuming that the theory is a QFT. In a QFT, however, the technical assumption is manifest. The other assumptions are quite minimal, but they are by no means satisfied by all interesting QFTs. In particular, one may give up the idea of particles states that decouple at large spatial separation, invalidating the concept of scattering experiments altogether. This is the case for conformal field theories (CFTs), which indeed enhance the Poincaré group to the conformal group. Another interesting possibility is to introduce symmetries that form a Lie *supergroup* rather than an ordinary Lie group.

Close to the identity, a supergroup is characterized by a *superalgebra* (or *graded algebra*). A superalgebra contains even and odd elements, which satisfy commutation and anti-commutation relations respectively. It turns out that there is only one way to enhance the Poincaré group with odd algebra elements. Haag, Lopuszanski and Sohnius [39] proved that the odd elements must be spin-$\frac{1}{2}$ representations of the Lorentz algebra, Q^L_α and $\bar{Q}_{\dot{\alpha}M}$. Their anti-commutator must give rise to the translation generator $P_{\alpha\dot{\alpha}}$. This is known as the *supersymmetry* algebra. In theories with linear supersymmetry, particles of integer and half-integer spin come as pairs with equal mass in *supermultiplets*. The fact that each boson is paired with an equal-mass fermion gives rise to cancellations in many important calculations. For this reason, supersymmetric theories have been of phenomenological interest for a very long time. Furthermore, supersymmetry is an essential part of all string theories that contain fermions. Let us give a precise statement of the Haag-Lopuszanski-Sohnius theorem.

The Haag-Lopuszanski-Sohnius theorem: Allow the set of algebraic symmetry transformations to form a Lie supergroup. Make all assumptions listed for the Coleman-Mandula theorem. In addition, assume that: [122]

- The odd operators $Q_{n,m} = Q_{\alpha_1 \dots \alpha_n \dot{\alpha}_1 \dots \dot{\alpha}_m}$ act in a Hilbert space with a positive definite metric.

- The Hermitian conjugates $\bar{Q}_{m,n} = \bar{Q}_{\alpha_1 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n}$ of $Q_{n,m}$ live in the superalgebra.

Then, the most general algebraic superalgebra is the $\mathcal{N}$-extended, centrally extended supersymmetry algebra. This superalgebra is spanned by the Poincaré generators, the supertranslations $(Q^M_\alpha, \bar{Q}_{\dot{\alpha}M})$, and the scalar generators B_i. The B_i generators make up an internal *R-symmetry* algebra which is the direct sum $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ of a semi-simple algebra $\mathfrak{a}_1$ and an Abelian algebra $\mathfrak{a}_2$. For the full list of anti-commutators in the general $D = 4$ supersymmetry algebra, see for example [122].

The theorems of Coleman-Mandula and Haag-Lopuszanski-Sohnius are very powerful. However, as we have emphasized throughout, they are applicable only for *algebraic* symmetries. There are numerous ways to add a non-trivial dynamical space-time symmetry to a relativistic field theory. We will encounter examples later in this chapter. It is natural to wonder whether one can give a similar accounting of the possible *dynamical* space-time symmetries. The rest of this thesis will be an attempt to answer this question. We will concentrate on the non-linearly realized global symmetries. It turns out that one can get very general results in two different ways. First, one can use the existence of an enhanced *soft limit* in scattering amplitudes, which always accompanies a non-linearly realized symmetry, as we will see in the next section. Second, we can make use of algebraic properties of non-linear realizations. These properties follow from the general theory of non-linear realizations, to be discussed in chapter 3.

2.5 Dynamics of Goldstone modes

Spontaneously broken symmetries have qualitatively different physical implications than symmetries of the physical Hilbert space. We have seen in section 2.2.4 that the latter may be used to define the particle multiplets of the theory in a convenient way. Spontaneously broken symmetries, being dynamical symmetries, do not lead to a similar classification of the particle spectrum. We can however, under the conditions discussed in section 2.3, identify a massless Goldstone modes for each broken symmetry. In addition, the interactions one can write down for the Goldstone modes are constrained by the broken and unbroken symmetries of the theory. This enables us to make very general statements, sometimes called *soft theorems* or *low-energy theorems*, about the way Goldstone modes couple to each other and to unpreferred "matter fields" which are not associated to a broken symmetry.

The most well-known example of a soft theorem is the *Adler zero*. Consider a theory with a spontaneously broken internal symmetry and its corresponding Goldstone boson ϕ. Then, the S-matrix element (up to factors of π and momentum conservation delta functions) for emitting a ϕ particle with on-shell momentum p in a $\alpha \to \beta$ scattering process is: [2–4, 126]

$$M_{\beta+\phi(p),\alpha} = \frac{1}{F} p_\mu N^\mu_{\beta\alpha} , \tag{2.88}$$

where $N^\mu_{\beta\alpha}$ is the regular contribution to the matrix element $\langle \beta | J^\mu(0) | \alpha \rangle$, with J^μ the Noether current of the ϕ symmetry. The important part of equation (2.88) is the factor p^μ. It ensures that the amplitude for emitting a ϕ particle goes to zero as the external momentum p^μ goes to zero. One often says that the scattering amplitude for this process has a vanishing *soft limit*. It is easy to motivate the Adler zero from our discussion in section 2.3. In the field basis $\phi_0 = 0$, the spontaneously broken symmetry transformation necessarily acts on the Goldstone field ϕ with a constant shift:

$$\delta\phi = c + \dots . \tag{2.89}$$

The shift requires that ϕ appears in the Lagrangian always with at least a single derivative.[7] In the literature, it is often said that ϕ must be *derivatively coupled*.

In fact, one can generalize (2.88) for symmetries that shift ϕ by polynomials in the coordinates rather than by a constant. In general, as we will see, a symmetry transformation that includes a term x^n leads to n powers of p^μ in (2.88). Such a symmetry with $n > 0$ is necessarily space-time, as it fails to commute with translations. We will have much more to say about such transformations in Chapter 3.

In this section, we will give an explanation of soft limits for higher-order symmetries and define *exceptional EFTs*, which have the highest-possible soft degree for a given derivative power counting. Exceptional EFTs will play an important part in the classification of non-linear space-time symmetries in Chapters 4 and 5. We will end this chapter by discussing two quintessential non-linear realizations: the Akulov-Volkov Goldstino and the DBI scalar.

2.5.1 Soft limits and extended shift symmetries

Consider a theory with a scalar field $\phi(x)$ which realizes the following n-th order non-linear symmetry transformation:

$$\delta^{(n)}\phi(x) = c_{\mu_1 \dots \mu_n} [x^{\mu_1} \dots x^{\mu_n} + f^{\mu_1 \dots \mu_n}(x, \phi, \partial\phi, \dots)] . \tag{2.90}$$

[7] Wess-Zumino terms, to be discussed in Chapter 3, are an exception to this statement

As we will see in Chapter 3, closure of the symmetry algebra requires that ϕ also realize an $(n-1)$-th order symmetry transformation of the same form as (2.90). Working our way backwards, we find that the scalar field realizes a shift symmetry $\delta^{(0)}\phi = c + \ldots$. Each independent transformation $\delta^{(n)}\phi$ leads to a current $(J^{(n)})^{\mu\mu_1\cdots\mu_n}$. These currents, however, are not all independent. There is an important off-shell relation between $J^{(n)}$ for $n > 0$ and the current for the shift symmetry $J^{(0)}$, discovered in [5] (see also [3]). This relation, together with the ordinary Adler zero, leads to enhanced soft degrees for theories with symmetries (2.90) for $n > 0$. Let us rewrite the whole chain of transformations (2.90) as:

$$\bar{\delta}\phi(x) = c_{(n)}(\alpha^{(n)}(x) + \alpha_A^{(n)}(x)O^A[\phi])\,, \tag{2.91}$$

where we have suppressed all Lorentz indices. Here, $\alpha_A^{(n)}$ and $\alpha^{(n)}$ are polynomials in x^μ and $O^A[\phi]$ is constructed out of ϕ and its space-time derivatives.

We can obtain a relation between $J^{(0)}$ and the other currents by applying a standard trick. One can calculate a Noether current for any global symmetry from the transformation of the action under a localized version of the symmetry. For instance, consider the following transformation:

$$\delta\phi = c(x)\Delta\phi(x)\,. \tag{2.92}$$

Assume that this is a symmetry transformation only for $c(x) =$ constant. Then, the transformation of the action should be proportional to a derivative of $c(x)$, as it vanishes in the limit $c(x) \to$ constant:

$$\delta S = \int d^dx\,\partial_\mu c(x) J^\mu(x, \phi, \partial\phi, \ldots)\,. \tag{2.93}$$

By definition, any variation of ϕ leaves S invariant when the equation of motion are satisfied. Therefore, after partial integration, we find $\partial_\mu J^\mu(x) = 0$ on-shell, which identifies J^μ as a Noether current. Now let us apply this trick to the transformation (2.91). We can write $\bar{\delta}\phi(x)$ as a special localized shift symmetry:

$$\bar{\delta}\phi(x) = \hat{a}(x)\,, \tag{2.94}$$

where $\hat{a}(x) = c_{(n)}(\alpha^{(n)}(x) + \alpha_A^{(n)}(x)O^A[\phi])$. Now, we can obtain the Noether currents either by applying the trick for a localized shift symmetry or by localizing the $c_{(n)}$ parameters of the n-th order transformations. We thus find:

$$\int d^dx\,\partial_\mu\hat{a}(x)\left(J^{(0)}\right)^\mu(x) = \int d^dx\,\partial c_{(n)} \cdot J^{(n)}(x)\,, \tag{2.95}$$

suppressing Lorentz indices for the n-th order case. In the limit $c_{(n)} \to$ constant, the integral on the right vanishes. Therefore, the integrand on the left becomes a total derivative. Writing out the derivative of $\hat{a}(x)$ and neglecting all $\partial c_{(n)}$ terms, we obtain:

$$\left(\partial\alpha^{(n)} + \partial\alpha_A^{(n)} O^A[\phi] + \alpha_A^{(n)} \partial O^A[\phi]\right) \cdot J = \partial_\mu \left(\big(\beta_I^{(n)}\big)^\mu O^I[\phi] \right), \tag{2.96}$$

where $\beta_I^{(n)}$ is a function of the coordinates and $O^I[\phi]$ is a local operator built out of ϕ and its derivatives. Equation (2.96) is the first of the off-shell relations. We find another one by inserting it back into (2.95):

$$\big(J^{(n)}\big)^\mu = \alpha_A^{(n)} O^A[\phi] J^\mu(x) - \big(\beta_I^{(n)}\big)^\mu O^I[\phi] \,. \tag{2.97}$$

Next, we follow [3] and assume that the identities (2.96) and (2.97) continue to hold in the quantum theory as operator equations. That is, we assume that they hold when squeezed in between the in- and out-states $|\beta\rangle$ and $\langle\alpha|$:

$$\langle\alpha|\left(\partial\alpha^{(n)} + \partial\alpha_A^{(n)} O^A[\phi] + \alpha_A^{(n)} \partial O^A[\phi]\right) \cdot J|\beta\rangle = \partial_\mu \langle\alpha|\big(\beta_I^{(n)}\big)^\mu O^I[\phi]|\beta\rangle \,, \tag{2.98}$$

Using a Ward identity for J^μ then leads to:

$$\partial\alpha^{(n)} \cdot \langle\alpha|J|\beta\rangle = -\partial \cdot \langle\alpha|\alpha_A^{(n)} O^A[\phi] J - \beta_I^{(n)} O^I[\phi]|\beta\rangle \,, \tag{2.99}$$

or, collecting the α_A and β_I polynomials and the corresponding local operators $O^A[\phi]$, $O^I[\phi]$:

$$\partial\alpha^{(n)} \cdot \langle\alpha|J(x)|\beta\rangle = \partial \cdot \langle\alpha|\gamma_C(x) O^C[\phi]|\beta\rangle \,. \tag{2.100}$$

We now need to elaborate on the inner products that appear in (2.100). Let us recall our discussion from section 2.3 on the non-perturbative pole structure of n-point functions. First, factor out the x-dependence in the inner products by using the translation operator:

$$e^{-ip\cdot x} \partial\alpha^{(n)} \cdot \langle\alpha|J(0)|\beta\rangle = \partial \cdot \big(\gamma_C(x) e^{-ip\cdot x}\big) \langle\alpha|O^C[\phi](0)|\beta\rangle \,. \tag{2.101}$$

Just like in section 2.3, the quantities $\langle\alpha|J(0)|\beta\rangle$ and $\langle\alpha|O^C[\phi](0)|\beta\rangle$ develop poles (as function of $p^\mu = P^\mu_\beta - P^\mu_\alpha$) at the mass squared of each physical particle, as long as the operator in the middle has a non-zero inner product in between the vacuum and the particle in-state. Let us denote the Goldstone boson in-states of momentum p^μ by $|\phi(p)\rangle$. The residues of the poles at zero mass due to the Goldstones are then given precisely by operator squeezed in between the vacuum and the single Goldstone in-states. By Lorentz covariance, we have:

$$\langle 0|J^\mu(x)|\phi(p)\rangle = ip^\mu F e^{-ip\cdot x}\,, \tag{2.102}$$

for some constant F. We find:

$$\begin{aligned}\langle\alpha|J^\mu(0)|\beta\rangle &= \frac{i}{p^2}\langle 0|J^\mu(0)|\phi(p)\rangle\langle\alpha+\phi(p)|\beta\rangle + N^\mu_{\alpha\beta}(p)\\ &= \frac{-p^\mu}{p^2}F\,\langle\alpha+\phi(p)|\beta\rangle + R^\mu(p)\,,\\ \langle\alpha|O^C[\phi](0)|\beta\rangle &= \frac{i}{p^2}\langle 0|O^C[\phi](0)|\phi(p)\rangle\langle\alpha+\phi(p)|\beta\rangle + N^C_{\alpha\beta}(p)\,,\end{aligned} \tag{2.103}$$

where $N^\mu_{\alpha\beta}(p)$ and $N^C_{\alpha\beta}(p)$ are, by assumption, regular remainder functions[8]. Let us contract the first equation in (2.103) with p_μ. Using the fact that J^μ is conserved, we then find:

$$p_\mu N^\mu_{\alpha\beta} = F\langle\alpha+\phi(p)|\beta\rangle = FM_{\alpha+\phi(p),\beta}\,. \tag{2.104}$$

In other words, we find the Adler zero relation $M_{\beta+\phi(p),\alpha} = \frac{1}{F}p_\mu N^\mu_{\beta\alpha}$.

Equations (2.101) and (2.103) lead to a relation between the remainders $N^\mu_{\alpha\beta}(p)$ and $N^C_{\alpha\beta}(p)$:

$$e^{-ip\cdot x}\partial\alpha^{(n)}\cdot N_{\alpha\beta}(p) = \partial\cdot\left(\gamma_C(x)e^{-ip\cdot x}\right)N^C_{\alpha\beta}\,. \tag{2.105}$$

Let us write out the n-th order polynomial $\alpha^{(n)}$ to find[9]:

$$e^{-ip\cdot x}\partial_\mu(x^{\mu_1}\dots x^{\mu_n})N^\mu_{\alpha\beta}(p) = \partial_\mu\left(\gamma_C^{\mu\mu_1\dots\mu_n}(x)e^{-ip\cdot x}\right)N^C_{\alpha\beta}\,. \tag{2.106}$$

Integrating this equation over all of space-time, one obtains:

$$p_\mu N^\mu_{\alpha\beta}(p)\partial^{\mu_1}\dots\partial^{\mu_n}\delta^{(d)}(p) = 0\,, \tag{2.107}$$

because the right-hand side of (2.106) is a total derivative. We can now work order-by-order in n to find out what (2.107) implies. At zeroth order, we find $\lim_{p\to 0} p_\mu N^\mu_{\alpha\beta} = 0$, which is just the statement that the remainder is regular. Taking a derivative of the zeroth order (2.107), we find:

$$0 = \partial^\nu(p_\mu N^\mu_{\alpha\beta})\delta^{(d)}(p) + p_\mu N^\mu_{\alpha\beta}\partial^\nu\delta^{(d)}(p) = 0\,, \tag{2.108}$$

which, using (2.106) at first order, implies $\lim_{p\to 0}\partial^\nu(p_\mu N^\mu_{\alpha\beta})$. Therefore, expanding the quantity $M_{\beta+\phi(p),\alpha}$ in powers of p^μ, the first $n+1$ terms vanish. In other words, the existence of an n-th order symmetry of the type (2.90) leads to an $(n+1)$ degree soft limit in the S-matrix element for emitting a Goldstone boson in the process $\beta\to\alpha$.

[8] This is always the case in the absence of cubic vertices. [3]

[9] To be precise, this equation only holds for the projection onto the Lorentz representation picked out by the parameter $c_{(n)}$.

2.5.2 Exceptional EFTs and on-shell reconstruction

Up to now, we have focused on the field-independent, coordinate-dependent shift in the transformation law (2.90). We have seen that an n-th order shift implies a soft degree $(n+1)$. This may be considered a rather trivial consequence of the fact that an n-th order shift in the coordinates requires the Goldstone fields to appear with $(n+1)$ derivatives. However, as we have indicated earlier, this is not always true for Wess-Zumino interactions (which are invariant only up to a total derivative) or when the field-dependent part of (2.90) is non-trivial. We will address the latter possibility in this section.

Consider a theory of a massless scalar, with the general Lagrangian: [2]

$$\mathcal{L} = (\partial\phi)^2 \sum_{n,m}^{\infty} \lambda_{m,n} \partial^m \phi^n \,. \tag{2.109}$$

Let us define the quantity $\rho = \frac{m}{n}$, which counts the number of derivatives per field apart from the $(\partial\phi)^2$ factor. A Lagrangian with a fixed ρ has the general form

$$\mathcal{L}_{(\rho)} = (\partial\phi)^2 \sum_{n}^{\infty} \lambda_n \partial^{\rho n} \phi^n = (\partial\phi)^2 F(\partial^m \phi^n) \,. \tag{2.110}$$

The quantity ρ is important because, for tree-level scattering, only diagrams coming from interactions with the same ρ can cancel each other to provide non-trivial soft-limits. Following [2], we will refer to ρ as the *derivative power counting* of a theory defined by $\mathcal{L}_\rho$.

As we have discussed before, for a given ρ a certain soft degree is a trivial consequence of having the right number of derivatives on the fields in every interaction. Specifically, a soft degree σ

$$\sigma \leq \frac{m+2}{n+2} = \frac{\rho n + 2}{n+2} \,, \tag{2.111}$$

for an interaction of the type (2.109) is trivial. Whenever a theory has a soft degree σ greater than this quantity, we say that its soft limit is *enhanced*.

Let us see how an enhanced soft limit $\sigma > \rho$ can come about. We assume that the soft degree σ is the consequence of an order $n = (\sigma - 1)$ shift symmetry, (2.90). Furthermore, we assume that the Lagrangian is strictly invariant under this transformation. We will address the possibility of invariance up to a total derivative in the next chapter. Clearly, none of the λ_n terms in the expansion (2.110) separately are invariant under the field-independent part of the transformation (2.90). Therefore, this transformation needs to be cancelled by one of the other terms in the Lagrangian. However, as these terms have a different number of fields n, the symmetry transformation must have a field-dependent part. To be precise, we want to cancel the transformation of the λ_n term under the shift symmetry with the field-dependent transformation of the $\lambda_{n'}$ term. This can happen if the field-dependent part of the transformation has a term with $n - n'$ fields.

However, now the field-dependent transformation of the λ_n term has to be cancelled by something else. Usually, this requires a term of the Lagrangian with more fields than λ_n. In general, therefore, we expect that a soft limit $\sigma > \rho$ leads to an infinite number of interactions all fixed by the symmetry in terms of a single coupling constant. One expects that such theories are sparse. Indeed (after imposing analyticity, crossing symmetry and unitarity), there is often only a single possible theory that realizes a particular choice (ρ, σ) with $\sigma > \rho$. Such a theory is therefore *reconstructible* given only its soft limits and derivative power counting. We will refer to a reconstructible theory with enhanced soft degree $\sigma > \rho$ as an *exceptional EFT*.

In Chapter 3, we will see that an n-th order symmetry of the type (2.90) is a consequence of having n *inverse Higgs relations* in the coset construction, which requires that the commutator of translations and the n-th order shift give rise to the $(n-1)$-th order shift. The field-dependent part of the transformation law of an exceptional theory is the consequence of non-vanishing commutation relations besides those required for imposing inverse Higgs constraints.

In the references [2–4, 9, 10], a classification of symmetries of the type (2.90) was achieved by reconstructing the EFTs from the *soft data* (ρ, σ). We will give a complementary classification using algebraic properties in chapters 4 and 5.

Chapter 3

The general theory of non-linear realizations

In the previous chapter, we examined some important effective field theories with special non-linear symmetries. The usefulness of that approach relies on the fact that the non-linear symmetries highly constrain the dynamics of the Goldstone degrees of freedom and their coupling to matter fields. As we have seen, non-linear symmetries allow one to derive important low-energy theorems, which explain the implications of the non-linear symmetries below their breaking scale. An important consequence at low energies is the existence of enhanced soft limits in scattering amplitudes of the Goldstone degrees of freedom. In this chapter, we will investigate how to obtain transformation laws for the Goldstone and matter fields and how to construct actions invariant under those transformations.

The general procedure for non-linear internal symmetries was developed long ago by Callan, Coleman, Wess & Zumino, (CCWZ) [49, 50] building on Weinberg's work on the non-linear realization of chiral symmetry. [41, 48] They showed that any given symmetry breaking pattern, defined by the coset $\frac{G}{H}$, leads to transformation laws for the Goldstone and matter fields which are unique up to field redefinitions. Furthermore, they showed how to construct the most general Lagrangians invariant under those transformation laws, using appropriately modified covariant derivatives. From now on, we will refer to their framework as the *coset construction*. The most general invariant action is a sum of the Lagrangians determined by the coset construction and *Wess-Zumino* terms, which are invariant only up to a total derivative. We will refer to the fact that the coset construction provides the most general invariant actions as *coset universality*. We will examine the coset construction for internal symmetries in section 3.1.

Volkov [54], Ivanov and Ogievetsky [55] generalized the coset construction to non-linear space-time symmetries. A new ingredient in the space-time case is the existence of *inverse Higgs constraints* (IHCs), which allow one to consistently eliminate some Goldstone fields in terms of others, thereby violating Goldstone's theorem. As we will show, the existence of inverse Higgs constraints is easy to understand as a consequence of degeneracy in the Goldstone modes that are induced by independent symmetry generators. [90] [80]This occurs when one Goldstone mode can be transformed into another by performing a local translation. The possibility of eliminating some Goldstone fields in favor of others allows one to introduce multiple symmetry generators per physical Goldstone field, giving rise to a richer structure of symmetry algebras than is possible for internal symmetries. However, it is only possible to impose the IHCs if the symmetry algebra satisfies some important conditions. [54] [141] As we will see, these conditions enable us to classify all possible non-linear symmetry algebras that can be realized on a particular choice of Goldstone fields, under minimal assumptions. [76] [77] [78] [79] We will cover the coset construction for space-time symmetries in detail in section 3.2, but leave the explanation of the EFT classification for the following chapter.

For space-time symmetries there is no formal proof that the coset construction provides the most general transformation laws and invariant actions. However, there are numerous non-trivial checks that lead us to believe that this is the case for space-time symmetries as well. In this chapter, we will take a look at the non-linear realization of the $D = 4$ conformal group and show that different basis choices lead to equivalent theories . [174] We will also address the question of universality for non-linear realizations of $\mathcal{N} = 1$ supersymmetry.

The coset construction has a straightforward generalization from ordinary space-time to superspace. [155] [156] [173] Just like Goldstone modes in ordinary space-time are degenerate when they can be transformed into each other by local translations, Goldstone modes in superspace are degenerate when they are related by a local supersymmetry transformation. [80] The existence of *superspace inverse Higgs constraints* constrains the allowed symmetry superalgebras. Once again, this enables us to classify all possible Goldstone EFTs under minimal assumptions. [80]

Particularly in the supersymmetric setting, the coset construction is a useful tool for constructing actions that describe extended objects such as superstrings and branes. [153, 154, 181] The superstring in $D = 10$ flat space in the Green-Schwarz formulation may be considered as a two-dimensional coset model for the symmetric space isomorphic to the $D = 10$ Type II supersymmetry group. In more general backgrounds, the target space description may require non-trivial p-form fluxes in order to solve the Einstein equations. In the worldsheet picture, complicated Wess-Zumino interactions of the symmetry breaking pattern must then be added to the action. These Wess-Zumino terms describe the couplings to the background gauge potentials. From the worldvolume point of view, they are necessary to preserve the local fermionic κ-symmetry. The coset construction is a crucial tool for explicitly constructing the non-trivial Wess-Zumino interactions.

In certain very special backgrounds, such as the $AdS_5 \times S_5$ background relevant to the AdS/CFT correspondence [75], the description of superstring actions as coset models allows one to construct infinite sets of conserved charges. The existence of an infinite number of charges can make calculations tractable in situations where neither side of the AdS/CFT is duality weakly coupled. Thus, the coset construction has become an important tool in the integrability approach to the AdS_3/CFT_2 correspondence (see for instance [154,194]). We will examine the formulation of superstring and brane actions as coset models in section 3.4.

3.1 Internal symmetries

We begin by considering the coset construction for internal symmetries, which was developed by CCWZ in [49,50]. Consider the symmetry breaking of the compact, semi-simple Lie group G down to a subgroup H. We want to know how to construct transformation laws under G which are linear only for elements of H. The groups G and H define a coset space $\frac{G}{H}$, which we will refer to as the *symmetry breaking pattern*. We require that G is an *internal* symmetry, which means that the overall symmetry group of the theory is a direct product of G and the Poincaré group. We parametrize an element g in the connected component of G as follows:

$$g = e^{c^i G_i} e^{u^a Z_a}\,, \tag{3.1}$$

where Z_a are the generators of the subgroup H and G_i are the remaining generators, which make up the coset $\frac{G}{H}$. In order to consistently break the group G to H, the algebra must take the form:

$$\begin{aligned}
&[Z_a, M_b] = f_{ab}{}^c M_c, \quad [G_i, G_j] = f_{ij}{}^k G_k + f_{ij}{}^a Z_a,\\
&[Z_a, G_i] = f_{ai}{}^j G_j\,.
\end{aligned} \tag{3.2}$$

In other words, the commutator of a broken generator with an unbroken one must equal a broken generator. Now consider a mapping $\gamma(x)$ from the space-time coordinates to an element of the coset space:

$$\gamma = e^{\phi^i(x) G_i}\,. \tag{3.3}$$

The function $\phi^i(x)$ is the *Goldstone* field associated to the symmetry generator G_i. Acting on the left with g on γ yields:

$$g \cdot \gamma = e^{\phi'^i(x) G_i} e^{h^a(\phi,g) Z_a} . \tag{3.4}$$

This equation defines the transformation law $\phi(x) \to \phi'^i(x)$ under g for the Goldstone fields $\phi^i(x)$. By acting on (3.4) with another group g_2, one finds that the transformation law $\phi(x) \to \phi'^i(x)$ satisfies the group law. The transformation becomes linear when $g = h$ is an element of H, that is:

$$\phi'^i(x) = D_G(h)^i{}_j \phi^j(x) , \tag{3.5}$$

where D_G is some representation of H.

Now consider a field multiplet ψ^A which transforms according to the representation D of H, $\psi'^A(x) = D^A{}_B(h)\psi^B(x)$. We can use the mapping $h^a(\phi, g)$ from (3.4) to define a transformation law for ψ^A under elements g of the full group:

$$\psi'^A(x) = D(h(\phi, g))^A{}_B \, \psi^B(x) , \tag{3.6}$$

where $h(\phi, g) = \exp h^a(\phi, g) Z_a$. Once again, this transformation law satisfies the group law and reduces to the ordinary transformation law when g is restricted to the unbroken group H. The full group G is then realized on the Goldstone fields ϕ^i and the matter fields ψ^A as a transformation that becomes linear when restricted to H, as desired. The primary result of CCWZ was that *any* realization of G that becomes linear on H can be transformed into (3.4) and (3.6) by an invertible local field redefinition, as long as G is an internal, compact, connected, and semisimple Lie group. [49] Let us see how to construct actions invariant under the symmetry transformations (3.4) and (3.6).

The field multiplet ψ^A transforms under $\frac{G}{H}$ as a field-dependent H transformation, so any H-invariant quantity built out of ψ^A is G-invariant. However, derivatives of ψ^A are not covariant due to the ϕ dependence of the transformation law. Similarly, neither the Goldstone fields ϕ^i nor their derivatives $\partial\phi^i$ transform covariantly. CCWZ proved that there is a unique set of covariant objects which contains derivatives of ϕ^i and ψ^A, up to invertible field redefinitions. Let us see how to construct them.

Consider the following Lie algebra-valued 1-form built out of the coset element γ:

$$\omega = \gamma^{-1} d\gamma , \tag{3.7}$$

where d is the exterior derivative on space-time. ω is the pullback to space-time of the Maurer-Cartan form on G by the mapping $\gamma(x)$. It satisfies the Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$. In the following, we simply refer to ω as the *Maurer-Cartan form* of the symmetry breaking pattern $\frac{G}{H}$.[1] We may decompose ω in terms of the symmetry generators G_i and Z_a:

$$\omega = \omega^a Z_a + \omega^i G_i = (\omega^a)_\mu dx^\mu Z_a + (\omega^i)_\mu dx^\mu G_i \,. \tag{3.8}$$

We will refer to the 1-forms multiplying the symmetry generators as *Maurer-Cartan components*. We now define covariant derivatives for ϕ^i and ψ^A as follows:

$$\hat{D}_\mu \phi^i = (\omega^i)_\mu, \quad \hat{D}_\mu \psi^A = \partial_\mu \psi^A + (\omega^a)_\mu D(Z_a)^A{}_B \psi^B \,, \tag{3.9}$$

where $D(Z_a)$ is the matrix associated to the generator Z_a in the H representation D of the multiplet ψ^A. These objects transform covariantly under the full group G:

$$(D_\mu \phi^i)'(x) = D_G(h)^i{}_j \phi^j(x), \quad (D_\mu \psi^A)'(x) = D(h)^A{}_B \psi^B(x) \,, \tag{3.10}$$

where once again $h = \exp h^a(\phi, g) Z_a$ and D_G is the H representation defined by equation (3.5). Therefore, the quantities ψ^A, $\hat{D}\phi^i$ and $\hat{D}\psi^A$ all transform under G as field-dependent H transformations. Any H-invariant Lagrangian built out of them is therefore automatically G-invariant.

3.1.1 Wess-Zumino terms

We have seen how the coset construction provides the most general invariant Lagrangian for the symmetry breaking pattern G/H. Of course, to find the most general invariant *action* we need in addition to know all terms which are not invariant but transform into a total derivative. Wess and Zumino showed [51] (in the context of pion physics) that such terms may be generated quantum mechanically, even when one does not include them initially. In the literature, operators which transform under a broken symmetry into a total derivative are often called *Wess-Zumino* or *Wess-Zumino-Witten* [52] terms. We will encounter Wess-Zumino terms throughout this thesis. Important examples are the Galileons [96] and the Lagrangians that govern the dynamics of supersymmetry-preserving branes and strings.

Wess-Zumino terms in d dimensions are always equal to integrals of invariant Lagrangians in $d+1$ dimensions. [52] Weinberg and D'Hoker [53] showed that the $d+1$-dimensional interactions are identified with elements of the $d+1$-th cohomology group of the manifold G/H, $H^{d+1}(G/H, \mathbb{R})$. In other words, they are the invariant and closed $d+1$-forms on G/H that are not themselves the exterior derivative of an invariant d-form. Let us briefly

[1] Note that the definition of the Maurer-Cartan form may change depending on the parametrization of the coset element.

review the arguments of Weinberg and D'Hoker. We will give an example of how one constructs these invariant forms after we have introduced the coset construction for space-time symmetries.

First, Weinberg and D'Hoker show that even when an operator transforms into a total derivative under the symmetry transformations, a *variation* of that operator with respect to a Goldstone field is strictly invariant. In order to see this, write the variation of the action as follows:

$$\delta S[\phi] = \int d^d x Tr\left[(\gamma^{-1}\delta\gamma)_{G^i} J[\phi, \partial\phi, \ldots]\right] . \tag{3.11}$$

where $\gamma(\phi)$ is the usual mapping from the Goldstone fields to elements of G/H and J is a general local functional of the Goldstones ϕ^i and their derivatives. The subscript in $(\gamma^{-1}\delta\gamma)_{G^i}$ indicates that we restrict to terms proportional to broken generators. It is possible to write a general variation in this way when the Goldstone fields always enter into the action by way of the mapping $\gamma(\phi(x))$, as is true in the coset construction.

At this stage, assume that the transformation laws of the Goldstone fields come from the coset construction, (3.4). Vary the transformation law with respect to the untransformed field ϕ^i to obtain the transformation of the bracketed quantity in (3.11):

$$\gamma^{-1}(\phi')\delta\gamma(\phi') = h(\phi, g)\cdot\gamma^{-1}(\phi)\delta\gamma(\phi)\cdot h^{-1}(\phi, g) - \delta h(\phi, g)\cdot h^{-1}(\phi, g) . \tag{3.12}$$

The transformation law of $(\gamma^{-1}\delta\gamma)_{G^i}$ reduces to the first term on the right-hand side. Then, by equating the variational derivatives of the transformed and untransformed actions, $\frac{\delta S[\phi']}{\delta\phi^i} = \frac{\delta S[\phi]}{\delta\phi^i}$, one obtains:

$$J(\phi') = h(\phi, g)\cdot J(\phi)\cdot h^{-1}(\phi, g) . \tag{3.13}$$

Together, (3.12) and (3.13) imply that (3.11) is the integral over a G-invariant Lagrangian density.

The next step is to write the invariant operator as the integral over an extended $d+1$-dimensional space. First, compactify the d-dimensional space-time to a d-sphere M_d. The image of the space-time in the manifold G/H under the mapping $\gamma(\phi(x))$ is then also a d-sphere. Assuming this d-sphere can be continuously shrunk to a point (i.e. assuming $\gamma(\phi(x))$ is in the trivial d-th homotopy class of G/H), we may construct a smooth interpolating function $\hat{\phi}^i(x, t_1)$ such that $\hat{\phi}^i(x, 1) = \phi^i(x)$ and $\hat{\phi}^i(x, 0) = 0$. Then, we may write $S[\phi]$ as:

$$S[\phi] = \int_{B_{d+1}} d^d x\, dt_1\, \mathrm{Tr}\left[\left(\gamma^{-1}(\hat{\phi}(x))\frac{\partial\gamma(\hat{\phi})}{\partial t_1}\right)_{G^i} J\right] , \tag{3.14}$$

where B_{d+1} is the $d+1$-ball formed by space-time and the new coordinate t_1. As we have explained, the integrand is a G-invariant density.

The last step is to smoothly extend $\hat{\phi}$ into other directions t^i such that x^μ and t^i provide a set of coordinates for the full coset space G/H. Then, one can write (3.14) as the integral over B_{d+1} of a closed G-invariant $d+1$-form F in G/H:

$$S[\phi] = \int_{B_{d+1}} F\,. \tag{3.15}$$

Now, when two invariant $d+1$-forms live F and F' in the same cohomology class of G/H, they differ by the exterior derivative of an *invariant* d-form H: $F'_{d+1} = F_{d+1} + dH$. That means we can write:

$$\int_{B_{d+1}} F = \int_{B_{d+1}} F' + \int_{M_d} H\,. \tag{3.16}$$

In other words, the difference in the actions can be written as a d-dimensional integral of an *invariant* density. Such terms are of course exactly what the usual coset construction provides. Thus, the non-trivial Wess-Zumino terms are classified by the cohomology classes of $d+1$-forms in G/H.

In this section, we have explained how one (formally) constructs a higher-dimensional invariant density out of a Wess-Zumino term. In practice, we will work the other way around and construct Wess-Zumino terms from closed, invariant $d+1$ forms that are not the exterior derivative of an invariant d-form. In the next section, we will explain how to find the Wess-Zumino terms of the Galileon algebra. In the last section of this chapter, we will see that Wess-Zumino terms play an important role in constructing superstring and brane actions.

3.2 Space-time symmetries

We now consider the case of broken space-time symmetries. A space-time symmetry group has generators which don't commute with the Poincaré algebra. We work in D dimensions and we assume that the full D-dimensional Poincaré group remains unbroken. The broken space-time symmetries therefore come in addition to the Poincaré group. As we have seen, the Coleman-Mandula theorem states that such symmetry groups cannot be realized linearly. However, there is no theorem that forbids non-linear realizations. The coset construction for space-time symmetries was developed by Volkov [54] and Ivanov & Ogievetsky. [55]

Because the overall symmetry group is no longer a simple direct product of Poincaré and an internal group, the G relevant for the coset construction includes the full Poincaré group in addition to the broken symmetries. The main subtlety in the coset construction for space-time symmetries is the role of the translation generators P_μ. Since they act non-linearly on the space-time coordinates in the passive formulation, they live in the coset $\frac{G}{H}$ rather than in H, even though translations act linearly on the fields. We parametrize G locally as follows:

$$g = e^{a^\mu P_\mu} e^{c^i G_i} e^{c^{\mu\nu} M_{\mu\nu}} e^{h^a Z_a} \,, \tag{3.17}$$

where we decompose the algebra of G into the broken generators G_i and unbroken ones Z_a, in addition to the Poincaré generators $M_{\mu\nu}$ and P_μ. We have suppressed the Lorentz indices of G_i and Z_a. In general, they can span an arbitrary representation of the Lorentz group. The commutation relations must be restricted to the following general form, for consistency of the coset construction:

$$\begin{aligned} &[Z_a, Z_b] = f_{ab}{}^c Z_c \,, \quad [G_i, Z_a] = f_{ia}{}^j G_j \,, \\ &[P_\mu, Z_a] = f_{\mu a}{}^\nu P_\nu \,. \end{aligned} \tag{3.18}$$

The remaining commutation relations, involving $[P_\mu, G_i]$ or $[G_j, G_j]$ are not restricted a priori. The choice of parametrization (3.17) is of course not a unique one. In fact, we will usually make a different choice where each non-linear H multiplet of Goldstones ϕ^i comes in a separate exponential, but let us work with (3.17) for the moment.

As before, we define a mapping $\gamma(x)$ from the space-time coordinates to the coset space by way of the Goldstone fields $\phi^i(x)$. Acting with a group element g defines the passive transformation law $\phi(x) \to \phi'(x')$ for the Goldstone fields and $x^\mu \to x'^\mu$ for space-time coordinates:

$$\begin{aligned} &\gamma(x) = e^{x^\mu P_\mu} e^{\phi^i(x) G_i}, \\ &g \cdot \gamma(x) = e^{x'^\mu P_\mu} e^{\phi'^i(x') G_i} e^{c^{\mu\nu}(\phi,g) M_{\mu\nu}} e^{h^a(\phi,g) Z_a} \,. \end{aligned} \tag{3.19}$$

Note the dependence on the transformed coordinate in $\phi'(x')$. The transformations of ϕ and x are, in general, *point transformations*, meaning they depend on the coordinates as well as the fields. The transformation law becomes linear when $g = h$ is an element of H:

$$h \cdot \phi(x) = \phi'^i(x') = D_G(h)^i{}_j \phi^j(x) \,, \tag{3.20}$$

for some representation D_G of H. The transformation law for the coordinates x then reduces to the standard Poincaré transformation. Note that D_G may act on the suppressed Lorentz indices of $\phi^i(x)$, since H includes the Lorentz group. Once again, the mappings $c^{\mu\nu}(\phi, g)$ and $h^a(\phi, g)$ determine the transformation law of the matter fields. Define $h(\phi, g) = e^{c^{\mu\nu}(\phi,g) M_{\mu\nu}} e^{h^a(\phi,g) Z_a}$. A matter field H multiplet ψ^A which transforms in the representation D of H transforms under the full group G as:

$$g \cdot \psi = \psi'^A(x') = D(h(\phi, g))^A{}_B \psi^B(x)\,, \tag{3.21}$$

where the index A runs over all indices of H in the representation of ψ^A, including the Lorentz indices.

The Maurer-Cartan form ω decomposes as follows:

$$\begin{aligned}\omega = \gamma^{-1} d\gamma &= \omega^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu} + \omega^i G_i + \omega^a Z_a \\ &= (\omega^\mu)_\nu dx^\nu P_\mu + (\omega^{\mu\nu})_\rho dx^\rho M_{\mu\nu} + (\omega^i)_\mu dx^\mu G_i + (\omega^a)_\mu dx^\mu Z_a\,.\end{aligned} \tag{3.22}$$

ω satisfies the Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$. Let us see how the Maurer-Cartan components transform. Using (3.19), we find for the non-linearly realized components ω_G (i.e. the translation and G_i components):

$$g \cdot \omega_G = \omega'_G(x') = h(\phi, g)\omega_G(x)h^{-1}(\phi, g)\,. \tag{3.23}$$

This is a linear transformation, so $g \cdot \omega_G = h(\phi, g) \cdot \omega_G = D(h(\phi, g))\omega_G$ for the representation D_G of H spanned by ω_G. In particular, $\omega^\mu = (\omega_P)^\mu$ transforms as a Lorentz vector with a field-dependent parameter:

$$\big(g \cdot \omega_P\big)^\mu(x') = \big(e^{c^{\rho\nu}(\phi,g)L_{\rho\nu}}\omega_P\big)^\mu(x)\,, \tag{3.24}$$

where $L_{\mu\nu}$ is the vector representation of Lorentz transformations. The linearly realized Lorentz and Z_a components, however, transform together like a gauge connection for H:

$$\begin{aligned}\omega'^a(x')Z_a + \omega'^{\mu\nu}M_{\mu\nu} &= (\omega^C)'(x')Z_C \\ &= h(\phi, g)\big(\omega^C(x)Z_C\big)h^{-1}(\phi, g) - h^{-1}(\phi, g)dh(\phi, g)\,,\end{aligned} \tag{3.25}$$

where Z_C stands for all generators of H, including $M_{\mu\nu}$.

The objects ω_G transform covariantly and are built out of derivatives of the Goldstones, so we can use them to build covariant derivatives. However, it is the full 1-form ω_G, not the component $(\omega_G)_\mu dx^\mu$ that has nice transformation properties. Since the coordinates transform non-trivially under G, we need to deal with the transformation of the basis 1-form dx^μ. Notice, however, that the Maurer-Cartan component of translations at zeroth order in the fields is simply a delta: $(\omega^\mu)_\nu = \delta^\mu{}_\nu + \mathcal{O}(\phi)$. We can therefore compute its inverse, $(\omega_\nu^{-1})^\mu$, at least perturbatively. We will soon see that $(\omega^\mu)_\nu$ is like a vielbein, so let us set $(\omega^\mu)_\nu = e^\mu{}_\nu$ and $(\omega_\nu^{-1})^\mu = (e^{-1})_\nu{}^\mu$. It is now clear that the following quantity $\hat{D}_\mu\phi^i$ transforms covariantly under G:

$$\hat{D}_\mu\phi^i := (e^{-1})_\mu{}^\nu(\omega^i)_\nu\,,$$
$$(\hat{D}_\mu\phi^i)'(x') = h(\phi, g)\cdot\hat{D}_\mu\phi^i(x)\,. \tag{3.26}$$

This is the covariant derivative for the Goldstone field ϕ^i. The covariant derivatives for matter fields are defined as follows:

$$\hat{D}_\mu\psi^A(x) = (e^{-1})_\mu{}^\nu\Big(\partial_\nu\psi^A(x) + (\omega^C(x))_\nu D(Z_C)^A{}_B\psi^B(x)\Big)\,. \tag{3.27}$$

The inhomogeneous transformation of $\partial_\mu\psi$ is cancelled by the transformation law (3.25) for ω_C. Thus, ω_C plays the role of a gauge connection for H, as expected.

We now have the objects $\hat{D}_\mu\phi^i$, $\hat{D}_\mu\psi^A$ and ψ^A, all of which transform under G as a field-dependent H transformation. The only thing left to deal with is the transformation of the space-time coordinates, defined by (3.19). Earlier we noted that ω_P transforms under G like a Lorentz vector with a field-dependent parameter, (3.24). Define the following volume form:

$$dV = (\omega_P)_0\wedge(\omega_P)_1\wedge\ldots\wedge(\omega_P)_{D-1} \tag{3.28}$$
$$= \det(e)dx^0\ldots dx^{D-1}\,. \tag{3.29}$$

dV is then invariant under G. Therefore, the general action obtained from the coset construction is:

$$S = \int d^Dx\,\det(e)\mathcal{L}\,, \tag{3.30}$$

where $\mathcal{L}$ is any H scalar built out of the quantities $\hat{D}_\mu\phi^i$, $\hat{D}_\mu\psi^A$ and ψ^A. In addition to the action obtained from the coset construction, there can be Wess-Zumino interactions for space-time symmetries, as we will see.

The EFTs for broken space-time symmetries that we encountered in the previous chapter have more broken symmetry generators than Goldstone fields, violating Goldstone's theorem. However, we have seen that in the coset construction one has to associate a Goldstone field $\phi^i(x)$ to each broken generator G_i. How can we reconcile these two observations? It turns out that not all Goldstone fields are essential to the non-linear realization. We can sometimes eliminate a Goldstone in favor of others by imposing constraints that are compatible with all symmetries. We refer to these as *inverse Higgs constraints* (IHCs) [55] and to the Goldstone fields that they eliminate as *inessential Goldstones.* [141] The existence of IHCs is an important part of why it is hard to prove coset universality for space-time symmetries. We will discuss IHCs in detail in the next subsection. After that, we will return to the question of coset universality.

3.2.1 Degenerate Goldstone modes

In the previous sections, we studied spontaneous symmetry breaking in the bottom-up approach of non-linear realizations. We saw that theories with non-linear space-time symmetries may violate Goldstone's theorem due to the possibility of imposing inverse Higgs constraints. In order to understand physically why IHCs exist, it will be useful to take a more top-down point of view. Consider a theory with a linearly realized symmetry group G, in the vacuum field configuration $|0\rangle$. We assume that $|0\rangle$ breaks the symmetries to the subgroup H. The broken generators G_i, when acting on $|0\rangle$, produce the following massless Goldstone modes:

$$\phi^i(x)G_i|0\rangle \, , \tag{3.31}$$

where $\phi^i(x)$ is a slowly-varying function. [90] We will refer to $\phi^i(x)$ as the Goldstone field. Note that once again we suppress any possible Lorentz indices on ϕ^i. When G_i generates a space-time symmetry, it has a non-trivial commutation relation with the generators of the Poincaré group. This opens up the possibility that (3.31) contains degeneracies. That is, there may be non-trivial solutions $\phi^i(x)$ to the following equation:

$$\phi^i(x)G_i|0\rangle = 0 \, . \tag{3.32}$$

Let us see why such degeneracies may exist for space-time symmetries. Act on (3.32) with the translation operator P_μ. On the Goldstone field, the translations are represented as $-i\partial_\mu$ and on the generators G_i, they act as defined by the symmetry algebra of G. The result is:

$$0 = (\partial_\mu \phi^i - f_{\mu j}{}^i \phi^j) G_i |0\rangle \, , \tag{3.33}$$

where $f_{\mu j}{}^i$ are the structure constants:

$$[P_\mu, G_i] = i f_{\mu i}{}^j G_j + \dots \, . \tag{3.34}$$

The ellipses stand for all generators apart from the broken generators G_i. Therefore, a non-trivial solution to (3.32) is:

$$\partial_\mu \phi^i - \phi^j f_{\mu j}{}^i = \mathcal{O}(\phi^2) \, , \tag{3.35}$$

This equation relates the linear combination of Goldstone field $\phi^j f_{\mu j}{}^i$ to the derivative of ϕ^i. Imposing it as an *inverse Higgs constraint* eliminates the linear combination $\phi^j f_{\mu j}{}^i$ from the theory. Clearly, such a solution only exists when $f_{\mu j}{}^i \neq 0$. In the following chapters, we will use this very important condition on the algebra to classify EFTs with extended space-time symmetries.

It should now be clear when inverse Higgs constraints can arise. The Goldstone modes (3.31) are nothing but localized G_i transformations. Even though the global actions of the G_i generators are independent, the local versions may not be.[2]

The $\mathcal{O}(\phi^2)$ terms are exactly those that enter into the covariant derivative of ϕ^i in the coset construction. Thus, in the coset construction one imposes the IHC by setting $\hat{D}_\mu \phi^i = 0$. The $\mathcal{O}(\phi^2)$ terms impose further conditions on the symmetry algebra (in addition to $f_{\mu j}{}^i \neq 0$) in order for (3.35) to be consistent. [141] As we will see, these conditions depend on how we parametrize the coset element. However, the condition $f_{\mu j}{}^i \neq 0$ is universal for any coset parametrization, as is obvious from the fact that we derived it without even using the coset construction.

3.2.2 Inverse Higgs constraints

To obtain the non-linear completion of (3.35), we need to return to the coset construction. Consider the symmetry breaking pattern $\frac{G}{H}$ where the coset is spanned by translations P_μ and the two broken generators G_1 and G_2. Assume that G_1 spans an arbitrary irreducible representation of H (which includes of course the Lorentz symmetry) and G_2 is another irrep such that we can have $[P_\mu, G_2] \supset G_1$. We indicate the H indices in the G_1 irrep collectively by Greek indices α, β, ..., and the indices of G_2 by a, b, ... We denote indices that run over all representations as I, J, ... With these conventions, define the following commutation relations in the symmetry algebra of G:

$$\begin{aligned}
&[(G_2)_a, (G_2)_b] = f_{ab}{}^{\alpha}(G_1)_\alpha + \dots , \quad [P_\mu, (G_2)_a] = f_{\mu a}{}^{\alpha}(G_1)_\alpha + \dots , \\
&[(G_2)_a, (G_1)_\alpha] = f_{a\alpha}{}^{\beta}(G_2)_\beta + f_{a\alpha}{}^{\mu}P_\mu + f_{a\alpha}{}^{b}(G_1)_b + \dots \coloneqq f_{a\alpha}{}^{I}G_I + \dots .
\end{aligned} \tag{3.36}$$

Now parametrize the coset element as follows:

$$\gamma = e^{x^\mu P_\mu} e^{\phi^1(x) G_1} e^{\phi_2(x) G_2} , \tag{3.37}$$

with H indices suppressed. Using the Baker-Campbell-Hausdorff formula in (3.22), we obtain the Maurer-Cartan component for G_1, which contains the

[2]The reference [90] contains an illuminating example of this for the symmetry breaking induced by a string in flat space-time, where the action of local translations and rotations in the transverse directions are degenerate.

following terms:

$$(\omega_1^\alpha)_\mu = \partial_\mu(\phi^1)^\alpha + (\phi^2)^a f_{\mu a}{}^1 - \tfrac{1}{2!}(\phi^2)^a\partial_\mu(\phi^2)^b f_{ab}{}^1 + \tfrac{1}{3!}(\phi^2)^a(\phi^2)^b\partial_\mu(\phi^2)^c f_{ac}^I f_{bI}{}^\alpha - \\ - \tfrac{1}{4!}(\phi^2)^a(\phi^2)^b(\phi^2)^c\partial_\mu(\phi^2)^d f_{ac}{}^I f_{bI}{}^J f_{cJ}{}^\alpha + \ldots \,. \tag{3.38}$$

The idea behind IHCs is to solve $\omega_1 = 0$ *algebraically* for ϕ^2. Only then can we consistently plug the solution back into the action. One can solve this equation algebraically only if no $\partial_\mu\phi^2$ terms appear. Thus, we obtain the following conditions on the symmetry algebra: [141]

$$f_{ab}{}^1 = f_{ac}^I f_{bI}{}^\alpha = f_{ac}{}^I f_{bI}{}^J f_{cJ}{}^\alpha = \ldots = 0\,. \tag{3.39}$$

We will use the first of these constraints in a number of places in the classification of space-time Goldstone EFTs. Note that these constraints depend heavily on the choice of coset element (3.37), as the entire structure of (3.38) will change. However, other parametrizations impose even stricter conditions.

One may worry whether the coset construction truly provides a universal action principle for any symmetry breaking pattern, given that the fundamental quantities depend so heavily on a choice of parametrization. Furthermore, there are situations where auxiliary Goldstone fields are eliminated by field equations rather than inverse Higgs constraints. However, it was shown in [141] that, at least prior to imposing IHCs, all choices of coset parametrization and algebra bases give rise to equivalent theories. Additionally, as long as the conditions (3.39) are satisfied, the transformation laws and invariant actions do not depend on how one eliminates the inessential fields. We will return to the question of coset universality in the next subsections, where we investigate two important case studies: the breaking of AdS isometries and the breaking of supersymmetry.

It is important to understand that imposing inverse Higgs conditions is never necessary. Without eliminating the inessential fields, the coset construction provides perfectly consistent EFT actions. However, note that (3.38) contains the universal term linear in the inessential field ϕ^2. This means that any term proportional to $(\hat{D}\phi^1)^2$ contains a mass term for ϕ^2. Thus, one may end up having to integrate out ϕ^2 anyway, as the Goldstone EFTs in any case are only valid in a limited energy range. Coset universality then guarantees that the theory after integrating out the inessential field is equivalent to a non-linear realization with inverse Higgs constraints.

3.2.3 Galileons and Inönu-Wigner contractions

We have now introduced the notions of inverse Higgs constraints, inessential Goldstone fields and Wess-Zumino interactions. In order to demonstrate how one deals with these concepts in practice, we will now look at the simplest class of EFTs where they all play a role: the Galileons. [3]

A (multi-field) Galileon theory is an EFT in d dimensions with a set of scalar fields π^i, where $i = 1, \ldots, n$. The fields π^i realize the following non-linear symmetry transformations:

$$\pi^i \to \pi^i + c^i + c^i_\mu x^\mu \,, \tag{3.40}$$

i.e. the theory has n independent first-order extended shift symmetries. In addition, we assume that the fields π^i form a multiplet of an unbroken internal $SO(n)$ symmetry. These transformations are based on the following algebra, involving the constant shift generators C^i, the $SO(n)$ generators J^{ij} and the Galileon boosts K^i_μ:

$$\begin{aligned} &[P_\mu, K^i_\nu] = \eta_{\mu\nu} C^i \,, \quad [J^{ij}, C^k] = (\delta^{ik} C^j - \delta^{jk} C^i) \,, \\ &[J^{ij}, K^k_\mu] = (\delta^{ik} B^j_\mu - \delta^{jk} B^i_\mu) \,. \end{aligned} \tag{3.41}$$

In addition, there is the d-dimensional Poincaré algebra. C^i and J^{ij} are Lorentz scalars and K^i_μ are vectors. Together, this makes up the algebra Gal(d, n).

The Galileons first attracted attention in the work of Dvali, Gabadadze and Porrati (DGP). [92] The DGP model is an action for a 3-brane embedded in a five-dimensional space consisting of two terms. The first is a standard bulk Einstein-Hilbert term, built out of the 5D metric and Ricci scalar. The second term is localized on the brane and built out of the induced metric and the associated 4D Ricci scalar. The effective action living on the brane looks like ordinary 4D gravity at short distances, but receives corrections on large scales due to the presence of a scalar field π with the interaction $\Box\pi(\partial\pi)^2$. [93] Below, we will identify this operator as the *cubic Galileon* in $d = 4$. It is clearly invariant up to a total derivative under a transformation of the type (3.40).

The Galileons have been studied in great detail over the last decade. We will not have the space here to cover all their interesting properties, but see for example [94–98]. For our purposes, the Galileons - and more generally the n-th order extended shift symmetries - are the foundation upon which more complicated EFTs are built. To that end, we will explain below the relation between the Galileons and DBI scalars. [96, 97]

[3] The Galileons were first considered as Wess-Zumino interactions in [96].

Inverse Higgs constraints and Wess-Zumino interactions

Let us examine the coset construction for a single Galileon in $d = 4$. We begin by defining the coset element g:

$$g = e^{x^\mu P_\mu} e^{\pi C + A_\mu K^\mu} . \tag{3.42}$$

The relevant Maurer-Cartan components are:

$$\omega_P^\mu = dx^\mu , \quad \omega_C = d\pi + A_\mu dx^\mu , \quad +\omega_K^\mu = dA^\mu . \tag{3.43}$$

Note that g is a mapping from the coordinates (x^μ, A^μ, π) to the coset space. We are not yet assuming that A^μ and π are fields living on the 4D Minkowski space-time. It should be clear that the relevant inverse Higgs constraint is $\omega_C = 0$. We will wait to impose this condition until after we pull back to space-time. Then, this constraint will eliminate $A_\mu(x)$ in terms of $C(x)$.

Let us construct the cubic Galileon from the DGP model using the coset construction for the symmetry breaking pattern $\mathrm{Gal}(4,1) \to \mathfrak{iso}3, 1$. The Maurer-Cartan components ω_C, ω_K, and ω_P are left-invariant under broken generators. As we have explained in section 3.1.1, Wess-Zumino interactions result from left-invariant 5-forms integrated over a space whose boundary is 4D Minkowski space. We can look for left-invariant 5-forms by wedging together the components of the Maurer-Cartan form.

Because we are looking for a term cubic in the fields, we need three wedgings of the forms ω_C or ω_K. The remaining two factors must then come from ω_P. To make the 5-form invariant under the linearly realized Lorentz symmetry, we must multiply this wedge product by a Lorentz-invariant tensor, i.e. $\eta_{\mu\nu}$ or $\epsilon_{\mu\nu\rho\sigma}$. The only possibility for a cubic Wess-Zumino interaction is then: [96]

$$\begin{aligned} \omega_3 &= \epsilon_{\mu\nu\rho\sigma} \omega_C \wedge \omega_B^\mu \wedge \omega_B^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma , \\ \omega_3 &= d\beta_3 = \epsilon_{\mu\nu\rho\sigma} d \left(\pi dA^\mu \wedge dA^\nu \wedge dx^\rho \wedge dx^\sigma - \frac{1}{3} A^2 dA^\mu dx^\nu dx^\rho dx^\sigma \right) . \end{aligned} \tag{3.44}$$

Importantly, we may write ω_3 as the exterior derivative of the 4-form β_3, which is not itself left-invariant. We must now integrate ω_3 over a slice of the coset space whose boundary is Minkowski space-time. In other words, we must integrate β_3 over M_4. At this stage, we also pull-back β_3 by the mappings defined by the Goldstone fields $A_\mu(x)$, $\pi(x)$. Then, we can impose the inverse Higgs constraint $\omega_C = 0$. After pulling back to space-time, it reads:

$$A_\mu = -\partial_\mu \pi . \tag{3.45}$$

All in all, we obtain:

$$S_3 = \int_{M_4} \Omega^\star(\beta_3)|_{\omega_C=0} \tag{3.46}$$

$$= \int_{M_4} d^4x \left(-2\pi((\partial \cdot A)^2 - \partial_\mu A^\nu \partial_\nu A^\mu) + 2A_\mu A^\mu \partial_\nu A^\nu \right)_{A_\mu = -\partial_\mu \pi}$$

$$= \int_{M_4} d^4x \, \Box\pi(\partial\pi)^2 \,, \tag{3.47}$$

where Ω is the mapping $A_\mu = A_\mu(x)$, $\pi = \pi(x)$ from M_4 to the coset.

There are four other single-field Galileon interactions in $d = 4$: the tadpole, the Klein-Gordon kinetic term, the quartic and quintic Galileons:

$$\mathcal{L}_4 = (\partial\pi)^2((\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2)\,, \tag{3.48}$$

$$\mathcal{L}_5 = (\partial\pi)^2((\Box\pi)^3 - 3\Box\pi(\partial_\mu\partial_\nu\pi)^2 + 2\partial_\mu\partial_\nu\pi\partial^\mu\partial^\sigma\pi\partial^\nu\partial_\sigma\pi)\,. \tag{3.49}$$

All of these interactions arise from left-invariant 5-forms in the coset space. [96] They have the important property that their field equations are second-order in time derivatives in π, even though the action depends on derivatives of higher than first-order. This means that the Galileon interactions define a consistent variational problem which does not suffer from an Ostrogradsky instability, as generic actions depending on the jet space of order $n \geq 2$ normally do. [94–98]

Inönu-Wigner contractions and small-field limits

The Galileon algebra and, more generally, the order n extended shift symmetries form the foundation for more complicated symmetry breaking patterns. One can see this from the fact that the Gal(d,n) algebra contains no non-vanishing commutation relations other than those required for inverse Higgs relations, and those that define the representation of generators under the Lorentz algebra and the internal $SO(n)$. All further contributions to commutation relations would add field-dependent terms to (3.40), leading to a transformation law of the general type (2.90). In this section, we want to explain how one explicitly relates more complicated theories (e.g. exceptional EFTs) to the baseline extended shift symmetries, at the level of the algebra and in the Lagrangian field theory.

The DBI scalar in $d = 4$ is defined by the following action:

$$S = \int d^4x \left(-\Lambda\sqrt{1 + (\partial\pi)^2} + \Lambda \right), \tag{3.50}$$

The scalar field $\pi(x)$ transforms as:

$$\pi(x) \to \pi(x) + c + c_\mu x^\mu + \pi(x) c^\mu \partial_\mu \pi(x)\,. \tag{3.51}$$

This transformation non-linearly realizes the translation and boost transformation of a $d = 5$ Poincaré algebra, $\mathfrak{iso}(4,1)$. The symmetry breaking pattern is therefore $\frac{\mathfrak{iso}(4,1)}{\mathfrak{so}(3,1)}$. Let us recall the d-dimensional Poincaré algebra (using anti-Hermitian generators J_{AB} and P_A):

$$\begin{aligned} &[J_{AB}, J_{CD}] = (\eta_{AC} J_{BD} - \eta_{BC} J_{AD} + \eta_{BD} J_{AC} - \eta_{AD} J_{BC})\,, \\ &[J_{AB}, P_C] = (\eta_{AC} P_B - \eta_{BC} P_A), \quad [P_A, P_B] = 0\,, \end{aligned} \tag{3.52}$$

where $A, B, \ldots = 0, \ldots, 4$. We will indicate the indices $A = 0, \ldots 3$ with the Greek indices $\mu, \nu, \ldots$. The coset element for $\frac{\mathfrak{iso}(4,1)}{\mathfrak{so}(3,1)}$ is:

$$g = e^{x \cdot P} e^{\pi(x) P_4} e^{A^{\mu 4}(x) J_{\mu 4}}\,, \tag{3.53}$$

Now, let us introduce a dummy parameter σ. We rescale the generators in the following way:

$$P_4 \to \sigma C, \quad J_{4\mu} \to \sigma K_\mu\,. \tag{3.54}$$

Upon sending $\sigma \to \infty$, we recover the Galileon algebra (3.41). We see that the Galileon and Poincaré algebras are related by an *Inönu-Wigner contraction*. Equivalently, we can say that Poincaré algebras are analytic deformations of Galileon algebras. It is easy to extend this procedure to multiple fields and different dimensions [96].

To see the effect of rescaling P_4 and $J_{\mu 4}$ on the transformation laws, let us simultaneously rescale $\pi = x^4$ and $A^\mu = A^{\mu 4}$ by the inverse factor:

$$\pi(x) \to \frac{1}{\sigma} \tilde{\pi}(x), \quad A^\mu(x) \to \frac{1}{\sigma} \tilde{A}^\mu(x)\,, \tag{3.55}$$

so that the coset element becomes:

$$g = e^{x \cdot P} e^{\tilde{\pi}(x) C} e^{\tilde{A}^\mu(x) K_\mu}\,, \tag{3.56}$$

Now, defining the symmetry parameters $\tilde{c} = \sigma c$, $\tilde{c}^\mu = \sigma c^\mu$ and inserting into (3.51), we find:

$$\tilde{\pi}(x) \to \tilde{\pi}(x) + \tilde{c} + \tilde{c}^\mu x_\mu\,. \tag{3.57}$$

We see that the the field-dependent term drops out and we recover (3.40). The procedure of rescaling the Goldstone fields is known as taking a *small-field limit*. We can just as well take the limit directly on the action (3.50).

We then obtain simply a Klein-Gordon kinetic term for $\pi(x)$, which is the "quadratic Galileon".

By taking a small-field limit on higher-derivative invariants of the Poincaré algebra, it is possible to reconstruct all of the Wess-Zumino interactions of the Galileon algebra. [97] Similarly, we can connect the conformal algebra in the AdS and conformal bases by an Inönu-Wigner contraction. We then recover the conformal Galileons by means of a small-field limit on the action for a probe brane in anti-de Sitter space. We will treat these cases in more detail in the next section.

In 3.3.3, we will show that the story of this section carries over without much modification to supersymmetric Goldstone EFTs: the supersymmetric Galileons - and order n supersymmetric extended shifts - provide the foundation for more complicated EFTs (such as those that describe supersymmetry preserving branes propagating in higher dimensions). Those EFTs can be then related to the supersymmetric Galileons again by taking a small-field limit on the action or an Inönu-Wigner contraction on the symmetry algebra.

3.2.4 Coset universality and space-time symmetries: the AdS and conformal bases

Earlier in this Chapter, we emphasized that there is no proof of coset universality beyond the case of internal, compact and semi-simple symmetries. While it is possible to relax semi-simplicity in the proof of CCWZ, it is not straightforward to generalize to non-compact space-time symmetry groups. However, we note that there are no known counterexamples to coset universality, despite some effort to discover them. In order to help the reader appreciate this, we will examine the following two case studies: the non-linear realizations of the $D = 4$ conformal algebra and the $\mathcal{N} = 1$ supersymmetry algebra. In both cases, several non-linear realizations exist, but they all map into each other by highly non-trivial field redefinitions. The mapping of the conformal group realizations was studied in the papers [25,174], and the mapping of supersymmetry realizations in [26–30,116,117]. The work of [25,174] was then generalized in [141]. The coset models for $SO(D,2)/SO(d,1)$ and $\mathcal{N}$-extended supersymmetry will appear again numerous times throughout this thesis. The following two subsections should also serve as an introduction to the simplest cases of these models and to their non-linear symmetries.

Let us begin with the symmetry breaking pattern of the $D = 4$ conformal group $SO(4,2)$ down to the Poincaré group $ISO(3,1)$. This coset has two common realizations. The first of these, called the *DBI realization* [25] naturally describes the dynamics of a probe 3-brane in AdS_5 space, with the action:

$$S_{DBI} = \int d^4x \bigg(- \lambda e^{-4\pi/\ell} \sqrt{1 + e^{2\pi/l} (\partial\pi)^2} + \lambda e^{-4\pi/\ell} \bigg) . \tag{3.58}$$

This is the static gauge DBI action for the probe brane, which we discussed in the previous chapter. It inherits the isometries of the ambient AdS_5 space. However, only the $ISO(3,1)$ subgroup remains linearly realized. The transverse isometries are broken because they come together with a compensating gauge transformation that puts the system back in the static gauge. The dilaton field π has the following symmetry transformations:

$$\begin{aligned} \delta_c \pi &= c \bigg(1 - \tfrac{1}{\ell} x^\mu \partial_\mu \phi \bigg) , \\ \delta_v \pi &= v_\mu x^\mu + \partial_\mu \bigg(\tfrac{\ell}{2} (e^{2\pi/\ell} - 1) v^\mu + \tfrac{1}{2\ell} v^\mu x^2 - \tfrac{1}{\ell} v^\nu x_\nu x^\mu \bigg) . \end{aligned} \tag{3.59}$$

The second realization of the symmetry breaking pattern is called the *Weyl realization*, where the fundamental quantity is the effective metric $e^{-2\bar{\pi}} \eta_{\mu\nu}$. Invariant actions are built out of curvature tensors of this object. The dilaton $\bar{\pi}$ now transforms as follows:

$$\begin{aligned} \delta_{\hat{c}} \bar{\pi} &= \hat{c} \bigg(1 - x^\mu \partial_\mu \bar{\pi} \bigg) , \\ \delta_{\hat{v}} \bar{\pi} &= \hat{v}^\mu \bigg(- 2x_\mu - x^2 \partial_\mu \bar{\pi} + x_\mu (x \cdot \partial \bar{\pi}) \bigg) . \end{aligned} \tag{3.60}$$

It is not straightforward to find redefinitions of the coordinates and the fields that map these symmetry transformation laws into each other. Let us see how these mappings follow from the coset construction. The DBI and Weyl realizations follow from different basis choices for the $SO(4,2)$ Lie algebra. In the so-called *AdS basis*, the $SO(d,2)$ algebra is spanned by the generators P_A, K_A, D, and M_{AB} where A, B are d-dimensional space-time indices. The commutation relations are:

$$\begin{aligned} [P_A, D] &= P_A , \\ [K_A, D] &= -K_A + P_A , \\ [P_A, K_B] &= 2M_{AB} + 2\eta_{AB} D , \\ [K_A, K_B] &= 2M_{AB} , \end{aligned} \qquad \begin{aligned} [M_{AB}, P_C] &= \eta_{AC} P_B - \eta_{BC} P_A , \\ [M_{AB}, K_C] &= \eta_{AC} K_B - \eta_{BC} K_A , \\ [M_{AB}, M_{CD}] &= 2\eta_{C[A} M_{B]D} - 2\eta_{B[D} M_{C]A} . \end{aligned}$$

The *conformal basis* is defined from the above by the redefinition:

$$\bar{K}_A = K_A - \tfrac{1}{2} P_A . \tag{3.61}$$

For general co-dimension branes, we would split up these indices into $D = 4$ Greek indices μ, ν, ... and the remaining $d - D$ indices I, J, ... For co-dimension 1, however, we have $D = d$, so we simply relabel the A indices with Greek letters. Then, in the DBI realization using the AdS basis, parametrize the coset as follows:

$$g_{DBI} = e^{x^\mu P_\mu} e^{\pi D} e^{\Lambda^\mu K_\mu} \,. \tag{3.62}$$

From this definition, one can calculate the Maurer-Cartan form $g_{DBI}^{-1}\, dg_{DBI}$. The result appears in [25]. The Maurer-Cartan form defines the following inverse Higgs constraint, which eliminates the inessential vector field Λ^μ:

$$\Lambda_\mu(x) = \left(\frac{\tan\sqrt{\Lambda^2/2}}{\sqrt{\Lambda^2}}\right)^{-1} \frac{\partial_\mu \pi e^{\pi/\ell}}{1 + \sqrt{1 + e^{2\pi/\ell}(\partial\pi)^2}} \,. \tag{3.63}$$

Likewise, the Weyl realization follows from the following coset parametrization in the conformal basis:

$$g_{Weyl} = e^{y^\mu P_\mu} e^{\bar{\pi} D} e^{\bar{\Lambda}^\mu \bar{K}_\mu} \,. \tag{3.64}$$

Once again, we can calculate the Maurer-Cartan form and obtain the inverse Higgs relation. The result appears in [25]. The Inverse Higgs constraint now reads:

$$\bar{\Lambda}_\mu(y) = \tfrac{1}{2} e^{\bar{\pi}} \partial_\mu \pi(y) \,. \tag{3.65}$$

Now we can obtain the mapping between the two realizations by equating the Maurer-Cartan forms: $g_{DBI}^{-1}\, dg_{DBI} = g_{Weyl}^{-1}\, dg_{Weyl}$. Solving the equation leads to the following remarkable relations between the DBI and Weyl coset variables:

$$\begin{aligned} y^\mu &= x^\mu + \ell e^{\pi(x)/\ell} \Lambda^\mu(x) \left(\frac{\tan\sqrt{\Lambda^2(x)/2}}{\sqrt{\Lambda^2(x)}}\right), \\ \bar{\pi}(y) &= \frac{\pi(x)}{\ell} + \log\left[1 + \left(\frac{\tan\sqrt{\Lambda^2/2}}{\sqrt{\Lambda^2}}\right)^2 \Lambda^2\right], \\ \bar{\Lambda}_\mu(y) &= \frac{1}{\ell}\left(\frac{\tan\sqrt{\Lambda^2/2}}{\sqrt{\Lambda^2}}\right) \Lambda_\mu(x) \,. \end{aligned} \tag{3.66}$$

These field redefinitions and coordinate transformations indeed provide a mapping between (3.59) and (3.60). Furthermore, they map the inverse Higgs constraints (3.59) and (3.60). In [141], it was shown that equating Maurer-Cartan forms before imposing inverse Higgs relations always results in this type of field redefinition that relates the transformation rules. However, the inverse Higgs constraints are not necessarily mapped by the same relations.

3.2.5 Coset universality and space-time symmetries: supersymmetry

We now turn our attention to the non-linear realization of $\mathcal{N} = 1$ supersymmetry in $D = 4$, with a linearly realized Poincaré group. Several realizations of this coset model appear in the literature. The most important one is the Volkov-Akulov model, [104] which makes use of a single Weyl fermion λ_α. While ordinary linear supersymmetry requires multiplets of fields with different spins, non-linear realizations only require a single *Goldstino*. Of course, this should be expected from our discussion in Chapter 1, where we explained that dynamical symmetries do not lead to particle classification. The $\mathcal{N} = 1$ supersymmetry algebra consists of the Poincaré algebra in addition to the fermionic generator Q_α which satisfies the anti-commutation relation:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 2P_{\alpha\dot{\beta}} \,. \tag{3.67}$$

The Volkov-Akulov Goldstino λ_α non-linearly realizes this algebra with the following transformation law:

$$\delta_\epsilon \lambda_\alpha(x) = f\epsilon_\alpha - \tfrac{i}{f}(\lambda\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda})\partial_\mu\lambda_\alpha \,. \tag{3.68}$$

The coset construction for this superalgebra gives rise to the following vielbein:

$$A_\mu{}^a = \delta_m{}^a - if^{-2}\partial_\mu\lambda\sigma^a\bar{\lambda} + if^{-2}\lambda\sigma^a\partial_\mu\bar{\lambda} \,. \tag{3.69}$$

The leading order invariant action is then given by:

$$\mathcal{L} = -\frac{f^2}{2}\det A \,. \tag{3.70}$$

This Lagrangian contains self-interaction terms for λ up to order λ^6 in $D = 4$. As an essential Goldstone field, λ is massless. The Lagrangian (3.70) gives rise to an enhanced soft-limit $\sigma = 2$ for λ due to the exceptional non-linear symmetry. [110, 111] Volkov and Akulov initially proposed (3.70) as a model for the neutrino, with the broken supersymmetry transformation explaining its (near-) masslessness. This idea was later found to disagree with experiment. [115] The Volkov-Akulov model lives on, however, as it exists at some level of description in any model with spontaneously broken supersymmetry.

The transformation of matter fields $A(x)$ of arbitrary spin that accompanies (3.68) is the following:

$$\delta_\epsilon A(x) = -if^{-1}\Big(\lambda(x)\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda}(x)\Big)\partial_\mu A(x) \,. \tag{3.71}$$

Together, (3.68) and (3.71) define the *standard realization* of non-linear supersymmetry.

Although non-linear supersymmetry does not require supermultiplets, it is still convenient to use the superfield formalism to describe the non-linear realization. The other models that we will explore in this section all make use of superfields. One constructs a superfield Λ_α out of λ_α as follows: [122]

$$\Lambda_\alpha(x,\theta,\bar{\theta}) = \exp(\theta Q + \bar{\theta}\bar{Q}) \cdot \lambda_\alpha(x)\,. \tag{3.72}$$

This θ polynomial transforms as a superfield under Q_α, while the lowest component $\Lambda_\alpha(x,\theta,\bar{\theta})| = \lambda_\alpha(x)$ retains the transformation law (3.68). Λ_α satisfies the constraints:

$$\begin{aligned} D_\alpha \Lambda_\beta &= f\epsilon_{\beta\alpha} + \frac{i}{f}(\sigma^\mu)_{\beta\dot{\beta}}\bar{\Lambda}^{\dot{\beta}}\partial_\mu\Lambda_\alpha\,, \\ \bar{D}_{\dot{\beta}}\Lambda_\alpha &= -\frac{i}{f}\Lambda^\beta(\sigma^\mu)_{\beta\dot{\beta}}\partial_\mu\Lambda_\alpha\,. \end{aligned} \tag{3.73}$$

In superspace, the Volkov-Akulov Lagrangian is:

$$\mathcal{L} = -\frac{1}{f^2}\int d^4\theta \Lambda^2\bar{\Lambda}^2\,. \tag{3.74}$$

We could have started the other way around and use these constraints to *define* Λ_α. Then we would find that its lowest components behaves exactly as a Volkov-Akulov fermion. The appearance of *constrained superfields* is universal for all non-linear realizations that make use of superspace.[4] In recent years, the description of non-linear supersymmetry using the *nilpotent* chiral superfield X has been popular. [106–109] On top of the chirality condition $\bar{D}_{\dot{\alpha}}X = 0$, X satisfies the non-linear constraint $X^2 = 0$. These two constraints are solved by:

$$X = \frac{G(y)^2}{2F(y)} + \sqrt{2}\theta G(y) + \theta^2 F(y)\,, \tag{3.75}$$

where G and F are functions of the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. Clearly, this solution makes sense only if $F \neq 0$, i.e. when supersymmetry is spontaneously broken. The supersymmetry transformation for $G(x)$ then becomes

[4]Of course, most of the superfield one encounters in the literature satisfy some kind of supersymmetry-covariant constraint. Usually, they project the superfield to an irreducible representation of supersymmetry, like the chiral constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$ does. The constraints we are talking about reduce the superfield to less than an irreducible multiplet. Clearly, such a constraint can only have a non-trivial solution when supersymmetry is spontaneously broken. We will refer to superfields like Φ as *irreducible superfields* and those that contain less than an irreducible supermultiplet as *constrained superfields*.

linear after inserting the solution to the auxiliary field equations for $F(x)$. Notice that the scalar superpartner of $G(y)$ (the "sgoldstino") is eliminated by the nilpotency condition. The fact that X does not contain a full multiplet of supersymmetry has been especially useful in cosmology, as it allows one to consider effective Lagrangians with spontaneously broken supersymmetry without worrying about stabilizing superfluous scalar excitations in the cosmological background. [167–169]

At leading order, the unique Lagrangian one can build out of X is:

$$\mathcal{L} = \int d^4\theta X\bar{X} + \left(f \int d^2\theta X + \text{c.c.}\right). \tag{3.76}$$

Our objective is to show the equivalence between (3.70) and (3.76). Many references have treated this question (see [26–30, 116, 117]). Let us follow the [30] and answer it completely in the superspace formalism.

Define the Samuel-Wess superfield [105] $\Gamma_\alpha(x, \theta, \bar{\theta})$ using the Volkov-Akulov superfield Λ_α:

$$\Gamma_\alpha = -2f \frac{D_\alpha \bar{D}^2(\Lambda^2\bar{\Lambda}^2)}{D^2\bar{D}^2(\Lambda^2\bar{\Lambda}^2)}. \tag{3.77}$$

The Samuel-Wess superfield satisfies constraints very similar to (3.73):

$$\begin{aligned} D_\alpha\Gamma_\beta &= f\epsilon_{\beta\alpha}, \\ \bar{D}_{\dot{\alpha}}\Gamma_\beta &= -\frac{2i}{f}\Gamma^\gamma(\sigma^\mu)_{\gamma\dot{\beta}}\partial_\mu\Gamma_\beta, \end{aligned} \tag{3.78}$$

which again could have served as a definition of Γ. The Samuel-Wess superfield can also be written in terms of the nilpotent superfield X:

$$\Gamma_\alpha = -2f \frac{D_\alpha X}{D^2 X}. \tag{3.79}$$

The relation between the Volkov-Akulov description and the nilpotent superfield description is provided by the equations (3.79) and (3.77). Note that the former definition for Γ still involves the auxiliary field F in X, which needs to be integrated out before these relations can be interpreted as field redefinitions. The reference [30] shows how to write (3.76) using (3.79). Upon integrating out the auxiliary field, the Lagrangian then reduces to (3.70).

We have introduced three different non-linear realizations of $\mathcal{N} = 1$ supersymmetry and shown how to relate them in superspace. One can find the explicit field redefinitions of the component fields that result from these relations in the references [26–29]. The existence of these field redefinitions is important evidence for coset universality.

In closing, let us comment on the universality of the matter field transformation (3.71). Given the superfield realization of Volkov-Akulov using Λ_α, how can we expect coupled matter fields (which should be carried by ordinary superfields Φ) to transform according to the standard realization? Ivanov and Kapustnikov [116, 117] that by performing a local supertranslation on Φ using $\lambda_\alpha(x)$ as a parameter, one finds a field redefinition that brings the components of Φ into the standard realization (3.71).

3.3 Supercosets

We will now extend our discussion of the coset construction for space-time symmetries to the case of linear $\mathcal{N} = 1$ supersymmetry in $D = 4$. As we will, see most of the previous discussion goes through in a straightforward way. However, there are two important new ingredients: *superspace* inverse Higgs constraints and covariant irreducibility conditions. The former are analogous to ordinary space-time IHCs. However, instead of eliminating inessential Goldstone fields, they eliminate superfluous superpartners by identifying them with other Goldstone fields. Just like ordinary space-time IHCs, superspace inverse Higgs constraints exist because independent global symmetries may become degenerate when they are localized in superspace. The covariant irreducibility conditions are the generalization of ordinary irreducibility conditions (like the usual chirality conditon $\bar{D}_{\dot{\alpha}}\Phi = 0$) in the presence of the non-linear symmetries. In practice, the covariant irreducibility conditions can be hard to find. However, we will see that they tend to simplify when one enhances the amount of non-linear symmetry in the EFT.

In this section and the following, we will follow the notation conventions of [122]. Space-time vector indices will be labeled m,n,a,b etc. Spinor indices are μ, ν,α, β, etc.

We assume the following set of linearly realized symmetries: the ordinary Poincaré algebra, generated by P_m and M_{mn}; a linear internal symmetry group, generated by Z_I; and $\mathcal{N} = 1$ supersymmetry, with the generators Q_μ which satisfy:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}}P_a = 2P_{\alpha\dot{\alpha}}\,. \tag{3.80}$$

Here, we have made use of two-component notation to write the matrix-valued vector $(\sigma^a)_{\alpha\dot{\alpha}}P_a$ as $P_{\alpha\dot{\alpha}}$. (See for instance [122, 124]) On top of the linearly realized symmetries we include the broken generators G_i that make up our coset G/H. We assign a superfield $\Phi^i(x, \theta, \bar{\theta})$ (with appropriate H indices) to each broken generator G_i. For consistency of the coset construction, the algebra must be restricted to the following general form:

$$[Z_I, Z_J] = f_{IJ}{}^K Z_K, \quad [G_i, Z_J] = f_{iJ}{}^j G_j, \tag{3.81}$$
$$[P_\mu, Z_I] = f_{\mu I}{}^\nu P_\nu, \quad [Q_\alpha, Z_I] = f_{\alpha I}{}^\beta Q_\beta, \tag{3.82}$$
$$\tag{3.83}$$

while the remaining (anti)-commutation relations are unconstrained at this stage.

At this stage, there are no constraints on Φ^i, so it forms a reducible representation of supersymmetry. For the moment, we will make use of the following parametrization for the coset element:

$$\gamma = e^{i(x^a P_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} e^{iZ^i G_i} . \tag{3.84}$$

Note that this will not be the most convenient parametrization when it comes to imposing superspace inverse Higgs constraints, as we will explain in a moment. The transformation law for the Goldstone superfields and the superspace coordinates is defined by:

$$g \cdot \gamma = e^{i(x'^a P_a + \theta'^\alpha Q_\alpha + \bar{\theta}'_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} e^{iZ'^i G_i} e^{ih^I(\Phi, g) Z_I} . \tag{3.85}$$

As usual, we define the Maurer-Cartan form ω:

$$\omega = \gamma^{-1} d\gamma \, , \tag{3.86}$$

where the exterior derivative is extended to the full superspace: $d = dx^m \partial_m + d\theta^\mu \partial_\mu + \bar{\theta}_{\dot{\mu}} \partial^{\dot{\mu}}$, where $\partial_\mu = \frac{\partial}{\partial \theta^\mu}$. We refer the reader to [122] for an introduction to differential forms in superspace. Expand the Maurer-Cartan form into Lie algebra generators:

$$\omega = (\omega_P^a) P_a + (\omega_Q^\alpha) Q_\alpha + (\omega_{\bar{Q}\dot{\alpha}}) \bar{Q}^{\dot{\alpha}} + (\omega^i) G_i + (\omega^I) Z_I + (\omega_M^{ab}) M_{ab} \, . \tag{3.87}$$

Each of the Maurer-Cartan components in general has component along all the superspace coordinate basis 1-forms, for example:

$$(\omega_P^a) = (\omega_P^a)_m dx^m + d\theta^\mu (\omega_P^a)_\mu + d\bar{\theta}_{\dot{\mu}} (\omega_P^a)^{\dot{\mu}} \, . \tag{3.88}$$

Much of the story of the previous sections now goes through. The Maurer-Cartan components proportional to broken generators and the coordinates transform under G as a Goldstone-dependent H transformation; the components proportional to linearly realized generators transform like gauge connections for H:

$$\begin{aligned}
(\omega_P^a)'(x',\theta',\bar\theta')P_a &= h\,(\omega_P^a)(x,\theta,\bar\theta)P_a\,h^{-1},\\
(\omega_Q^\alpha)'(x',\theta',\bar\theta')Q_\alpha &= h\,(\omega_Q^\alpha)(x)Q_\alpha\,h^{-1},\\
(\omega_{\bar Q\dot\alpha})'(x',\theta',\bar\theta')\bar Q^{\dot\alpha} &= h\,(\omega_{\bar Q\dot\alpha})(x,\theta,\bar\theta)\bar Q^{\dot\alpha}\,h^{-1},\\
(\omega^i)'(x',\theta',\bar\theta')G_i &= h\,\omega^i(x,\theta\,\bar\theta)Z_i\,h^{-1},\\
(\omega^I)'(x',\theta',\bar\theta')Z_I + (\omega_M)'^{ab}(x',\theta',\bar\theta')M_{ab} &= h\,[\omega^I(x,\theta,\bar\theta)(x,\theta,\bar\theta)Z_I\\
&\quad + \omega_M^{ab}(x,\theta,\bar\theta)M_{ab}]\,h^{-1} + h\,dh^{-1}\,,
\end{aligned} \tag{3.89}$$

where $h = h(\Phi, g) = e^{ih^a(\Phi,g)Z_a}$. The components (ω_P), (ω_Q) and $(\omega_{\bar Q})$ combine into a supervielbein $E_M{}^A$, such that:

$$\omega^a P_a + \omega_Q^\alpha Q_\alpha + \omega_{\bar Q\dot\alpha}\bar Q^{\dot\alpha} = dX^M E_M{}^A P_A\,. \tag{3.90}$$

M and A are combined spinor and vector indices. dX^M stands for the superspace coordinate basis 1-forms, i.e. $dX^m = dx^m$, $dX^\mu = d\theta^\mu$, $dX_{\dot\mu} = d\bar\theta_{\dot\mu}$. Similarly, P_A stands for the translation and supertranslation generators combined. Note that summation over the dotted indices always occurs with the up-down convention when the generalized indices are written down with down-up convention. In other words:

$$M^A N_A = M^a N_a + M^\alpha N_\alpha + M_{\dot\alpha} N^{\dot\alpha}\,. \tag{3.91}$$

We can again use the supervielbein to extract covariant derivatives out of the broken generator Maurer-Cartan components:

$$\hat D_A \Phi^i = \tilde E_A{}^N (\omega_i)_N\,, \tag{3.92}$$

where $\omega^i = dX^N(\omega_i)_N$. The quantity $\tilde E_A{}^N$ is the inverse vielbein: $\tilde E_A{}^N E_N{}^B = \delta_A{}^B$. The quantity $\hat D_A$ includes the covariantized versions of the superspace vector and spinor derivatives, i.e. ∂_a, D_α and $\bar D_{\dot\alpha}$. The covariant derivative of the matter field H multiplet $\Psi(x,\theta,\bar\theta)$ is defined as follows:

$$\hat D_A \Psi = E_A{}^N\left(\partial_N \Psi + (\omega^I)_N D(Z_I)\cdot\Psi + (\omega_M^{ab})_N D(M_{ab})\cdot\Psi\right) \tag{3.93}$$

where $D(Z_I)$ is the generator Z_I in the representation spanned by the H-multiplet Ψ.

Lastly, wedging together the eight (super)translation Maurer-Cartan components gives rise to an invariant integration measure, proportional to the superdeterminant (Berezinian) of the supervielbein:

$$\int d^4x\, d^4\theta\, \mathrm{Ber}(E_M{}^A) = \int d^4x\, d^4\theta\, E\,. \tag{3.94}$$

Any H-invariant density built out of covariant derivatives of Goldstone and matter fields, multiplied by the Berezinian of the supervielbein, is then also G-invariant.

3.3.1 Superspace inverse Higgs constraints

Just like in ordinary space-time, independent global symmetry generators may become degenerate when they are localized. We saw that this gives rise to the inverse Higgs phenomenon, whereby inessential Goldstone fields are eliminated after imposing covariant constraints. In the supersymmetric case, there are not just inessential Goldstone superfields, but also many superfluous superpartner fields that come along for the ride in the coset element (3.84). We would like to find a realization of the non-linear symmetries where several Goldstone fields live together in the same superfield. This is possible when the massless mode parametrized by the superpartner of a Goldstone Φ^1 is degenerate with another Goldstone mode Φ^2.

Let us work in a similar setup as before. We write the Goldstone modes as localized G/H transformations on the H-invariant vacuum field configuration $|0\rangle$:

$$\Phi^i(x,\theta,\bar{\theta})G_i|0\rangle\,, \tag{3.95}$$

where Φ^i is a slowly-varying function of superspace. There are degenerate modes when there are non-trivial solutions to the following equation:

$$\Phi^i(x,\theta,\bar{\theta})G_i|0\rangle = 0\,. \tag{3.96}$$

Again, let us act on this equation with a differential operator to see what form the solutions should take. The appropriate operator is $e^{-U}de^{U}$, where $U = i(x^aP_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})$. Here, the exterior derivative d acts on everything to the right and generators act only on each other, not on fields or coordinates. We then find:

$$\left[e^a(\partial_a\Phi^i - f_{aj}{}^i\Phi^j) + e^\alpha(D_\alpha\Phi^i - f_{\alpha j}{}^i\Phi^j) + e_{\dot{\alpha}}(\bar{D}^{\dot{\alpha}}\Phi^i - f^{\dot{\alpha}}{}_j{}^i\Phi^j)\right]G_i|0\rangle = 0\,. \tag{3.97}$$

We have switched here from the coordinate basis 1-forms to the so-called supersymmetric flat space basis [122] 1-forms e^a, e^α and $e_{\dot{\alpha}}$. In this basis, the exterior derivative takes the convenient form:

$$d = e^a\partial_a + e^\alpha D_\alpha + e_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\,, \tag{3.98}$$

so that each component of the exterior derivative of a superfield $d\Phi$ is a superfield. The structure constants in (3.97) are defined as follows:

$$[P_{\alpha\dot{\alpha}}, G_i] = -if_{\alpha\dot{\alpha}i}{}^j G_j + \dots, \quad [Q_\alpha, G_i]_\pm = if_{\alpha i}{}^j G_j + \dots$$
$$[\bar{Q}_{\dot{\alpha}}, G_i]_\pm = if_{\dot{\alpha}i}{}^j G_j + \dots \tag{3.99}$$

The ellipses on the right-hand side indicate unbroken generators. Working to first order in the fields, we may then set each of the bracketed terms to zero:

$$\begin{aligned} &\partial_a \Phi^i - f_{aj}{}^i \Phi^j = \mathcal{O}(\Phi^2), \\ &D_\alpha \Phi^i - f_{\alpha j}{}^i \Phi^j = \mathcal{O}(\Phi^2), \\ &\bar{D}^{\dot{\alpha}} \Phi^i - f^{\dot{\alpha}}{}_j{}^i \Phi^j = \mathcal{O}(\Phi^2)\,, \end{aligned} \tag{3.100}$$

The first of these equations is the linearized version of the ordinary IHcs. The second and third equations are the linearized superspace inverse Higgs constraints. We see that we can relate the linear combination of Goldstone modes $f_{\alpha j}{}^i \Phi^j$ to the superspace derivative Φ^i. At lowest order in θ and $\bar{\theta}$, this precisely relates the superpartner of the Goldstone associated to G_i to the $f_{\alpha j}{}^i \Phi^j$, as we have anticipated. We find that in order to relate the Goldstone superfield Φ^2 to $D_\alpha \Phi^1$, we must have $[Q_\alpha, G_2] \supset G_1$. Similarly, imposing $\Phi^2 \propto \bar{D}_{\dot{\alpha}} \Phi^1$ requires $[\bar{Q}_{\dot{\alpha}}, G_2] \supset G_1$.

Once again, to find the non-linear completion of the equations (3.100), we must make use of the coset construction. The inverse Higgs constraints of the ordinary coset construction generalize in a straightforward way to the supercosets. The algebraic relation $[Q_\alpha, G_2] \supset G_1$ leads to a term linear in Φ^2 in the supercoset derivative $\hat{D}_\alpha \Phi^1$, in addition to higher-order terms in the fields. Then, imposing the constraint $\hat{D}_\alpha \Phi^1 = 0$ leads to a relation $\Phi^2 \propto \bar{D}_{\dot{\alpha}} \Phi^1$, with the appropriate higher-order corrections. Similar statements hold of course for the barred spinor derivative $\hat{\bar{D}}_{\dot{\alpha}}$ and the covariant space-time derivative $\hat{D}_\mu$.

The higher-order corrections that appear in the coset covariant derivatives introduce consistency conditions on the algebra, in addition to the familiar relations of the type $[Q_\alpha, G_2] \supset G_1$. These conditions will not play an important role in the classification of exceptional EFTs, so we will not address them further. The form of the higher-order conditions will depend on the parametrization of the coset element, as we saw previously in 3.2.2.

3.3.2 Covariant irreducibility conditions

There is one last point to discuss regarding supercosets, before we move on to some concrete examples. In most cases, the Goldstone superfields Φ^i in (3.84) are, a priori, unconstrained functions of superspace. That means they carry reducible multiplets of supersymmetry. Ordinarily, we would reduce their field content to an irreducible supermultiplet by imposing one or a number of the canonical irreducibility conditions. For example, to reduce the field content of a complex scalar superfield Φ to that of a chiral supermultiplet, we impose the condition:

$$\bar{D}_{\dot{\alpha}}\Phi = 0\,. \tag{3.101}$$

However, as we have explained in the previous section, the spinor derivatives D_α and $\bar{D}_{\dot{\alpha}}$ and the ordinary space-time derivative $\partial_{\alpha\dot{\alpha}}$ are not always covariant with respect to the non-linear symmetry transformations. The constraint (3.101) is then not left invariant under one or some of the broken symmetries. This does not imply, however, that the non-linear transformations are actually incompatible with the irreducibility condition. Rather, it means that the condition (3.101) does not hold in the field basis where the non-linear field transformations are defined by (3.84). It can still be possible to perform a jet space redefinition such that (3.101) is covariant with respect to all non-linear symmetries, but the transformation laws are then not directly constructed from the coset construction. Alternatively, we can impose a generalized irreducibility that is manifestly covariant under all non-linear symmetries, but whose solutions are in one-to-one correspondence with a canonical irreducible superfield. We will pursue this strategy in the rest of this thesis, following for example [155, 156, 173].

Let us imagine we have a particular symmetry breaking pattern G/H which we would like to realize on a single chiral superfield. Our task is then to calculate the generalized covariant derivatives - $\hat{D}_\alpha$, $\hat{\bar{D}}_{\dot{\alpha}}$, and $\hat{\partial}_{\alpha\dot{\alpha}}$ - and to impose a condition of the form:

$$T_{\dot{\alpha}}[\hat{D}\Phi, \hat{\bar{D}}\Phi, \hat{\partial}\Phi, \ldots] = 0\,. \tag{3.102}$$

Here, $T_{\dot{\alpha}}$ is some dotted spinor built from manifestly covariant objects. We emphasize that in many cases the tensor T will not be a minimal generalization of the canonical version. We then attempt to reconstruct Φ out of a chiral superfield Ψ which satisfies the canonical condition (3.101):

$$\Phi = F[\Psi, D\Psi, \bar{D}\Psi, \partial\Psi, \ldots]\,. \tag{3.103}$$

We should then find that (3.102) holds as a consequence of (3.101). Additionally, we should ensure that (3.103) defines an invertible redefinition of the (super-) jet space. Because of this last point, the seemingly complicated and arbitrary process of finding suitable irreducibility conditions poses no additional problems for the question of coset universality. We can always map one set of irreducibility conditions into another by convoluting the redefinitions (3.103). In practice, however, it is difficult to prove invertibility. The examples in the literature known to us do not explicitly check invertibility. We comment on the issue here mostly to alleviate any concern that this inelegant process calls into question the viability of describing all non-linear realizations using supercosets.

3.3.3 Supersymmetric Galileons and Inönu-Wigner contractions

To illustrate the concepts of inverse Higgs constraints in superspace and irreducibility conditions, we will now take a look at one of the simplest possible examples: non-linear realizations of the supersymmetric Galileon algebra. We will explain how this algebra arises as an Inönu-Wigner contraction of the super-Poincaré algebra. [160] Equivalently, supersymmetric Galileons are the small-field limit of the SUSY DBI model. Thus, just like the purely bosonic Galileons of 3.2.3, the SUSY Galileons - and SUSY extended shift symmetries - provide the blueprint for more complicated EFTs.

Similarly to section 3.2.3, we define supersymmetric Galileons as the minimal algebra necessary for imposing a space-time inverse Higgs relation. In addition to the four-dimensional Poincaré algebra, defined by the generators $P_{\mu\dot{\mu}}$ and $M_{\mu\nu}$, we introduce a complex scalar generator G, a Weyl fermion S_ν and a vector $K_{\mu\dot{\mu}}$:

$$\{Q_\mu, S_\nu\} = 2\epsilon_{\mu\nu} G, \quad [P_{\mu\dot{\mu}}, G_{\nu\dot{\nu}}] = i\epsilon_{\mu\nu}\epsilon_{\dot{\mu}\dot{\nu}} G, \quad [\bar{Q}_{\dot{\mu}}, G_{\nu\dot{\nu}}] = i\epsilon_{\dot{\mu}\dot{\nu}} S_\nu \,. \tag{3.104}$$

Note that, following [122], we will use Greek letters from the middle of the alphabet (e.g. μ, ν, ...) for space-time spinor indices. Greek letters from the start of the alphabet (α, β, ...) are reserved for tangent space spinor indices. This tangent space will be defined by the supervielbein derived from the coset construction. Of course, the algebra (3.104) is very simple and the use of the coset construction to understand it may seem like overkill. We include this discussion mainly as a warmup to the more complicated cases that appear in the rest of the thesis.

Let us refer to the Goldstone superfields of G, $G_{\mu\dot{\mu}}$ and S_μ as Φ, $\Lambda^{\mu\dot{\mu}}$ and Ψ^μ respectively. The commutation relation $[P_{\mu\dot{\mu}}, G_{\nu\dot{\nu}}]$ allows for a space-time inverse Higgs relation to eliminate Λ in terms of Φ. Likewise, the bracket $\{Q_\mu, S_\nu\}$ suggests a superspace inverse Higgs relation between Φ and Ψ. Lastly, $[\bar{Q}_{\dot{\mu}}, G_{\nu\dot{\nu}}]$ allows an inverse Higgs constraint that links Λ to Ψ. Indeed, we will impose all of these relations below. The space-time IHC is related to the two superspace IHCs, as we will discuss in great detail in Chapter 5.

We will refer to the algebra (3.104) as $\mathfrak{sgal}(4,1\,|\,6,(1,0))$. The first two arguments refer to the $D=4$, $\mathcal{N}=1$ super-Poincaré subalgebra. The latter arguments represent that this algebra is based on the minimal $(1,0)$ supersymmetry algebra in $D=6$, as will become clear below.

Superspace inverse Higgs constraints and irreducibility conditions

We define a coset element as follows: [80]

$$\Omega = e^U e^V , \tag{3.105}$$

where

$$\begin{aligned} U &= \tfrac{i}{2}x^{\mu\dot\mu}P_{\mu\dot\mu} + i\theta^\mu Q_\mu + i\bar\theta_{\dot\mu}\bar Q^{\dot\mu} , \\ V &= i\Phi G + i\bar\Phi\bar G + i\Psi^\mu S_\mu + i\bar\Psi_{\dot\mu}\bar S^{\dot\mu} - \tfrac{i}{2}\Lambda^{\mu\dot\mu}G_{\mu\dot\mu} - \tfrac{i}{2}\bar\Lambda^{\mu\dot\mu}\bar G_{\mu\dot\mu} . \end{aligned} \tag{3.106}$$

The Maurer-Cartan form is given by:

$$\omega = -i\Omega^{-1}d\Omega = -ie^{-V}(e^{-U}de^U)e^V - ie^{-V}de^V . \tag{3.107}$$

We begin by computing $e^{-U}de^U$ which, by using the SUSY algebra $\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$, is given by

$$e^{-U}de^U = \tfrac{i}{2}P_{\mu\dot\mu}dx^{\mu\dot\mu} + id\theta^\mu Q_\mu + id\bar\theta_{\dot\mu}\bar Q^{\dot\mu} - P_{\mu\dot\mu}(d\theta^\mu\bar\theta^{\dot\mu} + d\bar\theta^{\dot\mu}\theta^\mu) . \tag{3.108}$$

In the supersymmetric flat space basis, the exterior derivative is expressed as

$$d = -\tfrac{1}{2}e^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + e^\alpha D_\alpha + e_{\dot\alpha}\bar D^{\dot\alpha} , \tag{3.109}$$

Expressing $e^{-U}de^U$ in terms of these basis one-forms, we obtain

$$e^{-U}de^U = \tfrac{i}{2}e^{\alpha\dot\alpha}P_{\alpha\dot\alpha} + ie^\alpha Q_\alpha + ie_{\dot\alpha}\bar Q^{\dot\alpha} . \tag{3.110}$$

It is then simple to show that:

$$\begin{aligned} e^{-V}(e^{-U}de^U)e^V = {} & \tfrac{i}{2}e^{\alpha\dot\alpha}P_{\alpha\dot\alpha} + ie^\alpha Q_\alpha + ie_{\dot\alpha}\bar Q^{\dot\alpha} + (2e^\alpha\Psi_\alpha + \tfrac{i}{4}e_{\alpha\dot\alpha}\Lambda^{\alpha\dot\alpha})G \\ & + (2e_{\dot\alpha}\bar\Psi^{\dot\alpha} + \tfrac{i}{4}e_{\alpha\dot\alpha}\bar\Lambda^{\alpha\dot\alpha})\bar G + \tfrac{i}{2}e_{\dot\beta}\Lambda^{\beta\dot\beta}S_\beta - \tfrac{i}{2}e^\alpha\bar\Lambda_{\alpha\dot\alpha}\bar S^{\dot\alpha} . \end{aligned} \tag{3.111}$$

The next factor is trivial, as all generators that appear in V commute amongst each other: $e^{-V}de^V = dV$. The full Maurer-Cartan form is then:

$$\begin{aligned} i\omega &= \tfrac{i}{2}e^{\alpha\dot\alpha}P_{\alpha\dot\alpha} + ie^\alpha Q_\alpha + ie_{\dot\alpha}\bar Q^{\dot\alpha} \\ &+ \Big[-\tfrac{i}{2}e_{\alpha\dot\alpha}(-\tfrac12\Lambda^{\alpha\dot\alpha}+\partial^{\alpha\dot\alpha}\Phi) + e^\alpha(2\Psi_\alpha + iD_\alpha\Phi) + ie_{\dot\alpha}\bar D^{\dot\alpha}\Phi\Big]G \\ &+ \Big[-\tfrac{i}{2}e_{\alpha\dot\alpha}(-\tfrac12\bar\Lambda^{\alpha\dot\alpha}+\partial^{\alpha\dot\alpha}\bar\Phi) + ie^\alpha D_\alpha\bar\Phi + e_{\dot\alpha}(2\bar\Psi^{\dot\alpha} + i\bar D^{\dot\alpha}\bar\Phi)\Big]\bar G \\ &+ \Big[-\tfrac{i}{2}e_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}\Psi^\beta + ie^\alpha D_\alpha\Psi^\beta + e_{\dot\beta}\big(\tfrac{i}{2}\Lambda^{\beta\dot\beta} + i\bar D^{\dot\beta}\Psi^\beta\big)\Big]S_\beta \\ &+ \Big[-\tfrac{i}{2}e_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}\bar\Psi^{\dot\beta} + e^\beta(\tfrac{i}{2}\bar\Lambda_\beta{}^{\dot\beta} + iD_\beta\bar\Psi^{\dot\beta}) + ie_{\dot\alpha}D^{\dot\alpha}\bar\Psi^{\dot\beta}\Big]\bar S_{\dot\beta} \\ &+ \Big[\tfrac{i}{4}e^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Lambda^{\beta\dot\beta} - \tfrac{i}{2}e^\alpha D_\alpha\Lambda^{\beta\dot\beta} - \tfrac{i}{2}e_{\dot\alpha}\bar D^{\dot\alpha}\Lambda^{\beta\dot\beta}\Big]G_{\beta\dot\beta} \\ &+ \Big[\tfrac{i}{4}e^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\bar\Lambda^{\beta\dot\beta} - \tfrac{i}{2}e^\alpha D_\alpha\bar\Lambda^{\beta\dot\beta} - \tfrac{i}{2}e_{\dot\alpha}\bar D^{\dot\alpha}\bar\Lambda^{\beta\dot\beta}\Big]\bar G_{\beta\dot\beta}\,. \end{aligned} \tag{3.112}$$

We explained in the first part of 3.3 that the covariant derivatives come from the product of the supervielbein and the Maurer-Cartan components, (3.92). For this algebra, the supervielbein is trivial and we can simply read off the full covariant derivatives $\hat D_A$. The ones relevant for the superspace inverse Higgs constraints are

$$\hat D_{\mu\dot\mu}\Phi = \partial_{\mu\dot\mu}\Phi - \tfrac12\Lambda_{\mu\dot\mu}, \quad \hat D_\mu\Phi = D_\mu\Phi - 2i\Psi_\mu, \quad \hat{\bar D}^{\dot\mu}\Psi^\nu = \bar D^{\dot\mu}\Psi^\nu + \tfrac12\Lambda^{\dot\mu\nu}\,. \tag{3.113}$$

The inverse Higgs constraints and their solutions are then:

$$\hat D_{\mu\dot\mu}\Phi = 0 \to \Lambda_{\mu\dot\mu} = 2\partial_{\mu\dot\mu}\Phi, \quad \hat D_\mu\Phi = 0 \to 2\Psi_\mu = -iD_\mu\Phi, \tag{3.114}$$

$$\hat{\bar D}^{\dot\mu}\Psi^\nu = 0 \to \Lambda^{\nu\dot\mu} = -2\bar D^{\dot\mu}\Psi^\nu\,. \tag{3.115}$$

Lastly, we need to impose the appropriate irreducibility conditions. From the solutions in (3.114), we find $\bar D_{\dot\mu}D_\mu\Phi = -2i\partial_{\mu\dot\mu}\Phi$. Using the algebra of ordinary $\mathcal{N}=1$ covariant derivatives $\{D_\mu, \bar D_{\dot\mu}\} = -2i\partial_{\mu\dot\mu}$, we find that we must have $\bar D_{\dot\mu}\Phi = 0$. This is a covariant condition since the ordinary barred spinor derivative $\bar D_{\dot\mu}\Phi$ coincides with the coset covariant version $\hat{\bar D}_{\dot\mu}\Phi$. This is precisely the canonical irreducibility condition for the chiral superfield. In this case, we can interpret it as a consistency condition following from imposing the superspace inverse Higgs constraints. For more complicated algebras, the irreducibility conditions are modified from their canonical expressions. We want to emphasize here that the choice of algebra has led to a natural essential Goldstone superfield. In Chapter 5, we will investigate systematically which algebras correspond to which irreducible supermultiplet of $\mathcal{N}=1$ supersymmetry.

Inönu-Wigner contractions and small-field limits

The algebra $\mathfrak{sgal}(4,1\,|\,6,(1,0))$ is related to the ordinary $(1,0)$ super-Poincaré algebra in $D=6$ by an Inönu-Wigner contraction. The relevant rescaling of generators is essentially the same as in 3.2.3, combined with a simultaneous rescaling of the odd (fermionic) components of the superalgebra. Let us begin by explaining what rescaling the fermionic generators would imply.

The $\mathcal{N}=1$ supersymmetry algebra is defined by:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2P_{\alpha\dot{\beta}}\,. \tag{3.116}$$

We now introduce the following contraction:

$$Q_\alpha \to \sigma S_\alpha, \quad \text{where } \sigma \to \infty\,. \tag{3.117}$$

The anti-commutator is now simply $\{S_\alpha, \bar{S}_{\dot{\beta}}\} = 0$.

Equivalently, we can take the non-linear realization of $\mathcal{N}=1$ supersymmetry, the Akulov-Volkov model, and take the small-field limit. This means that we rescale the Goldstino λ^α by the inverse factor:

$$\lambda^\alpha \to \lambda^\alpha/\sigma, \quad \text{where } \sigma \to \infty\,. \tag{3.118}$$

The Akulov-Volkov transformation law then becomes an ordinary shift symmetry:

$$\delta\lambda^\alpha = \epsilon^\alpha\,. \tag{3.119}$$

The implications of such a symmetry have been studied in, for instance, [111–113]. Similar to Galileons, a fermionic shift symmetry does not admit a unitary UV completion. It could, however, describe an alternative scenario of fermion compositeness. In [160], it was shown that the fermionic shift symmetry does not admit Wess-Zumino terms in $D=4$ beyond the ordinary Weyl kinetic term. This has important implications for the existence of a massive gravitino [114] in four dimensions.

Next, we will carry out the contraction of the full (centrally extended) $D=6$, $\mathcal{N}=(1,0)$ super-Poincaré algebra to $\mathfrak{sgal}(4,1\,|\,6,(1,0))$. The $(1,0)$ supersymmetry algebra has eight supercharges. These come in the form of a pair of $SU(2)$ Majorana-Weyl spinors Q_i $(i=1,2)$. Such a pair constitutes a minimal spinor of $D=6$. The spinors Q_i satisfy the condition:

$$Q_i = \epsilon_{ij}\,(Q_j)^C = \epsilon_{ij}\,B^{-1}(Q_j)^*\,, \tag{3.120}$$

where the superscript C denotes charge conjugation. The matrix B is related to the charge conjugation matrix C: $B = it_0 C\gamma^0$ (see e.g. [123]). The supersymmetry algebra is then defined by the following anti-commutation relation:

$$\{Q_{i\underline{\alpha}}, Q_{j\underline{\beta}}\} = -\tfrac{1}{2}(\gamma^A)_{\underline{\alpha}\underline{\beta}}\epsilon_{ij}P_A\,, \tag{3.121}$$

in addition to the standard $D = 6$ Poincaré algebra. Here, $\underline{\alpha}$ denotes a $D = 6$ spinor index, which has 4 components, and A denotes a $D = 6$ space-time index.

We can decompose the symplectic pair $Q_{i\underline{\alpha}}$ into two $D = 4$ Weyl spinors:

$$Q_{1\underline{\alpha}} = (Q_\alpha, -\overline{S}^{\dot{\alpha}})^T\,, \quad Q_{2\underline{\alpha}} = (S_\alpha, \overline{Q}^{\dot{\alpha}})\,. \tag{3.122}$$

Additionally, we decompose the $D = 6$ Poincaré subalgebra in terms of $D = 4$ representations. In two-component notation, we find:

$$[P_{\alpha\dot{\alpha}}, G_{\beta\dot{\beta}}] = i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}G, \quad [G_{\alpha\dot{\alpha}}, \bar{G}_{\beta\dot{\beta}}] = -i(\epsilon_{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}) + 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}M,$$
$$[\bar{G}, G_{\alpha\dot{\alpha}}] = 2iP_{\alpha\dot{\alpha}}, \quad [G, M] = G, \quad [G_{\alpha\dot{\alpha}}, M] = G_{\alpha\dot{\alpha}}, \tag{3.123}$$

The generator M is a generator of $U(1) \simeq SO(2)$, which rotates the codimensional directions into each other. The non-vanishing (anti)-commutation relations involving the fermions Q_α and $\bar{S}_{\dot{\alpha}}$ are:

$$\{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta}G, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \tag{3.124}$$
$$[Q_\alpha, \bar{G}_{\beta\dot{\beta}}] = i\epsilon_{\alpha\beta}\bar{S}_{\dot{\beta}}, \quad [S_\alpha, \bar{G}_{\beta\dot{\beta}}] = -i\epsilon_{\alpha\beta}\bar{Q}_{\dot{\beta}}. \tag{3.125}$$

This is the $\mathcal{N} = 2$-extended supersymmetry algebra in $D = 4$, together with the automorphism algebra $\mathfrak{so}(1, 5)$. Additionally, the algebra admits a $U(2)$ R-symmetry.

Let us introduce the rescaling parameter σ. Importantly, we cannot rescale the bosonic generators according to 3.2.3 without a simultaneous rescaling of the fermions. This would lead to a singular anti-commutation relation $\{Q, \bar{Q}\}$. Therefore, there is no $\mathcal{N} = 2$ extension of the $\mathfrak{gal}(4, 2)$ algebra. Instead, we need to rescale one of the fermions along with the bosons. Let us pick S_α, without loss of generality:

$$Q_\alpha \to Q_\alpha\,, \quad S_\alpha \to \sigma S_\alpha\,, \quad \text{where } \sigma \to \infty\,. \tag{3.126}$$

Simultaneously, we rescale the bosonic generators G and $G_{\alpha\dot{\alpha}}$. Finally, let us see what happens to the R-symmetry generators. We need to rescale the off-diagonal generators of $SU(2) \subset U(2)$, which we will denote by the complex scalar generator R.

$$R \to \sigma R\,, \qquad \bar{R} \to \sigma\bar{R}\,, \quad \text{where } \sigma \to \infty\,. \tag{3.127}$$

The non-vanishing (anti)-commutation relations are now:

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2P_{\alpha\dot\beta}\,, \quad [P_{\alpha\dot\alpha}, G_{\beta\dot\beta}] = i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}G\,,$$

$$\{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta}G\,, \quad [R, Q_\alpha] = S_\alpha\,, \tag{3.128}$$

$$[G, M] = G, \quad [G_{\alpha\dot\alpha}, M] = G_{\alpha\dot\alpha} \tag{3.129}$$

in addition, of course, to those that define the Poincaré group and the Lorentz representations of the generators. All the equations in (3.128) - apart from those that involve the $U(1) \simeq SO(2)$ generator M - are related to imposing superspace inverse Higgs relations. This is the algebra $\mathfrak{sgal}(4,1\,|\,6,(1,0))$, as desired.

Let us comment on the difference between (3.128) and (3.104). In (3.128), we have explicitly included the $U(1)$ generator M. We did not need to include this before, because this generator is linearly realized. It is the linear $SO(2)$ symmetry we included as part of the definition of 2-field Galileons in section 3.2.3. The generator R, conversely, must be either explicitly broken or non-linearly realized, because of the bracket $[R, Q_\alpha] = S_\alpha$. When non-linearly realized, it corresponds to a shift of the auxiliary field F in the chiral superfield. It will, therefore, be broken explicitly upon substituting the field equations for F. We will have more to say about the role of such symmetries in Chapter 5.

An interesting interaction for the coset (3.105) is the following: [166]

$$\mathcal{L}_4 = \int d^4\theta\, \Phi(\overline{D}_{\dot\alpha}\partial_\mu\overline{\Phi}\bar\sigma_\nu^{\dot\alpha\alpha}D_\alpha\partial_\rho\Phi)\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\overline{\Phi}\,. \tag{3.130}$$

where Φ is a chiral superfield. This is the supersymmetric version of the quartic bi-Galileon. In terms of its component fields ϕ and χ, it has the following symmetries:

$$\phi \to \phi + c + c_\mu x^\mu\,, \quad \chi_\alpha \to \chi_\alpha + \epsilon_\alpha\,, \tag{3.131}$$

which is a standard bi-Galileon symmetry combined with a shift symmetry of the fermion. The generator R is broken explicitly by the operator (3.130). Interestingly, (3.130) was derived in [166] by first constructing a new minimal supergravity version of the so-called *slotheon* [170], which is related to the Galileon by a decoupling procedure. We are not aware of a reference where all supersymmetric Galileon interactions are derived using a systematic Wess-Zumino formalism.

The contaction from super-Poincaré to supersymmetric Galileon algebras can of course be generalized to different dimensions and extended supersymmetry. One must be careful, however, to rescale the codimensional Lorentz boosts in the appropriate way. The reference [160] contains the contraction of minimal (Type I) $D = 10$ supersymmetry to $\mathfrak{sgal}(4,1\,|\,10,1)$.

3.4 Superstring and brane actions

Having studied the general theory of non-linear realizations in detail, we now want to give special attention to the actions of p-branes embedded in some ambient geometry. Such actions can have special linear and non-linear symmetries, inherited from the isometries of the embedding space. This makes it possible to construct and study p-brane actions using coset models. We will begin this section by explaining how p-brane actions inherit the symmetries of their embedding space. Some of these symmetries become non-linearly realized, depending on the formalism one uses to describe the brane action. Our explanation will focus on bosonic branes embedded in ordinary background geometries. However, the discussion should carry over to branes embedded in supermanifolds.

We will examine some important examples of p-brane actions. Firstly, we treat the Green-Schwarz superstring embedded in flat and then curved anti-de Sitter superspace. These are important coset models whose classical equations of motion are integrable. In the next section, we will see how the use of the coset construction for superstring actions allows one to find infinite sets of conserved charges relevant to integrability. Finally, we examine the $\mathcal{N} = 1$ preserving $D = 4$ brane actions discovered by Bagger & Galperin. [155, 156, 173] These actions will play a prominent role in our classification of $\mathcal{N} = 1$ preserving EFTs.

Consider a p-brane embedded in a d-dimensional space-time.[5] The brane is described by the $p + 1$ worldvolume coordinates x^α. Its embedding in space-time is given by the mappings $X^\mu(x)$ from the worldvolume to the space-time coordinates. The $X^\mu(x)$ are scalars on the worldvolume. The ambient space has a metric $G_{\mu\nu}(X)$ whose pull-back to the worldvolume is the brane induced metric $g_{\alpha\beta}$:

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{\partial x^\beta} G_{\mu\nu}(X(x)) \,. \tag{3.132}$$

Let us indicate the brane tangent vectors by $T^\mu{}_\alpha = \frac{\partial X^\mu}{\partial x^\alpha}$. The normal vector n^μ is defined by the conditions:

$$T^\mu{}_\alpha n^\mu G_{\mu\nu}(X) = 0, \quad n^\mu n^\nu G_{\mu\nu}(X) = 1 \,. \tag{3.133}$$

The brane action should have invariance under reparametrizations $x^\alpha \to x'^\alpha(x)$ (i.e. worldvolume diffeomorphisms). The infinitesimal action of this symmetry on the worldvolume scalar embedding functions is:

$$\delta X^\mu = \xi^\alpha(x)\partial_\alpha X^\mu(x) \,, \tag{3.134}$$

[5] We follow the discussion of [95].

where ξ^α is the parameter of the diffeomorphism. The most general action with these symmetries is:

$$S = \int d^d x \sqrt{-g}\, F(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, K_{\alpha\beta}, \nabla_\mu, \ldots)\,, \tag{3.135}$$

where F is some local function of the induced metric, its intrinsic curvatures and their covariant derivatives, and the *extrinsic curvature* $K_{\mu\nu}$ of the embedding. The leading order action consists of only the determinant of the induced metric, i.e. $F = 1$. We will refer to such actions as *Nambu-Goto* actions in what follows.

Now, if K^μ is a Killing vector of the ambient space metric, S will be invariant under the transformation:

$$\delta X^\mu = K^\mu(X)\,. \tag{3.136}$$

Let us now separate the embedding space coordinates X^μ into the groups X^α and X^i, where $\alpha = 0, 1, \ldots, p$ and $i = p+1, \ldots d$. Now we choose coordinates such that the brane is the hypersurface defined by $X^i =$ constant. Now we isolate the subalgebra of Killing vectors $K_I^\mu(X)$ that satisfy:

$$K_I^i(X) = 0\,. \tag{3.137}$$

The $K_I^\mu(X)$ preserve the chosen foliation of the ambient space. We denote the remaining Killing vectors $K_A^\mu(X)$. Now we make use of reparametrization invariance to choose the *static gauge*, where:

$$X^\alpha(x) = x^\alpha\,. \tag{3.138}$$

Furthermore, we relabel $X^i(x) = \pi^i(x)$. The $\pi^i(x)$ are the only remaining physical fields. The symmetry transformations (3.134) do not in general preserve the static gauge, so it is necessary to accompany them with a compensating gauge transformation. Therefore, in the static gauge the general symmetry transformation becomes:

$$\delta\pi^i(x) = a^A K_A^i(x,\pi) - a^I K_I^\alpha \partial_\alpha \pi^i(x,\pi) - a^A K_A^\alpha(x,\pi)\partial_\alpha \pi^i(x)\,. \tag{3.139}$$

We see that at most the subgroup generated by K_I^μ remains linearly realized in the static gauge. Before fixing the gauge, a larger subgroup of the ambient space isometries can be linearly realized. In both cases, the brane action can be considered a coset model. There is no mismatch between physical Goldstone modes in these two frameworks. On the gauge invariant side, a subset of the would-be Goldstone degrees of freedom is pure gauge. On the gauge-fixed side, the generators associated to that same subset are identified with the linearly realized translation generators. Furthermore, the Goldstone modes of the isotropy subgroup (which becomes non-linearly realized in the gauge-fixed description) are degenerate and can be eliminated by inverse Higgs constraints. We will encounter both gauge invariant and gauge-fixed descriptions in what follows.

3.4.1 Green-Schwarz superstrings

We now turn our attention to Type II superstrings, which are 2-dimensional extended objects living in $D = 10$ space-time. There are several inequivalent formulation of superstrings, each highlighting a different aspect of the theory. In the Green-Schwarz (GS) formulation, the symmetries of the target space are made manifest. [149, 150] We will first present the Green-Schwarz string action in flat space and then explain its interpretation as a coset model. [148] In the next subsection, we will examine the GS string in an $\mathrm{AdS}_5 \times \mathrm{S}_5$ background. We include the discussion of Green-Schwarz superstrings to provide an example of gauge invariant coset models (most of our attention in the following will be devoted to gauge-fixed models of extended objects) and to highlight a particularly beautiful application of the theory of non-linear realizations. As such, the sections 3.4.1 and 3.4.2 are not crucial to our main line of argumentation.

The GS superstring action in $D = 10$ flat space is a straightforward generalization of the Nambu-Goto action encountered in the previous subsection. It is built out of the worldsheet scalar fields $X^\mu(x)$ and the Grassmann-odd fields Θ^I, with $I = 1, 2$. Together, X^μ and Θ^I form the coordinates for the $D = 10$ superspace in which the string is embedded. The fermionic coordinates Θ^I are $D = 10$ Majorana-Weyl spinors [6]. The 10-dimensional chirality of Θ^I determines whether we are dealing with Type IIA or Type IIB superstrings. In Type IIA, the chiralities are opposite and in Type IIB they are the same. Consider the following linear combination of the superspace coordinates:

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^I \Gamma^\mu \partial_\alpha \Theta^I \,. \tag{3.140}$$

The Γ^μ are $D = 10$ gamma matrices. This quantity is invariant under $D = 10$ supersymmetry transformations, which act on X^μ and Θ^I as:

$$\delta\Theta^I = \epsilon^I, \quad \delta X^\mu = \bar{\epsilon}^I \Gamma^\mu \Theta^I \,. \tag{3.141}$$

[6] See [123] for a comprehensive treatment of gamma matrices and spinors in arbitrary dimensions.

This transformation law acts exactly as one would expect for coordinates of superspace. The transformation parameters ϵ^I are Majorana-Weyl spinors of the same chirality as Θ^I.

The GS superstring action now reads as follows:

$$S = -\frac{1}{\pi}\int d^2\sigma\,\sqrt{\det(\Pi_\alpha\cdot\Pi_\beta)} + \int \Omega_{WZ}\,,$$
$$\Omega_{WZ} = \frac{1}{\pi}\big((\Theta^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2)dX^\mu - \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2\big)\,. \tag{3.142}$$

The first term in the GS action is the supersymmetric generalization of the Nambu-Goto term. The second is a Wess-Zumino term of the coset model, as we will soon make clear. This is why it is written as the integral over a 2-form. It corresponds to the following invariant 3-form:

$$\Omega_3 = \frac{1}{\pi}(d\bar{\Theta}^1\Gamma_\mu d\Theta^1 - d\bar{\Theta}^2\Gamma_\mu d\Theta^2)(dX^\mu - \bar{\Theta}^I\Gamma^\mu d\Theta^I)\,. \tag{3.143}$$

Altogether, the GS action has the following symmetries:

- **World sheet diffeomorphisms**: Reparametrizations $\sigma^\alpha \to \sigma'^\alpha(\sigma)$ under which the fields transform as $X'^\mu(\sigma') = X^\mu(\sigma)$, $\Theta'^I(\sigma') = \Theta^I(\sigma)$.

- **Kappa symmetry**: A fermionic gauge symmetry under which:

$$\delta\Theta^1 = \bar{\kappa}^1 P_-,\quad \delta\Theta^2 = \bar{\kappa}^2 P_+\,,$$
$$\delta X^\mu = \bar{\Theta}^I\Gamma^\mu\delta\Theta^I\,,$$
$$P_\pm = \frac{1}{2}\left(1 \mp \frac{\epsilon^{\alpha\beta}\Pi^\mu_\alpha\Pi^\nu_\beta\Gamma_{\mu\nu}}{2\sqrt{-\det(\Pi_\alpha\cdot\Pi_\beta)}}\right)\,. \tag{3.144}$$

  The κ^I are independent $D = 10$ Majorana spinor gauge parameters of the appropriate chirality.

- **Global space-time supersymmetry**: The $D = 10$ supersymmetry acts on the fields X^μ and Θ^I as on coordinates of superspace:

$$\delta\Theta^I = \epsilon^I,\quad \delta X^\mu = \bar{\epsilon}^I\Gamma^\mu\Theta^I\,. \tag{3.145}$$

  Note that there is no manifest *worldsheet* supersymmetry. In the RNS formalism for Type II superstrings [132], however, a worldsheet supersymmetry is manifest. Unfortunately, in this formalism the ambient space symmetries are obscured.

- **Space-time Poincaré symmetry**: The 10-dimensional translations shift the coordinates $\delta X^\mu = a^\mu$ and Lorentz transformations act in the obvious way.

The local Kappa symmetry ensures that the fermionic and bosonic degrees of freedom match both before and after fixing the gauge symmetries. The presence of all these linear and non-linear symmetries suggests that the GS superstring is nothing but the coset model for the symmetry breaking pattern of the Type II super-Poincaré group in ten dimensions, modulo the ten-dimensional Lorentz group $SO(1,9)$. The model is then further constrained by the requirements of reparametrization invariance and Kappa symmetry. Note that the first term in (3.142) is invariant under the global symmetries and reparametrization invariance. The second term is a reparametrization invariant Wess-Zumino term for the coset space. Only this specific linear combination respects the Kappa symmetry.

To make the coset interpretation explicit, it is convenient to work in the Polyakov formalism. This will simplify the equations by getting rid of the cumbersome square root and determinant in (3.142). The Polyakov formalism requires an auxiliary worldsheet metric $h_{\alpha\beta}$, whose inverse is $h^{\alpha\beta}$. Now introduce some parametrization $g(X,\Theta)$ for the coset space. This gives rise to the Maurer-Cartan currents ω_α:

$$g^{-1}\partial_\alpha g = \omega_\alpha = (\omega_P^\mu)_\alpha P_\mu + (\omega_Q^I)_\alpha Q^I + (\omega_M^{\mu\nu})_\alpha M_{\mu\nu}\,. \tag{3.146}$$

In the usual exponential parametrization $g = e^{i(x^\alpha P_\alpha + \Theta^I Q_I)}$, the Maurer-Cartan component of translations is simply the supersymmetric invariant combination we found earlier, $(\omega_P^\alpha)_\mu = \Pi_\mu^\alpha$. Then GS action in the Polyakov formalism then reads:

$$\begin{aligned} S &= -\frac{1}{\pi}\int d^2\sigma\,\sqrt{h}h^{\alpha\beta}\,\mathrm{Str}(\omega_\alpha\omega_\beta)|_P + \int \Omega_{WZ}\,, \\ &= -\frac{1}{\pi}\int d^2\sigma\,\sqrt{h}h^{\alpha\beta}\,(\omega_P^\mu)_\alpha(\omega_P^\nu)_\beta\,\mathrm{Str}(P_\mu P_\nu) + \int \Omega_{WZ}\,. \end{aligned} \tag{3.147}$$

Here, Str(...) refers to the *supertrace* over the generators in a matrix representation of the symmetry algebra. In additional to the symmetries we enumerated earlier, the Polyakov GS action has a local Weyl symmetry $h_{\mu\nu} \to f(x)h_{\mu\nu}$. To recover the Nambu-Goto type action (3.142), calculate the equations of motion for the auxiliary metric and plug them back into the action. Note that the equations of motion only determine the metric up to a local rescaling, due to the Weyl symmetry. However, the rescaling drops out of (3.147).

3.4.2 Green-Schwarz-Metsaev-Tseytlin superstrings

In the previous subsection, we examined the Type II superstring in 10-dimensional flat space in the covariant GS formalism. The flat $D = 10$ target space represents the trivial $\mathbb{R}^{(1,9)}$ solution of the Type IIB supergravity equations of motion, which preserves all of the 32 supercharges. Another solution which preserves the maximal supersymmetry is the well-known $\text{AdS}_5 \times \text{S}_5$ background discovered in [151]. This solution describes the near-horizon geometry of D3-brane solutions in Type II supergravity, which plays a crucial role in the AdS / CFT correspondence. [75] The superstring on $\text{AdS}_5 \times \text{S}_5$ is sometimes called the Green-Schwarz-Metsaev-Tseytlin (GSMT) string, after Green-Schwarz and the authors of [153].

The $\text{AdS}_5 \times \text{S}_5$ space is the direct product of the coset spaces:

$$\text{AdS}_5 = \frac{SO(2,4)}{SO(1,4)}, \quad S^5 = \frac{SO(6)}{SO(5)} . \tag{3.148}$$

The isometry group generated by the Killing vectors on $\text{AdS}_5 \times \text{S}_5$ is then $SO(2,4) \times SO(6)$. Including the Killing *spinors* of the maximally supersymmetric Type II supergravity solution, the superisometry group is enhanced to $\text{PSU}(2,2|4)$. This is the $\mathcal{N} = 4$ superconformal group in $D = 4$, including its $SU(4)_R$ automorphism group. Let us highlight some important properties of its superalgebra, $\mathfrak{psu}(2,2|4)$, before we present the GSMT coset model.

The $\text{PSU}(2,2|4)$ superalgebra $\mathfrak{psu}(2,2|4)$ contains 32 supercharges, equal to the number of supersymmetries in Type II supergravity. The superalgebra is obtained from $\mathfrak{su}(2,2|4)$ as a quotient over its $\mathfrak{u}(1)$ center. The superalgebra $\mathfrak{su}(2,2|4)$, in turn, is spanned by supertraceless 8×8 supermatrices M that satisfy the reality condition:

$$M^\dagger H + HM = 0 , \tag{3.149}$$

where H is:

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} , \tag{3.150}$$

and:

$$\Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} . \tag{3.151}$$

The generators of $\mathfrak{psu}(2,2|4)$ may be separated into four sectors,

$$\mathfrak{psu}(2,2|4) = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 , \tag{3.152}$$

according to their charge under a $\mathbb{Z}_4$ automorphism[7]. This automorphism plays an important role in establishing the integrability of the GSMT superstring, as we will see. On a supermatrix representation M of $\mathfrak{su}(2,2|4)$, the automorphism acts as:

$$M \to \Omega(M) = -\mathcal{K} M^{st} \mathcal{K}{-1}\,, \tag{3.153}$$

where M^{st} refers to the *supertranspose* of the matrix M:

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \to M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix} \tag{3.154}$$

where m^t is the usual transpose of m. Furthermore, $\mathcal{K} = \mathrm{diag}(K, K)$, where K is defined by:

$$K = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} . \tag{3.155}$$

The elements M^k of $\mathfrak{g}_k$ in (3.152) then satisfy:

$$\Omega(M^k) = i^k M^k\,. \tag{3.156}$$

The elements of $\mathfrak{g}_0$ span the $\mathfrak{so}(1,4)\times\mathfrak{so}(5)$ bosonic subalgebra, which makes up the linearly realized symmetry group. The components $\mathfrak{g}_1$ and $\mathfrak{g}_3$ comprise the fermionic generators. Then, $\mathfrak{g}_2$ makes up the spontaneously broken bosonic sector.

With these generalities out of the way, we can present the GSMT Lagrangian : [154]

$$\mathcal{L} = -\frac{g}{2}\left[\gamma^{\alpha\beta}\mathrm{Str}(A^{(2)}_\alpha A^{(2)}_\beta) + \kappa\epsilon^{\alpha\beta}\mathrm{Str}(A^{(1)}_\alpha A^{(3)}_\beta)\right] . \tag{3.157}$$

where $A_\alpha = g^{-1}\partial_\alpha g$ is the Maurer-Cartan current and the bracketed superscripts refer to its components under the $\mathbb{Z}_4$ automorphism. Furthermore, we have combined the metric and its determinant into $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$.

One can show that for $\kappa = \pm 1$, (3.157) has a fermionic local Kappa symmetry which is realized as a right-multiplication:

$$g \cdot G = g' h\,, \tag{3.158}$$

where $G = \exp(\epsilon(\tau,\sigma))$ and $\epsilon(\tau,\sigma)$ is a local fermionic parameter which satisfies a constraint. In addition, the equations of motion derived from (3.157) admit a Lax representation if and only if $\kappa = \pm 1$.

[7] In fact, this automorphism exists for the entire superalgebra of supertraceless 8×8 supermatrices, $\mathfrak{sl}(4|4)$.

The second term in (3.157) is once again a Wess-Zumino term. It is equal to the integration of the following closed, invariant 3-form:

$$\Theta_3 = \mathrm{Str}(A^{(2)} \wedge A^{(3)} \wedge A^{(3)} - A^{(2)} \wedge A^{(1)} \wedge A^{(1)}) = \tfrac{1}{2}\mathrm{Str}(A^{(1)} \wedge A^{(3)}) \quad (3.159)$$

over a $D = 3$ space whose boundary is the string worldsheet.

Note that in order to do physics with (3.157), one has to specify a parametrization of the coset element g and calculate the Maurer-Cartan current, which in general is a cumbersome procedure. The Green-Schwarz type action for the superstring is known in a general background [140], including ones that have no interpretation as a coset model. However, the coset construction provides a clean formal expression in (3.157) from which one can deduce important properties of the theory. One of the advantages of the coset description is that it allows one to easily extract the conserved quantities of the action. These include an infinite set of commuting conserved currents, which means that the equations of motion that result from (3.157) are integrable.

3.4.3 Partial breaking of global supersymmetry in four dimensions

In this section, we will take a look at two quintessential examples of supersymmetric non-linear realizations, both related to the partial breaking of $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ in four dimensions. Spontaneously breaking a supersymmetry leads to at least a single Goldstino, which must live in a full supermultiplet in order to preserve $\mathcal{N} = 1$ supersymmetry. There are then several choices to complete the multiplet. For instance, we can include a complex scalar field to obtain a chiral superfield. Adding a vector field gives rise to a vector or Maxwell supermultiplet. This choice comes down to whether we choose to centrally extend the $\mathcal{N} = 2$ supersymmetry algebra, as we will soon discover. The main references for this section are [155, 156].

We first address an important subtlety of partially breaking supersymmetry. [157] The minimal $\mathcal{N} = 2$ supersymmetry algebra consists of two $D = 4$ Weyl spinors (S_α, Q_α) with the following anti-commutation relations:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}\,. \quad (3.160)$$

If we then assume that Q annihilates the vacuum state $|0\rangle$ (i.e. Q generates an unbroken symmetry), we must also conclude that the Hamiltonian $H \propto P^0$ annihilates the (spatial translation-invariant) vacuum, due to the anti-commutator $\{Q, \bar{Q}\}$:

$$H\,|0\rangle = 0\,. \quad (3.161)$$

Naively, this immediately implies that also $\{S, \bar{S}\} |0\rangle = 0$. Then, if the Hilbert space is positive definite, we must conclude:

$$S |0\rangle = \bar{S} |0\rangle = 0 \,, \tag{3.162}$$

which seems to suggest the second supersymmetry must be unbroken if the first supersymmetry is unbroken. However, as we have emphasized throughout this thesis, spontaneously broken symmetries do not always lead to well-defined quantum charges. We can only count on the existence of a *current algebra*, which will receive modifications from the breaking of supersymmetry such that the naive argument is circumvented. It is also possible that the Hilbert space is not positive definite. This happens, for instance in covariantly quantized supergravity [157] due to negative-norm components of the gravitino.

Even though the naive argument forbidding partial breaking may be circumvented, it does suggest that such theories are rather special. Indeed, the realizations of partial breaking we discuss below do not come from ordinary $\mathcal{N} = 2$ field theories living in four spatial dimensions. As we have explained above, they both describe the longitudinal modes of membranes. In the case of the chiral superfield, the theory describes the massless modes of a 3-brane solution in a minimally supersymmetric $D = 6$ gauge theory. [179] The theory is effectively four-dimensional because the massless modes are confined to the surface of the 3-brane. Likewise, the vector superfield describes a D3-brane solution of superstring theory. The full target space-time in this case is therefore ten-dimensional, but we can again consistently truncate to the massless modes which live on the surface of the D3-brane.

Partial breaking using a Maxwell superfield

Let us first examine the minimal case, where we do not centrally extend the $\mathcal{N} = 2$ algebra or include any of its R-symmetry group. In addition to the $D = 4$ Poincaré algebra, we have:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}} P_a, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}} P_a \,, \tag{3.163}$$

$$\{Q_\alpha, S_\beta\} = 0, \quad \{Q_\alpha, \bar{S}_{\dot{\beta}}\} = 0 \,. \tag{3.164}$$

Here, we will employ Latin letters for bosonic indices and Greek letters for spinorial indices. As usual, we will use letters from the beginning of the alphabet to denote tangent space indices and reserve letters from the middle of the alphabet for space-time. We intend to realize S_α non-linearly and keep Q_α unbroken. Therefore, we define the following coset element:

$$g = e^{i(x^a P_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} e^{i(\Psi^\alpha S_\alpha + \bar{\Psi}_{\dot{\alpha}} \bar{S}^{\dot{\alpha}})} \,. \tag{3.165}$$

The covariant derivatives $\hat{D}_\alpha$, $\hat{\bar{D}}_{\dot\alpha}$ and $\hat{D}_a$ are extracted from the Maurer-Cartan form $\gamma = g^{-1}dg$ according to the procedure discussed in 3.3. Among the terms proportional to (super)translation generators, only the space-time translation term is non-trivial:

$$(\omega_P^a) = dx^a + i(d\theta\sigma^a\bar\theta + \theta\sigma^a d\bar\theta + d\Psi\sigma\bar\Psi + \Psi\sigma d\bar\Psi)\,. \tag{3.166}$$

This defines the supervielbein component ${E_m}^a$, according to (3.90). We find:

$${E_m}^a = {\delta_m}^a + i(\partial_m\psi\sigma^a\bar\psi + \psi\sigma^a\partial_m\bar\psi)\,. \tag{3.167}$$

We can now extract covariant derivatives using (3.92). The result is: [156]

$$\begin{aligned}
\hat{D}_a\Psi_\beta &= {(E^{-1})_a}^m\partial_m\Psi_\beta\,,\\
\hat{D}_\alpha\Psi_\beta &= D_\alpha\Psi_\beta - i(D_\alpha\Psi\sigma^a\bar\Psi + D_\alpha\bar\Psi\bar\sigma^a\Psi){(E^{-1})_a}^m\partial_m\Psi_\beta\,,\\
\hat{\bar{D}}_{\dot\alpha}\Psi_\beta &= \bar{D}_{\dot\alpha} - i(\bar{D}_{\dot\alpha}\Psi\sigma^a\bar\Psi + \bar{D}_{\dot\alpha}\bar\Psi\bar\sigma^a\Psi){(E^{-1})_a}^m\partial_m\Psi_\beta\,,
\end{aligned} \tag{3.168}$$

where E^{-1} is the inverse of the supervielbein. These covariant derivatives realize a modified algebra:

$$\begin{aligned}
\{\hat{D}_\alpha, \hat{D}_\beta\} &= \mathcal{O}((\hat{D}\Psi)^3)\,,\\
\{\hat{D}_\alpha, \hat{\bar{D}}_{\dot\beta}\} &= 2i(\sigma^a)_{\alpha\dot\beta} + \mathcal{O}((\hat{D}\Psi)^3)\,,\\
[\hat{D}_\alpha, \hat{D}_a] &= \mathcal{O}((\hat{D}\Psi)^3)\,.
\end{aligned} \tag{3.169}$$

We must now make use of the covariant derivatives (3.168) to impose irreducibility conditions. We intend to identify Ψ_α, on the solution of the appropriate constraint equation, with a canonical Maxwell superfield W_α. [8] Such a superfield satsifies the constraint equations:

$$\bar{D}_{\dot\alpha}W_\alpha = 0, \quad D^\alpha W_\alpha + \bar{D}_{\dot\alpha}W^{\dot\alpha} = 0\,. \tag{3.170}$$

The first of these equations, the chirality conditions, is easy to make covariant with respect to the non-linear transformations. We simply replace the ordinary covariant derivative with the coset covariant derivative:

$$\hat{\bar{D}}_{\dot\alpha}\Psi_\alpha = 0\,. \tag{3.171}$$

[8] The Maxwell superfield is not the only irreducible spinor superfield. We are however led to this particular multiplet by choosing the minimal version of the $\mathcal{N}=2$ superalgebra. In Chapter 5, we will discuss in great detail how the algebra picks out a particular irreducible supermultiplet.

This constraint is compatible with the modified algebra of covariant derivatives (3.169). This constraint can made covariant in the minimal way because it is fixed by Lorentz invariance, as we will explain in Chapter 5.

The second constraint is rather more complicated. As Bagger & Galperin explain, the naive generalization $\hat{D}^\alpha\Psi_\alpha + c.c.$ is not compatible with the algebra of covariant derivatives (3.169). The only solution is then $\Psi = 0$. Instead, it must be replaced by a different constraint, which to fifth order in the fields reads:

$$\hat{D}^\alpha\Psi_\alpha - \frac{1}{2}\hat{D}^\alpha\Psi^\beta\hat{D}_\beta\Psi_\alpha\hat{D}^\gamma\Psi_\gamma + c.c. = \mathcal{O}(\Psi^5)\,. \tag{3.172}$$

The solution to this constraint may be written in terms of a canonical Maxwell superfield W_α:

$$\Psi_\alpha = W_\alpha + \frac{1}{4}\bar{D}^2(\bar{W}^2)W_\alpha - iW^\beta\bar{W}^{\dot{\beta}}\partial_{\beta\dot{\beta}}W_\alpha + \mathcal{O}(W^5)\,. \tag{3.173}$$

We will have more to say about the peculiar form of this constraint in Chapter 5.

Now that we have defined our covariant irreducibility conditions, the next step is to construct an action. Formally, all invariant actions are given by a sum of Wess-Zumino interactions and strictly invariant Lagrangians. We construct the latter by building scalars out of the covariant derivatives of Ψ_α, multiplying by the Berezinian of the supervielbein and integrating over $\mathcal{N} = 1$ superspace. In practice, however, building actions in this way is very involved for supercosets. Instead, Bagger-Galperin proceed by writing the transformation laws derived from the coset construction in terms of W_α, by way of equation (3.173). We can proceed by writing down the lowest-order Lorentz scalar W^2 and add higher-order interactions terms to find an interaction that is invariant up to some order in the fields. The result is, to order W^6:

$$\mathcal{L} = \frac{1}{4}\int d^4x\, d^2\theta W^2 + \frac{1}{4}\int d^4x\, d^4\theta\, W^2\bar{W}^2 + c.c + \mathcal{O}(W^6)\,. \tag{3.174}$$

If we truncate this action to only the gauge vector A_μ, we find that it coincides with the Born-Infeld action up to this order in the fields. [156] This confirms that the theory is related to the propagation of a D3-brane in superstring theory. Bagger-Galperin actually succeed in defining an invariant theory to all order in the fields, by means of a clever recursively-defined auxiliary multiplet.

We note that for our purposes in Chapter 5 - classifying the symmetry breaking patterns consistent with the Maxwell multiplet - it is more important to understand how to find covariant constraints than invariant actions.

In the next subsection, we will centrally extend the minimal $\mathcal{N} = 2$ superalgebra and attempt to break it to $\mathcal{N} = 1$. The additional symmetry generators force us to use a different irreducible multiplet of supersymmetry and to impose superspace inverse Higgs constraints.

Partial breaking using a chiral superfield

The $\mathcal{N} = 2$ algebra admits a central extension: we may add a complex scalar generator Z and fix the following anti-commutation relations:

$$\begin{aligned} &\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}} P_a, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}} P_a\,, \\ &\{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta} Z, \quad \{Q_\alpha, \bar{S}_{\dot{\beta}}\} = 0\,. \end{aligned} \tag{3.175}$$

The reader should compare this algebra to (3.104). The only difference resides in the $\{S, \bar{S}\}$ anti-commutation relation. This has no bearing on inverse Higgs relations (at least to linear order in the fields) so we will use the same structure of degeneracy conditions for this non-linear realization. Our essential Goldstone superfield is now a complex scalar Φ. The spinor Goldstino Ψ_α associated to S_α will be related to a derivative of Φ by superspace inverse Higgs conditions.

We may study (3.175) in its own right, deriving its covariant derivatives and transformation laws from the coset construction. However, it is more interesting to add a second level of degenerate Goldstone modes. We can do this by connecting the first level generator S_α to a vector K_a or a scalar R, by means of the following inverse Higgs commutation relations:

$$[K_a, \bar{Q}^{\dot{\alpha}}] = i(\bar{\sigma}_a)^{\dot{\alpha}\alpha} S_\alpha, \quad [R, \bar{Q}^{\dot{\alpha}}] = -\bar{S}^{\dot{\alpha}}, \quad [K_a, P_b] = i\eta_{ab} Z\,. \tag{3.176}$$

The first and second commutation relations allow us to impose the required inverse Higgs relations. The third bracket is enforced by the super-Jacobi identity involving the three generators $(Q_\alpha, \bar{Q}_{\dot{\alpha}}, K_a)$. This ensures that we may at the same time impose an ordinary space-time inverse Higgs relation to project out the lowest component field in the vector Goldstone superfield Λ_a associated to K_a. We will explore such structures in great detail in Chapter 5.

Because $\{S, \bar{S}\} \propto P$, we also need to include the following non-trivial brackets (as we will show in detail in Chapter 5):

$$[K_a, \bar{Z}] = 2iP_a \quad [K_a, \bar{S}^{\dot{\alpha}}] = -i(\bar{\sigma}_a)^{\dot{\alpha}\alpha} Q_\alpha \tag{3.177}$$

$$[T, S_\alpha] = Q_\alpha\,. \tag{3.178}$$

This algebra is equivalent to the minimal supersymmetry algebra in $D = 6$ enhanced with an $SU(2)$ R-symmetry.

Bagger-Galperin then proceed to define the coset element:

$$g = e^{i(x^a P_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} e^{i(\Phi Z + \bar{\Phi}\bar{Z} + \Psi^\alpha S_\alpha + \bar{\Psi}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}})} e^{i(\Lambda^a K_a + \bar{\Lambda}^a \bar{K}_a)} e^{i(TR + \bar{T}\bar{R})}\,, \quad (3.179)$$

and impose the following irreducibility and inverse Higgs conditions:

$$\hat{\bar{D}}_{\dot{\alpha}} \Phi = 0, \hat{D}_\alpha \Phi = 0, \hat{D}_a = 0, \quad (3.180)$$

$$\hat{D}_\alpha \Psi_\beta = 0, \hat{\bar{D}}_{\dot{\alpha}} \Psi_\beta = 0\,. \quad (3.181)$$

The first condition imposes chirality. The remaining spinorial derivative conditions impose the superspace inverse Higgs constraint, and the vector derivative condition takes care of space-time inverse Higgs. This means that there is only a single remaining independent superfield, which is Φ. Due to the covariant chirality condition, it is possible to write Φ (perturbatively) in terms of a canonical chiral superfield $\mathit{\Phi}$ [155]:

$$\Phi = \mathit{\Phi} - \frac{i}{4}(D\mathit{\Phi}\sigma^a \bar{D}\bar{\mathit{\Phi}})\partial_a \mathit{\Phi} - (\partial_a \mathit{\Phi})^2 \bar{\mathit{\Phi}} + \dots\,. \quad (3.182)$$

Using the solutions to the constraint equations and the transformation laws defined by the coset element (3.179), Bagger-Galperin then provide (up to sixth order in $\mathit{\Phi}$) the unique action that starts at quadratic order. It is more instructive, however, to report the action derived by Rocek and Tseytlin [163] using constrained superfield methods:

$$\mathcal{L} = \int d^4\theta\, \mathit{\Phi}\bar{\mathit{\Phi}} + \frac{\frac{1}{2}(D\mathit{\Phi})^2(\bar{D}\bar{\mathit{\Phi}})^2}{1 + A + \sqrt{(1+A)^2 - A\bar{A}}}\,, \quad (3.183)$$

where $A = \partial^{\alpha\dot{\alpha}}\mathit{\Phi}\partial_{\alpha\dot{\alpha}}\mathit{\Phi}$. One can check that the scalar component φ of $\mathit{\Phi}$ is subject to the ordinary DBI action in flat embedding space. See, for example, [164,165] where (3.183) was derived by explicitly supersymmetrizing the ordinary scalar DBI action. Thus, (3.183) provides the correct leading (in derivatives) action for the spontaneous breaking of $D = 6$ minimal supersymmetry, to all orders in the fields.

Chapter 4

Exceptional EFTs with Poincaré symmetry

4.1 Extended shift symmetries and IHCs

In Chapters 2 and 3, we discovered that spontaneously broken space-time symmetries may violate Goldstone's theorem. We arrived at this conclusion from several related directions. In section 3.2.1, we saw that Goldstone modes may be identified with a space-time dependent action of a broken global symmetry on the background field configuration. It can happen that the local action of one space-time symmetry transformation is degenerate with another, even if the two are independent as global transformations. In the theory of non-linear realizations, this degeneracy corresponds to inverse Higgs constraints, which consistently project out some Goldstone fields in terms of others. In section 2.5, we encountered theories with coordinate-dependent symmetries of the type (2.90), which may be realized at the same time as an ordinary shift symmetry without adding additional Goldstone fields. We then related the Noether current of the n-th order symmetry to that of the shift symmetry, implying that the two are not completely independent. Indeed, the off-shell relation (2.96) between the currents is a consequence of the local degeneracy of the corresponding symmetry transformations, as expressed by (2.94).

In this section, we will address the following question: given a set of independent fields $\phi(x)$, what is the most general set of symmetry generators that may be realized on $\phi(x)$ as non-linear transformations? We assume that our theory lives in $D = 4$ and has unbroken Poincaré symmetry, so that the fields $\phi(x)$ form a representation of the four-dimensional Lorentz algebra. Of course, we may ask the same question in different settings. Indeed, at the end of this chapter we will briefly examine non-linear realizations in anti-de Sitter space-time. In Chapter 5, we will address theories with unbroken $\mathcal{N} = 1$ supersymmetry. In answering this question, we will also constrain the commutation relations of the non-linearly realized generators with space-time translations. Fixing the spectrum of generators and their commutation relations with translations is the first step towards our larger aim: to classify the non-linear realizations.

Let us return to our discussion from section 3.2.1. There, we argued that a Lorentz irreducible Goldstone field may non-linearly realize several independent symmetries, if there are solutions to the equation

$$\phi^i(x) G_i |0\rangle = 0 \,. \tag{4.1}$$

Acting with the translation operator produces the consistency condition (3.33). Truncating to lowest order in the fields, the condition reads (3.35). A non-trivial solution therefore requires the algebraic relation ${f_{\mu j}}^i \neq 0$. Then, the linear combination of fields $\phi^j {f_{\mu j}}^i$ may be eliminated in favor of ϕ^i. This combination of Goldstone fields is associated to generators which contain the generator G_i in their commutation relation with translations.

We can of course repeat this trick and act with the translation operator P_μ on (3.32) a second time. Let us turn our attention to a particular Goldstone field ϕ^0, which we assume corresponds to a generator which satisfies $[P_\mu, G^{(0)}] = \ldots$, where the ellipses contain only linearly realized generators (i.e. in this case those belonging to Poincaré or some internal linear symmetry). It is never possible to project out such a Goldstone field with inverse Higgs relations. We will call such Goldstone fields *essential*, as opposed to *inessential* Goldstone fields which may be projected out. Acting with translations twice, we find:

$$\begin{aligned} \partial_\mu \phi^0(x) &= {f_{\mu i}}^0 \phi^i(x) + \mathcal{O}(\phi^2) \,, \\ \partial_\mu \partial_\nu \phi^0(x) &= {f_{\mu i}}^j {f_{\nu j}}^0 \phi^i(x) + \mathcal{O}(\phi^2) \,. \end{aligned} \tag{4.2}$$

Therefore, we may project out *two* independent linear combinations of Goldstone fields, if the algebra satisfies ${f_{\mu j}}^0 \neq 0$ and ${f_{\mu i}}^j {f_{\nu j}}^0 \neq 0$. The Goldstones eliminated by the first relation correspond to the linear combination of generators:

$$\sum_i G_i \, [P_\mu, G_i]|_{G^{(0)}} = G_i \, {f_{\mu i}}^0 \,, \tag{4.3}$$

which is just a sum over all generators which yield $G^{(0)}$ in their commutation relation with translations. We will refer to these as *first-order* generators. Similarly, the corresponding Goldstone fields are *first-order inessential*.

Inserting the first equation into the second, we also find:

$$\partial_\mu(f_{\nu i}{}^0\phi^i(x)) = f_{\mu i}{}^j f_{\nu j}{}^0\phi^i + \mathcal{O}(\phi^2)\,, \tag{4.4}$$

The second order generators are then:

$$\sum_i [P_\mu, G_i]|_{G_j}\, f_{\nu j}{}^0 = G_i\, f_{\nu i}{}^j f_{\mu j}{}^0\,, \tag{4.5}$$

which again is a sum over all generators G_i which satisfy $[P_\mu, G_i] \supset G^{(1)}$, where $G^{(1)}$ indicates the first-order generators.

Clearly, we may repeat this procedure an arbitrary number of times. At order n, we find that the n-th order generators are those which satisfy $[P_\mu, G^{(n)}] \supset G^{(n-1)}$. We may then eliminate the n-th order inessential Goldstone fields by a relation which reads, schematically:

$$\partial_\mu(f^{n-1}\phi(x)) = f^n\phi(x) + \mathcal{O}(\phi^2)\,. \tag{4.6}$$

The right- and left-hand sides of equations (4.2) contain, in general, several irreducible Lorentz representations. Let us assume that $G^{(0)}$ is an irreducible Lorentz representation of spin-s. The first-order generators then take spins $(s-1)$, s, or $(s+1)$. There can be two independent first-order spin-s generators, so that we have a degeneracy $(1,2,1)$. The second-order generators are spin $(s-2)$, $(s-1)$, s, $(s+1)$, or $(s+2)$. The degeneracy is then $(1,4,6,4,1)$. We are led to a tree of generators, depicted up to second order in figure 4.1. The dashed lines there indicate that the higher-order generator (lower on the figure) gives rise to the lower-order one when taking a commutation relation with translations. From now on, we will call a tree like 4.1 an *inverse Higgs tree*.

There are various consistency conditions which reduce the generator content depicted in 4.1. First of all, we observe that the left-hand side of the second equation (4.2) is symmetric with respect to exchanging $(\mu \leftrightarrow \nu)$. Therefore, we obtain a consistency condition $f_{[\mu i}{}^j f_{\nu]j}{}^0 = 0$. We obtain similar constraints by imposing more symmetrization conditions. It is clear that at order n in the inverse Higgs tree, the Lorentz representations that are compatible with the symmetry conditions are precisely those which appear at the n-th order of a Taylor expansion for a spin-s field. At second order, this reduces the degeneracy to at most $(1,2,4,2,1)$. We may obtain the same conditions (and more) from inspecting Jacobi identities involving two translation generators and an n-th order non-linearly realized one. For example, let us inspect one of the $(s-1)$ generators at second order, $G^{(2)}_{s-1}$. The Jacobi identity

$$[P_\mu, [P_\nu, G^{(2)}_{s-1}]] + [P_\nu, [G^{(2)}_{s-1}, P_\mu]] = 0\,, \tag{4.7}$$

Representations

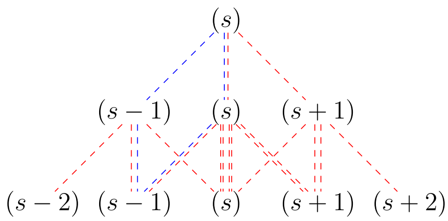

Figure 4.1: *The possible non-linear symmetries that can be realized on a spin-s Lorentz representation and their links via space-time translations.*

relates four coefficients to each other, depicted by the blue dashed lines in figure 4.1. This reduces the degeneracy at spin-$(s-1)$ to 2. From further Jacobi identities, we can again conclude that the generator content of the inverse Higgs tree at order n coincides with the Lorentz representations at n-th order in a Taylor expansion of a spin-s field. [76, 79] This is of course very reminiscent of the symmetry transformations (2.90) we examined in section 2.5. We will argue below that precisely these symmetry transformations result from the coset construction for a scalar essential Goldstone. However, the kinetic term is canonical in the same basis only if $[P_\mu, G^{(0)}] = 0$.

Note that the lines going upward from a particular n-th order generator to a $(n-1)$-th order generators always come together. For example, it is not possible to connect the vector at third order to the scalar at second order without also connecting it to the second order rank-2 symmetric tensor. The commutation relations implied by these lines are related to each other by Jacobi identities.

We can further simplify the algebra by making a convenient choice of basis. Let us focus for now on the case of a single scalar generator $G^{(0)}$ at order 0. The inverse Higgs tree, i.e. the Taylor expansion of $\phi(x)$, appears up to fourth order in figure 4.2. The dashed lines again indicate commutation relations with translations and the Young tableaux correspond to Lorentz representations. We have fixed the commutation relations so that $[P_\mu, G^{(n+1)}] \supset G^{(n)}$, but we have not yet ruled out other contributions to the right-hand side. We now show that one can always choose a basis such that $[P_\mu, G^{(n+1)}] = iG^{(n)} + \ldots$, where the ellipses contain only linearly realized generators. The commutation relations involving translations then admit a strict ordering in terms of levels in the inverse Higgs tree.

We will only include a finite number of generators from the inverse Higgs tree. Additionally, we assume that the zero-th order generator $G^{(0)}$ satisfies $[P_\mu, G^{(0)}] = \dots$ (i.e. only linearly realized generators appear on the right-hand side) and will only work with basis changes that preserve this choice. This means that the generators at the end points of the inverse Higgs tree never appear on the right-hand side of a commutation relation involving translations and a non-linearly realized symmetry. For example, if we assume that the inverse Higgs tree ends at the scalar $G^{(4)}$ at fourth order in 4.2, there is no vector G_μ such that $[P_\mu, G_\nu] \supset G^{(3)}$. If the highest order generator in the inverse Higgs tree is $G^{(n)}$, we will say that the tree itself has order n.

We note that it is impossible to make a basis change such that $[P_\mu, G^{(n)}] = \dots$ for $n > 0$, as this would render it impossible to impose degeneracy conditions. Let us consider all generators G_a in the algebra of a particular Lorentz representation. The index a runs from 1 to the number n of independent generators of that Lorentz representation. Then, commutation relations with translations are defined by an $n \times n$ matrix C_{aI}:

$$[P_\mu, G_a] = C_{aI}\, G^I \,, \tag{4.8}$$

where G^I $(I = 1, \dots, n)$ are the linear combinations of generators that attach to G_a generators in the inverse Higgs tree. For example, if G_a represent the vectors, then G^I will include the scalar at level 0, the combination of scalar and rank-2 symmetric tensor at level 2, etc. The matrix C_{aI} must have maximal rank, as otherwise it would be possible to define basis changes $\tilde{G}_a = D_{ab}\, G_b$ such that $[P_\mu, \tilde{G}_a] = \dots$ for some a. In other words, C_{aI} is diagonalizable. This means we can choose a basis such that $[P_\mu, G^{(n+1)}] = iG^{(n)} + \dots$, i.e. there exists a strict ordering of translation commutators by levels in the inverse Higgs tree. The same ideas generalize easily to the case of multiple scalar fields or multiple spin-$\frac{1}{2}$ essential Goldstone fermions. The algebra admits a strict ordering and the trees attached to the different zero-th order generators $G_{(i)}$ decouple from each other: $[P, G^{(n)}_{(i)}] = iG^{(n-1)}_{(i)}$.

Up to this point in this section, we have not directly used the coset construction. We did assume that the possibility of non-linearly realizing multiple symmetries on a single Goldstone field is due to a local degeneracy of independent global transformations. As we have explained in section 2.3, not all non-linearly realized symmetries are broken spontaneously. If there is no underlying linear realization of the symmetry, our starting point (3.32) is suspect. However, we can replace the assumption of local degeneracy with the assumption of coset universality. The algebraic condition $[P_\mu, G^{(n)}] \supset G^{(n-1)}$ that allows for inverse Higgs relations is of course the same that results from assuming local degeneracy.

To connect the results we have obtained here to the actual transformation laws realized by the Goldstone fields, we do need to refer to the coset

Scalar tree

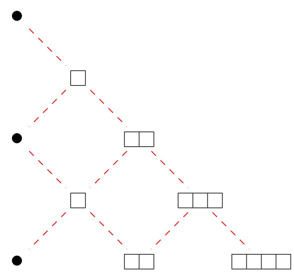

Figure 4.2: *The non-linear symmetries that can be realized on a scalar, and their space-time relations.*

construction. Consider a zero-th order scalar generator $G^{(0)}$ and its scalar Goldstone $\phi(x)$. From the coset construction, we immediately find that the transformation law of $\phi(x)$ under $G^{(0)}$ starts with a constant shift:

$$\delta_0 \phi(x) = c_0 + \dots , \tag{4.9}$$

where the ellipses contain any possible field-dependent terms. The condition $[P_\mu, G^{(n)}] = iG^{(n-1)} + \dots$ then tells us that, schematically:

$$\delta_{(n)} \phi(x) = c_{(n)} x^n + \dots . \tag{4.10}$$

In other words, the coset construction produces symmetry transformations of the type (2.90). However, the connection to our discussion of soft limits is not immediate. Whenever $[P_\mu, G^{(0)}] = P_\mu$, the coset construction does not generate a canonical kinetic term for $\phi(x)$ in the field basis where δ_0 takes the form (4.9).[1] After canonically normalizing, the transformation law contains field-dependence in every term. Indeed, the dilaton EFT does not show any special soft limits, see [32–34].

In Chapter 3, we saw that the coset construction - in addition to the constraint $[P_\mu, G^{(n)}] \supset G^{(n-1)}$ - requires higher-order conditions of the type (3.39) in order to impose inverse Higgs constraints. In principle, one could use these conditions to further reduce the Ansätze for non-linearly realized

[1]In this case, $G^{(0)}$ generates a scaling symmetry. It turns out that a non-linear realization of scaling symmetry always enhances to a non-linear realization of conformal symmetry. In other words, there is always a hidden generator of special conformal transformations.

algebras. In particular, they would constrain the form of the commutation relations involving two non-linearly realized generators, $[G^{(p)}, G^{(q)}]$, whereas up to now we have only constrained commutation relations involving translations. Equations (3.39), however, depend on the choice of parametrization of the coset element. This makes them possibly suspect to use for an exhaustive classification. A preferred exponential parametrization does exists which, in all known examples, is least restrictive for imposing inverse Higgs constraints. In practice, however, we have not come across a situation where the conditions (3.39) are necessary or useful for classifying non-linearly realized symmetries.

4.1.1 Canonical propagators

We have now fixed the generator content of the most general finite non-linearly realized algebra, assuming unbroken Poincaré invariance and given a choice of essential Goldstone field. By imposing Jacobi identities and consistency of the degeneracy conditions, and after picking a suitable basis, we were able to identify the inverse Higgs tree with the Taylor expansion of the essential field. We now introduce a minimal physical condition to further reduce the inverse Higgs tree: we assume that the coset construction produces a canonically normalized kinetic term for all Goldstone fields, in the field basis where the transformation laws take the standard form (4.10). If, in addition, there is no potential for the Goldstone fields, we arrive exactly at the spectrum of theories we considered in section 2.5. These are the EFTs whose S-matrix display the Adler zero and possibly higher-order soft limits dictated by the highest order transformation law (2.90) (i.e. by the order of the inverse Higgs tree).

Let us work with a single scalar essential Goldstone and assume that the canonical kinetic term is the operator with the fewest number of fields in the Lagrangian. Then, the kinetic term must be invariant under the field-independent part of the transformation law (4.10). This drastically cuts down on the allowed generators in figure 4.1. At order n, the field-independent part of the transformation is:

$$\delta_n \phi(x) = c^{(n)}_{\mu_1 \ldots \mu_n} x^{\mu_1} \ldots x^{\mu_n} + \ldots . \tag{4.11}$$

The canonical kinetic term $\phi \Box \phi$ is invariant, up to a total derivative, under this transformation only if $c^{(n)}$ is fully symmetric and traceless. This means that we reduce the inverse Higgs tree to the right-most diagonal line in figure 4.1. We have a single rank-n traceless and symmetric generator at order n.

There are certainly interesting EFTs of a single scalar which violate the assumptions we make in this section. We have already mentioned the theory of spontaneously broken conformal symmetry, in the previous section and 3.2.4. The codimension-1 case (i.e. the 5-dimensional AdS algebra) includes a potential in the vielbein term to achieve invariance under the transformation law (3.59). [2] It is the only single-scalar EFT with this property. The higher codimension AdS algebras (which require multiple essential scalar Goldstones) include generators which are not on the right-most diagonal line in 4.1. For example, the AdS_6 algebra includes the scalar generator at order 2, i.e. a transformation law of the type:

$$\delta_{(2)}\phi(x) = c_{(2)}x^2 + \dots \,. \tag{4.12}$$

We will argue below that these exceptions, based on anti-de Sitter space-time, are the only scalar EFTs which can make use of the off-diagonal generators and at the same time include canonical kinetic terms.

We can repeat the arguments for essential spin-$\frac{1}{2}$ Weyl fermion Goldstinos. In two-component notation, the general field-independent transformation law reads:

$$\delta_{(n)}\lambda_\alpha = (c_{(n)})_{\alpha\beta_1\dots\beta_n\dot\beta_1\dots\dot\beta_n}x^{\beta_1\dot\beta_1}\dots x^{\beta_1\dot\beta_1} + \dots \,, \tag{4.13}$$

where, a priori, the coefficients $c_{(n)}$ are not Lorentz irreps. They are, however, symmetric with respect to the pairwise exchange $(\beta_i \leftrightarrow \beta_j, \dot\beta_i \leftrightarrow \dot\beta_j)$. Now invariance, up to a total derivative, of the Weyl action requires that $c_{(n)}$ be fully symmetric in all dotted and undotted indices, respectively. This cuts the inverse Higgs tree down to a single spin-$\frac{1}{2}(n+1)$ Lorentz irrep at order n. The same conclusion holds in a theory of multiple essential spin-$\frac{1}{2}$ fermions.

The story is very similar for $U(1)$ gauge vectors. The non-linearly realized generators live in the Taylor expansion of a vector:

$$\delta_{(n)}A_\mu(x) = (c^{(n)})_{\mu\mu_1\dots\mu_n}x^{\mu_1}\dots x^{\mu_n} + \dots \,. \tag{4.14}$$

This implies that $(c^{(n)})_{\mu\mu_1\dots\mu_n}$ is either fully symmetric in all indices or symmetric in $(\mu_1,\dots\mu_n)$ and anti-symmetric under the exchange $(\mu \leftrightarrow \mu_i)$. In the case that $c^{(n)}$ is fully symmetric, the transformation (4.14) can be written as $\delta_{(n)}A_\mu(x) = \partial_\mu\alpha(x)$. In other words, all transformations of this type are part of the $U(1)$ gauge symmetry. This means that we always have to take into account an infinite sequence of generators when we study non-linear realizations in gauge-invariant theories. A convenient way to deal with this (for general p-form gauge theories), is to use the invariant field strength F_{p+1} as

[2] In the action (3.58), the potential is removed by a separately total derivative-invariant Wess-Zumino term.

fundamental field, subject to Bianchi identity, rather than the gauge potential A_p. This is the approach we will take when we look at SUSY-invariant theories in Chapter 5.

After taking into account the gauge symmetry, the transformation (4.14) still contains several Lorentz representations at each order in the inverse Higgs tree. Invariance of the gauge-invariant canonical kinetic term $F_{\mu\nu}F^{\mu\nu}$ leaves us again with only a single Lorentz-irreducible representation at each order. These are the generators which are anti-symmetric under the exchange $(\mu \leftrightarrow \mu_1)$, symmetric with respect to $(\mu_i \leftrightarrow \mu_j)$, and fully traceless.

4.1.2 Towards exceptional EFTs

Let us quickly recap what we have done so far. By making some minimal assumptions - local degeneracy of Goldstone modes/existence of inverse Higgs relations, compatibility with canonical normalization, and closure of the symmetry algebra - we were able to eliminate a large class of EFTs. The structure of non-linearly realized symmetry algebras is always identical to a Taylor expansion of the essential Goldstone fields. Invariance of canonical kinetic terms then reduces to a single non-linearly realized generator at each order. In fact, this structure is so simple that we can complete our classification simply by writing down a general Ansatz for the symmetry algebra and imposing closure with Jacobi identities. For the cases of single scalar, single vector or multiple fermion EFTs, we can then enumerate all solutions to the Jacobi identities up to arbitrary finite order in the inverse Higgs tree. Algebraic classifications of this type were first carried out by [76, 77] and [78]. In [79, 80], we emphasized the systematic and general approach outlined in this Chapter.

In section 4.2, we present the results of [76–79] alongside those from the complementary *soft bootstrap* approach of [2–4, 8]. The remaining discussion in this Chapter is based on the references [79, 80].

4.2 Classification of exceptional EFTs

In this section we carry out our procedure using a number of examples where we can exhaustively classify all algebras that can be non-linearly realized. We will employ two-component notation [124] (sometimes known as $SU(2) \times SU(2)$) throughout most of the following, since we will work exclusively in four space-time dimensions. We use the following convention for commutators between a tensor $T_{\alpha_1,\ldots\alpha_n\dot{\alpha}_1,\ldots\dot{\alpha}_m}$ and the Lorentz generators

$M_{\beta\gamma}$, $\bar{M}_{\dot{\beta}\dot{\gamma}}$

$$[T_{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_m}, M_{\beta\gamma}] = 2\, n!\, i\epsilon_{\alpha_1(\beta}T_{\gamma)\alpha_2\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_m}\,,$$
$$[T_{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_m}, \bar{M}_{\dot{\beta}\dot{\gamma}}] = 2\, m!\, i\epsilon_{\dot{\alpha}_1(\dot{\beta}}T_{|\alpha_1\ldots\alpha_n|\dot{\gamma})\dot{\alpha}_2\ldots\dot{\alpha}_m}\,, \tag{4.15}$$

where we have explicitly symmetrized in (β,γ) or $(\dot{\beta},\dot{\gamma})$ with weight one, where necessary. In these and all following equations, the symmetrization with weight one of groups of indices such as $\alpha_1,\ldots,\alpha_n$ will be implicit (and similarly for the dotted indices).

4.2.1 Single scalar Goldstone modes

We begin with a single scalar Goldstone where all non-linearly realized generators are fully symmetric and traceless, as argued in section 4.1. We denote the n^{th} order generator in the inverse Higgs tree by $G_n \equiv G_{\alpha_1,\ldots,\alpha_n\dot{\alpha}_1,\ldots\dot{\alpha}_n}$ where $n = 0, 1, \ldots Z$, i.e. we include generators up to a finite order Z with G_0 corresponding to the zeroth-order generator. These generators are fully symmetric in the sets $(\alpha_1,\ldots,\alpha_n)$, $(\dot{\alpha}_1,\ldots,\dot{\alpha}_n)$ since they correspond to symmetric traceless Lorentz tensors.

The appearance of non-linear generators in $[P_{\gamma\dot{\gamma}}, G_{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_n}] \equiv [P_{\gamma\dot{\gamma}}, G_n]$ is fixed by our above analysis of inverse Higgs trees while the commutator between two non-linear generators remains unconstrained. We have

$$\begin{aligned}[P_{\gamma\dot{\gamma}}, G_n] = {} & \tfrac{1}{2} i\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1}G_{\alpha_2\ldots\alpha_n\dot{\alpha}_2\ldots\dot{\alpha}_n} \\ & + iAP_{\gamma\dot{\gamma}} \quad \text{(only for } n=0) \\ & + B\epsilon_{\gamma\alpha_1}\bar{M}_{\dot{\gamma}\dot{\alpha}_1} - \bar{B}\epsilon_{\dot{\gamma}\dot{\alpha}_1}M_{\gamma\alpha_1}\,, \quad \text{(only for } n=1)\end{aligned} \tag{4.16}$$

where A and B are respectively real and complex parameters. The fact that B is complex suggests that there are two different Lorentz structures involving the Lorentz generators. In $SO(1,3)$ notation this is clearly the case, since we can write down both $M_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}M^{\rho\sigma}$ on the right-hand side when $n = 1$. We could have also added a term of the form $\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1}P_{\alpha_2\dot{\alpha}_2}$ in the $[P_{\gamma\dot{\gamma}}, G_2]$ commutator, but this can always be removed by a change of basis. The general form of the $[G_{\alpha_1\ldots\alpha_m\dot{\alpha}_1\ldots\dot{\alpha}_m}, G_{\beta_1\ldots\beta_n\dot{\beta}_1\ldots\dot{\beta}_n}] \equiv [G_m, G_n]$

commutators is:

$$[G_m, G_n] = \sum_{k=0}^{n} iC_k^{(m,n)} \prod_{q=1}^{k} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q} G_{\alpha_{k+1}\ldots\alpha_m\beta_{k+1}\ldots\beta_n\dot{\alpha}_{k+1}\ldots\dot{\alpha}_m\dot{\beta}_{k+1}\ldots\dot{\beta}_n}$$
$$+ iD^m \prod_{q=1}^{m-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q} P_{\alpha_m\dot{\alpha}_m} \quad (m = n+1)$$
$$+ \prod_{q=1}^{m-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q} (E^m \epsilon_{\alpha_m\beta_m}\bar{M}_{\dot{\alpha}_m\dot{\beta}_m} - \bar{E}^m \epsilon_{\dot{\alpha}_m\dot{\beta}_m} M_{\alpha_m\beta_m}) \,, (m = n) \tag{4.17}$$

where $C_k^{(m,n)}$ and D^m are real parameters and E^m are complex parameters. Note that $C_k^{(m,n)} = 0$ if $2k < (n + m - Z)$. We have also assumed in the above that $m \geq n$ without loss of generality.

We now constrain the form of these commutators using the remaining Jacobi identities. We consider the two cases of $Z \leq 2$ and $Z \geq 3$ separately. The former has been computed in [76, 77] (and was confirmed by our own analysis of this case).

- $Z \leq 2$

Up to and including two inverse Higgs relations, there are two branches of solutions depending on whether A vanishes or not. This distinguishes between the cases where the essential Goldstone realizes a shift or a scaling symmetry.

For $A = 0$ we also have $B = 0$ and G_0 generates a shift symmetry for the scalar. There are two distinct algebras up to first-order ($Z = 1$), with one arising as a singular contraction of the other. These correspond to the five-dimensional Poincaré algebra and its Galilean contraction. They are respectively non-linearly realized by the scalar DBI action [99, 100] and the Galileons [94]. At the level of transformation rules, the scalar DBI transformation rule has field-dependence while this is lost in the Galilean contraction, where the transformation rule is reduced to a first order extended shift symmetry (see section 3.2.3). As discussed, this field-dependence is responsible for the scalar DBI being an exceptional EFT. [3] Both EFTs have a quadratic scaling in soft scattering amplitudes, which is related to the fact that in each case we only need to impose a single inverse Higgs constraint (to remove the vector generator $G_{\alpha\dot{\alpha}}$). We refer the reader to [76, 77] for full details on the transformation rules and algebras. Schematically, the five-dimensional

Poincaré algebra reads:

$$[G_0, G_1] = aP, \qquad [G_1, G_1] = aM\,. \tag{4.18}$$

This will be a recurring theme in what follows[3].

In the presence of the second-order generator G_2, this set of generators again leads to two distinct algebras with one a contraction of the other. Both require the sub-algebra up to order $Z = 1$ to be that of the contracted five-dimensional Poincaré. The uncontracted $Z = 2$ algebra is that of the Special Galileon, [91, 103] which has non-vanishing commutators between non-linear generators. The contraction again loses this property, thereby reducing the G_2 transformation rule to a second order extended shift symmetry. The Special Galileon is an exceptional EFT due to the field-dependence in the transformation rule generated by G_2. Since in both cases we need to impose two inverse Higgs constraints, both algebras lead to EFTs with a cubic soft scaling in scattering amplitudes. Again, we refer the reader to [76, 77, 91, 103] for full details on the Special Galileon algebra. However, let us state that the non-zero commutators between non-linear generators are of the form[4]

$$[G_1, G_2] = bP, \qquad [G_2, G_2] = bM. \tag{4.19}$$

Note the close similarity between the structure of these commutators and those in (4.18).

For $A \neq 0$, G_0 is the generator of dilatations. Jacobi identities ensure that the algebra up to first-order is that of the four-dimensional conformal algebra. It is not possible to extend the conformal algebra with the addition of G_2 apart from in two space-time dimensions (see for [82] more details.). Due to the lack of shift symmetry and Adler's zero for the scalar, there is no sense in which the resulting EFT of the dilaton has an enhanced soft limit. We note that there are two well known bases for the conformal algebra but both give rise to identical EFTs, as we have discussed extensively in section 3.2.4.[5]. [25]

- $Z \geq 3$

We now turn our attention to the case with more than two inverse Higgs relations. We begin with the Jacobi identity which involves two copies of

[3] We note that this also includes scalar anti-DBI, where the non-linearly realized algebra has two time-like directions. Whether the exceptional EFT is scalar DBI or scalar anti-DBI depends only on the sign of the a.

[4] Again the parameter b can be positive or negative, similar to (anti-)DBI.

[5] This is not always guaranteed in the presence of inverse Higgs constraints. [141]

translations. Similar to the case with two inverse Higgs relations, this immediately fixes $A = B = 0$ leaving only $[P, G_n] = G_{n-1}$ as required to satisfy inverse Higgs relations. Next we consider the Jacobi identity with one copy of translations and two non-linear generators. By projecting onto different Lorentz structures, we find that all other parameters are also forced to vanish other than D^Z and $Re(E^Z)$, which are fixed to be proportional. We are therefore left with a single free parameter. The only non-vanishing commutators which contain non-linear generators are those required by inverse Higgs constraints, and the following, schematically

$$[G_{Z-1}, G_Z] = D^Z P, \qquad [G_Z, G_Z] = D^Z M\,. \tag{4.20}$$

Note that this structure is identical to the $Z = 1$ and $Z = 2$ cases above.

Finally, we consider the remaining Jacobi identities which involve three non-linear generators. Right away the Jacobi identity involving the generators (G_{Z-2}, G_{Z-1}, G_Z) fixes $D^Z = 0$ since we must have $[P, G_{Z-2}] \neq 0$ to reduce to the single scalar Goldstone. Therefore for $Z \geq 3$ all commutators between non-linear generators vanish. It follows that all symmetries reduce to extended shift symmetries and no further exceptional EFTs.

We have therefore proved, using only Lorentz invariance, the existence of inverse Higgs constraints, and Jacobi identities, that the only exceptional scalar EFTs are scalar DBI and the Special Galileon: exceptional theories with $\sigma > 3$ do not exist. We refer the reader to [3] for similar results derived using on-shell amplitudes methods.

4.2.2 Multiple scalar Goldstone modes

We now consider the case where there are $N > 1$ essential scalar Goldstones. Most of our discussion on the single scalar carries over to this case. In particular, the inverse Higgs trees attached to the different scalar zeroth-order generators decouple (so that $[P, G^{(i)}] \supset G^{(i)}$ where i labels each tree) and each attains the same structure as in the previous section: only symmetric, traceless generators at each order. We label the generators $G^i_n \equiv G^i_{\alpha_1,\ldots,\alpha_n\dot{\alpha}_1,\ldots\dot{\alpha}_n}$ according to the tree i they belong to with $i = 1, \ldots, N$ and their rank n within that tree. Now translations act as

$$\begin{aligned}[P_{\gamma\dot{\gamma}}, G^i_n] = {} & \tfrac{1}{2} i \epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1} G^i_{\alpha_2\ldots\alpha_n\dot{\alpha}_2\ldots\dot{\alpha}_n} \\ & + iA^i P_{\gamma\dot{\gamma}} \quad (\text{only for } n = 0) \\ & + B^i \epsilon_{\gamma\alpha_1}\bar{M}_{\dot{\gamma}\dot{\alpha}_1} - \bar{B}^i \epsilon_{\dot{\gamma}\dot{\alpha}_1} M_{\gamma\alpha_1} \quad (\text{only for } n = 1)\,,\end{aligned} \tag{4.21}$$

with each tree containing a finite number Z_i of non-linearly realized generators. The coefficient A^i may, without loss of generality, be set to zero for all but a single non-linear scalar generator, i.e. there can only be a single dilaton. The commutators $[G^i_n, G^j_m]$ are also very similar to the previous section, but coefficients now carry the appropriate extra indices

$$
\begin{aligned}
[G^i_m, G^j_n] = &\sum_{k=1}^{N}\sum_{w=0}^{n}\prod_{q=1}^{w}\epsilon_{\alpha_q\beta_q}\epsilon_{\dot\alpha_q\dot\beta_q} i D^{(i,m;j,n)k}_w G^k_{\alpha_{w+1}\ldots\alpha_m\beta_{w+1}\ldots\beta_n\dot\alpha_{w+1}\ldots\dot\alpha_m\dot\beta_{w+1}\ldots\dot\beta_n} \\
&+ iF^{ij,m}\prod_{q=1}^{m-1}\epsilon_{\alpha_q\beta_q}\epsilon_{\dot\alpha_q\dot\beta_q}P_{\alpha_m\dot\alpha_m} \quad \text{(only for } m = n+1) \\
&+ \prod_{q=1}^{m-1}\epsilon_{\alpha_q\beta_q}\epsilon_{\dot\alpha_q\dot\beta_q}(H^{ij,m}\epsilon_{\alpha_m\beta_m}\bar M_{\dot\alpha_m\dot\beta_m} - \bar H^{ij,m}\epsilon_{\dot\alpha_m\dot\beta_m}M_{\alpha_m\beta_m}) \quad (m = n) \\
&+ \prod_{q=1}^{m}\epsilon_{\alpha_q\beta_q}\epsilon_{\dot\alpha_q\dot\beta_q} iX^{ij,m} \quad \text{(only for } m = n)\,,
\end{aligned} \tag{4.22}
$$

where we have taken $m \geq n$. The parameters $D^{(i,m;j,n)k}_w$ and $F^{ij,m}$ are real, whereas $H^{ij,m}$ are complex in general. The linear scalar generators $X^{ij,m}$ are defined by this commutation relation. Since they are linearly realized, they commute with translations, form a sub-algebra, and their commutation relations with non-linear generators can only produce non-linear generators (see section 3.2.2).

When $m = n$, the right-hand side needs to be anti-symmetric under the simultaneous exchange of both the Lorentz indices on G^i_m and G^j_m, and the tree labels i and j. This imposes the conditions

$$
D^{(i,n;j,n)k}_w = -D^{(j,n;i,n)k}_w\,, \quad H^{ij,m} = H^{ji,m}\,, \quad X^{ij,m} = -X^{ji,m}\,. \tag{4.23}
$$

In particular, when there are two scalar essentials ($N = 2$), there is only a single linear scalar generator at each order: $X^{ij,m} \equiv X^m$. We also have $D^{(i,m;j,n)k}_w = 0$ when $2w < (n + m - Z_k)$.

We now consider the following cases separately: first we investigate the case where there are no inverse Higgs constraints i.e. $Z_i = 0$. We then consider the case where no tree involves more than a single additional non-linear generator i.e $Z_{max} = 1$. Finally, we consider the case where at least one essential Goldstone contains at least two additional non-linear generators in its inverse Higgs tree i.e. $Z_{max} \geq 2$. For these cases, we cannot enumerate all solutions to the Jacobi identities - there are infinitely many due to the possibility of non-linearly realizing generic internal symmetry groups - but we will derive general results on the structure of these algebras.

• $Z_{max} = 0$

In the case $Z_{max} = 0$ all generators other than translations and Lorentz transformations are scalars. We collectively denote them as (Y^i, D) for simplicity, where D is the generator of dilatations. We assume that D is a non-linear generator while Y^i includes both linear and non-linear generators. After imposing all the constraints from Jacobi identities we have

$$[P_{\alpha\dot{\alpha}}, Y^i] = 0, \quad [P_{\alpha\dot{\alpha}}, D] = iP_{\alpha\dot{\alpha}}\,, \tag{4.24}$$

and

$$[Y^i, Y^j] = iD^{ij}{}_k Y^k, \quad [D, Y^i] = iE^i{}_j Y^j\,, \tag{4.25}$$

with the constraints

$$D^{[ij}{}_k E^{k]}{}_l = 0, \qquad D^{[ij}{}_l D^{k]l}{}_m = 0\,. \tag{4.26}$$

In the presence of dilatations, each Y^i can therefore have a non-trivial scalar weight. In general the scalars of these theories are said to span a non-linear sigma-model. In the two-scalar cases they include the well known coset spaces[6]

$$\frac{SO(3)}{SO(2)}, \qquad \frac{SO(1,2)}{SO(2)}\,, \tag{4.27}$$

which appear often in the inflationary literature, e.g. as α-attractors. [84–86] Such non-linear sigma-models define an exceptional EFT since the two-derivative action, which includes interactions, is completely fixed by symmetry. Indeed, the transformation rules include field-dependent pieces.

• $Z_{max} = 1$

We now turn to the case where $Z_{max} = 1$. Here we find it useful to separate the calculation into two sub-cases: in the first we do not allow any non-linear generators in the dilaton's inverse Higgs tree (if the dilaton exists in the first place), while in the second case we do allow for that vector generator which we denote as K. Schematically, the Jacobi identity (P, P, G) imposes $[P, K] \propto D + M$ where D is the generator of dilatations. This means that K necessarily generates special conformal transformations. Because the case $Z_{max} = 1$ without the dilaton was considered in [76, 77], we will focus on what changes when the dilaton is included.

[6] They also include the algebra of the scaling superfluid presented in [82] and we refer the reader there for more details.

In the following the generators G^i are scalars which are connected to the non-linear vectors $G^i_{\alpha\dot{\alpha}}$. Furthermore, we have scalars X^I which do not fit into the previous two categories i.e. they can be linearly realized or correspond to scalar Goldstones with empty inverse Higgs trees.

Without special conformal transformations

In the first sub-case, after we have imposed all the constraints from Jacobi identities on the Ansatz (4.21)(4.22), the part of the algebra that does not involve dilatations reduces to:

$$[P_{\gamma\dot{\gamma}}, G^i_{\alpha\dot{\alpha}}] = \tfrac{1}{2} i\epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}} G^i\,, \quad [G^i_{\alpha\dot{\alpha}}, G^j] = iH^{ij} P_{\alpha\dot{\alpha}}$$
$$[G^i_{\alpha\dot{\alpha}}, G^j_{\beta\dot{\beta}}] = 4iH^{ij}(\epsilon_{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta}) + i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} Y^{ij}\,, \quad [X^I, X^J] = if^{IJ}{}_K X^K\,,$$
$$[X^I, G^i] = iB^{iI}{}_j G^j + iD^{iI}{}_J X^J, \quad [X^I, G^i_{\alpha\dot{\alpha}}] = iB^{iI}{}_j G^j_{\alpha\dot{\alpha}} + iC^{iI} P_{\alpha\dot{\alpha}}\,. \quad (4.28)$$

We arrive at this result by eliminating the generator D from the right-hand side of each of the above commutators. Then the calculation reduces to the case considered in [77]. We will consider the commutators involving D in a moment.

The $[G^i_{\alpha\dot{\alpha}}, G^j_{\beta\dot{\beta}}]$ commutator defines the scalars Y^{ij}. These generators are not independent from G^i and X^I. In general, a linear combination of G^i and X^I generators can appear on the right-hand side of the commutator for $i \neq j$[7]. There are several additional constraints on the coefficients in (4.28). The full list appears in [76][8] and we refer the reader there for full details. Here we simply comment on the structure of the solutions.

We start by assuming that no non-linear generators appear on the RHS of a commutator between a pair of non-linear generators. The matrix H^{ij} can be made diagonal by a basis change. Then, the non-zero elements can be made 1 or -1 by rescaling generators. Scalar Goldstones whose first-order vectors have a non-vanishing commutator take the DBI form while those with a vanishing commutator correspond to Galileons which can be coupled to the DBI scalars. The matrix H^{ij} fixes the commutation relations of the Y^{ij} with

[7]The commutation relations of the Y^{ij} generators are fixed by Jacobi identities, but that does not identify the linear combination of G^i and X^I generators that appears in the commutator. In particular, we can always add a central charge C. When linearly realized, this does not change the transformation laws or invariants derived from the coset construction.

[8]Note that the coefficients a^i_A and e^A_i of equation (4.1) in [76] are fixed (up to a basis change) by the inverse Higgs trees to be diagonal and zero, respectively. We have furthermore divided the scalar sectors in a different way, which is why we are able to remove the term $[G, G_{\alpha\dot{\alpha}}] \propto G_{\alpha\dot{\alpha}}$.

themselves and with $G^i_{\alpha\dot{\alpha}}$. In the case that $H^{ij} = \delta^{ij}$, the symmetry algebra contains a factor $ISO(1,3+N)$ and Y^{ij} generate the $SO(N)$ subgroup. This algebra is non-linearly realized by the multi-DBI exceptional EFT. [161]

We now consider the commutations relations between dilatations D and the other generators. We fix the dilatation weight of the translation generator to unity i.e. $[P_{\gamma\dot{\gamma}}, D] = iP_{\gamma\dot{\gamma}}$. The remaining commutation relations are

$$[D, G^i] = iB^i{}_j G^j, \quad [D, G^i_{\alpha\dot{\alpha}}] = i(B^i_j - \delta^i_j)G^j_{\alpha\dot{\alpha}} + iJ^i P_{\alpha\dot{\alpha}}\,,$$
$$[D, X^I] = iS^I{}_i G^i + iT^I{}_J X^J\,, \tag{4.29}$$

with constraints

$$T^{[ij]J} = 0, \quad S^{[ij]}{}_k = J^{[i}\delta^{j]}{}_k, \quad H^{ij} = -(B^{ik} - \delta^{ik})H^{kj}\,. \tag{4.30}$$

The pair of indices ij on the T and S coefficients is a special case of the general index I, as before. In addition to these, we have the usual constraint of the form (4.26) that relates the structure constants of the scalar subalgebra to the dilatation weights. Finally, the last constraint fixes the weights of the higher-dimensional translation and boost generators. Taking H^{ij} diagonal, it follows that the DBI scalars have zero scaling weight. The weights of the Galileon directions are not fixed by this equation.

We note that, similar to the $Z_{max} = 0$ case, there are many different solutions to the Jacobi identities for different choices of the generator content. However, the structure of those solutions is very simple: in every case the vectors generate the symmetry algebra of a higher-dimensional space with the scalars corresponding to DBI or Galileons. Furthermore, one can add a dilaton and some internal coset space G/H. The dilatation weights of the DBI scalars vanish and their representation under the internal coset space must satisfy the constraints of [76].

Let us explain in more detail why we can couple a DBI scalar to a Galileon. Consider the six-dimensional Poincaré algebra with generator content: $P_\mu, M_{\mu\nu}, P_4, P_5, M_{\mu 4}, M_{\mu 5}, M_{45}$ where $(4,5)$ refer to the two extra dimensions. In the usual construction of multi-DBI, P_4, P_5 correspond to the two essential scalars, while $M_{\mu 4}, M_{\mu 5}$ correspond to two inessential vectors which are eliminated by inverse Higgs constraints. M_{45} is a linearly realized $SO(2)$ between the two scalars. However, we can take a singular contraction by rescaling $P_5 \to \omega P_5, M_{\mu 5} \to \omega M_{\mu 5}$ and $M_{45} \to \omega M_{45}$ with $\omega \to \infty$ such that the scalar corresponding to P_4 is a DBI scalar with $[M_{\mu 4}, M_{\nu 4}] = M_{\mu\nu}$, $[P_4, M_{\mu 4}] = P_\mu$ while the scalar corresponding to P_5 reduces to a Galileon with $[M_{\mu 5}, M_{\nu 5}] = 0$, $[P_5, M_{\mu 5}] = 0$. We therefore have a DBI scalar coupled to a Galileon but let us stress that the presence of M_{45} is crucial since

$[M_{\mu 4}, M_{\nu 5}] = \eta_{\mu\nu} M_{45}$, both before and after the contraction. This algebra also appears in [77].

Existence of this algebra is a necessary condition for the existence of the corresponding EFT but more work is required to see if both the DBI and Galileon scalars can have standard kinetic terms which can be augmented by interactions. Indeed, the Galileon kinetic term is a Wess-Zumino and therefore a more thorough analysis than the coset construction is required. It would be very interesting to check if there is a sensible realisation of this algebra.

So far we have assumed that only linear generators appear on the RHS of commutators between a pair of non-linear generators. However, this is not forced upon us by Jacobi identities. Indeed, Jacobi identities allow for non-linear scalars to appear on the RHS of the commutator between a pair of non-linear vector generators. For example, consider an algebra with the generators $G^1, G^1_{\alpha\dot{\alpha}}, G^2, G^2_{\alpha\dot{\alpha}}$ and G^3 where $G^{1,2}$ are scalar generators corresponding to essential scalar Goldstones each with a tree containing a vector $G^{1,2}_{\alpha\dot{\alpha}}$:

$$[P_{\alpha_1\dot{\alpha}_1}, G^1_{\alpha_2\dot{\alpha}_2}] = i\epsilon_{\alpha_1\alpha_2}\epsilon_{\dot{\alpha}_1\dot{\alpha}_2}G^1, \quad [P_{\alpha_1\dot{\alpha}_1}, G^2_{\alpha_2\dot{\alpha}_2}] = i\epsilon_{\alpha_1\alpha_2}\epsilon_{\dot{\alpha}_1\dot{\alpha}_2}G^2, \qquad (4.31)$$

where G^3 is another scalar generator whose corresponding Goldstone has an empty inverse Higgs tree. The above structure from Jacobi identities allows for the commutator

$$[G^1_{\alpha_1\dot{\alpha}_1}, G^2_{\alpha_2\dot{\alpha}_2}] = i\epsilon_{\alpha_1\alpha_2}\epsilon_{\dot{\alpha}_1\dot{\alpha}_2}G^3\,, \qquad (4.32)$$

where G^3 is a central extension of the bi-Galileon algebra which cannot be eliminated by a basis change. This algebra satisfies Jacobi identities and can be non-linearly realized by a sensible EFT that is not equivalent to any of the cases mentioned elsewhere in the paper. A general analysis of such possibilities has not appeared in the literature.

With special conformal transformations

Next we consider the case where we include $K_{\alpha\dot{\alpha}}$. We immediately have

$$[P_{\gamma\dot{\gamma}}, K_{\alpha\dot{\alpha}}] = -i\epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}D + \tfrac{i}{2}\epsilon_{\gamma\alpha}\bar{M}_{\dot{\gamma}\dot{\alpha}} + \tfrac{i}{2}\epsilon_{\dot{\gamma}\dot{\alpha}}M_{\gamma\alpha}\,. \qquad (4.33)$$

The Jacobi identity (P, K, D) fixes the subalgebra spanned by D, $K_{\alpha\dot{\alpha}}$ and the Poincaré generators to the ordinary AdS_5 algebra. We have checked that it is not possible to extend this algebra with other non-linear vector generators. This is an unsurprising result, because the AdS_{4+n} algebra satisfies $Z_{max} = 2$ for $n > 1$. It is not possible to truncate these algebras to a $Z_{max} = 1$ component. We will return to the higher-dimensional Anti-de Sitter algebras in the following subsection.

- $Z_{max} \geq 2$

We now consider the case where at least one of the trees, say $i = 1$, includes a second-order non-linear generator i.e. $Z_1 \geq 2$. We do not assume anything about the other trees. However, their structure will be constrained by Jacobi identities. We will primarily concentrate on the cases where the RHS of commutators between a pair of non-linear generators contains a linear generator, $[G, G] = \text{linear} + \ldots$. We concentrate on these cases since they represent the natural extension of the DBI and Special Galileon algebras, which are the only exceptional ones in the single scalar case. The appearance of linear generators in the commutation relation of two broken generators is a necessary condition for the symmetry to have a passive form where it acts on the coordinates. When only linear generators appear in $[G, G]$, the coset is said to be symmetric.

Note, however, that there can still be exceptional EFTs where linear generators do not appear in the commutators of broken generators (as with e.g. non-linear sigma models). We outline some of the constraints on these algebras, but will not attempt to enumerate possibilities, as there are infinitely many solutions to the Jacobi identities.

We begin with the Jacobi identities involving two copies of translations and one non-linear generator since these Jacobi identities do not mix the different trees. It is simple to see that each tree that includes a second-order generator must have $A = B = 0$ i.e. an essential scalar generator can only generate dilatations if its tree has at most one non-linear generator (which of course corresponds to special conformal transformations).

We now move onto Jacobi identities involving one translation generator and two non-linear generators from any tree i.e. (P, G^i_m, G^j_n). This tells us that $B^i = 0$ for all trees, so any algebras involving at least one second-order, traceless generator cannot form an extension the conformal algebra. For this reason we will assume that the dilaton is not included for the moment and come back to that possibility later. The remaining constraints tell us the following:

- $H^{ij,Z_i} = 4iF^{ij,Z_j}$ if i and j label two trees with $Z_i = Z_j$. All other H and F coefficients are zero. This structure strongly resembles (4.20).

- The linear scalars $X^{ij,m}$ can only appear if $Z_i = Z_j = m$.

- The appearance of non-linear generators G^k in the commutators $[G^i, G^j]$ is also highly constrained. The only allowed structure is $[G^i_{Z^i}, G^j_m] \supset G^k_{m-Z^i}$ where $m \geq Z^i$ and k is an arbitrary tree label. Furthermore, the

(P,G,G) Jacobi identity tells us that $[G^i_{Z^i}, G^j_{m+1}] \supset G^k_{m+1-Z^i}$ whenever $[G^i_{Z^i}, G^j_m] \supset G^k_{m-Z^i}$, for the same k.

We now consider Jacobi identities with three non-linear generators, namely (G^i, G^j, G^k). We begin by taking the second-order generator $G^1_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}$ from the $i=1$ tree, together with two vectors $G^j_{\beta\dot{\beta}}$, $G^k_{\gamma\dot{\gamma}}$ from any trees in the algebra. From inspecting the terms proportional to $G^1_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}$ we obtain $H^{jk,1}=0$, telling us that any tree with $Z=1$ cannot realize the DBI structure (4.20). We also see that any scalar generator which has a non-zero commutator with $G^1_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}$ cannot appear in any commutator involving two vectors.

From this Jacobi identity we can also infer that it is impossible to couple several Special Galileons. Indeed, if we take $i=j=1$ and $Z_k=2$, there are two terms proportional to non-linear scalars given by

$$iF^{ik,2}\epsilon_{\alpha_1\gamma}\epsilon_{\dot{\alpha}_1\dot{\gamma}}\epsilon_{\alpha_2\beta}\epsilon_{\dot{\alpha}_2\dot{\beta}}G^j_0 - iF^{ij,2}\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_1\dot{\beta}}\epsilon_{\alpha_2\gamma}\epsilon_{\dot{\alpha}_2\dot{\gamma}}G^k_0 \tag{4.34}$$

with symmetrization over the α indices assumed, as usual. These terms only cancel when $i=j=k$. Therefore, at most one Special Galileon can exist at once. The same result, for the case of two scalar fields, was found from amplitude methods in [2–4].

The terms proportional to non-linear scalars impose important constraints as well. Taking the trees i and k to be the same, $i=k=1$, and j to be some tree that obeys $Z_j=1$, we find

$$F^{11,2} = -2D^{(12;j1)m}D^{(11;m1)j}\,. \tag{4.35}$$

The coefficient on the left-hand side $F^{11,2}$ determines whether the Special Galileon structure (4.20) is realized by the tree $i=1$. The coefficients on the right-hand side tell us whether the commutator $[G^1_2, G^j_1]$ (where the subscript refers to the number of Lorentz indices) contains a vector V which satisfies $[G^1_1, V] \propto G^j_0$. We will now show that no such vector V can exist. To do so we inspect the Jacobi identity involving three vectors, two of them from trees with $Z \geq 2$ and one of them from a tree with $Z=1$. Inserting the constraint $[G^i_{Z^i}, G^j_m] \supset G^k_{m-Z^i}$, we find

$$[G^i_1, G^j_1] \not\supset G^k_0 \quad (\text{if } Z_i = 1,\ Z_j = 2)\,, \tag{4.36}$$

which implies that the right-hand side of (4.35) is equal to zero.

We have therefore seen that if the EFT includes a Special Galileon, it must be the only one and can only couple to other scalars which have empty inverse Higgs trees. This rules out, for example, a Special Galileon coupled to a standard Galileon as well as multi-Special Galileon theories. This is in stark contrast to the $Z_{max} = 1$ case where we can have multi-DBI. Furthermore, if the EFT contains any scalar which has $Z \geq 3$, no exceptional algebras exist as one would expect from our single scalar analysis[9].

Off-diagonal generators and dilatons

An important caveat concerns our restriction to purely symmetric and traceless representations in inverse Higgs trees, i.e. from $Z_{max} = 2$ onward. As explained in section 3.2, these are the unique transformations that leave the kinetic terms invariant provided the latter are the lowest order terms in a derivative expansion. This is a general statement in the absence of a dilaton. In the presence of a dilaton, however, non-linearly realized symmetries may relate the dilaton potential to kinetic terms.

An interesting example of this possibility is provided by the AdS_{4+n} algebra, that can be written as

$$
\begin{aligned}
&[P_A, D] = P_A\,, && [M_{AB}, P_C] = \eta_{AC} P_B - \eta_{BC} P_A\,,\\
&[K_A, D] = -K_A + P_A\,, && [M_{AB}, K_C] = \eta_{AC} K_B - \eta_{BC} K_A\,,\\
&[P_A, K_B] = 2M_{AB} + 2\eta_{AB} D\,, && [M_{AB}, M_{CD}] = 2\eta_{C[A} M_{B]D} + 2\eta_{A[D} M_{CB}\,,\\
&[K_A, K_B] = 2M_{AB}\,, &&
\end{aligned}
$$

with $A = (\mu, i_2, \dots i_n)$. For $n = 1$, this only involves a dilaton Goldstone and its inverse Higgs vector of special conformal transformations, as discussed in section 4.2.1. However, when $n \geq 2$ this set-up is augmented with $n-1$ trees consisting of an axion Goldstone, its inverse Higgs vector of Lorentz boosts, as well as a $Z = 2$ scalar arising from special conformal transformations in the higher-dimensional directions[10]. It is discussed in [142] how the lowest order invariant that includes the kinetic terms also generates a potential term for the dilaton. This combination allows for the $Z = 2$ scalar in the axion trees, which was ruled out in the general discussion above under the assumption of having shift symmetries and no dilatations.

[9]This assumes that we only allow for linear generators on the RHS of commutators between a pair of non-linear generators. However, other algebras certainly exist. A simple example would be a $Z_{max} = 2$ generalisation of the $Z_{max} = 1$ discussion above where a scalar with an empty inverse Higgs tree corresponds to a central extension of an algebra which previously only gave rise to extended shift symmetries.

[10]This algebra allows for (at least) two inequivalent Inönu-Wigner contractions, leading either to the Poincaré or the Galileon algebra, as discussed in the single-field case in [97]. Importantly, the contraction that gives rise to Poincaré does not preserve the structure of the inverse Higgs relations.

Crucially, the combination of special conformal transformations in one inverse Higgs tree, and a $Z = 2$ symmetric traceless generator in another, was ruled out in the above. Moreover, the inclusion of any off-diagonal generators other than the $Z = 2$ scalar requires the $Z = 2$ symmetric traceless one, as discussed in section 3.2. This implies that the above exception based on conformal symmetry is the unique one; adding additional Goldstone modes to this can only give rise to higher-dimensional Anti-de Sitter algebras. However, these would not be exceptional EFTs with soft limits as defined 2.5, due to the different implications of scaling symmetries.

Finally, let us mention that we can still include the dilaton with an empty inverse Higgs tree, i.e. without special conformal transformations. Here the algebras are the ones discussed above but generators can have a non-vanishing scaling weight with generalisations of the (4.26) constraints.

4.2.3 Fermion Goldstone modes

We now study the case where the essential Goldstones are N spin-1/2 fermions χ^i_α, with $i = 1, \ldots, N$. Any higher-order generators we add to this algebra to realize more symmetries on the essential fermions must also be fermionic, as they are related to other non-linear generators by space-time translations. Moreover, since the anti-commutator between two fermionic generators can only give rise to bosonic generators (in this case only linear ones), the algebras at every order in the inverse Higgs tree will always form subalgebras. Note that this is very different to the bosonic case where there is much more freedom in a commutator between two non-linear generators, as illustrated by the discussion above. For this reason our analysis here will be exhaustive, in contrast to the multi-scalar case.

The inverse Higgs tree for each fermion decouples [79]. Section 4.1 tells us that if the essential fermions are to have canonical propagators, we can only add a single non-linear generator at order n in each inverse Higgs tree which has spin-$(n + 1/2)$. We again consider non-linear generators up to finite order Z_i, allowing for different top levels for each fermion, denoted $\chi^i_n \equiv \chi^i_{\alpha_1\ldots\alpha_{n+1}\dot\alpha_1\ldots\dot\alpha_n}$ with Hermitian conjugate $\bar\chi^i_n \equiv \bar\chi^i_{\alpha_1\ldots\alpha_n\dot\alpha_1\ldots\dot\alpha_{n+1}}$, where $n = 0, \ldots Z_i$. The inverse Higgs tree fixes the commutators between translations and non-linear generators to be

$$[P_{\gamma\dot\gamma}, \chi^i_n] = i\epsilon_{\gamma\alpha_1}\epsilon_{\dot\gamma\dot\alpha_1}\chi^i_{\alpha_2\ldots\alpha_{n+1}\dot\alpha_2\ldots\dot\alpha_n}\,, \tag{4.37}$$

while commutators between two non-linear generators are only constrained by the linear symmetries. We have

$$\{\chi^i_m, \chi^j_n\} = A^{(ij,m)} \prod_{q=1}^{m} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q} M_{\alpha_{m+1}\beta_{m+1}} \quad (m=n)$$
$$+ B^{(ij,m)} \prod_{q=1}^{m-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q}\epsilon_{\alpha_m\beta_m}\epsilon_{\alpha_{m+1}\beta_{m+1}} \bar{M}_{\dot{\alpha}_m\dot{\beta}_m} \quad (m=n)$$
$$+ C^{(ij,m)} \prod_{q=1}^{m-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q}\epsilon_{\alpha_m\beta_m} P_{\alpha_{m+1}\dot{\alpha}_m}, \quad (m=n+1) \tag{4.38}$$

with $m \geq n$ and complex parameters, and

$$\{\chi^i_m, \bar{\chi}^j_n\} = D^{(ij,m)} \prod_{q=1}^{m-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q}\epsilon_{\dot{\alpha}_m\dot{\beta}_m} M_{\alpha_m\alpha_{m+1}} \quad (m=n+1) \tag{4.39}$$
$$+ E^{(ij,n)} \prod_{q=1}^{n-1} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q}\epsilon_{\alpha_n\beta_n} \bar{M}_{\dot{\beta}_n\dot{\beta}_{n+1}} \quad (n=m+1) \tag{4.40}$$
$$+ F^{(ij,m)} \prod_{q=1}^{m} \epsilon_{\alpha_q\beta_q}\epsilon_{\dot{\alpha}_q\dot{\beta}_q} P_{\alpha_{m+1}\dot{\beta}_{m+1}}, \quad (m=n) \tag{4.41}$$

where again all parameters are complex. We now consider $Z_i = 0$ and $Z_{max} \geq 1$ separately.

- $Z_i = 0$

First consider the case where $Z_i = 0$, where the results are well known. The allowed algebras correspond to N-extended super-Poincaré and Inonu-Wigner contractions thereof. The only non-trivial and non-vanishing commutators in the uncontracted algebra are[11]

$$\{\chi^i_\alpha, \bar{\chi}^j_{\dot{\beta}}\} = 2\delta^{ij} P_{\alpha\dot{\beta}}\,. \tag{4.42}$$

At lowest order in derivatives, the EFT which non-linearly realizes the N-extended super-Poincaré algebra is that of multi Volkov-Akulov (VA). [104]

[11] The appearance of δ^{ij} is guaranteed by positivity in Hilbert space. This is a necessary requirement in any linear realisations of the symmetry algebra, but not in non-linear realisations as the currents don't integrate into well-defined charges in the quantum theory. Here we still assume the requirement of positivity in Hilbert space. This is a reasonable assumption if one anticipates that the non-linear realisations have a (partial) UV completion to a linearly realized theory, or to be a particular limit of such a theory.

The commutator (4.42) guarantees that the transformation rules for each fermion are field-dependent and therefore VA is an exceptional EFT with $\sigma = 1$ soft behaviour.

In addition there are many different contractions of this algebra that give rise to new ones. We can take the limit where $\{\chi^i_\alpha, \bar{\chi}^j_{\dot{\beta}}\} = 0$ for all i, j in which case all transformation rules reduce to shift symmetries for each fermion, see [160]. However, we do not have to perform this contraction for all N generators. We can do it to none, all or any other number in between. Indeed we can realize an EFT consisting of N_1 shift symmetric fermions and N_2 VA fermions with the only constraint that $N = N_1 + N_2$. The $N = 1$ contracted case was studied in detail in [160], where it was shown that in four-dimensions the only Wess-Zumino term one can write down is the fermion's kinetic term, i.e. all interactions need at least one derivative per field.

- $Z_{max} \geq 1$

We now consider adding higher-order non-linear generators with inverse Higgs relations. We allow for different top levels in each tree but we assume that at least one tree has $Z \geq 1$. We follow exactly the same process as we did previously: we use the Jacobi identities (P_μ, χ_m, χ_n), $(P_\mu, \chi_m, \bar{\chi}_n)$ and $(\chi_m, \chi_m, \bar{\chi}_n)$ and find that all free parameters must vanish. We therefore find that for $Z_{max} \geq 1$ the only non-trivial commutators are those required by inverse Higgs constraints, which results in extended shift symmetries for all the fermions. There are no other exceptional EFTs. This includes the fermionic generalisation of scalar multi-Galileons[12], which are invariant under shifts linear in the coordinates. This theory, for the case of a single fermion essential, was also discovered in [2] using soft amplitudes. We have therefore seen that field-dependent transformation rules for the essential fermions are incompatible with inverse Higgs constraints. The only exceptional fermion EFT is that of Volkov-Akulov and its multi-field extensions.

4.2.4 Vector Goldstone modes

After the above classifications for the cases of scalar or spin-1/2 fermion Goldstone modes, we would now like to discuss a number of aspects when turning to a vector instead.

At lowest order this involves the introduction of a vector generator that is spontaneously broken, with an associated vector Goldstone mode. In contrast to the fermion case, it turns out to be impossible to introduce a deformation

[12] See [**?**] for a discussion on bi-Galileons.

of this algebra with a non-vanishing RHS for the commutator of a vector with itself. [78] In other words, there are no exceptional algebras when just introducing an essential vector generator. This implies that the field transforms with a constant shift, ruling out a mass term. Without gauge symmetry, such theories will therefore generically propagate a ghost or an infinitely strongly coupled longitudinal mode; however, this conclusion can be circumvented in a number of ways, as we will see.

In the inverse Higgs tree, one can introduce three different non-linear generators at the first level, corresponding to an anti-symmetric tensor as well as a symmetric and traceless tensor and a trace. The latter two of these would belong to the possible gauge symmetry of the vector, which can be seen as an infinite sequence of non-linearly realized symmetries of the form

$$\delta A_\mu = u_\mu + u_{\mu\nu} x^\nu + u_{\mu\nu\rho} x^\nu x^\rho + \dots \,, \tag{4.43}$$

where the $u_{\mu\dots}$ parameters are symmetric and contain traces. Therefore, the first non-trivial extension of this symmetry under which the field strength transforms consists of the anti-symmetric component $\delta A_\mu = b_{\mu\nu} x^\nu + \dots$, where the dots indicate possible field-dependent terms. There are similar structures at higher powers of the coordinates that involve mixed symmetry tensors. However, these always require the two-form generator $B_{\mu\nu}$ to be included as well, and moreover the transformations up to and including first order generators always form a subalgebra. [78] It therefore suffices to investigate the implications of this algebra.

It was shown in [78] that it is impossible to have non-vanishing commutation relations between non-linear generators of this subalgebra. Therefore, there are no exceptional EFTs for a single gauge vector at this level, as was also found from an amplitude perspective in [2]. The only non-trivial possibility beyond gauge symmetry is therefore to have the field-independent transformation $\delta A_\mu = b_{\mu\nu} x^\nu$, which can be seen as the vector analogon of the scalar Galileon transformations. However, there are no corresponding interactions which do not introduce additional ghostly degrees of freedom. [171]

Discarding gauge symmetry, there are two interesting algebras that we would like to discuss. The first one only includes the symmetric and traceless non-linear generators $Z_{\mu\nu}$ at first level in the tree. This generator content allows for an exceptional algebra that is given by the complexified version of Poincaré i.e. $\mathcal{C}^4 \ltimes SU(4)$, see also [103]. This is the isometry group of an eight-dimensional ambient space with two invariant tensors, a Minkowski metric and a complex structure. The pull-back of these invariant tensors onto the worldvolume theory yields the covariant tensors

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu A_\rho \partial_\nu A^\rho \,, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \,, \tag{4.44}$$

which at the one-derivative-per-field level are the only building blocks for interactions. With these building blocks, however, the theory will (possibly unsurprisingly) propagate a ghost. However, this can be remedied by including a central extension of the algebra: one can add a scalar generator on the RHS of the commutator of the vector generator with translations. In other words, one can inverse Higgs the vector itself in terms of an essential scalar. The resulting symmetry breaking pattern is identical to that of the Special Galileon, [103] and so is the theory: due to the central extension one can add Wess-Zumino terms that give rise to a healthy kinetic term for the essential scalar Goldstone.

This construction therefore leads to a known theory, but sheds light on its symmetry and geometric interpretation: the Special Galileon algebra is a central extension of complexified Poincaré. Moreover, in the process we have introduced a vector that can be retained as a matter field with a specific transformation under the Special Galileon transformations. When writing the above vector as $A_\mu = \partial_\mu\phi + m\tilde{A}_\mu$ using the Stuckelberg trick, we introduce a gauge symmetry of the form

$$\delta\phi = -m\lambda\,, \qquad \delta\tilde{A}_\mu = \partial_\mu\lambda\,. \tag{4.45}$$

Moreover, the covariant tensors g and F depend on both fields in a specific way. In the limit $m \to 0$, these become (after an overall rescaling of the field strength)

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\partial_\rho\phi\partial_\nu\partial^\rho\phi\,, \qquad F_{\mu\nu} = \partial_{[\mu}\tilde{A}_{\nu]}\,. \tag{4.46}$$

Both are covariant tensors under the non-linear transformations. The transformation rules generated by $Z_{\mu\nu}$ read (up to a specific gauge transformation)[13]

$$\delta\phi = z^{\mu\nu}(x_\mu x_\nu + \partial_\mu\phi\partial_\nu\phi)\,, \qquad \delta\tilde{A}_\rho = 2z^{\mu\nu}(\partial_\mu\phi\partial_\nu\tilde{A}_\rho + \tilde{A}_\mu\partial_\nu\partial_\rho\phi)\,. \tag{4.48}$$

Note that the gauge vector forms a linear representation of the highest order generator of the essential scalar's inverse Higgs tree (it does not transform

[13]Note the close similarity to the DBI scalar coupled to a gauge vector, with transformations [?, ?]

$$\delta\phi = y^\mu(x_\mu + \phi\partial_\mu\phi)\,, \qquad \delta A_\mu = y^\nu(\phi\partial_\nu A_\mu + A_\nu\partial_\mu\phi)\,. \tag{4.47}$$

The gauge vector forms a linear representation of the vector generator of the scalar algebra (which is again the generator which sits at the highest level in the inverse Higgs tree) and requires specific couplings to the DBI scalar. The special pedigree of this theory can be seen from e.g. its higher-dimensional origin, its possible supersymmetrization and, given the present discussion, also from the perspective of soft limits, which would be $\sigma = 2$ and $\sigma = 0$ for the scalar and vector, respectively.

under the scalar and vector generators which sit at level 0 and 1 respectively). We therefore have a field content consisting of a scalar that is a Special Galileon Goldstone, and a gauge vector that transforms as a matter field. These symmetries require specific couplings between the fields in the resulting EFT: these have to be constructed from the above, and therefore are e.g. at lowest order given by

$$\sqrt{-g}g^{\mu\nu}F_{\nu\rho}g^{\rho\sigma}F_{\sigma\mu}\,. \tag{4.49}$$

Due to the non-linear symmetry, these must have soft limits of $\sigma = 3$ and $\sigma = 0$ for the scalar and vector, respectively. Interactions of this kind with exactly these soft limits were recently proposed in [2]. Therefore we expect that the above symmetry for the vector explains the non-trivial couplings found in that work.

To close the vector discussion, there is an alternative exceptional algebra when including only the anti-symmetric generator instead at the first level. This has an analogous geometric interpretation, in this case with two metric structures in the eight-dimensional space-time that break the isometries down to a double copy of four-dimensional Poincaré. [78] The covariant objects are $g_{\mu\nu}$ and the symmetric combination of $\partial_\mu A_\nu$. While this theory propagates a ghost (and does not allow for gauge symmetry or a central extension), it might be related to a sensible EFT by including a cosmological constant: in AdS space-time there is a theory of a massive vector with a double copy of the AdS isometries as non-linear symmetries, which arises in a specific decoupling limit of massive gravity. [81, 172] It would be very interesting to investigate the relations of this theory to its flat space-time counterpart discussed here.

Chapter 5

Exceptional EFTs with $\mathcal{N} = 1$ supersymmetry

5.1 Extended shift symmetries and SUSY IHCs

In Chapter 3, we saw how non-linearly realized space-time symmetries can evade Goldstone's theorem. The Goldstone bosons of independent broken symmetries may be related to each other if the symmetry transformations are locally degenerate. In 3.3.1, we explained how these statements carry over to supersymmetric theories. A priori, each broken symmetry generator G^i induces a massless mode in superspace, parametrized by a Goldstone superfield $\Phi^i(x, \theta, \bar{\theta})$. However, these modes are again related to each other if the symmetry transformations are locally degenerate in superspace. In 4.1, we extended this idea order by order for Poincaré-invariant theories, giving rise to *inverse Higgs trees*. We will now derive a similar generalization for $\mathcal{N} = 1$ supersymmetry in $D = 4$. Just like ordinary inverse Higgs trees are identical to Taylor expansions of the essential field, *superspace inverse Higgs trees* coincide with superspace expansions.

Let us begin as we did in 3.3.1, by looking for solutions to the equation:

$$\Phi^i(x, \theta, \bar{\theta}) G_i |0\rangle = 0 \,. \tag{5.1}$$

We found a first-order degeneracy by acting with the operator $e^{-U} d e^U$, which combines space-time and superspace derivatives and translation operators in a covariant manner. The operator U is defined as $U = i(x^a P_a + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})$ and the exterior derivative is:

$$d = e^a \partial_a + e^\alpha D_\alpha + e_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \,. \tag{5.2}$$

We then obtained (3.97), whose solution reads (3.100). We repeat the definition of the structure constants for convenience:

$$
\begin{aligned}
&[P_{\alpha\dot\alpha}, G_i] = -i f_{\alpha\dot\alpha i}{}^j G_j + \dots , \quad [Q_\alpha, G_i]_\pm = i f_{\alpha i}{}^j G_j + \dots , \\
&[\bar Q_{\dot\alpha}, G_i]_\pm = i f_{\dot\alpha i}{}^j G_j + \dots . \qquad (5.3)
\end{aligned}
$$

The equations (3.100) project out the following linear combinations of Goldstone superfields, in terms of superspace derivatives of Φ^i: $f_{\alpha\dot\alpha j}{}^i \Phi^j$, $f_{\alpha j}{}^i \Phi^j$, and $f^{\dot\alpha}{}_j{}^i \Phi^j$. As expected, these are just sums over all generators which contain G_i in their commutation relations with (respectively) space-time translations $P_{\alpha\dot\alpha}$, superspace translations Q_α, and their complex conjugate $\bar Q_{\dot\alpha}$.

Let us now pick a generator $G^{(0)}$, which satisfies:

$$
[P_{\alpha\dot\alpha}, G^{(0)}] = [Q_\alpha, G^{(0)}] = [\bar Q_{\dot\alpha}, G^{(0)}] = \dots , \qquad (5.4)
$$

where the ellipses, as usual, stand in for any possible unbroken realized symmetries (in this case: supersymmetry, Poincaré or some internal symmetry). Similarly to cases which are just Poincaré-invariant, the Goldstone superfield $\Phi^0(x, \theta, \bar\theta)$ can never be eliminated by inverse Higgs conditions and is therefore essential. In this Chapter, we will also use the terms *P-(in)essential* and *Q-(in)essential*. A P- or Q-inessential Goldstone satisfies the weaker requirement that its (anti)-commutation relation with $P_{\alpha\dot\alpha}$ or Q_α vanishes, modulo unbroken generators.

Acting with the operator $e^{-U} d e^U$ twice, and projecting the result onto $G^{(0)}|0\rangle$, we obtain a number of consistency conditions:

$$
\begin{aligned}
&D_\alpha \Phi^0 = f_{\alpha i}{}^0 \Phi^i , \qquad \bar D_{\dot\alpha} \Phi^0 = f_{\dot\alpha i}{}^0 \Phi^i , \\
&\partial_{\alpha\dot\alpha} \Phi^0 = f_{\alpha\dot\alpha i}{}^0 \Phi^i , \\
&D_\alpha D_\beta \Phi^0 = f_{\alpha i}{}^0 f_{\beta j}{}^i \Phi^j , \quad D_\alpha \bar D_{\dot\beta} \Phi^0 = f_{\alpha i}{}^0 f_{\dot\beta j}{}^i \Phi^j , \\
&\bar D_{\dot\alpha} D_\beta \Phi^0 = f_{\dot\alpha i}{}^0 f_{\beta j}{}^i \Phi^j , \quad \bar D_{\dot\alpha} \bar D_{\dot\beta} \Phi^0 = f_{\dot\alpha i}{}^0 f_{\dot\beta j}{}^i \Phi^j , \qquad (5.5)
\end{aligned}
$$

All of these equations are understood to hold to first order in the Goldstone fields Φ^i. For the non-linear completion, we have to refer to the coset construction.

The structure of the equations (5.5) is the same as (4.2). For example, let us focus on the equation involving $D_\alpha D_\beta \Phi^0$. On the right-hand side, we find the linear combination of Goldstone superfields $f_{\alpha i}{}^0 f_{\beta j}{}^i \Phi^j$. These Goldstones correspond to the following sum of generators:

$$
\sum_i G_i \, [Q_\alpha, G_i]|_{G_j} \, f_{\beta j}{}^0 = G_i \, f_{\alpha i}{}^j f_{\beta j}{}^0 . \qquad (5.6)
$$

This is a sum of generators G^i whose (anti)-commutation relation with Q_α yields a generator G^j: $[Q_\alpha, G_i] \supset G_j$. In turn, the G_j are the generators which yield $G^{(0)}$ when taking a bracket with Q_β. Again, the structure of a tree appears by repeatedly applying translation operators and derivatives. The same holds for the other equations (5.5). At each order n, we may project out - by means of the derivatives D, $\bar{D}$ and ∂ - a linear combination of generators whose bracket with (super)translations yields order $(n-1)$ generators.

There are many consistency conditions on the structure constants in (5.5) that result from the superspace derivative algebra:

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \quad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}}\,. \tag{5.7}$$

For example, the fact that the unbarred derivatives anti-commute tells us that the right-hand side of the corresponding equation in (5.5) is anti-symmetric under the exchange $(\alpha \leftrightarrow \beta)$:

$$f_{(\alpha i}{}^{0} f_{\beta) j}{}^{i} = 0\,. \tag{5.8}$$

The same sort of condition holds for the barred derivatives. Similarly, we can use $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}}$ to relate the third, fifth and sixth equations in (5.5) to each other. These conditions can also be derived from (super)-Jacobi identities. For example, the condition (5.8) also follows from the Jacobi identity:

$$[Q_\alpha, [Q_\beta, G_i]] + (-1)^{F_i}[Q_\beta, [G_i, Q_\alpha]] = 0\,, \tag{5.9}$$

where F_i is one or zero depending on whether G_i has even or odd grading. The square brackets here refer to the superbracket, i.e. they are anti-commutators when both of the generators have odd grading and commutators otherwise. The condition that follows from $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}}$ can similarly be derived from the Jacobi identity with the generators $(Q_\alpha, \bar{Q}_{\dot{\alpha}}, G_i)$.

Let us decompose into Lorentz irreps order-by-order and draw the inverse Higgs tree for a general (m, n) superfield, see figure 5.1. The solid blue lines indicate that the higher-order generator (i.e. lower on the picture) contains the lower-order one in its commutation relation with either Q or $\bar{Q}$. A dashed red line similarly indicates a commutation relation with $P_{\alpha\dot{\alpha}}$. In this figure, we have taken into account the (anti)-symmetry conditions from (5.5) or, equivalently, the Jacobi identities. This has, for example, reduced the degeneracy of spin-$\frac{m+n}{2}$ generators in the third line to 4, as we saw before in 4.1.

It should now be clear that each generator in the inverse Higgs tree can be identified with a term in the perturbative expansion of the essential (m, n)

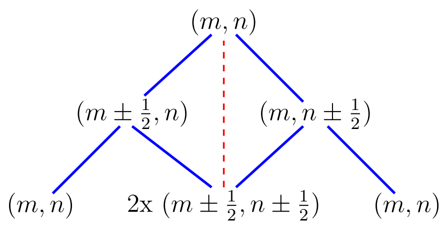

Figure 5.1: *The generators that can be non-linearly realized on a generic (m, n) supermultiplet thanks to superspace inverse Higgs constraints, and their relations under superspace and space-time translations. The solid blue lines heading north-west and north-east denote connections by $\bar{Q}$ and Q respectively while the red dashed lines denote connections by space-time translations.*

superfield $\Phi(x)$. The expansion is in powers of the space-time coordinates x^μ as well as the superspace coordinates θ, $\bar{\theta}$. In the following, we will often say that the inverse Higgs tree is identical to a *superspace expansion* of the essential superfield.

Just like we were able to do for essential scalar fields and Weyl fermions in Chapter 4, we can - in all cases that we will study - rearrange basis in our algebra so that the (super)translation brackets have a strict ordering. We assign each generator a (half-)integer label $G^{(i)/2}$. The zeroth-order generator $G^{(0)}$ is associated to the essential superfield $\Phi(x, \theta, \bar{\theta})$. Then, we obtain:

$$
\begin{aligned}
&[Q_\alpha, G^{(i)/2}] = G^{(i-1)/2} + \dots, \quad [\bar{Q}_{\dot{\alpha}}, G^{(i)/2}] = G^{(i-1)/2} + \dots, \\
&[P_{\alpha\dot{\alpha}}, G^{(i)/2}] = G^{(i-2)/2} + \dots, \qquad (5.10)
\end{aligned}
$$

where $[,]$ is the Lie superbracket. These statements are true under the same assumptions we made in Chapter 4: existence of an essential Goldstone field and finiteness of the superalgebra.

We note that the space-time translations $P_{\alpha\dot{\alpha}}$ move up the tree two steps. This is because the ordering is dictated by the Q and $\bar{Q}$ operators. If a generator G_i is connected to G_j by translations, i.e. $[P_\mu, G_i] \supset G_j$, we must also have that $[Q, [\bar{Q}, G_i]] \supset G_j$. This is what we concluded from consistency conditions (5.5) and from the Jacobi identity involving $(Q, \bar{Q}, G_i)$. The fact that space-time translations move up two levels implies, for example, that the Goldstone superfields $\Phi^{(1/2)}$ of $G^{(1/2)}$ are P-essential. From Chapter 3, we know that there must be a physical Goldstone boson/fermion $\phi^{(1/2)}$ associated to $G^{(1/2)}$. This is the lowest component of the $\Phi^{(1/2)}$. Then, $\phi^{(1/2)}$ is identified with the θ or $\bar{\theta}$ component of the essential superfield $\Phi(x, \theta, \bar{\theta})$ by the first-order superspace inverse Higgs relations.

Let us see what the structure of inverse Higgs relations implies for the transformation laws realized by the essential superfield. In the coset construction, the transformation law of each Goldstone under its own generator always starts with a constant shift. This is true for ordinary Goldstone spacetime fields and carries over to supersymmetric non-linear realizations. We therefore find:

$$\delta_{G^{(n)}} \Phi^{(n)}(x, \theta, \bar{\theta}) = \epsilon^n + \dots\,, \tag{5.11}$$

where the ellipses contain all field-dependent terms. Assume that we have $[Q_\alpha, G^{(n)/2}] = G^{(n-1)/2} + \dots$. Then, the degeneracy condition implies, schematically: $D_\alpha \Phi^{(n-1)/2} = \Phi^{(n)/2}$. Clearly this implies that $\Phi^{(n-1)/2}$ transforms with a constant shift at order θ^1:

$$\delta_{G^{(n)/2}} \Phi^{(n-1)/2}(x, \theta, \bar{\theta}) = \theta\epsilon^n + \dots\,, \tag{5.12}$$

We can extend this idea to the entire inverse Higgs tree. Each time we connect a new generator by Q_α, the essential superfield obtains a transformation law which starts with a shift at one higher power of θ^α, and similarly for $\bar{Q}$. Of course, there is no θ^3 or higher in $\mathcal{N} = 1$ supersymmetry, so we have to be careful. We can never build up the inverse Higgs tree with three subsequent Q_α relations. Rather, at least one $\bar{Q}$ relation must sit in between. Using the derivative algebra $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}}$, we see that the essential Goldstone then obtains a shift at one higher power of x^μ. We note that everything we say in this paragraph is essentially a restatement of what we learned from Jacobi identities and the consistency conditions on (5.5).

5.1.1 Covariant irreducibility conditions

In the last section, we examined superspace inverse Higgs trees for generic (m, n) supermultiplets. It is more interesting to study *irreducible* supermultiplets. We can obtain an irreducible supermultiplet from a generic one by imposing constraints that are covariant with respect to all the symmetries of the theory. An example is the chiral supermultiplet $\Phi(x, \theta, \bar{\theta})$, which satisfies the condition:

$$\bar{D}_{\dot{\alpha}} \Phi = 0\,. \tag{5.13}$$

The equations (5.5) need to be compatible with this irreducibility condition, as do the transformation laws (5.12). This dramatically reduces the generator content that can be realized on the essential superfield, as we will see in 5.3.1 and beyond.

In Chapter 2, we explained that irreducibility conditions are modified in the presence of non-linear symmetries. The superspace derivatives $(D_\alpha, \bar{D}_{\dot{\alpha}})$ are transformed by the coset construction to new operators $(\hat{D}_\alpha, \hat{\bar{D}}_{\dot{\alpha}})$. We must use these objects to impose irreducibility rather than the ordinary superspace derivatives. A generic constraint equation must take the following form:

$$T_{\alpha_1\ldots\dot{\alpha}_1\ldots}(\hat{\partial}\Phi, \hat{D}\Phi, \hat{\bar{D}}\Phi, \hat{D}^2\Phi, \hat{\bar{D}}^2\Phi, \ldots) = 0\,, \tag{5.14}$$

i.e. T is some tensor built out of the covariant derivatives from the coset construction.

In 3.4.3, we already saw that it is not always easy to find conditions that are equivalent to a canonical constraint of the type (5.13). There are often many covariant candidate terms that could appear in T. By imposing inverse Higgs conditions, however, we constrain some of these candidates to vanish. Thus, if we struggle to find irreducibility conditions for a particular symmetry breaking pattern G/H, we can try to extend the algebra to a higher order one G' in the inverse Higgs tree. The extended algebra allows for fewer independent candidate terms, so the problem of finding the right conditions simplifies. Then, we can rewrite the constraint equation obtained from G'/H in terms of G/H covariant derivatives.

This procedure will work as long as a realization of an extended algebra exists, because the canonical constraint equation has hidden covariance under all symmetries that can be realized on the supermultiplet. By using covariant derivatives of the extended algebra, we simply make this covariance manifest. We encountered an example of this idea in 3.4.3: the irreducibility conditions for partial breaking of $\mathcal{N} = 2$ on a Maxwell superfield are automatically symmetric under shifts of the auxiliary scalar field. We will comment further on irreducibility conditions case-by-case in the sections 5.3.1 to 5.3.3.

5.1.2 Canonical propagators

In Chapter 4, we required that the coset construction provide a canonical kinetic term for all propagating fields in the theory. As we want to continue to work only with theories that have a sensible perturbation theory, we will make the same assumption in this Chapter. In addition to propagating fields, off-shell supersymmetry multiplets contain auxiliary fields. We are interested in symmetries that can be realized on physical on-shell supermultiplets, so the auxiliary fields need to retain algebraic equations of motion. Furthermore, the field equations must contain a term linear in the auxiliary field, as otherwise they will impose an on-shell condition on the propagating fields.

Together, these conditions imply that the non-linearly realized symmetry must be compatible with the canonical *superspace* kinetic term for the superfield. In the next sections, we will study irreducible chiral, Maxwell and real linear superfields. Their superspace kinetic terms are:

- $\mathcal{L}_{free} = \int d^4\theta\, \Phi\bar{\Phi}$ for the chiral superfield,
- $\mathcal{L}_{free} = \int d^4\theta\, W^\alpha W_\alpha$ for the Maxwell superfield,
- $\mathcal{L}_{free} = \int d^4\theta\, L^2$ for the real linear superfield.

In Chapter 4, we saw that assuming a canonical kinetic term reduced the inverse Higgs tree to a single generator at each order. For supermultiplets, we will obtain a similar result. There remains a single generator at each order, dictated by Q and $\bar{Q}$. We will show this on a case-by-case basis starting in section 5.3.1.

The canonical superspace kinetic term for the chiral superfield contains at the component level a term proportional to $F\bar{F}$. Clearly, this term forbids a shift symmetry of the form $\delta F = c + \ldots$ if it is the term of lowest order in F. However, some of the algebras we have studied do contain generators that induce a shift of the auxiliary field. For example, this happened in the partial breaking of $\mathcal{N} = 2$ on a chiral superfield, see section 3.4.3. In that particular case it is consistent to plug the F field equations back into the action, even though the action does not contain the canonical term proportional to F^2.

This does not mean that we are missing any interesting EFTs by not including generators of auxiliary field shifts. On-shell, the shift symmetry is broken explicitly. Therefore, we can just as well describe the physical theory by a symmetry breaking pattern that does not include the generator that shifts F. This is only possible if the F-shift corresponds to an automorphism of the algebra. Conversely, if it is not an automorphism, one of two things must happen: either F becomes a propagating field, or the entire coset G/H is broken explicity by its field equations. Our classification will not miss any well-defined EFTs even if we do not take into account all such automorphisms. However, it is sometimes useful to include them, for example to simplify the search for irreducibility conditions.

5.2 Supersymmetric soft bootstrap

5.3 Classification of exceptional EFTs

It is time to present the classification of $\mathcal{N} = 1$ supersymmetric Goldstone EFTs. We will consider theories with an essential chiral, Maxwell, or real linear Goldstone superfield in sections 5.3.1, 5.3.2, and 5.3.3 respectively. In contrast to Chapter 4, we will always assume here that there is a single physical Goldstone superfield. The results of the algebraic classification are fully exhaustive - up to arbitrary finite order in the superspace inverse Higgs trees - for the chiral and Maxwell superfields. In case of the real linear superfield, we will be exhaustive only in our search for EFTs where the 2-form field can be dualized to a pseudoscalar. In the following sections, we will first discuss irreducibility conditions and then fix the superspace inverse Higgs trees for the essential superfield at hand. Similarly to Chapter 4, we then classify the possible Goldstone EFTs simply by demanding closure of the algebra with Jacobi identities. Along the way, we compare the results of the algebraic classification to those of the complementary soft bootstrap approach.

5.3.1 Chiral Goldstone superfields

Irreducibility conditions

The chiral superfield of $\mathcal{N} = 1$ supersymmetry is a complex scalar function of superspace $\Phi(x, \theta, \bar{\theta})$ which satisfies the condition $\bar{D}_{\dot{\alpha}}\Phi = 0$. The solution to this constraint can be written as follows:

$$\Phi(x, \theta, \bar{\theta}) = \Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\chi(y) + \theta^2 F(y)\,, \tag{5.15}$$

where $y_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} - 2i\theta_\alpha\bar{\theta}_{\dot{\alpha}}$. The physical fields that reside in the chiral superfield are thus a complex scalar $\phi(x)$, a Weyl fermion $\chi(x)$ and a complex scalar auxiliary field $F(x)$. In order to obtain a chiral superfield from the coset construction, we start with a zeroth-order complex scalar generator Z which satisfies:

$$[Q, Z] = [\bar{Q}, Z] = [P, Z] = \dots\,, \tag{5.16}$$

i.e. the brackets of Z with all supertranslation generators give rise to only unbroken symmetries. Then, we assign to Z the complex scalar Goldstone superfield Φ in the coset element. In the preferred exponential parametrization, we get:

$$g = e^{i\Phi(x,\theta,\bar{\theta})Z} e^{iG^{(1/2)}\Phi_{(1/2)}} \dots\,, \tag{5.17}$$

where the ellipses contain separate exponential factors for all higher-order generators. The physical EFT is obtained after imposing the relevant inverse Higgs relations and a chirality condition on Φ.

The chirality condition is modified from its canonical form by the non-linear symmetries. Because the correct constraint should remain a dotted spinor equation, the most general form it can take is the following:

$$T_{\dot{\alpha}\dot{\beta}}(\hat{D}\Phi, \hat{\bar{D}}\Phi, \hat{\partial}\Phi, \dots)\hat{D}^{\dot{\beta}}\Phi = 0\,. \tag{5.18}$$

The aim is to find a constraint of this type that is equivalent to the ordinary chirality condition, in the sense that there exists a redefinition that maps Φ - which is subject to the constraint (5.18) - to an ordinary chiral superfield Ψ. To be more precise, this redefinition only needs to exist when the (5.18) is evaluated on the solution of the inverse Higgs relations.

It is clear that all superfield configurations which satisfy the simpler constraint $\hat{\bar{D}}_{\dot{\alpha}}\Phi = 0$ are also solutions to (5.18). There can be further solutions if $T_{\dot{\alpha}\dot{\beta}}$ is a singular matrix. However, then Φ must be subject to an independent scalar constraint $\det(T_{\dot{\alpha}\dot{\beta}}) = 0$. Therefore, (5.18) can only be equivalent to the canonical chirality conditon $\bar{D}_{\dot{\alpha}}\Phi = 0$ if $T_{\dot{\alpha}\dot{\beta}}$ is invertible. In other words, we can replace the constraint with the minimal generalization of the ordinary chirality condition:

$$\hat{\bar{D}}_{\dot{\alpha}}\Phi = 0\,. \tag{5.19}$$

We cannot prove in generality that (5.19) is equivalent to the canonical chirality condition for all symmetry breaking patterns G/H. We have merely shown that if a consistent covariant constraint equation exists, it must be (5.19). We emphasize that in general the constraint equations will not be a simple covariantization of the canonical constraint. This is a special feature of the chiral superfield, essentially due to representation theory of the Lorentz group.

Superspace inverse Higgs tree

Let us investigate the consequence of the irreducibility condition (5.19) on the superspace inverse Higgs tree. At level $n = 1/2$ there are, a priori, two independent Weyl fermion generators χ_α and $\bar{\xi}^{\dot{\alpha}}$ compatible with the essential complex scalar. The relevant anti-commutation relations are:[1]

$$\{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta}Z + \dots\,, \quad \{\bar{Q}^{\dot{\alpha}}, \bar{\xi}^{\dot{\beta}}\} = 2\epsilon^{\dot{\alpha}\dot{\beta}}Z + \dots\,, \tag{5.20}$$

The inverse Higgs relation that projects out the Goldstone superfield of $\bar{\xi}$ would exactly coincide with (5.19). This constraint cannot, at the same time,

[1]Note that the full generator Z - rather than just its real $(Z + \bar{Z})$ or imaginary part $(Z - \bar{Z})$ - appears in these brackets. In case only a real (imaginary) part appears, the imaginary (real) part of Z will detach from the inverse Higgs tree. It will then induce only a constant shift symmetry on the imaginary (real) part of the scalar component field in Φ. In our view, such EFTs are naturally defined on the real linear supermultiplet rather than the chiral, so we will consider them in 5.3.3.

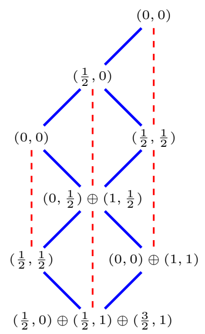

Figure 5.2: *The non-linear generators that can be realized on a chiral supermultiplet (left) and the subset that is consistent with canonical propagators (right).*

project out an inessential Goldstone and impose chirality on Φ. Therefore, we cannot include the generator $\bar{\xi}$ in the algebra. At level $n = 1/2$, we find only a single $(\frac{1}{2}, 0)$ Weyl fermion, see figure 5.2. We can also conclude this from the fact that both S and $\bar{\xi}$ are P-essential due to Jacobi identities. Therefore, any EFT with both of these symmetries requires at least two propagating Weyl fermions. As these cannot be accommodated by a single chiral superfield, one of the $n = 1/2$ Weyl fermions must be excluded.

In the previous sections, we have demonstrated that Jacobi identities imply many interrelations between (super)translation brackets. Excluding the generator $\bar{\xi}$ therefore has important implications at all orders in the inverse Higgs tree. Every generator that would be connected to $\bar{\xi}$ by a blue line in a figure such as 5.1 must be excluded.

At order $n = 1$, Lorentz invariance allows for a complex scalar $(0, 0)$, a complex vector $(\frac{1}{2}, \frac{1}{2})$ and a 2-form $(1, 0)$. These generators attach to S_α by Q_α, $\bar{Q}_{\dot{\alpha}}$ and Q_α respectively. A 2-form, however, does not appear in the Taylor expansion of any of the physical fields, so it cannot be included. Equivalently, it is ruled out at the level of the algebra by the Jacobi identity involving (Q, Q, B), where $B_{\alpha\beta}$ denotes the would-be 2-form generator. The relevant brackets are:

$$[Q_\alpha, R] = S_\alpha + \ldots, \quad [P_{\alpha\dot{\alpha}}, G_{\beta\dot{\beta}}] = i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \ldots, \quad [\bar{Q}_{\dot{\alpha}}, G_{\beta\dot{\beta}}] = i\epsilon_{\dot{\alpha}\dot{\beta}} S_\beta + \ldots . \tag{5.21}$$

These commutation relations are depicted in figure 5.2 on the third level of the left-hand side. The commutation relation of the vector $G_{\beta\dot{\beta}}$ with space-time translations is fixed by the Jacobi identity $(Q, \bar{Q}, G)$.

There are further Jacobi identities at higher order in the inverse Higgs tree. At order $n = \frac{3}{2}$ and beyond, the Jacobi identities with (Q, P, G) and $(\bar{Q}, P, G)$ impose constraints. From $n > 2$, we also have (P, P, G). In the end, one is left with all the generators that appear in Taylor expansions of the component fields in the chiral superfield, including the auxiliary field F. We depict the result up to order $n = \frac{5}{2}$ on the left of figure 5.2.

Canonical propagators

Let us see which generators are compatible with the canonical superspace kinetic term $\int d^4\theta\, \Phi\bar{\Phi}$. We argued in the previous section that a shift symmetry R of the highest component either requires a propagating field F, or R constitutes an automorphism generator. The former scenario is outside the scope of our classification and in the latter we can make sense of the EFT without explicitly including R. In figure 5.2, R is represented by the $(0, 0)$ on the left at level $n = 1$. We will discard this generator, so that we only have a complex $(\frac{1}{2}, \frac{1}{2})$ level $n = 1$, as seen on the right side of figure 4.2.

Furthermore, we know from Chapter 4 that only fully symmetric and traceless generators are compatible with the Klein-Gordon equation. Therefore, any generator connected to the essential $(0, 0)$ by a dashed red line (representing a commutation relation with space-time translations) must be a fully symmetric and traceless Lorentz irrep. A similar statement holds for the P-essential $(\frac{1}{2}, 0)$ generator at $n = 1/2$. In the end, the superspace inverse Higgs tree reduces to the right-hand side of figure 5.2. This figure represents the following general (anti)-commutation relations:

$$\begin{aligned}
\{Q_\gamma, S_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}}\} &= 2\epsilon_{\gamma\alpha_1} G_{\alpha_2\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}} + \ldots\,, \\
[\bar{Q}_{\dot{\gamma}}, G_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_N}] &= i\epsilon_{\dot{\gamma}\dot{\alpha}_1} S_{\alpha_1\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_N} + \ldots\,, \\
[P_{\gamma\dot{\gamma}}, S_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}}] &= i\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1} S_{\alpha_2\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_{N-1}} + \ldots\,, \\
[P_{\gamma\dot{\gamma}}, G_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_N}] &= i\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1} G_{\alpha_2\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_N} + \ldots\,.
\end{aligned} \tag{5.22}$$

In these expressions, a spin $(\frac{N}{2}, N - 12)$ fermionic (i.e. odd) generator is denoted $S_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}}$. A spin $(\frac{N}{2}, \frac{N}{2})$ bosonic generator is labeled by $G_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_N}$. Furthermore, we assume symmetry under exchange of any pair of α_i or $\dot{\alpha}_i$ indices.

Inverse Higgs tree and soft weights

We want to emphasize once more the relation between the inverse Higgs tree and the on-shell soft weights of the amplitudes. The coset construction generates precisely the transformation law that give rise to the Adler zero and possibly enhanced soft limits. We can read off the soft weight for the fermion and boson from the inverse Higgs tree. We can end the tree either at integer or half-integer order n. If n is integer, the scalar component field in the chiral multiplet realizes an order n extended shift symmetry, giving rise to soft weight $n + 1$. At the same time, the fermion has a soft weight one lower than the fermion, equal to n, because the superspace inverse Higgs tree alternates even and odd generators. If n is half-integer, both the Weyl fermion and the scalar realize an order n extended shift symmetry, giving rise to a soft weight $n + 1/2$ for both physical fields. In other words, we find that:

$$\begin{aligned} &\sigma_\phi = \sigma_\chi = n + 1/2\,, && \text{for half-integer } n\,, \\ &\sigma_\phi = \sigma_\chi + 1 = n + 1\,, && \text{for integer } n\,. \end{aligned} \tag{5.23}$$

This relation between the fermionic and bosonic soft weights was also derived in [9] using amplitudes methods.

Exceptional EFTs

We are now in a position where we can perform an exhaustive analysis of the possible algebras which can be non-linearly realized by the single chiral superfield. We remind the reader that the superspace inverse Higgs tree is merely a necessary structure to i) reduce the EFT to the single chiral superfield by incorporating the necessary superspace inverse Higgs constraints and ii) satisfy Jacobi identities involving two copies of (super)-translations, up to the presence of linear generators. If there are no linear generators on the RHS of commutators between (super)-translations and a non-linear generator, and all commutators between a pair of non-linear generators vanish, then all Jacobi identities have been satisfied. Algebras of this type were discussed in the introduction; they lead to extended shift symmetries for each component field. However, these are very easy to construct and indeed always exist at every level in the tree. We will be primarily interested in the other type of possible algebras where transformation rules can be field-dependent, thereby leading to exceptional EFTs.

$n = 0$

We begin with the most simple case: $n = 0$ without any additional generators. Given our above discussion on soft limits, here the complex scalar will have $\sigma_\phi = 1$ behaviour while the fermion has $\sigma_\chi = 0$. The fermion can therefore be seen as a matter field whose presence is only required to maintain linear SUSY. This of course includes the case where G commutes with all other generators thereby simply generating a constant shift on the complex scalar component ϕ. This leads to supersymmetric $P(X)$ theories [159]. Just as a standard $P(X)$ theory is the most simple Goldstone EFT one can write down arising when a global $U(1)$ symmetry is spontaneously broken, this is the most simple supersymmetric Goldstone EFT (in terms of algebras and symmetries that is; the leading order operators can be somewhat complicated [159]).

There are also slightly more complicated algebras at this level corresponding to supersymmetric non-linear sigma models characterised by the non-vanishing $[G, \bar{G}]$ commutator. In contrast to the purely shift symmetric case, the resulting EFTs can have field-dependent transformation rules and are therefore exceptional EFTs given our definition in this work. Indeed, the power counting in these theories is different to the naive expectation: even though we have $\sigma_\phi = 1$, the complex scalar can enter the action with fewer than one derivative per field. A simple example is the two-derivative action, which can be interpreted as a metric on the two-dimensional manifold spanned by the components of the scalar field. The non-linear generators G and $\bar{G}$ imply that this manifold has two transitively acting isometries. The only such manifolds are the maximally symmetric ones, i.e. the hyperbolic manifold $SU(1,1)/U(1)$ or the sphere $SO(3)/SO(2)$, which are well-known non-linear sigma models. We refer the reader to [9] and references therein for more details.

$n = 1/2$

We now consider the case where the tree terminates at $n = 1/2$ with a single additional non-linear generator S_α. The most general form of the commutators in addition to those of the linear realized super-Poincaré and the ones which define the Lorentz representation of the non-linear generators is

$$\begin{aligned}
&[P_{\alpha\dot{\alpha}}, G] = a_1 P_{\alpha\dot{\alpha}}, \quad [Q_\alpha, G] = a_2 Q_\alpha, \quad [\bar{Q}_{\dot{\alpha}}, G] = a_3 \bar{Q}_{\dot{\alpha}}, \\
&[P_{\alpha\dot{\alpha}}, S_\beta] = a_4 \epsilon_{\alpha\beta} \bar{Q}_{\dot{\alpha}}, \quad \{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta} G, + a_5 M_{\alpha\beta}, \\
&[G, \bar{G}] = a_6 G + a_7 \bar{G}, \quad [S_\alpha, G] = a_8 S_\alpha + a_9 Q_\alpha, \quad [\bar{S}_{\dot{\alpha}}, G] = a_{10} \bar{S}_{\dot{\alpha}} + a_{11} \bar{Q}_{\dot{\alpha}}, \\
&\{S_\alpha, S_\beta\} = a_{12} M_{\alpha\beta}, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = a_{13} P_{\alpha\dot{\alpha}}.
\end{aligned} \tag{5.24}$$

Note that we didn't allow for a commutator of the form $\{\bar{Q}_{\dot{\alpha}}, S_\alpha\} = a_{14} P_{\alpha\dot{\alpha}}$ since it can be set to zero by a change of basis. Now the Jacobi identities are very constraining, fixing all parameters to zero other than $a_{13} \equiv s$ which is unconstrained. If $s \neq 0$ we can set it to 2 by rescaling generators such that the algebra is that of $\mathcal{N} = 2$ SUSY augmented with the only inverse Higgs constraint[2]. In this case the component field χ takes the Volkov-Akulov (VA) form. [104] This is an exceptional algebra by virtue of having a non-vanishing commutator between non-linear generators. On the other hand, if $s = 0$ then S_α generates a constant shift on χ as studied in [160]. This is simply a contraction of the $s \neq 0$ algebra. In both cases G generates a constant shift on the complex scalar component field ϕ since by Jacobi identities G must commute with (super)-translations and with $\bar{G}$. We therefore have a shift symmetric complex scalar field coupled to either a VA or shift symmetric fermion field with the couplings fixed by linear SUSY. The soft weights at this level are $\sigma_\phi = \sigma_\chi = 1$. This discussion is unchanged if we add linear scalar generators[3]: they do not allow for additional exceptional algebras.

In terms of the low energy EFTs which can non-linearly realise these algebras, when $s = 2$ it is not clear if they are independent from those which sit at level $n = 1$ i.e. there could be symmetry enhancement. It was suggested in [155] that the symmetry is indeed enhanced to the case where the complex scalar has an additional symmetry but much more work is required to arrive at a definitive answer. However, for $s = 0$ there are invariants we can write down which do not exhibit symmetry enhancement. For example, the operator

$$\int d^4\theta \, \partial_{\alpha\dot{\alpha}}\Phi \partial_{\beta\dot{\beta}}\Phi \partial^{\alpha\dot{\alpha}}\bar{\Phi}\partial^{\beta\dot{\beta}}\bar{\Phi} \,, \tag{5.25}$$

for the chiral superfield Φ has a shift symmetry for its scalar and fermion components but does not exhibit enhancement to level $n = 1$.

$n = 1$

We now also include the complex vector $G_{\alpha\dot{\alpha}}$ taking us to level $n = 1$. Here the soft limits are $\sigma_\phi = 2$ and $\sigma_\chi = 1$. We play the same game as before: write down the most general commutators consistent with the superspace inverse Higgs tree and impose Jacobi identities to derive the algebras which can be

[2]We keep $s \geq 0$ to ensure positivity in Hilbert space. This is a necessary requirement in any linear realisations of the symmetry algebra, but not in non-linear realisations as the currents don't integrate into well-defined charges in the quantum theory. Here we still assume the requirement of positivity in Hilbert space. This is a reasonable assumption if one anticipates that the non-linear realisations have a (partial) UV completion to a linearly realized theory, or to be a particular limit of such a theory.

[3]Linearly realized scalar generators commute with the Poincaré factor but can appear on the RHS of the above commutators, can form their own sub-algebra and can have non-zero commutators with non-linear generators and super-translations.

non-linearly realized on the chiral superfield. This is a simple generalisation of the $n = 1/2$ case but since the full Ansatz for the commutators is quite involved, here we will just describe the results. As in the previous case, we allow for linear scalar generators which now turn out to be crucial in deriving exceptional algebras and EFTs. Note that in the Ansatz we do not allow for G or $\bar{G}$ to appear on the RHS of a commutator between a pair of non-linear generators which correspond to inessential Goldstones (S_α and $G_{\alpha\dot{\alpha}}$). This is necessary to ensure that the relevant superspace inverse Higgs constraints exists i.e. that the inessential Goldstones appear algebraically in the relevant covariant derivatives. We refer the reader to [141] for more details.

Given that in all cases the bosonic generators form a sub-algebra, we can use the results of Chapter 4 to fix these commutators. We refer the reader to [79] for more details but let us briefly outline the allowed structures. As in the $n = 1/2$ case, we find that the essential complex scalar cannot contain a component which transforms like a dilaton so the sub-algebra must correspond to that of the six-dimensional Poincaré group or contractions thereof. We can perform two distinct contractions thereby yielding three inequivalent algebras with their defining features the commutators between non-linear generators. The non-zero commutators which involve non-linear generators in the uncontracted six-dimensional Poincaré algebra are

$$[P_{\alpha\dot{\alpha}}, G_{\beta\dot{\beta}}] = i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}G, \quad [G_{\alpha\dot{\alpha}}, \bar{G}_{\beta\dot{\beta}}] = -i(\epsilon_{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}) + 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}M,$$
$$[\bar{G}, G_{\alpha\dot{\alpha}}] = 2iP_{\alpha\dot{\alpha}}, \quad [G, M] = G, \quad [G_{\alpha\dot{\alpha}}, M] = G_{\alpha\dot{\alpha}}, \tag{5.26}$$

where M is a real, linearly realized scalar generator. The non-linear realisation of this algebra is the two-scalar multi-DBI theory which has a neat probe brane interpretation. [161]

The obvious contraction we can do leads to the trivial algebra where all non-linear generators commute leaving only the commutators required by superspace inverse Higgs constraints (and the linearly realized bosonic sub-algebra). The low energy realisation of this algebra is that of bi-Galileons [162] and can be seen as taking the small-field limit for both components of the complex scalar. However, there is also a less obvious contraction we can perform where we retain non-vanishing commutators between non-linear generators. This contraction is somewhat difficult to understand in terms of these complex generators but is simple when using the more familiar generators P_A, M_{AB} where $A, B, \ldots$ are $SO(1,5)$ indices. In this case the linear scalar is $M_{45} \equiv M$ and the non-linear four-dimensional vectors are $M_{\mu 4} \equiv K_\mu$ and $M_{\mu 5} \equiv \hat{K}_\mu$, where μ is an $SO(1,3)$ index, which are related to the complex generators by

$$G = P_4 + iP_5, \quad G_{\alpha\dot{\alpha}} = K_{\alpha\dot{\alpha}} + i\hat{K}_{\alpha\dot{\alpha}}. \tag{5.27}$$

The relevant contraction corresponds to sending $P_5 \to \omega P_5, \hat{K}_\mu \to \omega \hat{K}_\mu$ and $M_{45} \to \omega M_{45}$ with $\omega \to \infty$. This contracted algebra is non-linearly realized by a DBI scalar coupled to a Galileon and can be seen as taking a small field limit for only one component of the complex scalar[4]. If we now switch back to the complex generators, since $[P_5, M_{\mu 5}] = 0$ we now have $[G, G_{\alpha\dot{\alpha}}] \neq 0$ in contrast to the fully uncontracted case. This will be important in what follows. We now take each of these sub-algebras in turn and ask which are consistent with linear SUSY and the required non-linear fermionic generator S_α.

If the bosonic sub-algebra is given by (5.26) then we find, perhaps unsurprisingly, that the most general algebra is that of six-dimensional super-Poincaré. In addition to the linearly realized super-Poincaré algebra and (5.26), the non-zero commutators are

$$\{Q_\alpha, S_\beta\} = 2\epsilon_{\alpha\beta} G, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \quad [Q_\alpha, \bar{G}_{\beta\dot{\beta}}] = i\epsilon_{\alpha\beta}\bar{S}_{\dot{\beta}},$$
$$[S_\alpha, \bar{G}_{\beta\dot{\beta}}] = -i\epsilon_{\alpha\beta}\bar{Q}_{\dot{\beta}}. \tag{5.28}$$

In the resulting low energy realisation, the complex scalar takes the multi-DBI form while the fermion takes the VA form. This theory has been very well studied in various contexts, see e.g. [155, 163].

If the bosonic algebra is the bi-Galileon one i.e. where the only non-vanishing commutators are those required by inverse Higgs constraints, we find that the supersymmetrisation also requires all commutators between non-linear generators to vanish. The only non-trivial commutators are therefore those required by superspace inverse Higgs constraints. This is simply a contraction of the six-dimensional Poincaré algebra and results in the six-dimensional supersymmetric Galileon algebra. Here the fermion is shift symmetric and a quartic Wess-Zumino interaction for this algebra was constructed in [166] (for more details see [9, 10, 160]). We presented the coset construction for this symmetry breaking pattern in 3.3.3.

Turning to the final bosonic sub-algebra, we find that it is impossible to supersymmetrise the theory of a DBI scalar coupled to a Galileon. Indeed, the Jacobi identities involving $(Q_\alpha, \bar{Q}_{\dot{\alpha}}, G_{\beta\dot{\beta}})$ and $(Q_\alpha, S_\beta, G_{\gamma\dot{\gamma}})$ fix $[G, G_{\alpha\dot{\alpha}}] = 0$ which is incompatible with this partly contracted algebra. We therefore conclude that there is only a single exceptional EFT for a chiral superfield with $\sigma_\phi = 2$, $\sigma_\chi = 1$ soft limits which is the VA-DBI system which non-linearly realizes the six-dimensional super-Poincaré algebra.

[4]This algebra also appeared in [77] and let us note that it is not clear if there exists a sensible realisation where both scalars have canonical kinetic terms. However, we will see in a moment that even if this theory existed, it cannot be supersymmetrised.

$n \geq 3/2$

When $n \geq 3/2$ we find that no exceptional EFTs are possible: the only non-trivial commutators are the ones required by superspace inverse Higgs constraints and lead to extended shift symmetries for the component fields. The situation for $n = 3/2$ is slightly different than for $n \geq 2$ so we will discuss these in turn but the results are qualitatively the same.

At $n = 3/2$, the bosonic sub-algebra must again be that of six-dimensional Poincaré, or contractions, since *i*) the fermionic generators do not allow for a dilaton as one component of the chiral superfield and *ii*) compared to $n = 1$ we haven't added any additional bosonic generators. However, we very quickly establish that this sub-algebra must be the fully contracted one i.e. both components of the complex scalar must transform as Galileons as opposed to DBI scalars.

To arrive at this conclusion we first use the $(P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}, S_{\gamma_1\gamma_2\dot{\gamma}})$ Jacobi identity to fix $[P_{\alpha\dot{\alpha}}, S_\beta] = 0$ and the $(P_{\alpha\dot{\alpha}}, S_\beta, \bar{S}_{\dot{\beta}})$ Jacobi identity to eliminate $G_{\alpha\dot{\alpha}}$ and $\bar{G}_{\alpha\dot{\alpha}}$ from the RHS of $\{S_\alpha, \bar{S}_{\dot{\alpha}}\}$. From the Jacobi identities involving two copies of (super)-translations and S_α we fix G to commute with all (super)-translations and remove the possibility of adding Lorentz generators to the RHS of $\{Q_\alpha, S_\beta\}$. The Jacobi identities involving one (super)-translation, G and either of the fermionic non-linear generators, and the $(Q_\alpha, S_\beta, \bar{S}_{\dot{\beta}})$ Jacobi, ensures that G commutes with these fermionic generators. From the $(G, Q_\alpha, S_{\beta_1\beta_2\dot{\beta}})$ and $(\bar{G}, Q_\alpha, S_{\beta_1\beta_2\dot{\beta}})$ Jacobi identities we then see that $[G, G_{\alpha\dot{\alpha}}] = [\bar{G}, G_{\alpha\dot{\alpha}}] = 0$ thereby telling us that the bosonic sub-algebra must be the fully contracted one. The remaining Jacobi identities tell us that all other commutators between non-linear generators must vanish leaving us with only extended shift symmetries. We have checked that this conclusion is unaltered if we allow for linear scalars generators beyond the one in the bosonic sub-algebra. So for $\sigma_\phi = \sigma_\chi = 2$ there are no exceptional EFTs.

The cases with $n \geq 2$ are slightly more straightforward given our results in Chapter 4. There we showed that if the essential Goldstone is a complex scalar, there are no exceptional EFTs with $\sigma_\phi \geq 3$. That is, if we include the $(\frac{1}{2}, \frac{1}{2})$ complex generator $G_{\alpha\dot{\alpha}}$ and the $(1, 1)$ complex generator $G_{\beta_1\beta_2\dot{\beta}_1\dot{\beta}_2}$, all non-linear generators must commute and give rise to only extended shift symmetries. In particular, there is no complex version of the Special Galileon, the algebra simply doesn't exist. Taking this as a starting point, we add the necessary superspace inverse Higgs commutators and use Jacobi identities to show that all non-linear generators, bosonic and fermionic, must commute amongst themselves. The calculation follows in a similar spirit to those described above and is valid for any finite $n \geq 2$.

Brief summary

Just like in Chapter 4, we have seen that exceptional EFTs are hard to come by: there are only a small number of non-linearly realized algebras which allow for field-dependent transformation rules on a chiral superfield. Here we summarise the main results of this section:

- The structure of the chiral superfield's superspace inverse Higgs tree tells us that the soft weights of the component fields are either equal or the complex scalar's can be one higher. The soft weights are fixed by the level of the inverse Higgs tree and given by (5.23).

- The most simple exceptional EFTs are non-linear sigma models characterised by $[G, \bar{G}] \neq 0$. Here the scalar has a $\sigma_\phi = 1$ soft weight whereas the fermion must have $\sigma_\chi = 0$. Indeed, whenever we include the generator S_α, which is necessary for $\sigma_\chi \geq 1$, we find $[G, \bar{G}] = 0$.

- In addition to non-linear sigma models, the only possible exceptional EFTs have $\sigma_\chi = 1$ and $\sigma_\phi = 1$ or 2. Even though an exceptional algebra exists at level $n = 1/2$, we expect that there is no realisation with the corresponding properties, i.e. all EFTs one can derive will actually realise the unique $n = 1$ exceptional algebra of six-dimensional super-Poincaré. The contraction of this algebra gives rise to supersymmetric Galileons.

- All other algebras, at any other finite level in the tree, lead to field-independent extended shift symmetries. In particular, when both parts of the complex scalar have equivalent inverse Higgs trees, it is impossible to realise superconformal algebras on the single chiral superfield. We will relax the assumption of equivalent inverse Higgs trees in section 5.3.3. Furthermore, one cannot supersymmetrise the Special Galileon, at least in four dimensions.

- For leading values of the soft weights our results are completely compatible with the on-shell approach of [9].

5.3.2 Maxwell Goldstone superfields

Irreducibility condition I

We now turn our attention to EFTs that contain a single Maxwell superfield $W_\alpha(x, \theta, \bar{\theta})$, also known as a *field strength multiplet*. W_α contains a Weyl fermion χ_α, a 2-form field strength $F_{\alpha\beta}$ - which is subject to Bianchi identity

- and a real (auxiliary) scalar D. This field content is the result of the following irreducibility conditions:

$$D^\alpha W_\alpha + \text{c.c.} = 0, \quad \bar{D}_{\dot\alpha} W_\alpha = 0\,, \tag{5.29}$$

whose solution reads:

$$W_\alpha = \chi_\alpha(y) + i\theta_\alpha D(y) + i\theta^\beta F_{\beta\alpha}(y) + i\theta^2 \partial_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}(y)\,. \tag{5.30}$$

Similarly to the scalar chiral superfield, the chirality condition $\bar{D}_{\dot\alpha} W_\alpha = 0$ imposes the dependence on $y_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} - 2i\theta_\alpha\bar\theta_{\dot\alpha}$ and the absence of further $\bar\theta$ terms. The first equation in (5.29) imposes $D = \bar{D}$ and the Bianchi identity $dF = 0$.

We can obtain a Maxwell superfield from the coset construction by starting with a Weyl fermion zeroth-order generator G_α, which must satisfy schematically:

$$\{Q, G\} = \{\bar{Q}, G\} = [P, G] = \dots\,. \tag{5.31}$$

Then we add higher-order generators with the appropriate commutation relations with supertranslations.

We will separate our discussion of irreducibility conditions into two parts. Let us first investigate the covariant generalization of the chirality condition $\bar{D}_{\dot\alpha} W_\alpha = 0$. A general complex spin $(\frac{1}{2}, \frac{1}{2})$ condition takes the form:

$$T_{\alpha\beta\dot\alpha\dot\beta}\,\hat{\bar{D}}^{\dot\beta} W^\beta = 0\,, \tag{5.32}$$

where T - evaluated on the inverse Higgs solution - is again a tensor built out of covariant derivatives of W_α. All solutions to (5.32) - other than those which satisfy $\hat{\bar{D}}^\beta W^\beta = 0$ - are subject to additional independent constraints of different Lorentz representations, related to matrix singularity of T. We therefore find that the appropriate generalization of the chirality constraint is the minimal one:

$$\hat{\bar{D}}_{\dot\alpha} W_\alpha = 0\,. \tag{5.33}$$

As before, one needs to investigate on a case-by-case basis whether (5.33) is really equivalent to the canonical chirality condition, at least after imposing inverse Higgs relations. The generalization of the first equation in (5.29) is not as straightforward as the chirality condition. We will return to this question in a moment.

Let us first pause to explain why we chose to consider Maxwell Goldstone EFTs. Often, one uses a vector superfield V to parametrize the same physical multiplet. A vector superfield, despite its name, is a scalar which is subject to a reality condition: $V = \bar{V}$. In addition, one assumes a gauge redundancy of the form:

$$V \to V + \Phi + \bar{\Phi}\,, \tag{5.34}$$

where Φ is an arbitrary chiral superfield. It is possible to gauge fix part of this redundancy such that V contains only a gauge vector $A_{\alpha\dot{\alpha}}$, a Weyl fermion χ_α and a real (auxiliary) scalar field D. This is of course equivalent to the field content of the Maxwell superfield. Indeed, a superspace relation between the two is:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V\,. \tag{5.35}$$

We use the Maxwell superfield precisely because the gauge redundancy of V requires us to include infinitely many generators from the inverse Higgs tree, corresponding to a subset of the large gauge transformations. The Maxwell superfield contains the gauge-invariant field strength $F_{\alpha\beta}$ rather than the potential $A_{\alpha\dot{\alpha}}$.

Finally, we note that the Maxwell multiplet is not the only irreducible spinor superfield. A chiral spinor ϕ_α admits the following gauge redundancy, parametrized by a vector superfield K:

$$\phi_\alpha \to \phi_\alpha + \bar{D}^2 D_\alpha K\,, \tag{5.36}$$

or, equivalently, by a Maxwell superfield $K_\alpha = \bar{D}^2 D_\alpha K$. After gauge fixing, ϕ_α contains a spinor χ_α, a real scalar a and the 2-form potential $B_{\alpha\beta}$. These fields make up an irreducible physical supermultiplet. Again, there exists another superfield which contains instead the invariant 3-form field strength and is therefore not subject to any gauge redundancy. This is the *real linear* superfield, which we will investigate in section 5.3.3.

Superspace inverse Higgs tree

In figure 5.3, we have depicted on the left the superspace inverse Higgs tree of the chiral spinor superfield W_α, up to level $n = 2$. The chirality condition reads $\hat{\bar{D}}_{\dot{\alpha}} W_\alpha = 0$, so the barred spinor derivative is not available to impose inverse Higgs relations. This means that at level $n = \frac{1}{2}$ we can only connect generators with the operator Q_α, leading to a complex scalar and a two-form, $(0,0) \oplus (1,0)$. At level $n = 1$, Lorentz symmetry allows for two independent $(\frac{1}{2}, 0)$ generators, connected to either the $(0,0)$ or $(1,0)$ by Q_α, but Jacobi identities only allow for one. Using $\bar{Q}_{\dot{\alpha}}$ we also find a $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ at $n = 1$. These generators correspond to the two independent shifts of the essential fermion, linear in the coordinates. All further Goldstone modes at higher levels are P-inessential. The tree on the left therefore reduces to a superspace expansion of the chiral spinor, as expected.

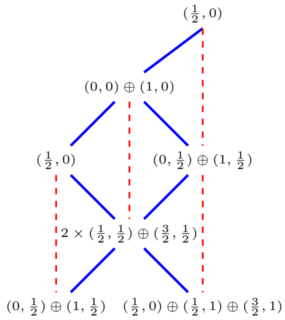

Figure 5.3: *The non-linear generators that can be realized on the chiral spinor (left) and the subset that is consistent with canonical propagators and all irreducibility conditions (right).*

Irreducibility condition II

The second irreducibility condition that defines the canonical Maxwell superfield reads:

$$D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0\,. \tag{5.37}$$

This condition imposes reality of the scalar field in the chiral spinor, $D = \bar{D}$. Furthermore, it imposes Bianchi identity on the 2-form $F_{\alpha\beta}$, which should therefore be considered a gauge field strength for a $U(1)$ gauge field $A_{\alpha\dot{\alpha}}$.

As we have seen, this irreducibility condition is harder to covariantize than the chirality condition. In 3.4.3, we studied the partial breaking of $\mathcal{N} = 2$ supersymmetry using the Maxwell superfield, following the reference [156]. In this setup, there is a single non-linearly realized generator S_α, with the anti-commutation relation:

$$\{S_\alpha, \bar{S}_{\dot{\beta}}\} = 2P_{\alpha\dot{\beta}}\,. \tag{5.38}$$

It was found that the correct generalization for this symmetry breaking pattern, up to fifth order in the fields, is:

$$\hat{D}^\alpha W_\alpha - \frac{1}{2}\hat{D}^\gamma W_\gamma \hat{D}_{(\alpha} W_{\beta)} \hat{D}^{(\alpha} W^{\beta)} + \text{c.c.} + \ldots = 0\,. \tag{5.39}$$

The solutions to this constraint equation can be written in terms of a canonical Maxwell superfield. On the contrary, the naive generalization $\hat{D}^\alpha W_\alpha +$ c.c. $= 0$ has only the trivial solution $W_\alpha = 0$.

The covariant irreducibility condition takes this complicated form because there are many different covariant objects one can build out of W_α by acting with $\hat{D}$ and $\hat{\bar{D}}$. Equation (5.39) makes use of three independent tensors at this order in the fields: the symmetric part of the tensor $D_\alpha W_\beta$, the real part of its trace, and the imaginary part of its trace.

We have explained before that irreducibility condition have many hidden covariances, corresponding to all the different symmetry transformations compatible with the physical supermultiplet. We can make the non-linear covariances manifest by extending the algebra to a higher level in the inverse Higgs tree. At the same time, extending the algebra introduces inverse Higgs relations, reducing the number of independent covariant objects at a given order in the fields and derivatives. The $\mathcal{N} = 2$ supersymmetry algebra allows for the following extension:

$$[Q_\alpha, C] = S_\alpha \, . \tag{5.40}$$

where C is a real scalar. This is just part of the R-symmetry automorphism group of $\mathcal{N} = 2$ supersymmetry, which lives at order $n = \frac{1}{2}$ in the inverse Higgs tree. The Goldstone superfield of C is projected out by the inverse Higgs relation:

$$\hat{D}^\alpha W_\alpha - \text{c.c.} = 0 \, . \tag{5.41}$$

This object therefore cannot appear in the irreducibility condition. Indeed, we have verified that (5.39) reduces to the naive generalization $\hat{D}^\alpha W_\alpha + \text{c.c.} = 0$ in terms of covariant derivatives of the extended algebra.[5] Note that the generator C corresponds to a shift of the auxiliary field. By assumption, this generator will not become a true symmetry of the physical EFT, but it is very useful to include it in order to find consistent irreducibility conditions.

A further extension involving the $n = \frac{1}{2}$ $(1, 0)$ generator would also remove the symmetry part of $\hat{D}_\alpha W_\beta$, reducing to a single covariant object at first order in the fields and derivatives. However, this extension is not compatible with the $\mathcal{N} = 2$ supersymmetry algebra, as we will find out below on the basis of Jacobi identities.

Canonical propagators

We depict the consequences of both irreducibility conditions as well as the restriction to canonical propagators (in the field basis picked out by the coset

[5] To be more precise, the condition (5.39) factorizes into $(\hat{D}^\alpha W_\alpha + \text{c.c.})T = 0$, where T is a scalar built from covariant objects. This indicates that (5.39) has a second branch of solutions which cannot be written in terms of a canonical Maxwell superfield.

construction) on the right in figure 5.3. We have reduced the inverse Higgs tree to a single generator at each order.

Let us explain how one gets from the left of 5.3 to the right. The second irreducibility condition reduces the complex $(0,0)$ at $n = \frac{1}{2}$ to a real $(0,0)$. This then also truncates all generators attached to its imaginary part, like the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ at $n = 1$, and so on.

The restriction to canonical propagators then removes the remaining real $(0,0)$ at $n = \frac{1}{2}$, because it corresponds to a shift of the auxiliary field (we have already explained that it is sometimes possible and useful to include this generator, but never necessary). The remaining P-essential component Goldstone fields are the essential Weyl fermion and the 2-form. The 2-form is subject to Bianchi identity, so it only allows for generators consistent with the canonical $U(1)$ gauge kinetic term $F_{\mu\nu}F^{\mu\nu}$. As we have explained in Chapter 4, this leaves only a single independent shift of $F_{\mu\nu}$ at each order in the coordinates, corresponding to a hook generator.

In the end, the right side of 5.3 depicts the following general (anti)-commutation relations:

$$\begin{aligned}
[Q_\gamma, G_{\alpha_1\ldots\alpha_{n+3/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}}] &= -i\epsilon_{\gamma\alpha_{n+3/2}} S_{\alpha_1\ldots\alpha_{n+1/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}} + \ldots\,, \\
\{\bar{Q}_{\dot{\gamma}}, S_{\alpha_1\ldots\alpha_{n+1}\dot{\alpha}_1\ldots\dot{\alpha}_n}\} &= -\epsilon_{\dot{\gamma}\dot{\alpha}_n} G_{\alpha_1\ldots\alpha_{n+1}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1}} + \ldots\,, \\
[P_{\gamma\dot{\gamma}}, G_{\alpha_1\ldots\alpha_{n+3/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}}] &= \tfrac{i}{2}\epsilon_{\gamma\alpha_{n+3/2}}\epsilon_{\dot{\gamma}\dot{\alpha}_{n-1/2}} G_{\alpha_1\ldots\alpha_{n+1/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-3/2}} + \ldots\,, \\
[P_{\gamma\dot{\gamma}}, S_{\alpha_1\ldots\alpha_{n+1}\dot{\alpha}_1\ldots\dot{\alpha}_n}] &= \tfrac{i}{2}\epsilon_{\gamma\alpha_{n+1}}\epsilon_{\dot{\gamma}\dot{\alpha}_n} S_{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_{n-1}} + \ldots\,.
\end{aligned} \tag{5.42}$$

In these expressions, the sets of indices $(\alpha_1, \ldots, \alpha_n)$ and $(\dot{\alpha}_1, \ldots, \dot{\alpha}_n)$ are fully symmetric on the right- and left-hand sides. The generators $S_{\alpha_1\ldots\alpha_{n+1}\dot{\alpha}_1\ldots\dot{\alpha}_n}$ are spin-$(n+\frac{1}{2})$ fermions, whereas $G_{\alpha_1\ldots\alpha_{n+3/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}}$ are $(\frac{n+3/2}{2}, \frac{n-1/2}{2})$ bosonic hook generators. Note that n denotes the level in the inverse Higgs tree, which goes up by half-integer steps and starts at $n = 0$ for the spin-$\frac{1}{2}$ essential Weyl fermion. The remaining structure (5.42) is simple enough to admit an exhaustive classification of all exceptional EFTs by simply demanding closure of the algebra by Jacobi identities.

Inverse Higgs tree and soft weights

We want to emphasize again that one can read off the relation between bosonic and fermionic soft weights from the superspace inverse Higgs tree 5.3, i.e. the algebra (5.42). This time, the bosonic soft weight is always equal to or one less than the fermionic soft weight, as the zeroth-order generator is a fermion. In other words, we find:

$$\sigma_\chi = \sigma_A + 1 = n + 1 \quad \text{for integer } n, \tag{5.43}$$
$$\sigma_\chi = \sigma_A = n + \tfrac{1}{2} \quad \text{for half-integer } n\,, \tag{5.44}$$

which was also derived by amplitudes methods in [9].

Exceptional EFTs

With the superspace inverse Higgs tree at hand, we can now classify the possible exceptional algebras. We will separate our discussion into three sections: the lowest level case $n = 0$ with no superspace inverse Higgs constraints, $n = 1/2$, and finally any finite $n \geq 1$. As it turns out, the Maxwell superfield allows for only one exceptional algebra: the non-linear realisation of $\mathcal{N} = 2$ supersymmetry by a VA fermion coupled to a BI vector described by Bagger and Galperin in [156].

$n = 0$

When $n = 0$, the only non-linearly realized generator is the spinor S_α and therefore the Ansatz for the commutators is very simple. Jacobi identities tell us that the only non-trivial commutator involving non-linear generators is

$$\{S_\alpha, \bar{S}_{\dot\alpha}\} = sP_{\alpha\dot\alpha}\,, \tag{5.45}$$

which for $s = 2$ leads to $\mathcal{N} = 2$ supersymmetry when combined with the other commutators. This is an exceptional algebra and is non-linearly realized by the exceptional EFT of a VA fermion coupled to a BI vector. As is now well-known [9, 78], the BI vector has a vanishing soft weight and can therefore be considered as a mater field required to maintain linear SUSY. This is in comparison to the role of the fermion in $P(X)$ theories of the chiral superfield discussed in section 5.3.1. The coset construction for this case was worked out in [156]. The $s = 0$ case is simply a contraction of the $\mathcal{N} = 2$ algebra and is non-linearly realized by a shift symmetric fermion coupled to a gauge vector in a linearly supersymmetric manner. The transformation rules here are now field-independent.

$n = 1/2$

At level $n = 1/2$, we find the real scalar generator a and the 2-form $G_{\alpha_1\alpha_2}$. The real scalar generator is projected out by the requirement of canonical propagators, but we will relax this assumption for a moment and include this automorphism generator. If we only include a and omit $G_{\alpha_1\alpha_2}$, Jacobi identities tell us that the only extension of the $\mathcal{N}=2$ algebra has

$$[Q_\alpha, a] = S_\alpha, \quad [S_\alpha, a] = Q_\alpha\,, \tag{5.46}$$

whereas if we include $G_{\alpha_1\alpha_2}$ as well, no exceptional algebras exist[6]. That is, in the presence of $G_{\alpha_1\alpha_2}$, the only non-trivial commutators are those required by superspace inverse Higgs constraints i.e.

$$\{Q_\alpha, a\} = S_\alpha, \quad [Q_\alpha, G_{\beta_1\beta_2}] = \epsilon_{\alpha\beta_1} S_{\beta_2}. \tag{5.47}$$

Here the field strength transforms with a constant shift under the 2-form parameter and therefore the vector has a Galileon type symmetry: a shift linear the space-time coordinates without field dependence. Interestingly, unlike for the scalar Galileon, there are no self-interactions for this Galileon gauge vector which do not introduce additional degrees of freedom. [171]

$n \geq 1$

We will now proceed further in the inverse Higgs tree, to level $n = 1$ and beyond. We make use of the superspace inverse Higgs relations (5.42) and write down a general Ansatz for the remaining (anti)-commutators. Again the answer is very long and complicated so to keep things readable we will outline how we did the calculation.

As we have done for the chiral superfield, we will start with just the bosonic sub-algebra which is spanned by the Poincaré generators and the non-linear generators $G_{\alpha_1\ldots\alpha_{n+3/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}}$. For $n = 1$ we have already seen that the bosonic sub-algebra must be trivial but there are possible exceptional structures at higher levels. In [78] it was shown that any vector symmetry of the form $\delta A_{\alpha_1\dot{\alpha}} = b_{\alpha_1}{}^{\alpha_2} x_{\alpha_2\dot{\alpha}}$ cannot be augmented with field-dependent pieces in the presence of the $U(1)$ gauge symmetry. Since this symmetry therefore only generates a constant shift on the field strength we will take $[P_{\gamma\dot{\gamma}}, G_{\beta_1\beta_2}] = 0$ as a starting point. Jacobi identities then tell us that the commutators between translations and any non-linear bosonic generator are fixed by the inverse Higgs relations i.e. the third equation in (5.42) with ellipses equal to zero, up to a basis changes.

[6] At the purely bosonic level there is a consistent exceptional algebra where the 2-form generator commutes with itself, into itself, just like the Lorentz generators. However, this algebra is not compatible with the Bianchi identity for the field strength and so cannot be realized on the gauge vector. One can see this by working out the transformation rules using the coset construction, or by reintroducing the gauge symmetries in the algebra computation as an infinite set of generators, realized on an essential vector, then checking closure of the algebra. See also [78].

Following the general recipe outlined in Chapter 4, we now inspect the Jacobi identities involving one translation and two bosonic non-linear generators : $(P, G^n, \bar{G}^m)$ and (P, G^n, G^m) where again m, n are half-integer. The former implies that the commutator $[G^m, \bar{G}^n] = 0$ for any m and n while the latter reduces the commutators schematically to

$$[G^{z_b}, G^{z_b}] = cM, \quad [G^{z_b}, G^{z_b-1}] = cP, \tag{5.48}$$

where z_b indicates the finite level at which the bosonic part of the tree terminates, M and P refer to Lorentz generators and space-time translations respectively, and c is an unconstrained coefficient. These structures are very familiar from Chapter 4 [79], for example the DBI algebra has precisely this structure. Note that Jacobi identities also allow for the 2-form generator $G_{\alpha\dot{\alpha}}$ to appear on the RHS of the first of these commutators, however its presence would spoil the inverse Higgs constraints since they would no longer be algebraic in the relevant inessential Goldstones. We encountered a similar scenario in section 5.3.1. We now consider the Jacobi identity involving three non-linear generators $(G^{z_b}, G^{z_b-1}, \bar{G}^n)$ which fixes $c = 0$ since for $n > 1$ there is always at least one bosonic generator which does not commute with translations due to the inverse Higgs relations. The only non-trivial commutators involving non-linear generators in the bosonic sub-algebra are therefore those required by inverse Higgs.

We now include the fermionic generators with the superspace inverse Higgs relations (5.42). It is easy to see that the Jacobi identities involving two (super)-translations and one non-linear generator ensure that the ellipses in these commutators vanish i.e. we cannot include linearly realized generators on the RHS. We also see that other commutators between (super)-translations and fermionic generators, which are not required by the superspace inverse Higgs constraints, i.e. $\{Q, S^n\}$ must also vanish.

The only other commutators we need to fix involve two non-linear generators with at least one of these being fermionic. There is a natural way to proceed through the remaining Jacobi identities, making use of the result that the bosonic sub-algebra is trivial. We begin, for example, with the $(\bar{Q}, G^n, S^m)$ Jacobi identity which contains a single non-trivial term given by

$$\{\bar{Q}_{\dot{\alpha}}, [G_{\alpha_1\ldots\alpha_{n+3/2}\dot{\alpha}_1\ldots\dot{\alpha}_{n-1/2}}, S_{\beta_1\ldots\beta_{m+1}\dot{\beta}_1\ldots\dot{\beta}_m}]\} = 0\,, \tag{5.49}$$

which is very constraining of the RHS of $[G^n, S^m]$. Proceeding in a similar fashion with the other Jacobi identities involving one supertranslation we find that schematically we can only have

$$\{S^{z_f}, \bar{S}^{z_f}\} = aP, \quad \{S^{z_f}, S^{z_f}\} = bM, \quad \{S^{z_f}, S^{z_f-1}\} = bP\,, \tag{5.50}$$

where z_f is the finite level at which the fermionic part of the tree terminates. Again we have also imposed the extra condition that all inessential Goldstones appear algebraically in the relevant covariant derivatives. Now we see that the Poincaré factor and the fermionic generators form a sub-algebra. Therefore, we can use our results of Chapter 4 [79] where we showed that the only exceptional algebra was that of the VA theory, i.e. only the zeroth order generator can form an exceptional algebra. This requires the tree to terminate at this level. Indeed, in the presence of any other fermionic generators no exceptional algebras are possible. Since in this part we are concentrating on $n \geq 1$ where we have at least two non-linear fermionic generators, we must now set $a = b = 0$.

We have therefore proven, to arbitrarily high finite level in the inverse Higgs tree, that the only exceptional linearly supersymmetric EFT that can be realized on a single Maxwell superfield is the VA/BI theory which non-linearly realizes $\mathcal{N} = 2$ SUSY [156] with $\sigma_\chi = 1$, $\sigma_A = 0$ soft weights.

Brief summary

Let us very briefly summarise the main results for the Maxwell superfield:

- The superspace inverse Higgs tree allows us to read off the soft weights of the fermion and gauge vector of the Maxwell superfield. The results are given in equations (5.43) and (5.44).
- The only exceptional EFT in this case corresponds to a non-linear realisation of $\mathcal{N} = 2$ SUSY and is realized by a VA fermion coupled to a BI vector. The soft weights are $\sigma_\chi = 1$ and $\sigma_A = 0$.
- All other algebras lead to field-independent non-linear symmetries i.e. extended shift symmetries. We have shown this to all finite levels in the superspace inverse Higgs tree.
- The covariant irreducibility constraints that have been imposed on the Maxwell supermultiplet can be understood via superspace inverse Higgs constraints in terms of algebras which live at a higher level in the tree. The constraints then take a simple form.

5.3.3 Real linear Goldstone superfields

Irreducibility conditions

The real linear superfield is a real scalar superfield $L = \bar{L}$ subject to the additional constraint:

$$D^2 L = \bar{D}^2 L = 0 \,. \tag{5.51}$$

Its component fields are a real scalar field $a(x)$, Weyl fermion $\chi(x)$ and a vector field $A(x)$ with the constraint:

$$\partial_{\alpha\dot{\alpha}} A^{\alpha\dot{\alpha}} = 0\,. \tag{5.52}$$

This constraint is equivalent to the Bianchi identify for a dual 3-form field strength, $H = \star A$. By virtue of (5.52) we obtain $H = dB$. Therefore, the constrained vector field A is equivalent to a 2-form gauge potential.

The superspace expansion then reads:

$$\begin{aligned} L = a(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) - \theta^{\alpha}\bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) - \tfrac{i}{2}\theta^2\bar{\theta}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\chi_{\alpha}(x) \\ + \tfrac{i}{2}\bar{\theta}^2\theta^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \tfrac{1}{2}\theta^2\bar{\theta}^2\Box a(x)\,. \end{aligned} \tag{5.53}$$

A free 2-form B is on-shell equivalent to a real, shift-symmetric pseudoscalar. To see this, let us recall the standard dualization procedure. The action for a free real 2-form $B_{\mu\nu}$ reads:

$$\mathcal{L} = H_{\mu\nu\rho}H^{\mu\nu\rho}\,. \tag{5.54}$$

where H is the 3-form field strength, $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$. We can equivalently consider $H_{\mu\nu\rho}$ as a fundamental 3-form field, and impose Bianchi identity by means of a Lagrange multiplier scalar field b. The action then reads:

$$\mathcal{L} = H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{12\sqrt{6}}\, b\,\epsilon^{\mu\nu\rho\sigma}\,\partial_{[\mu}H_{\nu\rho\sigma]}\,. \tag{5.55}$$

The field equation for b then imposes the Bianchi condition on H, i.e.: $\epsilon^{\mu\nu\rho\sigma}\partial_{[\mu}H_{\nu\rho\sigma]} = 0$, so that the action reduces to (5.54). If, instead of solving the field equations for b, we integrate out $H_{\mu\nu\rho}$ by its equations of motion, we obtain the dual action:

$$\mathcal{L} = \partial_{\mu}b\,\partial^{\mu}b\,, \tag{5.56}$$

which is just the free action for a real scalar field. It is possible to dualize more general actions than (5.54). For instance, we can minimally couple $H_{\mu\nu\rho}$ to a fundamental 3-form potential. This gauges the global 2-form shift symmetry of (5.54) and introduces a potential for $b(x)$ in the dual picture. [146]

Upon dualizing the 2-form B that resides in a real linear superfield, we obtain the field content of a physical chiral multiplet: two scalars $a(x)$ and $b(x)$, and a Weyl fermion $\chi(x)$. Indeed, it is often possible to dualize the real linear superfield to a chiral superfield directly in superspace, see for instance [163]. This does not mean the real linear and chiral superfields are strictly equivalent in all situations. For example, the real linear superfield can only describe actions with unbroken $U(1)$ R-symmetry, whereas this

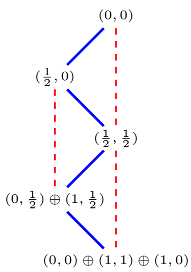

Figure 5.4: *The non-linear generators that can be realized on a real linear supermultiplet (left) and the subset that is consistent with the presence of physical theories with canonical propagators (right). In general, the bosonic generators at non-zero levels are complex but with only the real part connected to the zeroth level by space-time translations.*

automorphism can be broken by the chiral superfield. Additionally, it is not clear whether the dualization procedure can work when the action includes higher-derivative terms for the 2-form $B_{\mu\nu}$.

In the next section, we will describe the inverse Higgs trees for real linear superfields in full generality. However, when we look for exceptional algebras, we will restrict to those symmetry breaking patterns which are compatible with dualizing the 2-form degree of freedom. As we will see, this restricts us to algebras which admit a particular central extension.

Because the physical field content we will work with is equivalent to a chiral multiplet, the reader may wonder what the difference is between section 5.3.1 and the current discussion. In section 5.3.1, we assumed that both scalar fields realize symmetries of the same order in the space-time coordinates. We will relax this assumption in what follows. While the symmetry breaking patterns we discuss here can be realized on a chiral superfield - up to a central extension and following superspace dualization - the real linear superfield is the more natural object to describe these mixed-level systems. For informative examples, see for example [175] or [163].

Superspace inverse Higgs trees

The superspace inverse Higgs tree for a real linear multiplet starts at order $n = 0$ with a Hermitian scalar, which we will call D. This $(0, 0)$ must satisfy $[Q, D] = [\bar{Q}, D] = [P, D] = \ldots$, as usual. At order $n = \frac{1}{2}$, we can attach a single Weyl fermion S_α, by means of the following anti-commutation relation:

$$\{Q_\alpha, S_\beta\} = -\epsilon_{\alpha\beta} D + \dots \,. \tag{5.57}$$

Then, at order $n = 1$, Lorentz symmetry allows for a $(1, 0)$, a complex $(0, 0)$ and a complex $(\frac{1}{2}, \frac{1}{2})$ to attach to S_α. The former two would be connected by Q_α, the latter by $\bar{Q}_{\dot\alpha}$. Jacob identities remove the 2-form (i.e. the $(1, 0)$) and the scalar is killed by the irreducibility conditions. Note that there is no 2-form directly in the superspace expansion of L, but a constrained vector. Let us rearrange the complex vector at order $n = 1$ in terms of two Hermitian vectors $K_{\alpha\dot\alpha}$ and $\tilde{K}_{\alpha\dot\alpha}$. The relevant non-vanishing commutation relations are:

$$\begin{aligned}
&[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}] = -i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D + \dots \,, \\
&[\bar{Q}_{\dot\alpha}, K_{\beta\dot\beta}] = i\epsilon_{\dot\alpha\dot\beta} S_\beta + \dots \,, \\
&[\bar{Q}_{\dot\alpha}, \tilde{K}_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} S_\beta + \dots \,.
\end{aligned} \tag{5.58}$$

It is clear from these commutation relations that only the Goldstone field of $K_{\alpha\dot\alpha}$ is P-inessential. It leads to a shift of the essential scalar field, linear in the space-time coordinates. The generator $\tilde{K}_{\alpha\dot\alpha}$ instead corresponds to a constant shift of the constrained vector.

At level $n = \frac{3}{2}$, we find a $(1, \frac{1}{2})$ and a $(0, \frac{1}{2})$, both connected to the previous level by Q_α. When we include these $n = \frac{3}{2}$ generators, Jacobi identities dictate that the full complex vector at level $n = \frac{1}{2}$ be present. We have pictured the superspace inverse Higgs tree, prior to imposing compatibility with canonical kinetic terms, on the left of figure 5.4.

Canonical propagators

The canonical kinetic term for the real linear superfield reads:

$$\mathcal{L}_{free} = \int d^4\theta \, L^2 \,. \tag{5.59}$$

At the component level, this term includes a Weyl kinetic term for the fermion χ_α, the Klein-Gordon Lagrangian for the scalar field a and the term H^2 (where $H = dB$) for $B_{\alpha\beta}$. We have drawn the inverse Higgs tree compatible with the free action on the right in figure 5.4.

The result is very similar to the chiral superfield inverse Higgs tree 5.2. The only difference results from the fact that the $(\frac{1}{2}, 0)$ generator at level $n = 1/2$ is connected to a complex scalar field in 5.2 and to a real scalar field in 5.4. This amounts to the following central extension of the symmetry algebra:

$$\{Q_\alpha, S_\beta\} = \ldots + i\epsilon_{\alpha\beta} Z \,, \tag{5.60}$$

where Z is a Hermitian scalar generator. Jacobi identities then imply the further modification:

$$[P_{\alpha\dot{\alpha}}, \tilde{K}_{\beta\dot{\beta}}] = \ldots + i\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} Z \,, \tag{5.61}$$

and so on.

The picture 5.4 represents the following (anti)-commutation relations:

$$\begin{aligned}
&\{Q_\gamma, S_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}}\} = -\epsilon_{\gamma\alpha_1} G_{\alpha_2\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}} + \ldots \,, \\
&[\bar{Q}_{\dot{\gamma}}, G_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_N}] = -i\epsilon_{\dot{\gamma}\dot{\alpha}_1} S_{\alpha_1\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_N} + \ldots \,, \\
&[P_{\gamma\dot{\gamma}}, S_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_{N-1}}] = \tfrac{1}{2} i\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1} S_{\alpha_2\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_{N-1}} + \ldots \,, \\
&[P_{\gamma\dot{\gamma}}, G_{\alpha_1\ldots\alpha_N\dot{\alpha}_1\ldots\dot{\alpha}_N}] = \tfrac{1}{2} i\epsilon_{\gamma\alpha_1}\epsilon_{\dot{\gamma}\dot{\alpha}_1} G_{\alpha_2\ldots\alpha_N\dot{\alpha}_2\ldots\dot{\alpha}_N} + \ldots \,,
\end{aligned} \tag{5.62}$$

where the sets $(\alpha_1, \ldots \alpha_N)$ and $(\dot{\alpha}_1, \ldots, \dot{\alpha}_N)$ are assumed symmetric under exchange on the left- and right-hand sides.

Exceptional EFTs

In contrast to the previous two cases, here we will not perform a general analysis. Rather we will study certain cases of interest to illustrate that our general techniques can indeed be applied to a real linear superfield. Below we consider two cases: *i*) tree truncated at level $n = 1$ with a real vector generator and *ii*) tree truncated at level $n = 2$ with the complex vector generator at $n = 1$ (as required by Jacobi identities) and a real symmetric, traceless rank-2 generator (in addition to the fermionic generators in between). In the following, we only consider systems which can be dualised to the chiral superfield (or rather, those cases where the algebra does not rule out the dualisation). We leave an exhaustive classification that relaxes this assumption to future work.

$n = 1$

We begin at level $n = 1$ where the non-linear generators are $(D, S_\alpha, K_{\alpha\dot{\alpha}})$, with K Hermitian[7]. In addition to generators that define the Lorentz representation of each generator, the most general form of the commutators

[7] After dualising to the chiral superfield, this is an example of an algebra where the two parts of the complex scalar zeroth order generator have different inverse Higgs trees.

is

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2P_{\alpha\dot\alpha}, \quad \{S_\alpha, \bar{S}_{\dot\alpha}\} = sP_{\alpha\dot\alpha} + a_1 K_{\alpha\dot\alpha}, \quad \{S_\alpha, S_\beta\} = a_2 M_{\alpha\beta} \\
\{Q_\alpha, S_\beta\} &= -\epsilon_{\alpha\beta} D + a_3 M_{\alpha\beta} + i\epsilon_{\alpha\beta} M', \quad [Q_\alpha, K_{\beta\dot\beta}] = i\epsilon_{\alpha\beta}\bar{S}_{\dot\beta} + a_4 \epsilon_{\alpha\beta}\bar{Q}_{\dot\beta}, \\
[P_{\alpha\dot\alpha}, S_\beta] &= a_5 \epsilon_{\alpha\beta}\bar{Q}_{\dot\alpha}, \quad [P_{\alpha\dot\alpha}, D] = i a_6 P_{\alpha\dot\alpha}, \quad [Q_\alpha, D] = a_7 Q_\alpha, \\
[K_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= a_8 \epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta} - \bar{a}_8 \epsilon_{\dot\alpha\dot\beta} M_{\alpha\beta} + i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} M'', \\
[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= -i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D + i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} M^{(3)} + a_9 \epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta} - \bar{a}_9 \epsilon_{\dot\alpha\dot\beta} M_{\alpha\beta}, \\
[D, S_\alpha] &= a_{10} Q_\alpha + a_{11} S_\alpha, \quad [D, K_{\alpha\dot\alpha}] = i a_{12} P_{\alpha\dot\alpha} + i a_{13} K_{\alpha\dot\alpha}, \\
[S_\alpha, K_{\beta\dot\beta}] &= a_{14} \epsilon_{\alpha\beta}\bar{Q}_{\dot\beta} + a_{15} \epsilon_{\alpha\beta}\bar{S}_{\dot\beta}.
\end{aligned} \tag{5.63}$$

Note that we allow for the most general linear internal symmetries by introducing the scalar generators M', M'', and $M^{(3)}$ and again we have set $\{\bar{Q}_{\dot\alpha}, S_\alpha\} = 0$ without loss of generality by a basis change. Now Jacobi identities allow for only the M' linear scalar to exist and reduce the number of free parameters to two which we denote as s and m. We have

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2P_{\alpha\dot\alpha}, \quad \{S_\alpha, \bar{S}_{\dot\alpha}\} = sP_{\alpha\dot\alpha} - 2mK_{\alpha\dot\alpha}, \\
\{Q_\alpha, S_\beta\} &= -\epsilon_{\alpha\beta} D + mM_{\alpha\beta} + i\epsilon_{\alpha\beta} M', \quad [Q_\alpha, K_{\beta\dot\beta}] = i\epsilon_{\alpha\beta}\bar{S}_{\dot\beta} \\
[P_{\alpha\dot\alpha}, S_\beta] &= -im\epsilon_{\alpha\beta}\bar{Q}_{\dot\alpha}, \quad [P_{\alpha\dot\alpha}, D] = imP_{\alpha\dot\alpha}, \quad [Q_\alpha, D] = i\frac{m}{2} Q_\alpha, \\
[K_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= i\frac{s}{2}\epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta} + i\frac{s}{2}\epsilon_{\dot\alpha\dot\beta} M_{\alpha\beta}, \quad [S_\alpha, K_{\beta\dot\beta}] = -i\frac{s}{2}\epsilon_{\alpha\beta}\bar{Q}_{\dot\beta}, \\
[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}] &= -i\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D + i\frac{m}{2}\epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta} + i\frac{m}{2}\epsilon_{\dot\alpha\dot\beta} M_{\alpha\beta}, \\
[D, S_\alpha] &= i\frac{m}{2} S_\alpha, \quad [D, K_{\alpha\dot\alpha}] = -isP_{\alpha\dot\alpha} + imK_{\alpha\dot\alpha} \\
[M', Q_\alpha] &= -\frac{3m}{2} Q_\alpha, \quad [M', S_\alpha] = \frac{3m}{2} S_\alpha.
\end{aligned} \tag{5.64}$$

Let us now discuss these algebras in terms of s and m.

First of all, when $m \neq 0$ this is the AdS_5 superalgebra. In this case the parameter s turns out to be unphysical. Indeed we can make a simple change of basis from (K, P) to $(\hat{K}, P)$ where $\hat{K}_{\alpha\dot\beta} = K_{\alpha\dot\beta} - \frac{s}{2m} P_{\alpha\dot\beta}$ to set $s = 0$. When $s \neq 0$ this basis is usually referred to as the "AdS" basis while with $s = 0$ we have the "conformal basis" [174]. Therefore, the only actual parameter is the AdS radius $R = 1/m$. In terms of the bosonic sector these two bases were considered in [25] where it was shown that the two different realisations in terms of a single scalar degree of freedom (the vector associated to special conformal transformations is removed by an inverse Higgs constraint) are equivalent EFTs, as expected. The scalar in these theories has a vanishing soft weight [32–34]. As we explained in the introduction, this is compatible with our superspace inverse Higgs tree since in this case once we canonically normalise the scalar, all transformation rules become field-dependent.

The coset construction for this symmetry breaking pattern i.e. the AdS_5 superalgebra broken down the four-dimensional super-Poincaré algebra was studied in [158, 175] (see also [176] for a curved space generalization). The authors constructed the leading action for a supersymmetric 3-brane in AdS_5, utilising a real linear superfield L. Their Lagrangian transforms as a total derivative under a subset of the non-linear symmetries. After dualising the 2-form in L to a scalar, their Lagrangian realizes an additional shift symmetry that is not visible in the inverse Higgs tree. This allows for a different starting point where the essential generator is a complex scalar, but only its real part realizes non-linear symmetries in addition to the constant shift symmetries. This is because there is only a real vector generator at level-1 and therefore only a single scalar degree of freedom can support additional transformations. This reflects the fact that the real linear superfield can be dualised to a chiral superfield. The bosonic sector is then a dilaton (which realizes the conformal symmetries) coupled to an axion.

The flat limit of the bulk space-time corresponds to taking $m = 0$. In this case we cannot perform the aforementioned basis change and hence the second parameter s distinguishes between two different algebras. The case $s = 2$ is the flat limit of the AdS superalgebra and hence corresponds to the super-Poincaré algebra in $D = 5$. However, in this limit one often has symmetry enhancement to $D = 6$ super-Poincaré rather than $D = 5$ thanks to the dualised 2-form field which obtains a field-dependent transformation, see [155, 163]. This is related to the fact that no supersymmetric scalar 3-brane exists in $D = 5$ [177, 178]. The resulting EFT is equivalent to the scalar DBI-VA system we discussed in section 5.3.1.

Finally, we have the $m = s = 0$ case which yields the $D = 5$ supersymmetric Galileon algebra. The authors of [10] conjectured that this algebra has non-trivial quartic and quintic Wess-Zumino terms (in addition to the interaction constructed in [166]), which also realise a second shift symmetry. It is clear from our analysis that this Galileon/axion (the axion comes from dualising the 2-form) system is naturally described by a real linear superfield. We see from the algebra that when $s = m = 0$ we have $\{S_\alpha, \bar{S}_{\dot{\alpha}}\} = 0$ and therefore the fermion is no longer of the VA type but becomes shift symmetric.

$n = 2$

We now consider level $n = 2$ where the non-linear generators are $(D, S_\alpha, K_{\alpha\dot{\alpha}}, \tilde{K}_{\alpha\dot{\alpha}}, \psi_{\alpha_1\alpha_2\dot{\alpha}}, G_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2})$. As we saw above, in the presence of ψ

we need to include both K and $\tilde{K}$ however we keep G real. Rather than performing a full analysis, we ask if the lowest component of the superfield can be a Special Galileon [101] with a $\sigma_\phi = 3$ soft weight and a field-dependent transformation rule. We find, thanks to our results in Chapter 4 [79], that this is not possible. Indeed, since we are forced to include the full complex vector, after dualisation both scalar degrees of freedom must be Galileons i.e. both have a connection to a vector at level $n = 1$ by space-time translations. This implies that both have a transformation rule which starts out linear in the space-time coordinates. Now we are also asking for the lowest component to be a Special Galileon. However, we have already showed in Chapter 4 that we cannot couple a Special Galileon to a Galileon: there is no corresponding symmetry breaking pattern. Now since the bosonic sector is always a sub-algebra this conclusion is robust against adding the relevant fermionic generators. We therefore conclude that the lowest component of the real linear superfield cannot be of the Special Galileon form[8]. The only remaining possibility is that a Special Galileon exists, but that this algebra is not compatible with dualisation (i.e. the central extension). This would imply that the 2-form forms an integral part of the Goldstone EFT. We leave the classification of such possibilities to future work.

Brief summary

Again let us provide a brief summary of our main results with regards to the real linear superfield:

- The superspace inverse Higgs tree becomes particularly simple after imposing both irreducibility conditions and the existence of canonical propagators, and differs from the chiral case only by having a real (instead of a complex) scalar generator at the lowest level. If we truncate the tree at a half-integer level, all bosons other than the zeroth order must be complex. However, if we truncate at an integer level, the highest generator can also be real. Moreover, the gauge symmetry of the 2-form gauge potential sitting inside the constrained vector decouple from the tree.
- We have not performed an exhaustive classification, but demonstrated that the algebras up to and including $n = 1$ correspond to super-AdS in $D = 5$ and super-Poincaré in $D = 6$. We can perform a contraction of the latter leading to a supersymmetric Galileon algebra.

[8]Note that we can couple a Special Galileon to an axion but we see from the tree that this theory cannot be supersymmetrised since the presence of ψ demands that the axion becomes a Galileon.

- At $n = 2$ we have shown that the lowest order scalar cannot be a Special Galileon with a field-dependent transformation rule if we dualise the 2-form. Indeed, then the second scalar would be a Galileon which cannot be coupled to a Special Galileon [79]. The only way out, which is an interesting avenue for future work, is to not dualise the 2-form.

Chapter 6

Conclusion

In the Chapters 4 and 5, we have presented a thorough exploration of non-linear realizations in linearly Poincaré-invariant and $\mathcal{N} = 1$ supersymmetric backgrounds. The upshot of our results is that theories with non-trivial (field-dependent) non-linear transformation laws are very scarce indeed. This holds especially when the transformation laws are the result of inverse Higgs relations. Using algebraic methods, one can obtain an exhaustive list of all possible symmetry breaking patterns in certain cases. In Poincaré-invariant theories [1], the following statements are fully general:

- For a single scalar Goldstone field, all possible non-linear symmetry transformations are: I) (extended) shift symmetries, II) dilatations and special conformal transformations, III) $ISO(1,4)$ or $ISO(2,3)$ DBI transformations, and IV) the Special Galileon. [101, 103]
- For any number of spin-$\frac{1}{2}$ Goldstinos, we find: I) (extended) fermionic shift symmetries, II) ($\mathcal{N}$-extended) supersymmetry.
- For a single *gauge* vector Goldstone field, no extended shift symmetries [171] or exceptional symmetry breaking patterns [78] are possible. This rules out, in particular, that a hidden symmetry underlies the Born-Infeld action.

We have also examined the case where there are multiple independent Goldstone scalar fields. In this setup, it is not possible to enumerate all symmetry breaking patterns consistent with the theory of non-linear realizations and Jacobi identities. A systematic exploration of such possibilities is lacking. However, we have ruled out finding a fundamentally new structure by

[1] Here we mean that the *full* linearly realized space-time symmetry group is Poincaré, not that it admits a Poincaré subgroup. We only allow for internal symmetry groups which combine with the Poincaré group in a direct product.

coupling additional Goldstone scalar fields, at least as far as the relation to space-time symmetry generators is concerned. The appearance of space-time generators in commutation relations is limited to the DBI, conformal, or Special Galileon structures, just like in the single-scalar case. One then couple these systems to each other, or to Goldstones that non-linearly realize some internal symmetry group. For example, it is possible to couple a flat-space DBI scalar to a Galileon. [77] We can, however, rule out the following in general: I) coupling several Special Galileons, II) or coupling Special Galileons to DBI scalars (flat or curved space).

To derive these statements, we have assumed coset universality (as defined in Chapter 3) and that there are a finite number of independent non-linearly realized transformations. Then, we make use of a simple three-step procedure: I) fixing the commutation relations of all non-linear generators with translations (fixing the *inverse Higgs tree*), II) truncating to the subset of generators compatible with canonical kinetic terms [2], and III) fixing the remaining structure with Jacobi identities. We have presented this systematic procedure in [79, 80], building on the work of [76, 77].

In the case of linearly $\mathcal{N} = 1$ supersymmetric theories, we have performed an exhaustive classification in the following cases:

- For a single chiral Goldstone superfield, the possible symmetries are: I) (extended) super-shift symmeties, which shift the scalar, fermion and auxiliary field by some power of the coordinates, II) minimal $D = 6$ supersymmetry on flat or AdS space (these transformations include an $ISO(1, 5)$ or $SO(2, 5)$ DBI scalar and an Akulov-Volkov fermion). There is no supersymmetric version of the Special Galileon.

- For a single Maxwell superfield, the possibilities are I) (extended) super-shift symmetries, and II) minimal $\mathcal{N} = 2$ supersymmetry in $D = 4$. There is no special symmetry underlying the super-Born Infeld action - at least not one of the type considered here - in contradiction to what was anticipated by Bagger and Galperin. [156] This is consistent with the results for a single gauge vector Goldstone field in Poincaré theories. [78, 171]

In addition, the real linear superfield allows for non-linear realizations of the $D = 5$ supersymmetric AdS algebra [175] and the $D = 5$ super-Poincaré algebra. However, the latter is on-shell equivalent to a dual theory of a chiral superfield that non-linearly realizes minimal $D = 6$ supersymmetry. The two-form (constrained vector) in the real linear theory dualizes to a

[2] In the field basis where the zeroth-order transformation begins with a constant shift.

scalar field that realizes the additional non-linear symmetry. We have not been fully exhaustive for the real linear superfield, but we can rule out the existence of other exceptional EFTs up to level $n = 2$ in the inverse Higgs tree (i.e. symmetry transformations that start with a shift at second order in the coordinate). This, therefore, rules out the existence of a supersymmetric special Galileon. We obtain these results by following essentially the same three-step procedure as in the Poincaré background, as we have explained in Chapter 5.

These results coming from the algebraic method extend upon those from the soft bootstrap amplitudes method [2–4,9,10], and are in agreement where applicable. The algebraic method is usually simpler and therefore easier to extend to more general cases. However, the soft bootstrap method also provides a classification of the *interactions* - up to a given order in fields and derivatives - that respect a certain symmetry breaking pattern, rather than just the pattern itself. [3] In principle, one can easily classify invariant interactions using the coset formalism, from only the leading terms that appear in covariant derivatives. The structure of inverse Higgs relations and irreducibility conditions immediately tells us which covariant objects are available. However, this leaves out the Wess-Zumino terms, which are harder to classify.

Exceptional EFTs and positivity bounds

In recent years, the *swampland* program [182] has received much attention. This refers to the broad idea of constraining low-energy EFTs by general consistency requirements coming from the UV. One can, for example, constrain low-energy EFTs by the requirement that any UV completion should have a unitary, local , Lorentz-invariant S-matrix. This leads in particular to so-called *positivity constraints.* [111, 183–187] EFTs that do not satisfy positivity constraints are said to live in the swampland: they do not admit a consistent UV completion (in terms of a field theory living in the same number of dimensions)

It turns out that positivity constraints imply that in some cases there is a *unique* exceptional theory. This is the case when we include only a single scalar or fermion Goldstone in Poincaré-invariant theories. The only exceptional EFTs satisfying positivity are the DBI scalar (the anti-DBI scalar, which is related to the DBI algebra by a sign flip, is ruled out) and Volkov-Akulov fermion, respectively. This immediately implies that the only exceptional EFT in $\mathcal{N} = 1$ supersymmetry, using a single chiral superfield, that

[3] More precisely, it tells us what interactions *can* exist, rather than are actually realized as terms in a Lagrangian.

satisfies positivity is the $D = 6$ super-DBI action of Bagger and Galperin. This indicates how special the exceptional EFTs are. To our knowledge, a satisfying explanation of this fact is lacking.

Possibilities for future research

It is easy to imagine extensions of our approach, generalizing to different background symmetry groups. Some work in this direction appears in the literature. For classifications of non-linearly realized symmetries in the cosmological background, using Lagrangian and algebraic methods, see [82, 83]. A similar classification to the one presented in this thesis for AdS backgrounds appears in [81].

The study of soft amplitudes has achieved great results in understanding the UV behavior of extended supergravities. [11–15] Extended supergravity theories feature remarkable cancellations in loop diagrams, such that they often remain UV finite at very high loop order. [17–19] This has sometimes been explained by the absence of the appropriate counterterms [16] which would need to exist to cancel any UV divergences. Supergravity theories in $D = 4$ admit a more and more constrained multiplet structure as one extends the amount $\mathcal{N}$ of supersymmetry. While $\mathcal{N} = 1$ allows for the same chiral and vector multiplets that exist in global supersymmetry, the maximal $\mathcal{N} = 8$ supergravity theory has a completely fixed multiplet structure.

The scalar fields that enter into these multiplets non-linearly realize a global symmetry. For example, the scalar fields in $\mathcal{N} = 8$ supergravity parametrize the coset $E7(7)/SU(8)$. Therefore, their scattering amplitudes feature enhanced soft limits. If a candidate counterterm destroys the soft behavior of the scalar fields, and there is no other counterterm to restore the soft limit, we know that it cannot be supersymmetrized. The soft bootstrap method can therefore be used to classify counterterms in extended supergravities. [11–15] It would be interesting to investigate whether our approach, based on the structure of inverse Higgs trees from the theory of non-linear realizations, has anything to say about the counterterm structure. A linearized description of extended supergravity should include linearized diffeomorphisms and gauge transformations in a superspace inverse Higgs tree with $E7(7)/SU(8)$ scalars at the top, connected by global supersymmetry transformations. One should be able to count covariant objects from that inverse Higgs tree and classify counterterms using the covariant derivative (calculated perturbatively).

Another field of research where non-linear realizations have played a role recently is the study of integrable deformations. These are special deformations that preserve an infinite set of commuting symmetries. In other words, they preserve the property of integrability. A particularly interesting class are so-called $T\bar{T}$ deformations of $D = 2$ quantum field theories. Such a deformation adds to the Lagrangian of a theory the irrelevant local operator $T\bar{T}$, which is constructed from the energy-momentum tensor. One can add such a deformation either discretely or following an infinitesimal flow along the parameter t. In the latter case, the Lagrangian satisfies the differential equation:

$$\frac{\partial \mathcal{L}_t}{\partial t} = \det(T_{\mu\nu})\,. \tag{6.1}$$

[190] One can then study properties of the deformed action using the undeformed one. At the classical level, for example, there exists a mapping of solutions to the equations of motion between deformed and undeformed theories. [191, 192]

If one follows the flow defined by this equation starting at the free action for n scalars, $\mathcal{L} = (\partial\phi^i)(\partial\phi^j)\delta^{ij}$, one arrives at the Nambu-Goto action for a string in $D = 2+n$ target space, i.e. the scalar DBI action of codimension-n. Stated the inverted way, following the flow from finite t to $t = 0$ leads to an Inönu-Wigner contraction of the co-dimension-n Poincaré action to the $\mathfrak{gal}(d, n)$ algebra for n Galileons.

There are many other examples where deforming a free action by $T\bar{T}$ leads to a theory with a special non-linearly realized symmetry. For example, deforming an action of free fermions in $D = 2$ leads to the Volkov-Akulov action. When deforming a free action for a scalar supermultiplet, one couples the DBI scalar to the Volkov-Akulov fermion, leading to a theory of partially broken supersymmetry similar to the Bagger-Galperin model in $D = 4$. [193, 195] It would be very interesting to investigate whether there is a connection between integrable deformations and the theory of non-linear realizations. .

Samenvatting

Symmetrie is een cruciaal concept in de natuurkunde. Een object is symmetrisch wanneer men een transformatie kan uitoefenen op het object zonder het wezenlijk te veranderen. Zo is een bol symmetrisch onder rotaties. De wetten van de natuur kunnen ook symmetrie vertonen. De definitie van een symmetrie van de natuurwetten verschilt iets van die van een symmetrie van een object. De natuurwetten zijn symmetrisch wanneer men een transformatie kan uitoefenen op alle objecten in de wereld tegelijk, zodanig dat de getransformeerde toestand van de wereld nog steeds voldoet aan alle wetten van de natuur.

In 1632 stelde Galileo het *relativiteitsprincipe* voor. Dit principe stelt dat de natuurwetten gelijk zijn in alle *inertiale referentiestelsels*. Het relativiteitsprincipe is een voorbeeld van een symmetrie van de natuurwetten. Er bestaan namelijk vanwege het relativiteitsprincipe transformaties die elke volgens de natuurwetten legale toestand van de wereld transformeert naar een andere eveneens legale toestand. Dit zijn de transformaties die inertiaalstelsels met elkaar verbinden, de zogenaamde *Galileïsche transformaties*. Newtons wetten van de mechanica voldoen aan het relativiteitsprincipe van Galileo.

Binnen de mechanica van Newton bestaan belangrijke *behoudswetten*. De belangrijkste daarvan zijn de behoudswetten van *energie* en *impuls*. De behoudswetten in de Newtoniaanse mechanica zijn het gevolg van symmetrie. Impuls is behouden vanwege symmetrie onder ruimtelijke translaties in de wetten van Newton, energie vanwege translaties in tijd. Volgens de *eerste stelling van Noether* bestaat er voor elke symmetrie van de natuurwetten een corresponderende behoudswet, en vice versa. [35]

De volgende grote ontdekking over symmetrie kwam voort uit Maxwells theorie van elektromagnetisme. De vergelijkingen van Maxwell leiden tot een opmerkelijke conclusie: elektromagnetische golven in het vacuüm propageren altijd met dezelfde snelheid c, de *lichtsnelheid*. Dit lijkt in tegenstelling te zijn met het relativiteitsprincipe. Maxwells vergelijkingen hebben wel een andere symmetrie, de *Lorentz transformaties*. Deze zetten elektrische en magnetische velden in elkaar om, zodanig dat een elektromagnetische golf voor en na een Lorentz transformatie met snelheid c voortbeweegt.

Einstein gaf de correcte interpretatie van de Lorentz transformaties in 1905. Hij maakte twee minimale aannames: 1) het relativiteitsprincipe is geldig, 2) de snelheid van het licht is c in alle inertiaalstelsels. Einstein kon daarmee op een eenvoudige manier laten zien dat de transformaties van Lorentz, niet die van Galileo, het ene inertiaalstelsel omzetten in het andere. In Einsteins theorie van *speciale relativiteit* zijn de Lorentz transformaties dus een volwaardige symmetriegroep van de natuur. De transformaties van Lorentz vermengen de tijd t met de ruimtelijke coördinaten $\vec{x}$ en vice versa. In de speciale relativiteit zien we ruimte en tijd niet als gescheiden concepten, maar als onderdelen van een groter geheel, de *ruimtetijd*. De theorie van Einstein heeft daarmee drastisch veranderd hoe men denkt over ruimte en tijd. Einstein kwam tot deze fundamentele inzichten door symmetrie voorop te stellen. Het relativiteitsprincipe was volgens Einstein van groter belang dan de mechanicawetten van Newton.

Na Einsteins succes met de speciale relativiteitstheorie begonnen natuurkundigen theorieën te *definiëren* aan de hand van hun symmetrie. Het standaardmodel van de deeltjesfysica berust bijvoorbeeld op de symmetriegroep $SU(3) \times SU(2) \times U(1)$. Dit is een voorbeeld van een *ijksymmetrie*, een symmetrie wiens transformaties afhangen van een arbitraire functie over de ruimte-tijd coördinaten x^μ. Voor elke soortgelijke functie bestaat er een vectordeeltje, een *ijkboson*. De groep $SU(3) \times SU(2) \times U(1)$ staat 12 vrije functies toe en dus 12 ijkbosonen. Deze deeltjes zijn de *dragers* van de drie krachten in het standaardmodel. De zogenaamde *pure Yang-Mills* theorie van $SU(3) \times SU(2) \times U(1)$ impliceert echter dat alle ijkbosonen massaloos zijn. Dit is niet wat experimenteel geobserveerd wordt. Dit probleem kan worden opgelost door de ijkbosonen te koppelen aan een scalair veld (het *Higgs veld*) zodanig dat een effectieve massaterm voor de ijkbosonen verschijnt. Dit is alleen mogelijk wanneer het scalair veld zelf ook transformeert onder de ijkgroep. Tevens moet het scalairveld een *vacuüm verwachtingswaarde*. Deze twee voorwaarden samen impliceren dat de groep $SU(2) \times U(1)$ (deels) gebroken wordt door de vacuümtoestand. Dit is een voorbeeld van *spontane symmetriebreking*. De Higgs boson, een excitatie van het Higgs veld, werd in 2012 ontdekt door CERN. [67]

Goldstone, Salam en Weinberg [63] bewezen dat voor elke spontaan gebroken symmetrie een massaloos scalair deeltje bestaat. In het geval van een ijksymmetrie zoals $SU(2) \times U(1)$ kan dit deeltje worden opgevat als de longitudinale excitatie van een massieve vector. Als een *globale* symmetrie spontaan breekt, is het massaloze scalaire deeltje te observeren. Dit is een *Goldstone boson*. In het standaardmodel bestaat bij benadering een *chirale sym-*

metrie. Deze symmetry wordt spontaan gebroken in het vacuüm, waardoor Goldstone bosonen ontstaan, de *pionen*. Weinberg en Nambu [41, 42, 44–47] slaagden erin de sterke kernkracht te beschrijven aan de hand van een theorie waar pionen de dragers van de kracht zijn. De interacties van de pionen met de kerndeeltjes en met elkaar worden bepaald door de spontaan gebroken chirale symmetrie. De symmetrie werkt opeen speciale manier op de pionen: deze is *niet-lineair gerealizeerd*. De methodes die Weinberg en Nambu gebruikten om de sterke kernkracht te beschrijven met Goldstone bosonen zijn verder uitgewerkt door Callan, Coleman, Wess en Zumino (CCWZ). Zij ontwikkelden een algemene theorie die de interacties van Goldstone bosonen op basis van de niet-lineaire symmetrie vastlegt.

Vanwege het grote succes van symmetrieprincipes in de natuurkunde is het belangrijk te begrijpen welke symmetrieën zouden kunnen bestaan. Coleman en Mandula probeerden die vraag te beantwoorden in 1967. Zij leidden af, onder zeer algemene aannames, dat de symmetriegroep van de natuur een direct product moet zijn van de Poincaré transformaties van Einsteins speciale relativiteit en een interne symmetriegroep zoals de chirale symmetrie en de ijkgroep van het standaardmodel. Er is geen ruimte voor *hybride symmetrieën* die de symmetrie van de ruimte-tijd combineren met een interne symmetrie.

Het is mogelijk een deel van de aannames van Coleman en Mandula los te laten. Haag, Lopuszanski en Sohnius verruimden de aanpak van Coleman en Mandula door symmetrieën toe te laten die een *super-Liegroep* vormen in plaats van een standaard Liegroep. Een super-Liegroep kan lokaal worden beschreven door een super-Lie-algebra, waarin de standaard commutatierelaties worden uitgebreid door anti-commutatierelaties. Ook onder de aannames van Haag, Lopuszanski en Sohnius zijn de mogelijkheden beperkt: de symmetrie van Einsteins speciale relativiteit kan alleen worden uitgebreid tot *supersymmetrie*. Tot op heden is geen experimenteel bewijs gevonden voor supersymmetrie in de natuur. Het werk van Coleman en Mandula is ook niet van toepassing op zogeheten *dynamische symmetrieën*. Dit impliceert onder andere dat het mogelijk is om de Poincaré symmetrie van de speciale relativiteit uit te breiden met een *niet-lineair gerealizeerde* ruimte-tijd of hybride symmetrie.

In deze scriptie zullen we onderzoeken welke vormen van niet-lineair gerealizeerde symmetrie mogelijk zijn, op basis van de theorieën van Coleman/Mandula, Haag/Lopuszanski/Sohnius en CCWZ. We leggen de nadruk op niet-lineaire ruimte-tijd of hybride symmetrie. We zullen gebruik maken van algebraïsche methoden. Soortgelijke vragen zijn ook behandeld op basis van verstrooïngsamplitudes. We zullen in de scriptie onze resultaten met dergelijk werk vergelijken.

Uit ons onderzoek blijkt dat niet-lineaire ruimte-tijd symmetrieën erg zeldzaam zijn. In sommige gevallen (afhankelijk van hoeveel en welk type Goldstone velden we gebruiken) kunnen we een volledige lijst geven van alle mogelijkheden. Over theorieën met lineair gerealiseerde Poincaré symmetrie in vier dimensies kunnen we het volgende concluderen:

- Voor een enkel scalair Goldstone veld zijn de mogelijkheden: I) *extended shift symmetrie*, II) dilatatie symmetrie en speciale hoekgetrouwe transformaties, III) de vijf-dimensionale (anti-)Poincaré groep $ISO(1,4)$ of AdS groep $ISO(2,3)$ en de speciale Galileon.

- Voor een willekeurig aantal spin-$\frac{1}{2}$ Goldstino's zijn I) extended shift symmetrie en II) ($\mathcal{N}$-extended)) supersymmetrie mogelijk.

- Voor een enkele ijkvector Goldstone zijn geen extended shift symmetrie of veldafhankelijke (exceptionele) symmetrieën mogelijk.

We hebben deze conclusies afgeleid onder de aanname van *coset universality*. Dit houdt in dat I) de coset constructie, de methode can CCWZ, de unieke niet-lineair gerealiseerde transformaties genereert voor elk patroon van spontaan gebroken symmetrie en II) dat elke spontaan gebroken symmetrie op basis van de CCWZ methode kan worden beschreven. Deze aanname is bewezen voor het geval van interne niet-lineair gerealiseerde symmetrieën in een Poincaré theorie. In deze scriptie zullen we bestuderen in hoeverre we de aanname ook kunnen vertrouwen of verifiëren voor ruimte-tijd en hybride symmetrieën.

We hebben dezelfde aanpak gebruikt om niet-lineaire symmetrieën ook te bestuderen in lineair supersymmetrische theorieën. Voor $\mathcal{N} = 1$ in vier dimensies concluderen we het volgende:

- In het geval van een enkel chiraal Goldstone *superveld* zijn de mogelijkheden: I) (extended) *super*-shift symmetrie. Een dergelijke symmetrie shift het scalaire veld, het fermion, en/of het auxiliaire veld in een chiral superveld. II) minimale $D = 6$ supersymmetrie in vlakke, dan wel AdS ruimte-tijd. Een supersymmetrische versie van de speciale Galileon bestaat echter niet.

- Voor een enkel Maxwell Goldstone superveld is het volgende mogelijk: I) (extended) super-shift symmetrie, II) minimale $\mathcal{N} = 2$ supersymmetrie in vier dimensies.

We beginnen de scriptie met een algemene technische introductie tot symmetrie in de (kwantum)veldentheorie. Hier leggen we onder meer op technisch niveau uit wat het verschil is tussen globale en ijksymmetrieën, ruimte-tijd en interne symmetrieën, algebraïsche en dynamische symmetrieën. Daarna, in hoofdstuk 3, geven we een uitleg van de algemene theorie van CCWZ, de extensie daarvan naar ruimte-tijd symmetrieën, en het inverse Higgs effect. In de laatste twee hoofdstukken geven we een uitgebreide presentatie van onze algebraïsche methode en de resultaten voor Poincaré en supersymmetrische theorieën in vier dimensies.

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